Epistemic Logic: Completeness of Modal Logics

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Abstract

This work is a formalization of epistemic logic with countably many agents. It includes proofs of soundness and completeness for the axiom system K. The completeness proof is based on the textbook "Reasoning About Knowledge" by Fagin, Halpern, Moses and Vardi (MIT Press 1995) [2]. The extensions of system K (T, KB, K4, S4, S5) and their completeness proofs are based on the textbook "Modal Logic" by Blackburn, de Rijke and Venema (Cambridge University Press 2001) [1].

Contents

theory *Maximal-Consistent-Sets* **imports** *HOL−Cardinals.Cardinal-Order-Relation*
begin

context *wo-rel* **begin**

lemma *underS-bound*: ‹$a \in underS\ n \Longrightarrow b \in underS\ n \Longrightarrow a \in under\ b \vee b \in under\ a$›
  ⟨*proof*⟩

lemma *finite-underS-bound*:
  assumes ‹*finite X*› ‹$X \subseteq underS\ n$› ‹$X \neq \{\}$›
  shows ‹$\exists\, a \in X.\ \forall\, b \in X.\ b \in under\ a$›
  ⟨*proof*⟩

lemma *finite-bound-under*:
  assumes ‹*finite p*› ‹$p \subseteq (\bigcup n \in Field\ r.\ f\ n)$›
  shows ‹$\exists\, m.\ p \subseteq (\bigcup n \in under\ m.\ f\ n)$›
    ⟨*proof*⟩

end

locale *MCS-Lim-Ord* =
  fixes *r* :: ‹*'a rel*›
  assumes *WELL*: ‹*Well-order r*›
    and *isLimOrd-r*: ‹*isLimOrd r*›
  fixes *consistent* :: ‹$'a\ set \Rightarrow bool$›
  assumes *consistent-hereditary*: ‹$consistent\ S \Longrightarrow S' \subseteq S \Longrightarrow consistent\ S'$›
    and *inconsistent-finite*: ‹$\bigwedge S.\ \neg\ consistent\ S \Longrightarrow \exists\, S' \subseteq S.\ finite\ S' \wedge \neg\ consistent\ S'$›
begin

definition *extendS* :: ‹$'a\ set \Rightarrow 'a \Rightarrow 'a\ set \Rightarrow 'a\ set$› **where**
  ‹$extendS\ S\ n\ prev \equiv if\ consistent\ (\{n\} \cup prev)\ then\ \{n\} \cup prev\ else\ prev$›

definition *extendL* :: ‹$('a \Rightarrow 'a\ set) \Rightarrow 'a \Rightarrow 'a\ set$› **where**
  ‹$extendL\ rec\ n \equiv \bigcup m \in underS\ r\ n.\ rec\ m$›

definition *extend* :: ‹$'a\ set \Rightarrow 'a \Rightarrow 'a\ set$› **where**
  ‹$extend\ S\ n \equiv worecZSL\ r\ S\ (extendS\ S)\ extendL\ n$›

lemma *wo-rel-r*: ‹*wo-rel r*›
  ⟨*proof*⟩

lemma *adm-woL-extendL*: ‹*adm-woL r extendL*›
  ⟨*proof*⟩

definition *Extend* :: ‹$'a\ set \Rightarrow 'a\ set$› **where**

‹*Extend S* ≡ ⋃*n* ∈ *Field r*. *extend S n*›

lemma *extend-subset*: ‹*n* ∈ *Field r* ⟹ *S* ⊆ *extend S n*›
⟨*proof*⟩

lemma *Extend-subset′*: ‹*Field r* ≠ {} ⟹ *S* ⊆ *Extend S*›
⟨*proof*⟩

lemma *extend-underS*: ‹*m* ∈ *underS r n* ⟹ *extend S m* ⊆ *extend S n*›
⟨*proof*⟩

lemma *extend-under*: ‹*m* ∈ *under r n* ⟹ *extend S m* ⊆ *extend S n*›
⟨*proof*⟩

lemma *consistent-extend*:
assumes ‹*consistent S*›
shows ‹*consistent* (*extend S n*)›
⟨*proof*⟩

lemma *consistent-Extend*:
assumes ‹*consistent S*›
shows ‹*consistent* (*Extend S*)›
⟨*proof*⟩

definition *maximal′* :: ‹′*a set* ⇒ *bool*› **where**
‹*maximal′ S* ≡ ∀ *p* ∈ *Field r*. *consistent* ({*p*} ∪ *S*) ⟶ *p* ∈ *S*›

lemma *Extend-bound*: ‹*n* ∈ *Field r* ⟹ *extend S n* ⊆ *Extend S*›
⟨*proof*⟩

lemma *maximal′-Extend*: ‹*maximal′* (*Extend S*)›
⟨*proof*⟩

end

locale *MCS* =
fixes *consistent* :: ‹′*a set* ⇒ *bool*›
assumes *infinite-UNIV*: ‹*infinite* (*UNIV* :: ′*a set*)›
and ‹*consistent S* ⟹ *S′* ⊆ *S* ⟹ *consistent S′*›
and ‹⋀*S*. ¬ *consistent S* ⟹ ∃ *S′* ⊆ *S*. *finite S′* ∧ ¬ *consistent S′*›

sublocale *MCS* ⊆ *MCS-Lim-Ord* ‹|*UNIV*|›
⟨*proof*⟩

context *MCS* **begin**

lemma *Extend-subset*: ‹*S* ⊆ *Extend S*›
⟨*proof*⟩

definition *maximal* :: ‹*'a set* ⇒ *bool*› **where**
‹*maximal S* ≡ ∀ *p. consistent* ({*p*} ∪ *S*) ⟶ *p* ∈ *S*›

lemma *maximal-maximal'*: ‹*maximal S* ⟷ *maximal' S*›
⟨*proof*⟩

lemma *maximal-Extend*: ‹*maximal* (*Extend S*)›
⟨*proof*⟩

end

end

theory *Epistemic-Logic* **imports** *Maximal-Consistent-Sets* **begin**

1 Syntax

type-synonym *id* = *string*

datatype *'i fm*
= *FF* (‹⊥›)
| *Pro id*
| *Dis* ‹*'i fm*› ‹*'i fm*› (**infixr** ‹∨› *60*)
| *Con* ‹*'i fm*› ‹*'i fm*› (**infixr** ‹∧› *65*)
| *Imp* ‹*'i fm*› ‹*'i fm*› (**infixr** ‹⟶› *55*)
| *K 'i* ‹*'i fm*›

abbreviation *TT* (‹⊤›) **where**
‹*TT* ≡ ⊥ ⟶ ⊥›

abbreviation *Neg* (‹¬ -› [*70*] *70*) **where**
‹*Neg p* ≡ *p* ⟶ ⊥›

abbreviation ‹*L i p* ≡ ¬ *K i* (¬ *p*)›

2 Semantics

record (*'i*, *'w*) *frame* =
𝒲 :: ‹*'w set*›
𝒦 :: ‹*'i* ⇒ *'w* ⇒ *'w set*›

record (*'i*, *'w*) *kripke* =
‹(*'i*, *'w*) *frame*› +
π :: ‹*'w* ⇒ *id* ⇒ *bool*›

primrec *semantics* :: ‹(*'i*, *'w*) *kripke* ⇒ *'w* ⇒ *'i fm* ⇒ *bool*› (‹-, - ⊨ -› [*50*, *50*,

50] *50*) **where**
‹$M,\ w \models \bot \longleftrightarrow \mathit{False}$›
| ‹$M,\ w \models \mathit{Pro}\ x \longleftrightarrow \pi\ M\ w\ x$›
| ‹$M,\ w \models p \lor q \longleftrightarrow M,\ w \models p \lor M,\ w \models q$›
| ‹$M,\ w \models p \land q \longleftrightarrow M,\ w \models p \land M,\ w \models q$›
| ‹$M,\ w \models p \longrightarrow q \longleftrightarrow M,\ w \models p \longrightarrow M,\ w \models q$›
| ‹$M,\ w \models K\ i\ p \longleftrightarrow (\forall\, v \in \mathcal{W}\ M \cap \mathcal{K}\ M\ i\ w.\ M,\ v \models p)$›

abbreviation *validStar* :: ‹$(('i,\ 'w)\ \mathit{kripke} \Rightarrow \mathit{bool}) \Rightarrow \mathit{'i\ fm\ set} \Rightarrow \mathit{'i\ fm} \Rightarrow \mathit{bool}$›
(‹-; - $\models\!\!\models\star$ -› [*50*, *50*, *50*] *50*) **where**
‹$P;\ G \models\!\!\models\star\ p \equiv \forall\, M.\ P\ M \longrightarrow$
$(\forall\, w \in \mathcal{W}\ M.\ (\forall\, q \in G.\ M,\ w \models q) \longrightarrow M,\ w \models p)$›

3 S5 Axioms

definition *reflexive* :: ‹$('i,\ 'w,\ 'c)\ \mathit{frame\text{-}scheme} \Rightarrow \mathit{bool}$› **where**
‹$\mathit{reflexive}\ M \equiv \forall\, i.\ \forall\, w \in \mathcal{W}\ M.\ w \in \mathcal{K}\ M\ i\ w$›

definition *symmetric* :: ‹$('i,\ 'w,\ 'c)\ \mathit{frame\text{-}scheme} \Rightarrow \mathit{bool}$› **where**
‹$\mathit{symmetric}\ M \equiv \forall\, i.\ \forall\, v \in \mathcal{W}\ M.\ \forall\, w \in \mathcal{W}\ M.\ v \in \mathcal{K}\ M\ i\ w \longleftrightarrow w \in \mathcal{K}\ M\ i\ v$›

definition *transitive* :: ‹$('i,\ 'w,\ 'c)\ \mathit{frame\text{-}scheme} \Rightarrow \mathit{bool}$› **where**
‹$\mathit{transitive}\ M \equiv \forall\, i.\ \forall\, u \in \mathcal{W}\ M.\ \forall\, v \in \mathcal{W}\ M.\ \forall\, w \in \mathcal{W}\ M.$
$w \in \mathcal{K}\ M\ i\ v \land u \in \mathcal{K}\ M\ i\ w \longrightarrow u \in \mathcal{K}\ M\ i\ v$›

abbreviation *refltrans* :: ‹$('i,\ 'w,\ 'c)\ \mathit{frame\text{-}scheme} \Rightarrow \mathit{bool}$› **where**
‹$\mathit{refltrans}\ M \equiv \mathit{reflexive}\ M \land \mathit{transitive}\ M$›

abbreviation *equivalence* :: ‹$('i,\ 'w,\ 'c)\ \mathit{frame\text{-}scheme} \Rightarrow \mathit{bool}$› **where**
‹$\mathit{equivalence}\ M \equiv \mathit{reflexive}\ M \land \mathit{symmetric}\ M \land \mathit{transitive}\ M$›

definition *Euclidean* :: ‹$('i,\ 'w,\ 'c)\ \mathit{frame\text{-}scheme} \Rightarrow \mathit{bool}$› **where**
‹$\mathit{Euclidean}\ M \equiv \forall\, i.\ \forall\, u \in \mathcal{W}\ M.\ \forall\, v \in \mathcal{W}\ M.\ \forall\, w \in \mathcal{W}\ M.$
$v \in \mathcal{K}\ M\ i\ u \longrightarrow w \in \mathcal{K}\ M\ i\ u \longrightarrow w \in \mathcal{K}\ M\ i\ v$›

lemma *Imp-intro* [*intro*]: ‹$(M,\ w \models p \Longrightarrow M,\ w \models q) \Longrightarrow M,\ w \models p \longrightarrow q$›
⟨*proof*⟩

theorem *distribution*: ‹$M,\ w \models K\ i\ p \land K\ i\ (p \longrightarrow q) \longrightarrow K\ i\ q$›
⟨*proof*⟩

theorem *generalization*:
fixes M :: ‹$('i,\ 'w)\ \mathit{kripke}$›
assumes ‹$\forall\,(M :: ('i,\ 'w)\ \mathit{kripke}).\ \forall\, w \in \mathcal{W}\ M.\ M,\ w \models p$› ‹$w \in \mathcal{W}\ M$›
shows ‹$M,\ w \models K\ i\ p$›
⟨*proof*⟩

theorem *truth*:
assumes ‹$\mathit{reflexive}\ M$› ‹$w \in \mathcal{W}\ M$›

shows ‹M, w ⊨ K i p ⟶ p›
⟨*proof*⟩

theorem *pos-introspection*:
assumes ‹*transitive* M› ‹w ∈ 𝒲 M›
shows ‹M, w ⊨ K i p ⟶ K i (K i p)›
⟨*proof*⟩

theorem *neg-introspection*:
assumes ‹*symmetric* M› ‹*transitive* M› ‹w ∈ 𝒲 M›
shows ‹M, w ⊨ ¬ K i p ⟶ K i (¬ K i p)›
⟨*proof*⟩

4 Normal Modal Logic

primrec *eval* :: ‹(*id* ⇒ *bool*) ⇒ ('i *fm* ⇒ *bool*) ⇒ 'i *fm* ⇒ *bool*› **where**
‹*eval* - - ⊥ = *False*›
| ‹*eval* g - (*Pro* x) = g x›
| ‹*eval* g h (p ∨ q) = (*eval* g h p ∨ *eval* g h q)›
| ‹*eval* g h (p ∧ q) = (*eval* g h p ∧ *eval* g h q)›
| ‹*eval* g h (p ⟶ q) = (*eval* g h p ⟶ *eval* g h q)›
| ‹*eval* - h (K i p) = h (K i p)›

abbreviation ‹*tautology* p ≡ ∀ g h. *eval* g h p›

inductive *AK* :: ‹('i *fm* ⇒ *bool*) ⇒ 'i *fm* ⇒ *bool*› (‹- ⊢ -› [*50*, *50*] *50*)
for A :: ‹'i *fm* ⇒ *bool*› **where**
A1: ‹*tautology* p ⟹ A ⊢ p›
| *A2*: ‹A ⊢ K i p ∧ K i (p ⟶ q) ⟶ K i q›
| *Ax*: ‹A p ⟹ A ⊢ p›
| *R1*: ‹A ⊢ p ⟹ A ⊢ p ⟶ q ⟹ A ⊢ q›
| *R2*: ‹A ⊢ p ⟹ A ⊢ K i p›

primrec *imply* :: ‹'i *fm list* ⇒ 'i *fm* ⇒ 'i *fm*› (**infixr** ‹⇝› *56*) **where**
‹([] ⇝ q) = q›
| ‹(p # ps ⇝ q) = (p ⟶ ps ⇝ q)›

abbreviation *AK-assms* (‹-; - ⊢ -› [*50*, *50*, *50*] *50*) **where**
‹A; G ⊢ p ≡ ∃ qs. *set* qs ⊆ G ∧ (A ⊢ qs ⇝ p)›

5 Soundness

lemma *eval-semantics*:
‹*eval* (*pi* w) (λq. ⦇𝒲 = W, 𝒦 = r, π = *pi*⦈, w ⊨ q) p = (⦇𝒲 = W, 𝒦 = r, π = *pi*⦈, w ⊨ p)›
⟨*proof*⟩

lemma *tautology*:

assumes ‹$tautology\ p$›
shows ‹$M,\ w \models p$›
⟨*proof*⟩

theorem *soundness*:
assumes ‹$\bigwedge M\ w\ p.\ A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models p$›
shows ‹$A \vdash p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models p$›
⟨*proof*⟩

6 Derived rules

lemma *K-A2′*: ‹$A \vdash K\ i\ (p \longrightarrow q) \longrightarrow K\ i\ p \longrightarrow K\ i\ q$›
⟨*proof*⟩

lemma *K-map*:
assumes ‹$A \vdash p \longrightarrow q$›
shows ‹$A \vdash K\ i\ p \longrightarrow K\ i\ q$›
⟨*proof*⟩

lemma *K-LK*: ‹$A \vdash (L\ i\ (\neg\ p) \longrightarrow \neg\ K\ i\ p)$›
⟨*proof*⟩

lemma *K-imply-head*: ‹$A \vdash (p\ \#\ ps \leadsto p)$›
⟨*proof*⟩

lemma *K-imply-Cons*:
assumes ‹$A \vdash ps \leadsto q$›
shows ‹$A \vdash p\ \#\ ps \leadsto q$›
⟨*proof*⟩

lemma *K-right-mp*:
assumes ‹$A \vdash ps \leadsto p$› ‹$A \vdash ps \leadsto (p \longrightarrow q)$›
shows ‹$A \vdash ps \leadsto q$›
⟨*proof*⟩

lemma *tautology-imply-superset*:
assumes ‹$set\ ps \subseteq set\ qs$›
shows ‹$tautology\ (ps \leadsto r \longrightarrow qs \leadsto r)$›
⟨*proof*⟩

lemma *K-imply-weaken*:
assumes ‹$A \vdash ps \leadsto q$› ‹$set\ ps \subseteq set\ ps'$›
shows ‹$A \vdash ps' \leadsto q$›
⟨*proof*⟩

lemma *imply-append*: ‹$(ps\ @\ ps' \leadsto q) = (ps \leadsto ps' \leadsto q)$›
⟨*proof*⟩

lemma *K-ImpI*:

assumes ‹$A \vdash p \# G \leadsto q$›
shows ‹$A \vdash G \leadsto (p \longrightarrow q)$›
⟨*proof*⟩

lemma *K-Boole*:
assumes ‹$A \vdash (\neg\ p) \# G \leadsto \bot$›
shows ‹$A \vdash G \leadsto p$›
⟨*proof*⟩

lemma *K-DisE*:
assumes ‹$A \vdash p \# G \leadsto r$› ‹$A \vdash q \# G \leadsto r$› ‹$A \vdash G \leadsto p \vee q$›
shows ‹$A \vdash G \leadsto r$›
⟨*proof*⟩

lemma *K-mp*: ‹$A \vdash p \# (p \longrightarrow q) \# G \leadsto q$›
⟨*proof*⟩

lemma *K-swap*:
assumes ‹$A \vdash p \# q \# G \leadsto r$›
shows ‹$A \vdash q \# p \# G \leadsto r$›
⟨*proof*⟩

lemma *K-DisL*:
assumes ‹$A \vdash p \# ps \leadsto q$› ‹$A \vdash p' \# ps \leadsto q$›
shows ‹$A \vdash (p \vee p') \# ps \leadsto q$›
⟨*proof*⟩

lemma *K-distrib-K-imp*:
assumes ‹$A \vdash K\ i\ (G \leadsto q)$›
shows ‹$A \vdash map\ (K\ i)\ G \leadsto K\ i\ q$›
⟨*proof*⟩

lemma *K-trans*: ‹$A \vdash (p \longrightarrow q) \longrightarrow (q \longrightarrow r) \longrightarrow p \longrightarrow r$›
⟨*proof*⟩

lemma *K-L-dual*: ‹$A \vdash \neg\ L\ i\ (\neg\ p) \longrightarrow K\ i\ p$›
⟨*proof*⟩

7 Strong Soundness

corollary *soundness-imply*:
assumes ‹$\bigwedge M\ w\ p.\ A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p$›
shows ‹$A \vdash ps \leadsto p \Longrightarrow P; set\ ps \models\!\!\star\ p$›
⟨*proof*⟩

theorem *strong-soundness*:
assumes ‹$\bigwedge M\ w\ p.\ A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p$›
shows ‹$A; G \vdash p \Longrightarrow P; G \models\!\!\star\ p$›
⟨*proof*⟩

8 Completeness

8.1 Consistent sets

definition *consistent* :: ‹('*i fm* ⇒ *bool*) ⇒ '*i fm set* ⇒ *bool*› **where**
‹*consistent A S* ≡ ¬ (*A*; *S* ⊢ ⊥)›

lemma *inconsistent-subset*:
assumes ‹*consistent A V*› ‹¬ *consistent A* ({*p*} ∪ *V*)›
obtains *V'* **where** ‹*set V'* ⊆ *V*› ‹*A* ⊢ *p* # *V'* ⇝ ⊥›
⟨*proof*⟩

lemma *consistent-consequent*:
assumes ‹*consistent A V*› ‹*p* ∈ *V*› ‹*A* ⊢ *p* ⟶ *q*›
shows ‹*consistent A* ({*q*} ∪ *V*)›
⟨*proof*⟩

lemma *consistent-consequent'*:
assumes ‹*consistent A V*› ‹*p* ∈ *V*› ‹*tautology* (*p* ⟶ *q*)›
shows ‹*consistent A* ({*q*} ∪ *V*)›
⟨*proof*⟩

lemma *consistent-disjuncts*:
assumes ‹*consistent A V*› ‹(*p* ∨ *q*) ∈ *V*›
shows ‹*consistent A* ({*p*} ∪ *V*) ∨ *consistent A* ({*q*} ∪ *V*)›
⟨*proof*⟩

lemma *exists-finite-inconsistent*:
assumes ‹¬ *consistent A* ({¬ *p*} ∪ *V*)›
obtains *W* **where** ‹{¬ *p*} ∪ *W* ⊆ {¬ *p*} ∪ *V*› ‹(¬ *p*) ∉ *W*› ‹*finite W*› ‹¬ *consistent A* ({¬ *p*} ∪ *W*)›
⟨*proof*⟩

lemma *inconsistent-imply*:
assumes ‹¬ *consistent A* ({¬ *p*} ∪ *set G*)›
shows ‹*A* ⊢ *G* ⇝ *p*›
⟨*proof*⟩

8.2 Maximal consistent sets

lemma *fm-any-size*: ‹∃ *p* :: '*i fm*. *size p* = *n*›
⟨*proof*⟩

lemma *infinite-UNIV-fm*: ‹*infinite* (*UNIV* :: '*i fm set*)›
⟨*proof*⟩

interpretation *MCS* ‹*consistent A*› **for** *A* :: ‹'*i fm* ⇒ *bool*›
⟨*proof*⟩

theorem *deriv-in-maximal*:

assumes ‹*consistent A V*› ‹*maximal A V*› ‹$A \vdash p$›
shows ‹$p \in V$›
⟨*proof*⟩

theorem *exactly-one-in-maximal*:
assumes ‹*consistent A V*› ‹*maximal A V*›
shows ‹$p \in V \longleftrightarrow (\neg\, p) \notin V$›
⟨*proof*⟩

theorem *consequent-in-maximal*:
assumes ‹*consistent A V*› ‹*maximal A V*› ‹$p \in V$› ‹$(p \longrightarrow q) \in V$›
shows ‹$q \in V$›
⟨*proof*⟩

theorem *ax-in-maximal*:
assumes ‹*consistent A V*› ‹*maximal A V*› ‹*A p*›
shows ‹$p \in V$›
⟨*proof*⟩

theorem *mcs-properties*:
assumes ‹*consistent A V*› **and** ‹*maximal A V*›
shows ‹$A \vdash p \Longrightarrow p \in V$›
and ‹$p \in V \longleftrightarrow (\neg\, p) \notin V$›
and ‹$p \in V \Longrightarrow (p \longrightarrow q) \in V \Longrightarrow q \in V$›
⟨*proof*⟩

lemma *maximal-extension*:
fixes *V* :: ‹*'i fm set*›
assumes ‹*consistent A V*›
obtains *W* **where** ‹$V \subseteq W$› ‹*consistent A W*› ‹*maximal A W*›
⟨*proof*⟩

8.3 Canonical model

abbreviation *pi* :: ‹$'i\ fm\ set \Rightarrow id \Rightarrow bool$› **where**
‹$pi\ V\ x \equiv Pro\ x \in V$›

abbreviation *known* :: ‹$'i\ fm\ set \Rightarrow 'i \Rightarrow 'i\ fm\ set$› **where**
‹$known\ V\ i \equiv \{p.\ K\ i\ p \in V\}$›

abbreviation *reach* :: ‹$('i\ fm \Rightarrow bool) \Rightarrow 'i \Rightarrow 'i\ fm\ set \Rightarrow 'i\ fm\ set\ set$› **where**
‹$reach\ A\ i\ V \equiv \{W.\ known\ V\ i \subseteq W\}$›

abbreviation *mcss* :: ‹$('i\ fm \Rightarrow bool) \Rightarrow 'i\ fm\ set\ set$› **where**
‹$mcss\ A \equiv \{W.\ consistent\ A\ W \wedge maximal\ A\ W\}$›

abbreviation *canonical* :: ‹$('i\ fm \Rightarrow bool) \Rightarrow ('i, 'i\ fm\ set)\ kripke$› **where**
‹$canonical\ A \equiv (\!|\mathcal{W} = mcss\ A,\ \mathcal{K} = reach\ A,\ \pi = pi|\!)$›

lemma *truth-lemma*:
  fixes p :: ‹$'i$ *fm*›
  assumes ‹*consistent* A V› **and** ‹*maximal* A V›
  shows ‹$p \in V \longleftrightarrow$ *canonical* A, $V \models p$›
  ⟨*proof*⟩

lemma *canonical-model*:
  assumes ‹*consistent* A S› **and** ‹$p \in S$›
  defines ‹$V \equiv$ *Extend* A S› **and** ‹$M \equiv$ *canonical* A›
  shows ‹M, $V \models p$› **and** ‹*consistent* A V› **and** ‹*maximal* A V›
⟨*proof*⟩

8.4 Completeness

abbreviation *valid* :: ‹(($'i$, $'i$ *fm set*) *kripke* ⇒ *bool*) ⇒ $'i$ *fm set* ⇒ $'i$ *fm* ⇒ *bool*›
  (‹-; - ⊫ -› [*50*, *50*, *50*] *50*)
  where ‹P; $G \Vdash p \equiv P$; G ⊫⋆ p›

theorem *strong-completeness*:
  assumes ‹P; $G \Vdash p$› **and** ‹P (*canonical* A)›
  shows ‹A; $G \vdash p$›
⟨*proof*⟩

corollary *completeness*:
  assumes ‹P; {} $\Vdash p$› **and** ‹P (*canonical* A)›
  shows ‹$A \vdash p$›
  ⟨*proof*⟩

corollary *completeness*$_A$:
  assumes ‹(λ-. *True*); {} $\Vdash p$›
  shows ‹$A \vdash p$›
  ⟨*proof*⟩

9 System K

abbreviation *SystemK* (‹- $\vdash_K$ -› [*50*] *50*) **where**
  ‹$G \vdash_K p \equiv$ (λ-. *False*); $G \vdash p$›

lemma *strong-soundness*$_K$: ‹$G \vdash_K p \Longrightarrow P$; G ⊫⋆ p›
  ⟨*proof*⟩

abbreviation *validK* (‹- $\Vdash_K$ -› [*50*, *50*] *50*) **where**
  ‹$G \Vdash_K p \equiv$ (λ-. *True*); $G \Vdash p$›

lemma *strong-completeness*$_K$: ‹$G \Vdash_K p \Longrightarrow G \vdash_K p$›
  ⟨*proof*⟩

theorem *main*$_K$: ‹$G \Vdash_K p \longleftrightarrow G \vdash_K p$›
  ⟨*proof*⟩

corollary ‹$G \models_K p \Longrightarrow (\lambda\text{-}.\ True);\ G \models\star\ p$›
⟨*proof*⟩

10 System T

Also known as System M

inductive *AxT* :: ‹$'i\ fm \Rightarrow bool$› **where**
‹$AxT\ (K\ i\ p \longrightarrow p)$›

abbreviation *SystemT* (‹- $\vdash_T$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_T p \equiv AxT;\ G \vdash p$›

lemma *soundness-AxT*: ‹$AxT\ p \Longrightarrow reflexive\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p$›
⟨*proof*⟩

lemma *strong-soundness$_T$*: ‹$G \vdash_T p \Longrightarrow reflexive;\ G \models\star\ p$›
⟨*proof*⟩

lemma *AxT-reflexive*:
assumes ‹$AxT \leq A$› **and** ‹*consistent* A V› **and** ‹*maximal* A V›
shows ‹$V \in reach\ A\ i\ V$›
⟨*proof*⟩

lemma *reflexive$_T$*:
assumes ‹$AxT \leq A$›
shows ‹*reflexive* (*canonical* A)›
⟨*proof*⟩

abbreviation *validT* (‹- $\models_T$ -› [*50*, *50*] *50*) **where**
‹$G \models_T p \equiv reflexive;\ G \models p$›

lemma *strong-completeness$_T$*: ‹$G \models_T p \Longrightarrow G \vdash_T p$›
⟨*proof*⟩

theorem *main$_T$*: ‹$G \models_T p \longleftrightarrow G \vdash_T p$›
⟨*proof*⟩

corollary ‹$G \models_T p \longrightarrow reflexive;\ G \models\star\ p$›
⟨*proof*⟩

11 System KB

inductive *AxB* :: ‹$'i\ fm \Rightarrow bool$› **where**
‹$AxB\ (p \longrightarrow K\ i\ (L\ i\ p))$›

abbreviation *SystemKB* (‹- $\vdash_{KB}$ -› [*50*, *50*] *50*) **where**

‹$G \vdash_{KB} p \equiv AxB; G \vdash p$›

lemma *soundness-AxB*: ‹$AxB\ p \Longrightarrow symmetric\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p$›
⟨*proof*⟩

lemma *strong-soundness$_{KB}$*: ‹$G \vdash_{KB} p \Longrightarrow symmetric; G \models\star p$›
⟨*proof*⟩

lemma *AxB-symmetric′*:
assumes ‹$AxB \leq A$› ‹*consistent A V*› ‹*maximal A V*› ‹*consistent A W*› ‹*maximal A W*›
and ‹$W \in reach\ A\ i\ V$›
shows ‹$V \in reach\ A\ i\ W$›
⟨*proof*⟩

lemma *symmetric$_{KB}$*:
assumes ‹$AxB \leq A$›
shows ‹*symmetric* (*canonical A*)›
⟨*proof*⟩

abbreviation *validKB* (‹- $\models_{KB}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{KB} p \equiv symmetric; G \models p$›

lemma *strong-completeness$_{KB}$*: ‹$G \models_{KB} p \Longrightarrow G \vdash_{KB} p$›
⟨*proof*⟩

theorem *main$_{KB}$*: ‹$G \models_{KB} p \longleftrightarrow G \vdash_{KB} p$›
⟨*proof*⟩

corollary ‹$G \models_{KB} p \longrightarrow symmetric; G \models\star p$›
⟨*proof*⟩

12 System K4

inductive *Ax4* :: ‹*′i fm* ⇒ *bool*› **where**
‹$Ax4\ (K\ i\ p \longrightarrow K\ i\ (K\ i\ p))$›

abbreviation *SystemK4* (‹- $\vdash_{K4}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{K4} p \equiv Ax4; G \vdash p$›

lemma *soundness-Ax4*: ‹$Ax4\ p \Longrightarrow transitive\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p$›
⟨*proof*⟩

lemma *strong-soundness$_{K4}$*: ‹$G \vdash_{K4} p \Longrightarrow transitive; G \models\star p$›
⟨*proof*⟩

lemma *Ax4-transitive*:
assumes ‹$Ax4 \leq A$› ‹*consistent A V*› ‹*maximal A V*›
and ‹$W \in reach\ A\ i\ V$› ‹$U \in reach\ A\ i\ W$›

shows ‹$U \in$ *reach A i V*›
⟨*proof*⟩

lemma *transitive*$_{K4}$:
assumes ‹*Ax4* $\leq A$›
shows ‹*transitive* (*canonical A*)›
⟨*proof*⟩

abbreviation *validK4* (‹- $\models_{K4}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{K4} p \equiv$ *transitive*; $G \models p$›

lemma *strong-completeness*$_{K4}$: ‹$G \models_{K4} p \Longrightarrow G \vdash_{K4} p$›
⟨*proof*⟩

theorem *main*$_{K4}$: ‹$G \models_{K4} p \longleftrightarrow G \vdash_{K4} p$›
⟨*proof*⟩

corollary ‹$G \models_{K4} p \longrightarrow$ *transitive*; $G \models\star p$›
⟨*proof*⟩

13 System K5

inductive *Ax5* :: ‹*'i fm* ⇒ *bool*› **where**
‹*Ax5* ($L\ i\ p \longrightarrow K\ i\ (L\ i\ p)$)›

abbreviation *SystemK5* (‹- $\vdash_{K5}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{K5} p \equiv$ *Ax5*; $G \vdash p$›

lemma *soundness-Ax5*: ‹*Ax5* $p \Longrightarrow$ *Euclidean* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p$›
⟨*proof*⟩

lemma *strong-soundness*$_{K5}$: ‹$G \vdash_{K5} p \Longrightarrow$ *Euclidean*; $G \models\star p$›
⟨*proof*⟩

lemma *Ax5-Euclidean*:
assumes ‹*Ax5* $\leq A$›
‹*consistent A U*› ‹*maximal A U*›
‹*consistent A V*› ‹*maximal A V*›
‹*consistent A W*› ‹*maximal A W*›
and ‹$V \in$ *reach A i U*› ‹$W \in$ *reach A i U*›
shows ‹$W \in$ *reach A i V*›
⟨*proof*⟩

lemma *Euclidean*$_{K5}$:
assumes ‹*Ax5* $\leq A$›
shows ‹*Euclidean* (*canonical A*)›
⟨*proof*⟩

abbreviation *validK5* (‹- $\models_{K5}$ -› [*50*, *50*] *50*) **where**

‹$G \models_{K5}\ p \equiv$ *Euclidean*; $G \models\ p$›

lemma *strong-completeness*$_{K5}$: ‹$G \models_{K5}\ p \Longrightarrow G \vdash_{K5}\ p$›
⟨*proof*⟩

theorem *main*$_{K5}$: ‹$G \models_{K5}\ p \longleftrightarrow G \vdash_{K5}\ p$›
⟨*proof*⟩

corollary ‹$G \models_{K5}\ p \longrightarrow$ *Euclidean*; $G \models\star\ p$›
⟨*proof*⟩

14 System S4

abbreviation *Or* :: ‹$('a \Rightarrow bool) \Rightarrow ('a \Rightarrow bool) \Rightarrow 'a \Rightarrow bool$› (**infixl** ‹⊕› *65*) **where**
‹$(A \oplus A')\ p \equiv A\ p \vee A'\ p$›

abbreviation *SystemS4* (‹- $\vdash_{S4}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{S4}\ p \equiv AxT \oplus Ax4$; $G \vdash\ p$›

lemma *soundness-AxT4*: ‹$(AxT \oplus Ax4)\ p \Longrightarrow$ *reflexive* $M \wedge$ *transitive* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models p$›
⟨*proof*⟩

lemma *strong-soundness*$_{S4}$: ‹$G \vdash_{S4}\ p \Longrightarrow$ *refltrans*; $G \models\star\ p$›
⟨*proof*⟩

abbreviation *validS4* (‹- $\models_{S4}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{S4}\ p \equiv$ *refltrans*; $G \models\ p$›

lemma *strong-completeness*$_{S4}$: ‹$G \models_{S4}\ p \Longrightarrow G \vdash_{S4}\ p$›
⟨*proof*⟩

theorem *main*$_{S4}$: ‹$G \models_{S4}\ p \longleftrightarrow G \vdash_{S4}\ p$›
⟨*proof*⟩

corollary ‹$G \models_{S4}\ p \longrightarrow$ *refltrans*; $G \models\star\ p$›
⟨*proof*⟩

15 System S5

15.1 T + B + 4

abbreviation *SystemS5* (‹- $\vdash_{S5}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{S5}\ p \equiv AxT \oplus AxB \oplus Ax4$; $G \vdash\ p$›

abbreviation *AxTB4* :: ‹*'i fm* ⇒ *bool*› **where**
‹$AxTB4 \equiv AxT \oplus AxB \oplus Ax4$›

lemma *soundness-AxTB4*: ‹*AxTB4* $p \Longrightarrow$ *equivalence* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M$, $w \models p$›
⟨*proof*⟩

lemma *strong-soundness*$_{S5}$: ‹$G \vdash_{S5} p \Longrightarrow$ *equivalence*; $G \models\star\ p$›
⟨*proof*⟩

abbreviation *validS5* (‹- $\Vdash_{S5}$ -› [*50*, *50*] *50*) **where**
‹$G \Vdash_{S5} p \equiv$ *equivalence*; $G \Vdash p$›

lemma *strong-completeness*$_{S5}$: ‹$G \Vdash_{S5} p \Longrightarrow G \vdash_{S5} p$›
⟨*proof*⟩

theorem *main*$_{S5}$: ‹$G \Vdash_{S5} p \longleftrightarrow G \vdash_{S5} p$›
⟨*proof*⟩

corollary ‹$G \Vdash_{S5} p \longrightarrow$ *equivalence*; $G \models\star\ p$›
⟨*proof*⟩

15.2 T + 5

abbreviation *SystemS5'* (‹- $\vdash_{S5}''$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{S5}' p \equiv AxT \oplus Ax5$; $G \vdash p$›

abbreviation *AxT5* :: ‹*'i fm* ⇒ *bool*› **where**
‹$AxT5 \equiv AxT \oplus Ax5$›

lemma *symm-trans-Euclid*: ‹*symmetric* $M \Longrightarrow$ *transitive* $M \Longrightarrow$ *Euclidean* M›
⟨*proof*⟩

lemma *soundness-AxT5*: ‹*AxT5* $p \Longrightarrow$ *equivalence* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M$, w $\models p$›
⟨*proof*⟩

lemma *strong-soundness*$_{S5}$*'*: ‹$G \vdash_{S5}' p \Longrightarrow$ *equivalence*; $G \models\star\ p$›
⟨*proof*⟩

lemma *refl-Euclid-equiv*: ‹*reflexive* $M \Longrightarrow$ *Euclidean* $M \Longrightarrow$ *equivalence* M›
⟨*proof*⟩

lemma *strong-completeness*$_{S5}$*'*: ‹$G \Vdash_{S5} p \Longrightarrow G \vdash_{S5}' p$›
⟨*proof*⟩

theorem *main*$_{S5}$*'*: ‹$G \Vdash_{S5} p \longleftrightarrow G \vdash_{S5}' p$›
⟨*proof*⟩

15.3 Equivalence between systems

15.3.1 Axiom 5 from B and 4

lemma *K4-L*:
  assumes ‹$Ax4 \le A$›
  shows ‹$A \vdash L\ i\ (L\ i\ p) \longrightarrow L\ i\ p$›
⟨*proof*⟩

lemma *KB4-5*:
  assumes ‹$AxB \le A$› ‹$Ax4 \le A$›
  shows ‹$A \vdash L\ i\ p \longrightarrow K\ i\ (L\ i\ p)$›
⟨*proof*⟩

15.3.2 Axioms B and 4 from T and 5

lemma *T-L*:
  assumes ‹$AxT \le A$›
  shows ‹$A \vdash p \longrightarrow L\ i\ p$›
⟨*proof*⟩

lemma *S5′-B*:
  assumes ‹$AxT \le A$› ‹$Ax5 \le A$›
  shows ‹$A \vdash p \longrightarrow K\ i\ (L\ i\ p)$›
⟨*proof*⟩

lemma *K5-L*:
  assumes ‹$Ax5 \le A$›
  shows ‹$A \vdash L\ i\ (K\ i\ p) \longrightarrow K\ i\ p$›
⟨*proof*⟩

lemma *S5′-4*:
  assumes ‹$AxT \le A$› ‹$Ax5 \le A$›
  shows ‹$A \vdash K\ i\ p \longrightarrow K\ i\ (K\ i\ p)$›
⟨*proof*⟩

lemma *S5-S5′*: ‹$AxTB4 \vdash p \Longrightarrow AxT5 \vdash p$›
⟨*proof*⟩

lemma *S5′-S5*: ‹$AxT5 \vdash p \Longrightarrow AxTB4 \vdash p$›
⟨*proof*⟩

corollary *S5-S5′-assms*: ‹$G \vdash_{S5}\ p \longleftrightarrow G \vdash_{S5'}\ p$›
  ⟨*proof*⟩

16 Acknowledgements

The formalization is inspired by Berghofer's formalization of Henkin-style completeness.

- Stefan Berghofer: First-Order Logic According to Fitting. https://www.isa-afp.org/entries/FOL-Fitting.shtml

end

References

[1] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2001.

[2] R. Fagin, J. Halpern, Y. Moses, and M. Vardi. *Reasoning About Knowledge*. MIT Press, 1995.