The Transcendence of e

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Abstract

This work contains a formalisation of the proof that Euler's number e is transcendental. The proof follows the standard approach of assuming that e is algebraic and then using a specific integer polynomial to derive two inconsistent bounds, leading to a contradiction.

This approach can be found in many different sources; this formalisation mostly follows a PlanetMath article [1] by Roger Lipsett.

Contents

1 Proof of the Transcendence of e

theory *E-Transcendental*
imports
HOL−Complex-Analysis.Complex-Analysis
HOL−Number-Theory.Number-Theory
HOL−Computational-Algebra.Polynomial
Polynomial-Interpolation.Ring-Hom-Poly
begin

hide-const (**open**) *UnivPoly.coeff UnivPoly.up-ring.monom*
hide-const (**open**) *Module.smult Coset.order*

1.1 Various auxiliary facts

lemma *fact-dvd-pochhammer*:
assumes $m \leq n + 1$
shows *fact m dvd pochhammer* (*int n* − *int m* + *1*) *m*
⟨*proof*⟩

lemma *prime-elem-int-not-dvd-neg1-power*:

prime-elem $(p :: int) \Longrightarrow \neg p$ *dvd* (-1) ^ n
⟨*proof*⟩

lemma *nat-fact* [*simp*]: *nat* (*fact* n) = *fact* n
⟨*proof*⟩

lemma *prime-dvd-fact-iff-int*:
p *dvd fact* $n \longleftrightarrow p \leq$ *int* n **if** *prime* p
⟨*proof*⟩

lemma *power-over-fact-tendsto-0*:
$(\lambda n.\ (x :: real)$ ^ n / *fact* $n) \longrightarrow 0$
⟨*proof*⟩

lemma *power-over-fact-tendsto-0'*:
$(\lambda n.\ c * (x :: real)$ ^ n / *fact* $n) \longrightarrow 0$
⟨*proof*⟩

1.2 General facts about polynomials

lemma *fact-dvd-higher-pderiv*:
[:*fact* n :: *int*:] *dvd* (*pderiv* ^^ n) p
⟨*proof*⟩

lemma *fact-dvd-poly-higher-pderiv-aux*:
(*fact* n :: *int*) *dvd poly* ((*pderiv* ^^ n) p) x
⟨*proof*⟩

lemma *fact-dvd-poly-higher-pderiv-aux'*:
$m \leq n \Longrightarrow$ (*fact* m :: *int*) *dvd poly* ((*pderiv* ^^ n) p) x
⟨*proof*⟩

1.3 Main proof

lemma *lindemann-weierstrass-integral*:
fixes u :: *complex* **and** f :: *complex poly*
defines $df \equiv \lambda n.$ (*pderiv* ^^ n) f
defines $m \equiv$ *degree* f
defines $I \equiv \lambda f\ u.$ *exp* $u * (\sum j \leq$ *degree* f. *poly* ((*pderiv* ^^ j) f) 0) $-$
($\sum j \leq$ *degree* f. *poly* ((*pderiv* ^^ j) f) u)
shows (($\lambda t.$ *exp* $(u - t) *$ *poly* f t) *has-contour-integral* I f u) (*linepath* 0 u)
⟨*proof*⟩

locale *lindemann-weierstrass-aux* =
fixes f :: *complex poly*
begin

definition I :: *complex* $\Rightarrow$ *complex* **where**
$I\ u =$ *exp* $u * (\sum j \leq$ *degree* f. *poly* ((*pderiv* ^^ j) f) 0) $-$
($\sum j \leq$ *degree* f. *poly* ((*pderiv* ^^ j) f) u)

lemma *lindemann-weierstrass-integral-bound*:
  fixes u :: *complex*
  assumes $C \geq 0$ $\bigwedge t.\ t \in$ *closed-segment 0 u* $\Longrightarrow$ *norm* (*poly f t*) $\leq C$
  shows *norm* (I u) $\leq$ *norm u* $*$ *exp* (*norm u*) $*$ C
⟨*proof*⟩

end

lemma *poly-higher-pderiv-aux1*:
  fixes c :: $'a$ :: *idom*
  assumes $k < n$
  shows *poly* ((*pderiv* ^^ k) ([:$-c$, *1*:] ^ n $*$ p)) c = *0*
  ⟨*proof*⟩

lemma *poly-higher-pderiv-aux1'*:
  fixes c :: $'a$ :: *idom*
  assumes $k < n$ [:$-c$, *1*:] ^ n *dvd* p
  shows *poly* ((*pderiv* ^^ k) p) c = *0*
⟨*proof*⟩

lemma *poly-higher-pderiv-aux2*:
  fixes c :: $'a$ :: {*idom*, *semiring-char-0*}
  shows *poly* ((*pderiv* ^^ n) ([:$-c$, *1*:] ^ n $*$ p)) c = *fact n* $*$ *poly p c*
⟨*proof*⟩

lemma *poly-higher-pderiv-aux3*:
  fixes c :: $'a$ :: {*idom*,*semiring-char-0*}
  assumes $k \geq n$
  shows $\exists\, q$. *poly* ((*pderiv* ^^ k) ([:$-c$, *1*:] ^ n $*$ p)) c = *fact n* $*$ *poly q c*
  ⟨*proof*⟩

lemma *poly-higher-pderiv-aux3'*:
  fixes c :: $'a$ :: {*idom*, *semiring-char-0*}
  assumes $k \geq n$ [:$-c$, *1*:] ^ n *dvd* p
  shows *fact n dvd poly* ((*pderiv* ^^ k) p) c
⟨*proof*⟩

lemma *e-transcendental-aux-bound*:
  obtains C **where** $C \geq 0$
    $\bigwedge x.\ x \in$ *closed-segment 0* (*of-nat n*) $\Longrightarrow$
      norm ($\prod k \in \{1..n\}$. (x − *of-nat k* :: *complex*)) $\leq C$
⟨*proof*⟩

theorem *e-transcendental-complex*: ¬ *algebraic* (*exp 1* :: *complex*)
⟨*proof*⟩

corollary *e-transcendental-real*: ¬ *algebraic* (*exp 1* :: *real*)

$\langle proof \rangle$

end

References

[1] R. Lipsett. Planetmath. http://planetmath.org/proofoflindemannweierstrasstheoremandthateandpiaretranscendental, 2007.