Definition and Elementary Properties of Ultrametric Spaces

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Abstract

An ultrametric space is a metric space in which the triangle inequality is strengthened by using the maximum instead of the sum. More formally, such a space is equipped with a real-valued application *dist*, called distance, verifying the four following conditions.

$$dist\ x\ y \geq 0$$
$$dist\ x\ y = dist\ y\ x$$
$$dist\ x\ y = 0 \longleftrightarrow x = y$$
$$dist\ x\ z \leq max\ (dist\ x\ y)\ (dist\ y\ z)$$

In this entry, we present an elementary formalization of these spaces relying on axiomatic type classes. The connection with standard metric spaces is obtained through a subclass relationship, and fundamental properties of ultrametric spaces are formally established.

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1 Definition

⟨*ML*⟩

class *ultrametric-space* = *uniformity-dist* + *open-uniformity* +
assumes *dist-eq-0-iff* [*simp*]: ‹*dist x y* = *0* ⟷ *x* = *y*›
and *ultrametric-dist-triangle2*: ‹*dist x y* ≤ *max* (*dist x z*) (*dist y z*)›
begin

subclass *metric-space*
⟨*proof*⟩

end

⟨*ML*⟩

class *complete-ultrametric-space* = *ultrametric-space* +
assumes *Cauchy-convergent*: ‹*Cauchy X* ⟹ *convergent X*›
begin

subclass *complete-space* ⟨*proof*⟩

end

2 Properties on Balls

In ultrametric space, balls satisfy very strong properties.

context *ultrametric-space* **begin**

lemma *ultrametric-dist-triangle*: ‹$dist\ x\ z \leq max\ (dist\ x\ y)\ (dist\ y\ z)$›
⟨*proof*⟩

lemma *ultrametric-dist-triangle3*: ‹$dist\ x\ y \leq max\ (dist\ a\ x)\ (dist\ a\ y)$›
⟨*proof*⟩

end

2.1 Balls are centered everywhere

context fixes *x* :: ‹*'a* :: *ultrametric-space*› **begin**

The best way to do this would be to work in the context *ultrametric-space*. Unfortunately, *ball*, *cball*, etc. are not defined inside the context *metric-space* but through a sort constraint.

lemma *ultrametric-every-point-of-ball-is-centre* :
‹$ball\ y\ r = ball\ x\ r$› **if** ‹$y \in ball\ x\ r$›
⟨*proof*⟩

lemma *ultrametric-every-point-of-cball-is-centre* :
‹$cball\ y\ r = cball\ x\ r$› **if** ‹$y \in cball\ x\ r$›
⟨*proof*⟩

end

2.2 Balls are "clopen"

Balls are both open and closed.

context fixes *x* :: ‹*'a* :: *ultrametric-space*› **begin**

lemma *ultrametric-open-cball* [*intro*, *simp*] : ‹*open* (*cball x r*)› **if** ‹$0 < r$›
⟨*proof*⟩

lemma ‹*closed* (*cball y r*)› ⟨*proof*⟩

lemma *ultrametric-closed-ball* [*intro*, *simp*]: ‹*closed* (*ball x r*)› **if** ‹$0 \leq r$›
⟨*proof*⟩

lemma *ultrametric-open-sphere* [*intro*, *simp*] : ‹$0 < r \Longrightarrow$ *open* (*sphere* *x r*)›
⟨*proof*⟩

lemma *closed-sphere* [*intro*, *simp*] : ‹*closed* (*sphere y r*)›
⟨*proof*⟩

end

2.3 Balls are disjoint or contained

context fixes *x* :: ‹*'a* :: *ultrametric-space*› **begin**

lemma *ultrametric-ball-ball-disjoint-or-subset*:
‹*ball x r* ∩ *ball y s* = {} ∨ *ball x r* ⊆ *ball y s* ∨
ball y s ⊆ *ball x r*›
⟨*proof*⟩

lemma *ultrametric-ball-cball-disjoint-or-subset*:
‹*ball x r* ∩ *cball y s* = {} ∨ *ball x r* ⊆ *cball y s* ∨
cball y s ⊆ *ball x r*›
⟨*proof*⟩

corollary *ultrametric-cball-ball-disjoint-or-subset*:
‹*cball x r* ∩ *ball y s* = {} ∨ *cball x r* ⊆ *ball y s* ∨
ball y s ⊆ *cball x r*›
⟨*proof*⟩

lemma *ultrametric-cball-cball-disjoint-or-subset*:
‹*cball x r* ∩ *cball y s* = {} ∨ *cball x r* ⊆ *cball y s* ∨
cball y s ⊆ *cball x r*›
⟨*proof*⟩

end

2.4 Distance to a Ball

context fixes *a* :: ‹*'a* :: *ultrametric-space*› **begin**

lemma *ultrametric-equal-distance-to-ball*:
‹*dist a y* = *dist a z*› **if** ‹*a* ∉ *ball x r*› ‹*y* ∈ *ball x r*› ‹*z* ∈ *ball x r*›
⟨*proof*⟩

lemma *ultrametric-equal-distance-to-cball*:
‹*dist a y* = *dist a z*› **if** ‹*a* ∉ *cball x r*› ‹*y* ∈ *cball x r*› ‹*z* ∈ *cball x r*›
⟨*proof*⟩

end

context fixes x :: ‹$'a$ :: *ultrametric-space*› **begin**

lemma *ultrametric-equal-distance-between-ball-ball*:
‹$ball\ x\ r \cap ball\ y\ s = \{\} \Longrightarrow$
$\exists d.\ \forall a \in ball\ x\ r.\ \forall b \in ball\ y\ s.\ dist\ a\ b = d$›
⟨*proof*⟩

lemma *ultrametric-equal-distance-between-ball-cball*:
‹$ball\ x\ r \cap cball\ y\ s = \{\} \Longrightarrow$
$\exists d.\ \forall a \in ball\ x\ r.\ \forall b \in cball\ y\ s.\ dist\ a\ b = d$›
⟨*proof*⟩

lemma *ultrametric-equal-distance-between-cball-ball*:
‹$cball\ x\ r \cap ball\ y\ s = \{\} \Longrightarrow$
$\exists d.\ \forall a \in cball\ x\ r.\ \forall b \in ball\ y\ s.\ dist\ a\ b = d$›
⟨*proof*⟩

lemma *ultrametric-equal-distance-between-cball-cball*:
‹$cball\ x\ r \cap cball\ y\ s = \{\} \Longrightarrow$
$\exists d.\ \forall a \in cball\ x\ r.\ \forall b \in cball\ y\ s.\ dist\ a\ b = d$›
⟨*proof*⟩

end

3 Additional Properties

Here are a few other interesting properties.

3.1 Cauchy Sequences

lemma (**in** *ultrametric-space*) *ultrametric-dist-triangle-generalized*:
‹$n < m \Longrightarrow dist\ (\sigma\ n)\ (\sigma\ m) \leq (MAX\ l \in \{n..m - 1\}.\ dist\ (\sigma\ l)\ (\sigma\ (Suc\ l)))$›
⟨*proof*⟩

lemma (**in** *ultrametric-space*) *ultrametric-Cauchy-iff*:
‹$Cauchy\ \sigma \longleftrightarrow (\lambda n.\ dist\ (\sigma\ (Suc\ n))\ (\sigma\ n)) \longrightarrow 0$›
⟨*proof*⟩

3.2 Isosceles Triangle Principle

lemma (**in** *ultrametric-space*) *ultrametric-isosceles-triangle-principle*
:
‹$dist\ x\ z = max\ (dist\ x\ y)\ (dist\ y\ z)$› **if** ‹$dist\ x\ y \neq dist\ y\ z$›
⟨*proof*⟩

3.3 Distance to a convergent Sequence

lemma *ultrametric-dist-to-convergent-sequence-is-eventually-const* :
fixes σ :: ‹*nat* $\Rightarrow$ *'a* :: *ultrametric-space*›
assumes ‹$\sigma \longrightarrow \Sigma$› **and** ‹$x \neq \Sigma$›
shows ‹$\exists N. \forall n \geq N.$ *dist* (σ *n*) x = *dist* Σ x›
⟨*proof*⟩

3.4 Diameter

lemma *ultrametric-diameter* : ‹*diameter* S = (*SUP* $y \in S$. *dist* x y)›
if ‹*bounded* S› **and** ‹$x \in S$› **for** x :: ‹*'a* :: *ultrametric-space*›
⟨*proof*⟩

3.5 Totally disconnected

lemma *ultrametric-totally-disconnected* :
‹$\exists x.\ S = \{x\}$› **if** ‹$S \neq \{\}$› ‹*connected* S›
for S :: ‹*'a* :: *ultrametric-space set*›
⟨*proof*⟩