Ergodic theory in Isabelle

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Abstract

Ergodic theory is the branch of mathematics that studies the behaviour of measure preserving transformations, in finite or infinite measure. It interacts both with probability theory (mainly through measure theory) and with geometry as a lot of interesting examples are from geometric origin. We implement the first definitions and theorems of ergodic theory, including notably Poincaré recurrence theorem for finite measure preserving systems (together with the notion of conservativity in general), induced maps, Kac's theorem, Birkhoff theorem (arguably the most important theorem in ergodic theory), and variations around it such as conservativity of the corresponding skew product, or Atkinson lemma, and Kingman theorem. Using this material, we formalize completely the proof of the main theorems of [GK15] and [Gou18].

Contents

1 SG Libary complements

theory *SG-Library-Complement*
imports *HOL−Probability.Probability*
begin

In this file are included many statements that were useful to me, but belong rather naturally to existing theories. In a perfect world, some of these statements would get included into these files.

I tried to indicate to which of these classical theories the statements could be added.

lemma *compl-compl-eq-id* [*simp*]:
$UNIV - (UNIV - s) = s$
⟨*proof*⟩

notation *sym-diff* (**infixl** ‹Δ› *70*)

1.1 Set-Interval.thy

The next two lemmas belong naturally to `Set_Interval.thy`, next to `UN_le_add_shift`. They are not trivially equivalent to the corresponding lemmas with large inequalities, due to the difference when $n = 0$.

lemma *UN-le-eq-Un0-strict*:
 $(\bigcup i<n+1::nat.\ M\ i) = (\bigcup i\in\{1..<n+1\}.\ M\ i) \cup M\ 0$ (**is** $?A = ?B$)
⟨*proof*⟩

I use repeatedly this one, but I could not find it directly

lemma *union-insert-0*:
 $(\bigcup n::nat.\ A\ n) = A\ 0 \cup (\bigcup n\in\{1..\}.\ A\ n)$
⟨*proof*⟩

Next one could be close to `sum.nat_group`

lemma *sum-arith-progression*:
 $(\sum r<(N::nat).\ (\sum i<a.\ f\ (i*N+r))) = (\sum j<a*N.\ f\ j)$
⟨*proof*⟩

1.2 Miscellanous basic results

lemma *ind-from-1* [*case-names 1 Suc*, *consumes 1*]:
 assumes $n > 0$
 assumes $P\ 1$
  and $\bigwedge n.\ n > 0 \Longrightarrow P\ n \Longrightarrow P\ (Suc\ n)$
 shows $P\ n$
⟨*proof*⟩

This lemma is certainly available somewhere, but I couldn't locate it

lemma *tends-to-real-e*:
 fixes $u::nat \Rightarrow real$
 assumes $u \longrightarrow l\ e>0$
 shows $\exists N.\ \forall n>N.\ abs(u\ n\ -l) < e$
 ⟨*proof*⟩

lemma *nat-mod-cong*:
 assumes $a = b+(c::nat)$
  $a\ mod\ n = b\ mod\ n$
 shows $c\ mod\ n = 0$
⟨*proof*⟩

lemma *funpow-add'*: $(f\ \hat{}\hat{}\ (m + n))\ x = (f\ \hat{}\hat{}\ m)\ ((f\ \hat{}\hat{}\ n)\ x)$
⟨*proof*⟩

The next two lemmas are not directly equivalent, since f might not be injective.

lemma *abs-Max-sum*:
 fixes $A::real\ set$

assumes *finite A* $A \neq \{\}$
shows $abs(Max\ A) \leq (\sum a \in A.\ abs(a))$
⟨*proof*⟩

lemma *abs-Max-sum2*:
fixes f::- $\Rightarrow$ *real*
assumes *finite A* $A \neq \{\}$
shows $abs(Max\ (f`A)) \leq (\sum a \in A.\ abs(f\ a))$
⟨*proof*⟩

1.3 Conditionally-Complete-Lattices.thy

lemma *mono-cInf*:
fixes f :: *'a::conditionally-complete-lattice* $\Rightarrow$ *'b::conditionally-complete-lattice*
assumes *mono f* $A \neq \{\}$ *bdd-below A*
shows $f(Inf\ A) \leq Inf\ (f`A)$
⟨*proof*⟩

lemma *mono-bij-cInf*:
fixes f :: *'a::conditionally-complete-linorder* $\Rightarrow$ *'b::conditionally-complete-linorder*
assumes *mono f bij f* $A \neq \{\}$ *bdd-below A*
shows $f\ (Inf\ A) = Inf\ (f`A)$
⟨*proof*⟩

1.4 Topological-spaces.thy

lemma *open-less-abs* [*simp*]:
open $\{x.\ (C::real) < abs\ x\}$
⟨*proof*⟩

lemma *closed-le-abs* [*simp*]:
closed $\{x.\ (C::real) \leq abs\ x\}$
⟨*proof*⟩

The next statements come from the same statements for true subsequences

lemma *eventually-weak-subseq*:
fixes u::*nat* $\Rightarrow$ *nat*
assumes $(\lambda n.\ real(u\ n)) \longrightarrow \infty$ *eventually P sequentially*
shows *eventually* $(\lambda n.\ P\ (u\ n))$ *sequentially*
⟨*proof*⟩

lemma *filterlim-weak-subseq*:
fixes u::*nat* $\Rightarrow$ *nat*
assumes $(\lambda n.\ real(u\ n)) \longrightarrow \infty$
shows *LIM n sequentially. u n*:> *at-top*
⟨*proof*⟩

lemma *limit-along-weak-subseq*:
fixes u::*nat* $\Rightarrow$ *nat* **and** v::*nat* $\Rightarrow$ -

assumes $(\lambda n.\ real(u\ n)) \longrightarrow \infty\ v \longrightarrow l$
shows $(\lambda\ n.\ v(u\ n)) \longrightarrow l$
⟨*proof*⟩

lemma *frontier-indist-le*:
assumes $x \in frontier\ \{y.\ infdist\ y\ S \leq r\}$
shows $infdist\ x\ S = r$
⟨*proof*⟩

1.5 Limits

The next lemmas are not very natural, but I needed them several times

lemma *tendsto-shift-1-over-n* [*tendsto-intros*]:
fixes $f::nat \Rightarrow real$
assumes $(\lambda n.\ f\ n\ /\ n) \longrightarrow l$
shows $(\lambda n.\ f\ (n+k)\ /\ n) \longrightarrow l$
⟨*proof*⟩

lemma *tendsto-shift-1-over-n'* [*tendsto-intros*]:
fixes $f::nat \Rightarrow real$
assumes $(\lambda n.\ f\ n\ /\ n) \longrightarrow l$
shows $(\lambda n.\ f\ (n-k)\ /\ n) \longrightarrow l$
⟨*proof*⟩

declare *LIMSEQ-realpow-zero* [*tendsto-intros*]

1.6 Topology-Euclidean-Space

A (more usable) variation around `continuous_on_closure_sequentially`. The assumption that the spaces are metric spaces is definitely too strong, but sufficient for most applications.

lemma *continuous-on-closure-sequentially'*:
fixes $f::'a::metric\text{-}space \Rightarrow 'b::metric\text{-}space$
assumes $continuous\text{-}on\ (closure\ C)\ f$
$\bigwedge(n::nat).\ u\ n \in C$
$u \longrightarrow l$
shows $(\lambda n.\ f\ (u\ n)) \longrightarrow f\ l$
⟨*proof*⟩

1.7 Convexity

lemma *convex-on-mean-ineq*:
fixes $f::real \Rightarrow real$
assumes $convex\text{-}on\ A\ f\ x \in A\ y \in A$
shows $f\ ((x+y)/2) \leq (f\ x + f\ y)\ /\ 2$
⟨*proof*⟩

lemma *convex-on-closure*:

fixes C :: $'a$::*real-normed-vector set*
assumes *convex C*
convex-on C f
continuous-on (*closure C*) *f*
shows *convex-on* (*closure C*) *f*
⟨*proof*⟩

lemma *convex-on-norm* [*simp*]:
convex-on UNIV (λ(*x*::$'a$::*real-normed-vector*). *norm x*)
⟨*proof*⟩

lemma *continuous-abs-powr* [*continuous-intros*]:
assumes $p > 0$
shows *continuous-on UNIV* (λ(*x*::*real*). |*x*| *powr p*)
⟨*proof*⟩

lemma *continuous-mult-sgn* [*continuous-intros*]:
fixes *f*::*real* ⇒ *real*
assumes *continuous-on UNIV f f 0* = *0*
shows *continuous-on UNIV* (λ*x*. *sgn x* ∗ *f x*)
⟨*proof*⟩

lemma *DERIV-abs-powr* [*derivative-intros*]:
assumes *p* > (*1*::*real*)
shows *DERIV* (λ*x*. |*x*| *powr p*) *x* :> *p* ∗ *sgn x* ∗ |*x*| *powr* (*p* − *1*)
⟨*proof*⟩

lemma *convex-abs-powr*:
assumes $p \geq 1$
shows *convex-on UNIV* (λ*x*::*real*. |*x*| *powr p*)
⟨*proof*⟩

lemma *convex-powr*:
assumes $p \geq 1$
shows *convex-on* {*0*..} (λ*x*::*real*. *x powr p*)
⟨*proof*⟩

lemma *convex-powr'*:
assumes $p > 0$ $p \leq 1$
shows *convex-on* {*0*..} (λ*x*::*real*. − (*x powr p*))
⟨*proof*⟩

lemma *convex-fx-plus-fy-ineq*:
fixes *f*::*real* ⇒ *real*
assumes *convex-on* {*0*..} *f*
$x \geq 0$ $y \geq 0$ $f\ 0 = 0$
shows $f\ x + f\ y \leq f\ (x+y)$
⟨*proof*⟩

lemma *x-plus-y-p-le-xp-plus-yp*:
  fixes $p\ x\ y::real$
  assumes $p > 0\ p \leq 1\ x \geq 0\ y \geq 0$
  shows $(x + y)$ *powr* $p \leq x$ *powr* $p + y$ *powr* p
⟨*proof*⟩

1.8 Nonnegative-extended-real.thy

lemma *x-plus-top-ennreal* [*simp*]:
  $x + \top = (\top::ennreal)$
⟨*proof*⟩

lemma *ennreal-ge-nat-imp-PInf*:
  fixes $x::ennreal$
  assumes $\bigwedge N.\ x \geq$ *of-nat* N
  shows $x = \infty$
⟨*proof*⟩

lemma *ennreal-archimedean*:
  assumes $x \neq (\infty::ennreal)$
  shows $\exists\, n::nat.\ x \leq n$
  ⟨*proof*⟩

lemma *e2ennreal-mult*:
  fixes $a\ b::ereal$
  assumes $a \geq 0$
  shows *e2ennreal*$(a * b)$ = *e2ennreal* a * *e2ennreal* b
⟨*proof*⟩

lemma *e2ennreal-mult′*:
  fixes $a\ b::ereal$
  assumes $b \geq 0$
  shows *e2ennreal*$(a * b)$ = *e2ennreal* a * *e2ennreal* b
⟨*proof*⟩

lemma *SUP-real-ennreal*:
  assumes $A \neq \{\}$ *bdd-above* $(f`A)$
  shows $(SUP\ a\in A.\ ennreal\ (f\ a)) = ennreal(SUP\ a\in A.\ f\ a)$
⟨*proof*⟩

lemma *e2ennreal-Liminf*:
  $F \neq bot \Longrightarrow e2ennreal\ (Liminf\ F\ f) = Liminf\ F\ (\lambda n.\ e2ennreal\ (f\ n))$
  ⟨*proof*⟩

lemma *e2ennreal-eq-infty*[*simp*]: $0 \leq x \Longrightarrow e2ennreal\ x = top \longleftrightarrow x = \infty$
  ⟨*proof*⟩

lemma *ennreal-Inf-cmult*:
  assumes $c>(0::real)$

shows $Inf\ \{ennreal\ c * x\ |x.\ P\ x\} = ennreal\ c * Inf\ \{x.\ P\ x\}$
⟨*proof*⟩

lemma *continuous-on-const-minus-ennreal*:
fixes $f :: 'a :: topological\text{-}space \Rightarrow ennreal$
shows $continuous\text{-}on\ A\ f \Longrightarrow continuous\text{-}on\ A\ (\lambda x.\ a - f\ x)$
including *ennreal.lifting*
⟨*proof*⟩

lemma *const-minus-Liminf-ennreal*:
fixes $a :: ennreal$
shows $F \neq bot \Longrightarrow a - Liminf\ F\ f = Limsup\ F\ (\lambda x.\ a - f\ x)$
⟨*proof*⟩

lemma *tendsto-cmult-ennreal* [*tendsto-intros*]:
fixes $c\ l::ennreal$
assumes $\neg(c = \infty \wedge l = 0)$
$(f \longrightarrow l)\ F$
shows $((\lambda x.\ c * f\ x) \longrightarrow c * l)\ F$
⟨*proof*⟩

1.9 Indicator-Function.thy

There is something weird with `sum_mult_indicator`: it is defined both in Indicator.thy and BochnerIntegration.thy, with a different meaning. I am surprised there is no name collision... Here, I am using the version from BochnerIntegration.

lemma *sum-indicator-eq-card2*:
assumes *finite I*
shows $(\sum i{\in}I.\ (indicator\ (P\ i)\ x)::nat) = card\ \{i{\in}I.\ x \in P\ i\}$
⟨*proof*⟩

lemma *disjoint-family-indicator-le-1*:
assumes *disjoint-family-on A I*
shows $(\sum\ i{\in}\ I.\ indicator\ (A\ i)\ x) \leq (1::'a::\ \{comm\text{-}monoid\text{-}add,zero\text{-}less\text{-}one\})$
⟨*proof*⟩

1.10 sigma-algebra.thy

lemma *algebra-intersection*:
assumes $algebra\ \Omega\ A$
$algebra\ \Omega\ B$
shows $algebra\ \Omega\ (A \cap B)$
⟨*proof*⟩

lemma *sigma-algebra-intersection*:
assumes $sigma\text{-}algebra\ \Omega\ A$
$sigma\text{-}algebra\ \Omega\ B$

shows *sigma-algebra* Ω $(A \cap B)$
$\langle proof \rangle$

lemma *subalgebra-M-M* [*simp*]:
subalgebra M M
$\langle proof \rangle$

The next one is `disjoint_family_Suc` with inclusions reversed.

lemma *disjoint-family-Suc2*:
assumes *Suc*: $\bigwedge n.\ A\ (Suc\ n) \subseteq A\ n$
shows *disjoint-family* $(\lambda i.\ A\ i - A\ (Suc\ i))$
$\langle proof \rangle$

1.11 Measure-Space.thy

lemma *AE-equal-sum*:
assumes $\bigwedge i.\ AE\ x\ in\ M.\ f\ i\ x = g\ i\ x$
shows $AE\ x\ in\ M.\ (\sum i \in I.\ f\ i\ x) = (\sum i \in I.\ g\ i\ x)$
$\langle proof \rangle$

lemma *emeasure-pos-unionE*:
assumes $\bigwedge\ (N::nat).\ A\ N \in sets\ M$
$emeasure\ M\ (\bigcup N.\ A\ N) > 0$
shows $\exists N.\ emeasure\ M\ (A\ N) > 0$
$\langle proof \rangle$

lemma (**in** *prob-space*) *emeasure-intersection*:
fixes $e::nat \Rightarrow real$
assumes [*measurable*]: $\bigwedge n.\ U\ n \in sets\ M$
and [*simp*]: $\bigwedge n.\ 0 \le e\ n\ summable\ e$
and *ge*: $\bigwedge n.\ emeasure\ M\ (U\ n) \ge 1 - (e\ n)$
shows $emeasure\ M\ (\bigcap n.\ U\ n) \ge 1 - (\sum n.\ e\ n)$
$\langle proof \rangle$

lemma *null-sym-diff-transitive*:
assumes $A\ \Delta\ B \in null\text{-}sets\ M\ B\ \Delta\ C \in null\text{-}sets\ M$
and [*measurable*]: $A \in sets\ M\ C \in sets\ M$
shows $A\ \Delta\ C \in null\text{-}sets\ M$
$\langle proof \rangle$

lemma *Delta-null-of-null-is-null*:
assumes $B \in sets\ M\ A\ \Delta\ B \in null\text{-}sets\ M\ A \in null\text{-}sets\ M$
shows $B \in null\text{-}sets\ M$
$\langle proof \rangle$

lemma *Delta-null-same-emeasure*:
assumes $A\ \Delta\ B \in null\text{-}sets\ M$ **and** [*measurable*]: $A \in sets\ M\ B \in sets\ M$
shows $emeasure\ M\ A = emeasure\ M\ B$
$\langle proof \rangle$

lemma *AE-upper-bound-inf-ereal*:
fixes $F\ G{::}'a \Rightarrow ereal$
assumes $\bigwedge e.\ (e{::}real) > 0 \Longrightarrow AE\ x\ in\ M.\ F\ x \leq G\ x + e$
shows $AE\ x\ in\ M.\ F\ x \leq G\ x$
$\langle proof \rangle$

Egorov theorem asserts that, if a sequence of functions converges almost everywhere to a limit, then the convergence is uniform on a subset of close to full measure. The first step in the proof is the following lemma, often useful by itself, asserting the same result for predicates: if a property $P_n x$ is eventually true for almost every x, then there exists N such that $P_n x$ is true for all $n \geq N$ and all x in a set of close to full measure.

lemma (**in** *finite-measure*) *Egorov-lemma*:
assumes [*measurable*]: $\bigwedge n.\ (P\ n) \in measurable\ M\ (count\text{-}space\ UNIV)$
and $AE\ x\ in\ M.\ eventually\ (\lambda n.\ P\ n\ x)\ sequentially$
$epsilon > 0$
shows $\exists\, U\ N.\ U \in sets\ M \wedge (\forall\, n \geq N.\ \forall\, x \in U.\ P\ n\ x) \wedge emeasure\ M\ (space\ M - U) < epsilon$
$\langle proof \rangle$

The next lemma asserts that, in an uncountable family of disjoint sets, then there is one set with zero measure (and in fact uncountably many). It is often applied to the boundaries of r-neighborhoods of a given set, to show that one could choose r for which this boundary has zero measure (this shows up often in relation with weak convergence).

lemma (**in** *finite-measure*) *uncountable-disjoint-family-then-exists-zero-measure*:
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow A\ i \in sets\ M$
and *uncountable I*
disjoint-family-on A I
shows $\exists\, i{\in}I.\ measure\ M\ (A\ i) = 0$
$\langle proof \rangle$

The next statements are useful measurability statements.

lemma *measurable-Inf* [*measurable*]:
assumes [*measurable*]: $\bigwedge(n{::}nat).\ P\ n \in measurable\ M\ (count\text{-}space\ UNIV)$
shows $(\lambda x.\ Inf\ \{n.\ P\ n\ x\}) \in measurable\ M\ (count\text{-}space\ UNIV)$ (**is** $?f \in$ -)
$\langle proof \rangle$

lemma *measurable-T-iter* [*measurable*]:
fixes $f{::}'a \Rightarrow nat$
assumes [*measurable*]: $T \in measurable\ M\ M$
$f \in measurable\ M\ (count\text{-}space\ UNIV)$
shows $(\lambda x.\ (T \widehat{\ }\widehat{\ } (f\ x))\ x) \in measurable\ M\ M$
$\langle proof \rangle$

lemma *measurable-infdist* [*measurable*]:
$(\lambda x.\ infdist\ x\ S) \in borel\text{-}measurable\ borel$

⟨*proof*⟩

The next lemma shows that, in a sigma finite measure space, sets with large measure can be approximated by sets with large but finite measure.

lemma (**in** *sigma-finite-measure*) *approx-with-finite-emeasure*:
  assumes *W-meas*: $W \in \mathit{sets}\ M$
    and *W-inf*: $\mathit{emeasure}\ M\ W > C$
  obtains Z **where** $Z \in \mathit{sets}\ M\ Z \subseteq W\ \mathit{emeasure}\ M\ Z < \infty\ \mathit{emeasure}\ M\ Z > C$
⟨*proof*⟩

1.12 Nonnegative-Lebesgue-Integration.thy

The next lemma is a variant of `nn_integral_density`, with the density on the right instead of the left, as seems more common.

lemma *nn-integral-densityR*:
  assumes [*measurable*]: $f \in \mathit{borel\text{-}measurable}\ F\ g \in \mathit{borel\text{-}measurable}\ F$
  shows $(\int^+ x.\ f\ x * g\ x\ \partial F) = (\int^+ x.\ f\ x\ \partial(\mathit{density}\ F\ g))$
⟨*proof*⟩

lemma *not-AE-zero-int-ennreal-E*:
  fixes $f{::}'a \Rightarrow \mathit{ennreal}$
  assumes $(\int^+ x.\ f\ x\ \partial M) > 0$
    and [*measurable*]: $f \in \mathit{borel\text{-}measurable}\ M$
  shows $\exists A{\in}\mathit{sets}\ M.\ \exists e{::}\mathit{real}{>}0.\ \mathit{emeasure}\ M\ A > 0 \wedge (\forall x \in A.\ f\ x \geq e)$
⟨*proof*⟩

lemma (**in** *finite-measure*) *nn-integral-bounded-eq-bound-then-AE*:
  assumes $\mathit{AE}\ x\ \mathit{in}\ M.\ f\ x \leq \mathit{ennreal}\ c\ (\int^+ x.\ f\ x\ \partial M) = c * \mathit{emeasure}\ M\ (\mathit{space}\ M)$
    and [*measurable*]: $f \in \mathit{borel\text{-}measurable}\ M$
  shows $\mathit{AE}\ x\ \mathit{in}\ M.\ f\ x = c$
⟨*proof*⟩

lemma *null-sets-density*:
  assumes [*measurable*]: $h \in \mathit{borel\text{-}measurable}\ M$
    and $\mathit{AE}\ x\ \mathit{in}\ M.\ h\ x \neq 0$
  shows $\mathit{null\text{-}sets}\ (\mathit{density}\ M\ h) = \mathit{null\text{-}sets}\ M$
⟨*proof*⟩

The next proposition asserts that, if a function h is integrable, then its integral on any set with small enough measure is small. The good conceptual proof is by considering the distribution of the function h on $\mathbb{R}$ and looking at its tails. However, there is a less conceptual but more direct proof, based on dominated convergence and a proof by contradiction. This is the proof we give below.

proposition *integrable-small-integral-on-small-sets*:

fixes $h::'a \Rightarrow real$
assumes [*measurable*]: *integrable M h*
and $delta > 0$
shows $\exists\, epsilon>(0::real).\ \forall\, U \in sets\ M.\ emeasure\ M\ U < epsilon \longrightarrow abs\ (\int x\in U.\ h\ x\ \partial M) < delta$
⟨*proof*⟩

We also give the version for nonnegative ennreal valued functions. It follows from the previous one.

proposition *small-nn-integral-on-small-sets*:
fixes $h::'a \Rightarrow ennreal$
assumes [*measurable*]: $h \in borel\text{-}measurable\ M$
and $delta > (0::real)\ (\int^+ x.\ h\ x\ \partial M) \neq \infty$
shows $\exists\, epsilon>(0::real).\ \forall\, U \in sets\ M.\ emeasure\ M\ U < epsilon \longrightarrow (\int^+ x\in U.\ h\ x\ \partial M) < delta$
⟨*proof*⟩

1.13 Probability-measure.thy

The next lemmas ensure that, if sets have a probability close to 1, then their intersection also does.

lemma (**in** *prob-space*) *sum-measure-le-measure-inter*:
assumes $A \in sets\ M\ B \in sets\ M$
shows $prob\ A + prob\ B \leq 1 + prob\ (A \cap B)$
⟨*proof*⟩

lemma (**in** *prob-space*) *sum-measure-le-measure-inter3*:
assumes [*measurable*]: $A \in sets\ M\ B \in sets\ M\ C \in sets\ M$
shows $prob\ A + prob\ B + prob\ C \leq 2 + prob\ (A \cap B \cap C)$
⟨*proof*⟩

lemma (**in** *prob-space*) *sum-measure-le-measure-Inter*:
assumes [*measurable*]: $finite\ I\ I \neq \{\}\ \bigwedge i.\ i \in I \Longrightarrow A\ i \in sets\ M$
shows $(\sum i\in I.\ prob\ (A\ i)) \leq real(card\ I) - 1 + prob\ (\bigcap i\in I.\ A\ i)$
⟨*proof*⟩

A random variable gives a small mass to small neighborhoods of infinity.

lemma (**in** *prob-space*) *random-variable-small-tails*:
assumes $alpha > 0$ **and** [*measurable*]: $f \in borel\text{-}measurable\ M$
shows $\exists (C::real).\ prob\ \{x \in space\ M.\ abs(f\ x) \geq C\} < alpha \wedge C \geq K$
⟨*proof*⟩

1.14 Distribution-functions.thy

There is a locale called `finite_borel_measure` in `distribution-functions.thy`. However, it only deals with real measures, and real weak convergence. I will not need the weak convergence in more general settings, but still it seems more natural to me to do the proofs in the natural settings. Let me introduce the locale `finite_borel_measure'` for this, although it would be better to rename the locale in the library file.

locale *finite-borel-measure'* = *finite-measure M* **for** *M* :: (*'a*::*metric-space*) *measure* +
  assumes *M-is-borel* [*simp*, *measurable-cong*]: *sets M* = *sets borel*
begin

lemma *space-eq-univ* [*simp*]: *space M* = *UNIV*
  ⟨*proof*⟩

lemma *measurable-finite-borel* [*simp*]:
  $f \in$ *borel-measurable borel* $\Longrightarrow$ $f \in$ *borel-measurable M*
  ⟨*proof*⟩

Any closed set can be slightly enlarged to obtain a set whose boundary has 0 measure.

lemma *approx-closed-set-with-set-zero-measure-boundary*:
  assumes *closed S epsilon* > *0 S* ≠ {}
  shows ∃ *r*. *r* < *epsilon* ∧ *r* > *0* ∧ *measure M* {*x*. *infdist x S* = *r*} = *0* ∧ *measure M* {*x*. *infdist x S* ≤ *r*} < *measure M S* + *epsilon*
⟨*proof*⟩
end

sublocale *finite-borel-measure* ⊆ *finite-borel-measure'*
  ⟨*proof*⟩

1.15 Weak-convergence.thy

Since weak convergence is not implemented as a topology, the fact that the convergence of a sequence implies the convergence of a subsequence is not automatic. We prove it in the lemma below..

lemma *weak-conv-m-subseq*:
  assumes *weak-conv-m M-seq M strict-mono r*
  shows *weak-conv-m* (λ*n*. *M-seq* (*r n*)) *M*
⟨*proof*⟩

context
  fixes μ :: *nat* ⇒ *real measure*
    and *M* :: *real measure*
  assumes μ: ⋀*n*. *real-distribution* (μ *n*)
  assumes *M*: *real-distribution M*
  assumes μ*-to-M*: *weak-conv-m* μ *M*
begin

The measure of a closed set behaves upper semicontinuously with respect to weak convergence: if $\mu_n \to \mu$, then $\limsup \mu_n(F) \leq \mu(F)$ (and the inequality can be strict, think of the situation where μ is a Dirac mass at 0 and $F = \{0\}$, but μ_n has a density so that $\mu_n(\{0\}) = 0$).

lemma *closed-set-weak-conv-usc*:
  assumes *closed S measure M S* $<$ *l*
  shows *eventually* ($\lambda n.$ *measure* (μ *n*) *S* $<$ *l*) *sequentially*
⟨*proof*⟩

In the same way, the measure of an open set behaves lower semicontinuously with respect to weak convergence: if $\mu_n \to \mu$, then $\liminf \mu_n(U) \geq \mu(U)$ (and the inequality can be strict). This follows from the same statement for closed sets by passing to the complement.

lemma *open-set-weak-conv-lsc*:
  assumes *open S measure M S* $>$ *l*
  shows *eventually* ($\lambda n.$ *measure* (μ *n*) *S* $>$ *l*) *sequentially*
⟨*proof*⟩

end

end

theory *ME-Library-Complement*
  imports *HOL−Analysis.Analysis*
begin

1.16 The trivial measurable space

The trivial measurable space is the smallest possible σ-algebra, i.e. only the empty set and everything.

definition *trivial-measure* :: *'a set* ⇒ *'a measure* **where**
  trivial-measure X = *sigma X* {{}, *X*}

lemma *space-trivial-measure* [*simp*]: *space* (*trivial-measure X*) = *X*
  ⟨*proof*⟩

lemma *sets-trivial-measure*: *sets* (*trivial-measure X*) = {{}, *X*}
  ⟨*proof*⟩

lemma *measurable-trivial-measure*:
  assumes $f \in$ *space M* $\to X$ **and** f −‘ $X \cap$ *space M* $\in$ *sets M*
  shows $f \in M \to_M$ *trivial-measure X*
  ⟨*proof*⟩

lemma *measurable-trivial-measure-iff*:
  $f \in M \to_M$ *trivial-measure X* $\longleftrightarrow f \in$ *space M* $\to X \wedge f$ −‘ $X \cap$ *space M* $\in$ *sets M*
  ⟨*proof*⟩

1.17 Pullback algebras

The pullback algebra $f^{-1}(\Sigma)$ of a σ-algebra (Ω, Σ) is the smallest σ-algebra such that f is $f^{-1}(\Sigma) - -\Sigma$-measurable.

definition (**in** *sigma-algebra*) *pullback-algebra* :: $('b \Rightarrow 'a) \Rightarrow 'b\ set \Rightarrow 'b\ set\ set$
where
pullback-algebra $f\ \Omega' =$ *sigma-sets* $\Omega'\ \{f -`\ A \cap \Omega'\ |A.\ A \in M\}$

lemma *pullback-algebra-minimal*:
assumes $f \in M \rightarrow_M N$
shows *sets.pullback-algebra* $N\ f$ (*space* M) $\subseteq$ *sets* M
⟨*proof*⟩

lemma (**in** *sigma-algebra*) *in-pullback-algebra*: $A \in M \Longrightarrow f -`\ A \cap \Omega' \in$ *pullback-algebra* $f\ \Omega'$
⟨*proof*⟩

end

2 Subadditive and submultiplicative sequences

theory *Fekete*
imports *HOL−Analysis.Multivariate-Analysis*
begin

A real sequence is subadditive if $u_{n+m} \leq u_n + u_m$. This implies the convergence of u_n/n to $Inf\{u_n/n\} \in [-\infty, +\infty)$, a useful result known as Fekete lemma. We prove it below.

Taking logarithms, the same result applies to submultiplicative sequences. We illustrate it with the definition of the spectral radius as the limit of $\|x^n\|^{1/n}$, the convergence following from Fekete lemma.

2.1 Subadditive sequences

We define subadditive sequences, either from the start or eventually.

definition *subadditive*::(*nat*⇒*real*) ⇒ *bool*
where *subadditive* $u = (\forall\ m\ n.\ u\ (m+n) \leq u\ m + u\ n)$

lemma *subadditiveI*:
assumes $\bigwedge m\ n.\ u\ (m+n) \leq u\ m + u\ n$
shows *subadditive* u
⟨*proof*⟩

lemma *subadditiveD*:
assumes *subadditive* u
shows $u\ (m+n) \leq u\ m + u\ n$
⟨*proof*⟩

lemma *subadditive-un-le-nu1*:
  assumes *subadditive u*
    $n > 0$
  shows $u\ n \leq n * u\ 1$
⟨*proof*⟩

definition *eventually-subadditive*::(*nat*⇒*real*) ⇒ *nat* ⇒ *bool*
  where *eventually-subadditive u N0* = ($\forall m>N0.\ \forall n>N0.\ u\ (m+n) \leq u\ m + u\ n$)

lemma *eventually-subadditiveI*:
  assumes $\bigwedge m\ n.\ m > N0 \Longrightarrow n > N0 \Longrightarrow u\ (m+n) \leq u\ m + u\ n$
  shows *eventually-subadditive u N0*
⟨*proof*⟩

lemma *subadditive-imp-eventually-subadditive*:
  assumes *subadditive u*
  shows *eventually-subadditive u 0*
⟨*proof*⟩

The main inequality that will lead to convergence is given in the next lemma: given n, then eventually u_m/m is bounded by u_n/n, up to an arbitrarily small error. This is proved by doing the euclidean division of m by n and using the subadditivity. (the remainder in the euclidean division will give the error term.)

lemma *eventually-subadditive-ineq*:
  assumes *eventually-subadditive u N0* $e>0$ $n>N0$
  shows $\exists N>N0.\ \forall m\geq N.\ u\ m/m < u\ n/n + e$
⟨*proof*⟩

From the inequality above, we deduce the convergence of u_n/n to its infimum. As this infimum might be $-\infty$, we formulate this convergence in the extended reals. Then, we specialize it to the real situation, separating the cases where u_n/n is bounded below or not.

lemma *subadditive-converges-ereal'*:
  assumes *eventually-subadditive u N0*
  shows ($\lambda m.$ *ereal*($u\ m/m$)) $\longrightarrow$ *Inf* {*ereal*($u\ n/n$) | $n.\ n>N0$}
⟨*proof*⟩

lemma *subadditive-converges-ereal*:
  assumes *subadditive u*
  shows ($\lambda m.$ *ereal*($u\ m/m$)) $\longrightarrow$ *Inf* {*ereal*($u\ n/n$) | $n.\ n>0$}
⟨*proof*⟩

lemma *subadditive-converges-bounded'*:
  assumes *eventually-subadditive u N0*
    bdd-below {$u\ n/n$ | $n.\ n>N0$}

shows $(\lambda n.\ u\ n/n) \longrightarrow \mathit{Inf}\ \{u\ n/n \mid n.\ n{>}N0\}$
$\langle \mathit{proof} \rangle$

lemma *subadditive-converges-bounded*:
assumes *subadditive u*
$\mathit{bdd\text{-}below}\ \{u\ n/n \mid n.\ n{>}0\}$
shows $(\lambda n.\ u\ n/n) \longrightarrow \mathit{Inf}\ \{u\ n/n \mid n.\ n{>}0\}$
$\langle \mathit{proof} \rangle$

We reformulate the previous lemma in a more directly usable form, avoiding the infimum.

lemma *subadditive-converges-bounded''*:
assumes *subadditive u*
$\bigwedge n.\ n > 0 \Longrightarrow u\ n \geq n * (a{::}\mathit{real})$
shows $\exists\, l.\ (\lambda n.\ u\ n\ /\ n) \longrightarrow l \wedge (\forall\, n{>}0.\ u\ n \geq n * l)$
$\langle \mathit{proof} \rangle$

lemma *subadditive-converges-unbounded'*:
assumes *eventually-subadditive u N0*
$\neg\ (\mathit{bdd\text{-}below}\ \{u\ n/n \mid n.\ n{>}N0\})$
shows $(\lambda n.\ \mathit{ereal}(u\ n/n)) \longrightarrow -\infty$
$\langle \mathit{proof} \rangle$

lemma *subadditive-converges-unbounded*:
assumes *subadditive u*
$\neg\ (\mathit{bdd\text{-}below}\ \{u\ n/n \mid n.\ n{>}0\})$
shows $(\lambda n.\ \mathit{ereal}(u\ n/n)) \longrightarrow -\infty$
$\langle \mathit{proof} \rangle$

2.2 Superadditive sequences

While most applications involve subadditive sequences, one sometimes encounters superadditive sequences. We reformulate quickly some of the above results in this setting.

definition $\mathit{superadditive}{::}(\mathit{nat}{\Rightarrow}\mathit{real}) \Rightarrow \mathit{bool}$
where $\mathit{superadditive}\ u = (\forall\, m\ n.\ u\ (m{+}n) \geq u\ m + u\ n)$

lemma *subadditive-of-superadditive*:
assumes *superadditive u*
shows $\mathit{subadditive}\ (\lambda n.\ -u\ n)$
$\langle \mathit{proof} \rangle$

lemma *superadditive-un-ge-nu1*:
assumes *superadditive u*
$n > 0$
shows $u\ n \geq n * u\ 1$
$\langle \mathit{proof} \rangle$

lemma *superadditive-converges-bounded''*:
assumes *superadditive u*
$\bigwedge n.\ n > 0 \Longrightarrow u\ n \le n * (a{::}real)$
shows $\exists\, l.\ (\lambda n.\ u\ n\ /\ n) \longrightarrow l \wedge (\forall\, n{>}0.\ u\ n \le n * l)$
$\langle proof \rangle$

2.3 Almost additive sequences

One often encounters sequences which are both subadditive and superadditive, but only up to an additive constant. Adding or subtracting this constant, one can make the sequence genuinely subadditive or superadditive, and thus deduce results about its convergence, as follows. Such sequences appear notably when dealing with quasimorphisms.

lemma *almost-additive-converges*:
fixes $u{::}nat \Rightarrow real$
assumes $\bigwedge m\ n.\ abs(u(m{+}n) - u\ m - u\ n) \le C$
shows *convergent* $(\lambda n.\ u\ n/n)$
$abs(u\ k - k * lim\ (\lambda n.\ u\ n\ /\ n)) \le C$
$\langle proof \rangle$

2.4 Submultiplicative sequences, application to the spectral radius

In the same way as subadditive sequences, one may define submultiplicative sequences. Essentially, a sequence is submultiplicative if its logarithm is subadditive. A difference is that we allow a submultiplicative sequence to take the value 0, as this shows up in applications. This implies that we have to distinguish in the proofs the situations where the value 0 is taken or not. In the latter situation, we can use directly the results from the subadditive case to deduce convergence. In the former situation, convergence to 0 is obvious as the sequence vanishes eventually.

lemma *submultiplicative-converges*:
fixes $u{::}nat{\Rightarrow}real$
assumes $\bigwedge n.\ u\ n \ge 0$
$\bigwedge m\ n.\ u\ (m{+}n) \le u\ m * u\ n$
shows $(\lambda n.\ root\ n\ (u\ n)) \longrightarrow Inf\ \{root\ n\ (u\ n)\ |\ n.\ n{>}0\}$
$\langle proof \rangle$

An important application of submultiplicativity is to prove the existence of the spectral radius of a matrix, as the limit of $\|A^n\|^{1/n}$.

definition $spectral\text{-}radius{::}'a{::}real\text{-}normed\text{-}algebra\text{-}1 \Rightarrow real$
where $spectral\text{-}radius\ x = Inf\ \{root\ n\ (norm(x\hat{\ }n))|\ n.\ n{>}0\}$

lemma *spectral-radius-aux*:
fixes $x{::}'a{::}real\text{-}normed\text{-}algebra\text{-}1$
defines $V \equiv \{root\ n\ (norm(x\hat{\ }n))|\ n.\ n{>}0\}$

shows $\bigwedge t.\ t \in V \Longrightarrow t \geq$ *spectral-radius x*
$\bigwedge t.\ t \in V \Longrightarrow t \geq 0$
bdd-below V
$V \neq \{\}$
Inf $V \geq 0$
⟨*proof*⟩

lemma *spectral-radius-nonneg* [*simp*]:
spectral-radius $x \geq 0$
⟨*proof*⟩

lemma *spectral-radius-upper-bound* [*simp*]:
(*spectral-radius x*)$\hat{}n \leq$ *norm*($x\hat{}n$)
⟨*proof*⟩

lemma *spectral-radius-limit*:
($\lambda n.$ *root n* (*norm*($x\hat{}n$))) $\longrightarrow$ *spectral-radius x*
⟨*proof*⟩

end

3 Asymptotic densities

theory *Asymptotic-Density*
imports *SG-Library-Complement*
begin

The upper asymptotic density of a subset A of the integers is $\limsup Card(A \cap [0, n))/n \in [0, 1]$. It measures how big a set of integers is, at some times. In this paragraph, we establish the basic properties of this notion.

There is a corresponding notion of lower asymptotic density, with a liminf instead of a limsup, measuring how big a set is at all times. The corresponding properties are proved exactly in the same way.

3.1 Upper asymptotic densities

As limsups are only defined for sequences taking values in a complete lattice (here the extended reals), we define it in the extended reals and then go back to the reals. This is a little bit artificial, but it is not a real problem as in the applications we will never come back to this definition.

definition *upper-asymptotic-density*::*nat set* ⇒ *real*
where *upper-asymptotic-density A* = *real-of-ereal*(*limsup* ($\lambda n.$ *card*($A \cap \{..<n\}$)/n))

First basic property: the asymptotic density is between 0 and 1.

lemma *upper-asymptotic-density-in-01*:
ereal(*upper-asymptotic-density A*) = *limsup* ($\lambda n.$ *card*($A \cap \{..<n\}$)/n)

$upper\text{-}asymptotic\text{-}density\ A \leq 1$
$upper\text{-}asymptotic\text{-}density\ A \geq 0$
⟨*proof*⟩

The two next propositions give the usable characterization of the asymptotic density, in terms of the eventual cardinality of $A \cap [0, n)$. Note that the inequality is strict for one implication and large for the other.

proposition *upper-asymptotic-densityD*:
fixes $l::real$
assumes $upper\text{-}asymptotic\text{-}density\ A < l$
shows $eventually\ (\lambda n.\ card(A \cap \{..<n\}) < l * n)\ sequentially$
⟨*proof*⟩

proposition *upper-asymptotic-densityI*:
fixes $l::real$
assumes $eventually\ (\lambda n.\ card(A \cap \{..<n\}) \leq l * n)\ sequentially$
shows $upper\text{-}asymptotic\text{-}density\ A \leq l$
⟨*proof*⟩

The following trivial lemma is useful to control the asymptotic density of unions.

lemma *lem-ge-sum*:
fixes $l\ x\ y::real$
assumes $l>x+y$
shows $\exists\, lx\ ly.\ l = lx + ly \wedge lx > x \wedge ly > y$
⟨*proof*⟩

The asymptotic density of a union is bounded by the sum of the asymptotic densities.

lemma *upper-asymptotic-density-union*:
$upper\text{-}asymptotic\text{-}density\ (A \cup B) \leq upper\text{-}asymptotic\text{-}density\ A + upper\text{-}asymptotic\text{-}density$
B
⟨*proof*⟩

It follows that the asymptotic density is an increasing function for inclusion.

lemma *upper-asymptotic-density-subset*:
assumes $A \subseteq B$
shows $upper\text{-}asymptotic\text{-}density\ A \leq upper\text{-}asymptotic\text{-}density\ B$
⟨*proof*⟩

If a set has a density, then it is also its asymptotic density.

lemma *upper-asymptotic-density-lim*:
assumes $(\lambda n.\ card(A \cap \{..<n\})/n) \longrightarrow l$
shows $upper\text{-}asymptotic\text{-}density\ A = l$
⟨*proof*⟩

If two sets are equal up to something small, i.e. a set with zero upper density, then they have the same upper density.

lemma *upper-asymptotic-density-0-diff*:
assumes $A \subseteq B$ *upper-asymptotic-density* $(B-A) = 0$
shows *upper-asymptotic-density* $A =$ *upper-asymptotic-density* B
⟨*proof*⟩

lemma *upper-asymptotic-density-0-Delta*:
assumes *upper-asymptotic-density* $(A \Delta B) = 0$
shows *upper-asymptotic-density* $A =$ *upper-asymptotic-density* B
⟨*proof*⟩

Finite sets have vanishing upper asymptotic density.

lemma *upper-asymptotic-density-finite*:
assumes *finite* A
shows *upper-asymptotic-density* $A = 0$
⟨*proof*⟩

In particular, bounded intervals have zero upper density.

lemma *upper-asymptotic-density-bdd-interval* [*simp*]:
upper-asymptotic-density $\{\} = 0$
upper-asymptotic-density $\{..N\} = 0$
upper-asymptotic-density $\{..<N\} = 0$
upper-asymptotic-density $\{n..N\} = 0$
upper-asymptotic-density $\{n..<N\} = 0$
upper-asymptotic-density $\{n<..N\} = 0$
upper-asymptotic-density $\{n<..<N\} = 0$
⟨*proof*⟩

The density of a finite union is bounded by the sum of the densities.

lemma *upper-asymptotic-density-finite-Union*:
assumes *finite* I
shows *upper-asymptotic-density* $(\bigcup i{\in}I.\ A\ i) \leq (\sum i{\in}I.$ *upper-asymptotic-density* $(A\ i))$
⟨*proof*⟩

It is sometimes useful to compute the asymptotic density by shifting a little bit the set: this only makes a finite difference that vanishes when divided by n.

lemma *upper-asymptotic-density-shift*:
fixes k::*nat* **and** l::*int*
shows *ereal*(*upper-asymptotic-density* A) = *limsup* $(\lambda n.$ *card*$(A \cap \{k..nat(n+l)\})$ $/\ n)$
⟨*proof*⟩

Upper asymptotic density is measurable.

lemma *upper-asymptotic-density-meas* [*measurable*]:
assumes [*measurable*]: $\bigwedge(n::nat)$. *Measurable.pred* M $(P\ n)$
shows $(\lambda x.$ *upper-asymptotic-density* $\{n.\ P\ n\ x\}) \in$ *borel-measurable* M
⟨*proof*⟩

A finite union of sets with zero upper density still has zero upper density.

lemma *upper-asymptotic-density-zero-union*:
assumes $\textit{upper-asymptotic-density}\ A = 0\ \textit{upper-asymptotic-density}\ B = 0$
shows $\textit{upper-asymptotic-density}\ (A \cup B) = 0$
⟨*proof*⟩

lemma *upper-asymptotic-density-zero-finite-Union*:
assumes $\textit{finite}\ I \ \bigwedge i.\ i \in I \Longrightarrow \textit{upper-asymptotic-density}\ (A\ i) = 0$
shows $\textit{upper-asymptotic-density}\ (\bigcup i{\in}I.\ A\ i) = 0$
⟨*proof*⟩

The union of sets with small asymptotic densities can have a large density: think of $A_n = [0, n]$, it has density 0, but the union of the A_n has density 1. However, if one only wants a set which contains each A_n eventually, then one can obtain a "union" that has essentially the same density as each A_n. This is often used as a replacement for the diagonal argument in density arguments: if for each n one can find a set A_n with good properties and a controlled density, then their "union" will have the same properties (eventually) and a controlled density.

proposition *upper-asymptotic-density-incseq-Union*:
assumes $\bigwedge(n{::}\textit{nat}).\ \textit{upper-asymptotic-density}\ (A\ n) \leq l\ \textit{incseq}\ A$
shows $\exists B.\ \textit{upper-asymptotic-density}\ B \leq l \wedge (\forall n.\ \exists N.\ A\ n \cap \{N..\} \subseteq B)$
⟨*proof*⟩

When the sequence of sets is not increasing, one can only obtain a set whose density is bounded by the sum of the densities.

proposition *upper-asymptotic-density-Union*:
assumes $\textit{summable}\ (\lambda n.\ \textit{upper-asymptotic-density}\ (A\ n))$
shows $\exists B.\ \textit{upper-asymptotic-density}\ B \leq (\sum n.\ \textit{upper-asymptotic-density}\ (A\ n))$
$\wedge (\forall n.\ \exists N.\ A\ n \cap \{N..\} \subseteq B)$
⟨*proof*⟩

A particular case of the previous proposition, often useful, is when all sets have density zero.

proposition *upper-asymptotic-density-zero-Union*:
assumes $\bigwedge n{::}\textit{nat}.\ \textit{upper-asymptotic-density}\ (A\ n) = 0$
shows $\exists B.\ \textit{upper-asymptotic-density}\ B = 0 \wedge (\forall n.\ \exists N.\ A\ n \cap \{N..\} \subseteq B)$
⟨*proof*⟩

3.2 Lower asymptotic densities

The lower asymptotic density of a set of natural numbers is defined just as its upper asymptotic density but using a liminf instead of a limsup. Its properties are proved exactly in the same way.

definition $\textit{lower-asymptotic-density}{::}\textit{nat set} \Rightarrow \textit{real}$
where $\textit{lower-asymptotic-density}\ A = \textit{real-of-ereal}(\textit{liminf}\ (\lambda n.\ \textit{card}(A \cap \{..{<}n\})/n))$

lemma *lower-asymptotic-density-in-01*:
  $ereal(lower\text{-}asymptotic\text{-}density\ A) = liminf\ (\lambda n.\ card(A \cap \{..<n\})/n)$
  $lower\text{-}asymptotic\text{-}density\ A \leq 1$
  $lower\text{-}asymptotic\text{-}density\ A \geq 0$
⟨*proof*⟩

The lower asymptotic density is bounded by the upper one. When they coincide, $Card(A \cap [0, n))/n$ converges to this common value.

lemma *lower-asymptotic-density-le-upper*:
  $lower\text{-}asymptotic\text{-}density\ A \leq upper\text{-}asymptotic\text{-}density\ A$
⟨*proof*⟩

lemma *lower-asymptotic-density-eq-upper*:
  assumes $lower\text{-}asymptotic\text{-}density\ A = l$ $upper\text{-}asymptotic\text{-}density\ A = l$
  shows $(\lambda n.\ card(A \cap \{..<n\})/n) \longrightarrow l$
⟨*proof*⟩

In particular, when a set has a zero upper density, or a lower density one, then this implies the corresponding convergence of $Card(A \cap [0, n))/n$.

lemma *upper-asymptotic-density-zero-lim*:
  assumes $upper\text{-}asymptotic\text{-}density\ A = 0$
  shows $(\lambda n.\ card(A \cap \{..<n\})/n) \longrightarrow 0$
⟨*proof*⟩

lemma *lower-asymptotic-density-one-lim*:
  assumes $lower\text{-}asymptotic\text{-}density\ A = 1$
  shows $(\lambda n.\ card(A \cap \{..<n\})/n) \longrightarrow 1$
⟨*proof*⟩

The lower asymptotic density of a set is 1 minus the upper asymptotic density of its complement. Hence, most statements about one of them follow from statements about the other one, although we will rather give direct proofs as they are not more complicated.

lemma *lower-upper-asymptotic-density-complement*:
  $lower\text{-}asymptotic\text{-}density\ A = 1 - upper\text{-}asymptotic\text{-}density\ (UNIV - A)$
⟨*proof*⟩

proposition *lower-asymptotic-densityD*:
  fixes $l::real$
  assumes $lower\text{-}asymptotic\text{-}density\ A > l$
  shows $eventually\ (\lambda n.\ card(A \cap \{..<n\}) > l * n)\ sequentially$
⟨*proof*⟩

proposition *lower-asymptotic-densityI*:
  fixes $l::real$
  assumes $eventually\ (\lambda n.\ card(A \cap \{..<n\}) \geq l * n)\ sequentially$
  shows $lower\text{-}asymptotic\text{-}density\ A \geq l$

$\langle proof \rangle$

One can control the asymptotic density of an intersection in terms of the asymptotic density of each component

lemma *lower-asymptotic-density-intersection*:
$lower\text{-}asymptotic\text{-}density\ A + lower\text{-}asymptotic\text{-}density\ B \leq lower\text{-}asymptotic\text{-}density\ (A \cap B) + 1$
$\langle proof \rangle$

lemma *lower-asymptotic-density-subset*:
assumes $A \subseteq B$
shows $lower\text{-}asymptotic\text{-}density\ A \leq lower\text{-}asymptotic\text{-}density\ B$
$\langle proof \rangle$

lemma *lower-asymptotic-density-lim*:
assumes $(\lambda n.\ card(A \cap \{..<n\})/n) \longrightarrow l$
shows $lower\text{-}asymptotic\text{-}density\ A = l$
$\langle proof \rangle$

lemma *lower-asymptotic-density-finite*:
assumes *finite A*
shows $lower\text{-}asymptotic\text{-}density\ A = 0$
$\langle proof \rangle$

In particular, bounded intervals have zero lower density.

lemma *lower-asymptotic-density-bdd-interval* [*simp*]:
$lower\text{-}asymptotic\text{-}density\ \{\} = 0$
$lower\text{-}asymptotic\text{-}density\ \{..N\} = 0$
$lower\text{-}asymptotic\text{-}density\ \{..<N\} = 0$
$lower\text{-}asymptotic\text{-}density\ \{n..N\} = 0$
$lower\text{-}asymptotic\text{-}density\ \{n..<N\} = 0$
$lower\text{-}asymptotic\text{-}density\ \{n<..N\} = 0$
$lower\text{-}asymptotic\text{-}density\ \{n<..<N\} = 0$
$\langle proof \rangle$

Conversely, unbounded intervals have density 1.

lemma *lower-asymptotic-density-infinite-interval* [*simp*]:
$lower\text{-}asymptotic\text{-}density\ \{N..\} = 1$
$lower\text{-}asymptotic\text{-}density\ \{N<..\} = 1$
$lower\text{-}asymptotic\text{-}density\ UNIV = 1$
$\langle proof \rangle$

lemma *upper-asymptotic-density-infinite-interval* [*simp*]:
$upper\text{-}asymptotic\text{-}density\ \{N..\} = 1$
$upper\text{-}asymptotic\text{-}density\ \{N<..\} = 1$
$upper\text{-}asymptotic\text{-}density\ UNIV = 1$
$\langle proof \rangle$

The intersection of sets with lower density one still has lower density one.

lemma *lower-asymptotic-density-one-intersection*:
  assumes $lower\text{-}asymptotic\text{-}density\ A = 1\ lower\text{-}asymptotic\text{-}density\ B = 1$
  shows $lower\text{-}asymptotic\text{-}density\ (A \cap B) = 1$
$\langle proof \rangle$

lemma *lower-asymptotic-density-one-finite-Intersection*:
  assumes $finite\ I\ \bigwedge i.\ i \in I \Longrightarrow lower\text{-}asymptotic\text{-}density\ (A\ i) = 1$
  shows $lower\text{-}asymptotic\text{-}density\ (\bigcap i{\in}I.\ A\ i) = 1$
$\langle proof \rangle$

As for the upper asymptotic density, there is a modification of the intersection, akin to the diagonal argument in this context, for which the "intersection" of sets with large lower density still has large lower density.

proposition *lower-asymptotic-density-decseq-Inter*:
  assumes $\bigwedge(n{::}nat).\ lower\text{-}asymptotic\text{-}density\ (A\ n) \geq l\ decseq\ A$
  shows $\exists B.\ lower\text{-}asymptotic\text{-}density\ B \geq l \wedge (\forall n.\ \exists N.\ B \cap \{N..\} \subseteq A\ n)$
$\langle proof \rangle$

In the same way, the modified intersection of sets of density 1 still has density one, and is eventually contained in each of them.

proposition *lower-asymptotic-density-one-Inter*:
  assumes $\bigwedge n{::}nat.\ lower\text{-}asymptotic\text{-}density\ (A\ n) = 1$
  shows $\exists B.\ lower\text{-}asymptotic\text{-}density\ B = 1 \wedge (\forall n.\ \exists N.\ B \cap \{N..\} \subseteq A\ n)$
$\langle proof \rangle$

Sets with density 1 play an important role in relation to Cesaro convergence of nonnegative bounded sequences: such a sequence converges to 0 in Cesaro average if and only if it converges to 0 along a set of density 1.

The proof is not hard. Since the Cesaro average tends to 0, then given $\epsilon > 0$ the proportion of times where $u_n < \epsilon$ tends to 1, i.e., the set A_ϵ of such good times has density 1. A modified intersection (as constructed in Proposition `lower_asymptotic_density_one_Inter`) of these times has density 1, and u_n tends to 0 along this set.

theorem *cesaro-imp-density-one*:
  assumes $\bigwedge n.\ u\ n \geq (0{::}real)\ (\lambda n.\ (\sum i{<}n.\ u\ i)/n) \longrightarrow 0$
  shows $\exists A.\ lower\text{-}asymptotic\text{-}density\ A = 1 \wedge (\lambda n.\ u\ n * indicator\ A\ n) \longrightarrow 0$
$\langle proof \rangle$

The proof of the reverse implication is more direct: in the Cesaro sum, just bound the elements in A by a small ϵ, and the other ones by a uniform bound, to get a bound which is $o(n)$.

theorem *density-one-imp-cesaro*:
  assumes $\bigwedge n.\ u\ n \geq (0{::}real)\ \bigwedge n.\ u\ n \leq C$
    $lower\text{-}asymptotic\text{-}density\ A = 1$
    $(\lambda n.\ u\ n * indicator\ A\ n) \longrightarrow 0$
  shows $(\lambda n.\ (\sum i{<}n.\ u\ i)/n) \longrightarrow 0$

⟨*proof*⟩

end

4 Measure preserving or quasi-preserving maps

theory *Measure-Preserving-Transformations*
  imports *SG-Library-Complement*
begin

Ergodic theory in general is the study of the properties of measure preserving or quasi-preserving dynamical systems. In this section, we introduce the basic definitions in this respect.

4.1 The different classes of transformations

definition $quasi\text{-}measure\text{-}preserving::'a\ measure \Rightarrow 'b\ measure \Rightarrow ('a \Rightarrow 'b)\ set$
  where $quasi\text{-}measure\text{-}preserving\ M\ N$
    $= \{f \in measurable\ M\ N.\ \forall\ A \in sets\ N.\ (f -` A \cap space\ M \in null\text{-}sets\ M) = (A \in null\text{-}sets\ N)\}$

lemma $quasi\text{-}measure\text{-}preservingI$ [*intro*]:
  assumes $f \in measurable\ M\ N$
    $\bigwedge A.\ A \in sets\ N \Longrightarrow (f -` A \cap space\ M \in null\text{-}sets\ M) = (A \in null\text{-}sets\ N)$
  shows $f \in quasi\text{-}measure\text{-}preserving\ M\ N$
⟨*proof*⟩

lemma $quasi\text{-}measure\text{-}preservingE$:
  assumes $f \in quasi\text{-}measure\text{-}preserving\ M\ N$
  shows $f \in measurable\ M\ N$
    $\bigwedge A.\ A \in sets\ N \Longrightarrow (f -` A \cap space\ M \in null\text{-}sets\ M) = (A \in null\text{-}sets\ N)$
⟨*proof*⟩

lemma $id\text{-}quasi\text{-}measure\text{-}preserving$:
  $(\lambda x.\ x) \in quasi\text{-}measure\text{-}preserving\ M\ M$
⟨*proof*⟩

lemma $quasi\text{-}measure\text{-}preserving\text{-}composition$:
  assumes $f \in quasi\text{-}measure\text{-}preserving\ M\ N$
    $g \in quasi\text{-}measure\text{-}preserving\ N\ P$
  shows $(\lambda x.\ g(f\ x)) \in quasi\text{-}measure\text{-}preserving\ M\ P$
⟨*proof*⟩

lemma $quasi\text{-}measure\text{-}preserving\text{-}comp$:
  assumes $f \in quasi\text{-}measure\text{-}preserving\ M\ N$
    $g \in quasi\text{-}measure\text{-}preserving\ N\ P$
  shows $g\ o\ f \in quasi\text{-}measure\text{-}preserving\ M\ P$

⟨*proof*⟩

lemma *quasi-measure-preserving-AE*:
  assumes $f \in$ *quasi-measure-preserving M N*
    AE x in N. P x
  shows *AE x in M. P (f x)*
⟨*proof*⟩

lemma *quasi-measure-preserving-AE'*:
  assumes $f \in$ *quasi-measure-preserving M N*
    AE x in M. P (f x)
    $\{x \in$ *space N. P x*$\} \in$ *sets N*
  shows *AE x in N. P x*
⟨*proof*⟩

The push-forward under a quasi-measure preserving map f of a measure absolutely continuous with respect to M is absolutely continuous with respect to N.

lemma *quasi-measure-preserving-absolutely-continuous*:
  assumes $f \in$ *quasi-measure-preserving M N*
    $u \in$ *borel-measurable M*
  shows *absolutely-continuous N (distr (density M u) N f)*
⟨*proof*⟩

definition *measure-preserving*::*'a measure* $\Rightarrow$ *'b measure* $\Rightarrow$ *('a* $\Rightarrow$ *'b) set*
  where *measure-preserving M N*
    $= \{f \in$ *measurable M N.* $(\forall\ A \in$ *sets N. emeasure M* $(f-`A \cap$ *space M*$)$
$=$ *emeasure N A*$)\}$

lemma *measure-preservingE*:
  assumes $f \in$ *measure-preserving M N*
  shows $f \in$ *measurable M N*
    $\bigwedge A.\ A \in$ *sets N* $\Longrightarrow$ *emeasure M* $(f-`A \cap$ *space M*$)$ $=$ *emeasure N A*
⟨*proof*⟩

lemma *measure-preservingI* [*intro*]:
  assumes $f \in$ *measurable M N*
    $\bigwedge A.\ A \in$ *sets N* $\Longrightarrow$ *emeasure M* $(f-`A \cap$ *space M*$)$ $=$ *emeasure N A*
  shows $f \in$ *measure-preserving M N*
⟨*proof*⟩

lemma *measure-preserving-distr*:
  assumes $f \in$ *measure-preserving M N*
  shows *distr M N f* $=$ *N*
⟨*proof*⟩

lemma *measure-preserving-distr'*:
  assumes $f \in$ *measurable M N*
  shows $f \in$ *measure-preserving M (distr M N f)*

⟨*proof*⟩

lemma *measure-preserving-preserves-nn-integral*:
  assumes $T \in$ *measure-preserving M N*
    $f \in$ *borel-measurable N*
  shows $(\int^{+} x.\ f\ x\ \partial N) = (\int^{+} x.\ f\ (T\ x)\ \partial M)$
⟨*proof*⟩

lemma *measure-preserving-preserves-integral*:
  fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
  assumes $T \in$ *measure-preserving M N*
    and [*measurable*]: *integrable N f*
  shows *integrable M* $(\lambda x.\ f(T\ x))$ $(\int x.\ f\ x\ \partial N) = (\int x.\ f\ (T\ x)\ \partial M)$
⟨*proof*⟩

lemma *measure-preserving-preserves-integral'*:
  fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
  assumes $T \in$ *measure-preserving M N*
    and [*measurable*]: *integrable M* $(\lambda x.\ f\ (T\ x))$ $f \in$ *borel-measurable N*
  shows *integrable N f* $(\int x.\ f\ x\ \partial N) = (\int x.\ f\ (T\ x)\ \partial M)$
⟨*proof*⟩

lemma *id-measure-preserving*:
  $(\lambda x.\ x) \in$ *measure-preserving M M*
⟨*proof*⟩

lemma *measure-preserving-is-quasi-measure-preserving*:
  assumes $f \in$ *measure-preserving M N*
  shows $f \in$ *quasi-measure-preserving M N*
⟨*proof*⟩

lemma *measure-preserving-composition*:
  assumes $f \in$ *measure-preserving M N*
    $g \in$ *measure-preserving N P*
  shows $(\lambda x.\ g(f\ x)) \in$ *measure-preserving M P*
⟨*proof*⟩

lemma *measure-preserving-comp*:
  assumes $f \in$ *measure-preserving M N*
    $g \in$ *measure-preserving N P*
  shows $g\ o\ f \in$ *measure-preserving M P*
⟨*proof*⟩

lemma *measure-preserving-total-measure*:
  assumes $f \in$ *measure-preserving M N*
  shows *emeasure M* (*space M*) = *emeasure N* (*space N*)
⟨*proof*⟩

lemma *measure-preserving-finite-measure*:

assumes $f \in$ *measure-preserving M N*
shows *finite-measure M* $\longleftrightarrow$ *finite-measure N*
$\langle proof \rangle$

lemma *measure-preserving-prob-space*:
assumes $f \in$ *measure-preserving M N*
shows *prob-space M* $\longleftrightarrow$ *prob-space N*
$\langle proof \rangle$

locale *qmpt* = *sigma-finite-measure* +
fixes *T*
assumes *Tqm*: $T \in$ *quasi-measure-preserving M M*

locale *mpt* = *qmpt* +
assumes *Tm*: $T \in$ *measure-preserving M M*

locale *fmpt* = *mpt* + *finite-measure*

locale *pmpt* = *fmpt* + *prob-space*

lemma *qmpt-I*:
assumes *sigma-finite-measure M*
$T \in$ *measurable M M*
$\bigwedge A.\ A \in$ *sets M* $\Longrightarrow$ $((T-\text{`}A \cap$ *space M*$) \in$ *null-sets M*$) \longleftrightarrow (A \in$ *null-sets*
M$)$
shows *qmpt M T*
$\langle proof \rangle$

lemma *mpt-I*:
assumes *sigma-finite-measure M*
$T \in$ *measurable M M*
$\bigwedge A.\ A \in$ *sets M* $\Longrightarrow$ *emeasure M* $(T-\text{`}A \cap$ *space M*$)$ = *emeasure M A*
shows *mpt M T*
$\langle proof \rangle$

lemma *fmpt-I*:
assumes *finite-measure M*
$T \in$ *measurable M M*
$\bigwedge A.\ A \in$ *sets M* $\Longrightarrow$ *emeasure M* $(T-\text{`}A \cap$ *space M*$)$ = *emeasure M A*
shows *fmpt M T*
$\langle proof \rangle$

lemma *pmpt-I*:
assumes *prob-space M*
$T \in$ *measurable M M*
$\bigwedge A.\ A \in$ *sets M* $\Longrightarrow$ *emeasure M* $(T-\text{`}A \cap$ *space M*$)$ = *emeasure M A*
shows *pmpt M T*
$\langle proof \rangle$

4.2 Examples

lemma *fmpt-null-space*:
assumes *emeasure M (space M) = 0*
T ∈ measurable M M
shows *fmpt M T*
⟨*proof*⟩

lemma *fmpt-empty-space*:
assumes *space M = {}*
shows *fmpt M T*
⟨*proof*⟩

Translations are measure-preserving

lemma *mpt-translation*:
fixes *c :: 'a::euclidean-space*
shows *mpt lborel (λx. x + c)*
⟨*proof*⟩

Skew products are fibered maps of the form $(x, y) \mapsto (Tx, U(x, y))$. If the base map and the fiber maps all are measure preserving, so is the skew product.

lemma *pair-measure-null-product*:
assumes *emeasure M (space M) = 0*
shows *emeasure* $(M \bigotimes_M N)$ *(space* $(M \bigotimes_M N)$*) = 0*
⟨*proof*⟩

lemma *mpt-skew-product*:
assumes *mpt M T*
AE x in M. mpt N (U x)
and [*measurable*]: *(λ(x,y). U x y) ∈ measurable* $(M \bigotimes_M N)$ *N*
shows *mpt* $(M \bigotimes_M N)$ *(λ(x,y). (T x, U x y))*
⟨*proof*⟩

lemma *mpt-skew-product-real*:
fixes *f::'a ⇒ 'b::euclidean-space*
assumes *mpt M T* **and** [*measurable*]: *f ∈ borel-measurable M*
shows *mpt* $(M \bigotimes_M$ *lborel) (λ(x,y). (T x, y + f x))*
⟨*proof*⟩

4.3 Preimages restricted to *spaceM*

context *qmpt* **begin**

One is all the time lead to take the preimages of sets, and restrict them to `space M` where the dynamics is living. We introduce a shortcut for this notion.

definition *vimage-restr :: ('a ⇒ 'a) ⇒ 'a set ⇒ 'a set* (**infixr** ‹−−‘› *90*)
where

$f\ {-}{-}`\ A \equiv f{-}`\ (A \cap space\ M) \cap space\ M$

lemma *vrestr-eq* [*simp*]:
$a \in f{-}{-}`\ A \longleftrightarrow a \in space\ M \wedge f\ a \in A \cap space\ M$
⟨*proof*⟩

lemma *vrestr-intersec* [*simp*]:
$f{-}{-}`\ (A \cap B) = (f{-}{-}`A) \cap (f{-}{-}`\ B)$
⟨*proof*⟩

lemma *vrestr-union* [*simp*]:
$f{-}{-}`\ (A \cup B) = f{-}{-}`A \cup f{-}{-}`B$
⟨*proof*⟩

lemma *vrestr-difference* [*simp*]:
$f{-}{-}`(A{-}B) = f{-}{-}`A - f{-}{-}`B$
⟨*proof*⟩

lemma *vrestr-inclusion*:
$A \subseteq B \Longrightarrow f{-}{-}`A \subseteq f{-}{-}`B$
⟨*proof*⟩

lemma *vrestr-Union* [*simp*]:
$f\ {-}{-}`\ (\bigcup A) = (\bigcup X{\in}A.\ f\ {-}{-}`\ X)$
⟨*proof*⟩

lemma *vrestr-UN* [*simp*]:
$f\ {-}{-}`\ (\bigcup x{\in}A.\ B\ x) = (\bigcup x{\in}A.\ f\ {-}{-}`\ B\ x)$
⟨*proof*⟩

lemma *vrestr-Inter* [*simp*]:
assumes $A \neq \{\}$
shows $f\ {-}{-}`\ (\bigcap A) = (\bigcap X{\in}A.\ f\ {-}{-}`\ X)$
⟨*proof*⟩

lemma *vrestr-INT* [*simp*]:
assumes $A \neq \{\}$
shows $f\ {-}{-}`\ (\bigcap x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ f\ {-}{-}`\ B\ x)$
⟨*proof*⟩

lemma *vrestr-empty* [*simp*]:
$f{-}{-}`\{\} = \{\}$
⟨*proof*⟩

lemma *vrestr-sym-diff* [*simp*]:
$f{-}{-}`(A\ \Delta\ B) = (f{-}{-}`A)\ \Delta\ (f{-}{-}`B)$
⟨*proof*⟩

lemma *vrestr-image*:

assumes $x \in f{-}{-}`A$
shows $x \in space\ M\ f\ x \in space\ M\ f\ x \in A$
⟨*proof*⟩

lemma *vrestr-intersec-in-space*:
assumes $A \in sets\ M\ B \in sets\ M$
shows $A \cap f{-}{-}`B = A \cap f{-}`B$
⟨*proof*⟩

lemma *vrestr-compose*:
assumes $g \in measurable\ M\ M$
shows $(\lambda\ x.\ f(g\ x)){-}{-}`\ A = g{-}{-}`\ (f{-}{-}`\ A)$
⟨*proof*⟩

lemma *vrestr-comp*:
assumes $g \in measurable\ M\ M$
shows $(f\ o\ g){-}{-}`\ A = g{-}{-}`\ (f{-}{-}`\ A)$
⟨*proof*⟩

lemma *vrestr-of-set*:
assumes $g \in measurable\ M\ M$
shows $A \in sets\ M \Longrightarrow g{-}{-}`A = g{-}`A \cap space\ M$
⟨*proof*⟩

lemma *vrestr-meas* [*measurable* (*raw*)]:
assumes $g \in measurable\ M\ M$
$A \in sets\ M$
shows $g{-}{-}`A \in sets\ M$
⟨*proof*⟩

lemma *vrestr-same-emeasure-f*:
assumes $f \in measure\text{-}preserving\ M\ M$
$A \in sets\ M$
shows $emeasure\ M\ (f{-}{-}`A) = emeasure\ M\ A$
⟨*proof*⟩

lemma *vrestr-same-measure-f*:
assumes $f \in measure\text{-}preserving\ M\ M$
$A \in sets\ M$
shows $measure\ M\ (f{-}{-}`A) = measure\ M\ A$
⟨*proof*⟩

4.4 Basic properties of qmpt

lemma *T-meas* [*measurable* (*raw*)]:
$T \in measurable\ M\ M$
⟨*proof*⟩

lemma *Tn-quasi-measure-preserving*:

T^^$n \in$ *quasi-measure-preserving M M*
⟨*proof*⟩

lemma *Tn-meas* [*measurable* (*raw*)]:
T^^$n \in$ *measurable M M*
⟨*proof*⟩

lemma *T-vrestr-meas* [*measurable*]:
assumes $A \in$ *sets M*
shows T−−‘ $A \in$ *sets M*
(T^^n)−−‘ $A \in$ *sets M*
⟨*proof*⟩

We state the next lemma both with T^0 and with *id* as sometimes the simplifier simplifies T^0 to *id* before applying the first instance of the lemma.

lemma *T-vrestr-0* [*simp*]:
assumes $A \in$ *sets M*
shows (T^^*0*)−−‘$A = A$
id−−‘$A = A$
⟨*proof*⟩

lemma *T-vrestr-composed*:
assumes $A \in$ *sets M*
shows (T^^n)−−‘ (T^^m)−−‘ A = (T^^(n+m))−−‘ A
T−−‘ (T^^m)−−‘ A = (T^^(m+*1*))−−‘ A
(T^^m)−−‘ T−−‘ A = (T^^(m+*1*))−−‘ A
⟨*proof*⟩

In the next two lemmas, we give measurability statements that show up all the time for the usual preimage.

lemma *T-intersec-meas* [*measurable*]:
assumes [*measurable*]: $A \in$ *sets M* $B \in$ *sets M*
shows $A \cap T$−‘$B \in$ *sets M*
$A \cap$ (T^^n)−‘$B \in$ *sets M*
T−‘$A \cap B \in$ *sets M*
(T^^n)−‘$A \cap B \in$ *sets M*
$A \cap$ ($T \circ T$ ^^ n) −‘ $B \in$ *sets M*
($T \circ T$ ^^ n) −‘ $A \cap B \in$ *sets M*
⟨*proof*⟩

lemma *T-diff-meas* [*measurable*]:
assumes [*measurable*]: $A \in$ *sets M* $B \in$ *sets M*
shows $A - T$−‘$B \in$ *sets M*
A − (T^^n)−‘$B \in$ *sets M*
⟨*proof*⟩

lemma *T-spaceM-stable* [*simp*]:
assumes $x \in$ *space M*
shows $T\ x \in$ *space M*

$(T\widehat{\ }\widehat{\ }n)\ x \in space\ M$
⟨proof⟩

lemma *T-quasi-preserves-null*:
assumes $A \in sets\ M$
shows $A \in null\text{-}sets\ M \longleftrightarrow T--`\ A \in null\text{-}sets\ M$
$A \in null\text{-}sets\ M \longleftrightarrow (T\widehat{\ }\widehat{\ }n)--`\ A \in null\text{-}sets\ M$
⟨proof⟩

lemma *T-quasi-preserves*:
assumes $A \in sets\ M$
shows $emeasure\ M\ A = 0 \longleftrightarrow emeasure\ M\ (T--`\ A) = 0$
$emeasure\ M\ A = 0 \longleftrightarrow emeasure\ M\ ((T\widehat{\ }\widehat{\ }n)--`\ A) = 0$
⟨proof⟩

lemma *T-quasi-preserves-null2*:
assumes $A \in null\text{-}sets\ M$
shows $T--`\ A \in null\text{-}sets\ M$
$(T\widehat{\ }\widehat{\ }n)--`\ A \in null\text{-}sets\ M$
⟨proof⟩

lemma *T-composition-borel* [*measurable*]:
assumes $f \in borel\text{-}measurable\ M$
shows $(\lambda x.\ f(T\ x)) \in borel\text{-}measurable\ M\ (\lambda x.\ f((T\widehat{\ }\widehat{\ }k)\ x)) \in borel\text{-}measurable$
M
⟨proof⟩

lemma *T-AE-iterates*:
assumes $AE\ x\ in\ M.\ P\ x$
shows $AE\ x\ in\ M.\ \forall\, n.\ P\ ((T\widehat{\ }\widehat{\ }n)\ x)$
⟨proof⟩

lemma *qmpt-power*:
$qmpt\ M\ (T\widehat{\ }\widehat{\ }n)$
⟨proof⟩

lemma *T-Tn-T-compose*:
$T\ ((T\widehat{\ }\widehat{\ }n)\ x) = (T\widehat{\ }\widehat{\ }(Suc\ n))\ x$
$(T\widehat{\ }\widehat{\ }n)\ (T\ x) = (T\widehat{\ }\widehat{\ }(Suc\ n))\ x$
⟨proof⟩

lemma (**in** *qmpt*) *qmpt-density*:
assumes [*measurable*]: $h \in borel\text{-}measurable\ M$
and $AE\ x\ in\ M.\ h\ x \neq 0\ AE\ x\ in\ M.\ h\ x \neq \infty$
shows $qmpt\ (density\ M\ h)\ T$
⟨proof⟩

end

4.5 Basic properties of mpt

context *mpt*
begin

lemma *Tn-measure-preserving*:
  $T\widehat{\ \ }n \in$ *measure-preserving M M*
⟨*proof*⟩

lemma *T-integral-preserving*:
  fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
  assumes *integrable M f*
  shows *integrable M* $(\lambda x.\ f(T\ x))$ $(\int x.\ f(T\ x)\ \partial M) = (\int x.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *Tn-integral-preserving*:
  fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
  assumes *integrable M f*
  shows *integrable M* $(\lambda x.\ f((T\widehat{\ \ }n)\ x))$ $(\int x.\ f((T\widehat{\ \ }n)\ x)\ \partial M) = (\int x.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *T-nn-integral-preserving*:
  fixes $f :: 'a \Rightarrow ennreal$
  assumes $f \in$ *borel-measurable M*
  shows $(\int^+ x.\ f(T\ x)\ \partial M) = (\int^+ x.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *Tn-nn-integral-preserving*:
  fixes $f :: 'a \Rightarrow ennreal$
  assumes $f \in$ *borel-measurable M*
  shows $(\int^+ x.\ f((T\widehat{\ \ }n)\ x)\ \partial M) = (\int^+ x.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *mpt-power*:
  mpt M $(T\widehat{\ \ }n)$
⟨*proof*⟩

lemma *T-vrestr-same-emeasure*:
  assumes $A \in$ *sets M*
  shows *emeasure M* $(T{-}{-}`\ A)$ = *emeasure M A*
    emeasure M $((T\ \widehat{\ \ }\ n){-}{-}`A)$ = *emeasure M A*
⟨*proof*⟩

lemma *T-vrestr-same-measure*:
  assumes $A \in$ *sets M*
  shows *measure M* $(T{-}{-}`\ A)$ = *measure M A*
    measure M $((T\ \widehat{\ \ }\ n){-}{-}`A)$ = *measure M A*
⟨*proof*⟩

lemma (**in** *fmpt*) *fmpt-power*:

fmpt M $(T\widehat{\ }\widehat{\ }n)$
⟨*proof*⟩

end

4.6 Birkhoff sums

Birkhoff sums, obtained by summing a function along the orbit of a map, are basic objects to be understood in ergodic theory.

context *qmpt*
begin

definition *birkhoff-sum*::('*a* ⇒ '*b*::*comm-monoid-add*) ⇒ *nat* ⇒ '*a* ⇒ '*b*
  where *birkhoff-sum f n x* = $(\sum i{\in}\{..{<}n\}.\ f((T\widehat{\ }\widehat{\ }i)x))$

lemma *birkhoff-sum-meas* [*measurable*]:
  fixes *f*::'*a* ⇒ '*b*::{*second-countable-topology*, *topological-comm-monoid-add*}
  assumes *f* ∈ *borel-measurable M*
  shows *birkhoff-sum f n* ∈ *borel-measurable M*
⟨*proof*⟩

lemma *birkhoff-sum-1* [*simp*]:
  birkhoff-sum f 0 x = *0*
  birkhoff-sum f 1 x = *f x*
  birkhoff-sum f (*Suc 0*) *x* = *f x*
⟨*proof*⟩

lemma *birkhoff-sum-cocycle*:
  birkhoff-sum f (*n*+*m*) *x* = *birkhoff-sum f n x* + *birkhoff-sum f m* $((T\widehat{\ }\widehat{\ }n)x)$
⟨*proof*⟩

lemma *birkhoff-sum-mono*:
  fixes *f g*::- ⇒ *real*
  assumes ⋀*x*. *f x* ≤ *g x*
  shows *birkhoff-sum f n x* ≤ *birkhoff-sum g n x*
⟨*proof*⟩

lemma *birkhoff-sum-abs*:
  fixes *f*::- ⇒ '*b*::*real-normed-vector*
  shows *norm*(*birkhoff-sum f n x*) ≤ *birkhoff-sum* (λ*x*. *norm*(*f x*)) *n x*
⟨*proof*⟩

lemma *birkhoff-sum-add*:
  birkhoff-sum (λ*x*. *f x* + *g x*) *n x* = *birkhoff-sum f n x* + *birkhoff-sum g n x*
⟨*proof*⟩

lemma *birkhoff-sum-diff*:
  fixes *f g*::- ⇒ *real*

shows *birkhoff-sum* ($\lambda x.$ *f x* $-$ *g x*) *n x* $=$ *birkhoff-sum f n x* $-$ *birkhoff-sum g n x*
⟨*proof*⟩

lemma *birkhoff-sum-cmult*:
fixes *f*::- $\Rightarrow$ *real*
shows *birkhoff-sum* ($\lambda x.$ *c* $*$ *f x*) *n x* $=$ *c* $*$ *birkhoff-sum f n x*
⟨*proof*⟩

lemma *skew-product-real-iterates*:
fixes *f*::*'a* $\Rightarrow$ *real*
shows ((λ(*x*,*y*). (*T x*, *y* $+$ *f x*))^^*n*) (*x*,*y*) $=$ ((*T*^^*n*) *x*, *y* $+$ *birkhoff-sum f n x*)
⟨*proof*⟩

end

lemma (**in** *mpt*) *birkhoff-sum-integral*:
fixes *f* :: *'a* $\Rightarrow$ *'b*::{*banach*, *second-countable-topology*}
assumes [*measurable*]: *integrable M f*
shows *integrable M* (*birkhoff-sum f n*) ($\int$ *x. birkhoff-sum f n x* ∂M) $=$ *n* $*_R$ ($\int$ *x. f x* ∂M)
⟨*proof*⟩

lemma (**in** *mpt*) *birkhoff-sum-nn-integral*:
fixes *f* :: *'a* $\Rightarrow$ *ennreal*
assumes [*measurable*]: *f* $\in$ *borel-measurable M* **and** *pos*: $\bigwedge$*x. f x* $\geq$ *0*
shows ($\int^+$*x. birkhoff-sum f n x* ∂M) $=$ *n* $*$ ($\int^+$*x. f x* ∂M)
⟨*proof*⟩

4.7 Inverse map

context *qmpt* **begin**

definition
invertible-qmpt $\equiv$ (*bij T* $\wedge$ *inv T* $\in$ *measurable M M*)

definition
Tinv $\equiv$ *inv T*

lemma *T-Tinv-of-set*:
assumes *invertible-qmpt*
A $\in$ *sets M*
shows *T*$-$‘(*Tinv*$-$‘*A* $\cap$ *space M*) $\cap$ *space M* $=$ *A*
⟨*proof*⟩

lemma *Tinv-quasi-measure-preserving*:
assumes *invertible-qmpt*
shows *Tinv* $\in$ *quasi-measure-preserving M M*
⟨*proof*⟩

lemma *Tinv-qmpt*:
assumes *invertible-qmpt*
shows *qmpt M Tinv*
⟨*proof*⟩

end

lemma (**in** *mpt*) *Tinv-measure-preserving*:
assumes *invertible-qmpt*
shows *Tinv* ∈ *measure-preserving M M*
⟨*proof*⟩

lemma (**in** *mpt*) *Tinv-mpt*:
assumes *invertible-qmpt*
shows *mpt M Tinv*
⟨*proof*⟩

lemma (**in** *fmpt*) *Tinv-fmpt*:
assumes *invertible-qmpt*
shows *fmpt M Tinv*
⟨*proof*⟩

lemma (**in** *pmpt*) *Tinv-fmpt*:
assumes *invertible-qmpt*
shows *pmpt M Tinv*
⟨*proof*⟩

4.8 Factors

Factors of a system are quotients of this system, i.e., systems that can be obtained by a projection, forgetting some part of the dynamics. It is sometimes possible to transfer a result from a factor to the original system, making it possible to prove theorems by reduction to a simpler situation.

The dual notion, extension, is equally important and useful. We only mention factors below, as the results for extension readily follow by considering the original system as a factor of its extension.

In this paragraph, we define factors both in the qmpt and mpt categories, and prove their basic properties.

definition (**in** *qmpt*) *qmpt-factor*::(*'a* ⇒ *'b*) ⇒ (*'b measure*) ⇒ (*'b* ⇒ *'b*) ⇒ *bool*
where *qmpt-factor proj M2 T2* =
((*proj* ∈ *quasi-measure-preserving M M2*) ∧ (*AE x in M*. *proj* (*T x*) = *T2* (*proj x*)) ∧ *qmpt M2 T2*)

lemma (**in** *qmpt*) *qmpt-factorE*:
assumes *qmpt-factor proj M2 T2*
shows *proj* ∈ *quasi-measure-preserving M M2*

AE x in M. proj (T x) = T2 (proj x)
qmpt M2 T2
⟨*proof*⟩

lemma (**in** *qmpt*) *qmpt-factor-iterates*:
assumes *qmpt-factor proj M2 T2*
shows *AE x in M.* $\forall$ *n. proj ((T^^n) x) = (T2^^n) (proj x)*
⟨*proof*⟩

lemma (**in** *qmpt*) *qmpt-factorI*:
assumes *proj* $\in$ *quasi-measure-preserving M M2*
AE x in M. proj (T x) = T2 (proj x)
qmpt M2 T2
shows *qmpt-factor proj M2 T2*
⟨*proof*⟩

When there is a quasi-measure-preserving projection, then the quotient map automatically is quasi-measure-preserving. The same goes for measure-preservation below.

lemma (**in** *qmpt*) *qmpt-factorI'*:
assumes *proj* $\in$ *quasi-measure-preserving M M2*
AE x in M. proj (T x) = T2 (proj x)
sigma-finite-measure M2
T2 $\in$ *measurable M2 M2*
shows *qmpt-factor proj M2 T2*
⟨*proof*⟩

lemma *qmpt-factor-compose*:
assumes *qmpt M1 T1*
qmpt.qmpt-factor M1 T1 proj1 M2 T2
qmpt.qmpt-factor M2 T2 proj2 M3 T3
shows *qmpt.qmpt-factor M1 T1 (proj2 o proj1) M3 T3*
⟨*proof*⟩

The left shift on natural integers is a very natural dynamical system, that can be used to model many systems as we see below. For invertible systems, one uses rather all the integers.

definition *nat-left-shift*::(*nat* $\Rightarrow$ *'a*) $\Rightarrow$ (*nat* $\Rightarrow$ *'a*)
where *nat-left-shift x* = (λ*i. x (i+1)*)

lemma *nat-left-shift-continuous* [*intro*, *continuous-intros*]:
continuous-on UNIV nat-left-shift
⟨*proof*⟩

lemma *nat-left-shift-measurable* [*intro*, *measurable*]:
nat-left-shift $\in$ *measurable borel borel*
⟨*proof*⟩

definition *int-left-shift*::(*int* $\Rightarrow$ *'a*) $\Rightarrow$ (*int* $\Rightarrow$ *'a*)

where *int-left-shift* $x = (\lambda i.\ x\ (i+1))$

definition *int-right-shift*::$(int \Rightarrow 'a) \Rightarrow (int \Rightarrow 'a)$
where *int-right-shift* $x = (\lambda i.\ x\ (i-1))$

lemma *int-shift-continuous* [*intro*, *continuous-intros*]:
continuous-on UNIV int-left-shift
continuous-on UNIV int-right-shift
⟨*proof*⟩

lemma *int-shift-measurable* [*intro*, *measurable*]:
int-left-shift ∈ *measurable borel borel*
int-right-shift ∈ *measurable borel borel*
⟨*proof*⟩

lemma *int-shift-bij*:
bij int-left-shift inv int-left-shift = int-right-shift
bij int-right-shift inv int-right-shift = int-left-shift
⟨*proof*⟩

lemma (**in** *qmpt*) *qmpt-factor-projection*:
fixes f::$'a \Rightarrow ('b$::*second-countable-topology*)
assumes [*measurable*]: $f \in$ *borel-measurable* M
and *sigma-finite-measure* (*distr* M *borel* $(\lambda x\ n.\ f\ ((T \widehat{\ \ }\widehat{\ \ } n)\ x)))$
shows *qmpt-factor* $(\lambda x.\ (\lambda n.\ f\ ((T\widehat{\ \ }\widehat{\ \ }n) x)))$ (*distr* M *borel* $(\lambda x.\ (\lambda n.\ f\ ((T\widehat{\ \ }\widehat{\ \ }n) x))))$
nat-left-shift
⟨*proof*⟩

Let us now define factors of measure-preserving transformations, in the same way as above.

definition (**in** *mpt*) *mpt-factor*::$('a \Rightarrow 'b) \Rightarrow ('b\ measure) \Rightarrow ('b \Rightarrow 'b) \Rightarrow bool$
where *mpt-factor proj M2 T2* =
((*proj* ∈ *measure-preserving M M2*) ∧ (*AE x in M. proj* (*T x*) = *T2* (*proj x*))
∧ *mpt M2 T2*)

lemma (**in** *mpt*) *mpt-factor-is-qmpt-factor*:
assumes *mpt-factor proj M2 T2*
shows *qmpt-factor proj M2 T2*
⟨*proof*⟩

lemma (**in** *mpt*) *mpt-factorE*:
assumes *mpt-factor proj M2 T2*
shows *proj* ∈ *measure-preserving M M2*
AE x in M. proj (*T x*) = *T2* (*proj x*)
mpt M2 T2
⟨*proof*⟩

lemma (**in** *mpt*) *mpt-factorI*:
assumes *proj* ∈ *measure-preserving M M2*

AE x in M. proj (T x) = T2 (proj x)
mpt M2 T2
shows *mpt-factor proj M2 T2*
⟨*proof*⟩

When there is a measure-preserving projection commuting with the dynamics, and the dynamics above preserves the measure, then so does the dynamics below.

lemma (**in** *mpt*) *mpt-factorI′*:
assumes *proj* ∈ *measure-preserving M M2*
AE x in M. proj (T x) = T2 (proj x)
sigma-finite-measure M2
T2 ∈ *measurable M2 M2*
shows *mpt-factor proj M2 T2*
⟨*proof*⟩

lemma (**in** *fmpt*) *mpt-factorI′′*:
assumes *proj* ∈ *measure-preserving M M2*
AE x in M. proj (T x) = T2 (proj x)
T2 ∈ *measurable M2 M2*
shows *mpt-factor proj M2 T2*
⟨*proof*⟩

lemma (**in** *fmpt*) *fmpt-factor*:
assumes *mpt-factor proj M2 T2*
shows *fmpt M2 T2*
⟨*proof*⟩

lemma (**in** *pmpt*) *pmpt-factor*:
assumes *mpt-factor proj M2 T2*
shows *pmpt M2 T2*
⟨*proof*⟩

lemma *mpt-factor-compose*:
assumes *mpt M1 T1*
mpt.mpt-factor M1 T1 proj1 M2 T2
mpt.mpt-factor M2 T2 proj2 M3 T3
shows *mpt.mpt-factor M1 T1 (proj2 o proj1) M3 T3*
⟨*proof*⟩

Left shifts are naturally factors of finite measure preserving transformations.

lemma (**in** *mpt*) *mpt-factor-projection*:
fixes *f*::′*a* ⇒ (′*b*::*second-countable-topology*)
assumes [*measurable*]: *f* ∈ *borel-measurable M*
and *sigma-finite-measure* (*distr M borel* (λ*x n. f* ((*T* ^^ *n*) *x*)))
shows *mpt-factor* (λ*x*. (λ*n. f* ((*T*^^*n*)*x*))) (*distr M borel* (λ*x*. (λ*n. f* ((*T*^^*n*)*x*))))
nat-left-shift
⟨*proof*⟩

lemma (**in** *fmpt*) *fmpt-factor-projection*:
  fixes $f::'a \Rightarrow ('b::second\text{-}countable\text{-}topology)$
  assumes [*measurable*]: $f \in$ *borel-measurable M*
  shows *mpt-factor* $(\lambda x.\ (\lambda n.\ f\ ((T\widehat{\ }\widehat{\ }n)x)))$ (*distr M borel* $(\lambda x.\ (\lambda n.\ f\ ((T\widehat{\ }\widehat{\ }n)x))))$
nat-left-shift
⟨*proof*⟩

4.9 Natural extension

Many probability preserving dynamical systems are not invertible, while invertibility is often useful in proofs. The notion of natural extension is a solution to this problem: it shows that (essentially) any system has an extension which is invertible.

This extension is constructed by considering the space of orbits indexed by integer numbers, with the left shift acting on it. If one considers the orbits starting from time $-N$ (for some fixed N), then there is a natural measure on this space: such an orbit is parameterized by its starting point at time $-N$, hence one may use the original measure on this point. The invariance of the measure ensures that these measures are compatible with each other. Their projective limit (when N tends to infinity) is thus an invariant measure on the bilateral shift. The shift with this measure is the desired extension of the original system.

There is a difficulty in the above argument: one needs to make sure that the projective limit of a system of compatible measures is well defined. This requires some topological conditions on the measures (they should be inner regular, i.e., the measure of any set should be approximated from below by compact subsets – this is automatic on polish spaces). The existence of projective limits is proved in `Projective_Limits.thy` under the (sufficient) polish condition. We use this theory, so we need the underlying space to be a polish space and the measure to be a Borel measure. This is almost completely satisfactory.

What is not completely satisfactory is that the completion of a Borel measure on a polish space (i.e., we add all subsets of sets of measure 0 into the sigma algebra) does not fit into this setting, while this is an important framework in dynamical systems. It would readily follow once `Projective_Limits.thy` is extended to the more general inner regularity setting (the completion of a Borel measure on a polish space is always inner regular).

locale *polish-pmpt* = *pmpt* $M::('a::polish\text{-}space\ measure)$ T **for** M T
  + **assumes** *M-eq-borel*: *sets M* = *sets borel*
begin

definition *natural-extension-map*
  where *natural-extension-map* = $(int\text{-}left\text{-}shift::((int \Rightarrow 'a) \Rightarrow (int \Rightarrow 'a)))$

definition *natural-extension-measure*$::(int \Rightarrow 'a)$ *measure*

where *natural-extension-measure* =
projective-family.lim UNIV (λI. *distr M* (Π_M $i{\in}I$. *borel*) (λx. ($\lambda i{\in}I$. (T^^(*nat*(i– *Min I*))) x))) (λi. *borel*)

definition *natural-extension-proj*::(*int* ⇒ *'a*) ⇒ *'a*
where *natural-extension-proj* = (λx. x *0*)

theorem *natural-extension*:
pmpt natural-extension-measure natural-extension-map
qmpt.invertible-qmpt natural-extension-measure natural-extension-map
mpt.mpt-factor natural-extension-measure natural-extension-map natural-extension-proj M T
⟨*proof*⟩

end

end

5 Conservativity, recurrence

theory *Recurrence*
imports *Measure-Preserving-Transformations*
begin

A dynamical system is conservative if almost every point comes back close to its starting point. This is always the case if the measure is finite, not when it is infinite (think of the translation on $\mathbb{Z}$). In conservative systems, an important construction is the induced map: the first return map to a set of finite measure. It is measure-preserving and conservative if the original system is. This makes it possible to reduce statements about general conservative systems in infinite measure to statements about systems in finite measure, and as such is extremely useful.

5.1 Definition of conservativity

locale *conservative* = *qmpt* +
assumes *conservative*: ⋀*A*. *A* ∈ *sets M* ⟹ *emeasure M A* > *0* ⟹ ∃ *n*>*0*. *emeasure M* ((*T*^^*n*)–'*A* ∩ *A*) >*0*

lemma *conservativeI*:
assumes *qmpt M T*
⋀*A*. *A* ∈ *sets M* ⟹ *emeasure M A* > *0* ⟹ ∃ *n*>*0*. *emeasure M* ((*T*^^*n*)–'*A* ∩ *A*) >*0*
shows *conservative M T*
⟨*proof*⟩

To prove conservativity, it is in fact sufficient to show that the preimages of a set of positive measure intersect it, without any measure control. Indeed, in a non-conservative system, one can construct a set which does not satisfy this property.

lemma *conservativeI2*:
assumes *qmpt M T*
$\bigwedge A.\ A \in sets\ M \Longrightarrow emeasure\ M\ A > 0 \Longrightarrow \exists\, n{>}0.\ (T\widehat{\ \ }\widehat{\ \ }n){-}`A \cap A \neq \{\}$
shows *conservative M T*
⟨*proof*⟩

There is also a dual formulation, saying that conservativity follows from the fact that a set disjoint from all its preimages has to be null.

lemma *conservativeI3*:
assumes *qmpt M T*
$\bigwedge A.\ A \in sets\ M \Longrightarrow (\forall\, n{>}0.\ (T\widehat{\ \ }\widehat{\ \ }n){-}`A \cap A = \{\}) \Longrightarrow A \in null\text{-}sets\ M$
shows *conservative M T*
⟨*proof*⟩

The inverse of a conservative map is still conservative

lemma (**in** *conservative*) *conservative-Tinv*:
assumes *invertible-qmpt*
shows *conservative M Tinv*
⟨*proof*⟩

We introduce the locale of a conservative measure preserving map.

locale *conservative-mpt* = *mpt* + *conservative*

lemma *conservative-mptI*:
assumes *mpt M T*
$\bigwedge A.\ A \in sets\ M \Longrightarrow emeasure\ M\ A > 0 \Longrightarrow \exists\, n{>}0.\ (T\widehat{\ \ }\widehat{\ \ }n){-}`A \cap A \neq \{\}$
shows *conservative-mpt M T*
⟨*proof*⟩

The fact that finite measure preserving transformations are conservative, albeit easy, is extremely important. This result is known as Poincaré recurrence theorem.

sublocale *fmpt* ⊆ *conservative-mpt*
⟨*proof*⟩

The following fact that powers of conservative maps are also conservative is true, but nontrivial. It is proved as follows: consider a set A with positive measure, take a time n_1 such that $A_1 = T^{-n_1}A \cap A$ has positive measure, then a time n_2 such that $A_2 = T^{-n_2}A_1 \cap A$ has positive measure, and so on. It follows that $T^{-(n_i+n_{i+1}+\cdots+n_j)}A \cap A$ has positive measure for all $i < j$. Then, one can find $i < j$ such that $n_i + \cdots + n_j$ is a multiple of N.

proposition (**in** *conservative*) *conservative-power*:
conservative M $(T\widehat{\ \ }\widehat{\ \ }n)$
⟨*proof*⟩

proposition (**in** *conservative-mpt*) *conservative-mpt-power*:
  conservative-mpt M $(T\hat{}\hat{}n)$
$\langle proof \rangle$

The standard way to use conservativity is as follows: if a set is almost disjoint from all its preimages, then it is null:

lemma (**in** *conservative*) *ae-disjoint-then-null*:
  assumes $A \in$ *sets M*
    $\bigwedge n.\ n > 0 \Longrightarrow A \cap (T\hat{}\hat{}n)-\text{`}A \in$ *null-sets M*
  shows $A \in$ *null-sets M*
$\langle proof \rangle$

lemma (**in** *conservative*) *disjoint-then-null*:
  assumes $A \in$ *sets M*
    $\bigwedge n.\ n > 0 \Longrightarrow A \cap (T\hat{}\hat{}n)-\text{`}A = \{\}$
  shows $A \in$ *null-sets M*
$\langle proof \rangle$

Conservativity is preserved by replacing the measure by an equivalent one.

lemma (**in** *conservative*) *conservative-density*:
  assumes [*measurable*]: $h \in$ *borel-measurable M*
    and *AE x in M*. $h\ x \neq 0$ *AE x in M*. $h\ x \neq \infty$
  shows *conservative* (*density M h*) *T*
$\langle proof \rangle$

context *qmpt* **begin**

We introduce the recurrent subset of A, i.e., the set of points of A that return to A, and the infinitely recurrent subset, i.e., the set of points of A that return infinitely often to A. In conservative systems, both coincide with A almost everywhere.

definition *recurrent-subset*::$'a$ *set* $\Rightarrow$ $'a$ *set*
  where *recurrent-subset* $A = (\bigcup n \in \{1..\}.\ A \cap (T\hat{}\hat{}n)-\text{`}A)$

definition *recurrent-subset-infty*::$'a$ *set* $\Rightarrow$ $'a$ *set*
  where *recurrent-subset-infty* $A = A - (\bigcup n.\ (T\hat{}\hat{}n)-\text{`}\ (A -$ *recurrent-subset* $A))$

lemma *recurrent-subset-infty-inf-returns*:
  $x \in$ *recurrent-subset-infty* $A \longleftrightarrow (x \in A \wedge$ *infinite* $\{n.\ (T\hat{}\hat{}n)\ x \in A\})$
$\langle proof \rangle$

lemma *recurrent-subset-infty-series-infinite*:
  assumes $x \in$ *recurrent-subset-infty A*
  shows $(\sum n.$ *indicator* $A\ ((T\hat{}\hat{}n)\ x)) = (\infty$::*ennreal*)
$\langle proof \rangle$

lemma *recurrent-subset-infty-def'*:
recurrent-subset-infty $A = (\bigcap m.\ (\bigcup n{\in}\{m..\}.\ A \cap (T\widehat{\ }\widehat{\ }n){-}`A))$
⟨*proof*⟩

lemma *recurrent-subset-incl*:
recurrent-subset $A \subseteq A$
recurrent-subset-infty $A \subseteq A$
recurrent-subset-infty $A \subseteq$ *recurrent-subset* A
⟨*proof*⟩

lemma *recurrent-subset-meas* [*measurable*]:
assumes [*measurable*]: $A \in$ *sets* M
shows *recurrent-subset* $A \in$ *sets* M
recurrent-subset-infty $A \in$ *sets* M
⟨*proof*⟩

lemma *recurrent-subset-rel-incl*:
assumes $A \subseteq B$
shows *recurrent-subset* $A \subseteq$ *recurrent-subset* B
recurrent-subset-infty $A \subseteq$ *recurrent-subset-infty* B
⟨*proof*⟩

If a point belongs to the infinitely recurrent subset of A, then when they return to A its iterates also belong to the infinitely recurrent subset.

lemma *recurrent-subset-infty-returns*:
assumes $x \in$ *recurrent-subset-infty* A $(T\widehat{\ }\widehat{\ }n)\ x \in A$
shows $(T\widehat{\ }\widehat{\ }n)\ x \in$ *recurrent-subset-infty* A
⟨*proof*⟩

lemma *recurrent-subset-of-recurrent-subset*:
recurrent-subset-infty(*recurrent-subset-infty* A) = *recurrent-subset-infty* A
⟨*proof*⟩

The Poincare recurrence theorem states that almost every point of A returns (infinitely often) to A, i.e., the recurrent and infinitely recurrent subsets of A coincide almost everywhere with A. This is essentially trivial in conservative systems, as it is a reformulation of the definition of conservativity. (What is not trivial, and has been proved above, is that it is true in finite measure preserving systems, i.e., finite measure preserving systems are automatically conservative.)

theorem (**in** *conservative*) *Poincare-recurrence-thm*:
assumes [*measurable*]: $A \in$ *sets* M
shows $A -$ *recurrent-subset* $A \in$ *null-sets* M
$A -$ *recurrent-subset-infty* $A \in$ *null-sets* M
$A\ \Delta$ *recurrent-subset* $A \in$ *null-sets* M
$A\ \Delta$ *recurrent-subset-infty* $A \in$ *null-sets* M
emeasure M (*recurrent-subset* A) = *emeasure* M A
emeasure M (*recurrent-subset-infty* A) = *emeasure* M A

$AE\ x \in A\ in\ M.\ x \in recurrent\text{-}subset\text{-}infty\ A$
⟨*proof*⟩

A convenient way to use conservativity is given in the following theorem: if T is conservative, then the series $\sum_n f(T^n x)$ is infinite for almost every x with $fx > 0$. When f is an indicator function, this is the fact that, starting from B, one returns infinitely many times to B almost surely. The general case follows by approximating f from below by constants time indicators.

theorem (**in** *conservative*) *recurrence-series-infinite*:
  fixes $f::'a \Rightarrow ennreal$
  assumes [*measurable*]: $f \in borel\text{-}measurable\ M$
  shows $AE\ x\ in\ M.\ f\ x > 0 \longrightarrow (\sum n.\ f\ ((T\widehat{\ }\widehat{\ }n)\ x)) = \infty$
⟨*proof*⟩

5.2 The first return time

The first return time to a set A under the dynamics T is the smallest integer n such that $T^n(x) \in A$. The first return time is only well defined on the recurrent subset of A, elsewhere we set it to 0 for definiteness. We can partition A according to the value of the return time on it, thus defining the return partition of A.

definition $return\text{-}time\text{-}function::'a\ set \Rightarrow ('a \Rightarrow nat)$
  where $return\text{-}time\text{-}function\ A\ x = ($
    $if\ (x \in recurrent\text{-}subset\ A)\ then\ (Inf\ \{n::nat \in \{1..\}.\ (T\widehat{\ }\widehat{\ }n)\ x \in A\})$
    $else\ 0)$

definition $return\text{-}partition::'a\ set \Rightarrow nat \Rightarrow 'a\ set$
 where $return\text{-}partition\ A\ k = A \cap (T\widehat{\ }\widehat{\ }k){-}{-}`A - (\bigcup i\in\{0<..<k\}.\ (T\widehat{\ }\widehat{\ }i){-}{-}`A)$

Basic properties of the return partition.

lemma *return-partition-basics*:
  assumes *A-meas* [*measurable*]: $A \in sets\ M$
  shows [*measurable*]: $return\text{-}partition\ A\ n \in sets\ M$
    and $disjoint\text{-}family\ (\lambda n.\ return\text{-}partition\ A\ (n+1))$
      $(\bigcup n.\ return\text{-}partition\ A\ (n+1)) = recurrent\text{-}subset\ A$
⟨*proof*⟩

Basic properties of the return time, relationship with the return partition.

lemma *return-time0*:
  $(return\text{-}time\text{-}function\ A){-}`\{0\} = UNIV - recurrent\text{-}subset\ A$
⟨*proof*⟩

lemma *return-time-n*:
  assumes [*measurable*]: $A \in sets\ M$
  shows $(return\text{-}time\text{-}function\ A){-}`\{Suc\ n\} = return\text{-}partition\ A\ (Suc\ n)$
⟨*proof*⟩

The return time is measurable.

lemma *return-time-function-meas* [*measurable*]:
  assumes [*measurable*]: $A \in$ *sets M*
  shows *return-time-function* $A \in$ *measurable M* (*count-space UNIV*)
    return-time-function $A \in$ *borel-measurable M*
⟨*proof*⟩

A close cousin of the return time and the return partition is the first entrance set: we partition the space according to the first positive time where a point enters A.

definition *first-entrance-set*::*'a set* ⇒ *nat* ⇒ *'a set*
  where *first-entrance-set* $A\ n = (T\widehat{\ }\widehat{\ }n)$ −−' A − ($\bigcup$ $i<n$. $(T\widehat{\ }\widehat{\ }i)$−−'$A$)

lemma *first-entrance-meas* [*measurable*]:
  assumes [*measurable*]: $A \in$ *sets M*
    shows *first-entrance-set* $A\ n \in$ *sets M*
⟨*proof*⟩

lemma *first-entrance-disjoint*:
  disjoint-family (*first-entrance-set* A)
⟨*proof*⟩

There is an important dynamical phenomenon: if a point has first entrance time equal to n, then their preimages either have first entrance time equal to $n + 1$ (these are the preimages not in A) or they belong to A and have first return time equal to $n + 1$. When T preserves the measure, this gives an inductive control on the measure of the first entrance set, that will be used again and again in the proof of Kac's Formula. We formulate these (simple but extremely useful) facts now.

lemma *first-entrance-rec*:
  assumes [*measurable*]: $A \in$ *sets M*
  shows *first-entrance-set* A (*Suc* n) = T−−'(*first-entrance-set* $A\ n$) − A
⟨*proof*⟩

lemma *return-time-rec*:
  assumes $A \in$ *sets M*
  shows (*return-time-function* A)−'{*Suc* n} = T−−'(*first-entrance-set* $A\ n$) ∩ A
⟨*proof*⟩

5.3 Local time controls

The local time is the time that an orbit spends in a given set. Local time controls are basic to all the forthcoming developments.

definition *local-time*::*'a set* ⇒ *nat* ⇒ *'a* ⇒ *nat*
where *local-time* $A\ n\ x$ = *card* $\{i \in \{..<n\}.\ (T\widehat{\ }\widehat{\ }i)\ x \in A\}$

lemma *local-time-birkhoff*:
  local-time $A\ n\ x$ = *birkhoff-sum* (*indicator* A) $n\ x$

$\langle proof \rangle$

lemma *local-time-meas* [*measurable*]:
  assumes [*measurable*]: $A \in$ *sets M*
  shows *local-time A n* $\in$ *borel-measurable M*
$\langle proof \rangle$

lemma *local-time-cocycle*:
  local-time A n x + *local-time A m* $((T\hat{}\hat{}n)x)$ = *local-time A* $(n+m)$ *x*
$\langle proof \rangle$

lemma *local-time-incseq*:
  incseq (λn. *local-time A n x*)
$\langle proof \rangle$

lemma *local-time-Suc*:
  local-time A $(n+1)$ *x* = *local-time A n x* + *indicator A* $((T\hat{}\hat{}n)x)$
$\langle proof \rangle$

The local time is bounded by n: at most, one returns to A all the time!

lemma *local-time-bound*:
  local-time A n x $\leq n$
$\langle proof \rangle$

The fact that local times are unbounded will be the main technical tool in the proof of recurrence results or Kac formula below. In this direction, we prove more and more general results in the lemmas below.

We show that, in $T^{-n}(A)$, the number of visits to A tends to infinity in measure, when A has finite measure. In other words, the points in $T^{-n}(A)$ with local time $< k$ have a measure tending to 0 with k. The argument, by induction on k, goes as follows.

Consider the last return to A before time n, say at time $n-i$. It lands in the set S_i with retun time i. We get $T^{-n}A \subseteq \bigcup_{n<N} T^{-(n-i)}S_i \cup R$, where the union is disjoint and R is a set of measure $\mu(T^{-n}A) - \sum_{n<N} \mu(T^{-(n-i)}S_i) = \mu(A) - \sum_{n<N} \mu(S_i)$, which tends to 0 with N and that we may therefore discard. A point with local time $< k$ at time n in $T^{-n}A$ is then a point with local time $< k-1$ at time $n-i$ in $T^{-(n-i)}S_i \subseteq T^{-(n-i)}A$. Hence, we may conclude by the induction assumption that this has small measure.

lemma (**in** *conservative-mpt*) *local-time-unbounded1*:
  assumes *A-meas* [*measurable*]: $A \in$ *sets M*
    and *fin*: *emeasure M A* $< \infty$
  shows (λn. *emeasure M* $\{x \in (T\hat{}\hat{}n)--`A.$ *local-time A n x* $< k\}$) $\longrightarrow 0$
$\langle proof \rangle$

We deduce that local times to a set B also tend to infinity on $T^{-n}A$ if B is related to A, i.e., if points in A have some iterate in B. This is clearly a necessary condition for the lemmas to hold: otherwise, points of A that never visit B have a local time equal to B equal to 0, and so do all their preimages.

The lemmas are readily reduced to the previous one on the local time to A, since if one visits A then one visits B in finite time by assumption (uniformly bounded in the first lemma, uniformly bounded on a set of large measure in the second lemma).

lemma (**in** *conservative-mpt*) *local-time-unbounded2*:
  assumes *A-meas* [*measurable*]: $A \in$ *sets M*
    and *fin*: *emeasure M A* $< \infty$
    and *incl*: $A \subseteq (T\hat{}\hat{}i)--`B$
  shows (λn. *emeasure M* $\{x \in (T\hat{}\hat{}n)--`A.$ *local-time B n x* $< k\}) \longrightarrow 0$
⟨*proof*⟩

lemma (**in** *conservative-mpt*) *local-time-unbounded3*:
  assumes *A-meas*[*measurable*]: $A \in$ *sets M*
    and *B-meas*[*measurable*]: $B \in$ *sets M*
    and *fin*: *emeasure M A* $< \infty$
    and *incl*: $A - (\bigcup i.\ (T\hat{}\hat{}i)--`B) \in$ *null-sets M*
  shows (λn. *emeasure M* $\{x \in (T\hat{}\hat{}n)--`A.$ *local-time B n x* $< k\}) \longrightarrow 0$
⟨*proof*⟩

5.4 The induced map

The map induced by T on a set A is obtained by iterating T until one lands again in A. (Outside of A, we take the identity for definiteness.) It has very nice properties: if T is conservative, then the induced map T_A also is. If T is measure preserving, then so is T_A. (In particular, even if T preserves an infinite measure, T_A is a probability preserving map if A has measure 1 – this makes it possible to prove some statements in infinite measure by using results in finite measure systems). If T is invertible, then so is T_A. We prove all these properties in this paragraph.

definition *induced-map*::$'a$ *set* $\Rightarrow$ ($'a \Rightarrow 'a$)
  where *induced-map A* = (λ *x*. ($T\hat{}\hat{}$(*return-time-function A x*)) *x*)

The set A is stabilized by the induced map.

lemma *induced-map-stabilizes-A*:
  $x \in A \longleftrightarrow$ *induced-map A x* $\in A$
⟨*proof*⟩

lemma *induced-map-iterates-stabilize-A*:
  assumes $x \in A$
  shows ((*induced-map A*)$\hat{}\hat{}n$) $x \in A$
⟨*proof*⟩

lemma *induced-map-meas* [*measurable*]:
  assumes [*measurable*]: $A \in$ *sets M*

shows *induced-map A* $\in$ *measurable M M*
⟨*proof*⟩

The iterates of the induced map are given by a power of the original map, where the power is the Birkhoff sum (for the induced map) of the first return time. This is obvious, but useful.

lemma *induced-map-iterates*:
$((induced\text{-}map\ A)$^^$n)\ x = (T$^^$(\sum i < n.$ *return-time-function A* $((induced\text{-}map\ A$ ^^$i)\ x)))\ x$
⟨*proof*⟩

lemma *induced-map-stabilizes-recurrent-infty*:
assumes $x \in$ *recurrent-subset-infty A*
shows $((induced\text{-}map\ A)$^^$n)\ x \in$ *recurrent-subset-infty A*
⟨*proof*⟩

If $x \in A$, then its successive returns to A are exactly given by the iterations of the induced map.

lemma *induced-map-returns*:
assumes $x \in A$
shows $((T$^^$n)\ x \in A) \longleftrightarrow (\exists N \le n.\ n = (\sum i<N.$ *return-time-function A* $((induced\text{-}map\ A$ ^^ $i)\ x)))$
⟨*proof*⟩

If a map is conservative, then the induced map is still conservative. Note that this statement is not true if one replaces the word "conservative" with "qmpt": inducion only works well in conservative settings.

For instance, the right translation on $\mathbb{Z}$ is qmpt, but the induced map on $\mathbb{N}$ (again the right translation) is not, since the measure of $\{0\}$ is nonzero, while its preimage, the empty set, has zero measure.

To prove conservativity, given a subset B of A, there exists some time n such that $T^{-n}B \cap B$ has positive measure. But this time n corresponds to some returns to A for the induced map, so $T^{-n}B \cap B$ is included in $\bigcup_m T_A^{-m} B \cap B$, hence one of these sets must have positive measure.

The fact that the map is qmpt is then deduced from the conservativity.

proposition (**in** *conservative*) *induced-map-conservative*:
assumes *A-meas*: $A \in$ *sets M*
shows *conservative* (*restrict-space M A*) (*induced-map A*)
⟨*proof*⟩

Now, we want to prove that, if a map is conservative and measure preserving, then the induced map is also measure preserving. We first prove it for subsets W of A of finite measure, the general case will readily follow.

The argument uses the fact that the preimage of the set of points with first entrance time n is the union of the set of points with first entrance time $n+1$, and the points of A with first return $n+1$. Following the preimage of W

under this process, we will get the intersection of $T_A^{-1}W$ with the different elements of the return partition, and the points in $T^{-n}W$ whose first $n-1$ iterates do not meet A (and the measures of these sets add up to $\mu(W)$). To conclude, it suffices to show that the measure of points in $T^{-n}W$ whose first $n-1$ iterates do not meet A tends to 0. This follows from our local times estimates above.

lemma (**in** *conservative-mpt*) *induced-map-measure-preserving-aux*:
  assumes *A-meas* [*measurable*]: $A \in$ *sets M*
    and *W-meas* [*measurable*]: $W \in$ *sets M*
    and *incl*: $W \subseteq A$
    and *fin*: *emeasure M W* $< \infty$
  shows *emeasure M* ((*induced-map A*)−−'*W*) = *emeasure M W*
⟨*proof*⟩

lemma (**in** *conservative-mpt*) *induced-map-measure-preserving*:
  assumes *A-meas* [*measurable*]: $A \in$ *sets M*
    and *W-meas* [*measurable*]: $W \in$ *sets M*
  shows *emeasure M* ((*induced-map A*)−−'*W*) = *emeasure M W*
⟨*proof*⟩

We can now express the fact that induced maps preserve the measure.

theorem (**in** *conservative-mpt*) *induced-map-conservative-mpt*:
  assumes $A \in$ *sets M*
  shows *conservative-mpt* (*restrict-space M A*) (*induced-map A*)
⟨*proof*⟩

theorem (**in** *fmpt*) *induced-map-fmpt*:
  assumes $A \in$ *sets M*
  shows *fmpt* (*restrict-space M A*) (*induced-map A*)
⟨*proof*⟩

It will be useful to reformulate the fact that the recurrent subset has full measure in terms of the induced measure, to simplify the use of the induced map later on.

lemma (**in** *conservative*) *induced-map-recurrent-typical*:
  assumes *A-meas* [*measurable*]: $A \in$ *sets M*
  shows *AE z in* (*restrict-space M A*). $z \in$ *recurrent-subset A*
    AE z in (*restrict-space M A*). $z \in$ *recurrent-subset-infty A*
⟨*proof*⟩

5.5 Kac's theorem, and variants

Kac's theorem states that, for conservative maps, the integral of the return time to a subset A is equal to the measure of the space if the dynamics is ergodic, or of the space seen by A in the general case.

This result generalizes to any induced function, not just the return time, that we define now.

definition *induced-function*::$'a$ *set* $\Rightarrow$ $('a \Rightarrow 'b$::*comm-monoid-add*$) \Rightarrow ('a \Rightarrow 'b)$
where *induced-function A f* = $(\lambda x.\ (\sum i{\in}\{..<$ *return-time-function A x*$\}.\ f((T\widehat{\ \ }i)\ x)))$

By definition, the induced function is supported on the recurrent subset of A.

lemma *induced-function-support*:
fixes f::$'a \Rightarrow$ *ennreal*
shows *induced-function A f y* = *induced-function A f y* $*$ *indicator* ((*return-time-function A*)$-$‘$\{1..\}$) *y*
⟨*proof*⟩

Basic measurability statements.

lemma *induced-function-meas-ennreal* [*measurable*]:
fixes f::$'a \Rightarrow$ *ennreal*
assumes [*measurable*]: $f \in$ *borel-measurable M A* $\in$ *sets M*
shows *induced-function A f* $\in$ *borel-measurable M*
⟨*proof*⟩

lemma *induced-function-meas-real* [*measurable*]:
fixes f::$'a \Rightarrow$ *real*
assumes [*measurable*]: $f \in$ *borel-measurable M A* $\in$ *sets M*
shows *induced-function A f* $\in$ *borel-measurable M*
⟨*proof*⟩

The Birkhoff sums of the induced function for the induced map form a subsequence of the original Birkhoff sums for the original map, corresponding to the return times to A.

lemma (**in** *conservative*) *induced-function-birkhoff-sum*:
fixes f::$'a \Rightarrow$ *real*
assumes $A \in$ *sets M*
shows *birkhoff-sum f* (*qmpt.birkhoff-sum* (*induced-map A*) (*return-time-function A*) *n x*) *x*
= *qmpt.birkhoff-sum* (*induced-map A*) (*induced-function A f*) *n x*
⟨*proof*⟩

The next lemma is very simple (just a change of variables to reorder the indices in the double sum). However, the proof I give is very tedious: infinite sums on proper subsets are not handled well, hence I use integrals on products of discrete spaces instead, and go back and forth between the two notions – maybe there are better suited tools in the library, but I could not locate them...

This is the main combinatorial tool used in the proof of Kac's Formula.

lemma *kac-series-aux*:
fixes d:: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *ennreal*
shows $(\sum n.\ (\sum i{\le}n.\ d\ i\ n)) = (\sum n.\ d\ 0\ n) + (\sum n.\ (\sum i.\ d\ (i{+}1)\ (n{+}1{+}i)))$
(**is** - = *?R*)

$\langle proof \rangle$

end

context *conservative-mpt* **begin**

We prove Kac's Formula (in the general form for induced functions) first for functions taking values in ennreal (to avoid all summabilities issues). The result for real functions will follow by domination. First, we assume additionally that f is bounded and has a support of finite measure, the general case will follow readily by truncation.

The proof is again an instance of the fact that the preimage of the set of points with first entrance time n is the union of the set of points with first entrance time $n + 1$, and the points of A with first return $n + 1$. Keeping track of the integral of f on the different parts that appear in this argument, we will see that the integral of the induced function on the set of points with return time at most n is equal to the integral of the function, up to an error controlled by the measure of points in $T^{-n}(supp(f))$ with local time 0. Local time controls ensure that this contribution vanishes asymptotically.

lemma *induced-function-nn-integral-aux*:
  fixes $f::'a \Rightarrow$ *ennreal*
  assumes *A-meas* [*measurable*]: $A \in$ *sets* M
    and *f-meas* [*measurable*]: $f \in$ *borel-measurable* M
    and *f-bound*: $\bigwedge x.\ f\ x \leq$ *ennreal* $C\ 0 \leq C$
    and *f-supp*: *emeasure* M $\{x \in$ *space* $M.\ f\ x > 0\} < \infty$
  shows $(\int^{+} y.$ *induced-function* $A\ f\ y\ \partial M) = (\int^{+}\ x \in (\bigcup n.\ (T\widehat{\ }\widehat{\ }n)--`A).\ f\ x$
$\partial M)$
$\langle proof \rangle$

We remove the boundedness assumption on f and the finiteness assumption on its support by truncation (both in space and on the values of f).

theorem *induced-function-nn-integral*:
  fixes $f::'a \Rightarrow$ *ennreal*
  assumes *A-meas* [*measurable*]: $A \in$ *sets* M
    and *f-meas* [*measurable*]: $f \in$ *borel-measurable* M
  shows $(\int^{+} y.$ *induced-function* $A\ f\ y\ \partial M) = (\int^{+} x \in (\bigcup n.\ (T\widehat{\ }\widehat{\ }n)--`A).\ f\ x$
$\partial M)$
$\langle proof \rangle$

Taking the constant function equal to 1 in the previous statement, we obtain the usual Kac Formula.

theorem *kac-formula-nonergodic*:
  assumes *A-meas* [*measurable*]: $A \in$ *sets* M
  shows $(\int^{+} y.$ *return-time-function* $A\ y\ \partial M) =$ *emeasure* M $(\bigcup n.\ (T\widehat{\ }\widehat{\ }n)--`A)$
$\langle proof \rangle$

lemma (**in** *fmpt*) *return-time-integrable*:

assumes *A-meas* [*measurable*]: $A \in$ *sets M*
shows *integrable M* (*return-time-function A*)
⟨*proof*⟩

Now, we want to prove the same result but for real-valued integrable function. We first prove the statement for nonnegative functions by reducing to the nonnegative extended reals, and then for general functions by difference.

lemma *induced-function-integral-aux*:
fixes $f::'a \Rightarrow real$
assumes *A-meas* [*measurable*]: $A \in$ *sets M*
and *f-int* [*measurable*]: *integrable M f*
and *f-pos*: $\bigwedge x.\ f\ x \geq 0$
shows *integrable M* (*induced-function A f*)
$(\int y.\ \textit{induced-function}\ A\ f\ y\ \partial M) = (\int x \in (\bigcup n.\ (T\verb|^^|n)--\text{`}A).\ f\ x\ \partial M)$
⟨*proof*⟩

Here is the general version of Kac's Formula (for a general induced function, starting from a real-valued integrable function).

theorem *induced-function-integral-nonergodic*:
fixes $f::'a \Rightarrow real$
assumes [*measurable*]: $A \in$ *sets M integrable M f*
shows *integrable M* (*induced-function A f*)
$(\int y.\ \textit{induced-function}\ A\ f\ y\ \partial M) = (\int x \in (\bigcup n.\ (T\verb|^^|n)--\text{`}A).\ f\ x\ \partial M)$
⟨*proof*⟩

We can reformulate the previous statement in terms of induced measure.

lemma *induced-function-integral-restr-nonergodic*:
fixes $f::'a \Rightarrow real$
assumes [*measurable*]: $A \in$ *sets M integrable M f*
shows *integrable* (*restrict-space M A*) (*induced-function A f*)
$(\int y.\ \textit{induced-function}\ A\ f\ y\ \partial(\textit{restrict-space}\ M\ A)) = (\int x \in (\bigcup n.\ (T\verb|^^|n)--\text{`}A).\ f\ x\ \partial M)$
⟨*proof*⟩

end

end

6 The invariant sigma-algebra, Birkhoff theorem

theory *Invariants*
imports *Recurrence HOL−Probability.Conditional-Expectation*
begin

6.1 The sigma-algebra of invariant subsets

The invariant sigma-algebra of a qmpt is made of those sets that are invariant under the dynamics. When the transformation is ergodic, it is made of sets of zero or full measure. In general, the Birkhoff theorem is expressed in terms of the conditional expectation of an integrable function with respect to the invariant sigma-algebra.

context *qmpt* **begin**

We define the invariant sigma-algebra, as the sigma algebra of sets which are invariant under the dynamics, i.e., they coincide with their preimage under T.

definition *Invariants* **where** $Invariants = sigma\ (space\ M)\ \{A \in sets\ M.\ T-`A \cap space\ M = A\}$

For this definition to make sense, we need to check that it really defines a sigma-algebra: otherwise, the `sigma` operation would make garbage out of it. This is the content of the next lemma.

lemma *Invariants-sets*: $sets\ Invariants = \{A \in sets\ M.\ T-`A \cap space\ M = A\}$
⟨*proof*⟩

By definition, the invariant subalgebra is a subalgebra of the original algebra. This is expressed in the following lemmas.

lemma *Invariants-is-subalg*: *subalgebra M Invariants*
⟨*proof*⟩

lemma *Invariants-in-sets*:
assumes $A \in sets\ Invariants$
shows $A \in sets\ M$
⟨*proof*⟩

lemma *Invariants-measurable-func*:
assumes $f \in measurable\ Invariants\ N$
shows $f \in measurable\ M\ N$
⟨*proof*⟩

We give several trivial characterizations of invariant sets or functions.

lemma *Invariants-vrestr*:
assumes $A \in sets\ Invariants$
shows $T--`A = A$
⟨*proof*⟩

lemma *Invariants-points*:
assumes $A \in sets\ Invariants\ x \in A$
shows $T\ x \in A$
⟨*proof*⟩

lemma *Invariants-func-is-invariant*:
fixes *f*::- ⇒ *'b*::*t2-space*
assumes $f \in borel\text{-}measurable\ Invariants\ x \in space\ M$
shows $f\ (T\ x) = f\ x$

⟨*proof*⟩

lemma *Invariants-func-is-invariant-n*:
  fixes f::- ⇒ *'b::t2-space*
  assumes $f \in$ *borel-measurable Invariants* $x \in$ *space M*
  shows f $((T\widehat{\ }\widehat{\ }n)\ x) = f\ x$
⟨*proof*⟩

lemma *Invariants-func-charac*:
  assumes [*measurable*]: $f \in$ *measurable M N*
    and $\bigwedge x.\ x \in$ *space M* $\Longrightarrow f(T\ x) = f\ x$
  shows $f \in$ *measurable Invariants N*
⟨*proof*⟩

lemma *birkhoff-sum-of-invariants*:
  fixes f:: - ⇒ *real*
  assumes $f \in$ *borel-measurable Invariants* $x \in$ *space M*
  shows *birkhoff-sum f n x* $= n * f\ x$
⟨*proof*⟩

There are two possible definitions of the invariant sigma-algebra, competing in the literature: one could also use the sets such that $T^{-1}A$ coincides with A up to a measure 0 set. It turns out that this is equivalent to being invariant (in our sense) up to a measure 0 set. Therefore, for all interesting purposes, the two definitions would give the same results.

For the proof, we start from an almost invariant set, and build a genuinely invariant set that coincides with it by adding or throwing away null parts.

proposition *Invariants-quasi-Invariants-sets*:
  assumes [*measurable*]: $A \in$ *sets M*
  shows ($\exists B \in$ *sets Invariants.* $A\ \Delta\ B \in$ *null-sets M*) $\longleftrightarrow$ ($T{-}{-}$‘$A\ \Delta\ A \in$ *null-sets M*)
⟨*proof*⟩

In a conservative setting, it is enough to be included in its image or its preimage to be almost invariant: otherwise, since the difference set has disjoint preimages, and is therefore null by conservativity.

lemma (**in** *conservative*) *preimage-included-then-almost-invariant*:
  assumes [*measurable*]: $A \in$ *sets M* **and** $T{-}{-}$‘$A \subseteq A$
  shows $A\ \Delta\ (T{-}{-}$‘$A) \in$ *null-sets M*
⟨*proof*⟩

lemma (**in** *conservative*) *preimage-includes-then-almost-invariant*:
  assumes [*measurable*]: $A \in$ *sets M* **and** $A \subseteq T{-}{-}$‘A
  shows $A\ \Delta\ (T{-}{-}$‘$A) \in$ *null-sets M*
⟨*proof*⟩

The above properties for sets are also true for functions: if f and $f \circ T$ coincide almost everywhere, i.e., f is almost invariant, then f coincides almost everywhere with a true invariant function.

The idea of the proof is straightforward: throw away the orbits on which f is not really invariant (say this is the complement of the good set), and replace it by 0 there. However, this does not work directly: the good set is not invariant, some points may have a non-constant value of f on their orbit but reach the good set eventually. One can however define g to be equal to the eventual value of f along the orbit, if the orbit reaches the good set, and 0 elsewhere.

proposition *Invariants-quasi-Invariants-functions*:
  fixes f::- ⇒ *'b*::{*second-countable-topology*, *t2-space*}
  assumes *f-meas* [*measurable*]: $f \in$ *borel-measurable M*
  shows ($\exists g \in$ *borel-measurable Invariants. AE x in M.* $f\ x = g\ x$) $\longleftrightarrow$ (*AE x in M.* $f(T\ x) = f\ x$)
⟨*proof*⟩

In a conservative setting, it suffices to have an almost everywhere inequality to get an almost everywhere equality, as the set where there is strict inequality has 0 measure as its iterates are disjoint, by conservativity.

proposition (**in** *conservative*) *AE-decreasing-then-invariant*:
  fixes f::- ⇒ *'b*::{*linorder-topology*, *second-countable-topology*}
  assumes *AE x in M.* $f(T\ x) \leq f\ x$
    and [*measurable*]: $f \in$ *borel-measurable M*
  shows *AE x in M.* $f(T\ x) = f\ x$
⟨*proof*⟩

proposition (**in** *conservative*) *AE-increasing-then-invariant*:
  fixes f::- ⇒ *'b*::{*linorder-topology*, *second-countable-topology*}
  assumes *AE x in M.* $f(T\ x) \geq f\ x$
    and [*measurable*]: $f \in$ *borel-measurable M*
  shows *AE x in M.* $f(T\ x) = f\ x$
⟨*proof*⟩

For an invertible map, the invariants of T and T^{-1} are the same.

lemma *Invariants-Tinv*:
  assumes *invertible-qmpt*
  shows *qmpt.Invariants M Tinv = Invariants*
⟨*proof*⟩

end

sublocale *fmpt* ⊆ *finite-measure-subalgebra M Invariants*
  ⟨*proof*⟩

context *fmpt*
begin

The conditional expectation with respect to the invariant sigma-algebra is the same for f or $f \circ T$, essentially by definition.

lemma *Invariants-of-foTn*:
fixes $f::'a \Rightarrow real$
assumes [*measurable*]: *integrable M f*
shows *AE x in M. real-cond-exp M Invariants* $(f\ o\ (T\widehat{\ }\widehat{\ }n))\ x$ = *real-cond-exp M Invariants f x*
⟨*proof*⟩

lemma *Invariants-of-foT*:
fixes $f::'a \Rightarrow real$
assumes [*measurable*]: *integrable M f*
shows *AE x in M. real-cond-exp M Invariants f x* = *real-cond-exp M Invariants* $(f\ o\ T)\ x$
⟨*proof*⟩

lemma *birkhoff-sum-Invariants*:
fixes $f::'a \Rightarrow real$
assumes [*measurable*]: *integrable M f*
shows *AE x in M. real-cond-exp M Invariants* (*birkhoff-sum f n*) $x = n\ *$ *real-cond-exp M Invariants f x*
⟨*proof*⟩

end

6.2 Birkhoff theorem

6.2.1 Almost everywhere version of Birkhoff theorem

This paragraph is devoted to the proof of Birkhoff theorem, arguably the most fundamental result of ergodic theory. This theorem asserts that Birkhoff averages of an integrable function f converge almost surely, to the conditional expectation of f with respect to the invariant sigma algebra.

This result implies for instance the strong law of large numbers (in probability theory).

There are numerous proofs of this statement, but none is really easy. We follow the very efficient argument given in Katok-Hasselblatt. To help the reader, here is the same proof informally. The first part of the proof is formalized in `birkhoff_lemma1`, the second one in `birkhoff_lemma`, and the conclusion in `birkhoff_theorem`.

Start with an integrable function g. let $G_n(x) = \max_{k \leq n} S_k g(x)$. Then $\limsup S_n g/n \leq 0$ outside of A, the set where G_n tends to infinity. Moreover, $G_{n+1} - G_n \circ T$ is bounded by g, and tends to g on A. It follows from the dominated convergence theorem that $\int_A G_{n+1} - G_n \circ T \to \int_A g$. As $\int_A G_{n+1} - G_n \circ T = \int_A G_{n+1} - G_n \geq 0$, we obtain $\int_A g \geq 0$.

Apply now this result to the function $g = f - E(f|I) - \epsilon$, where $\epsilon > 0$ is fixed. Then $\int_A g = -\epsilon\mu(A)$, then have $\mu(A) = 0$. Thus, almost surely, $\limsup S_n g/n \leq 0$, i.e., $\limsup S_n f/n \leq E(f|I) + \epsilon$. Letting ϵ tend to 0 gives $\limsup S_n f/n \leq E(f|I)$.

Applying the same result to $-f$ gives $S_n f/n \to E(f|I)$.

context *fmpt*
begin

lemma *birkhoff-aux1*:
  fixes $f::'a \Rightarrow real$
  assumes [*measurable*]: *integrable M f*
  defines $A \equiv \{x \in space\ M.\ limsup\ (\lambda n.\ ereal(birkhoff\text{-}sum\ f\ n\ x)) = \infty\}$
    shows $A \in sets\ Invariants\ (\int x.\ f\ x * indicator\ A\ x\ \partial M) \geq 0$
⟨*proof*⟩

lemma *birkhoff-aux2*:
  fixes $f::'a \Rightarrow real$
  assumes [*measurable*]: *integrable M f*
  shows $AE\ x\ in\ M.\ limsup\ (\lambda n.\ ereal(birkhoff\text{-}sum\ f\ n\ x\ /\ n)) \leq real\text{-}cond\text{-}exp\ M$ *Invariants f x*
⟨*proof*⟩

theorem *birkhoff-theorem-AE-nonergodic*:
  fixes $f::'a \Rightarrow real$
  assumes *integrable M f*
  shows $AE\ x\ in\ M.\ (\lambda n.\ birkhoff\text{-}sum\ f\ n\ x\ /\ n) \longrightarrow real\text{-}cond\text{-}exp\ M\ Invariants$ *f x*
⟨*proof*⟩

If a function f is integrable, then $E(f \circ T - f|I) = E(f \circ T|I) - E(f|I) = 0$. Hence, $S_n(f \circ T - f)/n$ converges almost everywhere to 0, i.e., $f(T^n x)/n \to 0$. It is remarkable (and sometimes useful) that this holds under the weaker condition that $f \circ T - f$ is integrable (but not necessarily f), where this naive argument fails.

The reason is that the Birkhoff sum of $f \circ T - f$ is $f \circ T^n - f$. If n is such that x and $T^n(x)$ belong to a set where f is bounded, it follows that this Birkhoff sum is also bounded. Along such a sequence of times, $S_n(f \circ T - f)/n$ tends to 0. By Poincare recurrence theorem, there are such times for almost every points. As it also converges to $E(f \circ T - f|I)$, it follows that this function is almost everywhere 0. Then $f(T^n x)/n = S_n(f \circ T^n - f)/n - f/n$ tends almost surely to $E(f \circ T - f|I) = 0$.

lemma *limit-foTn-over-n*:
  fixes $f::'a \Rightarrow real$
  assumes [*measurable*]: $f \in borel\text{-}measurable\ M$
    and $integrable\ M\ (\lambda x.\ f(T\ x) - f\ x)$
  shows $AE\ x\ in\ M.\ real\text{-}cond\text{-}exp\ M\ Invariants\ (\lambda x.\ f(T\ x) - f\ x)\ x = 0$
        $AE\ x\ in\ M.\ (\lambda n.\ f((T\widehat{\ }\widehat{\ }n)\ x)\ /\ n) \longrightarrow 0$
⟨*proof*⟩

We specialize the previous statement to the case where f itself is integrable.

lemma *limit-foTn-over-n'*:
  fixes f::$'a \Rightarrow real$
  assumes [*measurable*]: *integrable M f*
  shows *AE x in M*. $(\lambda n.\ f((T\widehat{\ }\widehat{\ }n)\ x)\ /\ n) \longrightarrow 0$
⟨*proof*⟩

It is often useful to show that a function is cohomologous to a nicer function, i.e., to prove that a given f can be written as $f = g + u - u \circ T$ where g is nicer than f. We show below that any integrable function is cohomologous to a function which is arbitrarily close to $E(f|I)$. This is an improved version of Lemma 2.1 in [Benoist-Quint, Annals of maths, 2011]. Note that the function g to which f is cohomologous is very nice (and, in particular, integrable), but the transfer function is only measurable in this argument. The fact that the control on conditional expectation is nevertheless preserved throughout the argument follows from Lemma `limit_foTn_over_n` above.

We start with the lemma (and the proof) of [BQ2011]. It shows that, if a function has a conditional expectation with respect to invariants which is positive, then it is cohomologous to a nonnegative function. The argument is the clever remark that $g = \max(0, \inf_n S_n f)$ and $u = \min(0, \inf_n S_n f)$ work (where these expressions are well defined as $S_n f$ tends to infinity thanks to our assumption).

lemma *cohomologous-approx-cond-exp-aux*:
  fixes f::$'a \Rightarrow real$
  assumes [*measurable*]: *integrable M f*
    and *AE x in M*. *real-cond-exp M Invariants f x* > 0
   shows $\exists\, u\ g.\ u \in$ *borel-measurable M* $\wedge$ (*integrable M g*) $\wedge$ (*AE x in M*. $g\ x \geq 0 \wedge g\ x \leq max\ 0\ (f\ x)) \wedge (\forall x.\ f\ x = g\ x + u\ x - u\ (T\ x))$
⟨*proof*⟩

To deduce the stronger version that f is cohomologous to an arbitrarily good approximation of $E(f|I)$, we apply the previous lemma twice, to control successively the negative and the positive side. The sign control in the conclusion of the previous lemma implies that the second step does not spoil the first one.

lemma *cohomologous-approx-cond-exp*:
  fixes f::$'a \Rightarrow real$ **and** B::$'a \Rightarrow real$
  assumes [*measurable*]: *integrable M f B* $\in$ *borel-measurable M*
    and *AE x in M*. $B\ x > 0$
   shows $\exists\, g\ u.\ u \in$ *borel-measurable M*
      $\wedge$ *integrable M g*
      $\wedge$ $(\forall x.\ f\ x = g\ x + u\ x - u\ (T\ x))$
      $\wedge$ (*AE x in M*. *abs*($g\ x$ − *real-cond-exp M Invariants f x*) $\leq B\ x$)
⟨*proof*⟩

6.2.2 L^1 version of Birkhoff theorem

The L^1 convergence in Birkhoff theorem follows from the almost everywhere convergence and general considerations on L^1 convergence (Scheffe's lemma) explained in `AE_and_int_bound_implies_L1_conv2`. This argument works neatly for nonnegative functions, the general case reduces to this one by taking the positive and negative parts of a given function.

One could also prove it by truncation: for bounded functions, the L^1 convergence follows from the boundedness and almost sure convergence. The general case follows by density, but it is a little bit tedious to write as one need to make sure that the conditional expectation of the truncation converges to the conditional expectation of the original function. This is true in L^1 as the conditional expectation is a contraction in L^1, it follows almost everywhere after taking a subsequence. All in all, the argument based on Scheffe's lemma seems more economical.

lemma *birkhoff-lemma-L1*:
  fixes f::$'a \Rightarrow$ *real*
  assumes $\bigwedge x.\ f\ x \geq 0$
    and [*measurable*]: *integrable M f*
  shows $(\lambda n.\ \int^{+}x.\ norm(birkhoff\text{-}sum\ f\ n\ x\ /\ n\ -\ real\text{-}cond\text{-}exp\ M\ Invariants\ f\ x)\ \partial M) \longrightarrow 0$
⟨*proof*⟩

theorem *birkhoff-theorem-L1-nonergodic*:
  fixes f::$'a \Rightarrow$ *real*
  assumes [*measurable*]: *integrable M f*
  shows $(\lambda n.\ \int^{+}x.\ norm(birkhoff\text{-}sum\ f\ n\ x\ /\ n\ -\ real\text{-}cond\text{-}exp\ M\ Invariants\ f\ x)\ \partial M) \longrightarrow 0$
⟨*proof*⟩

6.2.3 Conservativity of skew products

The behaviour of skew-products of the form $(x, y) \mapsto (Tx, y + fx)$ is directly related to Birkhoff theorem, as the iterates involve the Birkhoff sums in the fiber. Birkhoff theorem implies that such a skew product is conservative when the function f has vanishing conditional expectation.

To prove the theorem, assume by contradiction that a set A with positive measure does not intersect its preimages. Replacing A with a smaller set C, we can assume that C is bounded in the y-direction, by a constant N, and also that all its nonempty vertical fibers, above the projection Cx, have a measure bounded from below. Then, by Birkhoff theorem, for any $r > 0$, most of the first n preimages of C are contained in the set $\{|y| \leq rn + N\}$, of measure $O(rn)$. Hence, they can not be disjoint if $r < \mu(C)$. To make this argument rigorous, one should only consider the preimages whose x-component belongs to a set Dx where the Birkhoff sums are small. This condition has a positive measure if $\mu(Cx) + \mu(Dx) > \mu(M)$, which one can ensure by taking Dx large enough.

theorem (**in** *fmpt*) *skew-product-conservative*:
  fixes $f::'a \Rightarrow real$
  assumes [*measurable*]: *integrable M f*
    and *AE x in M. real-cond-exp M Invariants f x* $=$ *0*
  shows *conservative-mpt* ($M \bigotimes_M$ *lborel*) ($\lambda(x,y)$. (T x, $y + f$ x))
⟨*proof*⟩

6.2.4 Oscillations around the limit in Birkhoff theorem

In this paragraph, we prove that, in Birkhoff theorem with vanishing limit, the Birkhoff sums are infinitely many times arbitrarily close to 0, both on the positive and the negative side.

In the ergodic case, this statement implies for instance that if the Birkhoff sums of an integrable function tend to infinity almost everywhere, then the integral of the function can not vanish, it has to be strictly positive (while Birkhoff theorem per se does not exclude the convergence to infinity, at a rate slower than linear). This converts a qualitative information (convergence to infinity at an unknown rate) to a quantitative information (linear convergence to infinity). This result (sometimes known as Atkinson's Lemma) has been reinvented many times, for instance by Kesten and by Guivarch. It plays an important role in the study of random products of matrices.

This is essentially a consequence of the conservativity of the corresponding skew-product, proved in `skew_product_conservative`. Indeed, this implies that, starting from a small set $X \times [-e/2, e/2]$, the skew-product comes back infinitely often to itself, which implies that the Birkhoff sums at these times are bounded by e.

To show that the Birkhoff sums come back to $[0, e]$ is a little bit more tricky. Argue by contradiction, and induce on $A \times [0, e/2]$ where A is the set of points where the Birkhoff sums don't come back to $[0, e]$. Then the second coordinate decreases strictly when one iterates the skew product, which is not compatible with conservativity.

lemma *birkhoff-sum-small-asymp-lemma*:
  assumes [*measurable*]: *integrable M f*
    and *AE x in M. real-cond-exp M Invariants f x* $=$ *0* *e*>(*0*::*real*)
  shows *AE x in M. infinite* {*n. birkhoff-sum f n x* $\in$ {*0..e*}}
⟨*proof*⟩

theorem *birkhoff-sum-small-asymp-pos-nonergodic*:
  assumes [*measurable*]: *integrable M f* **and** *e* $>$ (*0*::*real*)
  shows *AE x in M. infinite* {*n. birkhoff-sum f n x* $\in$ {*n* $*$ *real-cond-exp M Invariants f x* .. *n* $*$ *real-cond-exp M Invariants f x* $+$ *e*}}
⟨*proof*⟩

theorem *birkhoff-sum-small-asymp-neg-nonergodic*:
assumes [*measurable*]: *integrable M f* **and** $e > (0::real)$
shows *AE x in M. infinite* $\{n.$ *birkhoff-sum f n x* $\in \{n *$ *real-cond-exp M Invariants f x* $- e$ .. $n *$ *real-cond-exp M Invariants f x*$\}\}$
⟨*proof*⟩

6.2.5 Conditional expectation for the induced map

Thanks to Birkhoff theorem, one can relate conditional expectations with respect to the invariant sigma algebra, for a map and for a corresponding induced map, as follows.

proposition *Invariants-cond-exp-induced-map*:
fixes $f::'a \Rightarrow real$
assumes [*measurable*]: $A \in$ *sets M integrable M f*
defines $MA \equiv$ *restrict-space M A* **and** $TA \equiv$ *induced-map A* **and** $fA \equiv$ *induced-function A f*
shows *AE x in MA. real-cond-exp MA* (*qmpt.Invariants MA TA*) *fA x*
= *real-cond-exp M Invariants f x* $*$ *real-cond-exp MA* (*qmpt.Invariants MA TA*) (*return-time-function A*) *x*
⟨*proof*⟩

corollary *Invariants-cond-exp-induced-map-0*:
fixes $f::'a \Rightarrow real$
assumes [*measurable*]: $A \in$ *sets M integrable M f* **and** *AE x in M. real-cond-exp M Invariants f x* $= 0$
defines $MA \equiv$ *restrict-space M A* **and** $TA \equiv$ *induced-map A* **and** $fA \equiv$ *induced-function A f*
shows *AE x in MA. real-cond-exp MA* (*qmpt.Invariants MA TA*) *fA x* $= 0$
⟨*proof*⟩

end
end

7 Ergodicity

theory *Ergodicity*
imports *Invariants*
begin

A transformation is *ergodic* if any invariant set has zero measure or full measure. Ergodic transformations are, in a sense, extremal among measure preserving transformations. Hence, any transformation can be seen as an average of ergodic ones. This can be made precise by the notion of ergodic decomposition, only valid on standard measure spaces.

Many statements get nicer in the ergodic case, hence we will reformulate many of the previous results in this setting.

7.1 Ergodicity locales

locale *ergodic-qmpt* = *qmpt* +
  assumes *ergodic*: $\bigwedge A.\ A \in$ *sets Invariants* $\Longrightarrow$ ($A \in$ *null-sets* $M \vee$ *space* $M - A \in$ *null-sets* M)

locale *ergodic-mpt* = *mpt* + *ergodic-qmpt*

locale *ergodic-fmpt* = *ergodic-qmpt* + *fmpt*

locale *ergodic-pmpt* = *ergodic-qmpt* + *pmpt*

locale *ergodic-conservative* = *ergodic-qmpt* + *conservative*

locale *ergodic-conservative-mpt* = *ergodic-qmpt* + *conservative-mpt*

sublocale *ergodic-fmpt* $\subseteq$ *ergodic-mpt*
  ⟨*proof*⟩

sublocale *ergodic-pmpt* $\subseteq$ *ergodic-fmpt*
  ⟨*proof*⟩

sublocale *ergodic-fmpt* $\subseteq$ *ergodic-conservative-mpt*
  ⟨*proof*⟩

sublocale *ergodic-conservative-mpt* $\subseteq$ *ergodic-conservative*
  ⟨*proof*⟩

7.2 Behavior of sets in ergodic transformations

The main property of an ergodic transformation, essentially equivalent to the definition, is that a set which is almost invariant under the dynamics is null or conull.

lemma (**in** *ergodic-qmpt*) *AE-equal-preimage-then-null-or-conull*:
  assumes [*measurable*]: $A \in$ *sets* M **and** $A\ \Delta\ T{-}{-}\text{`}A \in$ *null-sets* M
  shows $A \in$ *null-sets* $M \vee$ *space* $M - A \in$ *null-sets* M
⟨*proof*⟩

The inverse of an ergodic transformation is also ergodic.

lemma (**in** *ergodic-qmpt*) *ergodic-Tinv*:
  assumes *invertible-qmpt*
  shows *ergodic-qmpt M Tinv*
⟨*proof*⟩

In the conservative case, instead of the almost invariance of a set, it suffices to assume that the preimage is contained in the set, or contains the set, to deduce that it is null or conull.

lemma (**in** *ergodic-conservative*) *preimage-included-then-null-or-conull*:

assumes $A \in$ *sets M* $T--`A \subseteq A$
shows $A \in$ *null-sets M* $\vee$ *space M* $-$ $A \in$ *null-sets M*
⟨*proof*⟩

lemma (**in** *ergodic-conservative*) *preimage-includes-then-null-or-conull*:
assumes $A \in$ *sets M* $T--`A \supseteq A$
shows $A \in$ *null-sets M* $\vee$ *space M* $-$ $A \in$ *null-sets M*
⟨*proof*⟩

lemma (**in** *ergodic-conservative*) *preimages-conull*:
assumes [*measurable*]: $A \in$ *sets M* **and** *emeasure M A* $>$ *0*
shows *space M* $-$ $(\bigcup n.\ (T\^\^n)--`A) \in$ *null-sets M*
space M Δ $(\bigcup n.\ (T\^\^n)--`A) \in$ *null-sets M*
⟨*proof*⟩

7.3 Behavior of functions in ergodic transformations

In the same way that invariant sets are null or conull, invariant functions are almost everywhere constant in an ergodic transformation. For real functions, one can consider the set where $\{f x \geq d\}$, it has measure 0 or 1 depending on d. Then f is almost surely equal to the maximal d such that this set has measure 1. For functions taking values in a general space, the argument is essentially the same, replacing intervals by a basis of the topology.

lemma (**in** *ergodic-qmpt*) *Invariant-func-is-AE-constant*:
fixes *f*::-⇒ *'b*::{*second-countable-topology*, *t1-space*}
assumes $f \in$ *borel-measurable Invariants*
shows $\exists y$. *AE x in M*. $f\ x = y$
⟨*proof*⟩

The same goes for functions which are only almost invariant, as they coindice almost everywhere with genuine invariant functions.

lemma (**in** *ergodic-qmpt*) *AE-Invariant-func-is-AE-constant*:
fixes *f*::- ⇒ *'b*::{*second-countable-topology*, *t2-space*}
assumes $f \in$ *borel-measurable M AE x in M*. $f(T\ x) = f\ x$
shows $\exists y$. *AE x in M*. $f\ x = y$
⟨*proof*⟩

In conservative systems, it suffices to have an inequality between f and $f \circ T$, since such a function is almost invariant.

lemma (**in** *ergodic-conservative*) *AE-decreasing-func-is-AE-constant*:
fixes *f*::- ⇒ *'b*::{*linorder-topology*, *second-countable-topology*}
assumes *AE x in M*. $f(T\ x) \leq f\ x$
and [*measurable*]: $f \in$ *borel-measurable M*
shows $\exists y$. *AE x in M*. $f\ x = y$
⟨*proof*⟩

lemma (**in** *ergodic-conservative*) *AE-increasing-func-is-AE-constant*:

fixes f::- $\Rightarrow$ $'b$::{*linorder-topology*, *second-countable-topology*}
assumes *AE x in M*. $f(T\ x) \geq f\ x$
and [*measurable*]: $f \in$ *borel-measurable M*
shows $\exists\, y$. *AE x in M*. $f\ x = y$
⟨*proof*⟩

When the function takes values in a Banach space, the value of the invariant (hence constant) function can be recovered by integrating the function.

lemma (**in** *ergodic-fmpt*) *Invariant-func-integral*:
fixes f::- $\Rightarrow$ $'b$::{*banach*, *second-countable-topology*}
assumes [*measurable*]: $f \in$ *borel-measurable Invariants*
shows *integrable M f*
AE x in M. $f\ x = (\int x.\ f\ x\ \partial M)/_R$ (*measure M* (*space M*))
⟨*proof*⟩

As the conditional expectation of a function and the original function have the same integral, it follows that the conditional expectation of a function with respect to the invariant sigma algebra is given by the average of the function.

lemma (**in** *ergodic-fmpt*) *Invariants-cond-exp-is-integral-fmpt*:
fixes f::- $\Rightarrow$ *real*
assumes *integrable M f*
shows *AE x in M*. *real-cond-exp M Invariants f x* $= (\int x.\ f\ x\ \partial M)$ / *measure M* (*space M*)
⟨*proof*⟩

lemma (**in** *ergodic-pmpt*) *Invariants-cond-exp-is-integral*:
fixes f::- $\Rightarrow$ *real*
assumes *integrable M f*
shows *AE x in M*. *real-cond-exp M Invariants f x* $= (\int x.\ f\ x\ \partial M)$
⟨*proof*⟩

7.4 Kac formula

We reformulate the different versions of Kac formula. They simplify as, for any set A with positive measure, the union $\bigcup T^{-n}A$ (which appears in all these statements) almost coincides with the whole space.

lemma (**in** *ergodic-conservative-mpt*) *local-time-unbounded*:
assumes [*measurable*]: $A \in$ *sets M* $B \in$ *sets M*
and *emeasure M A* $< \infty$ *emeasure M B* > 0
shows (λn. *emeasure M* $\{x \in (T\widehat{\ }\widehat{\ }n)$−−‘$A$. *local-time B n x* $< k\}$) $\longrightarrow 0$
⟨*proof*⟩

theorem (**in** *ergodic-conservative-mpt*) *kac-formula*:
assumes [*measurable*]: $A \in$ *sets M* **and** *emeasure M A* > 0
shows ($\int^+ y$. *return-time-function A y* ∂M) = *emeasure M* (*space M*)
⟨*proof*⟩

lemma (**in** *ergodic-conservative-mpt*) *induced-function-integral*:
fixes $f::'a \Rightarrow real$
assumes [*measurable*]: $A \in sets\ M$ *integrable M f* **and** *emeasure M A* > 0
shows *integrable M* (*induced-function A f*)
$(\int y.\ induced\text{-}function\ A\ f\ y\ \partial M) = (\int\ x.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma (**in** *ergodic-conservative-mpt*) *induced-function-integral-restr*:
fixes $f::'a \Rightarrow real$
assumes [*measurable*]: $A \in sets\ M$ *integrable M f* **and** *emeasure M A* > 0
shows *integrable* (*restrict-space M A*) (*induced-function A f*)
$(\int y.\ induced\text{-}function\ A\ f\ y\ \partial(restrict\text{-}space\ M\ A)) = (\int\ x.\ f\ x\ \partial M)$
⟨*proof*⟩

7.5 Birkhoff theorem

The general versions of Birkhoff theorem are formulated in terms of conditional expectations. In ergodic probability measure preserving transformations (the most common setting), they reduce to simpler versions that we state now, as the conditional expectations are simply the averages of the functions.

theorem (**in** *ergodic-pmpt*) *birkhoff-theorem-AE*:
fixes $f::'a \Rightarrow real$
assumes *integrable M f*
shows $AE\ x\ in\ M.\ (\lambda n.\ birkhoff\text{-}sum\ f\ n\ x\ /\ n) \longrightarrow (\int\ x.\ f\ x\ \partial M)$
⟨*proof*⟩

theorem (**in** *ergodic-pmpt*) *birkhoff-theorem-L1*:
fixes $f::'a \Rightarrow real$
assumes [*measurable*]: *integrable M f*
shows $(\lambda n.\ \int^+ x.\ norm(birkhoff\text{-}sum\ f\ n\ x\ /\ n - (\int\ x.\ f\ x\ \partial M))\ \partial M) \longrightarrow 0$
⟨*proof*⟩

theorem (**in** *ergodic-pmpt*) *birkhoff-sum-small-asymp-pos*:
fixes $f::'a \Rightarrow real$
assumes [*measurable*]: *integrable M f* **and** $e>0$
shows $AE\ x\ in\ M.\ infinite\ \{n.\ birkhoff\text{-}sum\ f\ n\ x \in \{n * (\int x.\ f\ x\ \partial M)\ ..\ n * (\int x.\ f\ x\ \partial M) + e\}\}$
⟨*proof*⟩

theorem (**in** *ergodic-pmpt*) *birkhoff-sum-small-asymp-neg*:
fixes $f::'a \Rightarrow real$
assumes [*measurable*]: *integrable M f* **and** $e>0$
shows $AE\ x\ in\ M.\ infinite\ \{n.\ birkhoff\text{-}sum\ f\ n\ x \in \{n * (\int x.\ f\ x\ \partial M) - e\ ..\ n * (\int x.\ f\ x\ \partial M)\}\}$
⟨*proof*⟩

lemma (**in** *ergodic-pmpt*) *birkhoff-positive-average*:
fixes f::$'a \Rightarrow$ *real*
assumes [*measurable*]: *integrable M f* **and** *AE x in M*. (λn. *birkhoff-sum f n x*) $\longrightarrow \infty$
shows ($\int$ x. f x ∂M) $>$ *0*
⟨*proof*⟩

lemma (**in** *ergodic-pmpt*) *birkhoff-negative-average*:
fixes f::$'a \Rightarrow$ *real*
assumes [*measurable*]: *integrable M f* **and** *AE x in M*. (λn. *birkhoff-sum f n x*) $\longrightarrow -\infty$
shows ($\int$ x. f x ∂M) $<$ *0*
⟨*proof*⟩

lemma (**in** *ergodic-pmpt*) *birkhoff-nonzero-average*:
fixes f::$'a \Rightarrow$ *real*
assumes [*measurable*]: *integrable M f* **and** *AE x in M*. (λn. *abs*(*birkhoff-sum f n x*)) $\longrightarrow \infty$
shows ($\int$ x. f x ∂M) $\neq$ *0*
⟨*proof*⟩

end

8 The shift operator on an infinite product measure

theory *Shift-Operator*
imports *Ergodicity ME-Library-Complement*
begin

Let P be an an infinite product of i.i.d. instances of the distribution M. Then the shift operator is the map

$$T(x_0, x_1, x_2, \ldots) = T(x_1, x_2, \ldots)\ .$$

In this section, we define this operator and show that it is ergodic using Kolmogorov's 0–1 law.

locale *shift-operator-ergodic* = *prob-space* +
fixes T :: (*nat* $\Rightarrow$ $'a$) $\Rightarrow$ (*nat* $\Rightarrow$ $'a$) **and** P :: (*nat* $\Rightarrow$ $'a$) *measure*
defines $T \equiv (\lambda f.\ f \circ$ *Suc*)
defines $P \equiv$ *PiM* (*UNIV* :: *nat set*) (λ-. M)
begin

sublocale P: *product-prob-space* λ-. *M UNIV*
⟨*proof*⟩

sublocale P: *prob-space P*
⟨*proof*⟩

lemma *measurable-T* [*measurable*]: $T \in P \rightarrow_M P$
⟨*proof*⟩

The n-th tail algebra $\mathcal{T}_n$ is, in some sense, the algebra in which we forget all information about all x_i with $i < n$. We simply change the product algebra of P by replacing the algebra for each $i < n$ with the trivial algebra that contains only the empty set and the entire space.

definition *tail-algebra* :: $nat \Rightarrow (nat \Rightarrow 'a)$ *measure*
where *tail-algebra n = PiM UNIV* ($\lambda i.$ *if* $i < n$ *then trivial-measure* (*space M*) *else M*)

lemma *tail-algebra-0* [*simp*]: *tail-algebra 0 = P*
⟨*proof*⟩

lemma *space-tail-algebra* [*simp*]: *space* (*tail-algebra n*) = *PiE UNIV* (λ-. *space M*)
⟨*proof*⟩

lemma *measurable-P-component* [*measurable*]: *P.random-variable M* ($\lambda f.\ f\ i$)
⟨*proof*⟩

lemma *P-component* [*simp*]: *distr P M* ($\lambda f.\ f\ i$) = *M*
⟨*proof*⟩

lemma *indep-vars*: *P.indep-vars* (λ-. *M*) ($\lambda i\ f.\ f\ i$) *UNIV*
⟨*proof*⟩

The shift operator takes us from $\mathcal{T}_n$ to $\mathcal{T}_{n+1}$ (it forgets the information about one more variable):

lemma *measurable-T-tail*: $T \in$ *tail-algebra* (*Suc n*) $\rightarrow_M$ *tail-algebra n*
⟨*proof*⟩

lemma *measurable-funpow-T*: T ^^ $n \in$ *tail-algebra* $(m + n) \rightarrow_M$ *tail-algebra m*
⟨*proof*⟩

lemma *measurable-funpow-T'*: T ^^ $n \in$ *tail-algebra n* $\rightarrow_M P$
⟨*proof*⟩

The shift operator is clearly measure-preserving:

lemma *measure-preserving*: $T \in$ *measure-preserving P P*
⟨*proof*⟩

sublocale *fmpt P T*
⟨*proof*⟩

lemma *indep-sets-vimage-algebra*:
P.indep-sets ($\lambda i.$ *sets* (*vimage-algebra* (*space P*) ($\lambda f.\ f\ i$) *M*)) *UNIV*

⟨*proof*⟩

We can now show that the tail algebra $\mathcal{T}_n$ is a subalgebra of the algebra generated by the algebras induced by all the variables x_i with $i \geq n$:

lemma *tail-algebra-subset*:
sets (*tail-algebra n*) $\subseteq$
sigma-sets (*space P*) ($\bigcup i \in \{n..\}$. *sets* (*vimage-algebra* (*space P*) (λf. *f i*) *M*))
⟨*proof*⟩

It now follows that the T-invariant events are a subset of the tail algebra induced by the variables:

lemma *Invariants-subset-tail-algebra*:
sets Invariants $\subseteq$ *P.tail-events* (λi. *sets* (*vimage-algebra* (*space P*) (λf. *f i*) *M*))
⟨*proof*⟩

A simple invocation of Kolmogorov's 0–1 law now proves that T is indeed ergodic:

sublocale *ergodic-fmpt P T*
⟨*proof*⟩

end

end

9 Subcocycles, subadditive ergodic theory

theory *Kingman*
imports *Ergodicity Fekete*
begin

Subadditive ergodic theory is the branch of ergodic theory devoted to the study of subadditive cocycles (named subcocycles in what follows), i.e., functions such that $u(n+m,x) \leq u(n,x) + u(m,T^n x)$ for all x and m,n.

For instance, Birkhoff sums are examples of such subadditive cocycles (in fact, they are additive), but more interesting examples are genuinely subadditive. The main result of the theory is Kingman's theorem, asserting the almost sure convergence of u_n/n (this is a generalization of Birkhoff theorem). If the asymptotic average $\lim \int u_n/n$ (which exists by subadditivity and Fekete lemma) is not $-\infty$, then the convergence takes also place in L^1. We prove all this below.

context *mpt*
begin

9.1 Definition and basic properties

definition *subcocycle*::(*nat* $\Rightarrow$ *'a* $\Rightarrow$ *real*) $\Rightarrow$ *bool*

where *subcocycle* $u = ((\forall\, n.\ integrable\ M\ (u\ n)) \wedge (\forall\, n\ m\ x.\ u\ (n{+}m)\ x \leq u\ n\ x + u\ m\ ((T\widehat{\ }\widehat{\ }n)\ x)))$

lemma *subcocycle-ineq*:
assumes *subcocycle u*
shows $u\ (n{+}m)\ x \leq u\ n\ x + u\ m\ ((T\widehat{\ }\widehat{\ }n)\ x)$
⟨*proof*⟩

lemma *subcocycle-0-nonneg*:
assumes *subcocycle u*
shows $u\ 0\ x \geq 0$
⟨*proof*⟩

lemma *subcocycle-integrable*:
assumes *subcocycle u*
shows *integrable M* (*u n*)
u n ∈ *borel-measurable M*
⟨*proof*⟩

lemma *subcocycle-birkhoff*:
assumes *integrable M f*
shows *subcocycle* (*birkhoff-sum f*)
⟨*proof*⟩

The set of subcocycles is stable under addition, multiplication by positive numbers, and max.

lemma *subcocycle-add*:
assumes *subcocycle u subcocycle v*
shows *subcocycle* $(\lambda n\ x.\ u\ n\ x + v\ n\ x)$
⟨*proof*⟩

lemma *subcocycle-cmult*:
assumes *subcocycle u* $c \geq 0$
shows *subcocycle* $(\lambda n\ x.\ c * u\ n\ x)$
⟨*proof*⟩

lemma *subcocycle-max*:
assumes *subcocycle u subcocycle v*
shows *subcocycle* $(\lambda n\ x.\ max\ (u\ n\ x)\ (v\ n\ x))$
⟨*proof*⟩

Applying inductively the subcocycle equation, it follows that a subcocycle is bounded by the Birkhoff sum of the subcocycle at time 1.

lemma *subcocycle-bounded-by-birkhoff1*:
assumes *subcocycle u* $n > 0$
shows $u\ n\ x \leq$ *birkhoff-sum* $(u\ 1)\ n\ x$
⟨*proof*⟩

It is often important to bound a cocycle $u_n(x)$ by the Birkhoff sums of

u_N/N. Compared to the trivial upper bound for u_1, there are additional boundary errors that make the estimate more cumbersome (but these terms only come from a N-neighborhood of 0 and n, so they are negligible if N is fixed and n tends to infinity.

lemma *subcocycle-bounded-by-birkhoffN*:
  assumes *subcocycle u* $n > 2*N$ $N>0$
  shows $u\ n\ x \le$ *birkhoff-sum* $(\lambda x.\ u\ N\ x\ /\ real\ N)\ (n - 2 * N)\ x$
$+ (\sum i<N.\ |u\ 1\ ((T\ \widehat{\ }\widehat{\ }\ i)\ x)|)$
$+ 2 * (\sum i<2*N.\ |u\ 1\ ((T\ \widehat{\ }\widehat{\ }\ (n - (2 * N - i)))\ x)|)$
⟨*proof*⟩

Many natural cocycles are only defined almost everywhere, and then the subadditivity property only makes sense almost everywhere. We will now show that such an a.e.-subcocycle coincides almost everywhere with a genuine subcocycle in the above sense. Then, all the results for subcocycles will apply to such a.e.-subcocycles. (Usually, in ergodic theory, subcocycles only satisfy the subadditivity property almost everywhere, but we have requested it everywhere for simplicity in the proofs.)

The subcocycle will be defined in a recursive way. This means that is can not be defined in a proof (since complicated function definitions are not available inside proofs). Since it is defined in terms of u, then u has to be available at the top level, which is most conveniently done using a context.

context
  fixes u::*nat* $\Rightarrow$ *'a* $\Rightarrow$ *real*
  assumes H: $\bigwedge m\ n.$ *AE x in M*. $u\ (n+m)\ x \le u\ n\ x + u\ m\ ((T\widehat{\ }\widehat{\ }n)\ x)$
$\bigwedge n.$ *integrable M* $(u\ n)$
begin

private fun v :: *nat* $\Rightarrow$ *'a* $\Rightarrow$ *real* **where** $v\ n\ x =$ (
  if $n = 0$ *then max* $(u\ 0\ x)\ 0$
  else if $n = 1$ *then* $u\ 1\ x$
  else min $(u\ n\ x)\ (Min\ ((\lambda k.\ v\ k\ x + v\ (n-k)\ ((T\widehat{\ }\widehat{\ }k)\ x))\ `\{0<..<n\})))$

private lemma *v0* [*simp*]:
  ‹$v\ 0\ x = max\ (u\ 0\ x)\ 0$›
  ⟨*proof*⟩ **lemma** *v1* [*simp*]:
  ‹$v\ (Suc\ 0)\ x = u\ 1\ x$›
  ⟨*proof*⟩ **lemma** *v2* [*simp*]:
  ‹$v\ n\ x = min\ (u\ n\ x)\ (Min\ ((\lambda k.\ v\ k\ x + v\ (n-k)\ ((T\widehat{\ }\widehat{\ }k)\ x))\ `\{0<..<n\}))$› **if**
‹$n \ge 2$›
  ⟨*proof*⟩

declare *v.simps* [*simp del*]

private lemma *integrable-v*:
  integrable M $(v\ n)$ **for** n
⟨*proof*⟩ **lemma** *u-eq-v-AE*:

AE x in M. v n x = u n x **for** *n*
⟨*proof*⟩ **lemma** *subcocycle-v*:
v (n+m) x ≤ v n x + v m ((T^^n) x)
⟨*proof*⟩

lemma *subcocycle-AE-in-context*:
∃ w. subcocycle w ∧ (AE x in M. ∀ n. w n x = u n x)
⟨*proof*⟩

end

lemma *subcocycle-AE*:
fixes *u::nat ⇒ 'a ⇒ real*
assumes *⋀m n. AE x in M. u (n+m) x ≤ u n x + u m ((T^^n) x)*
⋀n. integrable M (u n)
shows *∃ w. subcocycle w ∧ (AE x in M. ∀ n. w n x = u n x)*
⟨*proof*⟩

9.2 The asymptotic average

In this subsection, we define the asymptotic average of a subcocycle u, i.e., the limit of $\int u_n(x)/n$ (the convergence follows from subadditivity of $\int u_n$) and study its basic properties, especially in terms of operations on subcocycles. In general, it can be $-\infty$, so we define it in the extended reals.

definition *subcocycle-avg-ereal::(nat ⇒ 'a ⇒ real) ⇒ ereal* **where**
subcocycle-avg-ereal u = Inf {ereal((∫ x. u n x ∂M) / n) |n. n > 0}

lemma *subcocycle-avg-finite*:
subcocycle-avg-ereal u < ∞
⟨*proof*⟩

lemma *subcocycle-avg-subadditive*:
assumes *subcocycle u*
shows *subadditive (λn. (∫ x. u n x ∂M))*
⟨*proof*⟩

lemma *subcocycle-int-tendsto-avg-ereal*:
assumes *subcocycle u*
shows *(λn. (∫ x. u n x / n ∂M)) ⟶ subcocycle-avg-ereal u*
⟨*proof*⟩

The average behaves well under addition, scalar multiplication and max, trivially.

lemma *subcocycle-avg-ereal-add*:
assumes *subcocycle u subcocycle v*
shows *subcocycle-avg-ereal (λn x. u n x + v n x) = subcocycle-avg-ereal u + subcocycle-avg-ereal v*
⟨*proof*⟩

lemma *subcocycle-avg-ereal-cmult*:
  assumes *subcocycle u* $c \geq (0::real)$
  shows *subcocycle-avg-ereal* $(\lambda n\ x.\ c * u\ n\ x) = c *$ *subcocycle-avg-ereal u*
⟨*proof*⟩

lemma *subcocycle-avg-ereal-max*:
  assumes *subcocycle u subcocycle v*
  shows *subcocycle-avg-ereal* $(\lambda n\ x.\ max\ (u\ n\ x)\ (v\ n\ x)) \geq max$ (*subcocycle-avg-ereal u*) (*subcocycle-avg-ereal v*)
⟨*proof*⟩

For a Birkhoff sum, the average at each time is the same, equal to the average of the function, so the asymptotic average is also equal to this common value.

lemma *subcocycle-avg-ereal-birkhoff*:
  assumes *integrable M u*
  shows *subcocycle-avg-ereal* (*birkhoff-sum u*) = $(\int x.\ u\ x\ \partial M)$
⟨*proof*⟩

In nice situations, where one can avoid the use of ereal, the following definition is more convenient. The kind of statements we are after is as follows: if the ereal average is finite, then something holds, likely involving the real average.

In particular, we show in this setting what we have proved above under this new assumption: convergence (in real numbers) of the average to the asymptotic average, as well as good behavior under sum, scalar multiplication by positive numbers, max, formula for Birkhoff sums.

definition *subcocycle-avg*::$(nat \Rightarrow 'a \Rightarrow real) \Rightarrow real$ **where**
  subcocycle-avg u = *real-of-ereal*(*subcocycle-avg-ereal u*)

lemma *subcocycle-avg-real-ereal*:
  assumes *subcocycle-avg-ereal u* $> -\infty$
  shows *subcocycle-avg-ereal u* = *ereal*(*subcocycle-avg u*)
⟨*proof*⟩

lemma *subcocycle-int-tendsto-avg*:
  assumes *subcocycle u subcocycle-avg-ereal u* $> -\infty$
  shows $(\lambda n.\ (\int x.\ u\ n\ x\ /\ n\ \partial M)) \longrightarrow$ *subcocycle-avg u*
⟨*proof*⟩

lemma *subcocycle-avg-add*:
  assumes *subcocycle u subcocycle v subcocycle-avg-ereal u* $> -\infty$ *subcocycle-avg-ereal v* $> -\infty$
  shows *subcocycle-avg-ereal* $(\lambda n\ x.\ u\ n\ x + v\ n\ x) > -\infty$
    subcocycle-avg $(\lambda n\ x.\ u\ n\ x + v\ n\ x)$ = *subcocycle-avg u* + *subcocycle-avg v*
⟨*proof*⟩

lemma *subcocycle-avg-cmult*:

assumes *subcocycle u* $c \geq (0::real)$ *subcocycle-avg-ereal u* $> -\infty$
shows *subcocycle-avg-ereal* $(\lambda n\ x.\ c * u\ n\ x) > -\infty$
subcocycle-avg $(\lambda n\ x.\ c * u\ n\ x) = c *$ *subcocycle-avg u*
⟨*proof*⟩

lemma *subcocycle-avg-max*:
assumes *subcocycle u subcocycle v subcocycle-avg-ereal u* $> -\infty$ *subcocycle-avg-ereal v* $> -\infty$
shows *subcocycle-avg-ereal* $(\lambda n\ x.\ max\ (u\ n\ x)\ (v\ n\ x)) > -\infty$
subcocycle-avg $(\lambda n\ x.\ max\ (u\ n\ x)\ (v\ n\ x)) \geq max$ (*subcocycle-avg u*) (*subcocycle-avg v*)
⟨*proof*⟩

lemma *subcocycle-avg-birkhoff*:
assumes *integrable M u*
shows *subcocycle-avg-ereal* (*birkhoff-sum u*) $> -\infty$
subcocycle-avg (*birkhoff-sum u*) $= (\int x.\ u\ x\ \partial M)$
⟨*proof*⟩

end

9.3 Almost sure convergence of subcocycles

In this paragraph, we prove Kingman's theorem, i.e., the almost sure convergence of subcocycles. Their limit is almost surely invariant. There is no really easy proof. The one we use below is arguably the simplest known one, due to Steele (1989). The idea is to show that the limsup of the subcocycle is bounded by the liminf (which is almost surely constant along trajectories), by using subadditivity along time intervals where the liminf is almost reached, of length at most N. For some points, the liminf takes a large time $> N$ to be reached. We neglect those times, introducing an additional error that gets smaller with N, thanks to Birkhoff ergodic theorem applied to the set of bad points. The error is most easily managed if the subcocycle is assumed to be nonpositive, which one can assume in a first step. The general case is reduced to this one by replacing u_n with $u_n - S_n u_1 \leq 0$, and using Birkhoff theorem to control $S_n u_1$.

context *fmpt* **begin**

First, as explained above, we prove the theorem for nonpositive subcocycles.

lemma *kingman-theorem-AE-aux1*:
assumes *subcocycle u*
$\bigwedge x.\ u\ 1\ x \leq 0$
shows $\exists (g::'a \Rightarrow ereal).\ (g \in$ *borel-measurable Invariants* $\wedge (\forall x.\ g\ x < \infty) \wedge (AE\ x\ in\ M.\ (\lambda n.\ u\ n\ x\ /\ n) \longrightarrow g\ x))$
⟨*proof*⟩

We deduce it for general subcocycles, by reducing to nonpositive subcocycles by subtracting the Birkhoff sum of u_1 (for which the convergence follows from Birkhoff theorem).

theorem *kingman-theorem-AE-aux2*:
assumes *subcocycle u*
shows $\exists (g::'a \Rightarrow ereal).\ (g \in borel\text{-}measurable\ Invariants \wedge (\forall x.\ g\ x < \infty) \wedge (AE\ x\ in\ M.\ (\lambda n.\ u\ n\ x\ /\ n) \longrightarrow g\ x))$
$\langle proof \rangle$

For applications, it is convenient to have a limit which is really measurable with respect to the invariant sigma algebra and does not come from a hard to use abstract existence statement. Hence we introduce the following definition for the would-be limit – Kingman's theorem shows that it is indeed a limit.

We introduce the definition for any function, not only subcocycles, but it will only be usable for subcocycles. We introduce an if clause in the definition so that the limit is always measurable, even when u is not a subcocycle and there is no convergence.

definition *subcocycle-lim-ereal*::$(nat \Rightarrow 'a \Rightarrow real) \Rightarrow ('a \Rightarrow ereal)$
where *subcocycle-lim-ereal u* = (
if $(\exists (g::'a \Rightarrow ereal).\ (g \in borel\text{-}measurable\ Invariants \wedge$
$(\forall x.\ g\ x < \infty) \wedge (AE\ x\ in\ M.\ (\lambda n.\ u\ n\ x\ /\ n) \longrightarrow g\ x)))$
then $(SOME\ (g::'a \Rightarrow ereal).\ g \in borel\text{-}measurable\ Invariants \wedge$
$(\forall x.\ g\ x < \infty) \wedge (AE\ x\ in\ M.\ (\lambda n.\ u\ n\ x\ /\ n) \longrightarrow g\ x))$
else $(\lambda\text{-}.\ 0))$

definition *subcocycle-lim*::$(nat \Rightarrow 'a \Rightarrow real) \Rightarrow ('a \Rightarrow real)$
where *subcocycle-lim u* = $(\lambda x.\ real\text{-}of\text{-}ereal(subcocycle\text{-}lim\text{-}ereal\ u\ x))$

lemma *subcocycle-lim-meas-Inv* [*measurable*]:
subcocycle-lim-ereal u $\in$ *borel-measurable Invariants*
subcocycle-lim u $\in$ *borel-measurable Invariants*
$\langle proof \rangle$

lemma *subcocycle-lim-meas* [*measurable*]:
subcocycle-lim-ereal u $\in$ *borel-measurable M*
subcocycle-lim u $\in$ *borel-measurable M*
$\langle proof \rangle$

lemma *subcocycle-lim-ereal-not-PInf*:
subcocycle-lim-ereal u x $< \infty$
$\langle proof \rangle$

We reformulate the subadditive ergodic theorem of Kingman with this definition. From this point on, the technical definition of `subcocycle_lim_ereal` will never be used, only the following property will be relevant.

theorem *kingman-theorem-AE-nonergodic-ereal*:
assumes *subcocycle u*

shows *AE x in M*. (λ*n. u n x / n*) $\longrightarrow$ *subcocycle-lim-ereal u x*
⟨*proof*⟩

The subcocycle limit behaves well under addition, multiplication by a positive scalar, max, and is simply the conditional expectation with respect to invariants for Birkhoff sums, thanks to Birkhoff theorem.

lemma *subcocycle-lim-ereal-add*:
assumes *subcocycle u subcocycle v*
shows *AE x in M*. *subcocycle-lim-ereal* (λ*n x. u n x* + *v n x*) *x* = *subcocycle-lim-ereal u x* + *subcocycle-lim-ereal v x*
⟨*proof*⟩

lemma *subcocycle-lim-ereal-cmult*:
assumes *subcocycle u c*≥(*0::real*)
shows *AE x in M*. *subcocycle-lim-ereal* (λ*n x. c* ∗ *u n x*) *x* = *c* ∗ *subcocycle-lim-ereal u x*
⟨*proof*⟩

lemma *subcocycle-lim-ereal-max*:
assumes *subcocycle u subcocycle v*
shows *AE x in M*. *subcocycle-lim-ereal* (λ*n x. max* (*u n x*) (*v n x*)) *x*
= *max* (*subcocycle-lim-ereal u x*) (*subcocycle-lim-ereal v x*)
⟨*proof*⟩

lemma *subcocycle-lim-ereal-birkhoff*:
assumes *integrable M u*
shows *AE x in M*. *subcocycle-lim-ereal* (*birkhoff-sum u*) *x* = *ereal*(*real-cond-exp M Invariants u x*)
⟨*proof*⟩

9.4 L^1 and a.e. convergence of subcocycles with finite asymptotic average

In this subsection, we show that the almost sure convergence in Kingman theorem also takes place in L^1 if the limit is integrable, i.e., if the asymptotic average of the subcocycle is $> -\infty$. To deduce it from the almost sure convergence, we only need to show that there is no loss of mass, i.e., that the integral of the limit is not strictly larger than the limit of the integrals (thanks to Scheffe criterion). This follows from comparison to Birkhoff sums, for which we know that the average of the limit is the same as the average of the function.

First, we show that the subcocycle limit is bounded by the limit of the Birkhoff sums of u_N, i.e., its conditional expectation. This follows from the fact that u_n is bounded by the Birkhoff sum of u_N (up to negligible boundary terms).

lemma *subcocycle-lim-ereal-atmost-uN-invariants*:

assumes *subcocycle u* $N{>}(0{::}nat)$
shows *AE x in M. subcocycle-lim-ereal u x* $\leq$ *real-cond-exp M Invariants* ($\lambda x.$ *u N x / N*) *x*
⟨*proof*⟩

To apply Scheffe criterion, we need to deal with nonnegative functions, or equivalently with nonpositive functions after a change of sign. Hence, as in the proof of the almost sure version of Kingman theorem above, we first give the proof assuming that the subcocycle is nonpositive, and deduce the general statement by adding a suitable Birkhoff sum.

lemma *kingman-theorem-L1-aux*:
assumes *subcocycle u subcocycle-avg-ereal u* $> -\infty$ $\bigwedge x.$ *u 1 x* ≤ 0
shows *AE x in M.* ($\lambda n.$ *u n x / n*) $\longrightarrow$ *subcocycle-lim u x*
integrable M (*subcocycle-lim u*)
($\lambda n.$ ($\int^+ x.$ *abs*(*u n x / n* $-$ *subcocycle-lim u x*) ∂M)) $\longrightarrow 0$
⟨*proof*⟩

We can then remove the nonpositivity assumption, by subtracting the Birkhoff sums of u_1 to a general subcocycle u.

theorem *kingman-theorem-nonergodic*:
assumes *subcocycle u subcocycle-avg-ereal u* $> -\infty$
shows *AE x in M.* ($\lambda n.$ *u n x / n*) $\longrightarrow$ *subcocycle-lim u x*
integrable M (*subcocycle-lim u*)
($\lambda n.$ ($\int^+ x.$ *abs*(*u n x / n* $-$ *subcocycle-lim u x*) ∂M)) $\longrightarrow 0$
⟨*proof*⟩

From the almost sure convergence, we can prove the basic properties of the (real) subcocycle limit: relationship to the asymptotic average, behavior under sum, multiplication, max, behavior for Birkhoff sums.

lemma *subcocycle-lim-avg*:
assumes *subcocycle u subcocycle-avg-ereal u* $> -\infty$
shows ($\int x.$ *subcocycle-lim u x* ∂M) $=$ *subcocycle-avg u*
⟨*proof*⟩

lemma *subcocycle-lim-real-ereal*:
assumes *subcocycle u subcocycle-avg-ereal u* $> -\infty$
shows *AE x in M. subcocycle-lim-ereal u x* $=$ *ereal*(*subcocycle-lim u x*)
⟨*proof*⟩

lemma *subcocycle-lim-add*:
assumes *subcocycle u subcocycle v subcocycle-avg-ereal u* $> -\infty$ *subcocycle-avg-ereal v* $> -\infty$
shows *subcocycle-avg-ereal* ($\lambda n\ x.$ *u n x* $+$ *v n x*) $> -\infty$
AE x in M. subcocycle-lim ($\lambda n\ x.$ *u n x* $+$ *v n x*) *x* $=$ *subcocycle-lim u x* $+$ *subcocycle-lim v x*
⟨*proof*⟩

lemma *subcocycle-lim-cmult*:

assumes *subcocycle u subcocycle-avg-ereal u* $> -\infty$ $c \geq (0::real)$
shows *subcocycle-avg-ereal* $(\lambda n\ x.\ c * u\ n\ x) > -\infty$
AE x in M. subcocycle-lim $(\lambda n\ x.\ c * u\ n\ x)\ x = c *$ *subcocycle-lim u x*
⟨*proof*⟩

lemma *subcocycle-lim-max*:
assumes *subcocycle u subcocycle v subcocycle-avg-ereal u* $> -\infty$ *subcocycle-avg-ereal v* $> -\infty$
shows *subcocycle-avg-ereal* $(\lambda n\ x.\ max\ (u\ n\ x)\ (v\ n\ x)) > -\infty$
AE x in M. subcocycle-lim $(\lambda n\ x.\ max\ (u\ n\ x)\ (v\ n\ x))\ x = max$ (*subcocycle-lim u x*) (*subcocycle-lim v x*)
⟨*proof*⟩

lemma *subcocycle-lim-birkhoff*:
assumes *integrable M u*
shows *subcocycle-avg-ereal* (*birkhoff-sum u*) $> -\infty$
AE x in M. subcocycle-lim (*birkhoff-sum u*) *x* = *real-cond-exp M Invariants u x*
⟨*proof*⟩

9.5 Conditional expectations of subcocycles

In this subsection, we show that the conditional expectations of a subcocycle (with respect to the invariant subalgebra) also converge, with the same limit as the cocycle.

Note that the conditional expectation of a subcocycle u is still a subcocycle, with the same average at each step so with the same asymptotic average. Kingman theorem can be applied to it, and what we have to show is that the limit of this subcocycle is the same as the limit of the original subcocycle.

When the asymptotic average is $> -\infty$, both limits have the same integral, and moreover the domination of the subcocycle by the Birkhoff sums of u_n for fixed n (which converge to the conditional expectation of u_n) implies that one limit is smaller than the other. Hence, they coincide almost everywhere.

The case when the asymptotic average is $-\infty$ is deduced from the previous one by truncation.

First, we prove the result when the asymptotic average with finite.

theorem *kingman-theorem-nonergodic-invariant*:
assumes *subcocycle u subcocycle-avg-ereal u* $> -\infty$
shows *AE x in M.* $(\lambda n.$ *real-cond-exp M Invariants* $(u\ n)\ x\ /\ n) \longrightarrow$ *subcocycle-lim u x*
$(\lambda n.\ (\int^{+} x.\ abs($*real-cond-exp M Invariants* $(u\ n)\ x\ /\ n -$ *subcocycle-lim u x*$)\ \partial M)) \longrightarrow 0$
⟨*proof*⟩

Then, we extend it by truncation to the general case, i.e., to the asymptotic limit in extended reals.

theorem *kingman-theorem-AE-nonergodic-invariant-ereal*:
  assumes *subcocycle u*
  shows $AE\ x\ in\ M.\ (\lambda n.\ real\text{-}cond\text{-}exp\ M\ Invariants\ (u\ n)\ x\ /\ n) \longlongrightarrow subcocycle\text{-}lim\text{-}ereal\ u\ x$
⟨*proof*⟩

end

9.6 Subcocycles in the ergodic case

In this subsection, we describe how all the previous results simplify in the ergodic case. Indeed, subcocycle limits are almost surely constant, given by the asymptotic average.

context *ergodic-pmpt* **begin**

lemma *subcocycle-ergodic-lim-avg*:
  assumes *subcocycle u*
  shows $AE\ x\ in\ M.\ subcocycle\text{-}lim\text{-}ereal\ u\ x = subcocycle\text{-}avg\text{-}ereal\ u$
    $AE\ x\ in\ M.\ subcocycle\text{-}lim\ u\ x = subcocycle\text{-}avg\ u$
⟨*proof*⟩

theorem *kingman-theorem-AE-ereal*:
  assumes *subcocycle u*
  shows $AE\ x\ in\ M.\ (\lambda n.\ u\ n\ x\ /\ n) \longlongrightarrow subcocycle\text{-}avg\text{-}ereal\ u$
⟨*proof*⟩

theorem *kingman-theorem*:
  assumes $subcocycle\ u\ subcocycle\text{-}avg\text{-}ereal\ u > -\infty$
  shows $AE\ x\ in\ M.\ (\lambda n.\ u\ n\ x\ /\ n) \longlongrightarrow subcocycle\text{-}avg\ u$
    $(\lambda n.\ (\int^{+}x.\ abs(u\ n\ x\ /\ n - subcocycle\text{-}avg\ u)\ \partial M)) \longlongrightarrow 0$
⟨*proof*⟩

end

9.7 Subocycles for invertible maps

If T is invertible, then a subcocycle u_n for T gives rise to another subcocycle for T^{-1}. Intuitively, if u is subadditive along the time interval $[0, n)$, then it should also be subadditive along the time interval $[-n, 0)$. This is true, and formalized with the following statement.

proposition (**in** *mpt*) *subcocycle-u-Tinv*:
  assumes *subcocycle u*
    invertible-qmpt
  shows $mpt.subcocycle\ M\ Tinv\ (\lambda n\ x.\ u\ n\ (((Tinv)\widehat{\ \ }\widehat{\ \ }n)\ x))$
⟨*proof*⟩

The subcocycle averages for T and T^{-1} coincide.

proposition (**in** *mpt*) *subcocycle-avg-ereal-Tinv*:
assumes *subcocycle u*
invertible-qmpt
shows *mpt.subcocycle-avg-ereal M* ($\lambda n\ x.\ u\ n\ (((Tinv)\widehat{\ \ } n)\ x)) = subcocycle\text{-}avg\text{-}ereal$ *u*
⟨*proof*⟩

The asymptotic limit of the subcocycle is the same for T and T^{-1}. This is clear in the ergodic case, and follows from the ergodic decomposition in the general case (on a standard probability space). We give a direct proof below (on a general probability space) using the fact that the asymptotic limit is the same for the subcocycle conditioned by the invariant sigma-algebra, which is clearly the same for T and T^{-1} as it is constant along orbits.

proposition (**in** *fmpt*) *subcocycle-lim-ereal-Tinv*:
assumes *subcocycle u*
invertible-qmpt
shows *AE x in M. fmpt.subcocycle-lim-ereal M Tinv* ($\lambda n\ x.\ u\ n\ (((Tinv)\widehat{\ \ } n)\ x))\ x = subcocycle\text{-}lim\text{-}ereal\ u\ x$
⟨*proof*⟩

proposition (**in** *fmpt*) *subcocycle-lim-Tinv*:
assumes *subcocycle u*
invertible-qmpt
shows *AE x in M. fmpt.subcocycle-lim M Tinv* ($\lambda n\ x.\ u\ n\ (((Tinv)\widehat{\ \ } n)\ x))\ x = subcocycle\text{-}lim\ u\ x$
⟨*proof*⟩

end

10 Gouezel-Karlsson

theory *Gouezel-Karlsson*
imports *Asymptotic-Density Kingman*
begin

This section is devoted to the proof of the main ergodic result of the article "Subadditive and multiplicative ergodic theorems" by Gouezel and Karlsson [GK15]. It is a version of Kingman theorem ensuring that, for subadditive cocycles, there are almost surely many times n where the cocycle is nearly additive at *all* times between 0 and n.

This theorem is then used in this article to construct horofunctions characterizing the behavior at infinity of compositions of semi-contractions. This requires too many further notions to be implemented in current Isabelle/HOL, but the main ergodic result is completely proved below, in Theorem `Gouezel_Karlsson`, following the arguments in the paper (but in a slightly more general setting here as we are not making any ergodicity assumption).

To simplify the exposition, the theorem is proved assuming that the limit of the subcocycle vanishes almost everywhere, in the locale `Gouezel_Karlsson_Kingman`. The final result is proved by an easy reduction to this case.

The main steps of the proof are as follows:

- assume first that the map is invertible, and consider the inverse map and the corresponding inverse subcocycle. With combinatorial arguments that only work for this inverse subcocycle, we control the density of bad times given some allowed error $d > 0$, in a precise quantitative way, in Lemmas `upper_density_all_times` and `upper_density_large_k`. We put these estimates together in Lemma `upper_density_delta`.

- These estimates are then transfered to the original time direction and the original subcocycle in Lemma `upper_density_good_direction_invertible`. The fact that we have quantitative estimates in terms of asymptotic densities is central here, just having some infiniteness statement would not be enough.

- The invertibility assumption is removed in Lemma `upper_density_good_direction` by using the result in the natural extension.

- Finally, the main result is deduced in Lemma `infinite_AE` (still assuming that the asymptotic limit vanishes almost everywhere), and in full generality in Theorem `Gouezel_Karlsson_Kingman`.

lemma *upper-density-eventually-measure*:
  fixes a::*real*
  assumes [*measurable*]: $\bigwedge n.\ \{x \in \mathit{space}\ M.\ P\ x\ n\} \in \mathit{sets}\ M$
    and $\mathit{emeasure}\ M\ \{x \in \mathit{space}\ M.\ \mathit{upper\text{-}asymptotic\text{-}density}\ \{n.\ P\ x\ n\} < a\} > b$
  shows $\exists N.\ \mathit{emeasure}\ M\ \{x \in \mathit{space}\ M.\ \forall n \geq N.\ \mathit{card}\ (\{n.\ P\ x\ n\} \cap \{..<n\}) < a * n\} > b$
⟨*proof*⟩

locale *Gouezel-Karlsson-Kingman* = *pmpt* +
  fixes u::$\mathit{nat} \Rightarrow {}'a \Rightarrow \mathit{real}$
  assumes *subu*: *subcocycle u*
    and *subu-fin*: $\mathit{subcocycle\text{-}avg\text{-}ereal}\ u > -\infty$
    and *subu-0*: $\mathit{AE}\ x\ \mathit{in}\ M.\ \mathit{subcocycle\text{-}lim}\ u\ x = 0$
begin

lemma *int-u* [*measurable*]:
  integrable M (*u n*)
⟨*proof*⟩

Next lemma is Lemma 2.1 in [GK15].

lemma *upper-density-all-times*:

assumes $d > (0::real)$
shows $\exists\, c > (0::real).$
emeasure M $\{x \in$ *space* $M.$ *upper-asymptotic-density* $\{n.\ \exists\, l \in \{1..n\}.\ u\ n\ x - u\ (n{-}l)\ x \le -\ c * l\} < d\} > 1 - d$
⟨*proof*⟩

Next lemma is Lemma 2.2 in [GK15].

lemma *upper-density-large-k*:
assumes $d > (0::real)$ $d \le 1$
shows $\exists\, k::nat.$
emeasure M $\{x \in$ *space* $M.$ *upper-asymptotic-density* $\{n.\ \exists\, l \in \{k..n\}.\ u\ n\ x - u\ (n{-}l)\ x \le -\ d * l\} < d\} > 1 - d$
⟨*proof*⟩

The two previous lemmas are put together in the following lemma, corresponding to Lemma 2.3 in [GK15].

lemma *upper-density-delta*:
fixes $d::real$
assumes $d > 0$ $d \le 1$
shows $\exists\, delta::nat \Rightarrow real.\ (\forall\, l.\ delta\ l > 0) \land (delta \longrightarrow 0) \land$
emeasure M $\{x \in$ *space* $M.\ \forall\, (N::nat).$ *card* $\{n \in \{..<N\}.\ \exists\, l \in \{1..n\}.\ u\ n\ x - u\ (n{-}l)\ x \le -\ delta\ l * l\} \le d * N\} > 1 - d$
⟨*proof*⟩

We go back to the natural time direction, by using the previous result for the inverse map and the inverse subcocycle, and a change of variables argument. The price to pay is that the estimates we get are weaker: we have a control on a set of upper asymptotic density close to 1, while having a set of lower asymptotic density close to 1 as before would be stronger. This will nevertheless be sufficient for our purposes below.

lemma *upper-density-good-direction-invertible*:
assumes *invertible-qmpt*
$d > (0::real)$ $d \le 1$
shows $\exists\, delta::nat \Rightarrow real.\ (\forall\, l.\ delta\ l > 0) \land (delta \longrightarrow 0) \land$
emeasure M $\{x \in$ *space* $M.$ *upper-asymptotic-density* $\{n.\ \forall\, l \in \{1..n\}.\ u\ n\ x - u\ (n{-}l)\ ((T^{\frown\frown}l)\ x) > -\ delta\ l * l\} \ge 1{-}d\} \ge ennreal(1{-}d)$
⟨*proof*⟩

Now, we want to remove the invertibility assumption in the previous lemma. The idea is to go to the natural extension of the system, use the result there and project it back. However, if the system is not defined on a polish space, there is no reason why it should have a natural extension, so we have first to project the original system on a polish space on which the subcocycle is defined. This system is obtained by considering the joint distribution of the subcocycle and all its iterates (this is indeed a polish system, as a space of functions from $\mathbb{N}^2$ to $\mathbb{R}$).

lemma *upper-density-good-direction*:

assumes $d>(0::real)$ $d \leq 1$
shows $\exists$ *delta::nat⇒real.* ($\forall$ *l. delta l* > *0*) $\wedge$ (*delta* $\longrightarrow$ *0*) $\wedge$
emeasure M {*x* $\in$ *space M. upper-asymptotic-density* {*n.* $\forall$ *l* $\in$ {*1..n*}. *u n x* $-$ *u* (*n*$-$*l*) ((*T*^^*l*) *x*) > $-$ *delta l* $*$ *l*} $\geq$ *1*$-$*d*} $\geq$ *ennreal*(*1*$-$*d*)
⟨*proof*⟩

From the quantitative lemma above, we deduce the qualitative statement we are after, still in the setting of the locale.

lemma *infinite-AE*:
shows *AE x in M.* $\exists$ *delta::nat⇒real.* ($\forall$ *l. delta l* > *0*) $\wedge$ (*delta* $\longrightarrow$ *0*) $\wedge$
(*infinite* {*n.* $\forall$ *l* $\in$ {*1..n*}. *u n x* $-$ *u* (*n*$-$*l*) ((*T*^^*l*) *x*) > $-$ *delta l* $*$ *l*})
⟨*proof*⟩

end

Finally, we obtain the full statement, by reducing to the previous situation where the asymptotic average vanishes.

theorem (**in** *pmpt*) *Gouezel-Karlsson-Kingman*:
assumes *subcocycle u subcocycle-avg-ereal u* > $-\infty$
shows *AE x in M.* $\exists$ *delta::nat⇒real.* ($\forall$ *l. delta l* > *0*) $\wedge$ (*delta* $\longrightarrow$ *0*) $\wedge$
(*infinite* {*n.* $\forall$ *l* $\in$ {*1..n*}. *u n x* $-$ *u* (*n*$-$*l*) ((*T*^^*l*) *x*) $-$ *l* $*$ *subcocycle-lim u x* > $-$ *delta l* $*$ *l*})
⟨*proof*⟩

The previous theorem only contains a lower bound. The corresponding upper bound follows readily from Kingman's theorem. The next statement combines both upper and lower bounds.

theorem (**in** *pmpt*) *Gouezel-Karlsson-Kingman'*:
assumes *subcocycle u subcocycle-avg-ereal u* > $-\infty$
shows *AE x in M.* $\exists$ *delta::nat⇒real.* ($\forall$ *l. delta l* > *0*) $\wedge$ (*delta* $\longrightarrow$ *0*) $\wedge$
(*infinite* {*n.* $\forall$ *l* $\in$ {*1..n*}. *abs*(*u n x* $-$ *u* (*n*$-$*l*) ((*T*^^*l*) *x*) $-$ *l* $*$ *subcocycle-lim u x*) < *delta l* $*$ *l*})
⟨*proof*⟩

end

11 A theorem by Kohlberg and Neyman

theory *Kohlberg-Neyman-Karlsson*
imports *Fekete*
begin

In this section, we prove a theorem due to Kohlberg and Neyman: given a semicontraction T of a euclidean space, then $T^n(0)/n$ converges when $n \to \infty$. The proof we give is due to Karlsson. It mainly builds on subadditivity ideas. The geometry of the space is essentially not relevant except at the very end of the argument, where strict convexity comes into play.

We recall Fekete's lemma: if a sequence is subadditive (i.e., $u_{n+m} \leq u_n + u_m$), then u_n/n converges to its infimum. It is proved in a different file, but we recall the statement for self-containedness.

lemma *fekete*:
 fixes $u::nat \Rightarrow real$
 assumes $\bigwedge n\ m.\ u\ (m+n) \leq u\ m + u\ n$
 $bdd\text{-}below\ \{u\ n/n \mid n.\ n>0\}$
 shows $(\lambda n.\ u\ n/n) \longrightarrow Inf\ \{u\ n/n \mid n.\ n>0\}$
$\langle proof \rangle$

A real sequence tending to infinity has infinitely many high-scores, i.e., there are infinitely many times where it is larger than all its previous values.

lemma *high-scores*:
 fixes $u::nat \Rightarrow real$ **and** $i::nat$
 assumes $u \longrightarrow \infty$
 shows $\exists\, n \geq i.\ \forall\, l \leq n.\ u\ l \leq u\ n$
$\langle proof \rangle$

Hahn-Banach in euclidean spaces: given a vector u, there exists a unit norm vector v such that $\langle u, v\rangle = \|u\|$ (and we put a minus sign as we will use it in this form). This uses the fact that, in Isabelle/HOL, euclidean spaces have positive dimension by definition.

lemma *select-unit-norm*:
 fixes $u::'a::euclidean\text{-}space$
 shows $\exists\, v.\ norm\ v = 1 \wedge v \cdot u = -\ norm\ u$
$\langle proof \rangle$

We set up the assumption that we will use until the end of this file, in the following locale: we fix a semicontraction T of a euclidean space. Our goal will be to show that such a semicontraction has an asymptotic translation vector.

locale *Kohlberg-Neyman-Karlsson* =
 fixes $T::'a::euclidean\text{-}space \Rightarrow 'a$
 assumes *semicontract*: $dist\ (T\ x)\ (T\ y) \leq dist\ x\ y$
begin

The iterates of T are still semicontractions, by induction.

lemma *semicontract-Tn*:
 $dist\ ((T\widehat{\ }\widehat{\ }n)\ x)\ ((T\widehat{\ }\widehat{\ }n)\ y) \leq dist\ x\ y$
$\langle proof \rangle$

The main quantity we will use is the distance from the origin to its image under T^n. We denote it by u_n. The main point is that it is subadditive by semicontraction, hence it converges to a limit A given by $Inf\{u_n/n\}$, thanks to Fekete Lemma.

definition $u::nat \Rightarrow real$

where *u n = dist 0 ((T^^n) 0)*

definition *A::real*
where *A = Inf {u n/n | n. n>0}*

lemma *Apos: A ≥ 0*
⟨*proof*⟩

lemma *Alim:(λn. u n/n) ⟶ A*
⟨*proof*⟩

The main fact to prove the existence of an asymptotic translation vector for T is the following proposition: there exists a unit norm vector v such that $T^\ell(0)$ is in the half-space at distance $A\ell$ of the origin directed by v.

The idea of the proof is to find such a vector v_i that works (with a small error $\epsilon_i > 0$) for times up to a time n_i, and then take a limit by compactness (or weak compactness, but since we are in finite dimension, compactness works fine). Times n_i are chosen to be large high scores of the sequence $u_n - (A - \epsilon_i)n$, which tends to infinity since u_n/n tends to A.

proposition *half-space*:
∃ v. norm v = 1 ∧ (∀ l. v · (T ^^ l) 0 ≤ − A ∗ l)
⟨*proof*⟩

We can now show the existence of an asymptotic translation vector for T. It is the vector $-v$ of the previous proposition: the point $T^\ell(0)$ is in the half-space at distance $A\ell$ of the origin directed by v, and has norm $\sim A\ell$, hence it has to be essentially $-Av$ by strict convexity of the euclidean norm.

theorem *KNK-thm*:
convergent (λn. ((T^^n) 0) /$_R$ n)
⟨*proof*⟩

end

end

12 Transfer Operator

theory *Transfer-Operator*
imports *Recurrence*
begin

context *qmpt* **begin**

The map T acts on measures by push-forward. In particular, if $f d\mu$ is absolutely continuous with respect to the reference measure μ, then its push-forward $T_*(f d\mu)$ is absolutely continuous with respect to μ, and can therefore be written as $g d\mu$ for some function g. The map $f \mapsto g$, representing the action of T on the level of densities, is called the transfer operator associated to T and often denoted by $\hat{T}$.

We first define it on nonnegative functions, using Radon-Nikodym derivatives. Then, we extend it to general real-valued functions by separating it into positive and negative parts.

The theory presents many similarities with the theory of conditional expectations. Indeed, it is possible to make a theory encompassing the two. When the map is measure preserving, there is also a direct relationship: $(\hat{T} f) \circ T$ is the conditional expectation of f with respect to $T^{-1}B$ where B is the sigma-algebra. Instead of building a general theory, we copy the proofs for conditional expectations and adapt them where needed.

12.1 The transfer operator on nonnegative functions

definition *nn-transfer-operator* :: $('a \Rightarrow ennreal) \Rightarrow ('a \Rightarrow ennreal)$
where
nn-transfer-operator f = (*if* $f \in$ *borel-measurable* M *then* *RN-deriv* M (*distr* (*density* M f) M T)
else (λ-. *0*))

lemma *borel-measurable-nn-transfer-operator* [*measurable*]:
nn-transfer-operator $f \in$ *borel-measurable* M
⟨*proof*⟩

lemma *borel-measurable-nn-transfer-operator-iterates* [*measurable*]:
assumes [*measurable*]: $f \in$ *borel-measurable* M
shows (*nn-transfer-operator*^^n) $f \in$ *borel-measurable* M
⟨*proof*⟩

The next lemma is arguably the most fundamental property of the transfer operator: it is the adjoint of the composition by T. If one defined it as an abstract adjoint, it would be defined on the dual of L^∞, which is a large unwieldy space. The point is that it can be defined on genuine functions, using the push-forward point of view above. However, once we have this property, we can forget completely about the definition, since this property characterizes the transfer operator, as the second lemma below shows. From this point on, we will only work with it, and forget completely about the definition using Radon-Nikodym derivatives.

lemma *nn-transfer-operator-intg*:
assumes [*measurable*]: $f \in$ *borel-measurable* M $g \in$ *borel-measurable* M
shows ($\int^+ x.\ f\ x * $ *nn-transfer-operator* $g\ x\ \partial M$) = ($\int^+ x.\ f\ (T\ x) * g\ x\ \partial M$)
⟨*proof*⟩

lemma *nn-transfer-operator-intTn-g*:
assumes $f \in$ *borel-measurable* M $g \in$ *borel-measurable* M

shows ($\int^+ x.\ f\ x * (nn\text{-}transfer\text{-}operator\widehat{\ }\widehat{\ }n)\ g\ x\ \partial M) = (\int^+ x.\ f\ ((T\widehat{\ }\widehat{\ }n)\ x) * g\ x\ \partial M)$
$\langle proof \rangle$

lemma *nn-transfer-operator-intg-Tn*:
assumes $f \in borel\text{-}measurable\ M\ g \in borel\text{-}measurable\ M$
shows ($\int^+ x.\ (nn\text{-}transfer\text{-}operator\widehat{\ }\widehat{\ }n)\ g\ x * f\ x\ \partial M) = (\int^+ x.\ g\ x * f\ ((T\widehat{\ }\widehat{\ }n)\ x)\ \partial M)$
$\langle proof \rangle$

lemma *nn-transfer-operator-charact*:
assumes $\bigwedge A.\ A \in sets\ M \Longrightarrow (\int^+ x.\ indicator\ A\ x * g\ x\ \partial M) = (\int^+ x.\ indicator\ A\ (T\ x) * f\ x\ \partial M)$ **and**
[*measurable*]: $f \in borel\text{-}measurable\ M\ g \in borel\text{-}measurable\ M$
shows $AE\ x\ in\ M.\ nn\text{-}transfer\text{-}operator\ f\ x = g\ x$
$\langle proof \rangle$

When T is measure-preserving, $\hat{T}(f \circ T) = f$.

lemma (**in** *mpt*) *nn-transfer-operator-foT*:
assumes [*measurable*]: $f \in borel\text{-}measurable\ M$
shows $AE\ x\ in\ M.\ nn\text{-}transfer\text{-}operator\ (f\ o\ T)\ x = f\ x$
$\langle proof \rangle$

In general, one only has $\hat{T}(f \circ T \cdot g) = f \cdot \hat{T}g$.

lemma *nn-transfer-operator-foT-g*:
assumes [*measurable*]: $f \in borel\text{-}measurable\ M\ g \in borel\text{-}measurable\ M$
shows $AE\ x\ in\ M.\ nn\text{-}transfer\text{-}operator\ (\lambda x.\ f\ (T\ x) * g\ x)\ x = f\ x * nn\text{-}transfer\text{-}operator\ g\ x$
$\langle proof \rangle$

lemma *nn-transfer-operator-cmult*:
assumes [*measurable*]: $g \in borel\text{-}measurable\ M$
shows $AE\ x\ in\ M.\ nn\text{-}transfer\text{-}operator\ (\lambda x.\ c * g\ x)\ x = c * nn\text{-}transfer\text{-}operator\ g\ x$
$\langle proof \rangle$

lemma *nn-transfer-operator-zero*:
$AE\ x\ in\ M.\ nn\text{-}transfer\text{-}operator\ (\lambda x.\ 0)\ x = 0$
$\langle proof \rangle$

lemma *nn-transfer-operator-sum*:
assumes [*measurable*]: $f \in borel\text{-}measurable\ M\ g \in borel\text{-}measurable\ M$
shows $AE\ x\ in\ M.\ nn\text{-}transfer\text{-}operator\ (\lambda x.\ f\ x + g\ x)\ x = nn\text{-}transfer\text{-}operator\ f\ x + nn\text{-}transfer\text{-}operator\ g\ x$
$\langle proof \rangle$

lemma *nn-transfer-operator-cong*:
assumes $AE\ x\ in\ M.\ f\ x = g\ x$
and [*measurable*]: $f \in borel\text{-}measurable\ M\ g \in borel\text{-}measurable\ M$

shows $AE\ x\ in\ M.\ nn\text{-}transfer\text{-}operator\ f\ x = nn\text{-}transfer\text{-}operator\ g\ x$
⟨*proof*⟩

lemma *nn-transfer-operator-mono*:
assumes $AE\ x\ in\ M.\ f\ x \leq g\ x$
and [*measurable*]: $f \in borel\text{-}measurable\ M\ g \in borel\text{-}measurable\ M$
shows $AE\ x\ in\ M.\ nn\text{-}transfer\text{-}operator\ f\ x \leq nn\text{-}transfer\text{-}operator\ g\ x$
⟨*proof*⟩

12.2 The transfer operator on real functions

Once the transfer operator of positive functions is defined, the definition for real-valued functions follows readily, by taking the difference of positive and negative parts.

definition *real-transfer-operator* :: $('a \Rightarrow real) \Rightarrow ('a \Rightarrow real)$ **where**
real-transfer-operator f =
$(\lambda x.\ enn2real(nn\text{-}transfer\text{-}operator\ (\lambda x.\ ennreal\ (f\ x))\ x) - enn2real(nn\text{-}transfer\text{-}operator\ (\lambda x.\ ennreal\ (-f\ x))\ x))$

lemma *borel-measurable-transfer-operator* [*measurable*]:
$real\text{-}transfer\text{-}operator\ f \in borel\text{-}measurable\ M$
⟨*proof*⟩

lemma *borel-measurable-transfer-operator-iterates* [*measurable*]:
assumes [*measurable*]: $f \in borel\text{-}measurable\ M$
shows $(real\text{-}transfer\text{-}operator\widehat{\ }\widehat{\ }n)\ f \in borel\text{-}measurable\ M$
⟨*proof*⟩

lemma *real-transfer-operator-abs*:
assumes [*measurable*]: $f \in borel\text{-}measurable\ M$
shows $AE\ x\ in\ M.\ abs\ (real\text{-}transfer\text{-}operator\ f\ x) \leq nn\text{-}transfer\text{-}operator\ (\lambda x.\ ennreal\ (abs(f\ x)))\ x$
⟨*proof*⟩

The next lemma shows that the transfer operator as we have defined it satisfies the basic duality relation $\int \hat{T} f \cdot g = \int f \cdot g \circ T$. It follows from the same relation for nonnegative functions, and splitting into positive and negative parts.

Moreover, this relation characterizes the transfer operator. Hence, once this lemma is proved, we will never come back to the original definition of the transfer operator.

lemma *real-transfer-operator-intg-fpos*:
assumes $integrable\ M\ (\lambda x.\ f\ (T\ x) * g\ x)$ **and** *f-pos*[*simp*]: $\bigwedge x.\ f\ x \geq 0$ **and**
[*measurable*]: $f \in borel\text{-}measurable\ M\ g \in borel\text{-}measurable\ M$
shows $integrable\ M\ (\lambda x.\ f\ x * real\text{-}transfer\text{-}operator\ g\ x)$
$(\int x.\ f\ x * real\text{-}transfer\text{-}operator\ g\ x\ \partial M) = (\int x.\ f\ (T\ x) * g\ x\ \partial M)$
⟨*proof*⟩

lemma *real-transfer-operator-intg*:
  assumes *integrable* M ($\lambda x.$ f (T x) $*$ g x) **and**
    [*measurable*]: $f \in$ *borel-measurable* M $g \in$ *borel-measurable* M
  shows *integrable* M ($\lambda x.$ f x $*$ *real-transfer-operator* g x)
    ($\int$ $x.$ f x $*$ *real-transfer-operator* g x ∂M) $=$ ($\int$ $x.$ f (T x) $*$ g x ∂M)
⟨*proof*⟩

lemma *real-transfer-operator-int* [*intro*]:
  assumes *integrable* M f
  shows *integrable* M (*real-transfer-operator* f)
    ($\int x.$ *real-transfer-operator* f x ∂M) $=$ ($\int x.$ f x ∂M)
⟨*proof*⟩

lemma *real-transfer-operator-charact*:
  assumes $\bigwedge A.$ $A \in$ *sets* $M \Longrightarrow$ ($\int x.$ *indicator* A x $*$ g x ∂M) $=$ ($\int x.$ *indicator* A (T x) $*$ f x ∂M)
    and [*measurable*]: *integrable* M f *integrable* M g
  shows *AE* x *in* $M.$ *real-transfer-operator* f x $=$ g x
⟨*proof*⟩

lemma (**in** *mpt*) *real-transfer-operator-foT*:
  assumes *integrable* M f
  shows *AE* x *in* $M.$ *real-transfer-operator* (f *o* T) x $=$ f x
⟨*proof*⟩

lemma *real-transfer-operator-foT-g*:
  assumes [*measurable*]: $f \in$ *borel-measurable* M $g \in$ *borel-measurable* M *integrable* M ($\lambda x.$ f (T x) $*$ g x)
  shows *AE* x *in* $M.$ *real-transfer-operator* ($\lambda x.$ f (T x) $*$ g x) x $=$ f x $*$ *real-transfer-operator* g x
⟨*proof*⟩

lemma *real-transfer-operator-add* [*intro*]:
  assumes [*measurable*]: *integrable* M f *integrable* M g
  shows *AE* x *in* $M.$ *real-transfer-operator* ($\lambda x.$ f x $+$ g x) x $=$ *real-transfer-operator* f x $+$ *real-transfer-operator* g x
⟨*proof*⟩

lemma *real-transfer-operator-cong*:
  assumes *ae*: *AE* x *in* $M.$ f x $=$ g x **and** [*measurable*]: $f \in$ *borel-measurable* M g $\in$ *borel-measurable* M
  shows *AE* x *in* $M.$ *real-transfer-operator* f x $=$ *real-transfer-operator* g x
⟨*proof*⟩

lemma *real-transfer-operator-cmult* [*intro*, *simp*]:
  fixes *c*::*real*
  assumes *integrable* M f
  shows *AE* x *in* $M.$ *real-transfer-operator* ($\lambda x.$ c $*$ f x) x $=$ c $*$ *real-transfer-operator*

f x
⟨*proof*⟩

lemma *real-transfer-operator-cdiv* [*intro*, *simp*]:
fixes *c*::*real*
assumes *integrable M f*
shows *AE x in M*. *real-transfer-operator* (λx. *f x* / *c*) *x* = *real-transfer-operator f x* / *c*
⟨*proof*⟩

lemma *real-transfer-operator-diff* [*intro*, *simp*]:
assumes [*measurable*]: *integrable M f integrable M g*
shows *AE x in M*. *real-transfer-operator* (λx. *f x* − *g x*) *x* = *real-transfer-operator f x* − *real-transfer-operator g x*
⟨*proof*⟩

lemma *real-transfer-operator-pos* [*intro*]:
assumes *AE x in M*. *f x* ≥ *0* **and** [*measurable*]: *f* ∈ *borel-measurable M*
shows *AE x in M*. *real-transfer-operator f x* ≥ *0*
⟨*proof*⟩

lemma *real-transfer-operator-mono*:
assumes *AE x in M*. *f x* ≤ *g x* **and** [*measurable*]: *integrable M f integrable M g*
shows *AE x in M*. *real-transfer-operator f x* ≤ *real-transfer-operator g x*
⟨*proof*⟩

lemma *real-transfer-operator-sum* [*intro*, *simp*]:
fixes *f*::*'b* ⇒ *'a* ⇒ *real*
assumes [*measurable*]: $\bigwedge$*i*. *integrable M* (*f i*)
shows *AE x in M*. *real-transfer-operator* (λx. $\sum$ *i*∈*I*. *f i x*) *x* = ($\sum$ *i*∈*I*. *real-transfer-operator* (*f i*) *x*)
⟨*proof*⟩
end

12.3 Conservativity in terms of transfer operators

Conservativity amounts to the fact that $\sum f(T^n x) = \infty$ for almost every x such that $f(x) > 0$, if f is nonnegative (see Lemma `recurrent_series_infinite`). There is a dual formulation, in terms of transfer operators, asserting that $\sum \hat{T}^n f(x) = \infty$ for almost every x such that $f(x) > 0$. It is proved by duality, reducing to the previous statement.

theorem (**in** *conservative*) *recurrence-series-infinite-transfer-operator*:
assumes [*measurable*]: *f* ∈ *borel-measurable M*
shows *AE x in M*. *f x* > *0* ⟶ ($\sum$ *n*. (*nn-transfer-operator* ^^ *n*) *f x*) = ∞
⟨*proof*⟩

end

13 Normalizing sequences

theory *Normalizing-Sequences*
  imports *Transfer-Operator Asymptotic-Density*
begin

In this file, we prove the main result in [Gou18]: in a conservative system, if a renormalized sequence $S_n f/B_n$ converges in distribution towards a limit which is not a Dirac mass at 0, then B_n can not grow exponentially fast. We also prove the easier result that, in a probability preserving system, normalizing sequences grow at most polynomially.

13.1 Measure of the preimages of disjoint sets.

We start with a general result about conservative maps: If A_n are disjoint sets, and P is a finite mass measure which is absolutely continuous with respect to M, then $T^{-n}A_n$ is most often small: $P(T^{-n}A_n)$ tends to 0 in Cesaro average. The proof is written in terms of densities and positive transfer operators, so we first write it in ennreal.

theorem (**in** *conservative*) *disjoint-sets-emeasure-Cesaro-tendsto-zero*:
  fixes P::'*a measure* **and** A::*nat* $\Rightarrow$ '*a set*
  assumes [*measurable*]: $\bigwedge n.\ A\ n \in sets\ M$
    and *disjoint-family A*
      absolutely-continuous M P sets P = sets M
      emeasure P (*space M*) $\neq \infty$
  shows $(\lambda n.\ (\sum i{<}n.\ emeasure\ P\ (space\ M \cap (T\widehat{\ \ }i){-}`(A\ i)))/n) \longlongrightarrow 0$
⟨*proof*⟩

We state the previous theorem using measures instead of emeasures. This is clearly equivalent, but one has to play with ennreal carefully to prove it.

theorem (**in** *conservative*) *disjoint-sets-measure-Cesaro-tendsto-zero*:
  fixes P::'*a measure* **and** A::*nat* $\Rightarrow$ '*a set*
  assumes [*measurable*]: $\bigwedge n.\ A\ n \in sets\ M$
    and *disjoint-family A*
      absolutely-continuous M P sets P = sets M
      emeasure P (*space M*) $\neq \infty$
  shows $(\lambda n.\ (\sum i{<}n.\ measure\ P\ (space\ M \cap (T\widehat{\ \ }i){-}`(A\ i)))/n) \longlongrightarrow 0$
⟨*proof*⟩

As convergence to 0 in Cesaro mean is equivalent to convergence to 0 along a density one sequence, we obtain the equivalent formulation of the previous theorem.

theorem (**in** *conservative*) *disjoint-sets-measure-density-one-tendsto-zero*:
  fixes P::'*a measure* **and** A::*nat* $\Rightarrow$ '*a set*
  assumes [*measurable*]: $\bigwedge n.\ A\ n \in sets\ M$
    and *disjoint-family A*
      absolutely-continuous M P sets P = sets M

emeasure P (space M) $\neq \infty$
shows $\exists B.$ *lower-asymptotic-density B* $=$ *1* $\wedge$ ($\lambda n.$ *measure P (space M* $\cap$ *(T^^n)−'(A n)) * indicator B n)* $\longrightarrow$ *0*
⟨*proof*⟩

13.2 Normalizing sequences do not grow exponentially in conservative systems

We prove the main result in [Gou18]: in a conservative system, if a renormalized sequence $S_n f/B_n$ converges in distribution towards a limit which is not a Dirac mass at 0, then B_n can not grow exponentially fast. The proof is expressed in the following locale. The main theorem is Theorem `subexponential_growth` below. To prove it, we need several preliminary estimates.

We will use the fact that a real random variables which is not the Dirac mass at 0 gives positive mass to a set separated away from 0.

lemma (**in** *real-distribution*) *not-Dirac-0-imp-positive-mass-away-0*:
assumes *prob* {*0*} < *1*
shows $\exists a.$ *a* > *0* $\wedge$ *prob* {*x. abs(x)* > *a*} > *0*
⟨*proof*⟩

locale *conservative-limit* =
conservative M + *PS*: *prob-space P* + *PZ*: *real-distribution Z*
for *M::'a measure* **and** *P::'a measure* **and** *Z::real measure* +
fixes *f g::'a* $\Rightarrow$ *real* **and** *B::nat* $\Rightarrow$ *real*
assumes *PabsM*: *absolutely-continuous M P*
and *Bpos*: $\bigwedge n.$ *B n* > *0*
and *M* [*measurable*]: *f* $\in$ *borel-measurable M g* $\in$ *borel-measurable M sets P* = *sets M*
and *non-trivial*: *PZ.prob* {*0*} < *1*
and *conv*: *weak-conv-m* ($\lambda n.$ *distr P borel* ($\lambda x.$ (*g x* + *birkhoff-sum f n x*) / *B n*)) *Z*
begin

For measurability statements, we want every question about Z or P to reduce to a question about Borel sets of M. We add in the next lemma all the statements that are needed so that this happens automatically.

lemma *PSZ* [*simp*, *measurable-cong*]:
space P = *space M*
h $\in$ *borel-measurable P* $\longleftrightarrow$ *h* $\in$ *borel-measurable M*
A $\in$ *sets P* $\longleftrightarrow$ *A* $\in$ *sets M*
⟨*proof*⟩

The first nontrivial upper bound is the following lemma, asserting that B_{n+1} can not be much larger than $\max B_i$ for $i \leq n$. This is proved by saying that $S_{n+1} f = f + (S_n f) \circ T$, and we know that $S_n f$ is not too large on a set of very large measure, so the same goes for $(S_n f) \circ T$ by a non-singularity argument. Excepted that the measure P does not have to be nonsingular for the map T, so one has to tweak a little bit this idea, using transfer operators and conservativity. This is easier to do when the density of P is bounded by 1, so we first give the proof under this assumption, and then we reduce to this case by replacing M with $M + P$ in the second lemma below.

First, let us prove the lemma assuming that the density h of P is bounded by 1.

lemma *upper-bound-C-aux*:
  assumes $P = density\ M\ h\ \bigwedge x.\ h\ x \leq 1$
    and [*measurable*]: $h \in borel\text{-}measurable\ M$
  shows $\exists\, C \geq 1.\ \forall\, n.\ B\ (Suc\ n) \leq C * Max\ \{B\ i | i.\ i \leq n\}$
$\langle proof \rangle$

Then, we prove the lemma without further assumptions, reducing to the previous case by replacing m with $m+P$. We do this at the level of densities since the addition of measures is not defined in the library (and it would be problematic as measures carry their sigma-algebra, so what should one do when the sigma-algebras do not coincide?)

lemma *upper-bound-C*:
  $\exists\, C \geq 1.\ \forall\, n.\ B\ (Suc\ n) \leq C * Max\ \{B\ i | i.\ i \leq n\}$
$\langle proof \rangle$

The second main upper bound is the following. Again, it proves that $B_{n+1} \leq L \max_{i \leq n} B_i$, for some constant L, but with two differences. First, L only depends on the distribution of Z (which is stronger). Second, this estimate is only proved along a density 1 sequence of times (which is weaker). The first point implies that this lemma will also apply to T^j, with the same L, which amounts to replacing L by $L^{1/j}$, making it in practice arbitrarily close to 1. The second point is problematic at first sight, but for the exceptional times we will use the bound of the previous lemma so this will not really create problems.

For the proof, we split the sum $S_{n+1} f$ as $S_n f + f \circ T^n$. If B_{n+1} is much larger than B_n, we deduce that $S_n f$ is much smaller than $S_{n+1} f$ with large probability, which means that $f \circ T^n$ is larger than anything that has been seen before. Since preimages of distinct events have a measure that tends to 0 along a density 1 subsequence, this can only happen along a density 0 subsequence.

lemma *upper-bound-L*:
  fixes *a*::*real* **and** *L*::*real* **and** *alpha*::*real*
  assumes $a > 0\ alpha > 0\ L > 3$
    $PZ.prob\ \{x.\ abs\ (x) > 2 * a\} > 3 * alpha$
    $PZ.prob\ \{x.\ abs\ (x) \geq (L-1) * a\} < alpha$
  shows $\exists\, A.\ lower\text{-}asymptotic\text{-}density\ A = 1 \wedge (\forall\, n \in A.\ B\ (Suc\ n) \leq L * Max\ \{B\ i | i.\ i \leq n\})$

$\langle proof \rangle$

Now, we combine the two previous statements to prove the main theorem.

theorem *subexponential-growth*:
$(\lambda n.\ max\ 0\ (ln\ (B\ n)\ /n)) \longrightarrow 0$
$\langle proof \rangle$

end

13.3 Normalizing sequences grow at most polynomially in probability preserving systems

In probability preserving systems, normalizing sequences grow at most polynomially. The proof, also given in [Gou18], is considerably easier than the conservative case. We prove that $B_{n+1} \leq CB_n$ (more precisely, this only holds if B_{n+1} is large enough), by arguing that $S_{n+1}f = S_n f + f \circ T^n$, where $f \circ T^n$ is negligible if B_{n+1} is large thanks to the measure preservation. We also prove that $B_{2n} \leq EB_n$, by writing $S_{2n}f = S_n f + S_n f \circ T^n$ and arguing that the two terms on the right have the same distribution. Finally, combining these two estimates, the polynomial growth follows readily.

locale *pmpt-limit* =
pmpt M + PZ: *real-distribution Z*
for M::'*a measure* **and** Z::*real measure* +
fixes f::'*a* $\Rightarrow$ *real* **and** B::*nat* $\Rightarrow$ *real*
assumes *Bpos*: $\bigwedge n.\ B\ n > 0$
and *M* [*measurable*]: $f \in$ *borel-measurable M*
and *non-trivial*: *PZ.prob* $\{0\} < 1$
and *conv*: *weak-conv-m* (λn. *distr P borel* (λx. (*birkhoff-sum f n x*) / *B n*)) *Z*
begin

First, we prove that $B_{n+1} \leq CB_n$ if B_{n+1} is large enough.

lemma *upper-bound-CD*:
$\exists\, C\ D.\ (\forall\, n.\ B\ (Suc\ n) \leq D \vee B\ (Suc\ n) \leq C * B\ n) \wedge C \geq 1$
$\langle proof \rangle$

Second, we prove that $B_{2n} \leq EB_n$.

lemma *upper-bound-E*:
$\exists\, E.\ \forall\, n.\ B\ (2 * n) \leq E * B\ n$
$\langle proof \rangle$

Finally, we combine the estimates in the two lemmas above to show that B_n grows at most polynomially.

theorem *polynomial-growth*:
$\exists\, C\ K.\ \forall\, n{>}0.\ B\ n \leq C * (real\ n)\hat{}K$
$\langle proof \rangle$

end

end

References

[GK15] Sébastien Gouëzel and Anders Karlsson, *Subadditive and multiplicative ergodic theorems*, preprint, 2015.

[Gou18] Sébastien Gouëzel, *Growth of normalizing sequences in limit theorems for conservative maps*, preprint, 2018.