Disintegration Theorem

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Abstract

We formalize mixture and disintegraion of measures. This entry is a formalization of Chapter 14.D of the book by Baccelli et.al. [1]. The main result is the disintegration theorem: let (X, Σ_X) be a measurable space, (Y, Σ_Y) be a standard Borel space, ν be a σ-finite measure on $X \times Y$, and ν_X be the marginal measure on X defined by $\nu_X(A) = \nu(A \times Y)$. Assume that ν_X is σ-finite, then there exists a probability kernel κ from X to Y such that

$$\nu(A \times B) = \int_A \kappa_x(B)\nu_X(\mathrm{d}x),\ A \in \Sigma_X, B \in \Sigma_Y.$$

Such a probability kernel is unique ν_X-almost everywhere.

Contents

1 Lemmas

theory *Lemmas-Disintegration*
imports *Standard-Borel-Spaces.StandardBorel*
begin

1.1 Lemmas

lemma *semiring-of-sets-binary-product-sets*[*simp*]:
semiring-of-sets (*space* X × *space* Y) {a × b |a b. a ∈ *sets* X ∧ b ∈ *sets* Y}
⟨*proof*⟩

lemma *sets-pair-restrict-space*:
sets (*restrict-space* X A $\bigotimes_M$ *restrict-space* Y B) = *sets* (*restrict-space* (X $\bigotimes_M$ Y) (A × B))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *restrict-space-space*[*simp*]: *restrict-space* M (*space* M) = M
⟨*proof*⟩

lemma *atMostq-Int-stable*:
Int-stable {{..*r*} |*r*::*real*. *r* ∈ ℚ}
⟨*proof*⟩

lemma *rborel-eq-atMostq*:
borel = *sigma UNIV* { {..*r*} | *r*::*real*. *r* ∈ ℚ}
⟨*proof*⟩

corollary *rborel-eq-atMostq-sets*:
sets borel = *sigma-sets UNIV* {{..*r*} |*r*::*real*. *r* ∈ ℚ}
⟨*proof*⟩

lemma *mono-absolutely-continuous*:
assumes *sets* μ = *sets* ν $\bigwedge$*A*. A ∈ *sets* μ ⟹ μ A ≤ ν A
shows *absolutely-continuous* ν μ
⟨*proof*⟩

lemma *ex-measure-countable-space*:
assumes *countable* (*space* X)
and *sets* X = *Pow* (*space* X)
shows ∃ μ. *sets* μ = *sets* X ∧ (∀ *x*∈*space* X. μ {x} = *f x*)
⟨*proof*⟩

lemma *ex-prob-space-countable*:
assumes *space* X ≠ {} *countable* (*space* X)
and *sets* X = *Pow* (*space* X)
shows ∃ μ. *sets* μ = *sets* X ∧ *prob-space* μ
⟨*proof*⟩

lemma *AE-I''*:
assumes N ∈ *null-sets* M
and $\bigwedge$*x*. *x* ∈ *space* M ⟹ *x* ∉ N ⟹ P *x*
shows *AE x in M*. P *x*
⟨*proof*⟩

lemma *absolutely-continuous-trans*:
assumes *absolutely-continuous L M absolutely-continuous M N*
shows *absolutely-continuous L N*
⟨*proof*⟩

1.2 Equivalence of Measures

abbreviation *equivalence-measure* :: *'a measure* ⇒ *'a measure* ⇒ *bool* (**infix** ‹$\sim_M$› *60*)
where *equivalence-measure M N* ≡ *absolutely-continuous M N* ∧ *absolutely-continuous N M*

lemma *equivalence-measure-refl*: $M \sim_M M$
⟨*proof*⟩

lemma *equivalence-measure-sym*:
assumes $M \sim_M N$
shows $N \sim_M M$
⟨*proof*⟩

lemma *equivalence-measure-trans*:
assumes $M \sim_M N$ $N \sim_M L$
shows $M \sim_M L$
⟨*proof*⟩

lemma *equivalence-measureI*:
assumes *absolutely-continuous M N absolutely-continuous N M*
shows $M \sim_M N$
⟨*proof*⟩

end

2 Disintegration Theorem

theory *Disintegration*
imports *S-Finite-Measure-Monad.Kernels*
Lemmas-Disintegration
begin

2.1 Definition 14.D.2. (Mixture and Disintegration)

context *measure-kernel*
begin

definition *mixture-of* :: [(*'a* × *'b*) *measure*, *'a measure*] ⇒ *bool* **where**
mixture-of ν μ ⟷ *sets* ν = *sets* ($X \bigotimes_M Y$) ∧ *sets* μ = *sets X* ∧ (∀ *C*∈*sets* ($X \bigotimes_M Y$). ν *C* = ($\int^+ x.\ \int^+ y.$ *indicator C* (*x*,*y*) $\partial(\kappa\ x)\ \partial\mu$))

definition *disintegration* :: [(*'a* × *'b*) *measure*, *'a measure*] ⇒ *bool* **where**

disintegration ν μ $\longleftrightarrow$ *sets* ν = *sets* ($X \bigotimes_M Y$) $\wedge$ *sets* μ = *sets* X $\wedge$ ($\forall$ $A$$\in$*sets* X. $\forall$ $B$$\in$*sets* Y. ν ($A \times B$) = ($\int^+ x{\in}A.$ (κ x B) $\partial\mu$))

lemma *disintegrationI*:
assumes *sets* ν = *sets* ($X \bigotimes_M Y$) *sets* μ = *sets* X
and $\bigwedge A\ B.$ $A \in$ *sets* $X \Longrightarrow B \in$ *sets* $Y \Longrightarrow \nu$ ($A \times B$) = ($\int^+ x{\in}A.$ (κ x B) $\partial\mu$)
shows *disintegration* ν μ
⟨*proof*⟩

lemma *mixture-of-disintegration*:
assumes *mixture-of* ν μ
shows *disintegration* ν μ
⟨*proof*⟩

lemma
shows *mixture-of-sets-eq*: *mixture-of* ν μ $\Longrightarrow$ *sets* ν = *sets* ($X \bigotimes_M Y$) *mixture-of* ν μ $\Longrightarrow$ *sets* μ = *sets* X
and *mixture-of-space-eq*: *mixture-of* ν μ $\Longrightarrow$ *space* ν = *space* ($X \bigotimes_M Y$) *mixture-of* ν μ $\Longrightarrow$ *space* μ = *space* X
and *disintegration-sets-eq*: *disintegration* ν μ $\Longrightarrow$ *sets* ν = *sets* ($X \bigotimes_M Y$) *disintegration* ν μ $\Longrightarrow$ *sets* μ = *sets* X
and *disintegration-space-eq*: *disintegration* ν μ $\Longrightarrow$ *space* ν = *space* ($X \bigotimes_M Y$) *disintegration* ν μ $\Longrightarrow$ *space* μ = *space* X
⟨*proof*⟩

lemma
shows *mixture-ofD*: *mixture-of* ν μ $\Longrightarrow$ $C \in$ *sets* ($X \bigotimes_M Y$) $\Longrightarrow \nu$ C = ($\int^+ x.$ $\int^+ y.$ *indicator* C (x,y) $\partial(\kappa\ x)$ $\partial\mu$)
and *disintegrationD*: *disintegration* ν μ $\Longrightarrow$ $A \in$ *sets* $X \Longrightarrow B \in$ *sets* $Y \Longrightarrow \nu$ ($A \times B$) = ($\int^+ x{\in}A.$ (κ x B) $\partial\mu$)
⟨*proof*⟩

lemma *disintegration-restrict-space*:
assumes *disintegration* ν μ $A \cap$ *space* $X \in$ *sets* X
shows *measure-kernel.disintegration* (*restrict-space* X A) Y κ (*restrict-space* ν ($A \times$ *space* Y)) (*restrict-space* μ A)
⟨*proof*⟩

end

context *subprob-kernel*
begin
lemma *countable-disintegration-AE-unique*:
assumes *countable* (*space* Y) **and** [*measurable-cong*]:*sets* Y = *Pow* (*space* Y)
and *subprob-kernel* X Y κ' *sigma-finite-measure* μ
and *disintegration* ν μ *measure-kernel.disintegration* X Y κ' ν μ
shows *AE* x *in* μ. κ x = κ' x
⟨*proof*⟩

end

lemma(**in** *subprob-kernel*) *nu-mu-spaceY-le*:
  assumes *disintegration* ν μ $A \in$ *sets* X
  shows ν $(A \times$ *space* $Y) \leq \mu$ A
⟨*proof*⟩

context *prob-kernel*
begin

lemma *countable-disintegration-AE-unique-prob*:
  assumes *countable* (*space* Y) **and** [*measurable-cong*]:*sets* Y = *Pow* (*space* Y)
    and *prob-kernel* X Y κ' *sigma-finite-measure* μ
    and *disintegration* ν μ *measure-kernel.disintegration* X Y κ' ν μ
   shows *AE* x *in* μ. κ $x = \kappa'$ x
  ⟨*proof*⟩

end

2.2 Lemma 14.D.3.

lemma(**in** *prob-kernel*) *nu-mu-spaceY*:
  assumes *disintegration* ν μ $A \in$ *sets* X
  shows ν $(A \times$ *space* $Y) = \mu$ A
⟨*proof*⟩

corollary(**in** *subprob-kernel*) *nu-finite*:
  assumes *disintegration* ν μ *finite-measure* μ
  shows *finite-measure* ν
⟨*proof*⟩

corollary(**in** *subprob-kernel*) *nu-subprob-space*:
  assumes *disintegration* ν μ *subprob-space* μ
  shows *subprob-space* ν
⟨*proof*⟩

corollary(**in** *prob-kernel*) *nu-prob-space*:
  assumes *disintegration* ν μ *prob-space* μ
  shows *prob-space* ν
⟨*proof*⟩

lemma(**in** *subprob-kernel*) *nu-sigma-finite*:
  assumes *disintegration* ν μ *sigma-finite-measure* μ
  shows *sigma-finite-measure* ν
⟨*proof*⟩

2.3 Theorem 14.D.4. (Measure Mixture Theorem)

lemma(**in** *measure-kernel*) *exist-nu*:

assumes *sets* μ = *sets X*
shows $\exists\nu$. *disintegration* ν μ
⟨*proof*⟩

lemma(**in** *subprob-kernel*) *exist-unique-nu-sigma-finite′*:
assumes *sets* μ = *sets X sigma-finite-measure* μ
shows $\exists!\nu$. *disintegration* ν μ
⟨*proof*⟩

lemma(**in** *subprob-kernel*) *exist-unique-nu-sigma-finite*:
assumes *sets* μ = *sets X sigma-finite-measure* μ
shows $\exists!\nu$. *disintegration* ν μ $\wedge$ *sigma-finite-measure* ν
⟨*proof*⟩

lemma(**in** *subprob-kernel*) *exist-unique-nu-finite*:
assumes *sets* μ = *sets X finite-measure* μ
shows $\exists!\nu$. *disintegration* ν μ $\wedge$ *finite-measure* ν
⟨*proof*⟩

lemma(**in** *subprob-kernel*) *exist-unique-nu-sub-prob-space*:
assumes *sets* μ = *sets X subprob-space* μ
shows $\exists!\nu$. *disintegration* ν μ $\wedge$ *subprob-space* ν
⟨*proof*⟩

lemma(**in** *prob-kernel*) *exist-unique-nu-prob-space*:
assumes *sets* μ = *sets X prob-space* μ
shows $\exists!\nu$. *disintegration* ν μ $\wedge$ *prob-space* ν
⟨*proof*⟩

lemma(**in** *subprob-kernel*) *nn-integral-fst-finite′*:
assumes $f \in$ *borel-measurable* $(X \bigotimes_M Y)$ *disintegration* ν μ *finite-measure* μ
shows $(\int^+ z.\ f\ z\ \partial\nu) = (\int^+ x.\ \int^+ y.\ f\ (x,y)\ \partial(\kappa\ x)\ \partial\mu)$
⟨*proof*⟩

lemma(**in** *prob-kernel*) *nn-integral-fst*:
assumes $f \in$ *borel-measurable* $(X \bigotimes_M Y)$ *disintegration* ν μ *sigma-finite-measure* μ
shows $(\int^+ z.\ f\ z\ \partial\nu) = (\int^+ x.\ \int^+ y.\ f\ (x,y)\ \partial(\kappa\ x)\ \partial\mu)$
⟨*proof*⟩

lemma(**in** *prob-kernel*) *integrable-eq1*:
fixes *f* :: - ⇒ -::{*banach*, *second-countable-topology*}
assumes [*measurable*]:$f \in$ *borel-measurable* $(X \bigotimes_M Y)$
and *disintegration* ν μ *sigma-finite-measure* μ
shows $(\int^+ z.\ ennreal\ (norm\ (f\ z))\ \partial\nu) < \infty \longleftrightarrow (\int^+ x.\ \int^+ y.\ ennreal\ (norm\ (f\ (x,y)))\ \partial(\kappa\ x)\ \partial\mu) < \infty$
⟨*proof*⟩

lemma(**in** *prob-kernel*) *integrable-kernel-integrable*:
fixes f :: - ⇒ -::{*banach*, *second-countable-topology*}
assumes *integrable* ν f *disintegration* ν μ *sigma-finite-measure* μ
shows *AE x in* μ. *integrable* $(\kappa\ x)$ $(\lambda y.\ f\ (x,y))$
⟨*proof*⟩

lemma(**in** *prob-kernel*) *integrable-lebesgue-integral-integrable′*:
fixes f :: - ⇒ -::{*banach*, *second-countable-topology*}
assumes *integrable* ν f *disintegration* ν μ *sigma-finite-measure* μ
shows *integrable* μ $(\lambda x.\ \int y.\ f\ (x,y)\ \partial(\kappa\ x))$
⟨*proof*⟩

lemma(**in** *prob-kernel*) *integrable-lebesgue-integral-integrable*:
fixes f :: - ⇒- ⇒ -::{*banach*, *second-countable-topology*}
assumes *integrable* ν $(\lambda(x,y).\ f\ x\ y)$ *disintegration* ν μ *sigma-finite-measure* μ
shows *integrable* μ $(\lambda x.\ \int y.\ f\ x\ y\ \partial(\kappa\ x))$
⟨*proof*⟩

lemma(**in** *prob-kernel*) *integral-fst*:
fixes f :: - ⇒ -::{*banach*, *second-countable-topology*}
assumes *integrable* ν f *disintegration* ν μ *sigma-finite-measure* μ
shows $(\int z.\ f\ z\ \partial\nu) = (\int x.\ \int y.\ f\ (x,y)\ \partial(\kappa\ x)\ \partial\mu)$
⟨*proof*⟩

2.4 Marginal Measure

definition *marginal-measure-on* :: [′*a measure*, ′*b measure*, (′*a* × ′*b*) *measure*, ′*b set*] ⇒ ′*a measure* **where**
marginal-measure-on X Y ν B = *measure-of* (*space* X) (*sets* X) $(\lambda A.\ \nu\ (A \times B))$

abbreviation *marginal-measure* :: [′*a measure*, ′*b measure*, (′*a* × ′*b*) *measure*] ⇒ ′*a measure* **where**
marginal-measure X Y ν ≡ *marginal-measure-on* X Y ν (*space* Y)

lemma *space-marginal-measure*: *space* (*marginal-measure-on* X Y ν B) = *space* X
and *sets-marginal-measure*: *sets* (*marginal-measure-on* X Y ν B) = *sets* X
⟨*proof*⟩

lemma *emeasure-marginal-measure-on*:
assumes *sets* ν = *sets* $(X \bigotimes_M Y)$ $B \in$ *sets* Y $A \in$ *sets* X
shows *marginal-measure-on* X Y ν B $A = \nu\ (A \times B)$
⟨*proof*⟩

lemma *emeasure-marginal-measure*:
assumes *sets* ν = *sets* $(X \bigotimes_M Y)$ $A \in$ *sets* X
shows *marginal-measure* X Y ν $A = \nu$ (A × *space* Y)
⟨*proof*⟩

lemma *finite-measure-marginal-measure-on-finite*:

assumes *finite-measure* ν *sets* ν = *sets* $(X \bigotimes_M Y)$ $B \in$ *sets* Y
shows *finite-measure* (*marginal-measure-on* X Y ν B)
⟨*proof*⟩

lemma *finite-measure-marginal-measure-finite*:
assumes *finite-measure* ν *sets* ν = *sets* $(X \bigotimes_M Y)$
shows *finite-measure* (*marginal-measure* X Y ν)
⟨*proof*⟩

lemma *marginal-measure-restrict-space*:
assumes *sets* ν = *sets* $(X \bigotimes_M Y)$ $B \in$ *sets* Y
shows *marginal-measure* X (*restrict-space* Y B) (*restrict-space* ν (*space* $X \times B$))
= *marginal-measure-on* X Y ν B
⟨*proof*⟩

lemma *restrict-space-marginal-measure-on*:
assumes *sets* ν = *sets* $(X \bigotimes_M Y)$ $B \in$ *sets* Y $A \in$ *sets* X
shows *restrict-space* (*marginal-measure-on* X Y ν B) A = *marginal-measure-on* (*restrict-space* X A) Y (*restrict-space* ν ($A \times$ *space* Y)) B
⟨*proof*⟩

lemma *restrict-space-marginal-measure*:
assumes *sets* ν = *sets* $(X \bigotimes_M Y)$ $A \in$ *sets* X
shows *restrict-space* (*marginal-measure* X Y ν) A = *marginal-measure* (*restrict-space* X A) Y (*restrict-space* ν ($A \times$ *space* Y))
⟨*proof*⟩

lemma *marginal-measure-mono*:
assumes *sets* ν = *sets* $(X \bigotimes_M Y)$ $A \in$ *sets* Y $B \in$ *sets* Y $A \subseteq B$
shows *emeasure* (*marginal-measure-on* X Y ν A) $\leq$ *emeasure* (*marginal-measure-on* X Y ν B)
⟨*proof*⟩

lemma *marginal-measure-absolutely-countinuous*:
assumes *sets* ν = *sets* $(X \bigotimes_M Y)$ $A \in$ *sets* Y $B \in$ *sets* Y $A \subseteq B$
shows *absolutely-continuous* (*marginal-measure-on* X Y ν B) (*marginal-measure-on* X Y ν A)
⟨*proof*⟩

lemma *marginal-measure-absolutely-continuous'*:
assumes *sets* ν = *sets* $(X \bigotimes_M Y)$ $A \in$ *sets* Y
shows *absolutely-continuous* (*marginal-measure* X Y ν) (*marginal-measure-on* X Y ν A)
⟨*proof*⟩

2.5 Lemma 14.D.6.

locale *sigma-finite-measure-on-pair* =
fixes X :: *'a measure* **and** Y :: *'b measure* **and** ν :: (*'a* $\times$ *'b*) *measure*

assumes *nu-sets*[*measurable-cong*]: *sets* ν = *sets* ($X \bigotimes_M Y$)
and *sigma-finite*: *sigma-finite-measure* ν
begin

abbreviation $\nu x \equiv$ *marginal-measure* X Y ν

end

locale *projection-sigma-finite* =
fixes X :: $'a$ *measure* **and** Y :: $'b$ *measure* **and** ν :: $('a \times 'b)$ *measure*
assumes *nu-sets*[*measurable-cong*]: *sets* ν = *sets* ($X \bigotimes_M Y$)
and *marginal-sigma-finite*: *sigma-finite-measure* (*marginal-measure* X Y ν)
begin

sublocale νx : *sigma-finite-measure marginal-measure* X Y ν
⟨*proof*⟩

lemma ν-*sigma-finite*: *sigma-finite-measure* ν
⟨*proof*⟩

sublocale *sigma-finite-measure-on-pair*
⟨*proof*⟩

definition κ' :: $'a \Rightarrow 'b$ *set* $\Rightarrow$ *ennreal* **where**
κ' x B $\equiv$ *RN-deriv* νx (*marginal-measure-on* X Y ν B) x

lemma *kernel-measurable*[*measurable*]:
(λx. *RN-deriv* (*marginal-measure* X Y ν) (*marginal-measure-on* X Y ν B) x) $\in$ *borel-measurable* νx
⟨*proof*⟩

corollary κ'-*measurable*[*measurable*]:
(λx. κ' x B) $\in$ *borel-measurable* X
⟨*proof*⟩

lemma *kernel-RN-deriv*:
assumes $A \in$ *sets* X $B \in$ *sets* Y
shows ν ($A \times B$) = ($\int^+ x{\in}A$. κ' x B $\partial\nu x$)
⟨*proof*⟩

lemma *empty-Y-bot*:
assumes *space* Y = {}
shows $\nu = \bot$
⟨*proof*⟩

lemma *empty-Y-nux*:
assumes *space* Y = {}
shows νx A = *0*

⟨*proof*⟩

lemma *kernel-empty0-AE*:
  $AE\ x\ in\ \nu x.\ \kappa'\ x\ \{\} = 0$
  ⟨*proof*⟩

lemma *kernel-Y1-AE*:
  $AE\ x\ in\ \nu x.\ \kappa'\ x\ (space\ Y) = 1$
  ⟨*proof*⟩

lemma *kernel-suminf-AE*:
  assumes *disjoint-family F*
    and $\bigwedge i.\ F\ i \in sets\ Y$
  shows $AE\ x\ in\ \nu x.\ (\sum i.\ \ \kappa'\ x\ (F\ i)) = \kappa'\ x\ (\bigcup\ (range\ F))$
  ⟨*proof*⟩

lemma *kernel-finite-sum-AE*:
  assumes *disjoint-family-on F S finite S*
    and $\bigwedge i.\ i \in S \Longrightarrow F\ i \in sets\ Y$
  shows $AE\ x\ in\ \nu x.\ (\sum i{\in}S.\ \kappa'\ x\ (F\ i)) = \kappa'\ x\ (\bigcup i{\in}S.\ F\ i)$
⟨*proof*⟩

lemma *kernel-disjoint-sum-AE*:
  assumes $B \in sets\ Y\ C \in sets\ Y$
    and $B \cap C = \{\}$
  shows $AE\ x\ in\ \nu x.\ \kappa'\ x\ (B \cup C) = \kappa'\ x\ B + \kappa'\ x\ C$
⟨*proof*⟩

lemma *kernel-mono-AE*:
  assumes $B \in sets\ Y\ C \in sets\ Y$
    and $B \subseteq C$
  shows $AE\ x\ in\ \nu x.\ \kappa'\ x\ B \le \kappa'\ x\ C$
⟨*proof*⟩

lemma *kernel-incseq-AE*:
  assumes $range\ B \subseteq sets\ Y\ incseq\ B$
  shows $AE\ x\ in\ \nu x.\ incseq\ (\lambda n.\ \kappa'\ x\ (B\ n))$
  ⟨*proof*⟩

lemma *kernel-decseq-AE*:
  assumes $range\ B \subseteq sets\ Y\ decseq\ B$
  shows $AE\ x\ in\ \nu x.\ decseq\ (\lambda n.\ \kappa'\ x\ (B\ n))$
  ⟨*proof*⟩

corollary *kernel-01-AE*:
  assumes $B \in sets\ Y$
  shows $AE\ x\ in\ \nu x.\ 0 \le \kappa'\ x\ B \wedge \kappa'\ x\ B \le 1$
⟨*proof*⟩

lemma *kernel-get-0*: $0 \leq \kappa'\ x\ B$
⟨*proof*⟩

lemma *kernel-le1-AE*:
assumes $B \in sets\ Y$
shows $AE\ x\ in\ \nu x.\ \kappa'\ x\ B \leq 1$
⟨*proof*⟩

corollary *kernel-n-infty*:
assumes $B \in sets\ Y$
shows $AE\ x\ in\ \nu x.\ \kappa'\ x\ B \neq \top$
⟨*proof*⟩

corollary *kernel-le-infty*:
assumes $B \in sets\ Y$
shows $AE\ x\ in\ \nu x.\ \kappa'\ x\ B < \top$
⟨*proof*⟩

lemma *kernel-SUP-incseq*:
assumes $range\ B \subseteq sets\ Y\ incseq\ B$
shows $AE\ x\ in\ \nu x.\ \kappa'\ x\ (\bigcup\ (range\ B)) = (\bigsqcup n.\ \kappa'\ x\ (B\ n))$
⟨*proof*⟩

lemma *kernel-lim-incseq*:
assumes $range\ B \subseteq sets\ Y\ incseq\ B$
shows $AE\ x\ in\ \nu x.\ (\lambda n.\ \kappa'\ x\ (B\ n)) \longrightarrow \kappa'\ x\ (\bigcup\ (range\ B))$
⟨*proof*⟩

lemma *kernel-INF-decseq*:
assumes $range\ B \subseteq sets\ Y\ decseq\ B$
shows $AE\ x\ in\ \nu x.\ \kappa'\ x\ (\bigcap\ (range\ B)) = (\bigsqcap n.\ \kappa'\ x\ (B\ n))$
⟨*proof*⟩

lemma *kernel-lim-decseq*:
assumes $range\ B \subseteq sets\ Y\ decseq\ B$
shows $AE\ x\ in\ \nu x.\ (\lambda n.\ \kappa'\ x\ (B\ n)) \longrightarrow \kappa'\ x\ (\bigcap\ (range\ B))$
⟨*proof*⟩

end

lemma *qlim-eq-lim-mono-at-bot*:
fixes $g :: rat \Rightarrow 'a :: linorder\text{-}topology$
assumes $mono\ f\ (g \longrightarrow a)\ at\text{-}bot\ \bigwedge r{::}rat.\ f\ (real\text{-}of\text{-}rat\ r) = g\ r$
shows $(f \longrightarrow a)\ at\text{-}bot$
⟨*proof*⟩

lemma *qlim-eq-lim-mono-at-top*:
fixes $g :: rat \Rightarrow 'a :: linorder\text{-}topology$
assumes $mono\ f\ (g \longrightarrow a)\ at\text{-}top\ \bigwedge r{::}rat.\ f\ (real\text{-}of\text{-}rat\ r) = g\ r$

shows $(f \longrightarrow a)$ *at-top*
⟨*proof*⟩

2.6 Theorem 14.D.10. (Measure Disintegration Theorem)

locale *projection-sigma-finite-standard* = *projection-sigma-finite* + *standard-borel-ne* *Y*
begin

theorem *measure-disintegration*:
$\exists \kappa$. *prob-kernel X Y* $\kappa \wedge$ *measure-kernel.disintegration X Y* $\kappa\ \nu\ \nu x \wedge$
$(\forall \kappa''$. *prob-kernel X Y* $\kappa'' \longrightarrow$ *measure-kernel.disintegration X Y* $\kappa''\ \nu\ \nu x$ $\longrightarrow$ (*AE x in* νx. $\kappa\ x = \kappa''\ x$))
⟨*proof*⟩

end

2.7 Lemma 14.D.12.

lemma *ex-finite-density-measure*:
fixes *A* :: *nat* ⇒ -
assumes *A*: *range A* ⊆ *sets M* $\bigcup$ (*range A*) = *space M* $\bigwedge i$. *emeasure M* (*A i*) $\neq \infty$ *disjoint-family A*
defines $h \equiv (\lambda x.\ (\sum n.\ (1/2)\hat{}(Suc\ n) * (1\ /\ (1 + M\ (A\ n))) * indicator\ (A\ n)\ x))$
shows $h \in$ *borel-measurable M*
$\bigwedge x.\ x \in space\ M \Longrightarrow 0 < h\ x$
$\bigwedge x.\ x \in space\ M \Longrightarrow h\ x < 1$
finite-measure (*density M h*)
⟨*proof*⟩

lemma(**in** *sigma-finite-measure*) *finite-density-measure*:
obtains *h* **where** $h \in$ *borel-measurable M*
$\bigwedge x.\ x \in space\ M \Longrightarrow 0 < h\ x$
$\bigwedge x.\ x \in space\ M \Longrightarrow h\ x < 1$
finite-measure (*density M h*)
⟨*proof*⟩

2.8 Lemma 14.D.13.

lemma (**in** *measure-kernel*)
assumes *disintegration* $\nu\ \mu$
defines $\nu x \equiv$ *marginal-measure X Y* ν
shows *disintegration-absolutely-continuous*: *absolutely-continuous* $\mu\ \nu x$
and *disintegration-density*: νx = *density* μ ($\lambda x.\ \kappa\ x$ (*space Y*))
and *disintegration-absolutely-continuous-iff*:
absolutely-continuous $\nu x\ \mu \longleftrightarrow$ (*AE x in* μ. $\kappa\ x$ (*space Y*) > 0)
⟨*proof*⟩

2.9 Theorem 14.D.14.

locale *sigma-finite-measure-on-pair-standard* = *sigma-finite-measure-on-pair* + *standard-borel-ne Y*

sublocale *projection-sigma-finite-standard* ⊆ *sigma-finite-measure-on-pair-standard*
⟨*proof*⟩

context *sigma-finite-measure-on-pair-standard*
begin

lemma *measure-disintegration-extension*:
$\exists\mu\ \kappa.$ *finite-measure* $\mu \wedge$ *measure-kernel* $X\ Y\ \kappa \wedge$ *measure-kernel.disintegration* $X\ Y\ \kappa\ \nu\ \mu \wedge$
$(\forall x{\in}\mathit{space}\ X.\ \mathit{sigma\text{-}finite\text{-}measure}\ (\kappa\ x)) \wedge$
$(\forall x{\in}\mathit{space}\ X.\ \kappa\ x\ (\mathit{space}\ Y) > 0) \wedge$
$\mu \sim_M \nu x$ (**is** *?goal*)
⟨*proof*⟩

end

lemma(**in** *sigma-finite-measure-on-pair*) *measure-disintegration-extension-AE-unique*:
assumes *sigma-finite-measure* μ *sigma-finite-measure* μ'
measure-kernel $X\ Y\ \kappa$ *measure-kernel* $X\ Y\ \kappa'$
measure-kernel.disintegration $X\ Y\ \kappa\ \nu\ \mu$ *measure-kernel.disintegration* $X\ Y\ \kappa'\ \nu\ \mu'$
and *absolutely-continuous* $\mu\ \mu'$ $B \in$ *sets* Y
shows $AE\ x\ in\ \mu.\ \kappa'\ x\ B * \mathit{RN\text{-}deriv}\ \mu\ \mu'\ x = \kappa\ x\ B$
⟨*proof*⟩

end

References

[1] F. Baccelli, B. Blaszczyszyn, and M. Karray. *Random Measures, Point Processes, and Stochastic Geometry.* Inria, Jan. 2020.