

The Group Law for Elliptic Curves

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Abstract

We prove the group law for elliptic curves in Weierstrass form over fields of characteristic greater than 2. In addition to affine coordinates, we also formalize projective coordinates, which allow for more efficient computations. By specializing the abstract formalization to prime fields, we can apply the curve operations to parameters used in standard security protocols.

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1 Introduction

Elliptic curves play an important role in cryptography, since they allow to achieve a security level that is comparable to that of RSA, while requiring a smaller key size and less computation time. The primitive operation on elliptic curves is *point addition*. To ensure the proper functioning of cryptographic algorithms based on elliptic curves, such as Diffie-Hellman key exchange (ECDH) or digital signatures (ECDSA), it is important that the points on the curve form a group with respect to point addition.

Our formalization of elliptic curves is based on earlier work by Laurent Théry in Coq [4]. Like its Coq counterpart, the Isabelle formalization uses decision procedures for rings and fields based on reflection, which are executed using Isabelle’s code generator for efficiency reasons. The decision procedure for rings is due to Grégoire and Mahboubi [3] and was ported from Coq to Isabelle by Bernhard Haeupler.

The formalization exists in two flavours: one based on axiomatic type classes, and another one based on locales. While the axiomatic type class version is more concise, the locale version is more suitable for working with concrete rings or fields like prime fields.

2 Formalization using Axiomatic Type Classes

```
theory Elliptic-Axclass
imports HOL-Decision-Procs.Reflective-Field
begin
```

2.1 Affine Coordinates

```
datatype 'a point = Infinity | Point 'a 'a
```

```
class ell-field = field +
  assumes two-not-zero:  $2 \neq 0$ 
begin
```

```
definition nonsingular :: 'a  $\Rightarrow$  'a  $\Rightarrow$  bool where
  nonsingular a b =  $(4 * a^3 + 27 * b^2 \neq 0)$ 
```

```
definition on-curve :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a point  $\Rightarrow$  bool where
  on-curve a b p = (case p of
    Infinity  $\Rightarrow$  True
  | Point x y  $\Rightarrow$   $y^2 = x^3 + a * x + b$ )
```

```
definition add :: 'a  $\Rightarrow$  'a point  $\Rightarrow$  'a point  $\Rightarrow$  'a point where
  add a p1 p2 = (case p1 of
    Infinity  $\Rightarrow$  p2
  | Point x1 y1  $\Rightarrow$  (case p2 of
    Infinity  $\Rightarrow$  p1
  | Point x2 y2  $\Rightarrow$ 
    if  $x_1 = x_2$  then
      if  $y_1 = -y_2$  then Infinity
    else
      let
         $l = (3 * x_1^2 + a) / (2 * y_1);$ 
         $x_3 = l^2 - 2 * x_1$ 
      in
        Point x3  $(-y_1 - l * (x_3 - x_1))$ 
    else
```

let
 $l = (y_2 - y_1) / (x_2 - x_1);$
 $x_3 = l^2 - x_1 - x_2$
 in
 $Point\ x_3\ (-\ y_1 - l * (x_3 - x_1))$

definition *opp* :: 'a point \Rightarrow 'a point **where**

opp $p = (case\ p\ of$
 $Infinity \Rightarrow Infinity$
 $| Point\ x\ y \Rightarrow Point\ x\ (-\ y))$

end

lemma *on-curve-infinity* [*simp*]: *on-curve* $a\ b\ Infinity$
 $\langle proof \rangle$

lemma *opp-Infinity* [*simp*]: *opp* $Infinity = Infinity$
 $\langle proof \rangle$

lemma *opp-Point*: *opp* $(Point\ x\ y) = Point\ x\ (-\ y)$
 $\langle proof \rangle$

lemma *opp-opp*: *opp* $(opp\ p) = p$
 $\langle proof \rangle$

lemma *opp-closed*:
on-curve $a\ b\ p \implies on-curve\ a\ b\ (opp\ p)$
 $\langle proof \rangle$

lemma *curve-elt-opp*:
assumes $p_1 = Point\ x_1\ y_1$
and $p_2 = Point\ x_2\ y_2$
and *on-curve* $a\ b\ p_1$
and *on-curve* $a\ b\ p_2$
and $x_1 = x_2$
shows $p_1 = p_2 \vee p_1 = opp\ p_2$
 $\langle proof \rangle$

lemma *add-closed*:
assumes *on-curve* $a\ b\ p_1$ **and** *on-curve* $a\ b\ p_2$
shows *on-curve* $a\ b\ (add\ a\ p_1\ p_2)$
 $\langle proof \rangle$

lemma *add-case* [*consumes 2, case-names InfL InfR Opp Tan Gen*]:
assumes p : *on-curve* $a\ b\ p$
and q : *on-curve* $a\ b\ q$
and $R1$: $\bigwedge p. P\ Infinity\ p\ p$
and $R2$: $\bigwedge p. P\ p\ Infinity\ p$
and $R3$: $\bigwedge p. on-curve\ a\ b\ p \implies P\ p\ (opp\ p)\ Infinity$

and $R4: \bigwedge p_1 x_1 y_1 p_2 x_2 y_2 l.$
 $p_1 = \text{Point } x_1 y_1 \implies p_2 = \text{Point } x_2 y_2 \implies$
 $p_2 = \text{add } a p_1 p_1 \implies y_1 \neq 0 \implies$
 $l = (3 * x_1^2 + a) / (2 * y_1) \implies$
 $x_2 = l^2 - 2 * x_1 \implies$
 $y_2 = -y_1 - l * (x_2 - x_1) \implies$
 $P p_1 p_1 p_2$
and $R5: \bigwedge p_1 x_1 y_1 p_2 x_2 y_2 p_3 x_3 y_3 l.$
 $p_1 = \text{Point } x_1 y_1 \implies p_2 = \text{Point } x_2 y_2 \implies p_3 = \text{Point } x_3 y_3 \implies$
 $p_3 = \text{add } a p_1 p_2 \implies x_1 \neq x_2 \implies$
 $l = (y_2 - y_1) / (x_2 - x_1) \implies$
 $x_3 = l^2 - x_1 - x_2 \implies$
 $y_3 = -y_1 - l * (x_3 - x_1) \implies$
 $P p_1 p_2 p_3$
shows $P p q (\text{add } a p q)$
 $\langle \text{proof} \rangle$

lemma $eq\text{-opp-is-zero}: ((x::'a::ell\text{-field}) = -x) = (x = 0)$
 $\langle \text{proof} \rangle$

lemma $add\text{-casew}$ [consumes 2, case-names $InfL$ $InfR$ Opp Gen]:

assumes $p: \text{on-curve } a b p$
and $q: \text{on-curve } a b q$
and $R1: \bigwedge p. P \text{Infinity } p p$
and $R2: \bigwedge p. P p \text{Infinity } p$
and $R3: \bigwedge p. \text{on-curve } a b p \implies P p (\text{opp } p) \text{Infinity}$
and $R4: \bigwedge p_1 x_1 y_1 p_2 x_2 y_2 p_3 x_3 y_3 l.$
 $p_1 = \text{Point } x_1 y_1 \implies p_2 = \text{Point } x_2 y_2 \implies p_3 = \text{Point } x_3 y_3 \implies$
 $p_3 = \text{add } a p_1 p_2 \implies p_1 \neq \text{opp } p_2 \implies$
 $x_1 = x_2 \wedge y_1 = y_2 \wedge l = (3 * x_1^2 + a) / (2 * y_1) \vee$
 $x_1 \neq x_2 \wedge l = (y_2 - y_1) / (x_2 - x_1) \implies$
 $x_3 = l^2 - x_1 - x_2 \implies$
 $y_3 = -y_1 - l * (x_3 - x_1) \implies$
 $P p_1 p_2 p_3$
shows $P p q (\text{add } a p q)$
 $\langle \text{proof} \rangle$

definition

$is\text{-tangent } p q = (p \neq \text{Infinity} \wedge p = q \wedge p \neq \text{opp } q)$

definition

$is\text{-generic } p q =$
 $(p \neq \text{Infinity} \wedge q \neq \text{Infinity} \wedge$
 $p \neq q \wedge p \neq \text{opp } q)$

lemma $diff\text{-neq0}$:

$(a::'a::ring) \neq b \implies a - b \neq 0$
 $a \neq b \implies b - a \neq 0$
 $\langle \text{proof} \rangle$

lemma *minus2-not0*: $(-2::'a::\text{ell-field}) \neq 0$
<proof>

lemmas [*simp*] = *minus2-not0* [*simplified*]

declare *two-not-zero* [*simplified*, *simp add*]

lemma *spec1-assoc*:

assumes p_1 : *on-curve a b* p_1
and p_2 : *on-curve a b* p_2
and p_3 : *on-curve a b* p_3
and *is-generic* p_1 p_2
and *is-generic* p_2 p_3
and *is-generic* (*add a* p_1 p_2) p_3
and *is-generic* p_1 (*add a* p_2 p_3)
shows *add a* p_1 (*add a* p_2 p_3) = *add a* (*add a* p_1 p_2) p_3
<proof>

lemma *spec2-assoc*:

assumes p_1 : *on-curve a b* p_1
and p_2 : *on-curve a b* p_2
and p_3 : *on-curve a b* p_3
and *is-generic* p_1 p_2
and *is-tangent* p_2 p_3
and *is-generic* (*add a* p_1 p_2) p_3
and *is-generic* p_1 (*add a* p_2 p_3)
shows *add a* p_1 (*add a* p_2 p_3) = *add a* (*add a* p_1 p_2) p_3
<proof>

lemma *spec3-assoc*:

assumes p_1 : *on-curve a b* p_1
and p_2 : *on-curve a b* p_2
and p_3 : *on-curve a b* p_3
and *is-generic* p_1 p_2
and *is-tangent* p_2 p_3
and *is-generic* (*add a* p_1 p_2) p_3
and *is-tangent* p_1 (*add a* p_2 p_3)
shows *add a* p_1 (*add a* p_2 p_3) = *add a* (*add a* p_1 p_2) p_3
<proof>

lemma *add-0-l*: *add a Infinity p* = p
<proof>

lemma *add-0-r*: *add a p Infinity* = p
<proof>

lemma *add-opp*: *on-curve a b p* \implies *add a p (opp p)* = *Infinity*
<proof>

lemma *add-comm*:

assumes *on-curve a b p₁ on-curve a b p₂*

shows $\text{add } a \ p_1 \ p_2 = \text{add } a \ p_2 \ p_1$

<proof>

lemma *uniq-opp*:

assumes $\text{add } a \ p_1 \ p_2 = \text{Infinity}$

shows $p_2 = \text{opp } p_1$

<proof>

lemma *uniq-zero*:

assumes *ab: nonsingular a b*

and $p_1: \text{on-curve } a \ b \ p_1$

and $p_2: \text{on-curve } a \ b \ p_2$

and *add: add a p₁ p₂ = p₂*

shows $p_1 = \text{Infinity}$

<proof>

lemma *opp-add*:

assumes $p_1: \text{on-curve } a \ b \ p_1$

and $p_2: \text{on-curve } a \ b \ p_2$

shows $\text{opp } (\text{add } a \ p_1 \ p_2) = \text{add } a \ (\text{opp } p_1) \ (\text{opp } p_2)$

<proof>

lemma *compat-add-opp*:

assumes $p_1: \text{on-curve } a \ b \ p_1$

and $p_2: \text{on-curve } a \ b \ p_2$

and $\text{add } a \ p_1 \ p_2 = \text{add } a \ p_1 \ (\text{opp } p_2)$

and $p_1 \neq \text{opp } p_1$

shows $p_2 = \text{opp } p_2$

<proof>

lemma *compat-add-triple*:

assumes *ab: nonsingular a b*

and $p: \text{on-curve } a \ b \ p$

and $p \neq \text{opp } p$

and $\text{add } a \ p \ p \neq \text{opp } p$

shows $\text{add } a \ (\text{add } a \ p \ p) \ (\text{opp } p) = p$

<proof>

lemma *add-opp-double-opp*:

assumes *ab: nonsingular a b*

and $p_1: \text{on-curve } a \ b \ p_1$

and $p_2: \text{on-curve } a \ b \ p_2$

and $\text{add } a \ p_1 \ p_2 = \text{opp } p_1$

shows $p_2 = \text{add } a \ (\text{opp } p_1) \ (\text{opp } p_1)$

<proof>

lemma *cancel*:

assumes *ab*: *nonsingular a b*
and p_1 : *on-curve a b p₁*
and p_2 : *on-curve a b p₂*
and p_3 : *on-curve a b p₃*
and *eq*: $\text{add } a \ p_1 \ p_2 = \text{add } a \ p_1 \ p_3$
shows $p_2 = p_3$
<proof>

lemma *add-minus-id*:

assumes *ab*: *nonsingular a b*
and p_1 : *on-curve a b p₁*
and p_2 : *on-curve a b p₂*
shows $\text{add } a \ (\text{add } a \ p_1 \ p_2) \ (\text{opp } p_2) = p_1$
<proof>

lemma *add-shift-minus*:

assumes *ab*: *nonsingular a b*
and p_1 : *on-curve a b p₁*
and p_2 : *on-curve a b p₂*
and p_3 : *on-curve a b p₃*
and *eq*: $\text{add } a \ p_1 \ p_2 = p_3$
shows $p_1 = \text{add } a \ p_3 \ (\text{opp } p_2)$
<proof>

lemma *degen-assoc*:

assumes *ab*: *nonsingular a b*
and p_1 : *on-curve a b p₁*
and p_2 : *on-curve a b p₂*
and p_3 : *on-curve a b p₃*
and *H*:
 $(p_1 = \text{Infinity} \vee p_2 = \text{Infinity} \vee p_3 = \text{Infinity}) \vee$
 $(p_1 = \text{opp } p_2 \vee p_2 = \text{opp } p_3) \vee$
 $(\text{opp } p_1 = \text{add } a \ p_2 \ p_3 \vee \text{opp } p_3 = \text{add } a \ p_1 \ p_2)$
shows $\text{add } a \ p_1 \ (\text{add } a \ p_2 \ p_3) = \text{add } a \ (\text{add } a \ p_1 \ p_2) \ p_3$
<proof>

lemma *spec4-assoc*:

assumes *ab*: *nonsingular a b*
and p_1 : *on-curve a b p₁*
and p_2 : *on-curve a b p₂*
shows $\text{add } a \ p_1 \ (\text{add } a \ p_2 \ p_2) = \text{add } a \ (\text{add } a \ p_1 \ p_2) \ p_2$
<proof>

lemma *add-assoc*:

assumes *ab*: *nonsingular a b*
and p_1 : *on-curve a b p₁*
and p_2 : *on-curve a b p₂*
and p_3 : *on-curve a b p₃*

shows $\text{add } a \ p_1 \ (\text{add } a \ p_2 \ p_3) = \text{add } a \ (\text{add } a \ p_1 \ p_2) \ p_3$
 ⟨proof⟩

lemma *add-comm'*:

nonsingular $a \ b \implies$
on-curve $a \ b \ p_1 \implies \text{on-curve } a \ b \ p_2 \implies \text{on-curve } a \ b \ p_3 \implies$
 $\text{add } a \ p_2 \ (\text{add } a \ p_1 \ p_3) = \text{add } a \ p_1 \ (\text{add } a \ p_2 \ p_3)$
 ⟨proof⟩

primrec (in *ell-field*) *point-mult* :: 'a \Rightarrow nat \Rightarrow 'a *point* \Rightarrow 'a *point*
where

point-mult $a \ 0 \ p = \text{Infinity}$
 | *point-mult* $a \ (\text{Suc } n) \ p = \text{add } a \ p \ (\text{point-mult } a \ n \ p)$

lemma *point-mult-closed*: *on-curve* $a \ b \ p \implies \text{on-curve } a \ b \ (\text{point-mult } a \ n \ p)$
 ⟨proof⟩

lemma *point-mult-add*:

on-curve $a \ b \ p \implies \text{nonsingular } a \ b \implies$
 $\text{point-mult } a \ (m + n) \ p = \text{add } a \ (\text{point-mult } a \ m \ p) \ (\text{point-mult } a \ n \ p)$
 ⟨proof⟩

lemma *point-mult-mult*:

on-curve $a \ b \ p \implies \text{nonsingular } a \ b \implies$
 $\text{point-mult } a \ (m * n) \ p = \text{point-mult } a \ n \ (\text{point-mult } a \ m \ p)$
 ⟨proof⟩

lemma *point-mult2-eq-double*:

point-mult $a \ 2 \ p = \text{add } a \ p \ p$
 ⟨proof⟩

2.2 Projective Coordinates

type-synonym 'a *ppoint* = 'a \times 'a \times 'a

context *ell-field* **begin**

definition *pdouble* :: 'a \Rightarrow 'a *ppoint* \Rightarrow 'a *ppoint* **where**

pdouble $a \ p =$
 (let $(x, y, z) = p$
 in
 if $z = 0$ then p
 else
 let
 $l = 2 * y * z;$
 $m = 3 * x ^ 2 + a * z ^ 2$
 in
 $(l * (m ^ 2 - 4 * x * y * l),$
 $m * (6 * x * y * l - m ^ 2) -$

$$\begin{aligned} & 2 * y \wedge 2 * l \wedge 2, \\ & l \wedge 3)) \end{aligned}$$

definition *padd* :: 'a \Rightarrow 'a ppoint \Rightarrow 'a ppoint \Rightarrow 'a ppoint **where**

padd a p₁ p₂ =
 (let
 (x₁, y₁, z₁) = p₁;
 (x₂, y₂, z₂) = p₂
 in
 if z₁ = 0 then p₂
 else if z₂ = 0 then p₁
 else
 let
 d₁ = x₂ * z₁;
 d₂ = x₁ * z₂;
 l = d₁ - d₂;
 m = y₂ * z₁ - y₁ * z₂
 in
 if l = 0 then
 if m = 0 then pdouble a p₁
 else (0, 0, 0)
 else
 let h = m \wedge 2 * z₁ * z₂ - (d₁ + d₂) * l \wedge 2
 in
 (l * h,
 (d₂ * l \wedge 2 - h) * m - l \wedge 3 * y₁ * z₂,
 l \wedge 3 * z₁ * z₂))

definition *make-affine* :: 'a ppoint \Rightarrow 'a point **where**

make-affine p =
 (let (x, y, z) = p
 in if z = 0 then Infinity else Point (x / z) (y / z))

definition *on-curvep* :: 'a \Rightarrow 'a \Rightarrow 'a ppoint \Rightarrow bool **where**

on-curvep a b = ($\lambda(x, y, z). z \neq 0 \longrightarrow$
 y \wedge 2 * z = x \wedge 3 + a * x * z \wedge 2 + b * z \wedge 3)

end

lemma *on-curvep-infinity* [*simp*]: *on-curvep* a b (x, y, 0)
 <proof>

lemma *make-affine-infinity* [*simp*]: *make-affine* (x, y, 0) = Infinity
 <proof>

lemma *on-curvep-iff-on-curve*:

on-curvep a b p = *on-curve* a b (*make-affine* p)
 <proof>

lemma *pdouble-infinity* [simp]: $\text{pdouble } a (x, y, 0) = (x, y, 0)$
(proof)

lemma *padd-infinity-l* [simp]: $\text{padd } a (x, y, 0) p = p$
(proof)

lemma *pdouble-correct*:
 $\text{make-affine } (\text{pdouble } a p) = \text{add } a (\text{make-affine } p) (\text{make-affine } p)$
(proof)

lemma *padd-correct*:
assumes p_1 : *on-curvep* $a b p_1$ **and** p_2 : *on-curvep* $a b p_2$
shows $\text{make-affine } (\text{padd } a p_1 p_2) = \text{add } a (\text{make-affine } p_1) (\text{make-affine } p_2)$
(proof)

lemma *pdouble-closed*:
 $\text{on-curvep } a b p \implies \text{on-curvep } a b (\text{pdouble } a p)$
(proof)

lemma *padd-closed*:
 $\text{on-curvep } a b p_1 \implies \text{on-curvep } a b p_2 \implies \text{on-curvep } a b (\text{padd } a p_1 p_2)$
(proof)

primrec (in *ell-field*) *ppoint-mult* :: 'a \Rightarrow nat \Rightarrow 'a *ppoint* \Rightarrow 'a *ppoint*
where

$\text{ppoint-mult } a 0 p = (0, 0, 0)$
 $|\ \text{ppoint-mult } a (\text{Suc } n) p = \text{padd } a p (\text{ppoint-mult } a n p)$

lemma *ppoint-mult-closed* [simp]:
 $\text{on-curvep } a b p \implies \text{on-curvep } a b (\text{ppoint-mult } a n p)$
(proof)

lemma *ppoint-mult-correct*: $\text{on-curvep } a b p \implies$
 $\text{make-affine } (\text{ppoint-mult } a n p) = \text{point-mult } a n (\text{make-affine } p)$
(proof)

context *ell-field* **begin**

definition *proj-eq* :: 'a *ppoint* \Rightarrow 'a *ppoint* \Rightarrow bool **where**
 $\text{proj-eq} = (\lambda(x_1, y_1, z_1) (x_2, y_2, z_2).$
 $(z_1 = 0) = (z_2 = 0) \wedge x_1 * z_2 = x_2 * z_1 \wedge y_1 * z_2 = y_2 * z_1)$

end

lemma *proj-eq-refl*: $\text{proj-eq } p p$
(proof)

lemma *proj-eq-sym*: $\text{proj-eq } p p' \implies \text{proj-eq } p' p$
(proof)

lemma *proj-eq-trans*:

$in-carrierp\ p \implies in-carrierp\ p' \implies in-carrierp\ p'' \implies$
 $proj-eq\ p\ p' \implies proj-eq\ p'\ p'' \implies proj-eq\ p\ p''$
(proof)

lemma *make-affine-proj-eq-iff*:

$proj-eq\ p\ p' = (make-affine\ p = make-affine\ p')$
(proof)

lemma *pdouble-proj-eq-cong*:

$proj-eq\ p\ p' \implies proj-eq\ (pdouble\ a\ p)\ (pdouble\ a\ p')$
(proof)

lemma *padd-proj-eq-cong*:

$on-curvep\ a\ b\ p_1 \implies on-curvep\ a\ b\ p_1' \implies on-curvep\ a\ b\ p_2 \implies on-curvep\ a\ b\ p_2' \implies$
 $proj-eq\ p_1\ p_1' \implies proj-eq\ p_2\ p_2' \implies proj-eq\ (padd\ a\ p_1\ p_2)\ (padd\ a\ p_1'\ p_2')$
(proof)

end

3 Formalization using Locales

theory *Elliptic-Locale*

imports *HOL-Decision-Procs.Reflective-Field*

begin

3.1 Affine Coordinates

datatype *'a point* = *Infinity* | *Point 'a 'a*

locale *ell-field* = *field* +

assumes *two-not-zero*: $\langle 2 \rangle \neq \mathbf{0}$

begin

declare *two-not-zero* [*simplified, simp add*]

lemma *neg-equal-zero*:

assumes *x*: $x \in carrier\ R$

shows $(\ominus x = x) = (x = \mathbf{0})$

(proof)

lemmas *equal-neg-zero* = *trans* [*OF eq-commute neg-equal-zero*]

definition *nonsingular* :: *'a* \Rightarrow *'a* \Rightarrow *bool* **where**

nonsingular *a* *b* = $(\langle 4 \rangle \otimes a [\] (3::nat) \oplus \langle 27 \rangle \otimes b [\] (2::nat) \neq \mathbf{0})$

definition *on-curve* :: *'a* \Rightarrow *'a* \Rightarrow *'a point* \Rightarrow *bool* **where**

on-curve $a\ b\ p = (\text{case } p \text{ of}$
 Infinity $\Rightarrow \text{True}$
 | *Point* $x\ y \Rightarrow x \in \text{carrier } R \wedge y \in \text{carrier } R \wedge$
 $y \ [\uparrow] \ (2::\text{nat}) = x \ [\uparrow] \ (3::\text{nat}) \oplus a \otimes x \oplus b)$

definition *add* $:: 'a \Rightarrow 'a \text{ point} \Rightarrow 'a \text{ point} \Rightarrow 'a \text{ point}$ **where**

add $a\ p_1\ p_2 = (\text{case } p_1 \text{ of}$
 Infinity $\Rightarrow p_2$
 | *Point* $x_1\ y_1 \Rightarrow (\text{case } p_2 \text{ of}$
 Infinity $\Rightarrow p_1$
 | *Point* $x_2\ y_2 \Rightarrow$
 if $x_1 = x_2$ then
 if $y_1 = \ominus y_2$ then *Infinity*
 else
 let
 $l = (\llcorner 3 \gg \otimes x_1 \ [\uparrow] \ (2::\text{nat}) \oplus a) \otimes (\llcorner 2 \gg \otimes y_1);$
 $x_3 = l \ [\uparrow] \ (2::\text{nat}) \ominus \llcorner 2 \gg \otimes x_1$
 in
 Point $x_3 \ (\ominus y_1 \ominus l \otimes (x_3 \ominus x_1))$
 else
 let
 $l = (y_2 \ominus y_1) \otimes (x_2 \ominus x_1);$
 $x_3 = l \ [\uparrow] \ (2::\text{nat}) \ominus x_1 \ominus x_2$
 in
 Point $x_3 \ (\ominus y_1 \ominus l \otimes (x_3 \ominus x_1))))$

definition *opp* $:: 'a \text{ point} \Rightarrow 'a \text{ point}$ **where**

opp $p = (\text{case } p \text{ of}$
 Infinity $\Rightarrow \text{Infinity}$
 | *Point* $x\ y \Rightarrow \text{Point } x \ (\ominus y)$)

lemma *on-curve-infinity* [simp]: *on-curve* $a\ b$ *Infinity*
 ⟨proof⟩

lemma *opp-Infinity* [simp]: *opp* *Infinity* = *Infinity*
 ⟨proof⟩

lemma *opp-Point*: *opp* (*Point* $x\ y$) = *Point* $x \ (\ominus y)$
 ⟨proof⟩

lemma *opp-opp*: *on-curve* $a\ b\ p \Longrightarrow \text{opp} (\text{opp } p) = p$
 ⟨proof⟩

lemma *opp-closed*:
on-curve $a\ b\ p \Longrightarrow \text{on-curve } a\ b \ (\text{opp } p)$
 ⟨proof⟩

lemma *curve-elt-opp*:
 assumes $p_1 = \text{Point } x_1\ y_1$

and $p_2 = \text{Point } x_2 \ y_2$
and $\text{on-curve } a \ b \ p_1$
and $\text{on-curve } a \ b \ p_2$
and $x_1 = x_2$
shows $p_1 = p_2 \vee p_1 = \text{opp } p_2$
 ⟨proof⟩

lemma *add-closed*:
assumes $a \in \text{carrier } R$ **and** $b \in \text{carrier } R$
and $\text{on-curve } a \ b \ p_1$ **and** $\text{on-curve } a \ b \ p_2$
shows $\text{on-curve } a \ b \ (\text{add } a \ p_1 \ p_2)$
 ⟨proof⟩

lemma *add-case* [consumes 4, case-names *InfL InfR Opp Tan Gen*]:
assumes $a \in \text{carrier } R$
and $b \in \text{carrier } R$
and $p: \text{on-curve } a \ b \ p$
and $q: \text{on-curve } a \ b \ q$
and $R1: \bigwedge p. P \text{Infinity } p \ p$
and $R2: \bigwedge p. P \ p \ \text{Infinity } p$
and $R3: \bigwedge p. \text{on-curve } a \ b \ p \implies P \ p \ (\text{opp } p) \ \text{Infinity}$
and $R4: \bigwedge p_1 \ x_1 \ y_1 \ p_2 \ x_2 \ y_2 \ l.$
 $p_1 = \text{Point } x_1 \ y_1 \implies p_2 = \text{Point } x_2 \ y_2 \implies$
 $p_2 = \text{add } a \ p_1 \ p_1 \implies y_1 \neq \mathbf{0} \implies$
 $l = (\llbracket 3 \rrbracket \otimes x_1 \ [\uparrow] (2::\text{nat}) \oplus a) \otimes (\llbracket 2 \rrbracket \otimes y_1) \implies$
 $x_2 = l \ [\uparrow] (2::\text{nat}) \ominus \llbracket 2 \rrbracket \otimes x_1 \implies$
 $y_2 = \ominus y_1 \ominus l \otimes (x_2 \ominus x_1) \implies$
 $P \ p_1 \ p_1 \ p_2$
and $R5: \bigwedge p_1 \ x_1 \ y_1 \ p_2 \ x_2 \ y_2 \ p_3 \ x_3 \ y_3 \ l.$
 $p_1 = \text{Point } x_1 \ y_1 \implies p_2 = \text{Point } x_2 \ y_2 \implies p_3 = \text{Point } x_3 \ y_3 \implies$
 $p_3 = \text{add } a \ p_1 \ p_2 \implies x_1 \neq x_2 \implies$
 $l = (y_2 \ominus y_1) \otimes (x_2 \ominus x_1) \implies$
 $x_3 = l \ [\uparrow] (2::\text{nat}) \ominus x_1 \ominus x_2 \implies$
 $y_3 = \ominus y_1 \ominus l \otimes (x_3 \ominus x_1) \implies$
 $P \ p_1 \ p_2 \ p_3$
shows $P \ p \ q \ (\text{add } a \ p \ q)$
 ⟨proof⟩

lemma *add-casew* [consumes 4, case-names *InfL InfR Opp Gen*]:
assumes $a: a \in \text{carrier } R$
and $b: b \in \text{carrier } R$
and $p: \text{on-curve } a \ b \ p$
and $q: \text{on-curve } a \ b \ q$
and $R1: \bigwedge p. P \ \text{Infinity } p \ p$
and $R2: \bigwedge p. P \ p \ \text{Infinity } p$
and $R3: \bigwedge p. \text{on-curve } a \ b \ p \implies P \ p \ (\text{opp } p) \ \text{Infinity}$
and $R4: \bigwedge p_1 \ x_1 \ y_1 \ p_2 \ x_2 \ y_2 \ p_3 \ x_3 \ y_3 \ l.$
 $p_1 = \text{Point } x_1 \ y_1 \implies p_2 = \text{Point } x_2 \ y_2 \implies p_3 = \text{Point } x_3 \ y_3 \implies$
 $p_3 = \text{add } a \ p_1 \ p_2 \implies p_1 \neq \text{opp } p_2 \implies$

$$\begin{aligned}
x_1 = x_2 \wedge y_1 = y_2 \wedge l &= (\llbracket 3 \rrbracket \otimes x_1 \lceil \lceil (2::nat) \oplus a \rceil \otimes (\llbracket 2 \rrbracket \otimes y_1) \vee \\
x_1 \neq x_2 \wedge l &= (y_2 \ominus y_1) \otimes (x_2 \ominus x_1) \implies \\
x_3 &= l \lceil \lceil (2::nat) \ominus x_1 \ominus x_2 \implies \\
y_3 &= \ominus y_1 \ominus l \otimes (x_3 \ominus x_1) \implies \\
P &p_1 p_2 p_3 \\
\text{shows } P &p q \text{ (add a p q)} \\
\langle \text{proof} \rangle
\end{aligned}$$

definition

is-tangent $p q = (p \neq \text{Infinity} \wedge p = q \wedge p \neq \text{opp } q)$

definition

is-generic $p q =$
 $(p \neq \text{Infinity} \wedge q \neq \text{Infinity} \wedge$
 $p \neq q \wedge p \neq \text{opp } q)$

lemma spec1-assoc:

assumes $a: a \in \text{carrier } R$
and $b: b \in \text{carrier } R$
and $p_1: \text{on-curve } a b p_1$
and $p_2: \text{on-curve } a b p_2$
and $p_3: \text{on-curve } a b p_3$
and *is-generic* $p_1 p_2$
and *is-generic* $p_2 p_3$
and *is-generic* $(\text{add } a p_1 p_2) p_3$
and *is-generic* $p_1 (\text{add } a p_2 p_3)$
shows $\text{add } a p_1 (\text{add } a p_2 p_3) = \text{add } a (\text{add } a p_1 p_2) p_3$
 $\langle \text{proof} \rangle$

lemma spec2-assoc:

assumes $a: a \in \text{carrier } R$
and $b: b \in \text{carrier } R$
and $p_1: \text{on-curve } a b p_1$
and $p_2: \text{on-curve } a b p_2$
and $p_3: \text{on-curve } a b p_3$
and *is-generic* $p_1 p_2$
and *is-tangent* $p_2 p_3$
and *is-generic* $(\text{add } a p_1 p_2) p_3$
and *is-generic* $p_1 (\text{add } a p_2 p_3)$
shows $\text{add } a p_1 (\text{add } a p_2 p_3) = \text{add } a (\text{add } a p_1 p_2) p_3$
 $\langle \text{proof} \rangle$

lemma spec3-assoc:

assumes $a: a \in \text{carrier } R$
and $b: b \in \text{carrier } R$
and $p_1: \text{on-curve } a b p_1$
and $p_2: \text{on-curve } a b p_2$
and $p_3: \text{on-curve } a b p_3$
and *is-generic* $p_1 p_2$

and *is-tangent* $p_2 p_3$
and *is-generic* ($\text{add } a p_1 p_2$) p_3
and *is-tangent* p_1 ($\text{add } a p_2 p_3$)
shows $\text{add } a p_1 (\text{add } a p_2 p_3) = \text{add } a (\text{add } a p_1 p_2) p_3$
 ⟨*proof*⟩

lemma *add-0-l*: $\text{add } a \text{Infinity } p = p$
 ⟨*proof*⟩

lemma *add-0-r*: $\text{add } a p \text{Infinity} = p$
 ⟨*proof*⟩

lemma *add-opp*: $\text{on-curve } a b p \implies \text{add } a p (\text{opp } p) = \text{Infinity}$
 ⟨*proof*⟩

lemma *add-comm*:
assumes $a \in \text{carrier } R$ $b \in \text{carrier } R$ *on-curve* $a b p_1$ *on-curve* $a b p_2$
shows $\text{add } a p_1 p_2 = \text{add } a p_2 p_1$
 ⟨*proof*⟩

lemma *uniq-opp*:
assumes *on-curve* $a b p_2$
and $\text{add } a p_1 p_2 = \text{Infinity}$
shows $p_2 = \text{opp } p_1$
 ⟨*proof*⟩

lemma *uniq-zero*:
assumes $a: a \in \text{carrier } R$
and $b: b \in \text{carrier } R$
and $ab: \text{nonsingular } a b$
and $p_1: \text{on-curve } a b p_1$
and $p_2: \text{on-curve } a b p_2$
and $\text{add}: \text{add } a p_1 p_2 = p_2$
shows $p_1 = \text{Infinity}$
 ⟨*proof*⟩

lemma *opp-add*:
assumes $a: a \in \text{carrier } R$
and $b: b \in \text{carrier } R$
and $p_1: \text{on-curve } a b p_1$
and $p_2: \text{on-curve } a b p_2$
shows $\text{opp } (\text{add } a p_1 p_2) = \text{add } a (\text{opp } p_1) (\text{opp } p_2)$
 ⟨*proof*⟩

lemma *compat-add-opp*:
assumes $a: a \in \text{carrier } R$
and $b: b \in \text{carrier } R$
and $p_1: \text{on-curve } a b p_1$
and $p_2: \text{on-curve } a b p_2$

and $add\ a\ p_1\ p_2 = add\ a\ p_1\ (opp\ p_2)$
and $p_1 \neq opp\ p_1$
shows $p_2 = opp\ p_2$
<proof>

lemma *compat-add-triple*:
assumes $a: a \in carrier\ R$
and $b: b \in carrier\ R$
and $ab: nonsingular\ a\ b$
and $p: on-curve\ a\ b\ p$
and $p \neq opp\ p$
and $add\ a\ p\ p \neq opp\ p$
shows $add\ a\ (add\ a\ p\ p)\ (opp\ p) = p$
<proof>

lemma *add-opp-double-opp*:
assumes $a: a \in carrier\ R$
and $b: b \in carrier\ R$
and $ab: nonsingular\ a\ b$
and $p_1: on-curve\ a\ b\ p_1$
and $p_2: on-curve\ a\ b\ p_2$
and $add\ a\ p_1\ p_2 = opp\ p_1$
shows $p_2 = add\ a\ (opp\ p_1)\ (opp\ p_1)$
<proof>

lemma *cancel*:
assumes $a: a \in carrier\ R$
and $b: b \in carrier\ R$
and $ab: nonsingular\ a\ b$
and $p_1: on-curve\ a\ b\ p_1$
and $p_2: on-curve\ a\ b\ p_2$
and $p_3: on-curve\ a\ b\ p_3$
and $eq: add\ a\ p_1\ p_2 = add\ a\ p_1\ p_3$
shows $p_2 = p_3$
<proof>

lemma *add-minus-id*:
assumes $a: a \in carrier\ R$
and $b: b \in carrier\ R$
and $ab: nonsingular\ a\ b$
and $p_1: on-curve\ a\ b\ p_1$
and $p_2: on-curve\ a\ b\ p_2$
shows $add\ a\ (add\ a\ p_1\ p_2)\ (opp\ p_2) = p_1$
<proof>

lemma *add-shift-minus*:
assumes $a: a \in carrier\ R$
and $b: b \in carrier\ R$
and $ab: nonsingular\ a\ b$

and p_1 : *on-curve* a b p_1
and p_2 : *on-curve* a b p_2
and p_3 : *on-curve* a b p_3
and eq : $add\ a\ p_1\ p_2 = p_3$
shows $p_1 = add\ a\ p_3\ (opp\ p_2)$
 $\langle proof \rangle$

lemma *degen-assoc*:

assumes a : $a \in carrier\ R$
and b : $b \in carrier\ R$
and ab : *nonsingular* a b
and p_1 : *on-curve* a b p_1
and p_2 : *on-curve* a b p_2
and p_3 : *on-curve* a b p_3
and H :
 $(p_1 = Infinity \vee p_2 = Infinity \vee p_3 = Infinity) \vee$
 $(p_1 = opp\ p_2 \vee p_2 = opp\ p_3) \vee$
 $(opp\ p_1 = add\ a\ p_2\ p_3 \vee opp\ p_3 = add\ a\ p_1\ p_2)$
shows $add\ a\ p_1\ (add\ a\ p_2\ p_3) = add\ a\ (add\ a\ p_1\ p_2)\ p_3$
 $\langle proof \rangle$

lemma *spec4-assoc*:

assumes a : $a \in carrier\ R$
and b : $b \in carrier\ R$
and ab : *nonsingular* a b
and p_1 : *on-curve* a b p_1
and p_2 : *on-curve* a b p_2
shows $add\ a\ p_1\ (add\ a\ p_2\ p_2) = add\ a\ (add\ a\ p_1\ p_2)\ p_2$
 $\langle proof \rangle$

lemma *add-assoc*:

assumes a : $a \in carrier\ R$
and b : $b \in carrier\ R$
and ab : *nonsingular* a b
and p_1 : *on-curve* a b p_1
and p_2 : *on-curve* a b p_2
and p_3 : *on-curve* a b p_3
shows $add\ a\ p_1\ (add\ a\ p_2\ p_3) = add\ a\ (add\ a\ p_1\ p_2)\ p_3$
 $\langle proof \rangle$

lemma *add-comm'*:

$a \in carrier\ R \implies b \in carrier\ R \implies nonsingular\ a\ b \implies$
 $on-curve\ a\ b\ p_1 \implies on-curve\ a\ b\ p_2 \implies on-curve\ a\ b\ p_3 \implies$
 $add\ a\ p_2\ (add\ a\ p_1\ p_3) = add\ a\ p_1\ (add\ a\ p_2\ p_3)$
 $\langle proof \rangle$

primrec *point-mult* :: $'a \Rightarrow nat \Rightarrow 'a\ point \Rightarrow 'a\ point$

where

$point-mult\ a\ 0\ p = Infinity$

| $\text{point-mult } a \text{ (Suc } n) p = \text{add } a p \text{ (point-mult } a n p)$

lemma *point-mult-closed*: $a \in \text{carrier } R \implies b \in \text{carrier } R \implies$
 $\text{on-curve } a b p \implies \text{on-curve } a b \text{ (point-mult } a n p)$
 ⟨proof⟩

lemma *point-mult-add*:
 $a \in \text{carrier } R \implies b \in \text{carrier } R \implies \text{on-curve } a b p \implies \text{nonsingular } a b \implies$
 $\text{point-mult } a (m + n) p = \text{add } a \text{ (point-mult } a m p) \text{ (point-mult } a n p)$
 ⟨proof⟩

lemma *point-mult-mult*:
 $a \in \text{carrier } R \implies b \in \text{carrier } R \implies \text{on-curve } a b p \implies \text{nonsingular } a b \implies$
 $\text{point-mult } a (m * n) p = \text{point-mult } a n \text{ (point-mult } a m p)$
 ⟨proof⟩

lemma *point-mult2-eq-double*:
 $\text{point-mult } a 2 p = \text{add } a p p$
 ⟨proof⟩

end

3.2 Projective Coordinates

type-synonym $'a \text{ ppoint} = 'a \times 'a \times 'a$

definition (in *cring*) $\text{pdouble} :: 'a \Rightarrow 'a \text{ ppoint} \Rightarrow 'a \text{ ppoint}$ **where**
 $\text{pdouble } a p =$
 (let $(x, y, z) = p$
 in
 if $z = \mathbf{0}$ then p
 else
 let
 $l = \langle 2 \rangle \otimes y \otimes z;$
 $m = \langle 3 \rangle \otimes x [\uparrow] (2::\text{nat}) \oplus a \otimes z [\uparrow] (2::\text{nat})$
 in
 $(l \otimes (m [\uparrow] (2::\text{nat}) \ominus \langle 4 \rangle \otimes x \otimes y \otimes l),$
 $m \otimes (\langle 6 \rangle \otimes x \otimes y \otimes l \ominus m [\uparrow] (2::\text{nat})) \ominus$
 $\langle 2 \rangle \otimes y [\uparrow] (2::\text{nat}) \otimes l [\uparrow] (2::\text{nat}),$
 $l [\uparrow] (3::\text{nat}))$

definition (in *cring*) $\text{padd} :: 'a \Rightarrow 'a \text{ ppoint} \Rightarrow 'a \text{ ppoint} \Rightarrow 'a \text{ ppoint}$ **where**
 $\text{padd } a p_1 p_2 =$
 (let
 $(x_1, y_1, z_1) = p_1;$
 $(x_2, y_2, z_2) = p_2$
 in
 if $z_1 = \mathbf{0}$ then p_2
 else if $z_2 = \mathbf{0}$ then p_1

```

else
  let
    d1 = x2 ⊗ z1;
    d2 = x1 ⊗ z2;
    l = d1 ⊖ d2;
    m = y2 ⊗ z1 ⊖ y1 ⊗ z2
  in
    if l = 0 then
      if m = 0 then pdouble a p1
      else (0, 0, 0)
    else
      let h = m [∧] (2::nat) ⊗ z1 ⊗ z2 ⊖ (d1 ⊕ d2) ⊗ l [∧] (2::nat)
      in
        (l ⊗ h,
         (d2 ⊗ l [∧] (2::nat) ⊖ h) ⊗ m ⊖ l [∧] (3::nat) ⊗ y1 ⊗ z2,
         l [∧] (3::nat) ⊗ z1 ⊗ z2)

```

definition (in field) *make-affine* :: 'a ppoint ⇒ 'a point **where**

```

make-affine p =
  (let (x, y, z) = p
   in if z = 0 then Infinity else Point (x ⊗ z) (y ⊗ z))

```

definition (in cring) *in-carrierp* :: 'a ppoint ⇒ bool **where**

```

in-carrierp = (λ(x, y, z). x ∈ carrier R ∧ y ∈ carrier R ∧ z ∈ carrier R)

```

definition (in cring) *on-curvep* :: 'a ⇒ 'a ⇒ 'a ppoint ⇒ bool **where**

```

on-curvep a b = (λ(x, y, z).
  x ∈ carrier R ∧ y ∈ carrier R ∧ z ∈ carrier R ∧
  (z ≠ 0 →
   y [∧] (2::nat) ⊗ z = x [∧] (3::nat) ⊕ a ⊗ x ⊗ z [∧] (2::nat) ⊕ b ⊗ z [∧]
   (3::nat)))

```

lemma (in cring) *on-curvep-infinity* [simp]: on-curvep a b (x, y, **0**) = (x ∈ carrier R ∧ y ∈ carrier R)

⟨proof⟩

lemma (in field) *make-affine-infinity* [simp]: make-affine (x, y, **0**) = Infinity

⟨proof⟩

lemma (in cring) *on-curvep-imp-in-carrierp* [simp]: on-curvep a b p ⇒ in-carrierp p

⟨proof⟩

lemma (in ell-field) *on-curvep-iff-on-curve*:

```

assumes a ∈ carrier R b ∈ carrier R in-carrierp p
shows on-curvep a b p = on-curve a b (make-affine p)

```

⟨proof⟩

lemma (in cring) *pdouble-in-carrierp*:

$a \in \text{carrier } R \implies \text{in-carrierp } p \implies \text{in-carrierp } (\text{pdouble } a \ p)$
 ⟨proof⟩

lemma (in *cring*) *padd-in-carrierp*:

$a \in \text{carrier } R \implies \text{in-carrierp } p_1 \implies \text{in-carrierp } p_2 \implies \text{in-carrierp } (\text{padd } a \ p_1 \ p_2)$
 ⟨proof⟩

lemma (in *cring*) *pdouble-infinity [simp]*: $\text{pdouble } a \ (x, y, \mathbf{0}) = (x, y, \mathbf{0})$
 ⟨proof⟩

lemma (in *cring*) *padd-infinity-l [simp]*: $\text{padd } a \ (x, y, \mathbf{0}) \ p = p$
 ⟨proof⟩

lemma (in *ell-field*) *pdouble-correct*:

$a \in \text{carrier } R \implies \text{in-carrierp } p \implies$
 $\text{make-affine } (\text{pdouble } a \ p) = \text{add } a \ (\text{make-affine } p) \ (\text{make-affine } p)$
 ⟨proof⟩

lemma (in *ell-field*) *padd-correct*:

assumes $a: a \in \text{carrier } R$ **and** $b: b \in \text{carrier } R$
and $p_1: \text{on-curvep } a \ b \ p_1$ **and** $p_2: \text{on-curvep } a \ b \ p_2$
shows $\text{make-affine } (\text{padd } a \ p_1 \ p_2) = \text{add } a \ (\text{make-affine } p_1) \ (\text{make-affine } p_2)$
 ⟨proof⟩

lemma (in *ell-field*) *pdouble-closed*:

assumes $a \in \text{carrier } R$ $b \in \text{carrier } R$ $\text{on-curvep } a \ b \ p$
shows $\text{on-curvep } a \ b \ (\text{pdouble } a \ p)$
 ⟨proof⟩

lemma (in *ell-field*) *padd-closed*:

assumes $a \in \text{carrier } R$ $b \in \text{carrier } R$ $\text{on-curvep } a \ b \ p_1$ $\text{on-curvep } a \ b \ p_2$
shows $\text{on-curvep } a \ b \ (\text{padd } a \ p_1 \ p_2)$
 ⟨proof⟩

primrec (in *cring*) *ppoint-mult* :: $'a \Rightarrow \text{nat} \Rightarrow 'a \ \text{ppoint} \Rightarrow 'a \ \text{ppoint}$
where

$\text{ppoint-mult } a \ 0 \ p = (\mathbf{0}, \mathbf{0}, \mathbf{0})$
 $|\ \text{ppoint-mult } a \ (\text{Suc } n) \ p = \text{padd } a \ p \ (\text{ppoint-mult } a \ n \ p)$

lemma (in *ell-field*) *ppoint-mult-closed [simp]*:

$a \in \text{carrier } R \implies b \in \text{carrier } R \implies \text{on-curvep } a \ b \ p \implies \text{on-curvep } a \ b \ (\text{ppoint-mult } a \ n \ p)$
 ⟨proof⟩

lemma (in *ell-field*) *ppoint-mult-correct*: $a \in \text{carrier } R \implies b \in \text{carrier } R \implies$
 $\text{on-curvep } a \ b \ p \implies$

$\text{make-affine } (\text{ppoint-mult } a \ n \ p) = \text{point-mult } a \ n \ (\text{make-affine } p)$
 ⟨proof⟩

definition (in *cring*) *proj-eq* :: 'a ppoint \Rightarrow 'a ppoint \Rightarrow bool **where**
proj-eq = ($\lambda(x_1, y_1, z_1) (x_2, y_2, z_2).$
 $(z_1 = \mathbf{0}) = (z_2 = \mathbf{0}) \wedge x_1 \otimes z_2 = x_2 \otimes z_1 \wedge y_1 \otimes z_2 = y_2 \otimes z_1$)

lemma (in *cring*) *proj-eq-refl*: *proj-eq* *p* *p*
 \langle *proof* \rangle

lemma (in *cring*) *proj-eq-sym*: *proj-eq* *p* *p'* \Longrightarrow *proj-eq* *p'* *p*
 \langle *proof* \rangle

lemma (in *domain*) *proj-eq-trans*:
 $in-carrierp\ p \Longrightarrow in-carrierp\ p' \Longrightarrow in-carrierp\ p'' \Longrightarrow$
 $proj-eq\ p\ p' \Longrightarrow proj-eq\ p'\ p'' \Longrightarrow proj-eq\ p\ p''$
 \langle *proof* \rangle

lemma (in *field*) *make-affine-proj-eq-iff*:
 $in-carrierp\ p \Longrightarrow in-carrierp\ p' \Longrightarrow proj-eq\ p\ p' = (make-affine\ p = make-affine$
 $p')$
 \langle *proof* \rangle

lemma (in *ell-field*) *pdouble-proj-eq-cong*:
 $a \in carrier\ R \Longrightarrow in-carrierp\ p \Longrightarrow in-carrierp\ p' \Longrightarrow proj-eq\ p\ p' \Longrightarrow$
 $proj-eq\ (pdouble\ a\ p)\ (pdouble\ a\ p')$
 \langle *proof* \rangle

lemma (in *ell-field*) *padd-proj-eq-cong*:
 $a \in carrier\ R \Longrightarrow b \in carrier\ R \Longrightarrow on-curvep\ a\ b\ p_1 \Longrightarrow on-curvep\ a\ b\ p_1' \Longrightarrow$
 $on-curvep\ a\ b\ p_2 \Longrightarrow on-curvep\ a\ b\ p_2' \Longrightarrow proj-eq\ p_1\ p_1' \Longrightarrow proj-eq\ p_2\ p_2' \Longrightarrow$
 $proj-eq\ (padd\ a\ p_1\ p_2)\ (padd\ a\ p_1'\ p_2')$
 \langle *proof* \rangle

end

4 Validating the Specification

theory *Elliptic-Test*

imports

Elliptic-Locale

HOL-Number-Theory.Residues

begin

4.1 Specialized Definitions for Prime Fields

definition *mmult* :: int \Rightarrow int \Rightarrow int \Rightarrow int (infixl **₁ 70)
where $x\ **_m\ y = x * y \bmod m$

definition *madd* :: int \Rightarrow int \Rightarrow int \Rightarrow int (infixl ++₁ 65)
where $x\ ++_m\ y = (x + y) \bmod m$

definition $m_{sub} :: int \Rightarrow int \Rightarrow int \Rightarrow int$ (**infixl** $--_1$ 65)

where $x --_m y = (x - y) \bmod m$

definition $m_{pow} :: int \Rightarrow int \Rightarrow nat \Rightarrow int$ (**infixr** $\overset{\sim}{\sim}_1$ 80)

where $x \overset{\sim}{\sim}_m n = x \wedge n \bmod m$

lemma (**in** *residues*) *res-of-natural-eq*: $\langle n \rangle_{\mathbb{N}} = int\ n \bmod m$
<proof>

lemma (**in** *residues*) *res-of-integer-eq*: $\langle i \rangle = i \bmod m$
<proof>

lemma (**in** *residues*) *res-pow-eq*: $x \lceil (n :: nat) = x \wedge n \bmod m$
<proof>

lemma (**in** *residues*) *res-sub-eq*: $(x \bmod m) \ominus (y \bmod m) = (x \bmod m - y \bmod m) \bmod m$
<proof>

definition $m_{pdouble} :: int \Rightarrow int \Rightarrow int\ ppoint \Rightarrow int\ ppoint$ **where**

$m_{pdouble}\ m\ a\ p =$

(*let* $(x, y, z) = p$

in

if $z = 0$ *then* p

else

let

$l = 2 \bmod m **_m y **_m z;$

$n = 3 \bmod m **_m x \overset{\sim}{\sim}_m 2 ++_m a **_m z \overset{\sim}{\sim}_m 2$

in

$(l **_m (n \overset{\sim}{\sim}_m 2 --_m 4 \bmod m **_m x **_m y **_m l),$

$n **_m (6 \bmod m **_m x **_m y **_m l --_m n \overset{\sim}{\sim}_m 2) --_m$

$2 \bmod m **_m y \overset{\sim}{\sim}_m 2 **_m l \overset{\sim}{\sim}_m 2,$

$l \overset{\sim}{\sim}_m 3))$

definition $m_{padd} :: int \Rightarrow int \Rightarrow int\ ppoint \Rightarrow int\ ppoint \Rightarrow int\ ppoint$ **where**

$m_{padd}\ m\ a\ p_1\ p_2 =$

(*let*

$(x_1, y_1, z_1) = p_1;$

$(x_2, y_2, z_2) = p_2$

in

if $z_1 = 0$ *then* p_2

else if $z_2 = 0$ *then* p_1

else

let

$d_1 = x_2 **_m z_1;$

$d_2 = x_1 **_m z_2;$

$l = d_1 --_m d_2;$

$n = y_2 **_m z_1 --_m y_1 **_m z_2$

```

in
  if l = 0 then
    if n = 0 then mpdouble m a p1
    else (0, 0, 0)
  else
    let h = n  $\widetilde{m}^2$  **m z1 **m z2 --m (d1 ++m d2) **m l  $\widetilde{m}^2$ 
    in
      (l **m h,
       (d2 **m l  $\widetilde{m}^2$  --m h) **m n --m l  $\widetilde{m}^3$  **m y1 **m z2,
       l  $\widetilde{m}^3$  **m z1 **m z2)

```

lemma (in residues) pdouble-residue-eq: pdouble a p = mpdouble m a p
 ⟨proof⟩

lemma (in residues) padd-residue-eq: padd a p₁ p₂ = mpadd m a p₁ p₂
 ⟨proof⟩

fun fast-ppoint-mult :: int ⇒ int ⇒ nat ⇒ int ppoint ⇒ int ppoint
where

```

fast-ppoint-mult m a n p =
  (if n = 0 then (0, 0, 0)
   else if n mod 2 = 0 then mpdouble m a (fast-ppoint-mult m a (n div 2) p)
   else mpadd m a p (mpdouble m a (fast-ppoint-mult m a (n div 2) p)))

```

lemma fast-ppoint-mult-0 [simp]: fast-ppoint-mult m a 0 p = (0, 0, 0)
 ⟨proof⟩

lemma fast-ppoint-mult-even [simp]:
 n ≠ 0 ⇒ n mod 2 = 0 ⇒
 fast-ppoint-mult m a n p = mpdouble m a (fast-ppoint-mult m a (n div 2) p)
 ⟨proof⟩

lemma fast-ppoint-mult-odd [simp]:
 n ≠ 0 ⇒ n mod 2 ≠ 0 ⇒
 fast-ppoint-mult m a n p = mpadd m a p (mpdouble m a (fast-ppoint-mult m a
 (n div 2) p))
 ⟨proof⟩

declare fast-ppoint-mult.simps [simp del]

locale residues-prime-gt2 = residues-prime +
assumes gt2: 2 < p

sublocale residues-prime-gt2 < ell-field
 ⟨proof⟩

lemma (in residues-prime-gt2) fast-ppoint-mult-closed:
assumes a ∈ carrier R b ∈ carrier R on-curvep a b q
shows on-curvep a b (fast-ppoint-mult (int p) a n q)

$priv = 0x0037ADE9319A89F4DABDB3EF411AACCCA5123C61ACAB57B5393DCE47608172A095AA85A3$

definition $pubx :: int$ **where**

$pubx = 0x0015417E84DBF28C0AD3C278713349DC7DF153C897A1891BD98BAB4357C9ECBEE1E3BF42E$

definition $puby :: int$ **where**

$puby = 0x017CAE20B6641D2EEB695786D8C946146239D099E18E1D5A514C739D7CB4A10AD8A788015A$

definition $order :: nat$ **where**

$order = 0x01FF$

lemma $mon-curve\ m\ a\ b\ (Point\ gx\ gy)$

$\langle proof \rangle$

lemma $mmake-affine\ m\ (fast-ppoint-mult\ m\ a\ priv\ (gx,\ gy,\ 1)) = Point\ pubx\ puby$

$\langle proof \rangle$

lemma $mmake-affine\ m\ (fast-ppoint-mult\ m\ a\ order\ (gx,\ gy,\ 1)) = Infinity$

$\langle proof \rangle$

end

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