Formalizing Results on Directed Sets

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Abstract

Directed sets are of fundamental interest in domain theory and topology. In this paper, we formalize some results on directed sets in Isabelle/HOL, most notably: under the axiom of choice, a poset has a supremum for every directed set if and only if it does so for every chain; and a function between such posets preserves suprema of directed sets if and only if it preserves suprema of chains. The known pen-and-paper proofs of these results crucially use uncountable transfinite sequences, which are not directly implementable in Isabelle/HOL. We show how to emulate such proofs by utilizing Isabelle/HOL's ordinal and cardinal library. Thanks to the formalization, we relax some conditions for the above results.

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1 Introduction

A *directed set* is a set D equipped with a binary relation $\sqsubseteq$ such that any finite subset $X \subseteq D$ has an upper bound in D with respect to $\sqsubseteq$. The property is often equivalently stated that D is non-empty and any two elements $x, y \in D$ have a bound in D, assuming that $\sqsubseteq$ is transitive (as in posets).

Directed sets find uses in various fields of mathematics and computer science. In topology (see for example the textbook [7]), directed sets are used to generalize the set of natural numbers: sequences $\mathbb{N} \to A$ are generalized to *nets* $D \to A$, where D is an arbitrary directed set. For example, the usual result on metric spaces that continuous functions are precisely functions that preserve limits of sequences can be generalized in general topological spaces as: the continuous functions are precisely functions that preserve limits of nets. In domain theory [1], key ingredients are *directed-complete posets*, where every directed subset has a supremum in the poset, and *Scott-continuous functions* between posets, that is, functions that preserve suprema of directed sets. Thanks to their fixed-point properties (which we have formalized in Isabelle/HOL in a previous work [5]), directed-complete posets naturally appear in denotational semantics of languages with loops or fixed-point operators (see for example Scott domains [11, 13]). Directed sets also appear in reachability and coverability analyses of transition systems through the notion of ideals, that is, downward-closed directed sets. They allow effective representations of objects, making forward and backward analysis of well-structured transition systems – such as Petri nets – possible (see e.g., [6]).

Apparently milder generalizations of natural numbers are chains (totally ordered sets) or even well-ordered sets. In the mathematics literature, the following results are known (assuming the axiom of choice):

Theorem 1 **([4])** *A poset is directed-complete if (and only if) it has a supremum for every non-empty well-ordered subset.*

Theorem 2 **([9])** *Let f be a function between posets, each of which has a supremum for every non-empty chain. If f preserves suprema of non-empty chains, then it is Scott-continuous.*

The pen-and-paper proofs of these results use induction on cardinality, where the finite case is merely the base case. The core of the proof is a technical result called Iwamura's Lemma [8], where the countable case is merely an easy case, and the main part heavily uses transfinite sequences indexed by uncountable ordinals.

To formalize these results in Isabelle/HOL we extensively use the existing library for ordinals and cardinals [3], but we needed some delicate work in emulating the pen-and-paper proofs. In Isabelle/HOL, or any proof assistant based on higher-order logic (HOL), it is not possible to have a datatype for arbitrarily large ordinals; hence, it is not possible to directly formalize transfinite sequences. We show how to emulate transfinite sequences using the ordinal and cardinal library [3]. As far as the authors know, our work is the first to mechanize the proof of Theorems 1 and 2, as well as Iwamura's Lemma. We prove the two theorems for quasi-ordered sets, relaxing antisymmetry, and strengthen Theorem 2 so that chains are replaced by well-ordered sets and conditions on the codomain are completely dropped.

Related Work Systems based on Zermelo-Fraenkel set theory, such as Mizar [2] and Isabelle/ZF [10], have more direct support for ordinals and cardinals and should pose less challenge in mechanizing the above results. Nevertheless, a part of our contribution is in demonstrating that the power of (Isabelle/)HOL is strong enough to deal with uncountable transfinite sequences.

Except for the extra care for transfinite sequences, our proof of Iwamura's Lemma is largely based on the original proof from [8]. Markowsky presented a proof of Theorem 1 using Iwamura's Lemma [9, Corollary 1]. While he took a minimal-counterexample approach, we take a more constructive approach to build a well-ordered set of suprema. This construction was crucial to be reused in the proof of Theorem 2, which Markowsky claimed without a proof [9]. Another proof of Theorem 1 can be found in [4], without using Iwamura's Lemma, but still crucially using transfinite sequences.

This work has been published in the conference paper [14].

2 Preliminaries

2.1 Connecting Predicate-Based and Set-Based Relations

theory *Well-Order-Connection*
 imports
  Main
  Complete-Non-Orders.Well-Relations
begin

lemma *refl-on-relation-of*: *refl-on A* (*relation-of r A*) ⟷ *reflexive A r*
 by (*auto simp*: *refl-on-def reflexive-def relation-of-def*)

lemma *trans-relation-of*: *trans* (*relation-of r A*) ⟷ *transitive A r*
 by (*auto simp*: *trans-def relation-of-def transitive-def*)

lemma *preorder-on-relation-of*: *preorder-on A* (*relation-of r A*) ⟷ *quasi-ordered-set A r*
 by (*simp add*: *preorder-on-def refl-on-relation-of trans-relation-of quasi-ordered-set-def*)

lemma *antisym-relation-of*: *antisym* (*relation-of r A*) ⟷ *antisymmetric A r*
 by (*auto simp*: *antisym-def relation-of-def antisymmetric-def*)

lemma *partial-order-on-relation-of*:
 partial-order-on A (*relation-of r A*) ⟷ *partially-ordered-set A r*
 by (*auto simp*: *partial-order-on-def preorder-on-relation-of antisym-relation-of quasi-ordered-set-def partially-ordered-set-def*)

lemma *total-on-relation-of*: *total-on A* (*relation-of r A*) ⟷ *semiconnex A r*
 by (*auto simp*: *total-on-def relation-of-def semiconnex-def*)

lemma *linear-order-on-relation-of*:
 shows *linear-order-on* A (*relation-of* r A) $\longleftrightarrow$ *total-ordered-set* A r
 by (*auto simp*: *linear-order-on-def partial-order-on-relation-of total-on-relation-of total-ordered-set-def total-quasi-ordered-set-def partially-ordered-set-def connex-iff-semiconnex-reflexive*)

lemma *relation-of-sub-Id*: (*relation-of* r A $-$ *Id*) $=$ *relation-of* ($\lambda x\ y.\ r\ x\ y \wedge x \neq y$) A
 by (*auto simp*: *relation-of-def*)

lemma (**in** *antisymmetric*) *asympartp-iff-weak-neq*:
 shows $x \in A \Longrightarrow y \in A \Longrightarrow$ *asympartp* ($\sqsubseteq$) x y $\longleftrightarrow$ $x \sqsubseteq y \wedge x \neq y$
 by (*auto intro*!: *asympartpI antisym*)

lemma *wf-relation-of*: *wf* (*relation-of* r A) $=$ *well-founded* A r
 apply (*simp add*: *wf-eq-minimal relation-of-def well-founded-iff-ex-extremal Ball-def*)
 by (*metis* (*no-types*, *opaque-lifting*) *equals0I insert-Diff insert-not-empty subsetI subset-iff*)

lemma *well-order-on-relation-of*:
 shows *well-order-on* A (*relation-of* r A) $\longleftrightarrow$ *well-ordered-set* A r
 by (*auto simp*: *well-order-on-def linear-order-on-relation-of relation-of-sub-Id wf-relation-of well-ordered-iff-well-founded-total-ordered antisymmetric.asympartp-iff-weak-neq total-ordered-set-def cong*: *well-founded-cong*)

lemma (**in** *connex*) *Field-relation-of*: *Field* (*relation-of* ($\sqsubseteq$) A) $=$ A
 by (*auto simp*: *Field-def relation-of-def*)

lemma (**in** *well-ordered-set*) *Well-order-relation-of*:
 shows *Well-order* (*relation-of* ($\sqsubseteq$) A)
 by (*auto simp*: *Field-relation-of well-order-on-relation-of well-ordered-set-axioms*)

lemma *in-relation-of*: $(x,y) \in$ *relation-of* r A $\longleftrightarrow$ $x \in A \wedge y \in A \wedge r\ x\ y$
 by (*simp add*: *relation-of-def*)

lemma *relation-of-triv*: *relation-of* ($\lambda x\ y.\ (x,y) \in r$) *UNIV* $=$ r
 by (*auto simp*: *relation-of-def*)

lemma *Restr-eq-relation-of*: *Restr* R A $=$ *relation-of* ($\lambda x\ y.\ (x,y) \in R$) A
 by (*auto simp*: *relation-of-def*)

theorem *ex-well-order*: $\exists\, r.$ *well-ordered-set* A r
proof−
 from *well-order-on* **obtain** R **where** R: *well-order-on* A R **by** *auto*
 then have *well-order-on* A (*Restr* R A)
  by (*simp add*: *well-order-on-Field*[*OF* R] *Restr-Field*)
 then show *?thesis* **by** (*auto simp*: *Restr-eq-relation-of well-order-on-relation-of*)
qed

end
theory *Directed-Completeness*
  imports
    Complete-Non-Orders.Continuity
    Well-Order-Connection
    HOL−Cardinals.Cardinals
    HOL−Library.FuncSet
begin

2.2 Missing Lemmas

no-notation *disj* (**infixr** ‹|› *30*)

lemma *Sup-funpow-mono*:
  fixes *f* :: *'a* :: *complete-lattice* ⇒ *'a*
  assumes *mono*: *mono f*
  shows *mono* (⨆ *i*. *f* ^^ *i*)
  by (*intro monoI*, *auto intro*!: *Sup-mono dest*: *funpow-mono*[*OF mono*])

lemma *iso-imp-compat*:
  assumes *iso*: *iso r r' f* **shows** *compat r r' f*
  by (*simp add*: *compat-def iso iso-forward*)

lemma *iso-inv-into*:
  assumes *ISO*: *iso r r' f*
  shows *iso r' r* (*inv-into* (*Field r*) *f*)
  using *assms* **unfolding** *iso-def*
  using *bij-betw-inv-into inv-into-Field-embed-bij-betw* **by** *blast*

lemmas *iso-imp-compat-inv-into* = *iso-imp-compat*[*OF iso-inv-into*]

lemma *infinite-iff-natLeq*: *infinite A* ⟷ *natLeq* ≤*o* |*A*|
  using *infinite-iff-natLeq-ordLeq* **by** *blast*

As we cannot formalize transfinite sequences directly, we take the following approach: We just use A as the index set, and instead of the ordering on ordinals, we take the well-order that is chosen by the cardinality library to denote $|A|$.

definition *well-order-of* (‹('(⪯-'))› [*0*]*1000*) **where** (⪯$_A$) *x y* ≡ (*x*,*y*) ∈ |*A*|

abbreviation *well-order-le* (‹- ⪯- -› [*51*,*0*,*51*]*50*) **where** *x* ⪯$_A$ *y* ≡ (⪯$_A$) *x y*

abbreviation *well-order-less* (‹- ≺- -› [*51*,*0*,*51*]*50*) **where** *x* ≺$_A$ *y* ≡ *asympartp* (⪯$_A$) *x y*

lemmas *well-order-ofI* = *well-order-of-def*[*unfolded atomize-eq*, *THEN iffD2*]
lemmas *well-order-ofD* = *well-order-of-def*[*unfolded atomize-eq*, *THEN iffD1*]

lemma *carrier*: **assumes** $x \preceq_A y$ **shows** $x \in A$ **and** $y \in A$
using *assms* **by** (*auto dest*!: *well-order-ofD dest*: *FieldI1 FieldI2*)

lemma *relation-of*[*simp*]: *relation-of* $(\preceq_A)$ $A = |A|$
by (*auto simp*: *relation-of-def well-order-of-def dest*: *FieldI1 FieldI2*)

interpretation *well-order-of*: *well-ordered-set* A $(\preceq_A)$
apply (*fold well-order-on-relation-of*)
by *auto*

Thanks to the well-order theorem, one can have a sequence $\{A_\alpha\}_{\alpha<|A|}$ of subsets of A that satisfies the following three conditions:

- cardinality: $|A_\alpha| < |A|$ for every $\alpha < |A|$,
- monotonicity: $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta < |A|$, and
- range: if A is infinite, $A = \bigcup_{\alpha<|A|} A_\alpha$.

The following serves the purpose.

definition *Pre* (‹-≺› [*1000*]*1000*) **where** $A_\prec\ a \equiv \{b \in A.\ b \prec_A a\}$

lemma *Pre-eq-underS*: $A_\prec\ a$ = *underS* $|A|$ a
by (*auto simp*: *Pre-def underS-def well-order-ofD carrier well-order-of.antisym dest*!: *well-order-ofI*)

lemma *Pre-card*: **assumes** *aA*: $a \in A$ **shows** $|A_\prec\ a| <o |A|$
by (*auto simp*: *Pre-eq-underS aA intro*!: *card-of-underS*[*OF card-of-Card-order*])

lemma *Pre-carrier*: $A_\prec\ a \subseteq A$ **by** (*auto simp*: *Pre-def*)

lemma *Pre-mono*: *monotone-on* A $(\preceq_A)$ $(\subseteq)$ $(A_\prec)$
by (*auto intro*!: *monotone-onI simp*: *Pre-def dest*: *well-order-of.asym-trans well-order-of.asym.irrefl*)

lemma *extreme-imp-finite*:
assumes *e*: *extreme* A $(\preceq_A)$ e **shows** *finite* A
proof (*rule ccontr*)
assume *inf*: *infinite* A
from e **have** *eA*: $e \in A$ **by** *auto*
from e **have** $A = \{a \in A.\ a \preceq_A e\}$ **by** *auto*
also have $\ldots - \{e\} = A_\prec\ e$
using *eA* **by** (*auto simp*: *Pre-def dest*: *well-order-of.asympartp-iff-weak-neq*)
finally have *AeP*: $A - \{e\} = \ldots$ **.**
have *infinite* $(A - \{e\})$ **using** *infinite-remove*[*OF inf*]**.**
with *AeP* **have** *infP*: *infinite* $(A_\prec\ e)$ **by** *simp*
have A = *insert* e $(A_\prec\ e)$ **using** *eA* **by** (*fold AeP*, *auto*)
also have $|\ldots| =o |A_\prec\ e|$ **using** *infinite-card-of-insert*[*OF infP*]**.**
finally have $|A_\prec\ e| =o |A|$ **using** *ordIso-symmetric* **by** *auto*
with *Pre-card*[*OF eA*] *not-ordLess-ordIso*

```
  show False by auto
qed

lemma infinite-imp-ex-Pre:
  assumes inf: infinite A and xA: x ∈ A shows ∃ y ∈ A. x ∈ A≺ y
proof−
  from inf
  have ¬ extreme A (⪯A) x by (auto dest!: extreme-imp-finite)
  with xA obtain y where yA: y ∈ A and ¬ y ⪯A x by auto
  with xA have x ≺A y by (auto simp: well-order-of.not-weak-iff asympartpI)
  with yA show ?thesis by (auto simp: Pre-def xA)
qed

lemma infinite-imp-Un-Pre: assumes inf: infinite A shows ⋃(A≺ ‘ A) = A
proof (safe)
  fix x assume xA: x ∈ A
  show y ∈ A≺ x ⟹ y ∈ A for y using Pre-carrier[of A x] by auto
  from infinite-imp-ex-Pre[OF inf xA]
  show x ∈ ⋃ (A≺ ‘ A) by (auto simp: Pre-def)
qed
```

3 Iwamura's lemma

As the proof involves a number of (inductive) definitions, we build a locale for collecting those definitions and lemmas.

```
locale Iwamura-proof = related-set +
  assumes dir: directed-set A (⊑)
begin
```

Inside this locale, a related set $(A, \sqsubseteq)$ is fixed and assumed to be directed. The proof starts with declaring, using the axiom of choice, a function f that chooses a bound $f\ X \in A$ for every finite subset $X \subseteq A$. This function can be formalized using the SOME construction:

```
definition f where f X ≡ SOME z. z ∈ A ∧ bound X (⊑) z

lemma assumes XA: X ⊆ A and Xfin: finite X
  shows f-carrier: f X ∈ A and f-bound: bound X (⊑) (f X)
  using directed-setD[OF dir XA Xfin, unfolded Bex-def, THEN someI-ex]
  by (auto simp: f-def)
```

3.1 Uncountable Case

Actually, the main part of the proof of Iwamura's Lemma is about monotonically expanding an infinite subset (in particular A_α) of A into a directed one, without changing the cardinality. To this end, Iwamura's original proof introduces a function $F\colon PowA \to PowA$ that expands a set with upper bounds of *all finite subsets*. This approach is different from Markowsky's reproof (based on [12]) which uses nested transfinite induction to extend a set one element after another.

```
definition F where F X ≡ X ∪ f ‘ Fpow X

lemma F-carrier: X ⊆ A ⟹ F X ⊆ A
  and F-infl: X ⊆ F X
  and F-fin: finite X ⟹ finite (F X)
  by (auto simp: F-def Fpow-def f-carrier)

lemma F-card: assumes inf: infinite X shows |F X| =o |X|
proof−
  have |f ‘ Fpow X| ≤o |Fpow X| using card-of-image.
  thm card-of-Fpow-infinite
  also have |Fpow X| =o |X| using card-of-Fpow-infinite[OF inf].
  finally have |f ‘ Fpow X| ≤o |X|.
  with inf show ?thesis by (auto simp: F-def)
qed

lemma F-mono: mono F
proof(intro monoI)
  show X ⊆ Y ⟹ F X ⊆ F Y for X Y
    using Fpow-mono[of X Y] by (auto simp: F-def)
qed

lemma Fn-carrier: X ⊆ A ⟹ (F ^^ n) X ⊆ A
  and Fn-infl: X ⊆ (F ^^ n) X
  and Fn-fin: finite X ⟹ finite ((F ^^ n) X)
  and Fn-card: infinite X ⟹ |(F ^^ n) X| =o |X|
proof (atomize(full), induct n)
  case (Suc n)
  define Y where Y ≡ (F^^n) X
  then have ∗: (F ^^ Suc n) X = F Y by auto
  from Suc[folded Y-def]
  have infinite X ⟹ infinite Y ∧ |Y| =o |X|
    and finite X ⟹ finite Y
    and X ⊆ Y
    and X ⊆ A ⟹ Y ⊆ A by (auto simp: Y-def)
  with F-carrier[of Y] F-infl[of Y] F-card[of Y] F-fin[of Y]
  show ?case by (unfold ∗, auto del:subsetI dest:ordIso-transitive)
qed auto

lemma Fn-mono1: i ≤ j ⟹ (F ^^ i) X ⊆ (F ^^ j) X for i j
  using Fn-infl[of (F^^i) X j−i] funpow-add[of j−i i F]
  by auto
```

We take the ω-iteration of the monotone function F, namely:

```
definition Flim (‹F^ω›) where F^ω X ≡ ⋃ i. (F ^^ i) X

lemma Flim-mono: mono F^ω
```

proof−
  have F^ω = ($\bigsqcup$ *range* (($\frown\frown$) *F*)) **by** (*auto simp*: *Flim-def*)
  with *Sup-funpow-mono*[*OF F-mono*]
  show *?thesis* **by** *auto*
qed

lemma *Flim-infl*: $X \subseteq F^\omega\ X$
  using *Fn-infl* **by** (*auto simp*: *Flim-def*)

lemma *Flim-carrier*: **assumes** $X \subseteq A$ **shows** $F^\omega\ X \subseteq A$
  using *Fn-carrier*[*OF assms*] **by** (*auto simp*: *Flim-def*)

lemma *Flim-directed*: **assumes** $X \subseteq A$ **shows** *directed-set* (F^ω *X*) ($\sqsubseteq$)
proof (*safe intro*!: *directed-setI*)
  fix *Y* **assume** *YC*: $Y \subseteq F^\omega\ X$ **and** *finY*: *finite Y*
  from *finY YC* **have** $\exists$ *i*. $Y \subseteq (F \frown\frown i)\ X$
  proof (*induct*)
    case *empty*
    then show *?case* **by** *auto*
  next
    case (*insert y Y*)
    then obtain *i j* **where** *Yi*: $Y \subseteq (F \frown\frown i)\ X$ **and** $y \in (F \frown\frown j)\ X$ **by** (*auto simp*: *Flim-def*)
    with *Fn-mono1*[*OF max.cobounded1*[*of i j*], *of X*] *Fn-mono1*[*OF max.cobounded2*[*of j i*], *of X*]
    show *?case* **by** (*auto intro*!: *exI*[*of - max i j*])
  qed
  then obtain *i* **where** *Yi*: $Y \subseteq (F \frown\frown i)\ X$ **by** *auto*
  with *Fn-carrier*[*OF assms*] **have** *YA*: $Y \subseteq A$ **by** *auto*
  from *Yi finY* **have** $f\ Y \in (F \frown\frown Suc\ i)\ X$ **by** (*auto simp*: *F-def Fpow-def*)
  then have $f\ Y \in F^\omega\ X$ **by** (*auto simp*: *Flim-def simp del*: *funpow.simps*)
  with *f-bound*[*OF YA finY*]
  show $\exists z \in F^\omega\ X$. *bound Y* ($\sqsubseteq$) *z* **by** *auto*
qed

lemma *Flim-card*: **assumes** *infinite X* **shows** $|F^\omega\ X| =o\ |X|$
proof−
  from *assms* **have** *natX*: $|UNIV :: nat\ set| \leq o\ |X|$ **by** (*simp add*: *infinite-iff-card-of-nat*)
  have $|F^\omega\ X| \leq o\ |X|$
    apply (*unfold Flim-def*, *rule card-of-UNION-ordLeq-infinite*[*OF assms natX*])
    using *Fn-card*[*OF assms*] *ordIso-imp-ordLeq*
    by *auto*
  with *Flim-infl* **show** $|F^\omega\ X| =o\ |X|$ **by** (*simp add*: *ordIso-iff-ordLeq*)
qed

lemma *Flim-fin*: **assumes** *finite X* **shows** $|F^\omega\ X| \leq o$ *natLeq*
proof−
  have $|F^\omega\ X| \leq o\ |UNIV :: nat\ set|$
    apply (*unfold Flim-def*)

apply (*rule card-of-UNION-ordLeq-infinite*)
by (*auto simp*: *Fn-fin*[*OF assms*] *intro*!: *ordLess-imp-ordLeq*)
then show *?thesis* **using** *card-of-nat ordLeq-ordIso-trans* **by** *auto*
qed

lemma *mono-uncountable*: *monotone-on A* ($\preceq_A$) ($\subseteq$) ($F^\omega \circ A_\prec$)
using *monotone-on-o*[*OF Flim-mono Pre-mono*]
by (*auto simp*: *o-def*)

lemma *card-uncountable*:
assumes *aA*: $a \in A$ **and** *unc*: *natLeq* $<o$ $|A|$
shows $|F^\omega\ (A_\prec\ a)|$ $<o$ $|A|$
proof (*cases finite* ($A_\prec\ a$))
case *True*
note *Flim-fin*[*OF this*]
also note *unc*
finally show *?thesis*
using *unc not-ordLess-ordIso* **by** *auto*
next
case *False*
note *Flim-card*[*OF this*]
also note *Pre-card*[*OF aA*]
finally show *?thesis* **using** *unc not-ordLess-ordIso* **by** *auto*
qed

lemma *in-I-uncountable*:
assumes *aA*: $a \in A$ **and** *inf*: *infinite A*
shows $\exists\, a' \in A.\ a \in F^\omega\ (A_\prec\ a')$
using *infinite-imp-ex-Pre*[*OF inf aA*] *Flim-infl*
by *auto*

lemma *carrier-uncountable*:
shows $F^\omega\ (A_\prec\ a) \subseteq A$
using *Flim-carrier*[*OF Pre-carrier*]
by *auto*

lemma *range-uncountable*: **assumes** *inf*: *infinite A* **shows** $\bigcup((F^\omega \circ A_\prec)$ ‘ $A) =$
A
proof (*safe intro*!: *subset-antisym*)
fix *a* **assume** *aA*: $a \in A$
from *infinite-imp-ex-Pre*[*OF inf aA*] *Flim-infl*
show $a \in \bigcup\ ((F^\omega \circ A_\prec)$ ‘ $A)$ **by** *auto*
show $x \in (F^\omega \circ A_\prec)\ a \Longrightarrow x \in A$ **for** *x*
using *carrier-uncountable* **by** *auto*
qed

lemma *infl-uncountable*:
assumes *aA*: $a \in A$ **and** *bA*: $b \in A$ **and** *ab*: $a \prec_A b$
shows $a \in F^\omega\ (A_\prec\ b)$

using *assms Flim-infl*[*of* $A_{\prec}$ *b*]
by (*auto simp*: *Pre-def*)

3.2 Countable Case

context
assumes *countable*: $|A|$ $=o$ *natLeq*
begin

The assumption above means that there exists an order-isomorphism between $(\mathbb{N}, \leq)$ and $(A, \preceq_A)$.

definition *seq* :: *nat* ⇒ *'a* **where** *seq* ≡ *SOME f*. *iso natLeq* $|A|$ *f*

lemma *seq-iso*: *iso natLeq* $|A|$ *seq*
apply (*unfold seq-def*)
apply (*rule someI-ex*[*of iso natLeq* $|A|$])
using *countable*[*THEN ordIso-symmetric*]
apply (*unfold ordIso-def*) **by** *auto*

lemma *seq-bij-betw*: *bij-betw seq UNIV A*
using *seq-iso* **by** (*auto simp*: *iso-def Field-natLeq*)

This means that A has been indexed by $\mathbb{N}$.

lemma *range-seq*: *range seq* = *A*
using *seq-bij-betw bij-betw-imp-surj-on* **by** *force*

lemma *seq-mono*: *monotone* $(\leq)$ $(\preceq_A)$ *seq*
using *iso-imp-compat*[*OF seq-iso*]
by (*auto intro*!: *monotoneI well-order-ofI simp*: *compat-def natLeq-def*)

lemma *inv-seq-mono*: *monotone-on A* $(\preceq_A)$ $(\leq)$ (*inv seq*)
using *iso-imp-compat-inv-into*[*OF seq-iso*]
unfolding *Field-natLeq*
by (*auto intro*!: *monotone-onI simp*: *natLeq-def compat-def well-order-of-def*)

We turn the sequence into a sequence of directed subsets of A:

fun *Seq* :: *nat* ⇒ *'a set* **where**
Seq 0 = {*f* {}}
| *Seq* (*Suc n*) = *Seq n* ∪ {*seq n*, *f* (*Seq n* ∪ {*seq n*})}

lemma *seq-n-in-Seq-n*: *seq n* ∈ *Seq* (*Suc n*) **by** *auto*

lemma *Seq-finite*: *finite* (*Seq n*)
by (*induction n*) *auto*

lemma *Seq-card*: $|$*Seq n*$|$ $<o$ $|A|$
using *countable Seq-finite* **by** (*simp add*: *ordIso-natLeq-infinite1*)

lemma *Seq-carrier*: *Seq n* ⊆ *A*

```
proof(induction n)
  case 0
  show ?case by (auto intro!: f-carrier)
next
  case (Suc n)
  with range-seq have sgA: Seq n ∪ {seq n} ⊆ A by auto
  from Seq-finite f-carrier[OF sgA]
  have f (Seq n ∪ {seq n}) ∈ A by auto
  with sgA show ?case by auto
qed

lemma Seq-range: ⋃(range Seq) = A
proof (intro equalityI)
  from Seq-carrier show ⋃(range Seq) ⊆ A by auto
  show A ⊆ ⋃(range Seq)
  proof
    fix a assume aA: a ∈ A
    with seq-bij-betw obtain n where a = seq n
      by (metis bij-betw-inv-into-right)
    with seq-n-in-Seq-n show a ∈ ⋃(range Seq) by (auto intro!: exI[of - Suc n])
  qed
qed

lemma Seq-extremed:
  assumes refl: reflexive A (⊑) shows extremed (Seq n) (⊑)
proof −
  interpret reflexive using refl.
  show ?thesis
  proof(induction n)
    case 0
    show ?case by (auto intro!: extremedI extremeI f-carrier)
  next
    case (Suc n)
    show ?case
    proof (intro extremedI extremeI)
      show f (Seq n ∪ {seq n}) ∈ Seq (Suc n) by auto
      fix x assume xssn: x ∈ Seq (Suc n)
      show x ⊑ f (Seq n ∪ {seq n})
      proof(cases x ∈ Seq n ∪ {seq n})
        case True
        with f-bound[of Seq n ∪ {seq n}] range-seq Seq-finite[of n]
          Seq-carrier[of n]
        show ?thesis by (auto simp: bound-def)
      next
        case False
        with xssn have x: x = f (Seq n ∪ {seq n}) by auto
        from range-seq Seq-finite[of n] Seq-carrier[of n]
        show ?thesis by (auto simp: x intro!: f-carrier)
      qed
```

    qed
  qed
qed

lemma *Seq-directed*: **assumes** *refl*: *reflexive A* (⊑) **shows** *directed-set* (*Seq n*) (⊑)
 using *Seq-extremed*[*OF refl*] **by** (*simp add*: *directed-set-iff-extremed*[*OF Seq-finite*])

lemma *range-countable*: $\bigcup$((*Seq* ∘ *inv seq*) ' *A*) = *A*
  apply (*fold image-comp*)
  apply (*unfold bij-betw-imp-surj-on*[*OF bij-betw-inv-into*[*OF seq-bij-betw*]])
  using *Seq-range*.

lemma *Seq-mono*: *mono Seq*
proof (*intro monoI*)
  show $n \le m \Longrightarrow$ *Seq n* ⊆ *Seq m* **for** *n m* **by** (*induct rule*:*inc-induct*, *auto*)
qed

lemma *mono-countable*: *monotone-on A* ($\preceq_A$) (⊆) (*Seq* ∘ *inv seq*)
  by (*rule monotone-on-o*[*OF Seq-mono inv-seq-mono*]) *auto*

lemma *infl-countable*:
  assumes *aA*: *a* ∈ *A* **and** *bA*: *b* ∈ *A* **and** *ab*: $a \prec_A b$
  shows *a* ∈ *Seq* (*inv seq b*)
proof−
  from *aA seq-bij-betw seq-n-in-Seq-n*
  have *a*: *a* ∈ *Seq* (*Suc* (*inv seq a*)) **by** (*simp add*: *bij-betw-inv-into-right*)
  from *ab* **have** *inv seq a* < *inv seq b*
   by (*metis* (*mono-tags*, *lifting*) *aA well-order-of.asympartp-iff-weak-neq bA range-seq inv-seq-mono inv-into-injective not-le-imp-less ord.mono-onD verit-la-disequality*)
  then have *Suc* (*inv seq a*) ≤ *inv seq b* **by** *auto*
  from *a monoD*[*OF Seq-mono this*] **have** *a* ∈ *Seq* (*inv seq b*) **by** *auto*
  then show *?thesis* **by** *auto*
qed

end

To match the types, we use the inverse *inv seq* of the isomorphism isaseq. We define the final *I* as follows:

definition *I* **where** *I* ≡ *if* |*A*| =*o natLeq then Seq* ∘ *inv seq else* F^ω ∘ $A_\prec$

lemma *I-carrier*: *I a* ⊆ *A*
  using *Seq-carrier carrier-uncountable* **by** (*auto simp*: *I-def*)

lemma *I-directed*: **assumes** *reflexive A* (⊑) **shows** *directed-set* (*I a*) (⊑)
  using *Seq-directed*[*OF - assms*] *Flim-directed*[*OF Pre-carrier*]
  by (*auto simp*: *I-def*)

lemma *I-mono*: *monotone-on A* ($\preceq_A$) (⊆) *I*
  by (*auto simp*: *mono-uncountable mono-countable I-def*)

lemma *I-card*:
  assumes *inf*: *infinite A* **and** *aA*: *a* ∈ *A*
  shows |*I a*| <*o* |*A*|
proof (*cases* |*A*| =*o natLeq*)
  case *True*
  with *Seq-finite*[*OF this*] **show** *?thesis* **by** (*simp add*: *I-def inf*)
next
  case *F*: *False*
  with *inf* **have** *natLeq* <*o* |*A*|
   by (*auto simp*: *infinite-iff-natLeq ordLeq-iff-ordLess-or-ordIso ordIso-symmetric*)
  from *card-uncountable*[*OF aA this*] **show** *?thesis* **by** (*auto simp*: *I-def F*)
qed

lemma *I-range*: **assumes** *inf*: *infinite A* **shows** ⋃(*I*‘*A*) = *A*
  using *range-uncountable*[*OF inf*] *range-countable* **by** (*auto simp*: *I-def*)

lemma *I-infl*: **assumes** *a* ∈ *A b* ∈ *A a* $\prec_A$ *b* **shows** *a* ∈ *I b*
  using *infl-countable infl-uncountable assms* **by** (*auto simp*: *I-def*)

end

Now we close the locale *Iwamura-proof* and state the final result in the global scope.

theorem (**in** *reflexive*) *Iwamura*:
  assumes *dir*: *directed-set A* (⊑) **and** *inf*: *infinite A*
  shows ∃ *I*. (∀ *a* ∈ *A*. *directed-set* (*I a*) (⊑) ∧ |*I a*| <*o* |*A*|) ∧
    monotone-on A ($\preceq_A$) (⊆) *I* ∧ ⋃(*I*‘*A*) = *A*
proof−
  interpret *Iwamura-proof* **using** *dir* **by** *unfold-locales*
  show *?thesis* **using** *I-mono I-card*[*OF inf*] *I-directed I-range*[*OF inf*]
   by (*auto intro*!: *exI*[*of - I*])
qed

4 Directed Completeness and Scott-Continuity

abbreviation *nonempty A* ≡ *if A* = {} *then* ⊥ *else* ⊤

lemma (**in** *quasi-ordered-set*) *directed-completeness-lemma*:
  fixes *leB* (**infix** ‹⊴› *50*)
  assumes *comp*: (*nonempty* ⊓ *well-related-set*)−*complete A* (⊑) **and** *dir*: *directed-set D* (⊑) **and** *DA*: *D* ⊆ *A*
  shows ∃ *s*. *extreme-bound A* (⊑) *D s*
   and *well-related-set*−*continuous A* (⊑) *B* (⊴) *f* ⟹
     D ≠ {} ⟹ *extreme-bound A* (⊑) *D t* ⟹ *extreme-bound B* (⊴) (*f* ‘ *D*) (*f t*)
proof (*atomize*(*full*), *insert wf-ordLess dir DA*, *induct* |*D*| *arbitrary*: *D t rule*: *wf-induct-rule*)
  interpret *less-eq-symmetrize***.**

```
  case less
  note this(1)
  note IH = this[THEN conjunct1]
    and IH2 = this[THEN conjunct2, rule-format]
  note DA = ‹D ⊆ A›
  interpret D: quasi-ordered-set D (⊑) using quasi-ordered-subset[OF DA].
  note dir = ‹directed-set D (⊑)›
  show ?case
  proof(cases finite D)
    case True
    from directed-set-iff-extremed[OF True] dir
     obtain d where dD: d ∈ D and exd: extreme D (⊑) d by (auto simp: ex-
tremed-def)
    then have dd: d ⊑ d by (auto simp: extreme-def)
    show ?thesis
    proof(intro conjI allI impI exI[of - d])
      from extreme-imp-extreme-bound[OF exd DA]
      show exbd: extreme-bound A (⊑) D d by auto
      assume f: well-related-set−continuous A (⊑) B (⊴) f
        and Dt: extreme-bound A (⊑) D t and D0: D ≠ {}
      from f[THEN continuous-carrierD] have fAB: f ‘ A ⊆ B by auto
      from Dt have tA: t ∈ A by auto
      show extreme-bound B (⊴) (f ‘ D) (f t)
      proof (safe intro!: extreme-boundI)
        from fAB tA show f t ∈ B by auto
        fix x assume xD: x ∈ D
        from xD Dt have xt: x ⊑ t by auto
        have monotone-on A (⊑) (⊴) f
          by (auto intro!: continuous-imp-monotone-on[OF f] pair-well-related)
        from monotone-onD[OF this] xD DA tA xt
        show f x ⊴ f t by (auto simp: bound-empty extreme-def)
      next
        fix b assume bound (f ‘ D) (⊴) b and bB: b ∈ B
        with dD have fdb: f d ⊴ b by auto
        from Dt exbd have dt: d ∼ t by (auto simp: extreme-bound-iff)
        from dD DA have dA: d ∈ A by auto
        with extreme-bound-sym-trans[OF - extreme-bound-singleton[OF dA] dt tA]
        have extreme-bound A (⊑) {d} t by auto
        from dD DA f[THEN continuousD, OF well-related-singleton-refl - - this]
        have exfdt: extreme-bound B (⊴) {f d} (f t) by auto
        from fdb bB exfdt show f t ⊴ b by auto
      qed
    qed
  next
    case inf: False
    from D.Iwamura[OF dir inf]
    obtain I where Imono: monotone-on D (⪯_D) (⊆) I
      and Icard: ∀ a ∈ D. |I a| <o |D|
      and Idir: ∀ a ∈ D. directed-set (I a) (⊑)
```

and *Irange*: ⋃ (*I* ‘ *D*) = *D*
by *auto*
have ∀ *d* ∈ *D*. ∃ *s*. *extreme-bound A* (⊑) (*I d*) *s*
proof *safe*
fix *d* **assume** *dD*: *d* ∈ *D*
with *Irange DA* **have** *IdA*: *I d* ⊆ *A* **by** *auto*
with *IH Icard Idir dD range DA*
show ∃ *s*. *extreme-bound A* (⊑) (*I d*) *s* **by** *auto*
qed
from *bchoice*[*OF this*]
obtain *s* **where** *s*: ⋀*d*. *d* ∈ *D* ⟹ *extreme-bound A* (⊑) (*I d*) (*s d*) **by** *auto*
then have *sDA*: *s* ‘ *D* ⊆ *A* **by** *auto*
have *smono*: *monotone-on D* ($\preceq_D$) (⊑) *s*
proof (*intro monotone-onI*)
fix *x y* **assume** *xD*: *x* ∈ *D* **and** *yD*: *y* ∈ *D* **and** *xy*: *x* $\preceq_D$ *y*
show *s x* ⊑ *s y*
apply (*rule extreme-bound-subset*[*OF monotone-onD*[*OF Imono xD yD xy*],
of A])
using *s xD yD* **by** *auto*
qed
from *well-order-of.monotone-image-well-related*[*OF this*]
have *wsD*: *well-related-set* (*s* ‘ *D*) (⊑).
from *inf* **have** *sD0*: *nonempty* (*s* ‘ *D*) (⊑) **by** *auto*
from *completeD*[*OF comp sDA*] *wsD sD0*
obtain *x* **where** *x*: *extreme-bound A* (⊑) (*s* ‘ *D*) *x* **by** *auto*
show *?thesis*
proof (*intro conjI allI impI exI*[*of* - *x*])
show *Dx*: *extreme-bound A* (⊑) *D x*
proof (*intro smono exI*[*of* - *x*] *extreme-boundI*)
from *x* **show** *xA*: *x* ∈ *A* **by** *auto*
fix *d* **assume** *dD*: *d* ∈ *D*
with *Irange* **obtain** *d′* **where** *d′D*: *d′* ∈ *D* **and** *d* ∈ *I d′* **by** *auto*
with *s* **have** *1*: *d* ⊑ *s d′* **by** *auto*
from *x d′D* **have** *2*: ... ⊑ *x* **by** *auto*
from *trans*[*OF 1 2*] **show** *d* ⊑ *x* **using** *dD sDA d′D DA xA* **by** *auto*
next
fix *b* **assume** *bA*: *b* ∈ *A* **and** *Db*: *bound D* (⊑) *b*
have *bound* (*s* ‘ *D*) (⊑) *b*
proof *safe*
fix *d* **assume** *dD*: *d* ∈ *D*
from *dD Db Irange* **have** *bound* (*I d*) (⊑) *b* **by** *auto*
with *s dD bA* **show** *s d* ⊑ *b* **by** *auto*
qed
with *x bA* **show** *x* ⊑ *b* **by** *auto*
qed
assume *f*: *well-related-set−continuous A* (⊑) *B* (⊴) *f*
and *Dt*: *extreme-bound A* (⊑) *D t* **and** *D0*: *D* ≠ {}
from *Dt* **have** *tA*: *t* ∈ *A* **by** *auto*
have *fmono*: *monotone-on A* (⊑) (⊴) *f*

```
        by (auto intro!:continuous-imp-monotone-on[OF f] pair-well-related)
      show extreme-bound B (⊴) (f ‘ D) (f t)
      proof (safe intro!: extreme-boundI)
        from f tA show f t ∈ B by auto
        fix d assume dD: d ∈ D
        from dD Dt have dt: d ⊑ t by auto
       from dD Dt DA show f d ⊴ f t by (auto intro!: monotone-onD[OF fmono])
      next
        fix b assume fDb: bound (f ‘ D) (⊴) b and bB: b ∈ B
        from Dx Dt have x ∼ t by (auto intro!: sympartpI elim!: extreme-boundE)
        with extreme-bound-sym-trans[OF sDA x this tA]
        have extreme-bound A (⊑) (s ‘ D) t by auto
        from f[THEN continuousD, OF wsD - sDA this] D0
        have ft: extreme-bound B (⊴) (f ‘ s ‘ D) (f t) by auto
        have bound (f ‘ s ‘ D) (⊴) b
        proof (safe)
          fix d assume dD: d ∈ D
          from Irange dD have IdD: I d ⊆ D by auto
          with DA have IdA: I d ⊆ A by auto
          from directed-setD[OF Idir[rule-format, OF dD], of {}]
          have Idne: I d ≠ {} by auto
          have fsd: extreme-bound B (⊴) (f ‘ I d) (f (s d))
            apply (rule IH2[OF - - IdA f Idne s[OF dD]])
            using Icard Idir dD by auto
          from IdD have f ‘ I d ⊆ f ‘ D by auto
          from bound-subset[OF this fDb] fsd bB
          show f (s d) ⊴ b by auto
        qed
        with ft bB show f t ⊴ b by auto
      qed
    qed
  qed
qed
```

The next Theorem corresponds to Proposition 5.9 of [4], without antisymmetry on A.

```
theorem (in quasi-ordered-set) well-complete-iff-directed-complete:
  (nonempty ⊓ well-related-set)−complete A (⊑) ⟷ directed-set−complete A (⊑)
(is ?l ⟷ ?r)
proof (intro iffI)
  show ?l ⟹ ?r
    by (auto intro!: completeI dest!: directed-completeness-lemma(1))
  assume r: ?r
  show ?l
    apply (rule complete-subclass[OF r])
    using well-related-set.directed-set
    by auto
qed
```

The next Theorem corresponds to Corollary 3 of [9] without any assumptions on the codomain B and without antisymmetry on the domain A.

theorem (**in** *quasi-ordered-set*)
  fixes *leB* (**infix** ‹⊴› *50*)
  assumes *comp*: (*nonempty* ⊓ *well-related-set*)−*complete* *A* (⊑)
  shows *well-related-set*−*continuous* *A* (⊑) *B* (⊴) *f* ⟷ *directed-set*−*continuous* *A* (⊑) *B* (⊴) *f*
    (**is** *?l* ⟷ *?r*)
proof (*intro iffI*)
  assume *l*: *?l*
  show *?r*
    using *continuous-carrierD*[*OF l*]
    using *directed-completeness-lemma*(*2*)[*OF comp* - - *l*]
    by (*auto intro*!: *continuousI*)
next
  assume *r*: *?r*
  show *?l*
    apply (*rule continuous-subclass*[*OF* - *r*])
    using *well-related-set.directed-set* **by** *auto*
qed

end

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