

# The Transcendence of $e$

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## Abstract

This work contains a formalisation of the proof that Euler's number  $e$  is transcendental. The proof follows the standard approach of assuming that  $e$  is algebraic and then using a specific integer polynomial to derive two inconsistent bounds, leading to a contradiction.

This approach can be found in many different sources; this formalisation mostly follows a PlanetMath article [1] by Roger Lipsett.

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## 1 Proof of the Transcendence of $e$

```
theory E-Transcendental
  imports
    HOL-Analysis.Analysis
    HOL-Number-Theory.Number-Theory
    HOL-Computational-Algebra.Polynomial
begin
```

### 1.1 Various auxiliary facts

```
lemma fact-dvd-pochhammer:
  assumes  $m \leq n + 1$ 
  shows fact  $m$  dvd pochhammer (int  $n - int\ m + 1$ )  $m$ 
<proof>
```

```
lemma of-nat-eq-1-iff [simp]: of-nat  $x = (1 :: 'a :: semiring-char-0) \longleftrightarrow x = 1$ 
<proof>
```

```
lemma prime-elem-int-not-dvd-neg1-power:
```

*prime-elim*  $(p :: \text{int}) \implies \neg p \text{ dvd } (-1) \wedge n$   
*<proof>*

**lemma** *nat-fact [simp]*:  $\text{nat } (\text{fact } n) = \text{fact } n$   
*<proof>*

**lemma** *prime-dvd-fact-iff-int*:  
 $p \text{ dvd fact } n \iff p \leq \text{int } n \text{ if prime } p$   
*<proof>*

**lemma** *filterlim-minus-nat-at-top*:  
 $\text{filterlim } (\lambda n. n - k :: \text{nat}) \text{ at-top at-top}$   
*<proof>*

**lemma** *power-over-fact-tendsto-0*:  
 $(\lambda n. (x :: \text{real}) \wedge n / \text{fact } n) \longrightarrow 0$   
*<proof>*

**lemma** *power-over-fact-tendsto-0'*:  
 $(\lambda n. c * (x :: \text{real}) \wedge n / \text{fact } n) \longrightarrow 0$   
*<proof>*

## 1.2 Lifting integer polynomials

**lift-definition** *of-int-poly* ::  $\text{int poly} \Rightarrow 'a :: \text{comm-ring-1 poly}$  is  $\lambda g x. \text{of-int } (g x)$   
*<proof>*

**lemma** *coeff-of-int-poly [simp]*:  $\text{coeff } (\text{of-int-poly } p) n = \text{of-int } (\text{coeff } p n)$   
*<proof>*

**lemma** *of-int-poly-0 [simp]*:  $\text{of-int-poly } 0 = 0$   
*<proof>*

**lemma** *of-int-poly-pCons [simp]*:  $\text{of-int-poly } (p\text{Cons } c p) = p\text{Cons } (\text{of-int } c) (\text{of-int-poly } p)$   
*<proof>*

**lemma** *of-int-poly-smult [simp]*:  $\text{of-int-poly } (\text{smult } c p) = \text{smult } (\text{of-int } c) (\text{of-int-poly } p)$   
*<proof>*

**lemma** *of-int-poly-1 [simp]*:  $\text{of-int-poly } 1 = 1$   
*<proof>*

**lemma** *of-int-poly-add [simp]*:  $\text{of-int-poly } (p + q) = \text{of-int-poly } p + \text{of-int-poly } q$   
*<proof>*

**lemma** *of-int-poly-mult [simp]*:  $\text{of-int-poly } (p * q) = (\text{of-int-poly } p * \text{of-int-poly } q)$

*<proof>*

**lemma** *of-int-poly-sum* [simp]:  $of-int-poly (sum f A) = sum (\lambda x. of-int-poly (f x)) A$   
*<proof>*

**lemma** *of-int-poly-prod* [simp]:  $of-int-poly (prod f A) = prod (\lambda x. of-int-poly (f x)) A$   
*<proof>*

**lemma** *of-int-poly-power* [simp]:  $of-int-poly (p \wedge n) = of-int-poly p \wedge n$   
*<proof>*

**lemma** *of-int-poly-monom* [simp]:  $of-int-poly (monom c n) = monom (of-int c) n$   
*<proof>*

**lemma** *poly-of-int-poly* [simp]:  $poly (of-int-poly p) (of-int x) = of-int (poly p x)$   
*<proof>*

**lemma** *poly-of-int-poly-of-nat* [simp]:  $poly (of-int-poly p) (of-nat x) = of-int (poly p (int x))$   
*<proof>*

**lemma** *poly-of-int-poly-0* [simp]:  $poly (of-int-poly p) 0 = of-int (poly p 0)$   
*<proof>*

**lemma** *poly-of-int-poly-1* [simp]:  $poly (of-int-poly p) 1 = of-int (poly p 1)$   
*<proof>*

**lemma** *poly-of-int-poly-of-real* [simp]:  
 $poly (of-int-poly p) (of-real x) = of-real (poly (of-int-poly p) x)$   
*<proof>*

**lemma** *of-int-poly-eq-iff* [simp]:  
 $of-int-poly p = (of-int-poly q :: 'a :: \{comm-ring-1, ring-char-0\} poly) \iff p = q$   
*<proof>*

**lemma** *of-int-poly-eq-0-iff* [simp]:  
 $of-int-poly p = (0 :: 'a :: \{comm-ring-1, ring-char-0\} poly) \iff p = 0$   
*<proof>*

**lemma** *degree-of-int-poly* [simp]:  
 $degree (of-int-poly p :: 'a :: \{comm-ring-1, ring-char-0\} poly) = degree p$   
*<proof>*

**lemma** *pderiv-of-int-poly* [simp]:  $pderiv (of-int-poly p) = of-int-poly (pderiv p)$   
*<proof>*

**lemma** *higher-pderiv-of-int-poly* [simp]:  
 $(pderiv \hat{\hat{n}}) (of-int-poly p) = of-int-poly ((pderiv \hat{\hat{n}}) p)$   
 ⟨proof⟩

**lemma** *int-polyE*:  
 assumes  $\bigwedge n. coeff (p :: 'a :: \{comm-ring-1, ring-char-0\} poly) n \in \mathbb{Z}$   
 obtains  $p'$  where  $p = of-int-poly p'$   
 ⟨proof⟩

### 1.3 General facts about polynomials

**lemma** *pderiv-power*:  
 $pderiv (p \hat{n}) = smult (of-nat n) (p \hat{(n-1)} * pderiv p)$   
 ⟨proof⟩

**lemma** *degree-prod-sum-eq*:  
 $(\bigwedge x. x \in A \implies f x \neq 0) \implies$   
 $degree (prod f A :: 'a :: idom poly) = (\sum x \in A. degree (f x))$   
 ⟨proof⟩

**lemma** *pderiv-monom*:  
 $pderiv (monom c n) = monom (of-nat n * c) (n - 1)$   
 ⟨proof⟩

**lemma** *power-poly-const* [simp]:  $[:c:] \hat{n} = [:c \hat{n}:]$   
 ⟨proof⟩

**lemma** *monom-power*:  $monom c n \hat{k} = monom (c \hat{k}) (n * k)$   
 ⟨proof⟩

**lemma** *coeff-higher-pderiv*:  
 $coeff ((pderiv \hat{\hat{m}}) f) n = pochhammer (of-nat (Suc n)) m * coeff f (n + m)$   
 ⟨proof⟩

**lemma** *higher-pderiv-add*:  $(pderiv \hat{\hat{n}}) (p + q) = (pderiv \hat{\hat{n}}) p + (pderiv \hat{\hat{n}}) q$   
 ⟨proof⟩

**lemma** *higher-pderiv-smult*:  $(pderiv \hat{\hat{n}}) (smult c p) = smult c ((pderiv \hat{\hat{n}}) p)$   
 ⟨proof⟩

**lemma** *higher-pderiv-0* [simp]:  $(pderiv \hat{\hat{n}}) 0 = 0$   
 ⟨proof⟩

**lemma** *higher-pderiv-monom*:  
 $m \leq n + 1 \implies (pderiv \hat{\hat{m}}) (monom c n) = monom (pochhammer (int n - int m + 1) m * c) (n - m)$   
 ⟨proof⟩

**lemma** *higher-pderiv-monom-eq-zero*:

$m > n + 1 \implies (\text{pderiv } \hat{\hat{m}}) (\text{monom } c \ n) = 0$   
<proof>

**lemma** *higher-pderiv-sum*:  $(\text{pderiv } \hat{\hat{n}}) (\text{sum } f \ A) = (\sum x \in A. (\text{pderiv } \hat{\hat{n}}) (f \ x))$

<proof>

**lemma** *fact-dvd-higher-pderiv*:

$[\text{fact } n :: \text{int}] \text{ dvd } (\text{pderiv } \hat{\hat{n}}) \ p$   
<proof>

**lemma** *fact-dvd-poly-higher-pderiv-aux*:

$(\text{fact } n :: \text{int}) \text{ dvd poly } ((\text{pderiv } \hat{\hat{n}}) \ p) \ x$   
<proof>

**lemma** *fact-dvd-poly-higher-pderiv-aux'*:

$m \leq n \implies (\text{fact } m :: \text{int}) \text{ dvd poly } ((\text{pderiv } \hat{\hat{n}}) \ p) \ x$   
<proof>

**lemma** *algebraicE'*:

**assumes** *algebraic*  $(x :: 'a :: \text{field-char-0})$   
**obtains** *p where*  $p \neq 0 \text{ poly } (\text{of-int-poly } p) \ x = 0$   
<proof>

**lemma** *algebraicE'-nonzero*:

**assumes** *algebraic*  $(x :: 'a :: \text{field-char-0}) \ x \neq 0$   
**obtains** *p where*  $p \neq 0 \text{ coeff } p \ 0 \neq 0 \text{ poly } (\text{of-int-poly } p) \ x = 0$   
<proof>

**lemma** *algebraic-of-real-iff* [*simp*]:

$\text{algebraic } (\text{of-real } x :: 'a :: \{\text{real-algebra-1}, \text{field-char-0}\}) \longleftrightarrow \text{algebraic } x$   
<proof>

## 1.4 Main proof

**lemma** *lindemann-weierstrass-integral*:

**fixes**  $u :: \text{complex}$  **and**  $f :: \text{complex poly}$   
**defines**  $df \equiv \lambda n. (\text{pderiv } \hat{\hat{n}}) \ f$   
**defines**  $m \equiv \text{degree } f$   
**defines**  $I \equiv \lambda f \ u. \text{exp } u * (\sum j \leq \text{degree } f. \text{poly } ((\text{pderiv } \hat{\hat{j}}) \ f) \ 0) -$   
 $(\sum j \leq \text{degree } f. \text{poly } ((\text{pderiv } \hat{\hat{j}}) \ f) \ u)$   
**shows**  $(\lambda t. \text{exp } (u - t) * \text{poly } f \ t) \text{ has-contour-integral } I \ f \ u$  (*linpath*  $0 \ u$ )  
<proof>

**locale** *lindemann-weierstrass-aux* =

**fixes**  $f :: \text{complex poly}$   
**begin**

**definition**  $I :: \text{complex} \Rightarrow \text{complex}$  **where**

$$I u = \exp u * \left( \sum_{j \leq \text{degree } f} \text{poly } ((\text{pderiv } ^j f) 0) - \sum_{j \leq \text{degree } f} \text{poly } ((\text{pderiv } ^j f) u) \right)$$

**lemma** *lindemann-weierstrass-integral-bound*:

**fixes**  $u :: \text{complex}$

**assumes**  $C \geq 0 \wedge t. t \in \text{closed-segment } 0 u \implies \text{norm } (\text{poly } f t) \leq C$

**shows**  $\text{norm } (I u) \leq \text{norm } u * \exp (\text{norm } u) * C$

*<proof>*

**end**

**lemma** *poly-higher-pderiv-aux1*:

**fixes**  $c :: 'a :: \text{idom}$

**assumes**  $k < n$

**shows**  $\text{poly } ((\text{pderiv } ^k) ([: -c, 1:] ^n * p)) c = 0$

*<proof>*

**lemma** *poly-higher-pderiv-aux1'*:

**fixes**  $c :: 'a :: \text{idom}$

**assumes**  $k < n$   $[: -c, 1:] ^n \text{ dvd } p$

**shows**  $\text{poly } ((\text{pderiv } ^k) p) c = 0$

*<proof>*

**lemma** *poly-higher-pderiv-aux2*:

**fixes**  $c :: 'a :: \{\text{idom}, \text{semiring-char-0}\}$

**shows**  $\text{poly } ((\text{pderiv } ^n) ([: -c, 1:] ^n * p)) c = \text{fact } n * \text{poly } p c$

*<proof>*

**lemma** *poly-higher-pderiv-aux3*:

**fixes**  $c :: 'a :: \{\text{idom}, \text{semiring-char-0}\}$

**assumes**  $k \geq n$

**shows**  $\exists q. \text{poly } ((\text{pderiv } ^k) ([: -c, 1:] ^n * p)) c = \text{fact } n * \text{poly } q c$

*<proof>*

**lemma** *poly-higher-pderiv-aux3'*:

**fixes**  $c :: 'a :: \{\text{idom}, \text{semiring-char-0}\}$

**assumes**  $k \geq n$   $[: -c, 1:] ^n \text{ dvd } p$

**shows**  $\text{fact } n \text{ dvd } \text{poly } ((\text{pderiv } ^k) p) c$

*<proof>*

**lemma** *e-transcendental-aux-bound*:

**obtains**  $C$  **where**  $C \geq 0$

$\wedge x. x \in \text{closed-segment } 0 (\text{of-nat } n) \implies$

$\text{norm } (\prod_{k \in \{1..n\}}. (x - \text{of-nat } k :: \text{complex})) \leq C$

*<proof>*

**theorem** *e-transcendental-complex*:  $\neg \text{algebraic } (\exp 1 :: \text{complex})$

*<proof>*

**corollary** *e-transcendental-real:  $\neg$  algebraic (exp 1 :: real)*

*<proof>*

**end**

## References

- [1] R. Lipsett. Planetmath. <http://planetmath.org/proofofindemannweierstrasstheoremandthateandpiaretranscendental>, 2007.