Doob's Upcrossing Inequality and Martingale Convergence Theorem

Ata Keskin

March 29, 2024

Abstract

In this entry, we formalize Doob's upcrossing inequality and subsequently prove Doob's first martingale convergence theorem. The upcrossing inequality is a fundamental result in the study of martingales. It provides a bound on the expected number of times a submartingale crosses a certain threshold within a given interval. Doob's martingale convergence theorem states that, if we have a submartingale where the supremum over the mean of the positive parts is finite, then the limit process exists almost surely and is integrable. Equivalent statements for martingales and supermartingales are also provided as corollaries.

The proofs provided are based mostly on the formalization done in the Lean mathematical library [1,2].

Contents

1 Introduction

Martingales, in the context of stochastic processes, are encountered in various real-world scenarios where outcomes are influenced by past events but are not entirely predictable due to randomness or uncertainty. A martingale is a stochastic process in which the expected value of the next observation, given all past observations, is equal to the current observation.

One real-world example can be encountered in environmental monitoring, particularly in the study of river flow rates. Consider a hydrologist tasked with monitoring the flow rate of a river to understand its behavior over time. The flow rate of a river is influenced by various factors such as rainfall, snowmelt, groundwater levels, and human activities like dam releases or water diversions. These factors contribute to the variability and unpredictability of the flow rate. In this scenario, the flow rate of the river can be modeled as a martingale. The flow rate at any given time is influenced by past events but is not entirely predictable due to the random nature of rainfall and other factors.

One concept that comes up frequently in the study of martingales are upcrossings and downcrossings. Upcrossings and downcrossings are random variables representing when the value of a stochastic process leaves a fixed interval. Specifically, an upcrossing occurs when the process moves from below the lower bound of the interval to above the upper bound [4], indicating a potential upward trend or positive movement. Conversely, a downcrossing happens when the process crosses below the lower bound of the interval, suggesting a potential downward trend or negative movement. By analyzing the frequency and timing of these crossings, researchers can infer information about the underlying dynamics of the process and detect shifts in its behavior.

For instance, consider tracking the movement of a stock price over time. The process representing the stock's price might cross above a certain threshold (upcrossing) or below it (downcrossing) multiple times during a trading session. The number of such crossings provides insights into the volatility and the trend of the stock.

Doob's upcrossing inequality is a fundamental result in the study of martingales. It provides a bound on the expected number of upcrossings a submartingale undertakes before some point in time.

Let's consider our example concerning river flow rates again. In this context, upcrossings represent instances where the flow rate of the river rises above a certain threshold. For example, the flow rate might cross a threshold indicating flood risk. Downcrossings, on the other hand, represent instances where the flow rate decreases below a certain threshold. This could indicate drought conditions or low-flow periods.

Doob's first martingale convergence theorem gives sufficient conditions for a submartingale to converge to a random variable almost surely. The proof is based on controlling the rate of growth or fluctuations of the submartingale, which is where the *upcrossing inequality* comes into play. By bounding these fluctuations, we can ensure that the submartingale does not exhibit wild behavior or grow too quickly, which is essential for proving convergence.

Formally, the convergence theorem states that, if $(M_n)_{n\geq 0}$ is a submartingale with $\sup_n \mathbb{E}[M_n^+] < \infty$, where M_n^+ denotes the positive part of M_n, then the limit process $M_\infty := \lim_n M_n$ exists almost surely and is integrable. Furthermore, the limit process is measurable with respect to the smallest σ-algebra containing all of the σ-algebras in the filtration. In our formalization, we also show equivalent convergence statements for martingales and supermartingales. The theorem can be used to easily show convergence results for simple scenarios.

Consider the following example: Imagine a casino game where a player bets on the outcome of a random coin toss, where the coin comes up heads with odds $p \in [0, \frac{1}{2})$. Assume that the player goes bust when they have no money remaining. The player's wealth over time can be modeled as a supermartingale, where the value of their wealth at each time step depends only on the outcome of the previous coin toss. Doob's martingale convergence theorem assures us that the player will go bankrupt as the number of coin tosses increases.

The theorem that we have described here and formalized in the scope of our project is called *Doob's first martingale convergence theorem*. It is important to note that the convergence in this theorem is pointwise, not uniform, and is unrelated to convergence in mean square, or indeed in any L^p space. In order to obtain convergence in L^1 (i.e., convergence in mean), one requires uniform integrability of the random variables. In this form, the theorem is called *Doob's second martingale convergence theorem*. Since uniform integrability is not yet formalized in Isabelle/HOL, we have decided to confine our formalization to the first convergence theorem only.

2 Updates for the entry Martingales

This section contains the changes done for the entry Martingales [7]. We simplified the locale hierarchy by removing unnecessary locales and moving lemmas under more general locales where possible. We have to redefine almost all of the constants, in order to make sure we use the new locale hierarchy. The changes will be incorporated into the entry Martingales [7] and this file will be removed when the next Isabelle version rolls out.

theory *Martingales-Updates*
imports *Martingales.Martingale*
begin

2.1 Updates for *Martingales.Filtered-Measure*

lemma (**in** *filtered-measure*) *sets-F-subset*[*simp*]:
  assumes $t_0 \leq t$
  shows *sets* $(F\ t) \subseteq$ *sets* M
  ⟨*proof*⟩

locale *linearly-filtered-measure* = *filtered-measure* M F t_0 **for** M **and** F :: - :: {*linorder-topology*, *conditionally-complete-lattice*} ⇒ - **and** t_0

context *linearly-filtered-measure*
begin

— We define F_∞ to be the smallest σ-algebra containing all the σ-algebras in the filtration.

definition *F-infinity* :: *'a measure* **where**
  F-infinity = *sigma* (*space* M) ($\bigcup t \in \{t_0..\}$. *sets* $(F\ t)$)

notation *F-infinity* (‹F_∞›)

lemma *space-F-infinity*[*simp*]: *space* F_∞ = *space* M ⟨*proof*⟩

lemma *sets-F-infinity*: *sets* F_∞ = *sigma-sets* (*space* M) ($\bigcup t \in \{t_0..\}$. *sets* $(F\ t)$)
  ⟨*proof*⟩

lemma *subset-F-infinity*:
  assumes $t \geq t_0$
  shows $F\ t \subseteq F_\infty$ ⟨*proof*⟩

lemma *F-infinity-subset*: $F_\infty \subseteq M$
  ⟨*proof*⟩

lemma *F-infinity-measurableI*:
  assumes $t \geq t_0$ $f \in$ *borel-measurable* $(F\ t)$
  shows $f \in$ *borel-measurable* (F_∞)

⟨*proof*⟩

end

locale *nat-filtered-measure* = *linearly-filtered-measure M F 0* **for** *M* **and** *F* :: *nat* ⇒ -
locale *enat-filtered-measure* = *linearly-filtered-measure M F 0* **for** *M* **and** *F* :: *enat* ⇒ -
locale *real-filtered-measure* = *linearly-filtered-measure M F 0* **for** *M* **and** *F* :: *real* ⇒ -
locale *ennreal-filtered-measure* = *linearly-filtered-measure M F 0* **for** *M* **and** *F* :: *ennreal* ⇒ -

locale *nat-sigma-finite-filtered-measure* = *sigma-finite-filtered-measure M F 0* :: *nat* **for** *M F*
locale *enat-sigma-finite-filtered-measure* = *sigma-finite-filtered-measure M F 0* :: *enat* **for** *M F*
locale *real-sigma-finite-filtered-measure* = *sigma-finite-filtered-measure M F 0* :: *real* **for** *M F*
locale *ennreal-sigma-finite-filtered-measure* = *sigma-finite-filtered-measure M F 0* :: *ennreal* **for** *M F*

sublocale *nat-sigma-finite-filtered-measure* ⊆ *nat-filtered-measure* ⟨*proof*⟩
sublocale *enat-sigma-finite-filtered-measure* ⊆ *enat-filtered-measure* ⟨*proof*⟩
sublocale *real-sigma-finite-filtered-measure* ⊆ *real-filtered-measure* ⟨*proof*⟩
sublocale *ennreal-sigma-finite-filtered-measure* ⊆ *ennreal-filtered-measure* ⟨*proof*⟩

sublocale *nat-sigma-finite-filtered-measure* ⊆ *sigma-finite-subalgebra M F i* ⟨*proof*⟩
sublocale *enat-sigma-finite-filtered-measure* ⊆ *sigma-finite-subalgebra M F i* ⟨*proof*⟩
sublocale *real-sigma-finite-filtered-measure* ⊆ *sigma-finite-subalgebra M F* $|i|$ ⟨*proof*⟩

sublocale *ennreal-sigma-finite-filtered-measure* ⊆ *sigma-finite-subalgebra M F i* ⟨*proof*⟩

locale *finite-filtered-measure* = *filtered-measure* + *finite-measure*

sublocale *finite-filtered-measure* ⊆ *sigma-finite-filtered-measure*
⟨*proof*⟩

locale *nat-finite-filtered-measure* = *finite-filtered-measure M F 0* :: *nat* **for** *M F*
locale *enat-finite-filtered-measure* = *finite-filtered-measure M F 0* :: *enat* **for** *M F*
locale *real-finite-filtered-measure* = *finite-filtered-measure M F 0* :: *real* **for** *M F*
locale *ennreal-finite-filtered-measure* = *finite-filtered-measure M F 0* :: *ennreal* **for** *M F*

sublocale *nat-finite-filtered-measure* ⊆ *nat-sigma-finite-filtered-measure* ⟨*proof*⟩
sublocale *enat-finite-filtered-measure* ⊆ *enat-sigma-finite-filtered-measure* ⟨*proof*⟩
sublocale *real-finite-filtered-measure* ⊆ *real-sigma-finite-filtered-measure* ⟨*proof*⟩
sublocale *ennreal-finite-filtered-measure* ⊆ *ennreal-sigma-finite-filtered-measure*

⟨*proof*⟩

2.2 Updates for *Martingales.Stochastic-Process*

lemma (**in** *nat-filtered-measure*) *partial-sum-Suc-adapted*:
  assumes *adapted-process* $M\ F\ 0\ X$
  shows *adapted-process* $M\ F\ 0\ (\lambda n\ \xi.\ \sum i{<}n.\ X\ (Suc\ i)\ \xi)$
⟨*proof*⟩

lemma (**in** *enat-filtered-measure*) *partial-sum-eSuc-adapted*:
  assumes *adapted-process* $M\ F\ 0\ X$
  shows *adapted-process* $M\ F\ 0\ (\lambda n\ \xi.\ \sum i{<}n.\ X\ (eSuc\ i)\ \xi)$
⟨*proof*⟩

lemma (**in** *filtered-measure*) *adapted-process-sum*:
  assumes $\bigwedge i.\ i \in I \Longrightarrow$ *adapted-process* $M\ F\ t_0\ (X\ i)$
  shows *adapted-process* $M\ F\ t_0\ (\lambda k\ \xi.\ \sum i \in I.\ X\ i\ k\ \xi)$
⟨*proof*⟩

context *linearly-filtered-measure*
begin

definition Σ_P :: $('b \times 'a)$ *measure* **where** *predictable-sigma*: $\Sigma_P \equiv sigma\ (\{t_0..\} \times space\ M)\ (\{\{s{<}..t\} \times A \mid A\ s\ t.\ A \in F\ s \wedge t_0 \leq s \wedge s < t\} \cup \{\{t_0\} \times A \mid A.\ A \in F\ t_0\})$

lemma *space-predictable-sigma*[*simp*]: $space\ \Sigma_P = (\{t_0..\} \times space\ M)$ ⟨*proof*⟩

lemma *sets-predictable-sigma*: $sets\ \Sigma_P = sigma\text{-}sets\ (\{t_0..\} \times space\ M)\ (\{\{s{<}..t\} \times A \mid A\ s\ t.\ A \in F\ s \wedge t_0 \leq s \wedge s < t\} \cup \{\{t_0\} \times A \mid A.\ A \in F\ t_0\})$
  ⟨*proof*⟩

lemma *measurable-predictable-sigma-snd*:
  assumes *countable* $\mathcal{I}$ $\mathcal{I} \subseteq \{\{s{<}..t\} \mid s\ t.\ t_0 \leq s \wedge s < t\}$ $\{t_0{<}..\} \subseteq (\bigcup \mathcal{I})$
  shows $snd \in \Sigma_P \rightarrow_M F\ t_0$
⟨*proof*⟩

lemma *measurable-predictable-sigma-fst*:
  assumes *countable* $\mathcal{I}$ $\mathcal{I} \subseteq \{\{s{<}..t\} \mid s\ t.\ t_0 \leq s \wedge s < t\}$ $\{t_0{<}..\} \subseteq (\bigcup \mathcal{I})$
  shows $fst \in \Sigma_P \rightarrow_M borel$
⟨*proof*⟩

end

locale *predictable-process* = *linearly-filtered-measure* $M\ F\ t_0$ **for** $M\ F\ t_0$ **and** X :: - $\Rightarrow$ - $\Rightarrow$ - :: {*second-countable-topology*, *banach*} +
  assumes *predictable*: $(\lambda(t, x).\ X\ t\ x) \in borel\text{-}measurable\ \Sigma_P$
begin

lemmas *predictableD* = *measurable-sets*[*OF predictable*, *unfolded space-predictable-sigma*]

end

lemma (**in** *nat-filtered-measure*) *measurable-predictable-sigma-snd′*:
shows $snd \in \Sigma_P \rightarrow_M F\ 0$
⟨*proof*⟩

lemma (**in** *nat-filtered-measure*) *measurable-predictable-sigma-fst′*:
shows $fst \in \Sigma_P \rightarrow_M borel$
⟨*proof*⟩

lemma (**in** *enat-filtered-measure*) *measurable-predictable-sigma-snd′*:
shows $snd \in \Sigma_P \rightarrow_M F\ 0$
⟨*proof*⟩

lemma (**in** *enat-filtered-measure*) *measurable-predictable-sigma-fst′*:
shows $fst \in \Sigma_P \rightarrow_M borel$
⟨*proof*⟩

lemma (**in** *real-filtered-measure*) *measurable-predictable-sigma-snd′*:
shows $snd \in \Sigma_P \rightarrow_M F\ 0$
⟨*proof*⟩

lemma (**in** *real-filtered-measure*) *measurable-predictable-sigma-fst′*:
shows $fst \in \Sigma_P \rightarrow_M borel$
⟨*proof*⟩

lemma (**in** *ennreal-filtered-measure*) *measurable-predictable-sigma-snd′*:
shows $snd \in \Sigma_P \rightarrow_M F\ 0$
⟨*proof*⟩

lemma (**in** *ennreal-filtered-measure*) *measurable-predictable-sigma-fst′*:
shows $fst \in \Sigma_P \rightarrow_M borel$
⟨*proof*⟩

lemma (**in** *linearly-filtered-measure*) *predictable-process-const-fun*:
assumes $snd \in \Sigma_P \rightarrow_M F\ t_0$ $f \in$ *borel-measurable* $(F\ t_0)$
shows *predictable-process* $M\ F\ t_0\ (\lambda\text{-}.\ f)$
⟨*proof*⟩

lemma (**in** *nat-filtered-measure*) *predictable-process-const-fun′*[*intro*]:
assumes $f \in$ *borel-measurable* $(F\ 0)$
shows *predictable-process* $M\ F\ 0\ (\lambda\text{-}.\ f)$
⟨*proof*⟩

lemma (**in** *enat-filtered-measure*) *predictable-process-const-fun′*[*intro*]:
assumes $f \in$ *borel-measurable* $(F\ 0)$
shows *predictable-process* $M\ F\ 0\ (\lambda\text{-}.\ f)$

⟨*proof*⟩

lemma (**in** *real-filtered-measure*) *predictable-process-const-fun'*[*intro*]:
assumes $f \in$ *borel-measurable* (*F 0*)
shows *predictable-process M F 0* (λ-. *f*)
⟨*proof*⟩

lemma (**in** *ennreal-filtered-measure*) *predictable-process-const-fun'*[*intro*]:
assumes $f \in$ *borel-measurable* (*F 0*)
shows *predictable-process M F 0* (λ-. *f*)
⟨*proof*⟩

lemma (**in** *linearly-filtered-measure*) *predictable-process-const*:
assumes *fst* ∈ *borel-measurable* Σ_P *c* ∈ *borel-measurable borel*
shows *predictable-process M F* t_0 (λ*i* -. *c i*)
⟨*proof*⟩

lemma (**in** *linearly-filtered-measure*) *predictable-process-const-const*[*intro*]:
shows *predictable-process M F* t_0 (λ- -. *c*)
⟨*proof*⟩

lemma (**in** *nat-filtered-measure*) *predictable-process-const'*[*intro*]:
assumes *c* ∈ *borel-measurable borel*
shows *predictable-process M F 0* (λ*i* -. *c i*)
⟨*proof*⟩

lemma (**in** *enat-filtered-measure*) *predictable-process-const'*[*intro*]:
assumes *c* ∈ *borel-measurable borel*
shows *predictable-process M F 0* (λ*i* -. *c i*)
⟨*proof*⟩

lemma (**in** *real-filtered-measure*) *predictable-process-const'*[*intro*]:
assumes *c* ∈ *borel-measurable borel*
shows *predictable-process M F 0* (λ*i* -. *c i*)
⟨*proof*⟩

lemma (**in** *ennreal-filtered-measure*) *predictable-process-const'*[*intro*]:
assumes *c* ∈ *borel-measurable borel*
shows *predictable-process M F 0* (λ*i* -. *c i*)
⟨*proof*⟩

context *predictable-process*
begin

lemma *compose-predictable*:
assumes *fst* ∈ *borel-measurable* Σ_P *case-prod f* ∈ *borel-measurable borel*
shows *predictable-process M F* t_0 (λ*i* ξ. (*f i*) (*X i* ξ))
⟨*proof*⟩

lemma *norm-predictable*: *predictable-process* M F t_0 ($\lambda i\ \xi.$ *norm* ($X\ i\ \xi$)) ⟨*proof*⟩

lemma *scaleR-right-predictable*:
assumes *predictable-process* M F t_0 R
shows *predictable-process* M F t_0 ($\lambda i\ \xi.$ ($R\ i\ \xi$) $*_R$ ($X\ i\ \xi$))
⟨*proof*⟩

lemma *scaleR-right-const-fun-predictable*:
assumes *snd* $\in \Sigma_P \rightarrow_M F\ t_0$ $f \in$ *borel-measurable* ($F\ t_0$)
shows *predictable-process* M F t_0 ($\lambda i\ \xi.\ f\ \xi *_R (X\ i\ \xi)$)
⟨*proof*⟩

lemma *scaleR-right-const-predictable*:
assumes *fst* $\in$ *borel-measurable* Σ_P $c \in$ *borel-measurable borel*
shows *predictable-process* M F t_0 ($\lambda i\ \xi.\ c\ i *_R (X\ i\ \xi)$)
⟨*proof*⟩

lemma *scaleR-right-const′-predictable*: *predictable-process* M F t_0 ($\lambda i\ \xi.\ c *_R (X\ i$
$\xi)$)
⟨*proof*⟩

lemma *add-predictable*:
assumes *predictable-process* M F t_0 Y
shows *predictable-process* M F t_0 ($\lambda i\ \xi.\ X\ i\ \xi + Y\ i\ \xi$)
⟨*proof*⟩

lemma *diff-predictable*:
assumes *predictable-process* M F t_0 Y
shows *predictable-process* M F t_0 ($\lambda i\ \xi.\ X\ i\ \xi - Y\ i\ \xi$)
⟨*proof*⟩

lemma *uminus-predictable*: *predictable-process* M F t_0 ($-X$) ⟨*proof*⟩

end

sublocale *predictable-process* ⊆ *progressive-process*
⟨*proof*⟩

lemma (**in** *nat-filtered-measure*) *sets-in-filtration*:
assumes ($\bigcup i.\ \{i\} \times A\ i$) $\in \Sigma_P$
shows A (*Suc* i) $\in F\ i$ $A\ 0 \in F\ 0$
⟨*proof*⟩

lemma (**in** *nat-filtered-measure*) *predictable-implies-adapted-Suc*:
assumes *predictable-process* M F 0 X
shows *adapted-process* M F 0 ($\lambda i.\ X$ (*Suc* i))
⟨*proof*⟩

theorem (**in** *nat-filtered-measure*) *predictable-process-iff*: *predictable-process* M F 0

X ⟷ *adapted-process M F 0* (λ*i*. *X* (*Suc i*)) ∧ *X 0* ∈ *borel-measurable* (*F 0*)
⟨*proof*⟩

corollary (**in** *nat-filtered-measure*) *predictable-processI*[*intro*!]:
assumes *X 0* ∈ *borel-measurable* (*F 0*) ⋀*i*. *X* (*Suc i*) ∈ *borel-measurable* (*F i*)
shows *predictable-process M F 0 X*
⟨*proof*⟩

2.3 Updates for *Martingales.Martingale*

locale *martingale* = *sigma-finite-filtered-measure* + *adapted-process* +
assumes *integrable*: ⋀*i*. t_0 ≤ *i* ⟹ *integrable M* (*X i*)
and *martingale-property*: ⋀*i j*. t_0 ≤ *i* ⟹ *i* ≤ *j* ⟹ *AE* ξ *in M*. *X i* ξ = *cond-exp M* (*F i*) (*X j*) ξ

locale *martingale-order* = *martingale M F* t_0 *X* **for** *M F* t_0 **and** *X* :: - ⇒ - ⇒ - :: {*order-topology*, *ordered-real-vector*}
locale *martingale-linorder* = *martingale M F* t_0 *X* **for** *M F* t_0 **and** *X* :: - ⇒ - ⇒ - :: {*linorder-topology*, *ordered-real-vector*}
sublocale *martingale-linorder* ⊆ *martingale-order* ⟨*proof*⟩

lemma (**in** *sigma-finite-filtered-measure*) *martingale-const-fun*[*intro*]:
assumes *integrable M f f* ∈ *borel-measurable* (*F* t_0)
shows *martingale M F* t_0 (λ-. *f*)
⟨*proof*⟩

lemma (**in** *sigma-finite-filtered-measure*) *martingale-cond-exp*[*intro*]:
assumes *integrable M f*
shows *martingale M F* t_0 (λ*i*. *cond-exp M* (*F i*) *f*)
⟨*proof*⟩

corollary (**in** *sigma-finite-filtered-measure*) *martingale-zero*[*intro*]: *martingale M F* t_0 (λ- -. *0*) ⟨*proof*⟩

corollary (**in** *finite-filtered-measure*) *martingale-const*[*intro*]: *martingale M F* t_0 (λ- -. *c*) ⟨*proof*⟩

locale *submartingale* = *sigma-finite-filtered-measure M F* t_0 + *adapted-process M F* t_0 *X* **for** *M F* t_0 **and** *X* :: - ⇒ - ⇒ - :: {*order-topology*, *ordered-real-vector*} +
assumes *integrable*: ⋀*i*. t_0 ≤ *i* ⟹ *integrable M* (*X i*)
and *submartingale-property*: ⋀*i j*. t_0 ≤ *i* ⟹ *i* ≤ *j* ⟹ *AE* ξ *in M*. *X i* ξ ≤ *cond-exp M* (*F i*) (*X j*) ξ

locale *submartingale-linorder* = *submartingale M F* t_0 *X* **for** *M F* t_0 **and** *X* :: - ⇒ - ⇒ - :: {*linorder-topology*}

lemma (**in** *sigma-finite-filtered-measure*) *submartingale-const-fun*[*intro*]:
assumes *integrable M f f* ∈ *borel-measurable* (*F* t_0)
shows *submartingale M F* t_0 (λ-. *f*)

⟨*proof*⟩

lemma (**in** *sigma-finite-filtered-measure*) *submartingale-cond-exp*[*intro*]:
assumes *integrable M f*
shows *submartingale M F* t_0 (λi. *cond-exp M* (*F i*) *f*)
⟨*proof*⟩

corollary (**in** *finite-filtered-measure*) *submartingale-const*[*intro*]: *submartingale M F* t_0 (λ- -. *c*) ⟨*proof*⟩

sublocale *martingale-order* ⊆ *submartingale* ⟨*proof*⟩
sublocale *martingale-linorder* ⊆ *submartingale-linorder* ⟨*proof*⟩

locale *supermartingale* = *sigma-finite-filtered-measure M F* t_0 + *adapted-process M F* t_0 *X* **for** *M F* t_0 **and** *X* :: - ⇒ - ⇒ - :: {*order-topology*, *ordered-real-vector*} +
assumes *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable M* (*X i*)
and *supermartingale-property*: $\bigwedge i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow$ *AE* ξ *in M*. *X i* ξ ≥ *cond-exp M* (*F i*) (*X j*) ξ

locale *supermartingale-linorder* = *supermartingale M F* t_0 *X* **for** *M F* t_0 **and** *X* :: - ⇒ - ⇒ - :: {*linorder-topology*}

lemma (**in** *sigma-finite-filtered-measure*) *supermartingale-const-fun*[*intro*]:
assumes *integrable M f f* ∈ *borel-measurable* (*F* t_0)
shows *supermartingale M F* t_0 (λ-. *f*)
⟨*proof*⟩

lemma (**in** *sigma-finite-filtered-measure*) *supermartingale-cond-exp*[*intro*]:
assumes *integrable M f*
shows *supermartingale M F* t_0 (λi. *cond-exp M* (*F i*) *f*)
⟨*proof*⟩

corollary (**in** *finite-filtered-measure*) *supermartingale-const*[*intro*]: *supermartingale M F* t_0 (λ- -. *c*) ⟨*proof*⟩

sublocale *martingale-order* ⊆ *supermartingale* ⟨*proof*⟩
sublocale *martingale-linorder* ⊆ *supermartingale-linorder* ⟨*proof*⟩

lemma *martingale-iff*:
shows *martingale M F* t_0 *X* ⟷ *submartingale M F* t_0 *X* ∧ *supermartingale M F* t_0 *X*
⟨*proof*⟩

context *martingale*
begin

lemma *cond-exp-diff-eq-zero*:
assumes $t_0 \leq i$ $i \leq j$
shows *AE* ξ *in M*. *cond-exp M* (*F i*) ($\lambda\xi$. *X j* ξ − *X i* ξ) ξ = *0*

⟨*proof*⟩

lemma *set-integral-eq*:
assumes $A \in F\ i\ t_0 \leq i\ i \leq j$
shows *set-lebesgue-integral* $M\ A\ (X\ i)$ = *set-lebesgue-integral* $M\ A\ (X\ j)$
⟨*proof*⟩

lemma *scaleR-const*[*intro*]:
shows *martingale* $M\ F\ t_0\ (\lambda i\ x.\ c *_R X\ i\ x)$
⟨*proof*⟩

lemma *uminus*[*intro*]:
shows *martingale* $M\ F\ t_0\ (-\ X)$
⟨*proof*⟩

lemma *add*[*intro*]:
assumes *martingale* $M\ F\ t_0\ Y$
shows *martingale* $M\ F\ t_0\ (\lambda i\ \xi.\ X\ i\ \xi + Y\ i\ \xi)$
⟨*proof*⟩

lemma *diff*[*intro*]:
assumes *martingale* $M\ F\ t_0\ Y$
shows *martingale* $M\ F\ t_0\ (\lambda i\ x.\ X\ i\ x - Y\ i\ x)$
⟨*proof*⟩

end

lemma (**in** *sigma-finite-filtered-measure*) *martingale-of-cond-exp-diff-eq-zero*:
assumes *adapted*: *adapted-process* $M\ F\ t_0\ X$
and *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable* $M\ (X\ i)$
and *diff-zero*: $\bigwedge i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow$ *AE x in M*. *cond-exp* $M\ (F\ i)\ (\lambda\xi.\ X\ j\ \xi - X\ i\ \xi)\ x = 0$
shows *martingale* $M\ F\ t_0\ X$
⟨*proof*⟩

lemma (**in** *sigma-finite-filtered-measure*) *martingale-of-set-integral-eq*:
assumes *adapted*: *adapted-process* $M\ F\ t_0\ X$
and *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable* $M\ (X\ i)$
and $\bigwedge A\ i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow A \in F\ i \Longrightarrow$ *set-lebesgue-integral* $M\ A\ (X\ i)$ = *set-lebesgue-integral* $M\ A\ (X\ j)$
shows *martingale* $M\ F\ t_0\ X$
⟨*proof*⟩

context *submartingale*
begin

lemma *cond-exp-diff-nonneg*:
assumes $t_0 \leq i\ i \leq j$
shows *AE x in M*. *cond-exp* $M\ (F\ i)\ (\lambda\xi.\ X\ j\ \xi - X\ i\ \xi)\ x \geq 0$

⟨*proof*⟩

lemma *add*[*intro*]:
assumes *submartingale* M F t_0 Y
shows *submartingale* M F t_0 $(\lambda i\ \xi.\ X\ i\ \xi + Y\ i\ \xi)$
⟨*proof*⟩

lemma *diff*[*intro*]:
assumes *supermartingale* M F t_0 Y
shows *submartingale* M F t_0 $(\lambda i\ \xi.\ X\ i\ \xi - Y\ i\ \xi)$
⟨*proof*⟩

lemma *scaleR-nonneg*:
assumes $c \geq 0$
shows *submartingale* M F t_0 $(\lambda i\ \xi.\ c *_R X\ i\ \xi)$
⟨*proof*⟩

lemma *scaleR-le-zero*:
assumes $c \leq 0$
shows *supermartingale* M F t_0 $(\lambda i\ \xi.\ c *_R X\ i\ \xi)$
⟨*proof*⟩

lemma *uminus*[*intro*]:
shows *supermartingale* M F t_0 $(- X)$
⟨*proof*⟩

end

context *submartingale-linorder*
begin

lemma *set-integral-le*:
assumes $A \in F\ i$ $t_0 \leq i$ $i \leq j$
shows *set-lebesgue-integral* M A $(X\ i) \leq$ *set-lebesgue-integral* M A $(X\ j)$
⟨*proof*⟩

lemma *max*:
assumes *submartingale* M F t_0 Y
shows *submartingale* M F t_0 $(\lambda i\ \xi.\ max\ (X\ i\ \xi)\ (Y\ i\ \xi))$
⟨*proof*⟩

lemma *max-0*:
shows *submartingale* M F t_0 $(\lambda i\ \xi.\ max\ 0\ (X\ i\ \xi))$
⟨*proof*⟩

end

lemma (**in** *sigma-finite-filtered-measure*) *submartingale-of-cond-exp-diff-nonneg*:
assumes *adapted*: *adapted-process* M F t_0 X

and *integrable*: $\bigwedge i.\ t_0 \le i \Longrightarrow$ *integrable* M $(X\ i)$
and *diff-nonneg*: $\bigwedge i\ j.\ t_0 \le i \Longrightarrow i \le j \Longrightarrow$ *AE* x *in* M. *cond-exp* M $(F\ i)$ $(\lambda\xi.\ X\ j\ \xi - X\ i\ \xi)\ x \ge 0$
shows *submartingale* $M\ F\ t_0\ X$
⟨*proof*⟩

lemma (**in** *sigma-finite-filtered-measure*) *submartingale-of-set-integral-le*:
fixes X :: - ⇒ - ⇒ - :: {*linorder-topology*}
assumes *adapted*: *adapted-process* $M\ F\ t_0\ X$
and *integrable*: $\bigwedge i.\ t_0 \le i \Longrightarrow$ *integrable* M $(X\ i)$
and $\bigwedge A\ i\ j.\ t_0 \le i \Longrightarrow i \le j \Longrightarrow A \in F\ i \Longrightarrow$ *set-lebesgue-integral* $M\ A\ (X\ i) \le$ *set-lebesgue-integral* $M\ A\ (X\ j)$
shows *submartingale* $M\ F\ t_0\ X$
⟨*proof*⟩

context *supermartingale*
begin

lemma *cond-exp-diff-nonneg*:
assumes $t_0 \le i$ $i \le j$
shows *AE* x *in* M. *cond-exp* M $(F\ i)$ $(\lambda\xi.\ X\ i\ \xi - X\ j\ \xi)\ x \ge 0$
⟨*proof*⟩

lemma *add*[*intro*]:
assumes *supermartingale* $M\ F\ t_0\ Y$
shows *supermartingale* $M\ F\ t_0\ (\lambda i\ \xi.\ X\ i\ \xi + Y\ i\ \xi)$
⟨*proof*⟩

lemma *diff*[*intro*]:
assumes *submartingale* $M\ F\ t_0\ Y$
shows *supermartingale* $M\ F\ t_0\ (\lambda i\ \xi.\ X\ i\ \xi - Y\ i\ \xi)$
⟨*proof*⟩

lemma *scaleR-nonneg*:
assumes $c \ge 0$
shows *supermartingale* $M\ F\ t_0\ (\lambda i\ \xi.\ c *_R X\ i\ \xi)$
⟨*proof*⟩

lemma *scaleR-le-zero*:
assumes $c \le 0$
shows *submartingale* $M\ F\ t_0\ (\lambda i\ \xi.\ c *_R X\ i\ \xi)$
⟨*proof*⟩

lemma *uminus*[*intro*]:
shows *submartingale* $M\ F\ t_0\ (-\ X)$
⟨*proof*⟩

end

context *supermartingale-linorder*
begin

lemma *set-integral-ge*:
 assumes $A \in F\ i\ t_0 \leq i\ i \leq j$
 shows *set-lebesgue-integral* $M\ A\ (X\ i) \geq$ *set-lebesgue-integral* $M\ A\ (X\ j)$
 ⟨*proof*⟩

lemma *min*:
 assumes *supermartingale* $M\ F\ t_0\ Y$
 shows *supermartingale* $M\ F\ t_0\ (\lambda i\ \xi.\ min\ (X\ i\ \xi)\ (Y\ i\ \xi))$
⟨*proof*⟩

lemma *min-0*:
 shows *supermartingale* $M\ F\ t_0\ (\lambda i\ \xi.\ min\ 0\ (X\ i\ \xi))$
⟨*proof*⟩

end

lemma (**in** *sigma-finite-filtered-measure*) *supermartingale-of-cond-exp-diff-le-zero*:
 assumes *adapted*: *adapted-process* $M\ F\ t_0\ X$
 and *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable* $M\ (X\ i)$
 and *diff-le-zero*: $\bigwedge i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow AE\ x\ in\ M.$ *cond-exp* $M\ (F\ i)$ $(\lambda\xi.\ X\ j\ \xi - X\ i\ \xi)\ x \leq 0$
 shows *supermartingale* $M\ F\ t_0\ X$
⟨*proof*⟩

lemma (**in** *sigma-finite-filtered-measure*) *supermartingale-of-set-integral-ge*:
 fixes X :: - ⇒ - ⇒ - :: {*linorder-topology*}
 assumes *adapted*: *adapted-process* $M\ F\ t_0\ X$
 and *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable* $M\ (X\ i)$
 and $\bigwedge A\ i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow A \in F\ i \Longrightarrow$ *set-lebesgue-integral* $M\ A\ (X\ j) \leq$ *set-lebesgue-integral* $M\ A\ (X\ i)$
 shows *supermartingale* $M\ F\ t_0\ X$
⟨*proof*⟩

context *nat-sigma-finite-filtered-measure*
begin

lemma *predictable-const*:
 assumes *martingale* $M\ F\ 0\ X$
 and *predictable-process* $M\ F\ 0\ X$
 shows $AE\ \xi\ in\ M.\ X\ i\ \xi = X\ j\ \xi$
⟨*proof*⟩

lemma *martingale-of-set-integral-eq-Suc*:
 assumes *adapted*: *adapted-process* $M\ F\ 0\ X$
 and *integrable*: $\bigwedge i.$ *integrable* $M\ (X\ i)$
 and $\bigwedge A\ i.\ A \in F\ i \Longrightarrow$ *set-lebesgue-integral* $M\ A\ (X\ i) =$ *set-lebesgue-integral*

M A (*X* (*Suc i*))
shows *martingale M F 0 X*
⟨*proof*⟩

lemma *martingale-nat*:
assumes *adapted*: *adapted-process M F 0 X*
and *integrable*: $\bigwedge i$. *integrable M* (*X i*)
and $\bigwedge i$. *AE ξ in M*. *X i ξ* = *cond-exp M* (*F i*) (*X* (*Suc i*)) *ξ*
shows *martingale M F 0 X*
⟨*proof*⟩

lemma *martingale-of-cond-exp-diff-Suc-eq-zero*:
assumes *adapted*: *adapted-process M F 0 X*
and *integrable*: $\bigwedge i$. *integrable M* (*X i*)
and $\bigwedge i$. *AE ξ in M*. *cond-exp M* (*F i*) (λξ. *X* (*Suc i*) *ξ* − *X i ξ*) *ξ* = *0*
shows *martingale M F 0 X*
⟨*proof*⟩

end

context *nat-sigma-finite-filtered-measure*
begin

lemma *predictable-mono*:
assumes *submartingale M F 0 X*
and *predictable-process M F 0 X i* ≤ *j*
shows *AE ξ in M*. *X i ξ* ≤ *X j ξ*
⟨*proof*⟩

lemma *submartingale-of-set-integral-le-Suc*:
fixes *X* :: - ⇒ - ⇒ - :: {*linorder-topology*}
assumes *adapted*: *adapted-process M F 0 X*
and *integrable*: $\bigwedge i$. *integrable M* (*X i*)
and $\bigwedge$*A i*. *A* ∈ *F i* ⟹ *set-lebesgue-integral M A* (*X i*) ≤ *set-lebesgue-integral*
M A (*X* (*Suc i*))
shows *submartingale M F 0 X*
⟨*proof*⟩

lemma *submartingale-nat*:
fixes *X* :: - ⇒ - ⇒ - :: {*linorder-topology*}
assumes *adapted*: *adapted-process M F 0 X*
and *integrable*: $\bigwedge i$. *integrable M* (*X i*)
and $\bigwedge i$. *AE ξ in M*. *X i ξ* ≤ *cond-exp M* (*F i*) (*X* (*Suc i*)) *ξ*
shows *submartingale M F 0 X*
⟨*proof*⟩

lemma *submartingale-of-cond-exp-diff-Suc-nonneg*:
fixes *X* :: - ⇒ - ⇒ - :: {*linorder-topology*}
assumes *adapted*: *adapted-process M F 0 X*

and *integrable*: $\bigwedge i.$ *integrable M* $(X\ i)$
and $\bigwedge i.$ *AE* ξ *in M. cond-exp M* $(F\ i)$ $(\lambda\xi.\ X\ (Suc\ i)\ \xi - X\ i\ \xi)\ \xi \geq 0$
shows *submartingale M F 0 X*
⟨*proof*⟩

lemma *submartingale-partial-sum-scaleR*:
assumes *submartingale-linorder M F 0 X*
and *adapted-process M F 0 C* $\bigwedge i.$ *AE* ξ *in M.* $0 \leq C\ i\ \xi$ $\bigwedge i.$ *AE* ξ *in M.* $C\ i$ $\xi \leq R$
shows *submartingale M F 0* $(\lambda n\ \xi.\ \sum i{<}n.\ C\ i\ \xi *_R (X\ (Suc\ i)\ \xi - X\ i\ \xi))$
⟨*proof*⟩

lemma *submartingale-partial-sum-scaleR'*:
assumes *submartingale-linorder M F 0 X*
and *predictable-process M F 0 C* $\bigwedge i.$ *AE* ξ *in M.* $0 \leq C\ i\ \xi$ $\bigwedge i.$ *AE* ξ *in M.* C $i\ \xi \leq R$
shows *submartingale M F 0* $(\lambda n\ \xi.\ \sum i{<}n.\ C\ (Suc\ i)\ \xi *_R (X\ (Suc\ i)\ \xi - X\ i$ $\xi))$
⟨*proof*⟩

end

context *nat-sigma-finite-filtered-measure*
begin

lemma *predictable-mono'*:
assumes *supermartingale M F 0 X*
and *predictable-process M F 0 X* $i \leq j$
shows *AE* ξ *in M.* $X\ i\ \xi \geq X\ j\ \xi$
⟨*proof*⟩

lemma *supermartingale-of-set-integral-ge-Suc*:
fixes X :: - ⇒ - ⇒ - :: {*linorder-topology*}
assumes *adapted*: *adapted-process M F 0 X*
and *integrable*: $\bigwedge i.$ *integrable M* $(X\ i)$
and $\bigwedge A\ i.\ A \in F\ i \Longrightarrow$ *set-lebesgue-integral M A* $(X\ i) \geq$ *set-lebesgue-integral* *M A* $(X\ (Suc\ i))$
shows *supermartingale M F 0 X*
⟨*proof*⟩

lemma *supermartingale-nat*:
fixes X :: - ⇒ - ⇒ - :: {*linorder-topology*}
assumes *adapted*: *adapted-process M F 0 X*
and *integrable*: $\bigwedge i.$ *integrable M* $(X\ i)$
and $\bigwedge i.$ *AE* ξ *in M.* $X\ i\ \xi \geq$ *cond-exp M* $(F\ i)$ $(X\ (Suc\ i))\ \xi$
shows *supermartingale M F 0 X*
⟨*proof*⟩

lemma *supermartingale-of-cond-exp-diff-Suc-le-zero*:

fixes X :: - ⇒ - ⇒ - :: {*linorder-topology*}
assumes *adapted*: *adapted-process M F 0 X*
and *integrable*: $\bigwedge i$. *integrable M* (*X i*)
and $\bigwedge i$. *AE* ξ *in M*. *cond-exp M* (*F i*) ($\lambda\xi$. *X* (*Suc i*) ξ − *X i* ξ) $\xi \leq 0$
shows *supermartingale M F 0 X*
⟨*proof*⟩

end

end

3 Stopping Times and Hitting Times

In this section we formalize stopping times and hitting times. A stopping time is a random variable that represents the time at which a certain event occurs within a stochastic process. A hitting time, also known as first passage time or first hitting time, is a specific type of stopping time that represents the first time a stochastic process reaches a particular state or crosses a certain threshold.

theory *Stopping-Time*
imports *Martingales-Updates*
begin

3.1 Stopping Time

The formalization of stopping times here is simply a rewrite of the document *HOL−Probability.Stopping-Time* [5]. We have adapted the document to use the locales defined in our formalization of filtered measure spaces [6] [7]. This way we can omit the partial formalization of filtrations in the original document. Furthermore, we can include the initial time index t_0 that we introduced as well.

context *linearly-filtered-measure*
begin

— A stopping time is a measurable function from the measure space (possible events) into the time axis.

definition *stopping-time* :: ($'a \Rightarrow 'b$) ⇒ *bool* **where**
stopping-time T = (($T \in$ *space M* $\rightarrow \{t_0..\}$) $\wedge$ ($\forall t \geq t_0$. *Measurable.pred* (*F t*) (λx. $T\ x \leq t$)))

lemma *stopping-time-cong*:
assumes $\bigwedge t\ x.\ t \geq t_0 \Longrightarrow x \in$ *space* (*F t*) $\Longrightarrow T\ x = S\ x$
shows *stopping-time T* = *stopping-time S*
⟨*proof*⟩

lemma *stopping-time-ge-zero*:
assumes *stopping-time* T $\omega \in$ *space* M
shows $T\ \omega \geq t_0$
⟨*proof*⟩

lemma *stopping-timeD*:
assumes *stopping-time* T $t \geq t_0$
shows *Measurable.pred* $(F\ t)$ $(\lambda x.\ T\ x \leq t)$
⟨*proof*⟩

lemma *stopping-timeI*[*intro?*]:
assumes $\bigwedge x.\ x \in$ *space* $M \Longrightarrow T\ x \geq t_0$
$(\bigwedge t.\ t \geq t_0 \Longrightarrow$ *Measurable.pred* $(F\ t)$ $(\lambda x.\ T\ x \leq t))$
shows *stopping-time* T
⟨*proof*⟩

lemma *stopping-time-measurable*:
assumes *stopping-time* T
shows $T \in$ *borel-measurable* M
⟨*proof*⟩

lemma *stopping-time-const*:
assumes $t \geq t_0$
shows *stopping-time* $(\lambda x.\ t)$ ⟨*proof*⟩

lemma *stopping-time-min*:
assumes *stopping-time* T *stopping-time* S
shows *stopping-time* $(\lambda x.\ min\ (T\ x)\ (S\ x))$
⟨*proof*⟩

lemma *stopping-time-max*:
assumes *stopping-time* T *stopping-time* S
shows *stopping-time* $(\lambda x.\ max\ (T\ x)\ (S\ x))$
⟨*proof*⟩

3.2 σ-algebra of a Stopping Time

Moving on, we define the σ-algebra associated with a stopping time T. It contains all the information up to time T, the same way $F\ t$ contains all the information up to time t.

definition *pre-sigma* :: $('a \Rightarrow 'b) \Rightarrow 'a$ *measure* **where**
pre-sigma T = *sigma* (*space* M) $\{A \in$ *sets* $M.\ \forall t \geq t_0.\ \{\omega \in A.\ T\ \omega \leq t\} \in$ *sets* $(F\ t)\}$

lemma *measure-pre-sigma*[*simp*]: *emeasure* (*pre-sigma* T) = $(\lambda\text{-}.\ 0)$ ⟨*proof*⟩

lemma *sigma-algebra-pre-sigma*:
assumes *stopping-time* T

shows *sigma-algebra* (*space* M) $\{A \in$ *sets* $M.\ \forall\, t{\geq}t_0.\ \{\omega{\in}A.\ T\ \omega \leq t\} \in$ *sets* $(F$ $t)\}$
⟨*proof*⟩

lemma *space-pre-sigma*[*simp*]: *space* (*pre-sigma* T) = *space* M ⟨*proof*⟩

lemma *sets-pre-sigma*:
assumes *stopping-time* T
shows *sets* (*pre-sigma* T) = $\{A \in$ *sets* $M.\ \forall\, t{\geq}t_0.\ \{\omega{\in}A.\ T\ \omega \leq t\} \in F\ t\}$
⟨*proof*⟩

lemma *sets-pre-sigmaI*:
assumes *stopping-time* T
and $\bigwedge t.\ t \geq t_0 \Longrightarrow \{\omega \in A.\ T\ \omega \leq t\} \in F\ t$
shows $A \in$ *pre-sigma* T
⟨*proof*⟩

lemma *pred-pre-sigmaI*:
assumes *stopping-time* T
shows $(\bigwedge t.\ t \geq t_0 \Longrightarrow$ *Measurable.pred* $(F\ t)\ (\lambda\omega.\ P\ \omega \wedge T\ \omega \leq t)) \Longrightarrow$ *Measurable.pred* (*pre-sigma* T) P
⟨*proof*⟩

lemma *sets-pre-sigmaD*:
assumes *stopping-time* T $A \in$ *pre-sigma* T $t \geq t_0$
shows $\{\omega \in A.\ T\ \omega \leq t\} \in$ *sets* $(F\ t)$
⟨*proof*⟩

lemma *borel-measurable-stopping-time-pre-sigma*:
assumes *stopping-time* T
shows $T \in$ *borel-measurable* (*pre-sigma* T)
⟨*proof*⟩

lemma *mono-pre-sigma*:
assumes *stopping-time* T *stopping-time* S
and $\bigwedge x.\ x \in$ *space* $M \Longrightarrow T\ x \leq S\ x$
shows *pre-sigma* $T \subseteq$ *pre-sigma* S
⟨*proof*⟩

lemma *stopping-time-measurable-le*:
assumes *stopping-time* T $s \geq t_0$ $t \geq s$
shows *Measurable.pred* $(F\ t)\ (\lambda\omega.\ T\ \omega \leq s)$
⟨*proof*⟩

lemma *stopping-time-measurable-less*:
assumes *stopping-time* T $s \geq t_0$ $t \geq s$
shows *Measurable.pred* $(F\ t)\ (\lambda\omega.\ T\ \omega < s)$
⟨*proof*⟩

lemma *stopping-time-measurable-ge*:
  assumes *stopping-time* T $s \geq t_0$ $t \geq s$
  shows *Measurable.pred* (F t) ($\lambda\omega$. T $\omega \geq s$)
  ⟨*proof*⟩

lemma *stopping-time-measurable-gr*:
  assumes *stopping-time* T $s \geq t_0$ $t \geq s$
  shows *Measurable.pred* (F t) (λx. $s < T$ x)
  ⟨*proof*⟩

lemma *stopping-time-measurable-eq*:
  assumes *stopping-time* T $s \geq t_0$ $t \geq s$
  shows *Measurable.pred* (F t) ($\lambda\omega$. T $\omega = s$)
  ⟨*proof*⟩

lemma *stopping-time-less-stopping-time*:
  assumes *stopping-time* T *stopping-time* S
  shows *Measurable.pred* (*pre-sigma* T) ($\lambda\omega$. T $\omega < S$ ω)
⟨*proof*⟩

end

lemma (**in** *enat-filtered-measure*) *stopping-time-SUP-enat*:
  fixes T :: *nat* ⇒ (*'a* ⇒ *enat*)
  shows ($\bigwedge i$. *stopping-time* (T i)) ⟹ *stopping-time* (*SUP* i. T i)
  ⟨*proof*⟩

lemma (**in** *enat-filtered-measure*) *stopping-time-Inf-enat*:
  assumes $\bigwedge i$. *Measurable.pred* (F i) (P i)
  shows *stopping-time* ($\lambda\omega$. *Inf* {i. P i ω})
⟨*proof*⟩

lemma (**in** *nat-filtered-measure*) *stopping-time-Inf-nat*:
  assumes $\bigwedge i$. *Measurable.pred* (F i) (P i)
    $\bigwedge i\ \omega$. $\omega \in$ *space* M ⟹ $\exists n$. P n ω
  shows *stopping-time* ($\lambda\omega$. *Inf* {i. P i ω})
⟨*proof*⟩

definition *stopped-value* :: (*'b* ⇒ *'a* ⇒ *'c*) ⇒ (*'a* ⇒ *'b*) ⇒ (*'a* ⇒ *'c*) **where**
  stopped-value X τ ω = X (τ ω) ω

3.3 Hitting Time

Given a stochastic process X and a borel set A, *hitting-time* X A s t is the first time X is in A after time s and before time t. If X does not hit A after time s and before t then the hitting time is simply t. The definition presented here coincides with the definition of hitting times in mathlib [1].

context *linearly-filtered-measure*

begin

definition *hitting-time* :: $('b \Rightarrow 'a \Rightarrow 'c) \Rightarrow 'c\ set \Rightarrow 'b \Rightarrow 'b \Rightarrow ('a \Rightarrow 'b)$ **where**
$hitting\text{-}time\ X\ A\ s\ t = (\lambda\omega.\ if\ \exists\, i{\in}\{s..t\} \cap \{t_0..\}.\ X\ i\ \omega \in A\ then\ Inf\ (\{s..t\} \cap \{t_0..\} \cap \{i.\ X\ i\ \omega \in A\})\ else\ max\ t_0\ t)$

lemma *hitting-time-def'*:
$hitting\text{-}time\ X\ A\ s\ t = (\lambda\omega.\ Inf\ (insert\ (max\ t_0\ t)\ (\{s..t\} \cap \{t_0..\} \cap \{i.\ X\ i\ \omega \in A\})))$
⟨*proof*⟩

lemma *hitting-time-inj-on*:
assumes $inj\text{-}on\ f\ S\ \bigwedge\omega\ t.\ t \geq t_0 \Longrightarrow X\ t\ \omega \in S\ A \subseteq S$
shows $hitting\text{-}time\ X\ A = hitting\text{-}time\ (\lambda t\ \omega.\ f\ (X\ t\ \omega))\ (f\ `\ A)$
⟨*proof*⟩

lemma *hitting-time-translate*:
fixes *c* :: - :: *ab-group-add*
shows $hitting\text{-}time\ X\ A = hitting\text{-}time\ (\lambda n\ \omega.\ X\ n\ \omega + c)\ (((+)\ c)\ `\ A)$
⟨*proof*⟩

lemma *hitting-time-le*:
assumes $t \geq t_0$
shows $hitting\text{-}time\ X\ A\ s\ t\ \omega \leq t$
⟨*proof*⟩

lemma *hitting-time-ge*:
assumes $t \geq t_0\ s \leq t$
shows $s \leq hitting\text{-}time\ X\ A\ s\ t\ \omega$
⟨*proof*⟩

lemma *hitting-time-mono*:
assumes $t \geq t_0\ s \leq s'\ t \leq t'$
shows $hitting\text{-}time\ X\ A\ s\ t\ \omega \leq hitting\text{-}time\ X\ A\ s'\ t'\ \omega$
⟨*proof*⟩

end

context *nat-filtered-measure*
begin

— Hitting times are stopping times for adapted processes.

lemma *stopping-time-hitting-time*:
assumes $adapted\text{-}process\ M\ F\ 0\ X\ A \in borel$
shows $stopping\text{-}time\ (hitting\text{-}time\ X\ A\ s\ t)$
⟨*proof*⟩

lemma *stopping-time-hitting-time'*:

assumes *adapted-process M F 0 X A* ∈ *borel stopping-time s* $\bigwedge\omega$. *s* $\omega \leq t$
shows *stopping-time* ($\lambda\omega$. *hitting-time X A* (*s* ω) *t* ω)
⟨*proof*⟩

lemma *stopped-value-hitting-time-mem*:
assumes $j \in \{s..t\}$ *X j* $\omega \in A$
shows *stopped-value X* (*hitting-time X A s t*) $\omega \in A$
⟨*proof*⟩

lemma *hitting-time-le-iff*:
assumes $i < t$
shows *hitting-time X A s t* $\omega \leq i \longleftrightarrow (\exists j \in \{s..i\}.\ X\ j\ \omega \in A)$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *hitting-time-less-iff*:
assumes $i \leq t$
shows *hitting-time X A s t* $\omega < i \longleftrightarrow (\exists j \in \{s..{<}i\}.\ X\ j\ \omega \in A)$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *hitting-time-eq-hitting-time*:
assumes $t \leq t'$ $j \in \{s..t\}$ *X j* $\omega \in A$
shows *hitting-time X A s t* ω = *hitting-time X A s t'* ω (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

end

4 Doob's Upcrossing Inequality and Martingale Convergence Theorems

In this section we formalize upcrossings and downcrossings. Following this, we prove Doob's upcrossing inequality and first martingale convergence theorem.

theory *Upcrossing*
imports *Stopping-Time*
begin

lemma *real-embedding-borel-measurable*: *real* ∈ *borel-measurable borel* ⟨*proof*⟩

lemma *limsup-lower-bound*:
fixes *u*:: *nat* ⇒ *ereal*
assumes *limsup u* > *l*
shows $\exists N{>}k.\ u\ N > l$

⟨*proof*⟩

lemma *ereal-abs-max-min*: $|c| = max\ 0\ c - min\ 0\ c$ **for** *c* :: *ereal*
⟨*proof*⟩

4.1 Upcrossings and Downcrossings

Given a stochastic process X, real values a and b, and some point in time N, we would like to define a notion of "upcrossings" of X across the band $\{a..b\}$ which counts the number of times any realization of X crosses from below a to above b before time N. To make this heuristic rigorous, we inductively define the following hitting times.

context *nat-filtered-measure*
begin

context
fixes *X* :: *nat* ⇒ *'a* ⇒ *real*
and *a b* :: *real*
and *N* :: *nat*
begin

primrec *upcrossing* :: *nat* ⇒ *'a* ⇒ *nat* **where**
upcrossing 0 = (λω. *0*) |
upcrossing (*Suc n*) = (λω. *hitting-time X* {*b*..} (*hitting-time X* {..*a*} (*upcrossing n* ω) *N* ω) *N* ω)

definition *downcrossing* :: *nat* ⇒ *'a* ⇒ *nat* **where**
downcrossing n = (λω. *hitting-time X* {..*a*} (*upcrossing n* ω) *N* ω)

lemma *upcrossing-simps*:
upcrossing 0 = (λω. *0*)
upcrossing (*Suc n*) = (λω. *hitting-time X* {*b*..} (*downcrossing n* ω) *N* ω)
⟨*proof*⟩

lemma *downcrossing-simps*:
downcrossing 0 = *hitting-time X* {..*a*} *0 N*
downcrossing n = (λω. *hitting-time X* {..*a*} (*upcrossing n* ω) *N* ω)
⟨*proof*⟩

declare *upcrossing.simps*[*simp del*]

lemma *upcrossing-le*: *upcrossing n* ω ≤ *N*
⟨*proof*⟩

lemma *downcrossing-le*: *downcrossing n* ω ≤ *N*
⟨*proof*⟩

lemma *upcrossing-le-downcrossing*: *upcrossing n* ω ≤ *downcrossing n* ω

⟨*proof*⟩

lemma *downcrossing-le-upcrossing-Suc*: *downcrossing* n $\omega \leq$ *upcrossing* (*Suc* n) ω
⟨*proof*⟩

lemma *upcrossing-mono*:
assumes $n \leq m$
shows *upcrossing* n $\omega \leq$ *upcrossing* m ω
⟨*proof*⟩

lemma *downcrossing-mono*:
assumes $n \leq m$
shows *downcrossing* n $\omega \leq$ *downcrossing* m ω
⟨*proof*⟩

lemma *stopped-value-upcrossing*:
assumes *upcrossing* (*Suc* n) $\omega \neq N$
shows *stopped-value* X (*upcrossing* (*Suc* n)) $\omega \geq b$
⟨*proof*⟩

lemma *stopped-value-downcrossing*:
assumes *downcrossing* n $\omega \neq N$
shows *stopped-value* X (*downcrossing* n) $\omega \leq a$
⟨*proof*⟩

lemma *upcrossing-less-downcrossing*:
assumes $a < b$ *downcrossing* (*Suc* n) $\omega \neq N$
shows *upcrossing* (*Suc* n) $\omega <$ *downcrossing* (*Suc* n) ω
⟨*proof*⟩

lemma *downcrossing-less-upcrossing*:
assumes $a < b$ *upcrossing* (*Suc* n) $\omega \neq N$
shows *downcrossing* n $\omega <$ *upcrossing* (*Suc* n) ω
⟨*proof*⟩

lemma *upcrossing-less-Suc*:
assumes $a < b$ *upcrossing* n $\omega \neq N$
shows *upcrossing* n $\omega <$ *upcrossing* (*Suc* n) ω
⟨*proof*⟩

lemma *upcrossing-eq-bound*:
assumes $a < b$ $n \geq N$
shows *upcrossing* n $\omega = N$
⟨*proof*⟩

lemma *downcrossing-eq-bound*:
assumes $a < b$ $n \geq N$

shows *downcrossing n ω = N*
⟨*proof*⟩

lemma *stopping-time-crossings*:
assumes *adapted-process M F 0 X*
shows *stopping-time* (*upcrossing n*) *stopping-time* (*downcrossing n*)
⟨*proof*⟩

lemmas *stopping-time-upcrossing* = *stopping-time-crossings*(*1*)
lemmas *stopping-time-downcrossing* = *stopping-time-crossings*(*2*)

— We define *upcrossings-before* as the number of upcrossings which take place strictly before time *N*.

definition *upcrossings-before* :: *'a ⇒ nat* **where**
upcrossings-before = (*λω. Sup* {*n. upcrossing n ω < N*})

lemma *upcrossings-before-bdd-above*:
assumes *a < b*
shows *bdd-above* {*n. upcrossing n ω < N*}
⟨*proof*⟩

lemma *upcrossings-before-less*:
assumes *a < b 0 < N*
shows *upcrossings-before ω < N*
⟨*proof*⟩

lemma *upcrossings-before-less-implies-crossing-eq-bound*:
assumes *a < b upcrossings-before ω < n*
shows *upcrossing n ω = N*
downcrossing n ω = N
⟨*proof*⟩

lemma *upcrossings-before-le*:
assumes *a < b*
shows *upcrossings-before ω ≤ N*
⟨*proof*⟩

lemma *upcrossings-before-mem*:
assumes *a < b 0 < N*
shows *upcrossings-before ω ∈* {*n. upcrossing n ω < N*} ∩ {..<*N*}
⟨*proof*⟩

lemma *upcrossing-less-of-le-upcrossings-before*:
assumes *a < b 0 < N n ≤ upcrossings-before ω*
shows *upcrossing n ω < N*
⟨*proof*⟩

lemma *upcrossings-before-sum-def*:

assumes $a < b$
shows *upcrossings-before* $\omega = (\sum k{\in}\{1..N\}.$ *indicator* $\{n.$ *upcrossing* $n\ \omega < N\}$ $k)$
⟨*proof*⟩

lemma *upcrossings-before-measurable*:
assumes *adapted-process* $M\ F\ 0\ X\ a < b$
shows *upcrossings-before* $\in$ *borel-measurable* M
⟨*proof*⟩

lemma *upcrossings-before-measurable'*:
assumes *adapted-process* $M\ F\ 0\ X\ a < b$
shows $(\lambda\omega.$ *real* (*upcrossings-before* ω)) $\in$ *borel-measurable* M
⟨*proof*⟩

end

lemma *crossing-eq-crossing*:
assumes $N \leq N'$
and *downcrossing* $X\ a\ b\ N\ n\ \omega < N$
shows *upcrossing* $X\ a\ b\ N\ n\ \omega =$ *upcrossing* $X\ a\ b\ N'\ n\ \omega$
downcrossing $X\ a\ b\ N\ n\ \omega =$ *downcrossing* $X\ a\ b\ N'\ n\ \omega$
⟨*proof*⟩

lemma *crossing-eq-crossing'*:
assumes $N \leq N'$
and *upcrossing* $X\ a\ b\ N$ (*Suc* n) $\omega < N$
shows *upcrossing* $X\ a\ b\ N$ (*Suc* n) $\omega =$ *upcrossing* $X\ a\ b\ N'$ (*Suc* n) ω
downcrossing $X\ a\ b\ N\ n\ \omega =$ *downcrossing* $X\ a\ b\ N'\ n\ \omega$
⟨*proof*⟩

lemma *upcrossing-eq-upcrossing*:
assumes $N \leq N'$
and *upcrossing* $X\ a\ b\ N\ n\ \omega < N$
shows *upcrossing* $X\ a\ b\ N\ n\ \omega =$ *upcrossing* $X\ a\ b\ N'\ n\ \omega$
⟨*proof*⟩

lemma *upcrossings-before-zero*: *upcrossings-before* $X\ a\ b\ 0\ \omega = 0$
⟨*proof*⟩

lemma *upcrossings-before-less-exists-upcrossing*:
assumes $a < b$
and *upcrossing*: $N \leq L\ X\ L\ \omega < a\ L \leq U\ b < X\ U\ \omega$
shows *upcrossings-before* $X\ a\ b\ N\ \omega <$ *upcrossings-before* $X\ a\ b$ (*Suc* U) ω
⟨*proof*⟩

lemma *crossings-translate*:
upcrossing $X\ a\ b\ N =$ *upcrossing* $(\lambda n\ \omega.\ (X\ n\ \omega + c))\ (a + c)\ (b + c)\ N$
downcrossing $X\ a\ b\ N =$ *downcrossing* $(\lambda n\ \omega.\ (X\ n\ \omega + c))\ (a + c)\ (b + c)\ N$

⟨*proof*⟩

lemma *upcrossings-before-translate*:
upcrossings-before X a b N = upcrossings-before $(\lambda n\ \omega.\ (X\ n\ \omega + c))\ (a + c)\ (b + c)\ N$
⟨*proof*⟩

lemma *crossings-pos-eq*:
assumes $a < b$
shows *upcrossing X a b N = upcrossing* $(\lambda n\ \omega.\ max\ 0\ (X\ n\ \omega - a))\ 0\ (b - a)\ N$
downcrossing X a b N = downcrossing $(\lambda n\ \omega.\ max\ 0\ (X\ n\ \omega - a))\ 0\ (b - a)\ N$
⟨*proof*⟩

lemma *upcrossings-before-mono*:
assumes $a < b\ N \leq N'$
shows *upcrossings-before X a b N* $\omega \leq$ *upcrossings-before X a b* $N'\ \omega$
⟨*proof*⟩

lemma *upcrossings-before-pos-eq*:
assumes $a < b$
shows *upcrossings-before X a b N = upcrossings-before* $(\lambda n\ \omega.\ max\ 0\ (X\ n\ \omega - a))\ 0\ (b - a)\ N$
⟨*proof*⟩

definition *upcrossings* :: $(nat \Rightarrow 'a \Rightarrow real) \Rightarrow real \Rightarrow real \Rightarrow 'a \Rightarrow ennreal$ **where**
upcrossings X a b $= (\lambda\omega.\ (SUP\ N.\ ennreal\ (upcrossings\text{-}before\ X\ a\ b\ N\ \omega)))$

lemma *upcrossings-measurable*:
assumes *adapted-process M F 0 X* $a < b$
shows *upcrossings X a b* $\in$ *borel-measurable M*
⟨*proof*⟩

end

lemma (**in** *nat-finite-filtered-measure*) *integrable-upcrossings-before*:
assumes *adapted-process M F 0 X* $a < b$
shows *integrable M* $(\lambda\omega.\ real\ (upcrossings\text{-}before\ X\ a\ b\ N\ \omega))$
⟨*proof*⟩

4.2 Doob's Upcrossing Inequality

Doob's upcrossing inequality provides a bound on the expected number of upcrossings a submartingale completes before some point in time. The proof follows the proof presented in the paper *A Formalization of Doob's Martingale Convergence Theorems in mathlib* [1] [2].

context *nat-finite-filtered-measure*
begin

theorem *upcrossing-inequality*:
fixes *a b* :: *real* **and** *N* :: *nat*
assumes *submartingale M F 0 X*
shows $(b - a) * (\int \omega.\ real\ (upcrossings\text{-}before\ X\ a\ b\ N\ \omega)\ \partial M) \leq (\int \omega.\ max\ 0\ (X\ N\ \omega - a)\ \partial M)$
⟨*proof*⟩

theorem *upcrossing-inequality-Sup*:
fixes *a b* :: *real*
assumes *submartingale M F 0 X*
shows $(b - a) * (\int^{+} \omega.\ upcrossings\ X\ a\ b\ \omega\ \partial M) \leq (SUP\ N.\ (\int^{+} \omega.\ max\ 0\ (X\ N\ \omega - a)\ \partial M))$
⟨*proof*⟩

end

end

5 Doob's First Martingale Convergence Theorem

theory *Doob-Convergence*
imports *Upcrossing*
begin

context *nat-finite-filtered-measure*
begin

Doob's martingale convergence theorem states that, if we have a submartingale where the supremum over the mean of the positive parts is finite, then the limit process exists almost surely and is integrable. Furthermore, the limit process is measurable with respect to the smallest σ-algebra containing all of the σ-algebras in the filtration. The argumentation below is taken mostly from [3].

theorem *submartingale-convergence-AE*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
assumes *submartingale M F 0 X*
and $\bigwedge n.\ (\int \omega.\ max\ 0\ (X\ n\ \omega)\ \partial M) \leq C$
obtains X_{lim} **where** $AE\ \omega\ in\ M.\ (\lambda n.\ X\ n\ \omega) \longrightarrow X_{lim}\ \omega$
$integrable\ M\ X_{lim}$
$X_{lim} \in borel\text{-}measurable\ (F_{\infty})$
⟨*proof*⟩

corollary *supermartingale-convergence-AE*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
assumes *supermartingale M F 0 X*
and $\bigwedge n.\ (\int \omega.\ max\ 0\ (- X\ n\ \omega)\ \partial M) \leq C$
obtains X_{lim} **where** $AE\ \omega\ in\ M.\ (\lambda n.\ X\ n\ \omega) \longrightarrow X_{lim}\ \omega$

$integrable\ M\ X_{lim}$
$X_{lim} \in borel\text{-}measurable\ (F_\infty)$
⟨*proof*⟩

corollary *martingale-convergence-AE*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
assumes $martingale\ M\ F\ 0\ X$
and $\bigwedge n.\ (\int \omega.\ |X\ n\ \omega|\ \partial M) \leq C$
obtains X_{lim} **where** $AE\ \omega\ in\ M.\ (\lambda n.\ X\ n\ \omega) \longrightarrow X_{lim}\ \omega$
$integrable\ M\ X_{lim}$
$X_{lim} \in borel\text{-}measurable\ (F_\infty)$
⟨*proof*⟩

corollary *martingale-nonneg-convergence-AE*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
assumes $martingale\ M\ F\ 0\ X\ \bigwedge n.\ AE\ \omega\ in\ M.\ X\ n\ \omega \geq 0$
obtains X_{lim} **where** $AE\ \omega\ in\ M.\ (\lambda n.\ X\ n\ \omega) \longrightarrow X_{lim}\ \omega$
$integrable\ M\ X_{lim}$
$X_{lim} \in borel\text{-}measurable\ (F_\infty)$
⟨*proof*⟩

end

end

References

[1] R. Degenne and K. Ying. A Formalization of Doob's Martingale Convergence Theorems in mathlib. In *Proceedings of the 12th ACM SIGPLAN International Conference on Certified Programs and Proofs*. Association for Computing Machinery, New York, United States, 2022. arXiv:2212.05578.

[2] R. Degenne and K. Ying. probability.martingale.basic - mathlib, 2022. https://leanprover-community.github.io/mathlib_docs/probability/martingale/basic.html, Last Accessed: 15 Feb 2024.

[3] R. Durrett. *Probability: Theory and Examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2019. URL: https://books.google.de/books?id=b22MDwAAQBAJ.

[4] G. Grimmett and D. Stirzaker. *Probability and Random Processes*. Oxford University Press, 2020.

[5] J. Hölzl and A. Heller. Three Chapters of Measure Theory in Isabelle/HOL. In M. C. J. D. van Eekelen, H. Geuvers, J. Schmaltz, and F. Wiedijk, editors, *Interactive Theorem Proving (ITP 2011)*, volume 6898 of *LNCS*, pages 135–151, 2011. doi:10.1007/978-3-642-22863-6_12.

[6] A. Keskin. A Formalization of Martingales in Isabelle/HOL. Bachelor's thesis, Technical University of Munich, 2023. arXiv:2311.06188.

[7] A. Keskin. Martingales. *Archive of Formal Proofs*, November 2023. https://isa-afp.org/entries/Martingales.html, Formal proof development.