

The Transcendence of e

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Abstract

This work contains a formalisation of the proof that Euler's number e is transcendental. The proof follows the standard approach of assuming that e is algebraic and then using a specific integer polynomial to derive two inconsistent bounds, leading to a contradiction.

This approach can be found in many different sources; this formalisation mostly follows a PlanetMath article [1] by Roger Lipsett.

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1 Proof of the Transcendence of e

```
theory E-Transcendental
imports
  HOL-Complex-Analysis.Complex-Analysis
  HOL-Number-Theory.Number-Theory
  HOL-Computational-Algebra.Polynomial
begin

hide-const (open) UnivPoly.coeff UnivPoly.up-ring.monom
hide-const (open) Module.smult Coset.order
```

1.1 Various auxiliary facts

```
lemma fact-dvd-pochhammer:
  assumes m ≤ n + 1
  shows fact m dvd pochhammer (int n - int m + 1) m
  ⟨proof⟩
```

```
lemma prime-elem-int-not-dvd-neg1-power:
```

prime-elem ($p :: \text{int}$) $\implies \neg p \text{ dvd } (-1) \wedge n$
 $\langle \text{proof} \rangle$

lemma *nat-fact* [*simp*]: $\text{nat} (\text{fact } n) = \text{fact } n$
 $\langle \text{proof} \rangle$

lemma *prime-dvd-fact-iff-int*:
 $p \text{ dvd } \text{fact } n \longleftrightarrow p \leq \text{int } n$ **if** *prime* p
 $\langle \text{proof} \rangle$

lemma *power-over-fact-tends-to-0*:
 $(\lambda n. (x :: \text{real})^n / \text{fact } n) \longrightarrow 0$
 $\langle \text{proof} \rangle$

lemma *power-over-fact-tends-to-0'*:
 $(\lambda n. c * (x :: \text{real})^n / \text{fact } n) \longrightarrow 0$
 $\langle \text{proof} \rangle$

1.2 Lifting integer polynomials

lift-definition *of-int-poly* :: $\text{int poly} \Rightarrow 'a :: \text{comm-ring-1 poly}$ **is** $\lambda g x. \text{of-int } (g x)$
 $\langle \text{proof} \rangle$

lemma *coeff-of-int-poly* [*simp*]: $\text{coeff } (\text{of-int-poly } p) n = \text{of-int } (\text{coeff } p n)$
 $\langle \text{proof} \rangle$

lemma *of-int-poly-0* [*simp*]: $\text{of-int-poly } 0 = 0$
 $\langle \text{proof} \rangle$

lemma *of-int-poly-pCons* [*simp*]: $\text{of-int-poly } (\text{pCons } c p) = \text{pCons } (\text{of-int } c) (\text{of-int-poly } p)$
 $\langle \text{proof} \rangle$

lemma *of-int-poly-smult* [*simp*]: $\text{of-int-poly } (\text{smult } c p) = \text{smult } (\text{of-int } c) (\text{of-int-poly } p)$
 $\langle \text{proof} \rangle$

lemma *of-int-poly-1* [*simp*]: $\text{of-int-poly } 1 = 1$
 $\langle \text{proof} \rangle$

lemma *of-int-poly-add* [*simp*]: $\text{of-int-poly } (p + q) = \text{of-int-poly } p + \text{of-int-poly } q$
 $\langle \text{proof} \rangle$

lemma *of-int-poly-mult* [*simp*]: $\text{of-int-poly } (p * q) = (\text{of-int-poly } p * \text{of-int-poly } q)$
 $\langle \text{proof} \rangle$

lemma *of-int-poly-sum* [*simp*]: $\text{of-int-poly } (\text{sum } f A) = \text{sum } (\lambda x. \text{of-int-poly } (f x))_A$
 $\langle \text{proof} \rangle$

lemma *of-int-poly-prod* [simp]: *of-int-poly* (*prod f A*) = *prod* ($\lambda x.$ *of-int-poly* (*f x*))
A
{proof}

lemma *of-int-poly-power* [simp]: *of-int-poly* (*p ^ n*) = *of-int-poly* *p* ^ *n*
{proof}

lemma *of-int-poly-monom* [simp]: *of-int-poly* (*monom c n*) = *monom* (*of-int c*) *n*
{proof}

lemma *poly-of-int-poly* [simp]: *poly* (*of-int-poly p*) (*of-int x*) = *of-int* (*poly p x*)
{proof}

lemma *poly-of-int-poly-of-nat* [simp]: *poly* (*of-int-poly p*) (*of-nat x*) = *of-int* (*poly p* (*int x*))
{proof}

lemma *poly-of-int-poly-0* [simp]: *poly* (*of-int-poly p*) 0 = *of-int* (*poly p 0*)
{proof}

lemma *poly-of-int-poly-1* [simp]: *poly* (*of-int-poly p*) 1 = *of-int* (*poly p 1*)
{proof}

lemma *poly-of-int-poly-of-real* [simp]:
poly (*of-int-poly p*) (*of-real x*) = *of-real* (*poly* (*of-int-poly p*) *x*)
{proof}

lemma *of-int-poly-eq-iff* [simp]:
of-int-poly p = (*of-int-poly q* :: 'a :: {comm-ring-1, ring-char-0} poly) \longleftrightarrow *p* = *q*
{proof}

lemma *of-int-poly-eq-0-iff* [simp]:
of-int-poly p = (0 :: 'a :: {comm-ring-1, ring-char-0} poly) \longleftrightarrow *p* = 0
{proof}

lemma *degree-of-int-poly* [simp]:
degree (*of-int-poly p* :: 'a :: {comm-ring-1, ring-char-0} poly) = *degree p*
{proof}

lemma *pderiv-of-int-poly* [simp]: *pderiv* (*of-int-poly p*) = *of-int-poly* (*pderiv p*)
{proof}

lemma *higher-pderiv-of-int-poly* [simp]:
(*pderiv* ^ n) (*of-int-poly p*) = *of-int-poly* ((*pderiv* ^ n) *p*)
{proof}

lemma *int-polyE*:
assumes $\bigwedge n.$ *coeff* (*p* :: 'a :: {comm-ring-1, ring-char-0} poly) *n* $\in \mathbb{Z}$

obtains p' **where** $p = \text{of-int-poly } p'$
 $\langle \text{proof} \rangle$

1.3 General facts about polynomials

lemma *pderiv-power*:

$$\text{pderiv } (p \wedge n) = \text{smult } (\text{of-nat } n) (p \wedge (n - 1) * \text{pderiv } p)$$

$$\langle \text{proof} \rangle$$

lemma *degree-prod-sum-eq*:

$$(\bigwedge x. x \in A \implies f x \neq 0) \implies$$

$$\text{degree } (\text{prod } f A :: 'a :: \text{idom poly}) = (\sum_{x \in A.} \text{degree } (f x))$$

$$\langle \text{proof} \rangle$$

lemma *pderiv-monom*:

$$\text{pderiv } (\text{monom } c n) = \text{monom } (\text{of-nat } n * c) (n - 1)$$

$$\langle \text{proof} \rangle$$

lemma *power-poly-const [simp]*: $[:c:] \wedge n = [:c \wedge n:]$

$$\langle \text{proof} \rangle$$

lemma *monom-power*: $\text{monom } c n \wedge k = \text{monom } (c \wedge k) (n * k)$

$$\langle \text{proof} \rangle$$

lemma *coeff-higher-pderiv*:

$$\text{coeff } ((\text{pderiv } \wedge m) f) n = \text{pochhammer } (\text{of-nat } (\text{Suc } n)) m * \text{coeff } f (n + m)$$

$$\langle \text{proof} \rangle$$

lemma *higher-pderiv-add*: $(\text{pderiv } \wedge n) (p + q) = (\text{pderiv } \wedge n) p + (\text{pderiv } \wedge n) q$

$$\langle \text{proof} \rangle$$

lemma *higher-pderiv-smult*: $(\text{pderiv } \wedge n) (\text{smult } c p) = \text{smult } c ((\text{pderiv } \wedge n) p)$

$$\langle \text{proof} \rangle$$

lemma *higher-pderiv-0 [simp]*: $(\text{pderiv } \wedge n) 0 = 0$

$$\langle \text{proof} \rangle$$

lemma *higher-pderiv-monom*:

$$m \leq n + 1 \implies (\text{pderiv } \wedge m) (\text{monom } c n) = \text{monom } (\text{pochhammer } (\text{int } n - \text{int } m + 1) m * c) (n - m)$$

$$\langle \text{proof} \rangle$$

lemma *higher-pderiv-monom-eq-zero*:

$$m > n + 1 \implies (\text{pderiv } \wedge m) (\text{monom } c n) = 0$$

$$\langle \text{proof} \rangle$$

lemma *higher-pderiv-sum*: $(\text{pderiv } \wedge n) (\text{sum } f A) = (\sum_{x \in A.} (\text{pderiv } \wedge n) (f x))$

$\langle proof \rangle$

lemma *fact-dvd-higher-pderiv*:
 [:fact n :: int:] dvd (pderiv $\wedge\wedge$ n) p
 $\langle proof \rangle$

lemma *fact-dvd-poly-higher-pderiv-aux*:
 (fact n :: int) dvd poly ((pderiv $\wedge\wedge$ n) p) x
 $\langle proof \rangle$

lemma *fact-dvd-poly-higher-pderiv-aux'*:
 $m \leq n \implies$ (fact m :: int) dvd poly ((pderiv $\wedge\wedge$ n) p) x
 $\langle proof \rangle$

lemma *algebraicE'*:
 assumes algebraic (x :: 'a :: field-char-0)
 obtains p **where** p $\neq 0$ poly (of-int-poly p) x = 0
 $\langle proof \rangle$

lemma *algebraicE'-nonzero*:
 assumes algebraic (x :: 'a :: field-char-0) x $\neq 0$
 obtains p **where** p $\neq 0$ coeff p 0 $\neq 0$ poly (of-int-poly p) x = 0
 $\langle proof \rangle$

lemma *algebraic-of-real-iff* [simp]:
 algebraic (of-real x :: 'a :: {real-algebra-1,field-char-0}) \longleftrightarrow algebraic x
 $\langle proof \rangle$

1.4 Main proof

lemma *lindemann-weierstrass-integral*:
 fixes u :: complex **and** f :: complex poly
 defines df $\equiv \lambda n.$ (pderiv $\wedge\wedge$ n) f
 defines m \equiv degree f
 defines I $\equiv \lambda f u.$ exp u * ($\sum j \leq$ degree f. poly ((pderiv $\wedge\wedge$ j) f) 0) -
 ($\sum j \leq$ degree f. poly ((pderiv $\wedge\wedge$ j) f) u)
 shows (($\lambda t.$ exp (u - t) * poly f t) has-contour-integral I f u) (linepath 0 u)
 $\langle proof \rangle$

locale *lindemann-weierstrass-aux* =
 fixes f :: complex poly
 begin

definition I :: complex \Rightarrow complex **where**
 I u = exp u * ($\sum j \leq$ degree f. poly ((pderiv $\wedge\wedge$ j) f) 0) -
 ($\sum j \leq$ degree f. poly ((pderiv $\wedge\wedge$ j) f) u)

lemma *lindemann-weierstrass-integral-bound*:
 fixes u :: complex

```

assumes  $C \geq 0 \wedge t \in \text{closed-segment } 0 u \implies \text{norm} (\text{poly } f t) \leq C$ 
shows  $\text{norm } (I u) \leq \text{norm } u * \exp (\text{norm } u) * C$ 
⟨proof⟩

end

lemma poly-higher-pderiv-aux1:
fixes  $c :: 'a :: \text{idom}$ 
assumes  $k < n$ 
shows  $\text{poly } ((\text{pderiv} \wedge k) ([:-c, 1:] \wedge n * p)) c = 0$ 
⟨proof⟩

lemma poly-higher-pderiv-aux1':
fixes  $c :: 'a :: \text{idom}$ 
assumes  $k < n [:-c, 1:] \wedge n \text{ dvd } p$ 
shows  $\text{poly } ((\text{pderiv} \wedge k) p) c = 0$ 
⟨proof⟩

lemma poly-higher-pderiv-aux2:
fixes  $c :: 'a :: \{\text{idom}, \text{semiring-char-0}\}$ 
shows  $\text{poly } ((\text{pderiv} \wedge n) ([:-c, 1:] \wedge n * p)) c = \text{fact } n * \text{poly } p c$ 
⟨proof⟩

lemma poly-higher-pderiv-aux3:
fixes  $c :: 'a :: \{\text{idom}, \text{semiring-char-0}\}$ 
assumes  $k \geq n$ 
shows  $\exists q. \text{poly } ((\text{pderiv} \wedge k) ([:-c, 1:] \wedge n * p)) c = \text{fact } n * \text{poly } q c$ 
⟨proof⟩

lemma poly-higher-pderiv-aux3':
fixes  $c :: 'a :: \{\text{idom}, \text{semiring-char-0}\}$ 
assumes  $k \geq n [:-c, 1:] \wedge n \text{ dvd } p$ 
shows  $\text{fact } n \text{ dvd } \text{poly } ((\text{pderiv} \wedge k) p) c$ 
⟨proof⟩

lemma e-transcendental-aux-bound:
obtains  $C$  where  $C \geq 0$ 
 $\wedge x. x \in \text{closed-segment } 0 (\text{of-nat } n) \implies$ 
 $\text{norm } (\prod_{k \in \{1..n\}} (x - \text{of-nat } k :: \text{complex})) \leq C$ 
⟨proof⟩

theorem e-transcendental-complex:  $\neg \text{algebraic } (\exp 1 :: \text{complex})$ 
⟨proof⟩

corollary e-transcendental-real:  $\neg \text{algebraic } (\exp 1 :: \text{real})$ 
⟨proof⟩

end

```

References

- [1] R. Lipsett. Planetmath. <http://planetmath.org/proofoflindemannweierstrasstheoremandthatpiaretranscendental>, 2007.