# Dirichlet $L$-functions and Dirichlet's Theorem 

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#### Abstract

This article provides a formalisation of Dirichlet characters and Dirichlet $L$-functions including proofs of their basic properties - most notably their analyticity, their areas of convergence, and their nonvanishing for $\mathfrak{R}(s) \geq 1$. All of this is built in a very high-level style using Dirichlet series. The proof of the non-vanishing follows a very short and elegant proof by Newman [4], which we attempt to reproduce faithfully in a similar level of abstraction in Isabelle.

This also leads to a relatively short proof of Dirichlet's Theorem, which states that, if $h$ and $n$ are coprime, there are infinitely many primes $p$ with $p \equiv h(\bmod n)$.


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## 1 Multiplicative Characters of Finite Abelian Groups

theory Multiplicative-Characters<br>imports<br>Complex-Main<br>Finitely-Generated-Abelian-Groups.Finitely-Generated-Abelian-Groups<br>begin<br>notation integer-mod-group ( $Z$ )

### 1.1 Definition of characters

A (multiplicative) character is a completely multiplicative function from a group to the complex numbers. For simplicity, we restrict this to finite abelian groups here, which is the most interesting case.
Characters form a group where the identity is the principal character that maps all elements to 1 , multiplication is point-wise multiplication of the characters, and the inverse is the point-wise complex conjugate.
This group is often called the Pontryagin dual group and is isomorphic to the original group (in a non-natural way) while the double-dual group is naturally isomorphic to the original group.
To get extensionality of the characters, we also require characters to map anything that is not in the group to 0 .

```
definition principal-char :: ('a, 'b) monoid-scheme \(\Rightarrow\) ' \(a \Rightarrow\) complex where
    principal-char \(G a=(\) if \(a \in\) carrier \(G\) then 1 else 0\()\)
definition inv-character where
    inv-character \(\chi=(\lambda a . c n j(\chi a))\)
lemma inv-character-principal \([\) simp \(]\) : inv-character ( principal-char \(G\) ) \(=\) princi-
pal-char G
    \(\langle p r o o f\rangle\)
lemma inv-character-inv-character \([\operatorname{simp}]:\) inv-character \((\) inv-character \(\chi)=\chi\)
    \(\langle p r o o f\rangle\)
lemma eval-inv-character: inv-character \(\chi j=\operatorname{cnj}(\chi j)\)
    〈proof〉
bundle character-syntax
begin
notation principal-char ( \(\chi_{01}\) )
end
locale character \(=\) finite-comm-group +
    fixes \(\chi\) :: ' \(a \Rightarrow\) complex
```

assumes char-one-nz: $\chi \mathbf{1} \neq 0$
assumes char-eq- $0: \quad a \notin$ carrier $G \Longrightarrow \chi a=0$
assumes char-mult $[$ simp $]: a \in$ carrier $G \Longrightarrow b \in \operatorname{carrier} G \Longrightarrow \chi(a \otimes b)=\chi$
$a * \chi b$
begin

### 1.2 Basic properties

lemma char-one [simp]: $\chi \mathbf{1}=1$
〈proof〉
lemma char-power $[\operatorname{simp}]: a \in \operatorname{carrier} G \Longrightarrow \chi\left(a[\upharpoonleft k)=\chi a^{\wedge} k\right.$ $\langle$ proof $\rangle$
lemma char-root:
assumes $a \in$ carrier $G$
shows $\quad \chi a{ }^{\wedge}$ ord $a=1$
$\langle p r o o f\rangle$
lemma char-root':
assumes $a \in$ carrier $G$
shows $\quad \chi a{ }^{\wedge}$ order $G=1$
$\langle p r o o f\rangle$
lemma norm-char: norm $(\chi a)=($ if $a \in$ carrier $G$ then 1 else 0$)$ $\langle p r o o f\rangle$
lemma char-eq-0-iff: $\chi a=0 \longleftrightarrow a \notin$ carrier $G$ $\langle p r o o f\rangle$
lemma inv-character: character $G$ (inv-character $\chi)$ $\langle p r o o f\rangle$
lemma mult-inv-character: $\chi k *$ inv-character $\chi k=$ principal-char $G k$ $\langle p r o o f\rangle$
lemma
assumes $a \in$ carrier $G$
shows char-inv: $\chi($ inv $a)=\operatorname{cnj}(\chi a)$ and char-inv': $\chi(i n v a)=\operatorname{inverse}(\chi$ a)
$\langle p r o o f\rangle$
end
lemma (in finite-comm-group) character-principal [simp, intro]: character $G$ ( principal-char G)
$\langle p r o o f\rangle$
lemmas $[$ simp,intro $]=$ finite-comm-group.character-principal

```
lemma character-ext:
    assumes character G \chi character G \mp@subsup{\chi}{}{\prime}\bigwedgex.x\in carrier G\Longrightarrow\chix= \chi
    shows }\chi=\mp@subsup{\chi}{}{\prime
<proof\rangle
lemma character-mult [intro]:
    assumes character G \chi character G \chi'
    shows character G ( }\lambdax.\chix*\mp@subsup{\chi}{}{\prime}x
<proof\rangle
```

lemma character-inv-character-iff [simp]: character $G($ inv-character $\chi) \longleftrightarrow$ char-
acter $G \chi$
<proof〉
definition characters :: (' $a, ~ ' b)$ monoid-scheme $\Rightarrow\left({ }^{\prime} a \Rightarrow\right.$ complex) set where
characters $G=\{\chi$. character $G \chi\}$

## 1．3 The Character group

The characters of a finite abelian group $G$ form another group $\widehat{G}$ ，which is called its Pontryagin dual group．This generalises to the more general setting of locally compact abelian groups，but we restrict ourselves to the finite setting because it is much easier．

```
definition Characters :: ('a, 'b) monoid-scheme \(\Rightarrow\) ('a \(\Rightarrow\) complex) monoid
    where Characters \(G=\left(\right.\) carrier \(=\) characters \(G\), monoid.mult \(=\left(\begin{array}{l}\lambda \\ \chi_{1}\end{array} \chi_{2} k . \chi_{1}\right.\)
\(\left.k * \chi_{2} k\right)\),
    one \(=\) principal-char \(G\) D
lemma carrier-Characters: carrier (Characters \(G\) ) \(=\) characters \(G\)
    〈proof〉
lemma one-Characters: one (Characters \(G\) ) \(=\) principal-char \(G\)
    〈proof〉
lemma mult-Characters: monoid.mult (Characters \(G) \chi_{1} \chi_{2}=\left(\begin{array}{ll}\lambda a . & \chi_{1} a * \chi_{2} a\end{array}\right)\)
    \(\langle p r o o f\rangle\)
```

context finite-comm-group
begin
sublocale principal: character $G$ principal-char $G\langle p r o o f\rangle$
lemma finite-characters [intro]: finite (characters $G$ )
$\langle p r o o f\rangle$

```
lemma finite-comm-group-Characters [intro]: finite-comm-group (Characters G)
\langleproof\rangle
end
lemma (in character) character-in-order-1:
    assumes order G=1
    shows }\quad\chi=\mathrm{ principal-char }
\langleproof\rangle
lemma (in finite-comm-group) characters-in-order-1:
    assumes order G=1
    shows characters G}={\mathrm{ principal-char }G
    <proof>
lemma (in character) inv-Characters: inv Characters G \chi = inv-character \chi
<proof\rangle
lemma (in finite-comm-group) inv-Characters':
    \chi < \text { characters } G \Longrightarrow i n v \text { Characters } G \chi = i n v - c h a r a c t e r ~ \chi ~
    <proof\rangle
lemmas (in finite-comm-group) Characters-simps =
    carrier-Characters mult-Characters one-Characters inv-Characters'
lemma inv-Characters': \chi characters }G\Longrightarrow\mathrm{ inv Characters G }\chi=\mathrm{ inv-character
\chi
    <proof>
```


### 1.4 The isomorphism between a group and its dual

We start this section by inspecting the special case of a cyclic group. Here, any character is fixed by the value it assigns to the generating element of the cyclic group. This can then be used to construct a bijection between the nth unit roots and the elements of the character group - implying the other results.

```
lemma (in finite-cyclic-group)
    defines \(i\) c: induce-char \(\equiv(\lambda c::\) complex. ( \(\lambda a\). if \(a \in\) carrier \(G\) then \(c\) powi get-exp
gen a else 0))
    shows order-Characters: order (Characters \(G\) ) \(=\) order \(G\)
    and gen-fixes-char: \(\llbracket\) character \(G a ;\) character \(G b ; a\) gen \(=b\) gen \(\rrbracket a=b\)
    and unity-root-induce-char: \(z^{\wedge}\) order \(G=1 \Longrightarrow\) character \(G\) (induce-char \(z\) )
〈proof〉
```

Moreover, we can show that a character that assigns a "true" root of unity to the generating element of the group, generates the character group.
lemma (in finite-cyclic-group) finite-cyclic-group-Characters:
obtains $\chi$ where finite-cyclic-group (Characters $G$ ) $\chi$
$\langle p r o o f\rangle$
And as two cyclic groups of the same order are isomorphic it follows the isomorphism of a finite cyclic group and its dual.

```
lemma (in finite-cyclic-group) Characters-iso:
    G\cong Characters G
<proof\rangle
```

The character groups of two isomorphic groups are also isomorphic.

```
lemma (in finite-comm-group) iso-imp-iso-chars:
    assumes G\congH group H
    shows Characters G\cong Characters H
<proof\rangle
```

The following two lemmas characterize the way a character behaves in a direct group product: a character on the product induces characters on each of the factors. Also, any character on the direct product can be decomposed into a pointwise product of characters on the factors.

```
lemma DirProds-subchar:
    assumes finite-comm-group (DirProds Gs I)
    and x:x\in carrier (Characters (DirProds Gs I))
    and}i:i\in
    and I: finite I
    defines g:g\equiv(\lambdac. (\lambdai\inI. (\lambdaa.c ((\lambdai\inI. 1 1 Gs i)(i:=a)))))
    shows character (Gs i) (g x i)
<proof\rangle
lemma Characters-DirProds-single-prod:
    assumes finite-comm-group (DirProds Gs I)
    and x:x\in carrier (Characters (DirProds Gs I))
    and I: finite I
    defines g: g \equiv(\lambdaI. (\lambdac. (\lambdai\inI. (\lambdaa.c ((\lambdai\inI. 1 Gs i)
    shows (\lambdae. if e\incarrier(DirProds Gs I) then \prodi\inI. (gI x i) (e i) else 0) = x
(is ? g x = x)
\langleproof\rangle
```

This allows for the following: the character group of a direct product is isomorphic to the direct product of the character groups of the factors.

```
lemma (in finite-comm-group) Characters-DirProds-iso:
    assumes DirProds Gs I\congG group (DirProds Gs I) finite I
    shows DirProds (Characters \circ Gs) I\cong Characters G
<proof\rangle
```

As thus both the group and its character group can be decomposed into the same cyclic factors, the isomorphism follows for any finite abelian group.

```
theorem (in finite-comm-group) Characters-iso:
```

    shows \(G \cong\) Characters \(G\)
    ```
<proof\rangle
```

Hence, the orders are also equal.

```
corollary (in finite-comm-group) order-Characters:
    order (Characters G) = order G
    <proof>
corollary (in finite-comm-group) card-characters: card (characters G) = order G
    <proof\rangle
```


### 1.5 Non-trivial facts about characters

We characterize the character group of a quotient group as the group of characters that map all elements of the subgroup onto 1 .
lemma (in finite-comm-group) iso-Characters-FactGroup:
assumes $H$ : subgroup $H G$
shows $(\lambda \chi x$. if $x \in$ carrier $G$ then $\chi(H \#>x)$ else 0$) \in$
iso (Characters $(G$ Mod $H))(($ Characters $G) 0$ carrier $:=\{\chi \in$ characters
G. $\forall x \in H . \chi x=1\} \mid)$
$\langle p r o o f\rangle$

```
lemma (in finite-comm-group) is-iso-Characters-FactGroup:
    assumes \(H\) : subgroup \(H G\)
    shows Characters \((G\) Mod \(H) \cong(\) Characters \(G) \\) carrier \(:=\{\chi \in\) characters \(G\).
\(\forall x \in H . \chi x=1\}\) )
    \(\langle\) proof〉
```

In order to derive the number of extensions a character on a subgroup has to the entire group, we introduce the group homomorphism restrict-char that restricts a character to a given subgroup $H$.
definition restrict-char::'a set $\Rightarrow(' a \Rightarrow$ complex $) \Rightarrow\left({ }^{\prime} a \Rightarrow\right.$ complex $)$ where restrict-char $H \chi=(\lambda e$. if $e \in H$ then $\chi$ e else 0$)$

```
lemma (in finite-comm-group) restrict-char-hom:
    assumes subgroup \(H G\)
    shows group-hom (Characters \(G)(\) Characters \((G \|\) carrier \(:=H \mid))\) (restrict-char
H)
\(\langle p r o o f\rangle\)
```

The kernel is just the set of the characters that are 1 on all of $H$.

```
lemma (in finite-comm-group) restrict-char-kernel:
    assumes subgroup H G
    shows kernel (Characters G) (Characters (G(carrier :=HD)) (restrict-char H)
        = {\chi\incharacters G.}\forallx\inH.\chix=1
    <proof\rangle
```

Also, all of the characters on the subgroup are the image of some character on the whole group.

```
lemma (in finite-comm-group) restrict-char-image:
    assumes subgroup \(H G\)
    shows restrict-char \(H^{\prime}(\) carrier \((\) Characters \(G))=\) carrier (Characters \((G 0\) carrier
\(:=H())\) )
\(\langle p r o o f\rangle\)
```

It follows that any character on $H$ can be extended to a character on $G$.
lemma (in finite-comm-group) character-extension-exists:
assumes subgroup $H G$ character $(G($ carrier $:=H \|) \chi$
obtains $\chi^{\prime}$ where character $G \chi^{\prime}$ and $\bigwedge x . x \in H \Longrightarrow \chi^{\prime} x=\chi x$
$\langle p r o o f\rangle$
For two characters on a group $G$ the number of characters on subgroup $H$ that share the values with them is the same for both.

```
lemma (in finite-comm-group) character-restrict-card:
    assumes subgroup \(H G\) character \(G\) a character \(G b\)
    shows card \(\left\{\chi^{\prime} \in\right.\) characters \(\left.G . \forall x \in H . \chi^{\prime} x=a x\right\}=\operatorname{card}\left\{\chi^{\prime} \in\right.\) characters \(G\).
\(\left.\forall x \in H . \chi^{\prime} x=b x\right\}\)
\(\langle p r o o f\rangle\)
```

These lemmas allow to show that the number of extensions of a character on $H$ to a character on $G$ is just $|G| /|H|$.
theorem (in finite-comm-group) card-character-extensions:
assumes subgroup $H G$ character $(G($ carrier $:=H \mid) \chi$
shows card $\left\{\chi^{\prime} \in\right.$ characters $\left.G . \forall x \in H . \chi^{\prime} x=\chi x\right\} *$ card $H=$ order $G$
$\langle$ proof $\rangle$
Lastly, we can also show that for each $x \in H$ of order $n>1$ and each $n$-th root of unity $z$, there exists a character $\chi$ on $G$ such that $\chi(x)=z$.

```
lemma (in group) powi-get-exp-self:
    fixes z::complex
    assumes z^ n=1 x carrier G ord x = n n>1
    shows z powi get-exp x x = z
<proof\rangle
```

```
corollary (in finite-comm-group) character-with-value-exists:
```

corollary (in finite-comm-group) character-with-value-exists:
assumes }x\in\mathrm{ carrier }G\mathrm{ and }x\not=1\mathrm{ and z^ ord x=1
assumes }x\in\mathrm{ carrier }G\mathrm{ and }x\not=1\mathrm{ and z^ ord x=1
obtains \chi}\mathrm{ where character }G\chi\mathrm{ and }\chix=
obtains \chi}\mathrm{ where character }G\chi\mathrm{ and }\chix=
<proof\rangle

```
<proof\rangle
```

In particular, for any $x$ that is not the identity element, there exists a character $\chi$ such that $\chi(x) \neq 1$.
corollary (in finite-comm-group) character-neq-1-exists:
assumes $x \in$ carrier $G$ and $x \neq 1$
obtains $\chi$ where character $G \chi$ and $\chi x \neq 1$
〈proof〉

### 1.6 The first orthogonality relation

The entries of any non-principal character sum to 0 .
theorem (in character) sum-character:
( $\sum x \in$ carrier $\left.G . \chi x\right)=($ if $\chi=$ principal-char $G$ then of-nat $($ order $G)$ else 0) $\langle p r o o f\rangle$

```
corollary (in finite-comm-group) character-orthogonality1:
    assumes character G \chi and character G \chi'
    shows (\sumx\incarrier G. \chi x* cnj ( ( ' x)) =(if \chi = \chi' then of-nat (order G)
else 0)
<proof\rangle
```


### 1.7 The isomorphism between a group and its double dual

Lastly, we show that the double dual of a finite abelian group is naturally isomorphic to the original group via the obvious isomorphism $x \mapsto(\chi \mapsto$ $\chi(x))$. It is easy to see that this is a homomorphism and that it is injective. The fact $|\widehat{\widehat{G}}|=|\widehat{G}|=|G|$ then shows that it is also surjective.

```
context finite-comm-group
```

begin
definition double-dual-iso :: ' $a \Rightarrow\left({ }^{\prime} a \Rightarrow\right.$ complex $) \Rightarrow$ complex where double-dual-iso $x=(\lambda \chi$. if character $G \chi$ then $\chi x$ else 0$)$
lemma double-dual-iso-apply [simp]: character $G \chi \Longrightarrow$ double-dual-iso $x \chi=\chi x$ $\langle p r o o f\rangle$
lemma character-double-dual-iso [intro]:
assumes $x: x \in$ carrier $G$
shows character (Characters $G$ ) (double-dual-iso $x$ )
〈proof〉
lemma double-dual-iso-mult [simp]:
assumes $x \in$ carrier $G y \in$ carrier $G$
shows double-dual-iso $(x \otimes y)=$ double-dual-iso $x \otimes$ Characters (Characters $G$ ) double-dual-iso y
$\langle p r o o f\rangle$
lemma double-dual-iso-one [simp]:
double-dual-iso $\mathbf{1}=$ principal-char $($ Characters $G)$
$\langle p r o o f\rangle$
lemma inj-double-dual-iso: inj-on double-dual-iso (carrier $G$ )
$\langle p r o o f\rangle$
lemma double-dual-iso-eq-iff [simp]:

```
    \(x \in\) carrier \(G \Longrightarrow y \in\) carrier \(G \Longrightarrow\) double-dual-iso \(x=\) double-dual-iso \(y \longleftrightarrow\)
\(x=y\)
    \(\langle\) proof \(\rangle\)
```

theorem double-dual-iso: double-dual-iso $\in$ iso $G$ (Characters (Characters $G$ ))
$\langle p r o o f\rangle$
lemma double-dual-is-iso: Characters (Characters $G$ ) $\cong G$
$\langle p r o o f\rangle$

The second orthogonality relation follows from the first one via Pontryagin duality:
theorem sum-characters:
assumes $x: x \in$ carrier $G$
shows $\left(\sum \chi \in\right.$ characters $\left.G . \chi x\right)=($ if $x=\mathbf{1}$ then of-nat $($ order $G)$ else 0$)$
$\langle p r o o f\rangle$
corollary character-orthogonality2:
assumes $x \in$ carrier $G y \in$ carrier $G$
shows $\left(\sum \chi \in\right.$ characters $\left.G . \chi x * \operatorname{cnj}(\chi y)\right)=($ if $x=y$ then of-nat (order $G)$
else 0)
〈proof〉
end
no-notation integer-mod-group ( $Z$ )
end

## 2 Dirichlet Characters

theory Dirichlet-Characters imports<br>Multiplicative-Characters<br>HOL-Number-Theory.Residues<br>Dirichlet-Series.Multiplicative-Function<br>\section*{begin}

Dirichlet characters are essentially just the characters of the multiplicative group of integer residues $\mathbb{Z} \mathbb{Z} / n \mathbb{Z} \mathbb{Z}$ for some fixed $n$. For convenience, these residues are usually represented by natural numbers from 0 to $n-1$, and we extend the characters to all natural numbers periodically, so that $\chi(k$ $\bmod n)=\chi(k)$ holds.
Numbers that are not coprime to $n$ are not in the group and therefore are assigned 0 by all characters.

### 2.1 The multiplicative group of residues

definition residue-mult-group :: nat $\Rightarrow$ nat monoid where

```
    residue-mult-group n = \ carrier = totatives n, monoid.mult =( }\lambdaxy.(x*y
mod n), one = 1 \
definition principal-dchar :: nat }=>\mathrm{ nat }=>\mathrm{ complex where
    principal-dchar n = ( }\lambdak\mathrm{ . if coprime k n then 1 else 0)
lemma principal-dchar-coprime [simp]: coprime k n \Longrightarrow principal-dchar n k=1
    and principal-dchar-not-coprime [simp]: ᄀcoprime k n\Longrightarrow principal-dchar n k=
0
    <proof>
lemma principal-dchar-1 [simp]: principal-dchar n 1 = 1
    \langleproof\rangle
lemma principal-dchar-minus1 [simp]:
    assumes n>0
    shows principal-dchar n (n-Suc 0) =1
<proof\rangle
lemma mod-in-totatives: n>1\Longrightarrowa mod n \in totatives }n\longleftrightarrow\mathrm{ coprime a }
    <proof\rangle
bundle dcharacter-syntax
begin
notation principal-dchar (\chi01)
end
locale residues-nat =
    fixes n :: nat (structure) and G
    assumes n: n> 1
    defines }G\equiv\mathrm{ residue-mult-group n
begin
lemma order [simp]: order G = totient n
    <proof\rangle
lemma totatives-mod [simp]: x totatives }n\Longrightarrowx\operatorname{mod}n=
    <proof\rangle
lemma principal-dchar-minus1 [simp]: principal-dchar n (n-Suc 0) = 1
    <proof\rangle
sublocale finite-comm-group G
<proof\rangle
```


### 2.2 Definition of Dirichlet characters

The following two functions make the connection between Dirichlet characters and the multiplicative characters of the residue group.

```
definition \(c 2 d c::(\) nat \(\Rightarrow\) complex \() \Rightarrow(\) nat \(\Rightarrow\) complex \()\) where
    \(c 2 d c \chi=(\lambda x \cdot \chi(x \bmod n))\)
definition dc2c :: (nat \(\Rightarrow\) complex \() \Rightarrow(\) nat \(\Rightarrow\) complex \()\) where
    \(d c 2 c \chi=(\lambda x\). if \(x<n\) then \(\chi x\) else 0\()\)
lemma dc2c-c2dc [simp]:
    assumes character \(G \chi\)
    shows \(\quad d c 2 c(c 2 d c \chi)=\chi\)
\(\langle p r o o f\rangle\)
end
locale dcharacter \(=\) residues-nat +
    fixes \(\chi\) :: nat \(\Rightarrow\) complex
    assumes mult-aux: \(a \in\) totatives \(n \Longrightarrow b \in\) totatives \(n \Longrightarrow \chi(a * b)=\chi a * \chi\)
b
    assumes eq-zero: \(\neg\) coprime \(a n \Longrightarrow \chi a=0\)
    assumes periodic: \(\chi(a+n)=\chi a\)
    assumes one-not-zero: \(\chi 1 \neq 0\)
begin
lemma zero-eq-0 [simp]: \(\chi 0=0\)
    〈proof〉
lemma Suc-0 [simp]: \(\chi(\) Suc 0\()=1\)
    〈proof〉
lemma periodic-mult: \(\chi(a+m * n)=\chi a\)
\(\langle p r o o f\rangle\)
lemma minus-one-periodic [simp]:
    assumes \(k>0\)
    shows \(\quad \chi(k * n-1)=\chi(n-1)\)
\(\langle p r o o f\rangle\)
lemma cong:
    assumes \([a=b](\bmod n)\)
    shows \(\quad \chi a=\chi b\)
〈proof〉
lemma \(\bmod [\operatorname{simp}]: \chi(a \bmod n)=\chi a\)
    \(\langle p r o o f\rangle\)
lemma mult \([\) simp \(]: \chi(a * b)=\chi a * \chi b\)
\(\langle p r o o f\rangle\)
sublocale mult: completely-multiplicative-function \(\chi\)
    〈proof〉
```

```
lemma eq-zero-iff: \chi x=0\longleftrightarrow वcoprime x n
<proof\rangle
lemma minus-one': \chi (n-1) \in{-1,1}
\langleproof\rangle
lemma c2dc-dc2c [simp]:c2dc (dc2c \chi)=\chi
    <proof\rangle
lemma character-dc2c: character G (dc2c \chi)
    \langleproof\rangle
sublocale dc\mathcal{L}: character G dc2c \chi
    \langleproof\rangle
lemma dcharacter-inv-character [intro]: dcharacter n (inv-character \chi)
    <proof\rangle
lemma norm: norm ( }\chik)=(\mathrm{ if coprime k n then 1 else 0)
<proof\rangle
lemma norm-le-1: norm ( \chi k) \leq 1
    <proof>
end
definition dcharacters :: nat }=>\mathrm{ (nat }=>\mathrm{ complex) set where
    dcharacters }n={\chi.\mathrm{ dcharacter n }\chi
context residues-nat
begin
lemma character-dc2c:dcharacter n \chi \Longrightarrow character G (dc2c \chi)
    <proof\rangle
lemma dcharacter-c2dc:
    assumes character G \chi
    shows dcharacter n (c2dc \chi)
<proof\rangle
lemma principal-dchar-altdef: principal-dchar n = c2dc (principal-char G)
    \langleproof\rangle
sublocale principal: dcharacter n G principal-dchar n
    <proof>
lemma c2dc-principal [simp]:c2dc (principal-char G) = principal-dchar n
```

    \(\langle p r o o f\rangle\)
    lemma dc2c-principal $[$ simp $]: d c \mathcal{Z} c($ principal-dchar $n)=$ principal-char $G$
$\langle p r o o f\rangle$
lemma bij-betw-dcharacters-characters:
bij-betw dc2c (dcharacters $n$ ) (characters $G$ )
$\langle p r o o f\rangle$
lemma bij-betw-characters-dcharacters:
bij-betw c2dc (characters $G$ ) (dcharacters $n$ )
〈proof〉
lemma finite-dcharacters [intro]: finite (dcharacters $n$ )
$\langle p r o o f\rangle$
lemma card-dcharacters $[$ simp $]$ : card $($ dcharacters $n)=$ totient $n$
$\langle p r o o f\rangle$
end
lemma inv-character-eq-principal-dchar-iff [simp]:
inv-character $\chi=$ principal-dchar $n \longleftrightarrow \chi=$ principal-dchar $n$
〈proof〉

## 2．3 Sums of Dirichlet characters

lemma（in dcharacter）sum－dcharacter－totatives：
（ $\sum x \in$ totatives $\left.n . \chi x\right)=($ if $\chi=$ principal－dchar $n$ then of－nat（totient $n$ ）else 0） $\langle p r o o f\rangle$
lemma（in dcharacter）sum－dcharacter－block：
$\left(\sum x<n . \chi x\right)=($ if $\chi=$ principal－dchar $n$ then of－nat $($ totient $n)$ else 0$)$ $\langle p r o o f\rangle$
lemma（in dcharacter）sum－dcharacter－block＇：
sum $\chi\{$ Suc $0 . . n\}=($ if $\chi=$ principal－dchar $n$ then of－nat（totient $n$ ）else 0）
＜proof〉
lemma（in dcharacter）sum－lessThan－dcharacter：
assumes $\chi \neq$ principal－dchar $n$
shows $\left(\sum x<m . \chi x\right)=\left(\sum x<m \bmod n . \chi x\right)$
$\langle p r o o f\rangle$
lemma（in dcharacter）sum－dcharacter－lessThan－le：
assumes $\chi \neq$ principal－dchar $n$
shows norm $\left(\sum x<m . \chi x\right) \leq$ totient $n$

```
<proof\rangle
```

lemma (in dcharacter) sum-dcharacter-atMost-le:
assumes $\chi \neq$ principal-dchar $n$
shows norm $\left(\sum x \leq m . \chi x\right) \leq$ totient $n$
〈proof〉
lemma (in residues-nat) sum-dcharacters:
$\left(\sum \chi \in d\right.$ characters $\left.n . \chi x\right)=($ if $[x=1](\bmod n)$ then of-nat $($ totient $n)$ else 0$)$ $\langle p r o o f\rangle$

```
lemma (in dcharacter) even-dcharacter-linear-sum-eq-0 [simp]:
    assumes }\chi\not=\mathrm{ principal-dchar n and }\chi(n-1)=
    shows (\sumk=Suc 0..<n. of-nat k*\chik)=0
<proof\rangle
end
```


## 3 Dirichlet $L$-functions

theory Dirichlet-L-Functions<br>imports<br>Dirichlet-Characters<br>HOL-Library.Landau-Symbols<br>Zeta-Function.Zeta-Function<br>begin

We can now define the Dirichlet $L$-functions. These are essentially the functions in the complex plane that the Dirichlet series $\sum_{k=1}^{\infty} \chi(k) k^{-s}$ converge to, for some fixed Dirichlet character $\chi$.
First of all, we need to take care of a syntactical problem: The notation for vectors uses $\chi$ as syntax, which causes some annoyance to us, so we disable it locally.

### 3.1 Definition and basic properties

We now define Dirichlet $L$ functions as a finite linear combination of Hurwitz $\zeta$ functions. This has the advantage that we directly get the analytic continuation over the full domain and only need to prove that the series really converges to this definition whenever it does converge, which is not hard to do.

```
definition Dirichlet-L :: nat \(\Rightarrow\) ( nat \(\Rightarrow\) complex \() \Rightarrow\) complex \(\Rightarrow\) complex where
    Dirichlet-L \(m \chi s=\)
        (if \(s=1\) then
            if \(\chi=\) principal-dchar \(m\) then 0 else eval-fds \((f d s \chi) 1\)
        else
            of-nat \(m\) powr \(-s *\left(\sum k=1\right.\)..m. \(\chi k *\) hurwitz-zeta \((\) real \(k /\) real \(\left.\left.m) s\right)\right)\)
```

```
lemma Dirichlet-L-conv-hurwitz-zeta-nonprincipal:
    assumes s\not=1
    shows Dirichlet-L n \chi s=
    of-nat n powr -s*(\sumk=1..n. \chi k*hurwitz-zeta (real k/real n) s)
    <proof\rangle
```

Analyticity everywhere except 1 is trivial by the above definition, since the Hurwitz $\zeta$ function is analytic everywhere except 1. For $L$ functions of non principal characters, we will have to show the analyticity at 1 separately later.
lemma holomorphic-Dirichlet-L-weak:
assumes $m>01 \notin A$
shows Dirichlet-L $m \chi$ holomorphic-on $A$
$\langle p r o o f\rangle$
context dcharacter
begin
For a real value greater than 1, the formal Dirichlet series of an $L$ function for some character $\chi$ converges to the $L$ function.

```
lemma
    fixes s :: complex
    assumes s: Re s>1
    shows abs-summable-Dirichlet-L: summable ( }\lambda\mathrm{ n. norm ( }\chin*\mathrm{ of-nat n powr
-s))
    and summable-Dirichlet-L: summable ( }\lambdan.\chin*\mathrm{ of-nat n powr -s)
    and sums-Dirichlet-L: ( \lambdan. \chi n* n powr -s) sums Dirichlet-L n \chi s
    and Dirichlet-L-conv-eval-fds-weak: Dirichlet-L n \chi s = eval-fds (fds \chi)s
<proof>
lemma fds-abs-converges-weak:Re s>1\Longrightarrowfds-abs-converges (fds \chi)s
    <proof\rangle
```

lemma abs-conv-abscissa-weak: abs-conv-abscissa $($ fds $\chi) \leq 1$
〈proof〉

Dirichlet $L$ functions have the Euler product expansion

$$
L(\chi, s)=\prod_{p}\left(1-\frac{\chi(p)}{p^{-s}}\right)
$$

for all $s$ with $\mathfrak{R}(s)>1$.
lemma
fixes $s::$ complex assumes $s$ : Re $s>1$
shows Dirichlet-L-euler-product-LIMSEQ:
( $\lambda n . \Pi p \leq n$. if prime $p$ then inverse $(1-\chi p /$ nat-power $p$ s) else 1) $\longrightarrow$ Dirichlet-L $n \chi s$ (is ?th1)
and Dirichlet-L-abs-convergent-euler-product: abs-convergent-prod ( $\lambda$. if prime $p$ then inverse $(1-\chi p / p$ powr $s)$ else 1) (is ?th2)
$\langle p r o o f\rangle$
lemma Dirichlet-L-Re-gt-1-nonzero: assumes Re $s>1$
shows Dirichlet-L $n \chi s \neq 0$
$\langle p r o o f\rangle$
lemma sum-dcharacter-antimono-bound
fixes $x 0$ a $b$ :: real and $f f^{\prime}::$ real $\Rightarrow$ real
assumes nonprincipal: $\chi \neq \chi_{0}$
assumes $x 0: x 0 \geq 0$ and $a b: x 0 \leq a a<b$
assumes $f^{\prime}: \bigwedge x . x \geq x 0 \Longrightarrow\left(f\right.$ has-field-derivative $\left.f^{\prime} x\right)($ at $x)$
assumes f-nonneg: $\bigwedge x . x \geq x 0 \Longrightarrow f x \geq 0$ assumes $f^{\prime}$-nonpos: $\backslash x . x \geq x 0 \Longrightarrow f^{\prime} x \leq 0$ shows $\operatorname{norm}\left(\sum n \in\right.$ real $-‘\{a<. . b\} . \chi n *(f($ real $\left.n))\right) \leq 2 *$ real (totient $\left.n\right)$ * fa $\langle p r o o f\rangle$
lemma summable-dcharacter-antimono:
fixes $x 0$ a $b$ :: real and $f f^{\prime}::$ real $\Rightarrow$ real
assumes nonprincipal: $\chi \neq \chi_{0}$
assumes $f^{\prime}: \bigwedge x . x \geq x 0 \Longrightarrow\left(f\right.$ has-field-derivative $\left.f^{\prime} x\right)($ at $x)$
assumes $f$-nonneg: $\bigwedge x . x \geq x 0 \Longrightarrow f x \geq 0$
assumes $f^{\prime}$-nonpos: $\bigwedge x . x \geq x 0 \Longrightarrow f^{\prime} x \leq 0$
assumes lim: $(f \longrightarrow 0)$ at-top
shows summable $(\lambda n . \chi n * f n)$
$\langle p r o o f\rangle$
lemma conv-abscissa-le-0:
fixes $s$ :: real
assumes nonprincipal: $\chi \neq \chi_{0}$
shows conv-abscissa $(f d s \quad \chi) \leq 0$
<proof〉
lemma summable-Dirichlet- $L^{\prime}$ :
assumes nonprincipal: $\chi \neq \chi_{0}$
assumes $s$ : Re $s>0$
shows summable ( $\lambda n . \chi n *$ of-nat $n$ powr $-s$ )
$\langle p r o o f\rangle$

## lemma

assumes $\chi \neq \chi_{0}$
shows Dirichlet-L-conv-eval-fds: $\wedge$ s. Re $s>0 \Longrightarrow$ Dirichlet-L $n \chi s=$ eval-fds (fds $\chi) s$
and holomorphic-Dirichlet-L: Dirichlet-L $n$ र holomorphic-on A

```
<proof\rangle
```

lemma cnj-Dirichlet-L:
cnj $($ Dirichlet-L $n \chi s)=$ Dirichlet-L $n($ inv-character $\chi)(c n j s)$
$\langle p r o o f\rangle$ end
lemma holomorphic-Dirichlet-L [holomorphic-intros]:
assumes $n>1 \chi \neq$ principal-dchar $n \wedge$ dcharacter $n \chi \vee \chi=$ principal-dchar
$n \wedge 1 \notin A$
shows Dirichlet-L $n \chi$ holomorphic-on $A$
$\langle p r o o f\rangle$
lemma holomorphic-Dirichlet- $L^{\prime}$ [holomorphic-intros]:
assumes $n>1 f$ holomorphic-on $A$
$\chi \neq$ principal-dchar $n \wedge$ dcharacter $n \chi \vee \chi=$ principal-dchar $n \wedge(\forall x \in A$.
f $x \neq 1$ )
shows ( $\lambda$ s. Dirichlet-L $n \chi(f s))$ holomorphic-on $A$
$\langle p r o o f\rangle$
lemma continuous-on-Dirichlet-L:
assumes $n>1 \chi \neq$ principal-dchar $n \wedge$ dcharacter $n \chi \vee \chi=$ principal-dchar
$n \wedge 1 \notin A$
shows continuous-on $A$ (Dirichlet-L $n \chi$ )
〈proof〉
lemma continuous-on-Dirichlet- $L^{\prime}$ [continuous-intros]:
assumes continuous-on $A f n>1$
and $\chi \neq$ principal-dchar $n \wedge$ dcharacter $n \chi \vee \chi=$ principal-dchar $n \wedge$
$(\forall x \in A . f x \neq 1)$
shows continuous-on $A(\lambda x$. Dirichlet-L $n \chi(f x))$
$\langle p r o o f\rangle$
corollary continuous-Dirichlet-L [continuous-intros]:
$n>1 \Longrightarrow \chi \neq$ principal-dchar $n \wedge$ dcharacter $n \chi \vee \chi=$ principal-dchar $n \wedge s$
$\neq 1 \Longrightarrow$
continuous (at swithin A) (Dirichlet-L $n \chi$ )
〈proof〉
corollary continuous-Dirichlet- $L^{\prime}$ [continuous-intros]:
$n>1 \Longrightarrow$ continuous (at $s$ within $A) f \Longrightarrow$
$\chi \neq$ principal-dchar $n \wedge$ dcharacter $n \chi \vee \chi=$ principal-dchar $n \wedge f s \neq 1$
$\Longrightarrow$
continuous (at s within A) ( $\lambda x$. Dirichlet-L $n \chi(f x)$ )
$\langle p r o o f\rangle$
context residues-nat
begin

Applying the above to the $L\left(\chi_{0}, s\right)$ ，the $L$ function of the principal character，
we find that it differs from the Riemann $\zeta$ function only by multiplication with a constant that depends only on the modulus $n$. They therefore have the same analytic properties as the $\zeta$ function itself.

```
lemma Dirichlet-L-principal:
    fixes \(s\) :: complex
    shows Dirichlet-L \(n \chi_{0} s=\left(\prod p \mid\right.\) prime \(p \wedge p\) dvd \(n .(1-1 / p\) powr \(\left.s)\right) *\)
zeta s
    (is ?f \(s=? g s)\)
\(\langle\) proof \(\rangle\) end
```


### 3.2 The non-vanishing for $\mathfrak{R}(s) \geq 1$

```
lemma coprime-prime-exists:
    assumes n> (0 :: nat)
    obtains p}\mathrm{ where prime p coprime p n
<proof\rangle
```

The case of the principal character is trivial, since it differs from the Riemann $\zeta(s)$ only in a multiplicative factor that is clearly non-zero for $\mathfrak{R}(s) \geq 1$.

```
theorem (in residues-nat) Dirichlet-L-Re-ge-1-nonzero-principal:
```

    assumes Re \(s \geq 1 s \neq 1\)
    shows Dirichlet-L \(n\) (principal-dchar \(n\) ) \(s \neq 0\)
    $\langle p r o o f\rangle$

The proof for non-principal character is quite involved and is typically very complicated and technical in most textbooks. For instance, Apostol [1] proves the result separately for real and non-real characters, where the nonreal case is relatively short and nice, but the real case involves a number of complicated asymptotic estimates.
The following proof, on the other hand - like our proof of the analogous result for the Riemann $\zeta$ function - is based on Newman's book [4]. Newman gives a very short, concise, and high-level sketch that we aim to reproduce faithfully here.

```
context dcharacter
begin
theorem Dirichlet-L-Re-ge-1-nonzero-nonprincipal:
    assumes }\chi\not=\mp@subsup{\chi}{0}{}\mathrm{ and Re u}\geq
    shows Dirichlet-L n \chiu\not=0
<proof\rangle
    include dcharacter-syntax
    <proof\rangle
```


### 3.3 Asymptotic bounds on partial sums of Dirichlet $L$ functions

The following are some bounds on partial sums of the $L$-function of a character that are useful for asymptotic reasoning, particularly for Dirichlet's

## Theorem.

```
lemma sum-upto-dcharacter-le:
    assumes }\chi\not=\mp@subsup{\chi}{0}{
    shows norm (sum-upto \chi x) \leq totient n
<proof\rangle
```

```
lemma Dirichlet-L-minus-partial-sum-bound:
```

lemma Dirichlet-L-minus-partial-sum-bound:
fixes }s::\mathrm{ complex and }x\mathrm{ :: real
fixes }s::\mathrm{ complex and }x\mathrm{ :: real
assumes }\chi\not=\mp@subsup{\chi}{0}{}\mathrm{ and Re s>0 and }x>
assumes }\chi\not=\mp@subsup{\chi}{0}{}\mathrm{ and Re s>0 and }x>
defines }\sigma\equivRe
defines }\sigma\equivRe
shows norm (sum-upto ( }\lambdan.\chin*n\mathrm{ powr - s) x - Dirichlet-L n र s) }
shows norm (sum-upto ( }\lambdan.\chin*n\mathrm{ powr - s) x - Dirichlet-L n र s) }
real (totient n)*(2+\operatorname{cmod}s/\sigma)/x powr \sigma
real (totient n)*(2+\operatorname{cmod}s/\sigma)/x powr \sigma
<proof\rangle
<proof\rangle
lemma partial-Dirichlet-L-sum-bigo:
lemma partial-Dirichlet-L-sum-bigo:
fixes }s\mathrm{ :: complex and }x\mathrm{ :: real
fixes }s\mathrm{ :: complex and }x\mathrm{ :: real
assumes }\chi\not=\mp@subsup{\chi}{0}{}\operatorname{Re}s>
assumes }\chi\not=\mp@subsup{\chi}{0}{}\operatorname{Re}s>
shows (\lambdax. sum-upto (\lambdan. \chi n* n powr -s) x - Dirichlet-L n \chi s) \inO(\lambdax.
shows (\lambdax. sum-upto (\lambdan. \chi n* n powr -s) x - Dirichlet-L n \chi s) \inO(\lambdax.
x powr -s)
x powr -s)
<proof\rangleend

```
<proof\rangleend
```


### 3.4 Evaluation of $L(\chi, 0)$

context residues-nat
begin
lemma Dirichlet-L-0-principal $[$ simp $]$ : Dirichlet-L $n \chi_{0} 0=0$
$\langle p r o o f\rangle$

```
end
context dcharacter
begin
lemma Dirichlet-L-0-nonprincipal:
    assumes nonprincipal: }\chi\not=\mp@subsup{\chi}{0}{
    shows Dirichlet-L n\chi 0 = - (\sumk=1..<n. of-nat k*\chik)/ of-nat n
<proof\rangle
lemma Dirichlet-L-0-even [simp]:
    assumes }\chi(n-1)=
    shows Dirichlet-L n\chi 0=0
<proof\rangle
lemma Dirichlet-L-0:
    Dirichlet-L n \chi 0 = (if \chi (n-1) = 1 then 0 else - (\sumk=1..<n. of-nat k*\chi
k) / of-nat n)
    <proof\rangleend
```


### 3.5 Properties of $L(\chi, s)$ for real $\chi$

```
locale real-dcharacter \(=\) dcharacter +
    assumes real: \(\chi k \in \mathbb{R}\)
begin
lemma \(\operatorname{Im}-e q-0[\operatorname{simp}]: \operatorname{Im}(\chi k)=0\)
    \(\langle p r o o f\rangle\)
lemma of-real-Re [simp]: of-real \((\operatorname{Re}(\chi k))=\chi k\)
    \(\langle p r o o f\rangle\)
lemma char-cases: \(\chi k \in\{-1,0,1\}\)
\(\langle p r o o f\rangle\)
lemma \(c n j[\operatorname{simp}]: c n j(\chi k)=\chi k\)
    \(\langle p r o o f\rangle\)
lemma inv-character-id \([\) simp \(]\) : inv-character \(\chi=\chi\)
    〈proof〉
lemma Dirichlet-L-in-Reals:
    assumes \(s \in \mathbb{R}\)
    shows Dirichlet-L \(n \chi s \in \mathbb{R}\)
\(\langle p r o o f\rangle\)
```

The following property of real characters is used by Apostol to show the non-vanishing of $L(\chi, 1)$. We have already shown this in a much easier way, but this particular result is still of general interest.

```
lemma
    assumes k: k>0
    shows sum-char-divisors-ge: Re (\sumd|d dvd k. \chi d)\geq0(is Re(?A k)\geq0)
    and sum-char-divisors-square-ge: is-square k\LongrightarrowRe(\sumd|d dvd k.\chid)\geq1
<proof\rangle
end
end
```


## 4 Dirichlet's Theorem on primes in arithmetic progressions

theory Dirichlet-Theorem<br>imports<br>Dirichlet-L-Functions<br>Bertrands-Postulate.Bertrand<br>Landau-Symbols.Landau-More<br>begin

We can now turn to the proof of the main result: Dirichlet's theorem about
the infinitude of primes in arithmetic progressions.
There are previous proofs of this by John Harrison in HOL Light [3] and by Mario Carneiro in Metamath [2]. Both of them strive to prove Dirichlet's theorem with a minimum amount of auxiliary results and definitions, whereas our goal was to get a short and simple proof of Dirichlet's theorem built upon a large library of Analytic Number Theory.
At this point, we already have the key part - the non-vanishing of $L(1, \chi)$ - and the proof was relatively simple and straightforward due to the large amount of Complex Analysis and Analytic Number Theory we have available. The remainder will be a bit more concrete, but still reasonably concise. First, we need to re-frame some of the results from the AFP entry about Bertrand's postulate a little bit.

### 4.1 Auxiliary results

The AFP entry for Bertrand's postulate already provides a slightly stronger version of this for integer values of $x$, but we can easily extend this to real numbers to obtain a slightly nicer presentation.

```
lemma sum-upto-mangoldt-le:
    assumes }x\geq
    shows sum-upto mangoldt x\leq3/2*x
\langleproof\rangle
```

We can also, similarly, use the results from the Bertrand's postulate entry to show that the sum of $\ln p / p$ over all primes grows logarithmically.

## lemma Mertens-bigo:

$\left(\lambda x .\left(\sum p \mid\right.\right.$ prime $p \wedge$ real $\left.\left.p \leq x . \ln p / p\right)-\ln x\right) \in O(\lambda-.1)$
$\langle p r o o f\rangle$

### 4.2 The contribution of the non-principal characters

The estimates in the next two sections are partially inspired by John Harrison's proof of Dirichlet's Theorem [3].
We first estimate the growth of the partial sums of

$$
-L^{\prime}(1, \chi) / L(1, \chi)=\sum_{k=1}^{\infty} \chi(k) \frac{\Lambda(k)}{k}
$$

for a non-principal character $\chi$ and show that they are, in fact, bounded, which is ultimately a consequence of the non-vanishing of $L(1, \chi)$ for nonprincipal $\chi$.
context dcharacter
begin

```
context
    includes no-vec-lambda-notation dcharacter-syntax
    fixes }
    assumes nonprincipal: }\chi\not=\mp@subsup{\chi}{0}{
    defines L}\equiv\mathrm{ Dirichlet-L n ұ 1
begin
lemma Dirichlet-L-nonprincipal-mangoldt-bound-aux-strong:
    assumes x: x>0
    shows norm (L* sum-upto ( }\lambdak.\chik*\mathrm{ mangoldt k/k) x - sum-upto ( }\lambdak.\chi
* ln k ( k) x)
    <9 / 2* real (totient n)
<proof\rangle
lemma Dirichlet-L-nonprincipal-mangoldt-aux-bound:
    (\lambdax.L* sum-upto ( }\lambdak.\chik*\mathrm{ mangoldt k/k) x - sum-upto ( }\lambdak.\chik*\operatorname{ln}k/k
x) \inO(\lambda-. 1)
    <proof>
```

lemma nonprincipal-mangoldt-bound:
( $\lambda x$. sum-upto $(\lambda k . \chi k *$ mangoldt $k / k) x) \in O(\lambda$-. 1$)$ (is ?lhs $\in-)$
$\langle p r o o f\rangle$
end
end

### 4.3 The contribution of the principal character

Next, we turn to the analogous partial sum for the principal character and show that it grows logarithmically and therefore is the dominant contribution.

```
context residues-nat
begin
context
    includes no-vec-lambda-notation dcharacter-syntax
begin
lemma principal-dchar-sum-bound:
    (\lambdax. (\sump|prime p}\wedge real p\leqx. \chio p*(ln p/p))-\operatorname{ln}x)\inO(\lambda-. 1
<proof\rangle
lemma principal-dchar-sum-bound':
    (\lambdax. sum-upto ( }\lambdak.\mp@subsup{\chi}{0}{}k*\mathrm{ mangoldt k/k) x - Ln x) }\inO(\lambda-.1
<proof\rangle
```


### 4.4 The main result

We can now show the main result by extracting the primes we want using the orthogonality relation on characters, separating the principal part of the sum from the non-principal ones and then applying the above estimates.

```
lemma Dirichlet-strong:
    assumes coprime \(h n\)
    shows \(\left(\lambda x .\left(\sum p \mid \operatorname{prime} p \wedge[p=h](\bmod n) \wedge \operatorname{real} p \leq x \ln p / p\right)-\ln x /\right.\)
totient n)
    \(\in O(\lambda-1)(\) is \((\lambda x . \operatorname{sum}-(? A x)--) \in-)\)
\(\langle p r o o f\rangle\)
```

It is now obvious that the set of primes we are interested in is, in fact, infinite.

```
theorem Dirichlet:
    assumes coprime h n
    shows infinite {p. prime p}\wedge[p=h](\operatorname{mod}n)
<proof\rangle
```

In the future, one could extend this result to more precise estimates of the distribution of primes in arithmetic progressions in a similar way to the Prime Number Theorem.
end
end
end

## References

[1] T. M. Apostol. Introduction to Analytic Number Theory. Undergraduate Texts in Mathematics. Springer-Verlag, 1976.
[2] M. Carneiro. Formalization of the prime number theorem and dirichlet's theorem. In Proceedings of the 9th Conference on Intelligent Computer Mathematics (CICM 2016), pages 10-13, 2016.
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[4] D. Newman. Analytic Number Theory. Number 177 in Graduate Texts in Mathematics. Springer, 1998.

