# Disintegration Theorem

Michikazu Hirata

November 7, 2023

**Abstract**

We formalize mixture and disintegraion of measures. This entry is a formalization of Chapter 14.D of the book by Baccelli et.al. [1]. The main result is the disintegration theorem: let $(X, \Sigma_X)$ be a measurable space, $(Y, \Sigma_Y)$ be a standard Borel space, $\nu$ be a $\sigma$-finite measure on $X \times Y$, and $\nu_X$ be the marginal measure on $X$ defined by $\nu_X(A) = \nu(A \times Y)$. Assume that $\nu_X$ is $\sigma$-finite, then there exists a probability kernel $\kappa$ from $X$ to $Y$ such that

$$\nu(A \times B) = \int_A \kappa_x(B)\nu_X(\mathrm{d}x),\ A \in \Sigma_X, B \in \Sigma_Y.$$

Such a probability kernel is unique $\nu_X$-almost everywhere.

## Contents



# 1 Lemmas

**theory** *Lemmas-Disintegration*
**imports** *Standard-Borel-Spaces.StandardBorel*
**begin**

## 1.1 Lemmas

**lemma** *semiring-of-sets-binary-product-sets*[*simp*]:
*semiring-of-sets* (*space* $X \times$ *space* $Y$) $\{a \times b \mid a\ b.\ a \in$ *sets* $X \wedge b \in$ *sets* $Y\}$
⟨*proof*⟩

**lemma** *sets-pair-restrict-space*:
*sets* (*restrict-space* $X$ $A \bigotimes_M$ *restrict-space* $Y$ $B$) = *sets* (*restrict-space* ($X \bigotimes_M Y$) ($A \times B$))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

**lemma** *restrict-space-space*[*simp*]: *restrict-space* $M$ (*space* $M$) = $M$
⟨*proof*⟩

**lemma** *atMostq-Int-stable*:
*Int-stable* $\{\{..r\} \mid r::real.\ r \in \mathbb{Q}\}$
⟨*proof*⟩

**lemma** *rborel-eq-atMostq*:
*borel* = *sigma UNIV* $\{ \{..r\} \mid r::real.\ r \in \mathbb{Q}\}$
⟨*proof*⟩

**corollary** *rborel-eq-atMostq-sets*:
*sets borel* = *sigma-sets UNIV* $\{\{..r\} \mid r::real.\ r \in \mathbb{Q}\}$
⟨*proof*⟩

**lemma** *mono-absolutely-continuous*:
**assumes** *sets* $\mu$ = *sets* $\nu$ $\bigwedge A.\ A \in$ *sets* $\mu \Longrightarrow \mu\ A \le \nu\ A$
**shows** *absolutely-continuous* $\nu\ \mu$
⟨*proof*⟩

**lemma** *ex-measure-countable-space*:
**assumes** *countable* (*space* $X$)
**and** *sets* $X$ = *Pow* (*space* $X$)
**shows** $\exists\mu.$ *sets* $\mu$ = *sets* $X \wedge (\forall x{\in}$*space* $X.\ \mu\ \{x\} = f\ x)$
⟨*proof*⟩

**lemma** *ex-prob-space-countable*:
**assumes** *space* $X \neq \{\}$ *countable* (*space* $X$)
**and** *sets* $X$ = *Pow* (*space* $X$)
**shows** $\exists\mu.$ *sets* $\mu$ = *sets* $X \wedge$ *prob-space* $\mu$
⟨*proof*⟩

**lemma** *AE-I''*:
**assumes** $N \in$ *null-sets* $M$
**and** $\bigwedge x.\ x \in$ *space* $M \Longrightarrow x \notin N \Longrightarrow P\ x$
**shows** *AE* $x$ *in* $M.\ P\ x$
⟨*proof*⟩

**lemma** *absolutely-continuous-trans*:
**assumes** *absolutely-continuous L M absolutely-continuous M N*
**shows** *absolutely-continuous L N*
⟨*proof*⟩

### 1.2 Equivalence of Measures

**abbreviation** *equivalence-measure* :: $'a$ *measure* ⇒ $'a$ *measure* ⇒ *bool* (**infix** $\sim_M$ *60*)
**where** *equivalence-measure M N* ≡ *absolutely-continuous M N* ∧ *absolutely-continuous N M*

**lemma** *equivalence-measure-refl*: $M \sim_M M$
⟨*proof*⟩

**lemma** *equivalence-measure-sym*:
**assumes** $M \sim_M N$
**shows** $N \sim_M M$
⟨*proof*⟩

**lemma** *equivalence-measure-trans*:
**assumes** $M \sim_M N$ $N \sim_M L$
**shows** $M \sim_M L$
⟨*proof*⟩

**lemma** *equivalence-measureI*:
**assumes** *absolutely-continuous M N absolutely-continuous N M*
**shows** $M \sim_M N$
⟨*proof*⟩

**end**

# 2 Disintegration Theorem

**theory** *Disintegration*
**imports** *S-Finite-Measure-Monad.Kernels*
*Lemmas-Disintegration*
**begin**

### 2.1 Definition 14.D.2. (Mixture and Disintegration)

**context** *measure-kernel*
**begin**

**definition** *mixture-of* :: $[('a \times 'b)$ *measure*, $'a$ *measure*$]$ ⇒ *bool* **where**
*mixture-of* $\nu\ \mu \longleftrightarrow$ *sets* $\nu$ = *sets* $(X \bigotimes_M Y) \land$ *sets* $\mu$ = *sets* $X \land (\forall C \in$ *sets* $(X \bigotimes_M Y).\ \nu\ C = (\int^+ x.\ \int^+ y.$ *indicator* $C\ (x,y)\ \partial(\kappa\ x)\ \partial\mu))$

**definition** *disintegration* :: $[('a \times 'b)$ *measure*, $'a$ *measure*$]$ ⇒ *bool* **where**

*disintegration* $\nu$ $\mu$ $\longleftrightarrow$ *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ $\wedge$ *sets* $\mu$ = *sets* $X$ $\wedge$ ($\forall$ $A\in$*sets* $X$. $\forall$ $B\in$*sets* $Y$. $\nu$ $(A \times B) = (\int^+ x\in A.$ $(\kappa$ $x$ $B)$ $\partial\mu))$

**lemma** *disintegrationI*:
  **assumes** *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ *sets* $\mu$ = *sets* $X$
      **and** $\bigwedge A$ $B$. $A \in$ *sets* $X \Longrightarrow B \in$ *sets* $Y \Longrightarrow \nu$ $(A \times B) = (\int^+ x\in A.$ $(\kappa$ $x$ $B)$ $\partial\mu)$
    **shows** *disintegration* $\nu$ $\mu$
  $\langle$*proof*$\rangle$

**lemma** *mixture-of-disintegration*:
  **assumes** *mixture-of* $\nu$ $\mu$
  **shows** *disintegration* $\nu$ $\mu$
  $\langle$*proof*$\rangle$

**lemma**
 **shows** *mixture-of-sets-eq*: *mixture-of* $\nu$ $\mu \Longrightarrow$ *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ *mixture-of* $\nu$ $\mu \Longrightarrow$ *sets* $\mu$ = *sets* $X$
     **and** *mixture-of-space-eq*: *mixture-of* $\nu$ $\mu \Longrightarrow$ *space* $\nu$ = *space* $(X \bigotimes_M Y)$ *mixture-of* $\nu$ $\mu \Longrightarrow$ *space* $\mu$ = *space* $X$
    **and** *disintegration-sets-eq*: *disintegration* $\nu$ $\mu \Longrightarrow$ *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ *disintegration* $\nu$ $\mu \Longrightarrow$ *sets* $\mu$ = *sets* $X$
    **and** *disintegration-space-eq*: *disintegration* $\nu$ $\mu \Longrightarrow$ *space* $\nu$ = *space* $(X \bigotimes_M Y)$ *disintegration* $\nu$ $\mu \Longrightarrow$ *space* $\mu$ = *space* $X$
  $\langle$*proof*$\rangle$

**lemma**
  **shows** *mixture-ofD*: *mixture-of* $\nu$ $\mu \Longrightarrow C \in$ *sets* $(X \bigotimes_M Y) \Longrightarrow \nu$ $C = (\int^+ x.$ $\int^+ y.$ *indicator* $C$ $(x,y)$ $\partial(\kappa$ $x)$ $\partial\mu)$
    **and** *disintegrationD*: *disintegration* $\nu$ $\mu \Longrightarrow A \in$ *sets* $X \Longrightarrow B \in$ *sets* $Y \Longrightarrow \nu$ $(A \times B) = (\int^+ x\in A.$ $(\kappa$ $x$ $B)$ $\partial\mu)$
  $\langle$*proof*$\rangle$

**lemma** *disintegration-restrict-space*:
  **assumes** *disintegration* $\nu$ $\mu$ $A \cap$ *space* $X \in$ *sets* $X$
  **shows** *measure-kernel.disintegration* (*restrict-space* $X$ $A$) $Y$ $\kappa$ (*restrict-space* $\nu$ $(A \times$ *space* $Y)$) (*restrict-space* $\mu$ $A$)
$\langle$*proof*$\rangle$

**end**

**context** *subprob-kernel*
**begin**
**lemma** *countable-disintegration-AE-unique*:
  **assumes** *countable* (*space* $Y$) **and** [*measurable-cong*]:*sets* $Y$ = *Pow* (*space* $Y$)
      **and** *subprob-kernel* $X$ $Y$ $\kappa'$ *sigma-finite-measure* $\mu$
      **and** *disintegration* $\nu$ $\mu$ *measure-kernel.disintegration* $X$ $Y$ $\kappa'$ $\nu$ $\mu$
    **shows** *AE* $x$ *in* $\mu$. $\kappa$ $x = \kappa'$ $x$
$\langle$*proof*$\rangle$

**end**

**lemma**(**in** *subprob-kernel*) *nu-mu-spaceY-le*:
  **assumes** *disintegration* $\nu$ $\mu$ $A \in$ *sets* $X$
  **shows** $\nu$ $(A \times$ *space* $Y) \leq \mu$ $A$
⟨*proof*⟩

**context** *prob-kernel*
**begin**

**lemma** *countable-disintegration-AE-unique-prob*:
  **assumes** *countable* (*space* $Y$) **and** [*measurable-cong*]:*sets* $Y$ = *Pow* (*space* $Y$)
    **and** *prob-kernel* $X$ $Y$ $\kappa'$ *sigma-finite-measure* $\mu$
    **and** *disintegration* $\nu$ $\mu$ *measure-kernel.disintegration* $X$ $Y$ $\kappa'$ $\nu$ $\mu$
   **shows** *AE* $x$ *in* $\mu$. $\kappa$ $x = \kappa'$ $x$
 ⟨*proof*⟩

**end**

## 2.2 Lemma 14.D.3.

**lemma**(**in** *prob-kernel*) *nu-mu-spaceY*:
  **assumes** *disintegration* $\nu$ $\mu$ $A \in$ *sets* $X$
  **shows** $\nu$ $(A \times$ *space* $Y) = \mu$ $A$
⟨*proof*⟩

**corollary**(**in** *subprob-kernel*) *nu-finite*:
  **assumes** *disintegration* $\nu$ $\mu$ *finite-measure* $\mu$
  **shows** *finite-measure* $\nu$
⟨*proof*⟩

**corollary**(**in** *subprob-kernel*) *nu-subprob-space*:
  **assumes** *disintegration* $\nu$ $\mu$ *subprob-space* $\mu$
  **shows** *subprob-space* $\nu$
⟨*proof*⟩

**corollary**(**in** *prob-kernel*) *nu-prob-space*:
  **assumes** *disintegration* $\nu$ $\mu$ *prob-space* $\mu$
  **shows** *prob-space* $\nu$
⟨*proof*⟩

**lemma**(**in** *subprob-kernel*) *nu-sigma-finite*:
  **assumes** *disintegration* $\nu$ $\mu$ *sigma-finite-measure* $\mu$
  **shows** *sigma-finite-measure* $\nu$
⟨*proof*⟩

## 2.3 Theorem 14.D.4. (Measure Mixture Theorem)

**lemma**(**in** *measure-kernel*) *exist-nu*:

**assumes** *sets* $\mu$ = *sets* $X$
**shows** $\exists\nu$. *disintegration* $\nu$ $\mu$
$\langle proof \rangle$

**lemma**(**in** *subprob-kernel*) *exist-unique-nu-sigma-finite′*:
**assumes** *sets* $\mu$ = *sets* $X$ *sigma-finite-measure* $\mu$
**shows** $\exists!\nu$. *disintegration* $\nu$ $\mu$
$\langle proof \rangle$

**lemma**(**in** *subprob-kernel*) *exist-unique-nu-sigma-finite*:
**assumes** *sets* $\mu$ = *sets* $X$ *sigma-finite-measure* $\mu$
**shows** $\exists!\nu$. *disintegration* $\nu$ $\mu$ $\wedge$ *sigma-finite-measure* $\nu$
$\langle proof \rangle$

**lemma**(**in** *subprob-kernel*) *exist-unique-nu-finite*:
**assumes** *sets* $\mu$ = *sets* $X$ *finite-measure* $\mu$
**shows** $\exists!\nu$. *disintegration* $\nu$ $\mu$ $\wedge$ *finite-measure* $\nu$
$\langle proof \rangle$

**lemma**(**in** *subprob-kernel*) *exist-unique-nu-sub-prob-space*:
**assumes** *sets* $\mu$ = *sets* $X$ *subprob-space* $\mu$
**shows** $\exists!\nu$. *disintegration* $\nu$ $\mu$ $\wedge$ *subprob-space* $\nu$
$\langle proof \rangle$

**lemma**(**in** *prob-kernel*) *exist-unique-nu-prob-space*:
**assumes** *sets* $\mu$ = *sets* $X$ *prob-space* $\mu$
**shows** $\exists!\nu$. *disintegration* $\nu$ $\mu$ $\wedge$ *prob-space* $\nu$
$\langle proof \rangle$

**lemma**(**in** *subprob-kernel*) *nn-integral-fst-finite′*:
**assumes** $f \in$ *borel-measurable* $(X \bigotimes_M Y)$ *disintegration* $\nu$ $\mu$ *finite-measure* $\mu$
**shows** $(\int^+ z.\ f\ z\ \partial\nu) = (\int^+ x.\ \int^+ y.\ f\ (x,y)\ \partial(\kappa\ x)\ \partial\mu)$
$\langle proof \rangle$

**lemma**(**in** *prob-kernel*) *nn-integral-fst*:
**assumes** $f \in$ *borel-measurable* $(X \bigotimes_M Y)$ *disintegration* $\nu$ $\mu$ *sigma-finite-measure* $\mu$
**shows** $(\int^+ z.\ f\ z\ \partial\nu) = (\int^+ x.\ \int^+ y.\ f\ (x,y)\ \partial(\kappa\ x)\ \partial\mu)$
$\langle proof \rangle$

**lemma**(**in** *prob-kernel*) *integrable-eq1*:
**fixes** $f$ :: - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
**assumes** [*measurable*]:$f \in$ *borel-measurable* $(X \bigotimes_M Y)$
**and** *disintegration* $\nu$ $\mu$ *sigma-finite-measure* $\mu$
**shows** $(\int^+ z.$ *ennreal* (*norm* $(f\ z))\ \partial\nu) < \infty \longleftrightarrow (\int^+ x.\ \int^+ y.$ *ennreal* (*norm* $(f\ (x,y)))\ \partial(\kappa\ x)\ \partial\mu) < \infty$
$\langle proof \rangle$

**lemma**(**in** *prob-kernel*) *integrable-kernel-integrable*:
**fixes** $f$ :: - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
**assumes** *integrable* $\nu$ $f$ *disintegration* $\nu$ $\mu$ *sigma-finite-measure* $\mu$
**shows** *AE* $x$ *in* $\mu$. *integrable* $(\kappa\ x)$ $(\lambda y.\ f\ (x,y))$
⟨*proof*⟩

**lemma**(**in** *prob-kernel*) *integrable-lebesgue-integral-integrable'*:
**fixes** $f$ :: - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
**assumes** *integrable* $\nu$ $f$ *disintegration* $\nu$ $\mu$ *sigma-finite-measure* $\mu$
**shows** *integrable* $\mu$ $(\lambda x.\ \int y.\ f\ (x,y)\ \partial(\kappa\ x))$
⟨*proof*⟩

**lemma**(**in** *prob-kernel*) *integrable-lebesgue-integral-integrable*:
**fixes** $f$ :: - $\Rightarrow$- $\Rightarrow$ -::{*banach*, *second-countable-topology*}
**assumes** *integrable* $\nu$ $(\lambda(x,y).\ f\ x\ y)$ *disintegration* $\nu$ $\mu$ *sigma-finite-measure* $\mu$
**shows** *integrable* $\mu$ $(\lambda x.\ \int y.\ f\ x\ y\ \partial(\kappa\ x))$
⟨*proof*⟩

**lemma**(**in** *prob-kernel*) *integral-fst*:
**fixes** $f$ :: - $\Rightarrow$ -::{*banach*, *second-countable-topology*}
**assumes** *integrable* $\nu$ $f$ *disintegration* $\nu$ $\mu$ *sigma-finite-measure* $\mu$
**shows** $(\int z.\ f\ z\ \partial\nu) = (\int x.\ \int y.\ f\ (x,y)\ \partial(\kappa\ x)\ \partial\mu)$
⟨*proof*⟩

## 2.4 Marginal Measure

**definition** *marginal-measure-on* :: $['a\ measure,\ 'b\ measure,\ ('a \times 'b)\ measure,\ 'b\ set] \Rightarrow 'a\ measure$ **where**
*marginal-measure-on* $X\ Y\ \nu\ B$ = *measure-of* (*space* $X$) (*sets* $X$) $(\lambda A.\ \nu\ (A \times B))$

**abbreviation** *marginal-measure* :: $['a\ measure,\ 'b\ measure,\ ('a \times 'b)\ measure] \Rightarrow 'a\ measure$ **where**
*marginal-measure* $X\ Y\ \nu \equiv$ *marginal-measure-on* $X\ Y\ \nu$ (*space* $Y$)

**lemma** *space-marginal-measure*: *space* (*marginal-measure-on* $X\ Y\ \nu\ B$) = *space* $X$
**and** *sets-marginal-measure*: *sets* (*marginal-measure-on* $X\ Y\ \nu\ B$) = *sets* $X$
⟨*proof*⟩

**lemma** *emeasure-marginal-measure-on*:
**assumes** *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ $B \in$ *sets* $Y$ $A \in$ *sets* $X$
**shows** *marginal-measure-on* $X\ Y\ \nu\ B\ A = \nu\ (A \times B)$
⟨*proof*⟩

**lemma** *emeasure-marginal-measure*:
**assumes** *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ $A \in$ *sets* $X$
**shows** *marginal-measure* $X\ Y\ \nu\ A = \nu\ (A \times$ *space* $Y)$
⟨*proof*⟩

**lemma** *finite-measure-marginal-measure-on-finite*:

**assumes** *finite-measure* $\nu$ *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ $B \in$ *sets* $Y$
**shows** *finite-measure* (*marginal-measure-on* $X$ $Y$ $\nu$ $B$)
⟨*proof*⟩

**lemma** *finite-measure-marginal-measure-finite*:
**assumes** *finite-measure* $\nu$ *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$
**shows** *finite-measure* (*marginal-measure* $X$ $Y$ $\nu$)
⟨*proof*⟩

**lemma** *marginal-measure-restrict-space*:
**assumes** *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ $B \in$ *sets* $Y$
**shows** *marginal-measure* $X$ (*restrict-space* $Y$ $B$) (*restrict-space* $\nu$ (*space* $X \times B$)) = *marginal-measure-on* $X$ $Y$ $\nu$ $B$
⟨*proof*⟩

**lemma** *restrict-space-marginal-measure-on*:
**assumes** *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ $B \in$ *sets* $Y$ $A \in$ *sets* $X$
**shows** *restrict-space* (*marginal-measure-on* $X$ $Y$ $\nu$ $B$) $A$ = *marginal-measure-on* (*restrict-space* $X$ $A$) $Y$ (*restrict-space* $\nu$ ($A \times$ *space* $Y$)) $B$
⟨*proof*⟩

**lemma** *restrict-space-marginal-measure*:
**assumes** *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ $A \in$ *sets* $X$
**shows** *restrict-space* (*marginal-measure* $X$ $Y$ $\nu$) $A$ = *marginal-measure* (*restrict-space* $X$ $A$) $Y$ (*restrict-space* $\nu$ ($A \times$ *space* $Y$))
⟨*proof*⟩

**lemma** *marginal-measure-mono*:
**assumes** *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ $A \in$ *sets* $Y$ $B \in$ *sets* $Y$ $A \subseteq B$
**shows** *emeasure* (*marginal-measure-on* $X$ $Y$ $\nu$ $A$) $\leq$ *emeasure* (*marginal-measure-on* $X$ $Y$ $\nu$ $B$)
⟨*proof*⟩

**lemma** *marginal-measure-absolutely-countinuous*:
**assumes** *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ $A \in$ *sets* $Y$ $B \in$ *sets* $Y$ $A \subseteq B$
**shows** *absolutely-continuous* (*marginal-measure-on* $X$ $Y$ $\nu$ $B$) (*marginal-measure-on* $X$ $Y$ $\nu$ $A$)
⟨*proof*⟩

**lemma** *marginal-measure-absolutely-continuous'*:
**assumes** *sets* $\nu$ = *sets* $(X \bigotimes_M Y)$ $A \in$ *sets* $Y$
**shows** *absolutely-continuous* (*marginal-measure* $X$ $Y$ $\nu$) (*marginal-measure-on* $X$ $Y$ $\nu$ $A$)
⟨*proof*⟩

## 2.5 Lemma 14.D.6.

**locale** *sigma-finite-measure-on-pair* =
**fixes** $X$ :: *'a measure* **and** $Y$ :: *'b measure* **and** $\nu$ :: (*'a* $\times$ *'b*) *measure*

**assumes** *nu-sets*[*measurable-cong*]: *sets* $\nu$ = *sets* ($X \bigotimes_M Y$)
**and** *sigma-finite*: *sigma-finite-measure* $\nu$
**begin**

**abbreviation** $\nu x \equiv$ *marginal-measure* $X$ $Y$ $\nu$

**end**

**locale** *projection-sigma-finite* =
**fixes** $X$ :: *'a measure* **and** $Y$ :: *'b measure* **and** $\nu$ :: (*'a* $\times$ *'b*) *measure*
**assumes** *nu-sets*[*measurable-cong*]: *sets* $\nu$ = *sets* ($X \bigotimes_M Y$)
**and** *marginal-sigma-finite*: *sigma-finite-measure* (*marginal-measure* $X$ $Y$ $\nu$)
**begin**

**sublocale** $\nu x$ : *sigma-finite-measure marginal-measure* $X$ $Y$ $\nu$
⟨*proof*⟩

**lemma** $\nu$*-sigma-finite*: *sigma-finite-measure* $\nu$
⟨*proof*⟩

**sublocale** *sigma-finite-measure-on-pair*
⟨*proof*⟩

**definition** $\kappa'$ :: *'a* $\Rightarrow$ *'b set* $\Rightarrow$ *ennreal* **where**
$\kappa'$ $x$ $B$ $\equiv$ *RN-deriv* $\nu x$ (*marginal-measure-on* $X$ $Y$ $\nu$ $B$) $x$

**lemma** *kernel-measurable*[*measurable*]:
($\lambda x$. *RN-deriv* (*marginal-measure* $X$ $Y$ $\nu$) (*marginal-measure-on* $X$ $Y$ $\nu$ $B$) $x$) $\in$ *borel-measurable* $\nu x$
⟨*proof*⟩

**corollary** $\kappa'$*-measurable*[*measurable*]:
($\lambda x$. $\kappa'$ $x$ $B$) $\in$ *borel-measurable* $X$
⟨*proof*⟩

**lemma** *kernel-RN-deriv*:
**assumes** $A \in$ *sets* $X$ $B \in$ *sets* $Y$
**shows** $\nu$ ($A \times B$) = ($\int^+ x{\in}A.\ \kappa'\ x\ B\ \partial\nu x$)
⟨*proof*⟩

**lemma** *empty-Y-bot*:
**assumes** *space* $Y$ = {}
**shows** $\nu = \bot$
⟨*proof*⟩

**lemma** *empty-Y-nux*:
**assumes** *space* $Y$ = {}
**shows** $\nu x$ $A$ = *0*

⟨*proof*⟩

**lemma** *kernel-empty0-AE*:
  $AE\ x\ in\ \nu x.\ \kappa'\ x\ \{\} = 0$
  ⟨*proof*⟩

**lemma** *kernel-Y1-AE*:
  $AE\ x\ in\ \nu x.\ \kappa'\ x\ (space\ Y) = 1$
  ⟨*proof*⟩

**lemma** *kernel-suminf-AE*:
  **assumes** *disjoint-family F*
    **and** $\bigwedge i.\ F\ i \in sets\ Y$
  **shows** $AE\ x\ in\ \nu x.\ (\sum i.\ \ \kappa'\ x\ (F\ i)) = \kappa'\ x\ (\bigcup\ (range\ F))$
  ⟨*proof*⟩

**lemma** *kernel-finite-sum-AE*:
  **assumes** *disjoint-family-on F S finite S*
    **and** $\bigwedge i.\ i \in S \Longrightarrow F\ i \in sets\ Y$
  **shows** $AE\ x\ in\ \nu x.\ (\sum i{\in}S.\ \kappa'\ x\ (F\ i)) = \kappa'\ x\ (\bigcup i{\in}S.\ F\ i)$
⟨*proof*⟩

**lemma** *kernel-disjoint-sum-AE*:
  **assumes** $B \in sets\ Y\ C \in sets\ Y$
    **and** $B \cap C = \{\}$
  **shows** $AE\ x\ in\ \nu x.\ \kappa'\ x\ (B \cup C) = \kappa'\ x\ B + \kappa'\ x\ C$
⟨*proof*⟩

**lemma** *kernel-mono-AE*:
  **assumes** $B \in sets\ Y\ C \in sets\ Y$
    **and** $B \subseteq C$
  **shows** $AE\ x\ in\ \nu x.\ \kappa'\ x\ B \le \kappa'\ x\ C$
⟨*proof*⟩

**lemma** *kernel-incseq-AE*:
  **assumes** $range\ B \subseteq sets\ Y\ incseq\ B$
  **shows** $AE\ x\ in\ \nu x.\ incseq\ (\lambda n.\ \kappa'\ x\ (B\ n))$
  ⟨*proof*⟩

**lemma** *kernel-decseq-AE*:
  **assumes** $range\ B \subseteq sets\ Y\ decseq\ B$
  **shows** $AE\ x\ in\ \nu x.\ decseq\ (\lambda n.\ \kappa'\ x\ (B\ n))$
  ⟨*proof*⟩

**corollary** *kernel-01-AE*:
  **assumes** $B \in sets\ Y$
  **shows** $AE\ x\ in\ \nu x.\ 0 \le \kappa'\ x\ B \land \kappa'\ x\ B \le 1$
⟨*proof*⟩

**lemma** *kernel-get-0*: $0 \leq \kappa'\ x\ B$
$\langle proof \rangle$

**lemma** *kernel-le1-AE*:
**assumes** $B \in sets\ Y$
**shows** $AE\ x\ in\ \nu x.\ \kappa'\ x\ B \leq 1$
$\langle proof \rangle$

**corollary** *kernel-n-infty*:
**assumes** $B \in sets\ Y$
**shows** $AE\ x\ in\ \nu x.\ \kappa'\ x\ B \neq \top$
$\langle proof \rangle$

**corollary** *kernel-le-infty*:
**assumes** $B \in sets\ Y$
**shows** $AE\ x\ in\ \nu x.\ \kappa'\ x\ B < \top$
$\langle proof \rangle$

**lemma** *kernel-SUP-incseq*:
**assumes** $range\ B \subseteq sets\ Y\ incseq\ B$
**shows** $AE\ x\ in\ \nu x.\ \kappa'\ x\ (\bigcup\ (range\ B)) = (\bigsqcup n.\ \kappa'\ x\ (B\ n))$
$\langle proof \rangle$

**lemma** *kernel-lim-incseq*:
**assumes** $range\ B \subseteq sets\ Y\ incseq\ B$
**shows** $AE\ x\ in\ \nu x.\ (\lambda n.\ \kappa'\ x\ (B\ n)) \longlongrightarrow \kappa'\ x\ (\bigcup\ (range\ B))$
$\langle proof \rangle$

**lemma** *kernel-INF-decseq*:
**assumes** $range\ B \subseteq sets\ Y\ decseq\ B$
**shows** $AE\ x\ in\ \nu x.\ \kappa'\ x\ (\bigcap\ (range\ B)) = (\bigsqcap n.\ \kappa'\ x\ (B\ n))$
$\langle proof \rangle$

**lemma** *kernel-lim-decseq*:
**assumes** $range\ B \subseteq sets\ Y\ decseq\ B$
**shows** $AE\ x\ in\ \nu x.\ (\lambda n.\ \kappa'\ x\ (B\ n)) \longlongrightarrow \kappa'\ x\ (\bigcap\ (range\ B))$
$\langle proof \rangle$

**end**

## 2.6 Theorem 14.D.10. (Measure Disintegration Theorem)

**locale** *projection-sigma-finite-standard* = *projection-sigma-finite* + *standard-borel-ne* $Y$
**begin**

**theorem** *measure-disintegration*:
$\exists \kappa.\ prob\text{-}kernel\ X\ Y\ \kappa \wedge measure\text{-}kernel.disintegration\ X\ Y\ \kappa\ \nu\ \nu x\ \wedge$
$(\forall \kappa''.\ prob\text{-}kernel\ X\ Y\ \kappa'' \longrightarrow measure\text{-}kernel.disintegration\ X\ Y\ \kappa''\ \nu\ \nu x$

$\longrightarrow$ (*AE x in* $\nu x$. $\kappa$ *x* = $\kappa''$ *x*))
⟨*proof*⟩

**end**

## 2.7 Lemma 14.D.12.

**lemma** *ex-finite-density-measure*:
**fixes** *A* :: *nat* ⇒ -
**assumes** *A*: *range A* ⊆ *sets M* $\bigcup$ (*range A*) = *space M* $\bigwedge$*i*. *emeasure M* (*A i*) ≠ ∞ *disjoint-family A*
**defines** *h* ≡ ($\lambda$*x*. ($\sum$*n*. (*1/2*)^(*Suc n*) ∗ (*1* / (*1* + *M* (*A n*))) ∗ *indicator* (*A n*) *x*))
**shows** *h* ∈ *borel-measurable M*
$\bigwedge$*x*. *x* ∈ *space M* ⟹ *0* < *h x*
$\bigwedge$*x*. *x* ∈ *space M* ⟹ *h x* < *1*
*finite-measure* (*density M h*)
⟨*proof*⟩

**lemma**(**in** *sigma-finite-measure*) *finite-density-measure*:
**obtains** *h* **where** *h* ∈ *borel-measurable M*
$\bigwedge$*x*. *x* ∈ *space M* ⟹ *0* < *h x*
$\bigwedge$*x*. *x* ∈ *space M* ⟹ *h x* < *1*
*finite-measure* (*density M h*)
⟨*proof*⟩

## 2.8 Lemma 14.D.13.

**lemma** (**in** *measure-kernel*)
**assumes** *disintegration* $\nu$ $\mu$
**defines** $\nu x$ ≡ *marginal-measure X Y* $\nu$
**shows** *disintegration-absolutely-continuous*: *absolutely-continuous* $\mu$ $\nu x$
**and** *disintegration-density*: $\nu x$ = *density* $\mu$ ($\lambda$*x*. $\kappa$ *x* (*space Y*))
**and** *disintegration-absolutely-continuous-iff*:
*absolutely-continuous* $\nu x$ $\mu$ ⟷ (*AE x in* $\mu$. $\kappa$ *x* (*space Y*) > *0*)
⟨*proof*⟩

## 2.9 Theorem 14.D.14.

**locale** *sigma-finite-measure-on-pair-standard* = *sigma-finite-measure-on-pair* + *standard-borel-ne Y*

**sublocale** *projection-sigma-finite-standard* ⊆ *sigma-finite-measure-on-pair-standard*
⟨*proof*⟩

**context** *sigma-finite-measure-on-pair-standard*
**begin**

**lemma** *measure-disintegration-extension*:

$\exists\,\mu\ \kappa.$ *finite-measure* $\mu \wedge$ *measure-kernel* $X\ Y\ \kappa \wedge$ *measure-kernel.disintegration* $X\ Y\ \kappa\ \nu\ \mu \wedge$
$(\forall\, x \in$ *space* $X.$ *sigma-finite-measure* $(\kappa\ x)) \wedge$
$(\forall\, x \in$ *space* $X.\ \kappa\ x$ (*space* $Y) > 0) \wedge$
$\mu \sim_M \nu x$ (**is** *?goal*)
⟨*proof*⟩

**end**

**lemma**(**in** *sigma-finite-measure-on-pair*) *measure-disintegration-extension-AE-unique*:
**assumes** *sigma-finite-measure* $\mu$ *sigma-finite-measure* $\mu'$
*measure-kernel* $X\ Y\ \kappa$ *measure-kernel* $X\ Y\ \kappa'$
*measure-kernel.disintegration* $X\ Y\ \kappa\ \nu\ \mu$ *measure-kernel.disintegration* $X\ Y\ \kappa'\ \nu\ \mu'$
**and** *absolutely-continuous* $\mu\ \mu'\ B \in$ *sets* $Y$
**shows** *AE* $x$ *in* $\mu.\ \kappa'\ x\ B *$ *RN-deriv* $\mu\ \mu'\ x = \kappa\ x\ B$
⟨*proof*⟩

**end**

# References


[1] F. Baccelli, B. Blaszczyszyn, and M. Karray. *Random Measures, Point Processes, and Stochastic Geometry.* Inria, Jan. 2020.