Differential-Game-Logic

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Abstract

This formalization provides differential game logic (dGL), a logic for proving properties of hybrid game. In addition to the syntax and semantics, it formalizes a uniform substitution calculus for dGL. Church's uniform substitutions substitute a term or formula for a function or predicate symbol everywhere. The uniform substitutions for dGL also substitute hybrid games for a game symbol everywhere. We prove soundness of one-pass uniform substitutions and the axioms of differential game logic with respect to their denotational semantics. One-pass uniform substitutions are faster by postponing soundness-critical admissibility checks with a linear pass homomorphic application and regain soundness by a variable condition at the replacements.

The formalization is based on prior non-mechanized soundness proofs for dGL [1, 2, 4, 1, 3]. This AFP entry formalizes the mathematical proofs [4, 5] till Theorem 19.

Contents

This formalization provides *Differential Game Logic* dGL [5, 4] till Theorem 19, including the corresponding results from [2] till Lemma 13. Differential Game Logic originates from [1].

theory *Lib*
imports
Complex-Main
begin

1 Generic Mathematical Background Lemmas

lemma *finite-subset* [*simp*]: *finite M* $\Longrightarrow$ *finite* $\{x{\in}M.\ P\ x\}$
by *simp*

lemma *finite-powerset* [*simp*]: *finite M* $\Longrightarrow$ *finite* $\{S.\ S{\subseteq}M\}$
by *simp*

definition *fst-proj*:: $('a{*}'b)\ set \Rightarrow 'a\ set$
where *fst-proj M* $\equiv \{A.\ \exists\, B.\ (A,B){\in}M\}$

definition *snd-proj*:: $('a{*}'b)\ set \Rightarrow 'b\ set$
where *snd-proj M* $\equiv \{B.\ \exists\, A.\ (A,B){\in}M\}$

lemma *fst-proj-mem* [*simp*]: $(A \in$ *fst-proj M*$) = (\exists\, B.\ (A,B){\in}M)$
unfolding *fst-proj-def* **by** *auto*

lemma *snd-proj-mem* [*simp*]: $(B \in$ *snd-proj M*$) = (\exists\, A.\ (A,B){\in}M)$
unfolding *snd-proj-def* **by** *auto*

lemma *fst-proj-prop*: $\forall\, x{\in}$*fst-proj* $\{(A,B)|\ A\ B.\ P\ A \wedge R\ A\ B\}.\ P(x)$
unfolding *fst-proj-def* **by** *auto*

lemma *snd-proj-prop*: $\forall\, x{\in}$*snd-proj* $\{(A,B)\mid A\ B.\ P\ B \wedge R\ A\ B\}.\ P(x)$
unfolding *snd-proj-def* **by** *auto*

lemma *map-cons*: *map f* (*Cons x xs*) = *Cons* (*f x*) (*map f xs*)
by (*rule List.list.map*)
lemma *map-append*: *map f* (*append xs ys*) = *append* (*map f xs*) (*map f ys*)
by *simp*

Lockstep induction schema for two simultaneous least fixpoints. If the successor step and supremum step of two least fixpoint inflations preserve a relation, then that relation holds of the two respective least fixpoints.

lemma *lfp-lockstep-induct* [*case-names monof monog step union*]:
fixes *f* :: *'a*::*complete-lattice* $\Rightarrow$ *'a*
and *g* :: *'b*::*complete-lattice* $\Rightarrow$ *'b*
assumes *monof*: *mono f*

and *monog*: *mono g*
and *R-step*: $\bigwedge A\ B.\ A \leq lfp(f) \Longrightarrow B \leq lfp(g) \Longrightarrow R\ A\ B \Longrightarrow R\ (f(A))\ (g(B))$
and *R-Union*: $\bigwedge M{::}('a*'b)\ set.\ (\forall\,(A,B){\in}M.\ R\ A\ B) \Longrightarrow R\ (Sup\ (fst\text{-}proj\ M))\ (Sup\ (snd\text{-}proj\ M))$
shows *R* (*lfp f*) (*lfp g*)
proof−

let *?M* = $\{(A,B).\ A \leq lfp\ f \wedge B \leq lfp\ g \wedge R\ A\ B\}$
from *R-Union* **have** *supdoes*: *R* (*Sup* (*fst-proj ?M*)) (*Sup* (*snd-proj ?M*)) **by** *simp*
also have *Sup* (*fst-proj ?M*) = *lfp f* **and** *Sup* (*snd-proj ?M*) = *lfp g*
proof (*rule antisym*)
show *fle*: *Sup* (*fst-proj ?M*) ≤ *lfp f*
using *fst-proj-prop Sup-le-iff* **by** *fastforce*
then have *f* (*Sup* (*fst-proj ?M*)) ≤ *f* (*lfp f*)
by (*rule monof* [*THEN monoD*])
then have *fsup*: *f* (*Sup* (*fst-proj ?M*)) ≤ *lfp f*
using *monof* [*THEN lfp-unfold*] **by** *simp*

have *gle*: *Sup* (*snd-proj ?M*) ≤ *lfp g*
using *snd-proj-prop Sup-le-iff* **by** *fastforce*
then have *g* (*Sup* (*snd-proj ?M*)) ≤ *g* (*lfp g*)
by (*rule monog* [*THEN monoD*])
then have *gsup*: *g* (*Sup* (*snd-proj ?M*)) ≤ *lfp g*
using *monog* [*THEN lfp-unfold*] **by** *simp*

from *fsup* **and** *gsup* **have** *fgsup*: (*f*(*Sup*(*fst-proj ?M*)), *g*(*Sup*(*snd-proj ?M*))) ∈ *?M*
using *R-Union R-step Sup-le-iff*
using *calculation fle gle* **by** *blast*

from *fgsup* **have** *f* (*Sup* (*fst-proj ?M*)) ≤ *Sup* (*fst-proj ?M*)
using *Sup-upper* **by** (*metis* (*mono-tags*, *lifting*) *fst-proj-def mem-Collect-eq*)
then show *fge*: *lfp f* ≤ *Sup* (*fst-proj ?M*)
by (*rule lfp-lowerbound*)
show *Sup* (*snd-proj ?M*) = *lfp g*
proof (*rule antisym*)
show *Sup* (*snd-proj ?M*) ≤ *lfp g* **by** (*rule gle*)
from *fgsup* **have** *g* (*Sup* (*snd-proj ?M*)) ≤ *Sup* (*snd-proj ?M*)
using *Sup-upper* **by** (*metis* (*mono-tags*, *lifting*) *snd-proj-def mem-Collect-eq*)
then show *gge*: *lfp g* ≤ *Sup* (*snd-proj ?M*)
by (*rule lfp-lowerbound*)
qed
qed
then show *?thesis* **using** *supdoes* **by** *simp*
qed

lemma *sup-eq-all*: $(\bigwedge A.\ (A{\in}M \Longrightarrow f(A){=}g(A)))$

$\Longrightarrow$ *Sup* $\{f(A) \mid A.\ A{\in}M\}$ = *Sup* $\{g(A) \mid A.\ A{\in}M\}$
by *metis*

lemma *sup-corr-eq-chain*: $\bigwedge M{::}('a{::}complete\text{-}lattice{*}'a)\ set.\ (\forall\,(A,B){\in}M.\ f(A){=}g(B))$
$\Longrightarrow$ (*Sup* $\{f(A) \mid A.\ A{\in}fst\text{-}proj\ M\}$ = *Sup* $\{g(B) \mid B.\ B{\in}snd\text{-}proj\ M\}$)
by (*metis* (*mono-tags*, *lifting*) *case-prod-conv fst-proj-mem snd-proj-mem*)

end
theory *Identifiers*
imports *Complex-Main*
begin

1.1 Identifier Namespace Configuration

Different configurations are possible for the namespace of identifiers. Finite support is the only important aspect of it.

type-synonym *ident* = *char*

The identifier used for the replacement marker in uniform substitutions

abbreviation *dotid*:: *ident*
where *dotid* ≡ *CHR* ''.''

end
theory *Syntax*
imports
Complex-Main
Identifiers
begin

2 Syntax

Defines the syntax of Differential Game Logic as inductively defined data types. https://doi.org/10.1145/2817824 https://doi.org/10.1007/978-3-319-94205-6_15

2.1 Terms

Numeric literals

type-synonym *lit* = *real*

the set of all real variables

abbreviation *allidents*:: *ident set*
where *allidents* ≡ $\{x \mid x.\ True\}$

Variables and differential variables

datatype *variable* =

RVar ident
| *DVar ident*

datatype *trm* =
Var variable
| *Number lit*
| *Const ident*
| *Func ident trm*
| *Plus trm trm*
| *Times trm trm*
| *Differential trm*

2.2 Formulas and Hybrid Games

datatype *fml* =
Pred ident trm
| *Geq trm trm*
| *Not fml* (!)
| *And fml fml* (**infixr** &\& *8*)
| *Exists variable fml*
| *Diamond game fml* ((⟨ - ⟩ -) *20*)
and *game* =
Game ident
| *Assign variable trm* (**infixr** := *20*)
| *Test fml* (*?*)
| *Choice game game* (**infixr** ∪∪ *10*)
| *Compose game game* (**infixr** ;; *8*)
| *Loop game* (-∗∗)
| *Dual game* (-^*d*)
| *ODE ident trm*

Derived operators **definition** *Neg* ::*trm* ⇒ *trm*
where *Neg ϑ* = *Times* (*Number* (−*1*)) *ϑ*

definition *Minus* ::*trm* ⇒ *trm* ⇒ *trm*
where *Minus ϑ η* = *Plus ϑ* (*Neg η*)

definition *Or* :: *fml* ⇒ *fml* ⇒ *fml* (**infixr** || *7*)
where *Or P Q* = *Not* (*And* (*Not P*) (*Not Q*))

definition *Implies* :: *fml* ⇒ *fml* ⇒ *fml* (**infixr** → *10*)
where *Implies P Q* = *Or Q* (*Not P*)

definition *Equiv* :: *fml* ⇒ *fml* ⇒ *fml* (**infixr** ↔ *10*)
where *Equiv P Q* = *Or* (*And P Q*) (*And* (*Not P*) (*Not Q*))

definition *Forall* :: *variable* ⇒ *fml* ⇒ *fml*
where *Forall x P* = *Not* (*Exists x* (*Not P*))

definition *Equals* :: *trm* ⇒ *trm* ⇒ *fml*
where *Equals* ϑ ϑ′ = ((*Geq* ϑ ϑ′) && (*Geq* ϑ′ ϑ))

definition *Greater* :: *trm* ⇒ *trm* ⇒ *fml*
where *Greater* ϑ ϑ′ = ((*Geq* ϑ ϑ′) && (*Not* (*Geq* ϑ′ ϑ)))

Justification: determinacy theorem justifies this equivalent syntactic abbreviation for box modalities from diamond modalities Theorem 3.1 https://doi.org/10.1145/2817824

definition *Box* :: *game* ⇒ *fml* ⇒ *fml* (([[-]]-) *20*)
where *Box* α *P* = *Not* (*Diamond* α (*Not P*))

definition *TT* ::*fml*
where *TT* = *Geq* (*Number 0*) (*Number 0*)

definition *FF* ::*fml*
where *FF* = *Geq* (*Number 0*) (*Number 1*)

definition *Skip* ::*game*
where *Skip* = *Test TT*

Inference: premises, then conclusion

type-synonym *inference* = *fml list* ∗ *fml*

type-synonym *sequent* = *fml list* ∗ *fml list*

Rule: premises, then conclusion

type-synonym *rule* = *sequent list* ∗ *sequent*

2.3 Structural Induction

Induction principles for hybrid games owing to their mutually recursive definition with formulas

lemma *game-induct* [*case-names Game Assign ODE Test Choice Compose Loop Dual*]:
(⋀*a*. *P* (*Game a*))
⟹ (⋀*x* ϑ. *P* (*Assign x* ϑ))
⟹ (⋀*x* ϑ. *P* (*ODE x* ϑ))
⟹ (⋀φ. *P* (*?* φ))
⟹ (⋀α β. *P* α ⟹ *P* β ⟹ *P* (α ∪∪ β))
⟹ (⋀α β. *P* α ⟹ *P* β ⟹ *P* (α ;; β))
⟹ (⋀α. *P* α ⟹ *P* (α∗∗))
⟹ (⋀α. *P* α ⟹ *P* (α^*d*))
⟹ *P* α
by(*induction rule*: *game.induct*) (*auto*)

lemma *fml-induct* [*case-names Pred Geq Not And Exists Diamond*]:
(⋀*x* ϑ. *P* (*Pred x* ϑ))

$\Longrightarrow (\bigwedge\vartheta\ \eta.\ P\ (Geq\ \vartheta\ \eta))$
$\Longrightarrow (\bigwedge\varphi.\ P\ \varphi \Longrightarrow P\ (Not\ \varphi))$
$\Longrightarrow (\bigwedge\varphi\ \psi.\ P\ \varphi \Longrightarrow P\ \psi \Longrightarrow P\ (And\ \varphi\ \psi))$
$\Longrightarrow (\bigwedge x\ \varphi.\ P\ \varphi \Longrightarrow P\ (Exists\ x\ \varphi))$
$\Longrightarrow (\bigwedge\alpha\ \varphi.\ P\ \varphi \Longrightarrow P\ (Diamond\ \alpha\ \varphi))$
$\Longrightarrow P\ \varphi$
by (*induction rule*: *fml.induct*) (*auto*)

the set of all variables

abbreviation *allvars*:: *variable set*
  where *allvars* ≡ {*x*::*variable*. *True*}

end
theory *Denotational-Semantics*
imports
  HOL−Analysis.Derivative
  Syntax
begin

3 Denotational Semantics

Defines the denotational semantics of Differential Game Logic. https://doi.org/10.1145/2817824 https://doi.org/10.1007/978-3-319-94205-6_15

3.1 States

Vector of reals over ident

type-synonym *Rvec* = *variable* ⇒ *real*
type-synonym *state* = *Rvec*

the set of all worlds

definition *worlds*:: *state set*
  where *worlds* = {*ν*. *True*}

the set of all variables

abbreviation *allvars*:: *variable set*
  where *allvars* ≡ {*x*::*variable*. *True*}

the set of all real variables

abbreviation *allrvars*:: *variable set*
  where *allrvars* ≡ {*RVar x* | *x*. *True*}

the set of all differential variables

abbreviation *alldvars*:: *variable set*
  where *alldvars* ≡ {*DVar x* | *x*. *True*}

lemma *ident-finite*: *finite*({*x*::*ident*. *True*})
by *auto*

lemma *allvar-cases*: *allvars* = *allrvars* ∪ *alldvars*
using *variable.exhaust* **by** *blast*

lemma *rvar-finite*: *finite allrvars*
using *finite-imageI*[*OF ident-finite*, **where** *h*=‹λ*x*. *RVar x*›] **by** (*simp add*: *full-SetCompr-eq*)

lemma *dvar-finite*: *finite alldvars*
using *finite-imageI*[*OF ident-finite*, **where** *h*=‹λ*x*. *DVar x*›] **by** (*simp add*: *full-SetCompr-eq*)

lemma *allvars-finite* [*simp*]: *finite*(*allvars*)
using *allvar-cases dvar-finite rvar-finite* **by** (*metis finite-Un*)

definition *Vagree* :: *state* ⇒ *state* ⇒ *variable set* ⇒ *bool*
where *Vagree* ν ν′ *V* ≡ (∀ *i*. *i*∈*V* ⟶ ν(*i*) = ν′(*i*))

definition *Uvariation* :: *state* ⇒ *state* ⇒ *variable set* ⇒ *bool*
where *Uvariation* ν ν′ *U* ≡ (∀ *i*. ~(*i*∈*U*) ⟶ ν(*i*) = ν′(*i*))

lemma *Uvariation-Vagree* [*simp*]: *Uvariation* ν ν′ (−*V*) = *Vagree* ν ν′ *V*
unfolding *Vagree-def Uvariation-def* **by** *simp*

lemma *Vagree-refl* [*simp*]: *Vagree* ν ν *V*
by (*auto simp add*: *Vagree-def*)

lemma *Vagree-sym*: *Vagree* ν ν′ *V* = *Vagree* ν′ ν *V*
by (*auto simp add*: *Vagree-def*)

lemma *Vagree-sym-rel* [*sym*]: *Vagree* ν ν′ *V* ⟹ *Vagree* ν′ ν *V*
using *Vagree-sym* **by** *auto*

lemma *Vagree-union* [*trans*]: *Vagree* ν ν′ *V* ⟹ *Vagree* ν ν′ *W* ⟹ *Vagree* ν ν′ (*V*∪*W*)
by (*auto simp add*: *Vagree-def*)

lemma *Vagree-trans* [*trans*]: *Vagree* ν ν′ *V* ⟹ *Vagree* ν′ ν″ *W* ⟹ *Vagree* ν ν″ (*V*∩*W*)
by (*auto simp add*: *Vagree-def*)

lemma *Vagree-antimon* [*simp*]: *Vagree* ν ν′ *V* ∧ *W*⊆*V* ⟶ *Vagree* ν ν′ *W*
by (*auto simp add*: *Vagree-def*)

lemma *Vagree-empty* [*simp*]: *Vagree* ν ν' {}
by (*auto simp add*: *Vagree-def*)

lemma *Uvariation-empty* [*simp*]: *Uvariation* ν ν' {} = (ν=ν')
by (*auto simp add*: *Uvariation-def*)

lemma *Vagree-univ* [*simp*]: *Vagree* ν ν' *allvars* = (ν=ν')
by (*auto simp add*: *Vagree-def*)

lemma *Uvariation-univ* [*simp*]: *Uvariation* ν ν' *allvars*
by (*auto simp add*: *Uvariation-def*)

lemma *Vagree-and* [*simp*]: *Vagree* ν ν' V $\wedge$ *Vagree* ν ν' W $\longleftrightarrow$ *Vagree* ν ν' ($V \cup W$)
by (*auto simp add*: *Vagree-def*)

lemma *Vagree-or*: *Vagree* ν ν' V $\vee$ *Vagree* ν ν' W $\longrightarrow$ *Vagree* ν ν' ($V \cap W$)
by (*auto simp add*: *Vagree-def*)

lemma *Uvariation-refl* [*simp*]: *Uvariation* ν ν V
by (*auto simp add*: *Uvariation-def*)

lemma *Uvariation-sym*: *Uvariation* ω ν U = *Uvariation* ν ω U
unfolding *Uvariation-def* **by** *auto*

lemma *Uvariation-sym-rel* [*sym*]: *Uvariation* ω ν U $\Longrightarrow$ *Uvariation* ν ω U
using *Uvariation-sym* **by** *auto*

lemma *Uvariation-trans* [*trans*]: *Uvariation* ω ν U $\Longrightarrow$ *Uvariation* ν μ V $\Longrightarrow$ *Uvariation* ω μ ($U \cup V$)
unfolding *Uvariation-def* **by** *simp*

lemma *Uvariation-mon* [*simp*]: $V \supseteq U$ $\Longrightarrow$ *Uvariation* ω ν U $\Longrightarrow$ *Uvariation* ω ν V
unfolding *Uvariation-def* **by** *auto*

3.2 Interpretations

lemma *mon-mono*: *mono* r = (($\forall$ X Y. ($X \subseteq Y$ $\longrightarrow$ $r(X) \subseteq r(Y)$))))
unfolding *mono-def* **by** *simp*

interpretations of symbols in ident

type-synonym *interp-rep* =
(*ident* $\Rightarrow$ *real*) $\times$ (*ident* $\Rightarrow$ (*real* $\Rightarrow$ *real*)) $\times$ (*ident* $\Rightarrow$ (*real* $\Rightarrow$ *bool*)) $\times$ (*ident* $\Rightarrow$ (*state set* $\Rightarrow$ *state set*))

definition *is-interp* :: *interp-rep* $\Rightarrow$ *bool*
where *is-interp* $I \equiv$ *case* I *of* (-, -, -, G) $\Rightarrow$ ($\forall$ a. *mono* (G a))

typedef *interp* = {*I*:: *interp-rep*. *is-interp I*}
  morphisms *raw-interp well-interp*
proof
 show (λ*f*. *0*, λ*f x*. *0*, λ*p x*. *True*, λ*a*. λ*X*. *X*) ∈ {*I*. *is-interp I*} **unfolding** *is-interp-def mono-def* **by** *simp*
qed

setup-lifting *type-definition-interp*

lift-definition *Consts*::*interp* ⇒ *ident* ⇒ (*real*) **is** λ(*F0*, -, -, -). *F0* **.**
lift-definition *Funcs*:: *interp* ⇒ *ident* ⇒ (*real* ⇒ *real*) **is** λ(-, *F*, -, -). *F* **.**
lift-definition *Preds*:: *interp* ⇒ *ident* ⇒ (*real* ⇒ *bool*) **is** λ(-, -, *P*, -). *P* **.**
lift-definition *Games*:: *interp* ⇒ *ident* ⇒ (*state set* ⇒ *state set*) **is** λ(-, -, -, *G*). *G* **.**

make interpretations

lift-definition *mkinterp*:: (*ident* ⇒ *real*) × (*ident* ⇒ (*real* ⇒ *real*)) × (*ident* ⇒ (*real* ⇒ *bool*)) × (*ident* ⇒ (*state set* ⇒ *state set*))
⇒ *interp*
  is λ(*C*, *F*, *P*, *G*). *if* ∀ *a*. *mono* (*G a*) *then* (*C*, *F*, *P*, *G*) *else* (*C*, *F*, *P*, λ- -. {})
  by (*auto split*: *prod.splits simp*: *mono-def is-interp-def*)

lemma *Consts-mkinterp* [*simp*]: *Consts* (*mkinterp*(*C*,*F*,*P*,*G*)) = *C*
  apply (*transfer fixing*: *C F P G*)
  apply (*auto simp add*: *is-interp-def mono-def*)
  done

lemma *Funcs-mkinterp* [*simp*]: *Funcs* (*mkinterp*(*C*,*F*,*P*,*G*)) = *F*
  apply (*transfer fixing*: *C F P G*)
  apply (*auto simp add*: *is-interp-def mono-def*)
  done

lemma *Preds-mkinterp* [*simp*]: *Preds* (*mkinterp*(*C*,*F*,*P*,*G*)) = *P*
  apply (*transfer fixing*: *C F P G*)
  apply (*auto simp add*: *is-interp-def mono-def*)
  done

lemma *Games-mkinterp* [*simp*]: (⋀*a*. *mono* (*G a*)) ⟹ *Games* (*mkinterp*(*C*,*F*,*P*,*G*))
= *G*
  apply (*transfer fixing*: *C F P G*)
  apply (*auto simp add*: *is-interp-def mono-def*)
  done

lemma *mkinterp-eq* [*iff*]: (*Consts I* = *Consts J* ∧ *Funcs I* = *Funcs J* ∧ *Preds I*
= *Preds J* ∧ *Games I* = *Games J*) = (*I*=*J*)
  apply (*transfer fixing*: *C F P G*)
  apply (*auto simp add*: *is-interp-def mono-def*)
  done

lemma [*simp*]: $X \subseteq Y \Longrightarrow (Games\ I\ a)(X) \subseteq (Games\ I\ a)(Y)$
apply (*transfer fixing*: *a X Y*)
apply (*auto simp add*: *is-interp-def mono-def*)
apply (*blast*)
done

lifting-update *interp.lifting*
lifting-forget *interp.lifting*

3.3 Semantics

Semantic modification *repv* ω *x r* replaces the value of variable x in the state ω with r

definition *repv* :: *state* ⇒ *variable* ⇒ *real* ⇒ *state*
where *repv* ω *x r* = *fun-upd* ω *x r*

lemma *repv-def-correct*: *repv* ω *x r* = $(\lambda y.$ *if x* = *y then r else* $\omega(y))$
unfolding *repv-def* **by** *auto*

lemma *repv-access* [*simp*]: (*repv* ω *x r*)(*y*) = (*if* (*x*=*y*) *then r else* $\omega(y)$)
unfolding *repv-def* **by** *simp*

lemma *repv-self* [*simp*]: *repv* ω *x* $(\omega(x))$ = ω
unfolding *repv-def* **by** *auto*

lemma *Vagree-repv*: *Vagree* ω (*repv* ω *x d*) $(-\{x\})$
unfolding *repv-def Vagree-def* **by** *simp*

lemma *Vagree-repv-self*: *Vagree* ω (*repv* ω *x d*) $\{x\}$ = (*d*=$\omega(x)$)
unfolding *repv-def Vagree-def* **by** *auto*

lemma *Uvariation-repv*: *Uvariation* ω (*repv* ω *x d*) $\{x\}$
unfolding *repv-def Uvariation-def* **by** *simp*

Semantics of Terms **fun** *term-sem* :: *interp* ⇒ *trm* ⇒ (*state* ⇒ *real*)
where
term-sem I (*Var x*) = $(\lambda\omega.\ \omega(x))$
| *term-sem I* (*Number r*) = $(\lambda\omega.\ r)$
| *term-sem I* (*Const f*) = $(\lambda\omega.$ (*Consts I f*))
| *term-sem I* (*Func f* ϑ) = $(\lambda\omega.$ (*Funcs I f*)(*term-sem I* ϑ ω))
| *term-sem I* (*Plus* ϑ η) = $(\lambda\omega.$ *term-sem I* ϑ ω + *term-sem I* η ω)
| *term-sem I* (*Times* ϑ η) = $(\lambda\omega.$ *term-sem I* ϑ ω ∗ *term-sem I* η ω)
| *term-sem I* (*Differential* ϑ) = $(\lambda\omega.$ *sum*$(\lambda x.\ \omega(DVar\ x)$∗*deriv*$(\lambda X.$ *term-sem I* ϑ (*repv* ω (*RVar x*) *X*))$(\omega(RVar\ x)))(allidents))$

Solutions of Differential Equations **type-synonym** *solution* = *real* ⇒ *state*

definition *solves-ODE* :: *interp* ⇒ *solution* ⇒ *ident* ⇒ *trm* ⇒ *bool*
where *solves-ODE I F x ϑ* ≡ (∀ζ::*real*.
Vagree (*F*(0)) (*F*(ζ)) (−{*RVar x*, *DVar x*})
∧ *F*(ζ)(*DVar x*) = *deriv*(λ*t*. *F*(*t*)(*RVar x*))(ζ)
∧ *F*(ζ)(*DVar x*) = *term-sem I ϑ* (*F*(ζ)))

Semantics of Formulas and Games **fun** *fml-sem* :: *interp* ⇒ *fml* ⇒ (*state set*) **and**
game-sem :: *interp* ⇒ *game* ⇒ (*state set* ⇒ *state set*)
where
fml-sem I (*Pred p ϑ*) = {ω. (*Preds I p*)(*term-sem I ϑ* ω)}
| *fml-sem I* (*Geq ϑ η*) = {ω. *term-sem I ϑ* ω ≥ *term-sem I η* ω}
| *fml-sem I* (*Not* φ) = {ω. ω ∉ *fml-sem I* φ}
| *fml-sem I* (*And* φ ψ) = *fml-sem I* φ ∩ *fml-sem I* ψ
| *fml-sem I* (*Exists x* φ) = {ω. ∃*r*. (*repv* ω *x r*) ∈ *fml-sem I* φ}
| *fml-sem I* (*Diamond* α φ) = *game-sem I* α (*fml-sem I* φ)

| *game-sem I* (*Game a*) = (λ*X*. (*Games I a*)(*X*))
| *game-sem I* (*Assign x ϑ*) = (λ*X*. {ω. (*repv* ω *x* (*term-sem I ϑ* ω)) ∈ *X*})
| *game-sem I* (*Test* φ) = (λ*X*. *fml-sem I* φ ∩ *X*)
| *game-sem I* (*Choice* α β) = (λ*X*. *game-sem I* α *X* ∪ *game-sem I* β *X*)
| *game-sem I* (*Compose* α β) = (λ*X*. *game-sem I* α (*game-sem I* β *X*))
| *game-sem I* (*Loop* α) = (λ*X*. ⋂{*Z*. *X* ∪ *game-sem I* α *Z* ⊆ *Z*})
| *game-sem I* (*Dual* α) = (λ*X*. −(*game-sem I* α (−*X*)))
| *game-sem I* (*ODE x ϑ*) = (λ*X*. {ω. ∃*F T*. *Vagree* ω (*F*(0)) (−{*DVar x*}) ∧ *F*(*T*) ∈ *X* ∧ *solves-ODE I F x ϑ*})

Validity

definition *valid-in* :: *interp* ⇒ *fml* ⇒ *bool*
where *valid-in I* φ ≡ (∀ω. ω ∈ *fml-sem I* φ)

definition *valid* :: *fml* ⇒ *bool*
where *valid* φ ≡ (∀*I*.∀ω. ω ∈ *fml-sem I* φ)

lemma *valid-is-valid-in-all*: *valid* φ = (∀*I*. *valid-in I* φ)
unfolding *valid-def valid-in-def* **by** *auto*

definition *locally-sound* :: *inference* ⇒ *bool*
where *locally-sound R* ≡
(∀*I*. (∀*k*. 0≤*k* ⟶ *k*<*length* (*fst R*) ⟶ *valid-in I* (*nth* (*fst R*) *k*)) ⟶ *valid-in I* (*snd R*))

definition *sound* :: *inference* ⇒ *bool*
where *sound R* ≡
(∀*k*. 0≤*k* ⟶ *k*<*length* (*fst R*) ⟶ *valid* (*nth* (*fst R*) *k*)) ⟶ *valid* (*snd R*)

lemma *locally-sound-is-sound*: *locally-sound R* ⟹ *sound R*
unfolding *locally-sound-def sound-def* **using** *valid-is-valid-in-all* **by** *auto*

3.4 Monotone Semantics

lemma *monotone-Test* [*simp*]: $X \subseteq Y \Longrightarrow$ *game-sem I* (*Test* φ) $X \subseteq$ *game-sem I* (*Test* φ) Y
by *auto*

lemma *monotone* [*simp*]: $X \subseteq Y \Longrightarrow$ *game-sem I* α $X \subseteq$ *game-sem I* α Y
proof (*induction* α *arbitrary*: X Y *rule*: *game-induct*)
case (*Game a*)
then show *?case* **by** *simp*
next
case (*Assign x* ϑ)
then show *?case* **by** *auto*
next
case (*Test* φ)
then show *?case* **by** *auto*
next
case (*Choice* $\alpha 1$ $\alpha 2$)
then show *?case* **by** (*metis Un-mono game-sem.simps*(*4*))
next
case (*Compose* $\alpha 1$ $\alpha 2$)
then show *?case* **by** *auto*
next
case (*Loop* α)
then show *?case* **by** *auto*
next
case (*Dual* α)
then show *?case* **by** *auto*
next
case (*ODE x* ϑ)
then show *?case* **by** *auto*
qed

corollary *game-sem-mono* [*simp*]: *mono* (λX. *game-sem I* α X)
by (*simp add*: *mon-mono*)

corollary *game-union*: *game-sem I* α ($X \cup Y$) $\supseteq$ *game-sem I* α $X \cup$ *game-sem I* α Y
by *simp*

lemmas *game-sem-union* = *game-union*

3.5 Fixpoint Semantics Alternative for Loops

lemma *game-sem-loop-fixpoint-mono*: *mono* (λZ. $X \cup$ *game-sem I* α Z)
using *game-sem-mono* **by** (*metis Un-mono mon-mono order-refl*)

Consequence of Knaster-Tarski Theorem 3.5 of https://doi.org/10.1145/2817824

lemma *game-sem-loop*: *game-sem I* (*Loop* α) = (λX. *lfp*(λZ. $X \cup$ *game-sem I* α

Z))
proof–
have $\bigcap\{Z.\ X \cup \textit{game-sem}\ I\ \alpha\ Z \subseteq Z\} = \textit{lfp}(\lambda Z.\ X \cup \textit{game-sem}\ I\ \alpha\ Z)$ **by** (*simp add*: *lfp-def*)
then show *?thesis* **by** (*simp add*: *lfp-def*)
qed

corollary *game-sem-loop-back*: $(\lambda X.\ \textit{lfp}(\lambda Z.\ X \cup \textit{game-sem}\ I\ \alpha\ Z)) = \textit{game-sem}\ I\ (\textit{Loop}\ \alpha)$
using *game-sem-loop* **by** *simp*

corollary *game-sem-loop-iterate*: $\textit{game-sem}\ I\ (\textit{Loop}\ \alpha) = (\lambda X.\ X \cup \textit{game-sem}\ I\ \alpha\ (\textit{game-sem}\ I\ (\textit{Loop}\ \alpha)\ X))$
by (*metis* (*no-types*) *game-sem-loop game-sem-loop-fixpoint-mono lfp-fixpoint*)

corollary *game-sem-loop-unwind*: $\textit{game-sem}\ I\ (\textit{Loop}\ \alpha) = (\lambda X.\ X \cup \textit{game-sem}\ I\ (\textit{Compose}\ \alpha\ (\textit{Loop}\ \alpha))\ X)$
using *game-sem-loop-iterate* **by** (*metis game-sem.simps*(5))

corollary *game-sem-loop-unwind-reduce*: $(\lambda X.\ X \cup \textit{game-sem}\ I\ (\textit{Compose}\ \alpha\ (\textit{Loop}\ \alpha))\ X) = \textit{game-sem}\ I\ (\textit{Loop}\ \alpha)$
using *game-sem-loop-unwind* **by** (*rule sym*)

lemmas *lfp-ordinal-induct-set-cases* = *lfp-ordinal-induct-set* [*case-names mono step union*]

lemma *game-loop-induct* [*case-names step union*]:
$(\bigwedge Z.\ Z \subseteq \textit{game-sem}\ I\ (\textit{Loop}\ \alpha)\ X \Longrightarrow P(Z) \Longrightarrow P(X \cup \textit{game-sem}\ I\ \alpha\ Z))$
$\Longrightarrow (\bigwedge M.\ (\forall Z{\in}M.\ P(Z)) \Longrightarrow P(\textit{Sup}\ M))$
$\Longrightarrow P(\textit{game-sem}\ I\ (\textit{Loop}\ \alpha)\ X)$
proof–
assume *loopstep*: $\bigwedge Z.\ Z \subseteq \textit{game-sem}\ I\ (\textit{Loop}\ \alpha)\ X \Longrightarrow P(Z) \Longrightarrow P(X \cup \textit{game-sem}\ I\ \alpha\ Z)$
assume *loopsup*: $\bigwedge M.\ (\forall Z{\in}M.\ P(Z)) \Longrightarrow P(\textit{Sup}\ M)$
have $P(\textit{lfp}(\lambda Z.\ X \cup \textit{game-sem}\ I\ \alpha\ Z))$
proof (*induction rule*: *lfp-ordinal-induct*[**where** $f{=}\langle\lambda Z.\ X \cup \textit{game-sem}\ I\ \alpha\ Z\rangle$])
case *mono*
then show *?case* **using** *game-sem-loop-fixpoint-mono* **by** *simp*
next
case (*step S*)
then show *?case* **using** *loopstep*[**where** $Z{=}S$] *game-sem-loop*[**where** $I{=}I$ **and** $\alpha{=}\alpha$] **by** (*simp add*: *loopstep*)
next
case (*union M*)
then show *?case* **using** *loopsup game-sem-loop* **by** *auto*
qed
then show $P(\textit{game-sem}\ I\ (\textit{Loop}\ \alpha)\ X)$ **using** *game-sem-loop* **by** *simp*

qed

3.6 Some Simple Obvious Observations

lemma *fml-sem-not* [*simp*]: *fml-sem I* (*Not* φ) = $-$*fml-sem I* φ
by *auto*

lemma *fml-sem-not-not* [*simp*]: *fml-sem I* (*Not* (*Not* φ)) = *fml-sem I* φ
by *simp*

lemma *fml-sem-or* [*simp*]: *fml-sem I* (*Or* φ ψ) = *fml-sem I* φ $\cup$ *fml-sem I* ψ
unfolding *Or-def* **by** *auto*

lemma *fml-sem-implies* [*simp*]: *fml-sem I* (*Implies* φ ψ) = ($-$*fml-sem I* φ) $\cup$ *fml-sem I* ψ
unfolding *Implies-def* **by** *auto*

lemma *TT-valid* [*simp*]: *valid TT*
unfolding *valid-def TT-def* **by** *simp*

Semantic equivalence of formulas **definition** *fml-equiv*:: *fml* => *fml* => *bool*
where *fml-equiv* φ ψ $\equiv$ ($\forall$ *I*. *fml-sem I* φ = *fml-sem I* ψ)

Substitutionality for Equivalent Formulas

lemma *fml-equiv-subst*: *fml-equiv* φ ψ $\Longrightarrow$ *P* (*fml-sem I* φ) $\Longrightarrow$ *P* (*fml-sem I* ψ)
proof–
assume *a1*: *fml-equiv* φ ψ
assume *a2*: *P* (*fml-sem I* φ)
from *a1* **have** *fml-sem I* φ = *fml-sem I* ψ **using** *fml-equiv-def* **by** *blast*
then show *?thesis* **using** *forw-subst a2* **by** *simp*
qed

lemma *valid-fml-equiv*: *valid* (φ $\leftrightarrow$ ψ) = *fml-equiv* φ ψ
unfolding *valid-def Equiv-def Or-def fml-equiv-def* **by** *auto*

lemma *valid-in-equiv*: *valid-in I* (φ $\leftrightarrow$ ψ) = ((*fml-sem I* φ) = (*fml-sem I* ψ))
using *valid-in-def Equiv-def Or-def* **by** *auto*

lemma *valid-in-impl*: *valid-in I* (φ $\rightarrow$ ψ) = ((*fml-sem I* φ) $\subseteq$ (*fml-sem I* ψ))
unfolding *valid-in-def Implies-def Or-def* **by** *auto*

lemma *valid-equiv*: *valid* (φ $\leftrightarrow$ ψ) = ($\forall$ *I*. *fml-sem I* φ = *fml-sem I* ψ)
using *valid-fml-equiv fml-equiv-def* **by** *auto*

lemma *valid-impl*: *valid* (φ $\rightarrow$ ψ) = ($\forall$ *I*. (*fml-sem I* φ) $\subseteq$ (*fml-sem I* ψ))
unfolding *valid-def Implies-def Or-def* **by** *auto*

lemma *fml-sem-equals* [*simp*]: $(\omega \in \textit{fml-sem } I \ (\textit{Equals } \vartheta \ \eta)) = (\textit{term-sem } I \ \vartheta \ \omega = \textit{term-sem } I \ \eta \ \omega)$
unfolding *valid-def Equals-def Or-def* **by** *auto*

lemma *equiv-refl-valid* [*simp*]: *valid* $(\varphi \leftrightarrow \varphi)$
unfolding *valid-def Equiv-def Or-def* **by** *simp*

lemma *equal-refl-valid* [*simp*]: *valid* $(\textit{Equals } \vartheta \ \vartheta)$
unfolding *valid-def Equals-def Or-def* **by** *simp*

lemma *solves-ODE-alt* : *solves-ODE I F x* $\vartheta \equiv (\forall \zeta{::}\textit{real}.$
$\textit{Vagree } (F(0)) \ (F(\zeta)) \ (-\{\textit{RVar } x, \textit{DVar } x\})$
$\wedge \ F(\zeta)(\textit{DVar } x) = \textit{deriv}(\lambda t. \ F(t)(\textit{RVar } x))(\zeta)$
$\wedge \ F(\zeta) \in \textit{fml-sem } I \ (\textit{Equals } (\textit{Var } (\textit{DVar } x)) \ \vartheta))$
unfolding *solves-ODE-def* **using** *fml-sem-equals* **by** *simp*

Semantic equivalence of games **definition** *game-equiv*:: *game* => *game* => *bool*
where *game-equiv* $\alpha \ \beta \equiv (\forall I \ X. \ \textit{game-sem } I \ \alpha \ X = \textit{game-sem } I \ \beta \ X)$

Substitutionality for Equivalent Games

lemma *game-equiv-subst*: *game-equiv* $\alpha \ \beta \Longrightarrow P \ (\textit{game-sem } I \ \alpha \ X) \Longrightarrow P \ (\textit{game-sem } I \ \beta \ X)$
proof–
assume *a1*: *game-equiv* $\alpha \ \beta$
assume *a2*: $P \ (\textit{game-sem } I \ \alpha \ X)$
from *a1* **have** *game-sem I* $\alpha \ X = \textit{game-sem } I \ \beta \ X$ **using** *game-equiv-def* **by** *blast*
then show *?thesis* **using** *forw-subst a2* **by** *simp*
qed

lemma *game-equiv-subst-eq*: *game-equiv* $\alpha \ \beta \Longrightarrow P \ (\textit{game-sem } I \ \alpha \ X) == P \ (\textit{game-sem } I \ \beta \ X)$
by (*simp add*: *game-equiv-def*)

lemma *skip-id* [*simp*]: *game-sem I Skip X* $= X$
unfolding *Skip-def TT-def* **by** *auto*

lemma *loop-iterate-equiv*: *game-equiv* $(\textit{Loop } \alpha) \ (\textit{Choice Skip } (\textit{Compose } \alpha \ (\textit{Loop } \alpha)))$
unfolding *game-equiv-def*
proof (*clarify*)
fix *I X*
from *game-sem-loop-unwind-reduce* **have** $X \cup \textit{game-sem } I \ (\textit{Compose } \alpha \ (\textit{Loop } \alpha)) \ X = \textit{game-sem } I \ (\textit{Loop } \alpha) \ X$ **by** *metis*
then show *game-sem I* $(\textit{Loop } \alpha) \ X = \textit{game-sem } I \ (\textit{Choice Skip } (\textit{Compose } \alpha \ (\textit{Loop } \alpha))) \ X$ **using** *skip-id* **by** *auto*
qed

lemma *fml-equiv-valid*: *fml-equiv* φ $\psi \Longrightarrow$ *valid* φ = *valid* ψ
unfolding *valid-def* **using** *fml-equiv-subst* **by** *blast*

lemma *solves-Vagree*: *solves-ODE I F x* $\vartheta \Longrightarrow (\bigwedge\zeta.$ *Vagree* $(F(\zeta))$ $(F(0))$ $(-\{RVar\ x, DVar\ x\}))$
using *solves-ODE-def Vagree-sym-rel* **by** *blast*

lemma *solves-Vagree-trans*: *Uvariation* $(F(0))$ ω $U \Longrightarrow$ *solves-ODE I F x* $\vartheta \Longrightarrow$ *Uvariation* $(F(\zeta))$ ω $(U \cup \{RVar\ x, DVar\ x\})$
using *solves-Vagree Uvariation-Vagree solves-ODE-def*
by (*metis Uvariation-sym-rel Uvariation-trans double-complement*)

end
theory *Static-Semantics*
imports
Syntax
Denotational-Semantics
begin

4 Static Semantics

4.1 Semantically-defined Static Semantics

Auxiliary notions of projection of winning conditions upward projection: *restrictto X V* is extends X to the states that agree on V with some state in X, so variables outside V can assume arbitrary values.

definition *restrictto* :: *state set* ⇒ *variable set* ⇒ *state set*
where
restrictto X V = $\{\nu.\ \exists\omega.\ \omega \in X \wedge$ *Vagree* $\omega\ \nu\ V\}$

downward projection: *selectlike X* ν *V* selects state ν on V in X, so all variables of V are required to remain constant

definition *selectlike* :: *state set* ⇒ *state* ⇒ *variable set* ⇒ *state set*
where
selectlike X ν *V* = $\{\omega \in X.$ *Vagree* $\omega\ \nu\ V\}$

Free variables, semantically characterized. Free variables of a term

definition *FVT* :: *trm* ⇒ *variable set*
where
FVT t = $\{x.\ \exists I.\exists\nu.\exists\omega.$ *Vagree* $\nu\ \omega\ (-\{x\}) \wedge \neg$(*term-sem I t* ν = *term-sem I t* ω)$\}$

Free variables of a formula

definition *FVF* :: *fml* ⇒ *variable set*
where

FVF $\varphi = \{x.\ \exists I.\exists \nu.\exists \omega.$ *Vagree* $\nu\ \omega\ (-\{x\}) \wedge \nu \in$ *fml-sem* $I\ \varphi \wedge \omega \notin$ *fml-sem* $I\ \varphi\}$

Free variables of a hybrid game

definition *FVG* :: *game* ⇒ *variable set*
where
FVG $\alpha = \{x.\ \exists I.\exists \nu.\exists \omega.\exists X.$ *Vagree* $\nu\ \omega\ (-\{x\}) \wedge \nu \in$ *game-sem* $I\ \alpha$ (*restrictto* $X\ (-\{x\})) \wedge \omega \notin$ *game-sem* $I\ \alpha$ (*restrictto* $X\ (-\{x\}))\}$

Bound variables, semantically characterized. Bound variables of a hybrid game

definition *BVG* :: *game* ⇒ *variable set*
where
BVG $\alpha = \{x.\ \exists I.\exists \omega.\exists X.\ \omega \in$ *game-sem* $I\ \alpha\ X \wedge \omega \notin$ *game-sem* $I\ \alpha$ (*selectlike* $X\ \omega\ \{x\})\}$

4.2 Simple Observations

lemma *BVG-elem* [*simp*] :($x \in BVG\ \alpha$) = ($\exists I\ \omega\ X.\ \omega \in$ *game-sem* $I\ \alpha\ X \wedge \omega \notin$ *game-sem* $I\ \alpha$ (*selectlike* $X\ \omega\ \{x\}$))
unfolding *BVG-def* **by** *simp*

lemma *nonBVG-rule*: ($\bigwedge I\ \omega\ X.\ (\omega \in$ *game-sem* $I\ \alpha\ X$) = ($\omega \in$ *game-sem* $I\ \alpha$ (*selectlike* $X\ \omega\ \{x\}$)))
⟹ $x \notin BVG\ \alpha$
using *BVG-elem* **by** *simp*

lemma *nonBVG-inc-rule*: ($\bigwedge I\ \omega\ X.\ (\omega \in$ *game-sem* $I\ \alpha\ X$) ⟹ ($\omega \in$ *game-sem* $I\ \alpha$ (*selectlike* $X\ \omega\ \{x\}$)))
⟹ $x \notin BVG\ \alpha$
using *BVG-elem* **by** *simp*

lemma *FVT-finite*: *finite*(*FVT* t)
using *allvars-finite* **by** (*metis finite-subset mem-Collect-eq subsetI*)
lemma *FVF-finite*: *finite*(*FVF* e)
using *allvars-finite* **by** (*metis finite-subset mem-Collect-eq subsetI*)
lemma *FVG-finite*: *finite*(*FVG* a)
using *allvars-finite* **by** (*metis finite-subset mem-Collect-eq subsetI*)

end
theory *Coincidence*
imports
Lib
Syntax
Denotational-Semantics
Static-Semantics
HOL.Finite-Set
begin

5 Static Semantics Properties

5.1 Auxiliaries

The state interpolating *stateinterpol* ν ω S between ν and ω that is ν on S and ω elsewhere

definition *stateinterpol*:: *state* ⇒ *state* ⇒ *variable set* ⇒ *state*
where
stateinterpol ν ω S = (λx. *if* ($x \in S$) *then* $\nu(x)$ *else* $\omega(x)$)

definition *statediff*:: *state* ⇒ *state* ⇒ *variable set*
where *statediff* ν ω = $\{x.\ \nu(x) \neq \omega(x)\}$

lemma *nostatediff*: $x \notin$ *statediff* ν ω ⟹ $\nu(x) = \omega(x)$
by (*simp add*: *statediff-def*)

lemma *stateinterpol-empty*: *stateinterpol* ν ω $\{\}$ = ω
proof
fix x
have *empty*: $\bigwedge x.\ \neg(x \in \{\})$ **by** *auto*
show $\bigwedge x$. *stateinterpol* ν ω $\{\}$ x = ω x **using** *empty* **by** (*simp add*: *stateinterpol-def*)
qed

lemma *stateinterpol-left* [*simp*]: $x \in S$ ⟹ (*stateinterpol* ν ω S)(x)=$\nu(x)$
by (*simp add*: *stateinterpol-def*)

lemma *stateinterpol-right* [*simp*]: $x \notin S$ ⟹ (*stateinterpol* ν ω S)(x)=$\omega(x)$
by (*simp add*: *stateinterpol-def*)

lemma *Vagree-stateinterpol* [*simp*]: *Vagree* (*stateinterpol* ν ω S) ν S
and *Vagree* (*stateinterpol* ν ω S) ω ($-S$)
unfolding *Vagree-def* **by** *auto*

lemma *Vagree-ror*: *Vagree* ν ν' ($V \cap W$) ⟹ ($\exists \omega$. (*Vagree* ν ω V ∧ *Vagree* ω ν' W))
proof −
assume *Vagree* ν ν' ($V \cap W$)
hence $\forall x.\ x \in V \cap W \longrightarrow \nu(x) = \nu'(x)$ **by** (*simp add*: *Vagree-def*)
let *?w*=*stateinterpol* ν ν' V
have *l*: *Vagree* ν *?w* V **by** (*simp add*: *Vagree-def*)
have *r*: *Vagree* *?w* ν' W ∧ *Vagree* *?w* ν' W **by** (*simp add*: *Vagree-def stateinterpol-def* ‹$\forall x.\ x \in V \cap W \longrightarrow \nu\ x = \nu'\ x$›)
have *Vagree* ν *?w* V ∧ *Vagree* *?w* ν' W **using** *l* **and** *r* **by** *blast*
thus *?thesis* **by** *auto*
qed

Remark 8 https://doi.org/10.1007/978-3-319-94205-6_15 about simple properties of projections

lemma *restrictto-extends* [*simp*]: *restrictto X V* $\supseteq$ *X*
by (*auto simp add*: *restrictto-def*)

lemma *restrictto-compose* [*simp*]: *restrictto* (*restrictto X V*) *W* = *restrictto X* (*V*$\cap$*W*)
proof
show *restrictto* (*restrictto X V*) *W* $\subseteq$ *restrictto X* (*V*$\cap$*W*)
by (*auto simp add*: *restrictto-def Vagree-def*)
next
show *restrictto X* (*V*$\cap$*W*) $\subseteq$ *restrictto* (*restrictto X V*) *W*

proof −
obtain *rr* :: (*variable* $\Rightarrow$ *real*) *set* $\Rightarrow$ (*variable* $\Rightarrow$ *real*) *set* $\Rightarrow$ *variable* $\Rightarrow$ *real*
where
$\forall$ *x0 x1*. ($\exists$ *v2*. *v2* $\in$ *x1* $\wedge$ *v2* $\notin$ *x0*) = (*rr x0 x1* $\in$ *x1* $\wedge$ *rr x0 x1* $\notin$ *x0*)
by *moura*
then have *f1*: $\forall$ *F Fa*. *rr Fa F* $\in$ *F* $\wedge$ *rr Fa F* $\notin$ *Fa* $\vee$ *F* $\subseteq$ *Fa*
by (*meson subsetI*)
obtain *rra* :: (*variable* $\Rightarrow$ *real*) $\Rightarrow$ *variable set* $\Rightarrow$ (*variable* $\Rightarrow$ *real*) *set* $\Rightarrow$ *variable* $\Rightarrow$ *real* **where**
$\forall$ *x0 x1 x2*. ($\exists$ *v3*. *v3* $\in$ *x2* $\wedge$ *Vagree v3 x0 x1*) = (*rra x0 x1 x2* $\in$ *x2* $\wedge$ *Vagree* (*rra x0 x1 x2*) *x0 x1*)
by *moura*
then have *f2*: $\forall$ *F V f*. (*f* $\notin$ {*f*. $\exists$ *fa*. *fa* $\in$ *F* $\wedge$ *Vagree fa f V*} $\vee$ *rra f V F* $\in$ *F* $\wedge$ *Vagree* (*rra f V F*) *f V*) $\wedge$ (*f* $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ *F* $\wedge$ *Vagree fa f V*} $\vee$ ($\forall$ *fa*. *fa* $\notin$ *F* $\vee$ $\neg$ *Vagree fa f V*))
by *blast*
moreover
{ assume $\exists$ *f*. *f* $\in$ *X* $\wedge$ *Vagree f* (*v4-1 W V* (*rr* {*f*. $\exists$ *fa*. *fa* $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f V*} $\wedge$ *Vagree fa f W*} {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f* (*V* $\cap$ *W*)}) (*rra* (*rr* {*f*. $\exists$ *fa*. *fa* $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f V*} $\wedge$ *Vagree fa f W*} {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f* (*V* $\cap$ *W*)}) (*V* $\cap$ *W*) *X*)) *V*
moreover
{ assume $\exists$ *f*. *f* $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f V*} $\wedge$ *Vagree f* (*rr* {*f*. $\exists$ *fa*. *fa* $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f V*} $\wedge$ *Vagree fa f W*} {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f* (*V* $\cap$ *W*)}) *W*
then have *rr* {*f*. $\exists$ *fa*. *fa* $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f V*} $\wedge$ *Vagree fa f W*} {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f* (*V* $\cap$ *W*)} $\notin$ {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f* (*V* $\cap$ *W*)} $\vee$ *rr* {*f*. $\exists$ *fa*. *fa* $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f V*} $\wedge$ *Vagree fa f W*} {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f* (*V* $\cap$ *W*)} $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f V*} $\wedge$ *Vagree fa f W*}
by *blast*
then have {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f* (*V* $\cap$ *W*)} $\subseteq$ {*f*. $\exists$ *fa*. *fa* $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f V*} $\wedge$ *Vagree fa f W*}
using *f1* **by** *meson* **}**
ultimately have ($\neg$ *Vagree* (*rra* (*rr* {*f*. $\exists$ *fa*. *fa* $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f V*} $\wedge$ *Vagree fa f W*} {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f* (*V* $\cap$ *W*)}) (*V* $\cap$ *W*) *X*) (*v4-1 W V* (*rr* {*f*. $\exists$ *fa*. *fa* $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f V*} $\wedge$ *Vagree fa f W*} {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$ *Vagree fa f* (*V* $\cap$ *W*)}) (*rra* (*rr* {*f*. $\exists$ *fa*. *fa* $\in$ {*f*. $\exists$ *fa*. *fa* $\in$ *X* $\wedge$

Vagree fa f V} ∧ *Vagree fa f W*} {*f*. ∃*fa*. *fa* ∈ *X* ∧ *Vagree fa f* (*V* ∩ *W*)}) (*V* ∩ *W*) *X*)) *V* ∨ ¬ *Vagree* (*v4-1 W V* (*rr* {*f*. ∃*fa*. *fa* ∈ {*f*. ∃*fa*. *fa* ∈ *X* ∧ *Vagree fa f V*} ∧ *Vagree fa f W*} {*f*. ∃*fa*. *fa* ∈ *X* ∧ *Vagree fa f* (*V* ∩ *W*)}) (*rra* (*rr* {*f*. ∃*fa*. *fa* ∈ {*f*. ∃*fa*. *fa* ∈ *X* ∧ *Vagree fa f V*} ∧ *Vagree fa f W*} {*f*. ∃*fa*. *fa* ∈ *X* ∧ *Vagree fa f* (*V* ∩ *W*)}) (*V* ∩ *W*) *X*)) (*rr* {*f*. ∃*fa*. *fa* ∈ {*f*. ∃*fa*. *fa* ∈ *X* ∧ *Vagree fa f V*} ∧ *Vagree fa f W*} {*f*. ∃*fa*. *fa* ∈ *X* ∧ *Vagree fa f* (*V* ∩ *W*)}) *W*) ∨ {*f*. ∃*fa*. *fa* ∈ *X* ∧ *Vagree fa f* (*V* ∩ *W*)} ⊆ {*f*. ∃*fa*. *fa* ∈ {*f*. ∃*fa*. *fa* ∈ *X* ∧ *Vagree fa f V*} ∧ *Vagree fa f W*}
  using *f2* **by** *meson* **}**
 ultimately have {*f*. ∃*fa*. *fa* ∈ *X* ∧ *Vagree fa f* (*V* ∩ *W*)} ⊆ {*f*. ∃*fa*. *fa* ∈ {*f*. ∃*fa*. *fa* ∈ *X* ∧ *Vagree fa f V*} ∧ *Vagree fa f W*}
 using *f1* **by** (*meson Vagree-ror*)
 then show *?thesis*
 using *restrictto-def* **by** *presburger*
 qed
qed

lemma *restrictto-antimon* [*simp*]: *W*⊇*V* ⟹ *restrictto X W* ⊆ *restrictto X V*
proof −
 assume *W*⊇*V*
 then have ∃ *U*. *V*=*W*∩*U* **by** *auto*
 then obtain *U* **where** *V*=*W*∩*U* **by** *auto*
 hence *restrictto X V* = *restrictto* (*restrictto X W*) *U* **by** *simp*
 hence *restrictto X V* ⊇ *restrictto X W* **using** *restrictto-extends* **by** *blast*
 thus *?thesis* **by** *auto*
qed

lemma *restrictto-empty* [*simp*]: *X*≠{} ⟹ *restrictto X* {} = *worlds*
 by (*auto simp add*: *restrictto-def worlds-def*)

lemma *selectlike-shrinks* [*simp*]: *selectlike X ν V* ⊆ *X*
 by (*auto simp add*: *selectlike-def*)

lemma *selectlike-compose* [*simp*]: *selectlike* (*selectlike X ν V*) *ν W* = *selectlike X ν* (*V*∪*W*)
 by (*auto simp add*: *selectlike-def*)

lemma *selectlike-antimon* [*simp*]: *W*⊇*V* ⟹ *selectlike X ν W* ⊆ *selectlike X ν V*
 by (*auto simp add*: *selectlike-def*)

lemma *selectlike-empty* [*simp*]: *selectlike X ν* {} = *X*
 by (*auto simp add*: *selectlike-def*)

lemma *selectlike-self* [*simp*]: (*ν* ∈ *selectlike X ν V*) = (*ν*∈*X*)
 by (*auto simp add*: *selectlike-def*)

lemma *selectlike-complement* [*simp*]: *selectlike* (−*X*) *ν V* ⊆ −*selectlike X ν V*
 by (*auto simp add*: *selectlike-def*)

lemma *selectlike-union*: *selectlike* $(X \cup Y)\ \nu\ V = \mathit{selectlike}\ X\ \nu\ V \cup \mathit{selectlike}\ Y\ \nu\ V$
by (*auto simp add*: *selectlike-def*)

lemma *selectlike-Sup*: *selectlike* (*Sup M*) $\nu\ V = \mathit{Sup}\ \{\mathit{selectlike}\ X\ \nu\ V \mid X.\ X \in M\}$
using *selectlike-def* **by** *auto*

lemma *selectlike-equal-cond*: $(\mathit{selectlike}\ X\ \nu\ V = \mathit{selectlike}\ Y\ \nu\ V) = (\forall \mu.\ \mathit{Uvariation}\ \mu\ \nu\ (-V) \longrightarrow (\mu \in X) = (\mu \in Y))$
unfolding *selectlike-def* **using** *Uvariation-Vagree* **by** *auto*

lemma *selectlike-equal-cocond*: $(\mathit{selectlike}\ X\ \nu\ (-V) = \mathit{selectlike}\ Y\ \nu\ (-V)) = (\forall \mu.\ \mathit{Uvariation}\ \mu\ \nu\ V \longrightarrow (\mu \in X) = (\mu \in Y))$
using *selectlike-equal-cond*[**where** $V = \langle -V \rangle$] **by** *simp*

lemma *selectlike-equal-cocond-rule*: $(\bigwedge \mu.\ \mathit{Uvariation}\ \mu\ \nu\ (-V) \Longrightarrow (\mu \in X) = (\mu \in Y))$
$\Longrightarrow (\mathit{selectlike}\ X\ \nu\ V = \mathit{selectlike}\ Y\ \nu\ V)$
using *selectlike-equal-cond*[**where** $V = \langle V \rangle$] **by** *simp*

lemma *selectlike-equal-cocond-corule*: $(\bigwedge \mu.\ \mathit{Uvariation}\ \mu\ \nu\ V \Longrightarrow (\mu \in X) = (\mu \in Y))$
$\Longrightarrow (\mathit{selectlike}\ X\ \nu\ (-V) = \mathit{selectlike}\ Y\ \nu\ (-V))$
using *selectlike-equal-cond*[**where** $V = \langle -V \rangle$] **by** *simp*

lemma *co-selectlike*: $-(\mathit{selectlike}\ X\ \nu\ V) = (-X) \cup \{\omega.\ \neg \mathit{Vagree}\ \omega\ \nu\ V\}$
unfolding *selectlike-def* **by** *auto*

lemma *selectlike-co-selectlike*: $\mathit{selectlike}\ (-(\mathit{selectlike}\ X\ \nu\ V))\ \nu\ V = \mathit{selectlike}\ (-X)\ \nu\ V$
unfolding *selectlike-def* **by** *auto*

lemma *selectlike-Vagree*: $\mathit{Vagree}\ \nu\ \omega\ V \Longrightarrow \mathit{selectlike}\ X\ \nu\ V = \mathit{selectlike}\ X\ \omega\ V$
using *Vagree-def selectlike-def* **by** *auto*

lemma *similar-selectlike-mem*: $\mathit{Vagree}\ \nu\ \omega\ V \Longrightarrow (\nu \in \mathit{selectlike}\ X\ \omega\ V) = (\nu \in X)$
unfolding *selectlike-def* **using** *Vagree-sym-rel* **by** *blast*

lemma *BVG-nonelem* [*simp*] :$(x \notin \mathit{BVG}\ \alpha) = (\forall I\ \omega\ X.\ (\omega \in \mathit{game\text{-}sem}\ I\ \alpha\ X) = (\omega \in \mathit{game\text{-}sem}\ I\ \alpha\ (\mathit{selectlike}\ X\ \omega\ \{x\})))$
using *BVG-elem monotone selectlike-shrinks*
by (*metis subset-iff*)

statediff interoperability

lemma *Vagree-statediff* [*simp*]: $\mathit{Vagree}\ \omega\ \omega'\ S \Longrightarrow \mathit{statediff}\ \omega\ \omega' \subseteq -S$
by (*auto simp add*: *Vagree-def statediff-def*)

lemma *stateinterpol-diff* [*simp*]: $\mathit{stateinterpol}\ \nu\ \omega\ (\mathit{statediff}\ \nu\ \omega) = \nu$
proof

fix x
show *sp*: $(stateinterpol\ \nu\ \omega\ (statediff\ \nu\ \omega))(x) = \nu(x)$
proof (*cases* $x \in statediff\ \nu\ \omega$)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then show *?thesis* **by** (*simp add*: *stateinterpol-def nostatediff*)
qed
qed

lemma *stateinterpol-insert*: *Vagree* (*stateinterpol v w S*) (*stateinterpol v w* (*insert z S*)) $(-\{z\})$
by (*simp add*: *Vagree-def stateinterpol-def*)

lemma *stateinterpol-FVT* [*simp*]: $z \notin FVT(t) \Longrightarrow term\text{-}sem\ I\ t\ (stateinterpol\ \omega\ \omega'\ S) = term\text{-}sem\ I\ t\ (stateinterpol\ \omega\ \omega'\ (insert\ z\ S))$
proof −
assume *a*: $z \notin FVT(t)$
have *fvc*: $\bigwedge v.\ \bigwedge w.\ Vagree\ v\ w\ (-\{z\}) \Longrightarrow (term\text{-}sem\ I\ t\ v = term\text{-}sem\ I\ t\ w)$ **using** *a* **by** (*simp add*: *FVT-def*)
then show $term\text{-}sem\ I\ t\ (stateinterpol\ \omega\ \omega'\ S) = term\text{-}sem\ I\ t\ (stateinterpol\ \omega\ \omega'\ (insert\ z\ S))$
using *fvc* **and** *stateinterpol-insert* **by** *blast*
qed

5.2 Coincidence Lemmas

Coincidence for Terms Lemma 10 https://doi.org/10.1007/978-3-319-94205-6_15

theorem *coincidence-term*: $Vagree\ \omega\ \omega'\ (FVT\ \vartheta) \Longrightarrow term\text{-}sem\ I\ \vartheta\ \omega = term\text{-}sem\ I\ \vartheta\ \omega'$
proof −
assume *a*: $Vagree\ \omega\ \omega'\ (FVT\ \vartheta)$
have *isS*: $statediff\ \omega\ \omega' \subseteq -FVT(\vartheta)$ **using** *a* **and** *Vagree-statediff* **by** *simp*
have *gen*: $S \subseteq -FVT(\vartheta) \Longrightarrow (term\text{-}sem\ I\ \vartheta\ \omega' = term\text{-}sem\ I\ \vartheta\ (stateinterpol\ \omega\ \omega'\ S))$ **if** *finite S* **for** *S*
using *that*
proof (*induction S*)
case *empty*
show *?case* **by** (*simp add*: *stateinterpol-empty*)
next
case (*insert z S*)
thus *?case* **by** *auto*
qed
from *isS* **have** *finS*: *finite* (*statediff* $\omega\ \omega'$) **using** *allvars-finite* **by** (*metis FVT-finite UNIV-def finite-compl rev-finite-subset*)
show *?thesis* **using** *gen*[**where** $S = \langle statediff\ \omega\ \omega' \rangle$, *OF finS*, *OF isS*] **by** *simp*

qed

corollary *coincidence-term-cor*: *Uvariation* $\omega\ \omega'\ U \Longrightarrow (FVT\ \vartheta) \cap U = \{\} \Longrightarrow$ *term-sem I* $\vartheta\ \omega =$ *term-sem I* $\vartheta\ \omega'$
using *coincidence-term Uvariation-Vagree*
by (*metis Vagree-antimon disjoint-eq-subset-Compl double-compl*)

lemma *stateinterpol-FVF* [*simp*]: $z \notin FVF(e) \Longrightarrow$
$((stateinterpol\ \omega\ \omega'\ S) \in fml\text{-}sem\ I\ e \longleftrightarrow (stateinterpol\ \omega\ \omega'\ (insert\ z\ S)) \in fml\text{-}sem\ I\ e)$
proof −
assume *a*: $z \notin FVF(e)$
have *agr*: *Vagree* (*stateinterpol* $\omega\ \omega'\ S$) (*stateinterpol* $\omega\ \omega'$ (*insert z S*)) $(-\{z\})$
by (*simp add*: *Vagree-def stateinterpol-def*)
have *fvc*: $\bigwedge v.\ \bigwedge w.\ (Vagree\ v\ w\ (-\{z\}) \Longrightarrow (v \in fml\text{-}sem\ I\ e \Longrightarrow w \in fml\text{-}sem\ I\ e))$
using *a* **by** (*simp add*: *FVF-def*)
then have *fvce*: $\bigwedge v.\ \bigwedge w.\ (Vagree\ v\ w\ (-\{z\}) \Longrightarrow ((v \in fml\text{-}sem\ I\ e) = (w \in fml\text{-}sem\ I\ e)))$ **using** *Vagree-sym-rel* **by** *blast*
then show $(stateinterpol\ \omega\ \omega'\ S) \in fml\text{-}sem\ I\ e \longleftrightarrow (stateinterpol\ \omega\ \omega'\ (insert\ z\ S)) \in fml\text{-}sem\ I\ e$
using *agr* **by** *simp*
qed

Coincidence for Formulas Lemma 11 https://doi.org/10.1007/978-3-319-94205-6_15

theorem *coincidence-formula*: $Vagree\ \omega\ \omega'\ (FVF\ \varphi) \Longrightarrow (\omega \in fml\text{-}sem\ I\ \varphi \longleftrightarrow \omega' \in fml\text{-}sem\ I\ \varphi)$
proof −
assume *a*: *Vagree* $\omega\ \omega'\ (FVF\ \varphi)$
have *isS*: *statediff* $\omega\ \omega' \subseteq -FVF(\varphi)$ **using** *a* **and** *Vagree-statediff* **by** *simp*
have *gen*: $S \subseteq -FVF(\varphi) \Longrightarrow (\omega' \in fml\text{-}sem\ I\ \varphi \longleftrightarrow (stateinterpol\ \omega\ \omega'\ S) \in fml\text{-}sem\ I\ \varphi)$ **if** *finite S* **for** *S*
using *that*
proof (*induction S*)
case *empty*
show *?case* **by** (*simp add*: *stateinterpol-empty*)
next
case (*insert z S*)
thus *?case* **by** *auto*
qed
from *isS* **have** *finS*: *finite* (*statediff* $\omega\ \omega'$) **using** *allvars-finite* **by** (*metis FVF-finite UNIV-def finite-compl rev-finite-subset*)
show *?thesis* **using** *gen*[**where** $S = \langle statediff\ \omega\ \omega' \rangle$, *OF finS*, *OF isS*] **by** *simp*
qed

corollary *coincidence-formula-cor*: $Uvariation\ \omega\ \omega'\ U \Longrightarrow (FVF\ \varphi) \cap U = \{\} \Longrightarrow (\omega \in fml\text{-}sem\ I\ \varphi \longleftrightarrow \omega' \in fml\text{-}sem\ I\ \varphi)$

using *coincidence-formula Uvariation-Vagree*
by (*metis Uvariation-def disjoint-eq-subset-Compl inf.commute subsetCE*)

Coincidence for Games *Cignorabimus* α V is the set of all sets of variables that can be ignored for the coincidence game lemma

definition *Cignorabimus*:: *game* ⇒ *variable set* ⇒ *variable set set*
where
Cignorabimus α $V = \{M.\ \forall I.\forall \omega.\forall \omega'.\forall X.\ (Vagree\ \omega\ \omega'\ (-M) \longrightarrow (\omega \in game\text{-}sem\ I\ \alpha\ (restrictto\ X\ V)) \longrightarrow (\omega' \in game\text{-}sem\ I\ \alpha\ (restrictto\ X\ V)))\}$

lemma *Cignorabimus-finite* [*simp*]: *finite* (*Cignorabimus* α V)
unfolding *Cignorabimus-def* **using** *finite-powerset*[*OF allvars-finite*] *finite-subset*
using *Finite-Set.finite-subset* **by** *fastforce*

lemma *Cignorabimus-equiv* [*simp*]: *Cignorabimus* α $V = \{M.\ \forall I.\forall \omega.\forall \omega'.\forall X.\ (Vagree\ \omega\ \omega'\ (-M) \longrightarrow (\omega \in game\text{-}sem\ I\ \alpha\ (restrictto\ X\ V)) = (\omega' \in game\text{-}sem\ I\ \alpha\ (restrictto\ X\ V)))\}$
unfolding *Cignorabimus-def* **by** (*metis* (*no-types*, *lifting*) *Vagree-sym-rel*)

lemma *Cignorabimus-antimon* [*simp*]: $M \in$ *Cignorabimus* α $V \wedge N \subseteq M \Longrightarrow N \in$ *Cignorabimus* α V
unfolding *Cignorabimus-def*
using *Vagree-antimon* **by** *blast*

lemma *coempty*: $-\{\} = allvars$
by *simp*

lemma *Cignorabimus-empty* [*simp*]: $\{\} \in$ *Cignorabimus* α V
unfolding *Cignorabimus-def* **using** *coempty Vagree-univ*
by *simp*

Cignorabimus contains nonfree variables

lemma *Cignorabimus-init*: $V \supseteq FVG(\alpha) \Longrightarrow x \notin V \Longrightarrow \{x\} \in$ *Cignorabimus* α V
proof−
assume $V \supseteq FVG(\alpha)$
assume *a0*: $x \notin V$
hence *a1*: $x \notin FVG(\alpha)$ **using** ‹$FVG\ \alpha \subseteq V$› **by** *blast*
hence $\bigwedge I\ v\ w.\ Vagree\ v\ w\ (-\{x\}) \Longrightarrow (v \in game\text{-}sem\ I\ \alpha\ (restrictto\ X\ (-\{x\})) \longleftrightarrow w \in game\text{-}sem\ I\ \alpha\ (restrictto\ X\ (-\{x\})))$
by (*metis* (*mono-tags*, *lifting*) *CollectI FVG-def Vagree-sym-rel*)
show $\{x\} \in$ *Cignorabimus* α V
proof−
{
fix $I\ \omega\ \omega'\ X$
have $Vagree\ \omega\ \omega'\ (-\{x\}) \longrightarrow (\omega \in game\text{-}sem\ I\ \alpha\ (restrictto\ X\ V)) \longrightarrow (\omega' \in game\text{-}sem\ I\ \alpha\ (restrictto\ X\ V))$
proof
assume *a2*: $Vagree\ \omega\ \omega'\ (-\{x\})$

```
      show (ω∈game-sem I α (restrictto X V)) ⟶ (ω'∈game-sem I α (restrictto
X V))
       proof
         assume ω∈game-sem I α (restrictto X V)
        hence ω∈game-sem I α (restrictto (restrictto X V) (−{x})) by (simp add:
Int-absorb2 ‹x∉V›)
       hence ω'∈game-sem I α (restrictto (restrictto X V) (−{x})) using FVG-def
a1 a2 by blast
         hence ω'∈game-sem I α (restrictto X (V∩−{x})) by simp
         show ω'∈game-sem I α (restrictto X V) using a0
           by (metis Int-absorb2 ‹ω' ∈ game-sem I α (restrictto X (V ∩ − {x}))›
subset-Compl-singleton)
       qed
     qed
   }
   thus ?thesis
     unfolding Cignorabimus-def
     by auto
 qed
qed
```

Cignorabimus is closed under union

```
lemma Cignorabimus-union: M∈Cignorabimus α V ⟹ N∈Cignorabimus α V
⟹ (M∪N)∈Cignorabimus α V
proof−
 assume a1: M∈Cignorabimus α V
 assume a2: N∈Cignorabimus α V
 show (M∪N)∈Cignorabimus α V
 proof−
   {
     fix I ω ω' X
   assume a3: Vagree ω ω' (−(M∪N))
   have h1: ⋀I ω ω'.⋀X. (Vagree ω ω' (−M) ⟹ (ω∈game-sem I α (restrictto
X V)) ⟹ (ω'∈game-sem I α (restrictto X V))) using a1 by simp
   have h2: ⋀I ω ω'.⋀X. (Vagree ω ω' (−N) ⟹ (ω∈game-sem I α (restrictto
X V)) ⟹ (ω'∈game-sem I α (restrictto X V))) using a2 by simp
   let ?s = stateinterpol ω' ω M
   have v1: Vagree ω ?s (−(M∪N)) using a3 by (simp add: Vagree-def)
   have v2: Vagree ?s ω' (−(M∪N)) using a3 by (simp add: Vagree-def)
  have r1: ω∈game-sem I α (restrictto X V) ⟹ ?s∈game-sem I α (restrictto X
V)
     by (metis ComplD Vagree-def h1 stateinterpol-right)
   have r2: ?s∈game-sem I α (restrictto X V) ⟹ ω'∈game-sem I α (restrictto
X V)
     by (metis Vagree-ror compl-sup h1 h2 v2)
   have res: ω∈game-sem I α (restrictto X V) ⟹ ω'∈game-sem I α (restrictto
X V) using r1 r2 by blast
   }
   thus ?thesis
```

unfolding *Cignorabimus-def*
by *auto*
qed
qed

lemma *powersetup-induct* [*case-names Base Cup*]:
$\bigwedge C.\ (\bigwedge M.\ M{\in}C \Longrightarrow P\ M) \Longrightarrow$
$(\bigwedge S.\ (\bigwedge M.\ M{\in}S \Longrightarrow P\ M) \Longrightarrow P\ (\bigcup S)) \Longrightarrow$
$P\ (\bigcup C)$
by *simp*

lemma *Union-insert*: $\bigcup(\mathit{insert}\ x\ S) = x{\cup}\bigcup S$
by *simp*

lemma *powerset2up-induct* [*case-names Finite Nonempty Base Cup*]:
$(\mathit{finite}\ C) \Longrightarrow (C{\neq}\{\}) \Longrightarrow (\bigwedge M.\ M{\in}C \Longrightarrow P\ M) \Longrightarrow$
$(\bigwedge M\ N.\ P\ M \Longrightarrow P\ N \Longrightarrow P\ (M{\cup}N)) \Longrightarrow$
$P\ (\bigcup C)$
proof (*induction rule*: *finite-induct*)
case *empty*
then show *?case* **by** *simp*
next
case (*insert x F*)
then show *?case* **by** *force*
qed

lemma *Cignorabimus-step*: $(\bigwedge M.\ M{\in}S \Longrightarrow M{\in}\mathit{Cignorabimus}\ \alpha\ V) \Longrightarrow (\bigcup S){\in}\mathit{Cignorabimus}\ \alpha\ V$
proof (*cases* $S{=}\{\}$)
case *True*
then show *?thesis* **using** *Cignorabimus-empty* **by** *simp*
next
case *nonem*: *False*
then show $\bigcup S{\in}\mathit{Cignorabimus}\ \alpha\ V$ **if** $\bigwedge M.\ M{\in}S \Longrightarrow M{\in}\mathit{Cignorabimus}\ \alpha\ V$
and *nonemp*:$S{\neq}\{\}$ **for** S
proof (*induction rule*: *powerset2up-induct*)
case *Finite*
then show *?case* **using** *Cignorabimus-finite* **by** (*meson infinite-super subset-eq that*(*1*))
next
case *Nonempty*
then show *?case* **using** *nonemp* **by** *simp*
next
case (*Base M*)
then show *?case* **using** *that* **by** *simp*
next
case (*Cup S*)
then show *?case* **using** *that Cignorabimus-union* **by** *blast*

qed
qed

Lemma 12 https://doi.org/10.1007/978-3-319-94205-6_15

theorem *coincidence-game*: *Vagree* $\omega\ \omega'\ V \Longrightarrow V \supseteq FVG(\alpha) \Longrightarrow (\omega \in$ *game-sem* $I\ \alpha$ (*restrictto* $X\ V)) = (\omega' \in$ *game-sem* $I\ \alpha$ (*restrictto* $X\ V))$
proof−
assume *a1*: *Vagree* $\omega\ \omega'\ V$
assume *a2*: $V \supseteq FVG\ \alpha$
have *base*: $\{x\} \in$ *Cignorabimus* $\alpha\ V$ **if** *a3*: $x \notin V$ **and** *a4*: $V \supseteq FVG\ \alpha$ **for** $x\ V$
using *a3* **and** *a4* **and** *Cignorabimus-init* **by** *simp*
have *h*: $-V = \bigcup \{xx.\ \exists x.\ xx = \{x\} \wedge x \notin V\}$ **by** *auto*
have $(-V) \in$ *Cignorabimus* $\alpha\ V$ **using** *a2 base h* **using** *Cignorabimus-step*

proof −
have *f1*: $\forall v\ V.\ v \in V \vee \neg\ FVG\ \alpha \subseteq V \vee \{v\} \in$ *Cignorabimus* $\alpha\ V$
using *base* **by** *satx*
obtain *VV* :: *variable set* ⇒ *game* ⇒ *variable set set* ⇒ *variable set* **where**
f2: $\forall$ *x0 x1 x2*. $(\exists$ *v3*. *v3* $\in$ *x2* $\wedge$ *v3* $\notin$ *Cignorabimus x1 x0*$) =$ (*VV x0 x1 x2* $\in$ *x2* $\wedge$ *VV x0 x1 x2* $\notin$ *Cignorabimus x1 x0*)
by *moura*
obtain *vv* :: *variable set* ⇒ *variable* **where**
f3: $((\nexists v.\ VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\} = \{v\} \wedge v \notin V) \vee VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\} = \{vv\ (VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\})\} \wedge vv\ (VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\}) \notin V) \wedge ((\exists v.\ VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\} = \{v\} \wedge v \notin V) \vee (\forall v.\ VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\} \neq \{v\} \vee v \in V))$
by *moura*
moreover
{ assume $\{vv\ (VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\})\} \in$ *Cignorabimus* $\alpha\ V$
then have $(VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\} \neq \{vv\ (VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\})\} \vee vv\ (VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\}) \in V) \vee VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\} \notin \{\{v\}\ |v.\ v \notin V\} \vee VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\} \in$ *Cignorabimus* $\alpha\ V$
by *metis*
then have $(\exists v.\ VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\} = \{v\} \wedge v \notin V) \longrightarrow VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\} \notin \{\{v\}\ |v.\ v \notin V\} \vee VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\} \in$ *Cignorabimus* $\alpha\ V$
using *f3* **by** *blast* **}**
ultimately have $VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\} \notin \{\{v\}\ |v.\ v \notin V\} \vee VV\ V\ \alpha\ \{\{v\}\ |v.\ v \notin V\} \in$ *Cignorabimus* $\alpha\ V$
using *f1 a2* **by** *blast*
then have $\bigcup \{\{v\}\ |v.\ v \notin V\} \in$ *Cignorabimus* $\alpha\ V$
using *f2* **by** (*meson Cignorabimus-step*)
then show *?thesis*
using *h* **by** *presburger*
qed
from *this* **show** *?thesis* **by** (*simp add*: *a1*)
qed

corollary *coincidence-game-cor*: *Uvariation* $\omega\ \omega'\ U \Longrightarrow U \cap FVG(\alpha) = \{\} \Longrightarrow (\omega \in$ *game-sem* $I\ \alpha$ (*restrictto* $X\ (-U))) = (\omega' \in$ *game-sem* $I\ \alpha$ (*restrictto* $X\ (-U)))$

using *coincidence-game Uvariation-Vagree*
by (*metis Uvariation-Vagree coincidence-game compl-le-swap1 disjoint-eq-subset-Compl double-compl*)

5.3 Bound Effect Lemmas

Bignorabimus α V is the set of all sets of variables that can be ignored for boundeffect

definition *Bignorabimus*:: *game* $\Rightarrow$ *variable set set*
where
Bignorabimus $\alpha = \{M.\ \forall I.\forall \omega.\forall X.\ \omega\in$*game-sem* $I\ \alpha\ X \longleftrightarrow \omega\in$*game-sem* $I\ \alpha$ (*selectlike* $X\ \omega\ M$)$\}$

lemma *Bignorabimus-finite* [*simp*]: *finite* (*Bignorabimus* α)
unfolding *Bignorabimus-def* **using** *finite-powerset*[*OF allvars-finite*] *finite-subset* **using** *Finite-Set.finite-subset* **by** *fastforce*

lemma *Bignorabimus-single* [*simp*]: *game-sem* $I\ \alpha$ (*selectlike* $X\ \omega\ M) \subseteq$ *game-sem* $I\ \alpha\ X$
by (*meson monotone selectlike-shrinks subsetCE*)

lemma *Bignorabimus-equiv* [*simp*]: *Bignorabimus* $\alpha = \{M.\ \forall I.\forall \omega.\forall X.\ (\omega\in$*game-sem* $I\ \alpha\ X \longrightarrow \omega\in$*game-sem* $I\ \alpha$ (*selectlike* $X\ \omega\ M))\}$

proof −
obtain $VV :: (\text{variable set} \Rightarrow \text{bool}) \Rightarrow (\text{variable set} \Rightarrow \text{bool}) \Rightarrow \text{variable set}$ **where**
f1: $\forall p\ pa.\ (\neg\ p\ (VV\ pa\ p)) = pa\ (VV\ pa\ p) \vee$ *Collect* $p =$ *Collect* pa
by (*metis* (*no-types*) *Collect-cong*)
obtain $rr :: \text{variable set} \Rightarrow \text{game} \Rightarrow \text{variable} \Rightarrow \text{real}$ **and** $ii :: \text{variable set} \Rightarrow \text{game} \Rightarrow \text{interp}$ **and** $FF :: \text{variable set} \Rightarrow \text{game} \Rightarrow (\text{variable} \Rightarrow \text{real})\ \text{set}$ **where**
f2: $\forall x0\ x1.\ (\exists v2\ v3\ v4.\ (v3 \in$ *game-sem* $v2\ x1\ v4) \neq (v3 \in$ *game-sem* $v2\ x1$ (*selectlike* $v4\ v3\ x0$))) = ((*rr* $x0\ x1 \in$ *game-sem* (*ii* $x0\ x1$) $x1$ (*FF* $x0\ x1$)) $\neq$ (*rr* $x0\ x1 \in$ *game-sem* (*ii* $x0\ x1$) $x1$ (*selectlike* (*FF* $x0\ x1$) (*rr* $x0\ x1$) $x0$)))
by *moura*
have *fact*: $\{V.\ \forall i\ f\ F.\ (f \in$ *game-sem* $i\ \alpha\ F) = (f \in$ *game-sem* $i\ \alpha$ (*selectlike* $F\ f\ V))\} = \{V.\ \forall i\ f\ F.\ f \in$ *game-sem* $i\ \alpha\ F \longrightarrow f \in$ *game-sem* $i\ \alpha$ (*selectlike* $F\ f\ V)\}$
using *f1* **by** (*metis* (*no-types*, *hide-lams*) *Bignorabimus-single subsetCE*)
have *f3*: *rr* $(VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in$ *game-sem* $i\ \alpha\ F) = (f \in$ *game-sem* $i\ \alpha$ (*selectlike* $F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in$ *game-sem* $i\ \alpha\ F \longrightarrow f \in$ *game-sem* $i\ \alpha$ (*selectlike* $F\ f\ V)))\ \alpha \notin$ *game-sem* (*ii* $(VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in$ *game-sem* $i\ \alpha\ F) = (f \in$ *game-sem* $i\ \alpha$ (*selectlike* $F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in$ *game-sem* $i\ \alpha\ F \longrightarrow f \in$ *game-sem* $i\ \alpha$ (*selectlike* $F\ f\ V)))\ \alpha)\ \alpha$ (*selectlike* (*FF* $(VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in$ *game-sem* $i\ \alpha\ F) = (f \in$ *game-sem* $i\ \alpha$ (*selectlike* $F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in$ *game-sem* $i\ \alpha\ F \longrightarrow f \in$ *game-sem* $i\ \alpha$ (*selectlike* $F\ f\ V)))\ \alpha)$ (*rr* $(VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in$ *game-sem* $i\ \alpha\ F) = (f \in$ *game-sem* $i\ \alpha$ (*selectlike* $F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in$ *game-sem* $i\ \alpha\ F \longrightarrow f \in$ *game-sem* $i\ \alpha$ (*selectlike* $F\ f\ V)))\ \alpha)$ $(VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in$ *game-sem* $i\ \alpha\ F) = (f \in$ *game-sem* $i\ \alpha$ (*selectlike* $F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in$ *game-sem* $i\ \alpha\ F \longrightarrow f \in$ *game-sem* $i\ \alpha$ (*selectlike* $F\ f\ V))))) \vee rr\ (VV\ (\lambda V.\ \forall i$

$f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha \in \mathit{game\text{-}sem}\ (\mathit{ii}\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha)\ \alpha\ (FF\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha)$

using *Bignorabimus-single* **by** *blast*

have *f4*: $(\neg\ (\forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V))))))\ \vee\ (\forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))))))\ \wedge\ ((\forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V))))))\ \vee\ (rr\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha \notin \mathit{game\text{-}sem}\ (\mathit{ii}\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha)\ \alpha\ (FF\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha)) = (rr\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha \in \mathit{game\text{-}sem}\ (\mathit{ii}\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha)\ \alpha\ (\mathit{selectlike}\ (FF\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha)\ (rr\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha)\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V))))))$

using *f2* **by** *blast*

have $rr\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha \in \mathit{game\text{-}sem}\ (\mathit{ii}\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha)\ \alpha\ (FF\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha) \longrightarrow (\exists i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \wedge f \notin \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))))\ \vee\ rr\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha \in \mathit{game\text{-}sem}\ (\mathit{ii}\ (VV\ (\lambda V.\ \forall i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha)\ \alpha\ (\mathit{selectlike}\ (FF$

$(VV\ (\lambda V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))$
$(\lambda V.\ \forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ \alpha)$
$(rr\ (VV\ (\lambda V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f$
$V)))\ (\lambda V.\ \forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))$
$\alpha)\ (VV\ (\lambda V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f$
$V)))\ (\lambda V.\ \forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V))))$
by *blast*
moreover
{ assume $\exists\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \wedge f \notin \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ (VV$
$(\lambda V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.$
$\forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V))))$
then have $\neg\ (\forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f$
$(VV\ (\lambda V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))$
$(\lambda V.\ \forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))))$
by *blast*
moreover
{ assume $(\neg\ (\forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f$
$(VV\ (\lambda V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))$
$(\lambda V.\ \forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V))))))) \neq$
$(\forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ (VV\ (\lambda V.\ \forall\, i$
$f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall\, i\ f\ F.$
$f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V))))))$
then have $\{V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}$
$F\ f\ V))\} = \{V.\ \forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f$
$V)\}$
using *fact* **by** *simp* **}**
ultimately have $(\forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}$
$F\ f\ (VV\ (\lambda V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F$
$f\ V)))\ (\lambda V.\ \forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f$
$V))))))) \vee \{V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f$
$V))\} = \{V.\ \forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)\}$
by *satx* **}**
moreover
{ assume $\forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f$
$(VV\ (\lambda V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))$
$(\lambda V.\ \forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))))$
then have $(\neg\ (\forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f$
$(VV\ (\lambda V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))$
$(\lambda V.\ \forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V))))))) \neq$
$(\forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ (VV\ (\lambda V.\ \forall\, i$
$f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V)))\ (\lambda V.\ \forall\, i\ f\ F.$
$f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f\ V))))))$
by *blast*
then have $\{V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}$
$F\ f\ V))\} = \{V.\ \forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha\ (\mathit{selectlike}\ F\ f$
$V)\}$
using *fact* **by** *simp***}**
ultimately have $\{V.\ \forall\, i\ f\ F.\ (f \in \mathit{game\text{-}sem}\ i\ \alpha\ F) = (f \in \mathit{game\text{-}sem}\ i\ \alpha$
$(\mathit{selectlike}\ F\ f\ V))\} = \{V.\ \forall\, i\ f\ F.\ f \in \mathit{game\text{-}sem}\ i\ \alpha\ F \longrightarrow f \in \mathit{game\text{-}sem}\ i\ \alpha$

(*selectlike F f V*)}
using *f4 f3* **by** *satx*
then show *?thesis*
using *Bignorabimus-def* **by** *presburger*
qed

lemma *Bignorabimus-empty* [*simp*]: {} ∈ *Bignorabimus* α
unfolding *Bignorabimus-def* **using** *coempty selectlike-empty*
by *simp*

lemma *Bignorabimus-init*: *x*∉*BVG*(α) ⟹ {*x*}∈*Bignorabimus* α
unfolding *Bignorabimus-def BVG-def*
proof−
assume *x* ∉ {*x*. ∃ *I* ω *X*. ω ∈ *game-sem I* α *X* ∧ ω ∉ *game-sem I* α (*selectlike X* ω {*x*})}
hence ¬(∃ *I* ω *X*. ω ∈ *game-sem I* α *X* ∧ ω ∉ *game-sem I* α (*selectlike X* ω {*x*})) **by** *simp*
hence ∀ *I* ω *X*. (ω ∈ *game-sem I* α *X*) = (ω ∈ *game-sem I* α (*selectlike X* ω {*x*})) **using** *Bignorabimus-single* **by** *blast*
thus {*x*} ∈ {*M*. ∀ *I* ω *X*. (ω ∈ *game-sem I* α *X*) = (ω ∈ *game-sem I* α (*selectlike X* ω *M*))} **by** *simp*
qed

Bignorabimus is closed under union

lemma *Bignorabimus-union*: *M*∈*Bignorabimus* α ⟹ *N*∈*Bignorabimus* α ⟹ (*M*∪*N*)∈*Bignorabimus* α
proof −
assume *a1*: *M*∈*Bignorabimus* α
assume *a2*: *N*∈*Bignorabimus* α
have *h1*: ∀ *I*.∀ ω.∀ *X*. (ω∈*game-sem I* α *X*) ⟷ (ω∈*game-sem I* α (*selectlike X* ω *M*)) **using** *a1*
using *Bignorabimus-equiv Bignorabimus-single* **by** *blast*
have *h2*: ∀ *I*.∀ ω.∀ *X*. (ω∈*game-sem I* α *X*) ⟷ (ω∈*game-sem I* α (*selectlike X* ω *N*)) **using** *a2*
using *Bignorabimus-equiv Bignorabimus-single* **by** *blast*
have *c*: ∀ *I*.∀ ω.∀ *X*. (ω∈*game-sem I* α *X*) ⟷ (ω∈*game-sem I* α (*selectlike X* ω (*M*∪*N*))) **by** (*metis h1 h2 selectlike-compose*)
then show (*M*∪*N*)∈*Bignorabimus* α **unfolding** *Bignorabimus-def* **using** *c* **by** *fastforce*
qed

lemma *Bignorabimus-step*: (⋀*M*. *M*∈*S* ⟹ *M*∈*Bignorabimus* α) ⟹ (⋃ *S*)∈*Bignorabimus* α
proof (*cases S*={})
case *True*
then show *?thesis* **using** *Bignorabimus-empty* **by** *simp*
next
case *nonem*: *False*
then show ⋃ *S*∈*Bignorabimus* α **if** ⋀*M*. *M*∈*S* ⟹ *M*∈*Bignorabimus* α **and**

nonemp:$S \neq \{\}$ **for** S
proof (*induction rule*: *powerset2up-induct*)
case *Finite*
then show *?case* **using** *Bignorabimus-finite* **by** (*meson infinite-super subset-eq that*(*1*))
next
case *Nonempty*
then show *?case* **using** *nonemp* **by** *simp*
next
case (*Base M*)
then show *?case* **using** *that* **by** *simp*
next
case (*Cup S*)
then show *?case* **using** *that Bignorabimus-union* **by** *blast*
qed
qed

Lemma 13 https://doi.org/10.1007/978-3-319-94205-6_15

theorem *boundeffect*: $(\omega \in \textit{game-sem}\ I\ \alpha\ X) = (\omega \in \textit{game-sem}\ I\ \alpha\ (\textit{selectlike}\ X\ \omega\ (-BVG(\alpha))))$
proof−
have *base*: $\{x\} \in Bignorabimus\ \alpha$ **if** *a3*: $x \notin BVG\ \alpha$ **for** x **using** *a3* **and** *Bignorabimus-init* **by** *simp*
have *h*: $-BVG\ \alpha = \bigcup \{xx.\ \exists x.\ xx = \{x\} \wedge x \notin BVG\ \alpha\}$ **by** *blast*
have $(-BVG\ \alpha) \in Bignorabimus\ \alpha$ **using** *base h*

proof −
obtain $VV :: \textit{game} \Rightarrow \textit{variable set set} \Rightarrow \textit{variable set}$ **where**
f1: $\forall x0\ x1.\ (\exists v2.\ v2 \in x1 \wedge v2 \notin Bignorabimus\ x0) = (VV\ x0\ x1 \in x1 \wedge VV\ x0\ x1 \notin Bignorabimus\ x0)$
by *moura*
have $VV\ \alpha\ \{\{v\}\ |v.\ v \notin BVG\ \alpha\} \notin \{\{v\}\ |v.\ v \notin BVG\ \alpha\} \vee VV\ \alpha\ \{\{v\}\ |v.\ v \notin BVG\ \alpha\} \in Bignorabimus\ \alpha$
by *fastforce*
then have $\bigcup \{\{v\}\ |v.\ v \notin BVG\ \alpha\} \in Bignorabimus\ \alpha$
using *f1* **by** (*meson Bignorabimus-step*)
then show *?thesis*
using *h* **by** *presburger*
qed
from *this* **show** *?thesis* **using** *Bignorabimus-def* **by** *blast*
qed

corollary *boundeffect-cor*: $V \cap BVG(\alpha) = \{\} \Longrightarrow (\omega \in \textit{game-sem}\ I\ \alpha\ X) = (\omega \in \textit{game-sem}\ I\ \alpha\ (\textit{selectlike}\ X\ \omega\ V))$
using *boundeffect*
by (*metis disjoint-eq-subset-Compl selectlike-compose sup.absorb-iff2*)

5.4 Static Analysis Observations

lemma *BVG-equiv*: *game-equiv* α $\beta \Longrightarrow BVG(\alpha) = BVG(\beta)$
proof–
assume *a*: *game-equiv* α β
show $BVG(\alpha) = BVG(\beta)$ **unfolding** *BVG-def* **using** *game-equiv-subst-eq*[*OF a*] **by** *metis*
qed

lemmas *union-or* = *Set.Un-iff*

lemma *not-union-or*: $(x \notin A \cup B) = (x \notin A \wedge x \notin B)$
by *simp*

lemma *repv-selectlike-self*: (*repv* ω x $d \in$ *selectlike* X ω $\{x\}$) = ($d{=}\omega(x) \wedge \omega \in X$)
unfolding *selectlike-def* **using** *Vagree-repv-self Vagree-sym*
by (*metis* (*no-types*, *lifting*) *mem-Collect-eq repv-self*)

lemma *repv-selectlike-other*: $x{\neq}y \Longrightarrow$ (*repv* ω x $d \in$ *selectlike* X ω $\{y\}$) = (*repv* ω x $d \in X$)
proof
assume *a*: $x{\neq}y$
then have *h*: $\{y\} \subseteq -\{x\}$ **by** *simp*
show (*repv* ω x $d \in$ *selectlike* X ω $\{y\}$) $\Longrightarrow$ (*repv* ω x $d \in X$) **using** *a selectlike-def Vagree-repv*[*of* ω x d]
by *auto*
show (*repv* ω x $d \in X$) $\Longrightarrow$ (*repv* ω x $d \in$ *selectlike* X ω $\{y\}$)
using *selectlike-def*[**where** $X{=}X$ **and** $\nu{=}\omega$ **and** $V{=}\langle -\{x\}\rangle$] *Vagree-repv*[**where** $\omega{=}\omega$ **and** $x{=}x$ **and** $d{=}d$]
selectlike-antimon[**where** $X{=}X$ **and** $\nu{=}\omega$ **and** $V{=}\langle\{y\}\rangle$ **and** $W{=}\langle -\{x\}\rangle$, *OF h*] *Vagree-sym*[**where** $\nu{=}\langle$*repv* ω x $d\rangle$ **and** $V{=}\langle -\{x\}\rangle$]
by *auto*
qed

lemma *repv-selectlike-other-converse*: $x{\neq}y \Longrightarrow$ (*repv* ω x $d \in X$) = (*repv* ω x $d \in$ *selectlike* X ω $\{y\}$)
using *repv-selectlike-other HOL.eq-commute* **by** *blast*

lemma *BVG-assign-other*: $x{\neq}y \Longrightarrow y \notin BVG(Assign\ x\ \vartheta)$
using *repv-selectlike-other-converse*[**where** $x{=}x$ **and** $y{=}y$] **by** *simp*

lemma *BVG-assign-meta*: $(\bigwedge I\ \omega.$ *term-sem* $I\ \vartheta\ \omega = \omega(x)) \Longrightarrow BVG(Assign\ x\ \vartheta) = \{\}$
and *term-sem* $I\ \vartheta\ \omega \neq \omega(x) \Longrightarrow BVG(Assign\ x\ \vartheta) = \{x\}$

proof–
have *fact*: $BVG(Assign\ x\ \vartheta) \subseteq \{x\}$ **using** *BVG-assign-other* **by** (*metis singleton-iff subsetI*)

from *fact* **show** $(\bigwedge I\ \omega.\ \textit{term-sem}\ I\ \vartheta\ \omega = \omega(x)) \Longrightarrow BVG(\textit{Assign}\ x\ \vartheta) = \{\}$ **using** *BVG-def* **by** *simp*
have *h2*: $\exists I\ \omega.\ \textit{term-sem}\ I\ \vartheta\ \omega \neq \omega(x) \Longrightarrow x \in BVG(\textit{Assign}\ x\ \vartheta)$ **using** *repv-selectlike-self* **by** *auto*
from *fact* **show** *term-sem I ϑ ω ≠ ω(x)* $\Longrightarrow BVG(\textit{Assign}\ x\ \vartheta) = \{x\}$ **using** *BVG-elem h2* **by** *blast*
qed

lemma *BVG-assign*: $BVG(\textit{Assign}\ x\ \vartheta) = (\textit{if}\ (\forall I\ \omega.\ \textit{term-sem}\ I\ \vartheta\ \omega = \omega(x))\ \textit{then}\ \{\}\ \textit{else}\ \{x\})$
using *repv-selectlike-self repv-selectlike-other BVG-assign-other*
proof–
have *c0*: $BVG(\textit{Assign}\ x\ \vartheta) \subseteq \{x\}$ **using** *BVG-assign-other* **by** (*metis singletonI subsetI*)
have *c1*: $\forall I\ \omega.\ \textit{term-sem}\ I\ \vartheta\ \omega = \omega(x) \Longrightarrow BVG(\textit{Assign}\ x\ \vartheta) = \{\}$ **using** *BVG-assign-other* **by** *auto*
have *h2*: $\exists I\ \omega.\ \textit{term-sem}\ I\ \vartheta\ \omega \neq \omega(x) \Longrightarrow x \in BVG(\textit{Assign}\ x\ \vartheta)$ **using** *repv-selectlike-self* **by** *auto*
have *c2*: *∃ I ω. term-sem I ϑ ω ≠ ω(x)* $\Longrightarrow BVG(\textit{Assign}\ x\ \vartheta) = \{x\}$ **using** *c0 h2* **by** *blast*
from *c1* **and** *c2* **show** *?thesis* **by** *simp*
qed

lemma *BVG-ODE-other*: *y≠RVar x* $\Longrightarrow$ *y≠DVar x* $\Longrightarrow$ *y∉BVG(ODE x ϑ)*

proof–
assume *yx*: *y≠RVar x*
assume *yxp*: *y≠DVar x*
show *y∉BVG(ODE x ϑ)*
proof (*rule nonBVG-inc-rule*)
fix *I ω X*
assume *ω ∈ game-sem I (ODE x ϑ) X*
then have *∃ F T. Vagree ω (F(0)) (−{DVar x}) ∧ F(T) ∈ X ∧ solves-ODE I F x ϑ* **by** *simp*
then obtain *F T* **where** *Vagree ω (F(0)) (−{DVar x}) ∧ F(T) ∈ X ∧ solves-ODE I F x ϑ* **by** *blast*
then have *Vagree ω (F(0)) (−{DVar x}) ∧ F(T) ∈ (selectlike X ω {y}) ∧ solves-ODE I F x ϑ*
using *yx yxp solves-Vagree Vagree-def similar-selectlike-mem* **by** *auto*
then have *∃ F T. Vagree ω (F(0)) (−{DVar x}) ∧ F(T) ∈ (selectlike X ω {y}) ∧ solves-ODE I F x ϑ* **by** *blast*
then show *ω ∈ game-sem I (ODE x ϑ) (selectlike X ω {y})* **by** *simp*
qed
qed

This result could be strengthened to a conditional equality based on the RHS values

lemma *BVG-ODE*: *BVG(ODE x ϑ) ⊆ {RVar x,DVar x}*
using *BVG-ODE-other* **by** *blast*

lemma *BVG-test*: $BVG(Test\ \varphi) = \{\}$
unfolding *BVG-def game-sem.simps* **by** *auto*

lemma *BVG-choice*: $BVG(Choice\ \alpha\ \beta) \subseteq BVG(\alpha) \cup BVG(\beta)$
unfolding *BVG-def game-sem.simps* **using** *not-union-or* **by** *auto*

lemma *select-nonBV*: $x \notin BVG(\alpha) \Longrightarrow selectlike\ (game\text{-}sem\ I\ \alpha\ (selectlike\ X\ \omega\ \{x\}))\ \omega\ \{x\} = selectlike\ (game\text{-}sem\ I\ \alpha\ X)\ \omega\ \{x\}$
proof
show $selectlike\ (game\text{-}sem\ I\ \alpha\ (selectlike\ X\ \omega\ \{x\}))\ \omega\ \{x\} \subseteq selectlike\ (game\text{-}sem\ I\ \alpha\ X)\ \omega\ \{x\}$
using *game-sem-mono selectlike-shrinks selectlike-antimon Bignorabimus-single*
by (*metis selectlike-union sup.absorb-iff1*)
next
assume *nonbound*: $x \notin BVG(\alpha)$
then have *fact*: $\{x\} \cap BVG(\alpha) = \{\}$ **by** *auto*
show $selectlike\ (game\text{-}sem\ I\ \alpha\ X)\ \omega\ \{x\} \subseteq selectlike\ (game\text{-}sem\ I\ \alpha\ (selectlike\ X\ \omega\ \{x\}))\ \omega\ \{x\}$
proof
fix μ
assume $\mu \in selectlike\ (game\text{-}sem\ I\ \alpha\ X)\ \omega\ \{x\}$

then have $\mu \in selectlike\ (game\text{-}sem\ I\ \alpha\ (selectlike\ X\ \mu\ \{x\}))\ \omega\ \{x\}$
using *boundeffect-cor*[**where** $\omega{=}\mu$ **and** $V{=}\langle\{x\}\rangle$ **and** $\alpha{=}\alpha$, *OF fact*] *nonbound*
by (*metis ComplD ComplI co-selectlike not-union-or*)
then show $\mu \in selectlike\ (game\text{-}sem\ I\ \alpha\ (selectlike\ X\ \omega\ \{x\}))\ \omega\ \{x\}$ **using** *selectlike-Vagree selectlike-def* **by** *fastforce*
qed
qed

lemma *BVG-compose*: $BVG(Compose\ \alpha\ \beta) \subseteq BVG(\alpha) \cup BVG(\beta)$

proof
fix x
assume *a*: $x \in BVG(Compose\ \alpha\ \beta)$
show $x \in BVG\ \alpha \cup BVG\ \beta$
proof (*rule ccontr*)
assume $x \notin BVG\ \alpha \cup BVG(\beta)$
then have $n\beta$: $x \notin BVG(\beta)$
and $n\alpha$: $x \notin BVG(\alpha)$ **by** *simp-all*
from *a* **have** $\exists I.\exists\omega.\exists X.\ \omega \in game\text{-}sem\ I\ (Compose\ \alpha\ \beta)\ X \wedge \omega \notin game\text{-}sem\ I\ (Compose\ \alpha\ \beta)\ (selectlike\ X\ \omega\ \{x\})$ **by** *simp*
then obtain $I\ \omega\ X$ **where** *adef*: $\omega \in game\text{-}sem\ I\ (Compose\ \alpha\ \beta)\ X \wedge \omega \notin game\text{-}sem\ I\ (Compose\ \alpha\ \beta)\ (selectlike\ X\ \omega\ \{x\})$ **by** *blast*
from *adef* **have** *a1*: $\omega \in game\text{-}sem\ I\ \alpha\ (game\text{-}sem\ I\ \beta\ X)$ **by** *simp*

from *adef* **have** *a2*: $\omega \notin$ *game-sem I* α (*game-sem I* β (*selectlike X* ω $\{x\}$)) **by** *simp*
let *?Y* = *selectlike X* ω $\{x\}$
from *n*α **have** *n*α*c*: $\bigwedge I\ \omega\ X.\ (\omega \in$ *game-sem I* $\alpha\ X) = (\omega \in$ *game-sem I* α (*selectlike X* ω $\{x\}$)) **using** *BVG-nonelem* **by** *simp*
from *n*β **have** *n*β*c*: $\bigwedge I\ \omega\ X.\ (\omega \in$ *game-sem I* $\beta\ X) = (\omega \in$ *game-sem I* β (*selectlike X* ω $\{x\}$)) **using** *BVG-nonelem* **by** *simp*
have *c1*: $\omega \in$ *game-sem I* α (*selectlike* (*game-sem I* β *X*) ω $\{x\}$) **using** *a1* *n*α*c*[**where** *I*=*I* **and** ω=ω **and** *X*=‹*game-sem I* β *X*›] **by** *blast*
have *c2*: $\omega \notin$ *game-sem I* α (*selectlike* (*game-sem I* β *?Y*) ω $\{x\}$) **using** *a2* *n*α*c*[**where** *I*=*I* **and** ω=ω **and** *X*=‹*game-sem I* β *?Y*›] **by** *blast*
from *c2* **have** *c3*: $\omega \notin$ *game-sem I* α (*selectlike* (*game-sem I* β *X*) ω $\{x\}$) **using** *n*β *selectlike-Vagree*
proof−
have *selectlike* (*game-sem I* β *?Y*) ω $\{x\}$ = *selectlike* (*game-sem I* β *X*) ω $\{x\}$ **using** *n*β **by** (*rule select-nonBV*)
thus *?thesis* **using** *c2* **by** *simp*
qed
show *False* **using** *c1 c3 n*β*c*[**where** *I*=*I*] **by** *auto*
qed
qed

The converse inclusion does not hold generally, because $BVG(x := x{+}1;\ x{:=}\ x{-}1) = \{\} \neq BVG(x := x{+}1) \cup BVG(x := x{-}1) = \{x\}$

lemma *BVG*(*Compose* (*Assign x* (*Plus* (*Var x*) (*Number 1*))) (*Assign x* (*Plus* (*Var x*) (*Number* (−*1*)))))
≠ *BVG*(*Assign x* (*Plus* (*Var x*) (*Number 1*))) ∪ *BVG*(*Assign x* (*Plus* (*Var x*) (*Number* (−*1*))))
unfolding *BVG-def selectlike-def repv-def Vagree-def* **by** *auto*

lemma *BVG-loop*: *BVG*(*Loop* α) ⊆ *BVG*(α)
proof
fix *x*
assume *a*: *x*∈*BVG*(*Loop* α)
show *x* ∈ *BVG*(α)
proof (*rule ccontr*)
assume ¬ (*x* ∈ *BVG*(α))
then have *n*α: *x*∉*BVG* α **by** *simp*
from *n*α **have** *n*α*c*: $\bigwedge I\ \omega\ X.\ (\omega \in$ *game-sem I* $\alpha\ X) = (\omega \in$ *game-sem I* α (*selectlike X* ω $\{x\}$)) **using** *BVG-nonelem* **by** *simp*
have *x*∉*BVG*(*Loop* α)
proof (*rule nonBVG-rule*)
fix *I* ω *X*
let *?f* = $\lambda Z.\ X \cup$ *game-sem I* α *Z*
let *?g* = $\lambda Y.$ (*selectlike X* ω $\{x\}$) ∪ *game-sem I* α *Y*
let *?R* = $\lambda Z\ Y.$ *selectlike Z* ω $\{x\}$ = *selectlike Y* ω $\{x\}$
have *?R* (*lfp ?f*) (*lfp ?g*)
proof (*induction rule*: *lfp-lockstep-induct*[**where** *f*=‹*?f*› **and** *g*=‹*?g*› **and**

R=‹*?R*›])
case *monof*
then show *?case* **using** *game-sem-loop-fixpoint-mono* **by** *simp*
next
case *monog*
then show *?case* **using** *game-sem-loop-fixpoint-mono* **by** *simp*
next
case (*step A B*)
then have *IH*: *selectlike A ω* {*x*} = *selectlike B ω* {*x*} **by** *simp*
then show *?case*

proof–
have *selectlike* (*X* ∪ *game-sem I α A*) *ω* {*x*} = *selectlike X ω* {*x*} ∪ *selectlike* (*game-sem I α A*) *ω* {*x*} **using** *selectlike-union* **by** *simp*
also have ... = *selectlike X ω* {*x*} ∪ *selectlike* (*game-sem I α* (*selectlike A ω* {*x*})) *ω* {*x*} **using** *nα select-nonBV* **by** *blast*
also have ... = *selectlike X ω* {*x*} ∪ *selectlike* (*game-sem I α* (*selectlike B ω* {*x*})) *ω* {*x*} **using** *IH* **by** *simp*
also have ... = *selectlike* (*selectlike X ω* {*x*} ∪ *game-sem I α B*) *ω* {*x*} **using** *selectlike-union nα select-nonBV* **by** *auto*
finally show *selectlike* (*X* ∪ *game-sem I α A*) *ω* {*x*} = *selectlike* (*selectlike X ω* {*x*} ∪ *game-sem I α B*) *ω* {*x*} .
qed
next
case (*union M*)
then have *IH*: ∀ (*A*,*B*)∈*M*. *selectlike A ω* {*x*} = *selectlike B ω* {*x*} .
then show *?case*
using *fst-proj-mem*[**where** *M*=*M*] *snd-proj-mem*[**where** *M*=*M*]
selectlike-Sup[**where** *ν*=*ω* **and** *V*=‹{*x*}›] *sup-corr-eq-chain* **by** *simp*

qed
from *this* **show** (*ω* ∈ *game-sem I* (*Loop α*) *X*) = (*ω* ∈ *game-sem I* (*Loop α*) (*selectlike X ω* {*x*}))
by (*metis* (*mono-tags*) *game-sem.simps*(*6*) *lfp-def selectlike-self*)

qed
then show *False* **using** *a* **by** *blast*
qed
qed

lemma *BVG-dual*: *BVG*(*Dual α*) ⊆ *BVG*(*α*)

proof
fix *x*
assume *a*: *x*∈*BVG*(*Dual α*)
show *x*∈*BVG α*
proof–
from *a* **have** ∃ *I*.∃*ω*.∃ *X*. *ω* ∈ *game-sem I* (*Dual α*) *X* ∧ *ω* ∉ *game-sem I* (*Dual*

α) (*selectlike X* ω $\{x\}$) **by** *simp*
then obtain *I* ω *X* **where** *adef*: $\omega \in$ *game-sem I* (*Dual* α) *X* $\wedge$ $\omega \notin$ *game-sem I* (*Dual* α) (*selectlike X* ω $\{x\}$) **by** *blast*
from *adef* **have** *a1*: $\omega \notin$ *game-sem I* α (− *X*) **by** *simp*
from *adef* **have** *a2*: $\omega \in$ *game-sem I* α (− *selectlike X* ω $\{x\}$) **by** *simp*
let *?Y* = − *selectlike X* ω $\{x\}$
have *f1*: $\omega \in$ *game-sem I* α *?Y* **by** (*rule a2*)
have *f2*: $\omega \notin$ *game-sem I* α (*selectlike ?Y* ω $\{x\}$) **using** *a1 selectlike-co-selectlike*
by (*metis* (*no-types*, *lifting*) *selectlike-shrinks monotone dual-order.trans subset-Compl-singleton*)
show *x*∈*BVG*(α) **using** *f1 f2* **by** *auto*
qed
qed

end
theory *USubst*
imports
Complex-Main
Syntax
Static-Semantics
Coincidence
Denotational-Semantics
begin

6 Uniform Substitution

uniform substitution representation as tuple of partial maps from identifiers to type-compatible replacements.

type-synonym *usubst* =
(*ident* ⇀ *trm*) × (*ident* ⇀ *trm*) × (*ident* ⇀ *fml*) × (*ident* ⇀ *game*)

abbreviation *SConst*:: *usubst* ⇒ (*ident* ⇀ *trm*)
where *SConst* ≡ (λ(*F0*, -, -, -). *F0*)
abbreviation *SFuncs*:: *usubst* ⇒ (*ident* ⇀ *trm*)
where *SFuncs* ≡ (λ(-, *F*, -, -). *F*)
abbreviation *SPreds*:: *usubst* ⇒ (*ident* ⇀ *fml*)
where *SPreds* ≡ (λ(-, -, *P*, -). *P*)
abbreviation *SGames*:: *usubst* ⇒ (*ident* ⇀ *game*)
where *SGames* ≡ (λ(-, -, -, *G*). *G*)

crude approximation of size which is enough for termination arguments

definition *usubstsize*:: *usubst* ⇒ *nat*
where *usubstsize* σ = (*if* (*dom* (*SFuncs* σ) = {} ∧ *dom* (*SPreds* σ) = {}) *then 1 else 2*)

dot is some fixed constant function symbol that is reserved for the purposes of the substitution

definition *dot*:: *trm*

where *dot* = *Const* (*dotid*)

6.1 Strict Mechanism for Handling Substitution Partiality in Isabelle

Optional terms that result from a substitution, either actually a term or just none to indicate that the substitution clashed

type-synonym *trmo* = *trm option*

abbreviation *undeft*:: *trmo* **where** *undeft* ≡ *None*
abbreviation *Aterm*:: *trm* ⇒ *trmo* **where** *Aterm* ≡ *Some*

lemma *undeft-None*: *undeft=None* **by** *simp*
lemma *Aterm-Some*: *Aterm ϑ=Some ϑ* **by** *simp*

lemma *undeft-equiv*: (*ϑ≠undeft*) = (∃ *t*. *ϑ=Aterm t*)
by *simp*

Plus on defined terms, strict undeft otherwise

fun *Pluso* :: *trmo* ⇒ *trmo* ⇒ *trmo*
where
Pluso (*Aterm ϑ*) (*Aterm η*) = *Aterm*(*Plus ϑ η*)
| *Pluso undeft η* = *undeft*
| *Pluso ϑ undeft* = *undeft*

Times on defined terms, strict undeft otherwise

fun *Timeso* :: *trmo* ⇒ *trmo* ⇒ *trmo*
where
Timeso (*Aterm ϑ*) (*Aterm η*) = *Aterm*(*Times ϑ η*)
| *Timeso undeft η* = *undeft*
| *Timeso ϑ undeft* = *undeft*

fun *Differentialo* :: *trmo* ⇒ *trmo*
where
Differentialo (*Aterm ϑ*) = *Aterm*(*Differential ϑ*)
| *Differentialo undeft* = *undeft*

lemma *Pluso-undef*: (*Pluso ϑ η* = *undeft*) = (*ϑ=undeft* ∨ *η=undeft*) **by** (*metis Pluso.elims option.distinct*(*1*))
lemma *Timeso-undef*: (*Timeso ϑ η* = *undeft*) = (*ϑ=undeft* ∨ *η=undeft*) **by** (*metis Timeso.elims option.distinct*(*1*))
lemma *Differentialo-undef*: (*Differentialo ϑ* = *undeft*) = (*ϑ=undeft*) **by** (*metis Differentialo.elims option.distinct*(*1*))

type-synonym *fmlo* = *fml option*

abbreviation *undeff*:: *fmlo* **where** *undeff* ≡ *None*

abbreviation *Afml*:: *fml* ⇒ *fmlo* **where** *Afml* ≡ *Some*

type-synonym *gameo* = *game option*

abbreviation *undefg*:: *gameo* **where** *undefg* ≡ *None*
abbreviation *Agame*:: *game* ⇒ *gameo* **where** *Agame* ≡ *Some*

lemma *undeff-equiv*: (*φ*≠*undeff*) = (∃ *f*. *φ*=*Afml f*)
by *simp*

lemma *undefg-equiv*: (*α*≠*undefg*) = (∃ *g*. *α*=*Agame g*)
by *simp*

Geq on defined terms, strict undeft otherwise

fun *Geqo* :: *trmo* ⇒ *trmo* ⇒ *fmlo*
where
Geqo (*Aterm ϑ*) (*Aterm η*) = *Afml*(*Geq ϑ η*)
| *Geqo undeft η* = *undeff*
| *Geqo ϑ undeft* = *undeff*

Not on defined formulas, strict undeft otherwise

fun *Noto* :: *fmlo* ⇒ *fmlo*
where
Noto (*Afml φ*) = *Afml*(*Not φ*)
| *Noto undeff* = *undeff*

And on defined formulas, strict undeft otherwise

fun *Ando* :: *fmlo* ⇒ *fmlo* ⇒ *fmlo*
where
Ando (*Afml φ*) (*Afml ψ*) = *Afml*(*And φ ψ*)
| *Ando undeff ψ* = *undeff*
| *Ando φ undeff* = *undeff*

Exists on defined formulas, strict undeft otherwise

fun *Existso* :: *variable* ⇒ *fmlo* ⇒ *fmlo*
where
Existso x (*Afml φ*) = *Afml*(*Exists x φ*)
| *Existso x undeff* = *undeff*

Diamond on defined games/formulas, strict undeft otherwise

fun *Diamondo* :: *gameo* ⇒ *fmlo* ⇒ *fmlo*
where
Diamondo (*Agame α*) (*Afml φ*) = *Afml*(*Diamond α φ*)
| *Diamondo undefg φ* = *undeff*
| *Diamondo α undeff* = *undeff*

lemma *Geqo-undef*: (*Geqo ϑ η* = *undeff*) = (*ϑ*=*undeft* ∨ *η*=*undeft*)
by (*metis Geqo.elims option.distinct*(*1*))

lemma *Noto-undef*: (*Noto* φ = *undeff*) = (φ=*undeff*)
by (*metis Noto.elims option.distinct*(*1*))
lemma *Ando-undef*: (*Ando* φ ψ = *undeff*) = (φ=*undeff* $\vee$ ψ=*undeff*)
by (*metis Ando.elims option.distinct*(*1*))
lemma *Existso-undef*: (*Existso* *x* φ = *undeff*) = (φ=*undeff*)
by (*metis Existso.elims option.distinct*(*1*))
lemma *Diamondo-undef*: (*Diamondo* α φ = *undeff*) = (α=*undefg* $\vee$ φ=*undeff*)
by (*metis Diamondo.elims option.distinct*(*1*))

Assign on defined terms, strict undefg otherwise

fun *Assigno* :: *variable* ⇒ *trmo* ⇒ *gameo*
where
Assigno *x* (*Aterm* ϑ) = *Agame*(*Assign* *x* ϑ)
| *Assigno* *x* *undeft* = *undefg*

fun *ODEo* :: *ident* ⇒ *trmo* ⇒ *gameo*
where
ODEo *x* (*Aterm* ϑ) = *Agame*(*ODE* *x* ϑ)
| *ODEo* *x* *undeft* = *undefg*

Test on defined formulas, strict undefg otherwise

fun *Testo* :: *fmlo* ⇒ *gameo*
where
Testo (*Afml* φ) = *Agame*(*Test* φ)
| *Testo* *undeff* = *undefg*

Choice on defined games, strict undefg otherwise

fun *Choiceo* :: *gameo* ⇒ *gameo* ⇒ *gameo*
where
Choiceo (*Agame* α) (*Agame* β) = *Agame*(*Choice* α β)
| *Choiceo* α *undefg* = *undefg*
| *Choiceo* *undefg* β = *undefg*

Compose on defined games, strict undefg otherwise

fun *Composeo* :: *gameo* ⇒ *gameo* ⇒ *gameo*
where
Composeo (*Agame* α) (*Agame* β) = *Agame*(*Compose* α β)
| *Composeo* α *undefg* = *undefg*
| *Composeo* *undefg* β = *undefg*

Loop on defined games, strict undefg otherwise

fun *Loopo* :: *gameo* ⇒ *gameo*
where
Loopo (*Agame* α) = *Agame*(*Loop* α)
| *Loopo* *undefg* = *undefg*

Dual on defined games, strict undefg otherwise

fun *Dualo* :: *gameo* ⇒ *gameo*

where
  Dualo (*Agame* α) = *Agame*(*Dual* α)
| *Dualo undefg* = *undefg*

lemma *Assigno-undef*: (*Assigno x* ϑ = *undefg*) = (ϑ=*undeft*) **by** (*metis Assigno.elims option.distinct*(*1*))
lemma *ODEo-undef*: (*ODEo x* ϑ = *undefg*) = (ϑ=*undeft*) **by** (*metis ODEo.elims option.distinct*(*1*))
lemma *Testo-undef*: (*Testo* φ = *undefg*) = (φ=*undeff*) **by** (*metis Testo.elims option.distinct*(*1*))
lemma *Choiceo-undef*: (*Choiceo* α β = *undefg*) = (α=*undefg* $\vee$ β=*undefg*) **by** (*metis Choiceo.elims option.distinct*(*1*))
lemma *Composeo-undef*: (*Composeo* α β = *undefg*) = (α=*undefg* $\vee$ β=*undefg*) **by** (*metis Composeo.elims option.distinct*(*1*))
lemma *Loopo-undef*: (*Loopo* α = *undefg*) = (α=*undefg*) **by** (*metis Loopo.elims option.distinct*(*1*))
lemma *Dualo-undef*: (*Dualo* α = *undefg*) = (α=*undefg*) **by** (*metis Dualo.elims option.distinct*(*1*))

6.2 Recursive Application of One-Pass Uniform Substitution

dotsubstt ϑ is the dot substitution {. $^\sim$> ϑ} substituting a term for the . function symbol

definition *dotsubstt*:: *trm* $\Rightarrow$ *usubst*
  where *dotsubstt* ϑ = (
      (λf. (*if f*=*dotid then* (*Some*(ϑ)) *else None*)),
      (λ-. *None*),
      (λ-. *None*),
      (λ-. *None*)
)

definition *usappconst*:: *usubst* $\Rightarrow$ *variable set* $\Rightarrow$ *ident* $\Rightarrow$ (*trmo*)
where *usappconst* σ *U f* $\equiv$ (*case SConst* σ *f of Some r* $\Rightarrow$ *if FVT*(*r*)$\cap$*U*={} *then Aterm*(*r*) *else undeft* | *None* $\Rightarrow$ *Aterm*(*Const f*))

function *usubstappt*:: *usubst* $\Rightarrow$ *variable set* $\Rightarrow$ (*trm* $\Rightarrow$ *trmo*)
where
  usubstappt σ *U* (*Var x*) = *Aterm* (*Var x*)
| *usubstappt* σ *U* (*Number r*) = *Aterm* (*Number r*)
| *usubstappt* σ *U* (*Const f*) = *usappconst* σ *U f*
| *usubstappt* σ *U* (*Func f* ϑ) =
  (*case usubstappt* σ *U* ϑ *of undeft* $\Rightarrow$ *undeft*
                | *Aterm* $\sigma\vartheta$ $\Rightarrow$ (*case SFuncs* σ *f of Some r* $\Rightarrow$ *if FVT*(*r*)$\cap$*U*={} *then usubstappt*(*dotsubstt* $\sigma\vartheta$) {} *r else undeft* | *None* $\Rightarrow$ *Aterm*(*Func f* $\sigma\vartheta$)))
| *usubstappt* σ *U* (*Plus* ϑ η) = *Pluso* (*usubstappt* σ *U* ϑ) (*usubstappt* σ *U* η)
| *usubstappt* σ *U* (*Times* ϑ η) = *Timeso* (*usubstappt* σ *U* ϑ) (*usubstappt* σ *U* η)
| *usubstappt* σ *U* (*Differential* ϑ) = *Differentialo* (*usubstappt* σ *allvars* ϑ)
  by *pat-completeness auto*

termination
by (*relation measures* [$\lambda(\sigma,U,\vartheta)$. *usubstsize* σ , $\lambda(\sigma,U,\vartheta)$. *size* ϑ])
(*auto simp add*: *usubstsize-def dotsubstt-def*)

declare *Let-def* [*simp*]

function *usubstappf*:: *usubst* ⇒ *variable set* ⇒ (*fml* ⇒ *fmlo*)
and *usubstappp*:: *usubst* ⇒ *variable set* ⇒ (*game* ⇒ *variable set* × *gameo*)
where
usubstappf σ *U* (*Pred p* ϑ) =
(*case usubstappt* σ *U* ϑ *of undeft* ⇒ *undeff*
| *Aterm* $\sigma\vartheta$ ⇒ (*case SPreds* σ *p of Some r* ⇒ *if FVF*(*r*)∩*U*={}
then usubstappf(*dotsubstt* $\sigma\vartheta$) {} *r else undeff* | *None* ⇒ *Afml*(*Pred p* $\sigma\vartheta$)))
| *usubstappf* σ *U* (*Geq* ϑ η) = *Geqo* (*usubstappt* σ *U* ϑ) (*usubstappt* σ *U* η)
| *usubstappf* σ *U* (*Not* φ) = *Noto* (*usubstappf* σ *U* φ)
| *usubstappf* σ *U* (*And* φ ψ) = *Ando* (*usubstappf* σ *U* φ) (*usubstappf* σ *U* ψ)
| *usubstappf* σ *U* (*Exists x* φ) = *Existso x* (*usubstappf* σ (*U*∪{*x*}) φ)
| *usubstappf* σ *U* (*Diamond* α φ) = (*let* $V\alpha$ = *usubstappp* σ *U* α *in Diamondo*
(*snd* $V\alpha$) (*usubstappf* σ (*fst* $V\alpha$) φ))

| *usubstappp* σ *U* (*Game a*) =
(*case SGames* σ *a of Some r* ⇒ (*U*∪*BVG*(*r*),*Agame r*)
| *None* ⇒ (*allvars*,*Agame*(*Game a*)))
| *usubstappp* σ *U* (*Assign x* ϑ) = (*U*∪{*x*}, *Assigno x* (*usubstappt* σ *U* ϑ))
| *usubstappp* σ *U* (*Test* φ) = (*U*, *Testo* (*usubstappf* σ *U* φ))
| *usubstappp* σ *U* (*Choice* α β) =
(*let* $V\alpha$ = *usubstappp* σ *U* α *in*
let $W\beta$ = *usubstappp* σ *U* β *in*
(*fst* $V\alpha$∪*fst* $W\beta$, *Choiceo* (*snd* $V\alpha$) (*snd* $W\beta$)))
| *usubstappp* σ *U* (*Compose* α β) =
(*let* $V\alpha$ = *usubstappp* σ *U* α *in*
let $W\beta$ = *usubstappp* σ (*fst* $V\alpha$) β *in*
(*fst* $W\beta$, *Composeo* (*snd* $V\alpha$) (*snd* $W\beta$)))
| *usubstappp* σ *U* (*Loop* α) =
(*let V* = *fst*(*usubstappp* σ *U* α) *in*
(*V*, *Loopo* (*snd*(*usubstappp* σ *V* α))))
| *usubstappp* σ *U* (*Dual* α) =
(*let* $V\alpha$ = *usubstappp* σ *U* α *in* (*fst* $V\alpha$, *Dualo* (*snd* $V\alpha$)))
| *usubstappp* σ *U* (*ODE x* ϑ) = (*U*∪{*RVar x*,*DVar x*}, *ODEo x* (*usubstappt* σ
(*U*∪{*RVar x*,*DVar x*}) ϑ))
by *pat-completeness auto*
termination
by (*relation measures* [(λk. *usubstsize* (*case k of Inl*(σ,*U*,φ) ⇒ σ | *Inr*(σ,*U*,α) ⇒
σ)) , (λk. *case k of Inl* (σ,*U*,φ) ⇒ *size* φ | *Inr* (σ,*U*,α) ⇒ *size* α)])
(*auto simp add*: *usubstsize-def dotsubstt-def*)

Induction Principles for Uniform Substitutions

lemmas *usubstappt-induct* = *usubstappt.induct* [*case-names Var Number Const*

FuncMatch Plus Times Differential]
lemmas *usubstappfp-induct = usubstappf-usubstappp.induct* [*case-names Pred Geq Not And Exists Diamond Game Assign Test Choice Compose Loop Dual ODE*]

Simple Observations for Automation More automation for Case

lemma *usappconst-simp* [*simp*]: *SConst σ f = Some r ⟹ FVT(r)∩U={} ⟹ usappconst σ U f = Aterm(r)*
and *SConst σ f = None ⟹ usappconst σ U f = Aterm(Const f)*
and *SConst σ f = Some r ⟹ FVT(r)∩U≠{} ⟹ usappconst σ U f = undeft*
unfolding *usappconst-def* **by** *auto*

lemma *usappconst-conv*: *usappconst σ U f≠undeft ⟹*
SConst σ f = None ∨ (∃ r. SConst σ f = Some r ∧ FVT(r)∩U={})

proof−
assume *as*: *usappconst σ U f≠undeft*
show *SConst σ f = None ∨ (∃ r. SConst σ f = Some r ∧ FVT(r)∩U={})*
proof (*cases SConst σ f*)
case *None*
then show *?thesis*
by *auto*
next
case (*Some a*)
then show *?thesis* **using** *as usappconst-def*[**where** *σ=σ* **and** *U=U* **and** *f=f*] *option.distinct(1)* **by** *fastforce*
qed
qed

lemma *usubstappt-const* [*simp*]: *SConst σ f = Some r ⟹ FVT(r)∩U={} ⟹ usubstappt σ U (Const f) = Aterm(r)*
and *SConst σ f = None ⟹ usubstappt σ U (Const f) = Aterm(Const f)*
and *SConst σ f = Some r ⟹ FVT(r)∩U≠{} ⟹ usubstappt σ U (Const f) = undeft*
by (*auto simp add*: *usappconst-def*)

lemma *usubstappt-const-conv*: *usubstappt σ U (Const f)≠undeft ⟹*
SConst σ f = None ∨ (∃ r. SConst σ f = Some r ∧ FVT(r)∩U={})
using *usappconst-conv* **by** *auto*

lemma *usubstappt-func* [*simp*]: *SFuncs σ f = Some r ⟹ FVT(r)∩U={} ⟹ usubstappt σ U ϑ = Aterm σϑ ⟹*
usubstappt σ U (Func f ϑ) = usubstappt (dotsubstt σϑ) {} r
and *SFuncs σ f=None ⟹ usubstappt σ U ϑ = Aterm σϑ ⟹ usubstappt σ U (Func f ϑ) = Aterm(Func f σϑ)*
and *usubstappt σ U ϑ = undeft ⟹ usubstappt σ U (Func f ϑ) = undeft*
by *auto*

lemma *usubstappt-func2* [*simp*]: *SFuncs σ f = Some r ⟹ FVT(r)∩U≠{} ⟹*

usubstappt σ U (*Func* f ϑ) = *undeft*
by (*cases usubstappt* σ U ϑ) (*auto*)

lemma *usubstappt-func-conv*: *usubstappt* σ U (*Func* f ϑ) $\neq$ *undeft* $\Longrightarrow$
usubstappt σ U ϑ $\neq$ *undeft* $\wedge$
(*SFuncs* σ f = *None* $\vee$ ($\exists$ r. *SFuncs* σ f = *Some* r $\wedge$ $FVT(r) \cap U = \{\}$))
by (*metis* (*lifting*) *option.simps*(4) *undeft-equiv usubstappt.simps*(4) *usubstappt-func2*)

lemma *usubstappt-plus-conv*: *usubstappt* σ U (*Plus* ϑ η) $\neq$ *undeft* $\Longrightarrow$
usubstappt σ U ϑ $\neq$ *undeft* $\wedge$ *usubstappt* σ U η $\neq$ *undeft*
by (*simp add*: *Pluso-undef*)

lemma *usubstappt-times-conv*: *usubstappt* σ U (*Times* ϑ η) $\neq$ *undeft* $\Longrightarrow$
usubstappt σ U ϑ $\neq$ *undeft* $\wedge$ *usubstappt* σ U η $\neq$ *undeft*
by (*simp add*: *Timeso-undef*)

lemma *usubstappt-differential-conv*: *usubstappt* σ U (*Differential* ϑ) $\neq$ *undeft* $\Longrightarrow$
usubstappt σ *allvars* ϑ $\neq$ *undeft*
by (*simp add*: *Differentialo-undef*)

lemma *usubstappf-pred* [*simp*]: *SPreds* σ p = *Some* r $\Longrightarrow$ $FVF(r) \cap U = \{\}$ $\Longrightarrow$ *usubstappt* σ U ϑ = *Aterm* $\sigma\vartheta$ $\Longrightarrow$
usubstappf σ U (*Pred* p ϑ) = *usubstappf* (*dotsubstt* $\sigma\vartheta$) {} r
and *SPreds* σ p = *None* $\Longrightarrow$ *usubstappt* σ U ϑ = *Aterm* $\sigma\vartheta$ $\Longrightarrow$ *usubstappf* σ U (*Pred* p ϑ) = *Afml*(*Pred* p $\sigma\vartheta$)
and *usubstappt* σ U ϑ = *undeft* $\Longrightarrow$ *usubstappf* σ U (*Pred* p ϑ) = *undeff*
by *auto*

lemma *usubstappf-pred2* [*simp*]: *SPreds* σ p = *Some* r $\Longrightarrow$ $FVF(r) \cap U \neq \{\}$ $\Longrightarrow$ *usubstappf* σ U (*Pred* p ϑ) = *undeff*
by (*cases usubstappt* σ U ϑ) (*auto*)

lemma *usubstappf-pred-conv*: *usubstappf* σ U (*Pred* p ϑ) $\neq$ *undeff* $\Longrightarrow$
usubstappt σ U ϑ $\neq$ *undeft* $\wedge$
(*SPreds* σ p = *None* $\vee$ ($\exists$ r. *SPreds* σ p = *Some* r $\wedge$ $FVF(r) \cap U = \{\}$))
by (*metis* (*lifting*) *option.simps*(4) *undeff-equiv usubstappf.simps*(1) *usubstappf-pred2*)

lemma *usubstappf-geq*: *usubstappt* σ U ϑ $\neq$ *undeft* $\Longrightarrow$ *usubstappt* σ U η $\neq$ *undeft* $\Longrightarrow$
usubstappf σ U (*Geq* ϑ η) = *Afml*(*Geq* (*the* (*usubstappt* σ U ϑ)) (*the* (*usubstappt* σ U η)))
by *fastforce*

lemma *usubstappf-geq-conv*: *usubstappf* σ U (*Geq* ϑ η) $\neq$ *undeff* $\Longrightarrow$
usubstappt σ U ϑ $\neq$ *undeft* $\wedge$ *usubstappt* σ U η $\neq$ *undeft*
by (*simp add*: *Geqo-undef*)

lemma *usubstappf-geqr*: *usubstappf σ U (Geq ϑ η) ≠ undeff ⟹*
usubstappf σ U (Geq ϑ η) = Afml(Geq (the (usubstappt σ U ϑ)) (the (usubstappt σ U η)))
using *usubstappf-geq usubstappf-geq-conv* **by** *blast*

lemma *usubstappf-exists*: *usubstappf σ U (Exists x φ) ≠ undeff ⟹*
usubstappf σ U (Exists x φ) = Afml(Exists x (the (usubstappf σ (U∪{x}) φ)))
using *Existso-undef* **by** *auto*

lemma *usubstappp-game* [*simp*]: *SGames σ a = Some r ⟹ usubstappp σ U (Game a) = (U∪BVG(r),Agame(r))*
and *SGames σ a = None ⟹ usubstappp σ U (Game a) = (allvars,Agame(Game a))*
by *auto*

lemma *usubstappp-choice* [*simp*]: *usubstappp σ U (Choice α β) =*
(fst(usubstappp σ U α)∪fst(usubstappp σ U β), Choiceo (snd(usubstappp σ U α)) (snd(usubstappp σ U β)))
by *auto*

lemma *usubstappp-choice-conv* : *snd(usubstappp σ U (Choice α β)) ≠ undefg ⟹*
snd(usubstappp σ U α) ≠ undefg ∧ snd(usubstappp σ U β) ≠ undefg
by (*simp add*: *Choiceo-undef*)

lemma *usubstappp-compose* [*simp*]: *usubstappp σ U (Compose α β) =*
(fst(usubstappp σ (fst(usubstappp σ U α)) β), Composeo (snd(usubstappp σ U α)) (snd(usubstappp σ (fst(usubstappp σ U α)) β)))
by *simp*

lemma *usubstappp-loop*: *usubstappp σ U (Loop α) =*
(fst(usubstappp σ U α), Loopo (snd(usubstappp σ (fst(usubstappp σ U α)) α)))
by *auto*

lemma *usubstappp-dual* [*simp*]: *usubstappp σ U (Dual α) =*
(fst(usubstappp σ U α), Dualo (snd (usubstappp σ U α)))
by *simp*

7 Soundness of Uniform Substitution

7.1 USubst Application is a Function of Deterministic Result

lemma *usubstappt-det*: *usubstappt σ U ϑ ≠ undeft ⟹ usubstappt σ V ϑ ≠ undeft ⟹*
usubstappt σ U ϑ = usubstappt σ V ϑ
proof (*induction ϑ*)
case (*Var x*)
then show *?case* **by** *simp*
next
case (*Number x*)

then show *?case* **by** *simp*
next
case (*Const f*)
then show *?case*

proof −
have *f1*: *usubstappt σ U* (*Const f*) = (*case SConst σ f of None ⇒ Aterm* (*Const f*) | *Some t ⇒ if FVT t ∩ U* = {} *then Aterm t else undeft*)
by (*simp add*: *usappconst-def*)
have *f2*: ∀ *z f za*. *if za* = *undeft then* (*case za of None ⇒ z*::*trm option* | *Some x ⇒ f x*) = *z else* (*case za of None ⇒ z* | *Some x ⇒ f x*) = *f* (*the za*)
by *force*
then have *SConst σ f* ≠ *undeft* ⟶ (*if FVT* (*the* (*SConst σ f*)) ∩ *U* = {} *then Aterm* (*the* (*SConst σ f*)) *else undeft*) = *usappconst σ U f*
by (*simp add*: *usappconst-def*)
then have *f3*: *SConst σ f* ≠ *undeft* ⟶ *FVT* (*the* (*SConst σ f*)) ∩ *U* = {}
by (*metis Const.prems*(*1*) *usubstappt.simps*(*3*))
have *SConst σ f* ≠ *undeft* ⟶ (*if FVT* (*the* (*SConst σ f*)) ∩ *V* = {} *then Aterm* (*the* (*SConst σ f*)) *else undeft*) = *usappconst σ V f*
using *f2 usappconst-def* **by** *presburger*
then have *SConst σ f* ≠ *undeft* ⟶ *FVT* (*the* (*SConst σ f*)) ∩ *V* = {}
by (*metis* (*no-types*) *Const.prems*(*2*) *usubstappt.simps*(*3*))
then have *f4*: *SConst σ f* ≠ *undeft* ⟶ *usubstappt σ U* (*Const f*) = *usappconst σ V f*
using *f3 f2 f1 usappconst-def* **by** *presburger*
{ assume *usubstappt σ U* (*Const f*) ≠ *usubstappt σ V* (*Const f*)
then have *usubstappt σ U* (*Const f*) ≠ (*case SConst σ f of None ⇒ Aterm* (*Const f*) | *Some t ⇒ if FVT t ∩ V* = {} *then Aterm t else undeft*)
by (*simp add*: *usappconst-def*)
then have *SConst σ f* ≠ *undeft*
using *f2 f1* **by** (*metis* (*no-types*))
then have *?thesis*
using *f4* **by** *simp* **}**
then show *?thesis*
by *blast*
qed
next
case (*Func f ϑ*)
then show *?case* **using** *usubstappt-func*

proof −
have *f1*: (*case usubstappt σ U ϑ of None ⇒ undeft* | *Some t ⇒* (*case SFuncs σ f of None ⇒ Aterm* (*trm.Func f t*) | *Some ta ⇒ if FVT ta ∩ U* = {} *then usubstappt* (*dotsubstt t*) {} *ta else undeft*)) ≠ *undeft*
using *Func*(*2*) **by** *auto*
have *f2*: ∀ *z f za*. *if za* = *undeft then* (*case za of None ⇒ z*::*trm option* | *Some x ⇒ f x*) = *z else* (*case za of None ⇒ z* | *Some x ⇒ f x*) = *f* (*the za*)
by *force*

then have *f3*: *usubstappt σ U ϑ ≠ undeft*
using *f1* **by** *meson*
have (*case usubstappt σ V ϑ of None ⇒ undeft | Some t ⇒ (case SFuncs σ f of None ⇒ Aterm (trm.Func f t) | Some ta ⇒ if FVT ta ∩ V = {} then usubstappt (dotsubstt t) {} ta else undeft)) ≠ undeft*
using *Func(3)* **by** *auto*
then have *f4*: *usubstappt σ V ϑ ≠ undeft*
using *f2* **by** *meson*
then have *f5*: *usubstappt σ U (trm.Func f ϑ) = (case SFuncs σ f of None ⇒ Aterm (trm.Func f (the (usubstappt σ V ϑ))) | Some t ⇒ if FVT t ∩ U = {} then usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} t else undeft)*
using *f3 f2 Func(1) usubstappt.simps(4)* **by** *presburger*
have *SFuncs σ f ≠ undeft ⟶ (if FVT (the (SFuncs σ f)) ∩ U = {} then usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SFuncs σ f)) else undeft) = usubstappt σ U (trm.Func f ϑ)*
using *f4 f3 f2 Func(1) usubstappt.simps(4)* **by** *presburger*
then have *f6*: *SFuncs σ f ≠ undeft ⟶ FVT (the (SFuncs σ f)) ∩ U = {}*
by (*metis Func(2)*)
have *f7*: (*case usubstappt σ V ϑ of None ⇒ undeft | Some t ⇒ (case SFuncs σ f of None ⇒ Aterm (trm.Func f t) | Some ta ⇒ if FVT ta ∩ V = {} then usubstappt (dotsubstt t) {} ta else undeft)) = (case SFuncs σ f of None ⇒ Aterm (trm.Func f (the (usubstappt σ V ϑ))) | Some t ⇒ if FVT t ∩ V = {} then usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} t else undeft)*
using *f4 f2* **by** *presburger*
then have *SFuncs σ f ≠ undeft ⟶ (if FVT (the (SFuncs σ f)) ∩ V = {} then usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SFuncs σ f)) else undeft) = usubstappt σ V (trm.Func f ϑ)*
using *f2* **by** *simp*
then have *f8*: *SFuncs σ f ≠ undeft ⟶ FVT (the (SFuncs σ f)) ∩ V = {}*
by (*metis* (*full-types*) *Func(3)*)
{ assume *usubstappt σ U (trm.Func f ϑ) ≠ usubstappt σ V (trm.Func f ϑ)*
moreover
{ assume (*case SFuncs σ f of None ⇒ Aterm (trm.Func f (the (usubstappt σ V ϑ))) | Some t ⇒ if FVT t ∩ V = {} then usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} t else undeft) ≠ Aterm (trm.Func f (the (usubstappt σ V ϑ)))*
then have *SFuncs σ f ≠ undeft*
using *f2* **by** *meson* **}**
ultimately have *SFuncs σ f ≠ undeft*
using *f7 f5* **by** *fastforce*
then have *?thesis*
using *f8 f7 f6 f5 f2* **by** *simp* **}**
then show *?thesis*
by *blast*
qed
next
case (*Plus ϑ1 ϑ2*)
then show *?case* **using** *Pluso-undef* **by** *auto*
next
case (*Times ϑ1 ϑ2*)

then show *?case* **using** *Timeso-undef* **by** *auto*
next
case (*Differential* ϑ)
then show *?case* **using** *Differentialo-undef* **by** *auto*
qed

lemma *usubstappf-and-usubstappp-det*:
shows *usubstappf* σ U $\varphi \neq$ *undeff* $\Longrightarrow$ *usubstappf* σ V $\varphi \neq$ *undeff* $\Longrightarrow$ *usubstappf* σ U φ = *usubstappf* σ V φ
and *snd*(*usubstappp* σ U α) $\neq$ *undefg* $\Longrightarrow$ *snd*(*usubstappp* σ V α) $\neq$ *undefg* $\Longrightarrow$ *snd*(*usubstappp* σ U α) = *snd*(*usubstappp* σ V α)
proof (*induction* φ **and** α *arbitrary*: U V **and** U V)
case (*Pred p* ϑ)
then show *?case* **using** *usubstappt-det usubstappf-pred*

proof −
have *f1*: (*case usubstappt* σ U ϑ *of None* ⇒ *undeff* | *Some t* ⇒ (*case SPreds* σ *p of None* ⇒ *Afml* (*Pred p t*) | *Some f* ⇒ *if FVF f* ∩ U = {} *then usubstappf* (*dotsubstt t*) {} *f else undeff*)) ≠ *undeff*
using *Pred.prems*(*1*) **by** *auto*
have *f2*: ∀ *z f za*. *if za* = *undeft then* (*case za of None* ⇒ *z*::*fml option* | *Some x* ⇒ *f x*) = *z else* (*case za of None* ⇒ *z* | *Some x* ⇒ *f x*) = *f* (*the za*)
by (*simp add*: *option.case-eq-if*)
then have *f3*: (*case usubstappt* σ U ϑ *of None* ⇒ *undeff* | *Some t* ⇒ (*case SPreds* σ *p of None* ⇒ *Afml* (*Pred p t*) | *Some f* ⇒ *if FVF f* ∩ U = {} *then usubstappf* (*dotsubstt t*) {} *f else undeff*)) = (*case SPreds* σ *p of None* ⇒ *Afml* (*Pred p* (*the* (*usubstappt* σ U ϑ))) | *Some f* ⇒ *if FVF f* ∩ U = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ U ϑ))) {} *f else undeff*)
using *f1* **by** *meson*
have *f4*: (*case usubstappt* σ V ϑ *of None* ⇒ *undeff* | *Some t* ⇒ (*case SPreds* σ *p of None* ⇒ *Afml* (*Pred p t*) | *Some f* ⇒ *if FVF f* ∩ V = {} *then usubstappf* (*dotsubstt t*) {} *f else undeff*)) ≠ *undeff*
using *Pred.prems*(*2*) **by** *auto*
then have *f5*: *usubstappt* σ U ϑ = *usubstappt* σ V ϑ
using *f2 f1* **by** (*meson usubstappt-det*)
then have *f6*: *usubstappf* σ U (*Pred p* ϑ) = (*case SPreds* σ *p of None* ⇒ *Afml* (*Pred p* (*the* (*usubstappt* σ V ϑ))) | *Some f* ⇒ *if FVF f* ∩ U = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ V ϑ))) {} *f else undeff*)
using *f3 usubstappf.simps*(*1*) **by** *presburger*
have *f7*: ∀ *z f za*. *if za* = *undeff then* (*case za of None* ⇒ *z*::*fml option* | *Some x* ⇒ *f x*) = *z else* (*case za of None* ⇒ *z* | *Some x* ⇒ *f x*) = *f* (*the za*)
by (*simp add*: *option.case-eq-if*)
have *f8*: (*case usubstappt* σ V ϑ *of None* ⇒ *undeff* | *Some t* ⇒ (*case SPreds* σ *p of None* ⇒ *Afml* (*Pred p t*) | *Some f* ⇒ *if FVF f* ∩ V = {} *then usubstappf* (*dotsubstt t*) {} *f else undeff*)) = (*case SPreds* σ *p of None* ⇒ *Afml* (*Pred p* (*the* (*usubstappt* σ V ϑ))) | *Some f* ⇒ *if FVF f* ∩ V = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ V ϑ))) {} *f else undeff*)

using *f4 f2* **by** *meson*
then have *f9*: *SPreds σ p = undeff ⟶ usubstappf σ U (Pred p ϑ) = usubstappf σ V (Pred p ϑ)*
using *f6* **by** *fastforce*
{ assume *usubstappf σ U (Pred p ϑ) ≠ usubstappf σ V (Pred p ϑ)*
then have *usubstappf σ U (Pred p ϑ) ≠ (case SPreds σ p of None ⇒ Afml (Pred p (the (usubstappt σ V ϑ))) | Some f ⇒ if FVF f ∩ V = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} f else undeff)*
using *f8* **by** *simp*
moreover
{ assume *usubstappf σ U (Pred p ϑ) ≠ (if FVF (the (SPreds σ p)) ∩ V = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p)) else undeff)*
moreover
{ assume *usubstappf σ U (Pred p ϑ) ≠ usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p))*
moreover
{ assume *usubstappf σ U (Pred p ϑ) ≠ (if FVF (the (SPreds σ p)) ∩ U = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p)) else undeff)*
then have *(case SPreds σ p of None ⇒ Afml (Pred p (the (usubstappt σ V ϑ))) | Some f ⇒ if FVF f ∩ U = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} f else undeff) ≠ (if FVF (the (SPreds σ p)) ∩ U = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p)) else undeff)*
using *f6* **by** *force*
then have *SPreds σ p = undeff*
using *f7* **by** *(metis (no-types, lifting) Pred.prems(1) calculation f5 option.collapse usubstappf-pred usubstappf-pred2 usubstappf-pred-conv)* **}**
moreover
{ assume *(if FVF (the (SPreds σ p)) ∩ U = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p)) else undeff) ≠ usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p))*
then have *(if FVF (the (SPreds σ p)) ∩ U = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p)) else undeff) ≠ (case SPreds σ p of None ⇒ Afml (Pred p (the (usubstappt σ U ϑ))) | Some f ⇒ if FVF f ∩ U = {} then usubstappf (dotsubstt (the (usubstappt σ U ϑ))) {} f else undeff)*
using *f3 Pred.prems(1)* **by** *auto*
then have *(if FVF (the (SPreds σ p)) ∩ U = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p)) else undeff) ≠ (case SPreds σ p of None ⇒ Afml (Pred p (the (usubstappt σ V ϑ))) | Some f ⇒ if FVF f ∩ U = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} f else undeff)*
using *f5* **using** *Pred.prems(1)* ‹*(if FVF (the (SPreds σ p)) ∩ U = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p)) else undeff) ≠ usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p))*› *f6* **by** *auto*
then have *(case SPreds σ p of None ⇒ Afml (Pred p (the (usubstappt σ V ϑ))) | Some f ⇒ if FVF f ∩ U = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} f else undeff) ≠ (if FVF (the (SPreds σ p)) ∩ U = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p)) else undeff)*

by *simp*
then have *SPreds σ p = undeff*
using *f7* **proof** −
show *?thesis*
using ‹(*case SPreds σ p of None ⇒ Afml (Pred p (the (usubstappt σ V ϑ)))*
| *Some f ⇒ if FVF f ∩ U = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ)))*
{} f else undeff) ≠ (if FVF (the (SPreds σ p)) ∩ U = {} then usubstappf (dotsubstt
(the (usubstappt σ V ϑ))) {} (the (SPreds σ p)) else undeff)› calculation(2) f6 **by**
presburger
qed}
ultimately have *SPreds σ p = undeff*
by *fastforce* **}**
moreover
{ assume *(if FVF (the (SPreds σ p)) ∩ V = {} then usubstappf (dotsubstt (the*
(usubstappt σ V ϑ))) {} (the (SPreds σ p)) else undeff) ≠ usubstappf (dotsubstt
(the (usubstappt σ V ϑ))) {} (the (SPreds σ p))
then have *(case SPreds σ p of None ⇒ Afml (Pred p (the (usubstappt σ V*
ϑ))) | Some f ⇒ if FVF f ∩ V = {} then usubstappf (dotsubstt (the (usubstappt σ
V ϑ))) {} f else undeff) ≠ (if FVF (the (SPreds σ p)) ∩ V = {} then usubstappf
(dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p)) else undeff)
using *f8 Pred.prems(2)* **by** *auto*
then have *SPreds σ p = undeff*
using *f7* **by** *(metis (no-types, lifting) Pred.prems(2) ‹(if FVF (the (SPreds*
σ p)) ∩ V = {} then usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {} (the
(SPreds σ p)) else undeff) ≠ usubstappf (dotsubstt (the (usubstappt σ V ϑ))) {}
*(the (SPreds σ p))› option.collapse usubstappf-pred2)***}**
ultimately have *SPreds σ p = undeff*
by *fastforce* **}**
moreover
{ assume *(case SPreds σ p of None ⇒ Afml (Pred p (the (usubstappt σ V*
ϑ))) | Some f ⇒ if FVF f ∩ V = {} then usubstappf (dotsubstt (the (usubstappt σ
V ϑ))) {} f else undeff) ≠ (if FVF (the (SPreds σ p)) ∩ V = {} then usubstappf
(dotsubstt (the (usubstappt σ V ϑ))) {} (the (SPreds σ p)) else undeff)
then have *SPreds σ p = undeff*
using *f7* **by** *(metis (no-types, lifting) Pred.prems(1) Pred.prems(2) ‹usub-*
stappf σ U (Pred p ϑ) ≠ usubstappf σ V (Pred p ϑ)› calculation(2) option.collapse
*usubstappf-pred usubstappf-pred-conv)***}**
ultimately have *?thesis*
using *f9* **by** *fastforce* **}**
then show *?thesis*
by *blast*
qed
next
case *(Geq ϑ η)*
then show *?case* **using** *usubstappt-det* **by** *(metis Geqo-undef usubstappf.simps(2))*
next
case *(Not x)*
then show *?case* **by** *(metis Noto.simps(2) usubstappf.simps(3))*
next

case (*And x1 x2*)
then show *?case* **by** (*metis Ando-undef usubstappf.simps*(*4*))
next
case (*Exists x1 x2*)
then show *?case* **by** (*metis Existso-undef usubstappf.simps*(*5*))
next
case (*Diamond x1 x2*)
then show *?case* **by** (*metis Diamondo-undef usubstappf.simps*(*6*))
next
case (*Game a*)
then show *?case* **by** (*cases SGames* σ *a*) (*auto*)
next
case (*Assign x* ϑ)
then show *?case* **using** *usubstappt-det* **by** (*metis Assigno-undef snd-conv usubstappp.simps*(*2*))
next
case (*ODE x* ϑ)
then show *?case* **using** *usubstappt-det* **by** (*metis ODEo-undef snd-conv usubstappp.simps*(*8*))
next
case (*Test* φ)
then show *?case* **by** (*metis Testo-undef snd-conv usubstappp.simps*(*3*))
next
case (*Choice* α β)
then show *?case* **by** (*metis Choiceo-undef snd-conv usubstappp.simps*(*4*))
next
case (*Compose* α β)
then show *?case* **by** (*metis Composeo-undef snd-conv usubstappp.simps*(*5*))
next
case (*Loop* α)
then show *?case* **by** (*metis Loopo-undef snd-conv usubstappp.simps*(*6*))
next
case (*Dual* α)
then show *?case* **by** (*metis Dualo-undef snd-conv usubstappp.simps*(*7*))
qed

lemma *usubstappf-det*: *usubstappf* σ *U* $\varphi \neq$ *undeff* $\Longrightarrow$ *usubstappf* σ *V* $\varphi \neq$ *undeff* $\Longrightarrow$ *usubstappf* σ *U* φ = *usubstappf* σ *V* φ
using *usubstappf-and-usubstappp-det* **by** *simp*

lemma *usubstappp-det*: *snd*(*usubstappp* σ *U* α) $\neq$ *undefg* $\Longrightarrow$ *snd*(*usubstappp* σ *V* α) $\neq$ *undefg* $\Longrightarrow$ *snd*(*usubstappp* σ *U* α) = *snd*(*usubstappp* σ *V* α)
using *usubstappf-and-usubstappp-det* **by** *simp*

7.2 Uniform Substitutions are Antimonotone in Taboos

lemma *usubst-taboos-mon*: *fst*(*usubstappp* σ *U* α) $\supseteq$ *U*
proof (*induction* α *arbitrary*: *U rule*: *game-induct*)
case (*Game a*)

then show *?case* **by** (*cases SGames* σ *a*) (*auto*)
next
case (*Assign x* ϑ)
then show *?case* **by** *fastforce*
next
case (*ODE x* ϑ)
then show *?case* **by** *fastforce*
next
case (*Test* φ)
then show *?case* **by** *fastforce*
next
case (*Choice* α β)
then show *?case* **by** *fastforce*
next
case (*Compose* α β)
then show *?case* **by** *fastforce*
next
case (*Loop* α)
then show *?case* **by** *fastforce*
next
case (*Dual* α)
then show *?case* **by** *fastforce*
qed

lemma *fst-pair* [*simp*]: *fst* (*a*,*b*) = *a*
by *simp*

lemma *snd-pair* [*simp*]: *snd* (*a*,*b*) = *b*
by *simp*

lemma *usubstappt-antimon*: $V \subseteq U \Longrightarrow$ *usubstappt* σ *U* $\vartheta \neq$ *undeft* $\Longrightarrow$
usubstappt σ *U* ϑ = *usubstappt* σ *V* ϑ
proof (*induction* ϑ)
case (*Var x*)
then show *?case* **by** *simp*
next
case (*Number x*)
then show *?case* **by** *simp*
next
case (*Const f*)
then show *?case*

proof −
have *f1*: *usubstappt* σ *U* (*Const f*) = (*case SConst* σ *f of None* ⇒ *Aterm* (*Const f*) | *Some t* ⇒ *if FVT t* ∩ *U* = {} *then Aterm t else undeft*)
by (*simp add*: *usappconst-def*)
have *f2*: ∀ *z f za*. *if za* = *undeft then* (*case za of None* ⇒ *z*::*trm option* | *Some x* ⇒ *f x*) = *z else* (*case za of None* ⇒ *z* | *Some x* ⇒ *f x*) = *f* (*the za*)

by *force*
then have *SConst σ f ≠ undeft ⟶ (if FVT (the (SConst σ f)) ∩ U = {} then Aterm (the (SConst σ f)) else undeft) = usappconst σ U f*
using *usappconst-def* **by** *presburger*
then have *f3*: *SConst σ f ≠ undeft ⟶ FVT (the (SConst σ f)) ∩ U = {}*
by (*metis* (*no-types*) *Const.prems*(*2*) *usubstappt.simps*(*3*))
have *f4*: *∀ V Va. (V ∩ Va = {}) = (∀ v. (v::variable) ∈ V ⟶ (∀ va. va ∈ Va ⟶ v ≠ va))*
by *blast*
{ assume *usubstappt σ U (Const f) ≠ usubstappt σ V (Const f)*
then have *usubstappt σ U (Const f) ≠ (case SConst σ f of None ⇒ Aterm (Const f) | Some t ⇒ if FVT t ∩ V = {} then Aterm t else undeft)*
by (*simp add*: *usappconst-def*)
then have *SConst σ f ≠ undeft*
using *f2 f1* **by** *metis*
then have *SConst σ f ≠ undeft ∧ FVT (the (SConst σ f)) ∩ V = {}*
using *f4 f3* **by** (*meson Const.prems*(*1*) *subsetD*)
then have *?thesis*
using *f3 f2 usappconst-def usubstappt.simps*(*3*) **by** *presburger* **}**
then show *?thesis*
by *blast*
qed
next
case (*Func f ϑ*)
then show *?case* **using** *usubstappt-func*

proof −
have *f1*: (*case usubstappt σ U ϑ of None ⇒ undeft | Some t ⇒ (case SFuncs σ f of None ⇒ Aterm (trm.Func f t) | Some ta ⇒ if FVT ta ∩ U = {} then usubstappt (dotsubstt t) {} ta else undeft*)) *≠ undeft*
using *Func.prems*(*2*) **by** *fastforce*
have *f2*: *∀ z f za. if za = undeft then (case za of None ⇒ z::trm option | Some x ⇒ f x) = z else (case za of None ⇒ z | Some x ⇒ f x) = f (the za)*
by *fastforce*
then have *f3*: *usubstappt σ U ϑ ≠ undeft*
using *f1* **by** *meson*
then have *f4*: (*case usubstappt σ U ϑ of None ⇒ undeft | Some t ⇒ (case SFuncs σ f of None ⇒ Aterm (trm.Func f t) | Some ta ⇒ if FVT ta ∩ U = {} then usubstappt (dotsubstt t) {} ta else undeft*)) = (*case SFuncs σ f of None ⇒ Aterm (trm.Func f (the (usubstappt σ U ϑ))) | Some t ⇒ if FVT t ∩ U = {} then usubstappt (dotsubstt (the (usubstappt σ U ϑ))) {} t else undeft*)
using *f2* **by** *presburger*
have *f5*: *usubstappt σ U ϑ = undeft ∨ usubstappt σ U ϑ = usubstappt σ V ϑ*
using *Func.IH Func.prems*(*1*) **by** *fastforce*
have *SFuncs σ f ≠ undeft ⟶ (if FVT (the (SFuncs σ f)) ∩ U = {} then usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SFuncs σ f)) else undeft) = (case SFuncs σ f of None ⇒ Aterm (trm.Func f (the (usubstappt σ V ϑ))) | Some t ⇒ if FVT t ∩ U = {} then usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} t else undeft)*

using *f2* **by** *presburger*
then have *SFuncs σ f ≠ undeft ⟶ (if FVT (the (SFuncs σ f)) ∩ U = {} then usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SFuncs σ f)) else undeft) = usubstappt σ U (trm.Func f ϑ)*
using *f5 f4 f3 usubstappt.simps(4)* **by** *presburger*
then have *f6*: *SFuncs σ f ≠ undeft ⟶ FVT (the (SFuncs σ f)) ∩ U = {}*
by *(metis (no-types) Func.prems(2))*
then have *f7*: *SFuncs σ f ≠ undeft ⟶ V ⊆ − FVT (the (SFuncs σ f))*
using *Func.prems(1)* **by** *blast*
have *f8*: *(case usubstappt σ V ϑ of None ⇒ undeft | Some t ⇒ (case SFuncs σ f of None ⇒ Aterm (trm.Func f t) | Some ta ⇒ if FVT ta ∩ V = {} then usubstappt (dotsubstt t) {} ta else undeft)) = (case SFuncs σ f of None ⇒ Aterm (trm.Func f (the (usubstappt σ V ϑ))) | Some t ⇒ if FVT t ∩ V = {} then usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} t else undeft)*
using *f5 f3 f2* **by** *presburger*
have *SFuncs σ f ≠ undeft ⟶ usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SFuncs σ f)) = (case SFuncs σ f of None ⇒ Aterm (trm.Func f (the (usubstappt σ V ϑ))) | Some t ⇒ if FVT t ∩ U = {} then usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} t else undeft)*
using *f6 f2* **by** *presburger*
then have *f9*: *SFuncs σ f ≠ undeft ⟶ usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} (the (SFuncs σ f)) = usubstappt σ U (trm.Func f ϑ)*
using *f5 f4 f3 usubstappt.simps(4)* **by** *presburger*
{ assume *usubstappt σ U (trm.Func f ϑ) ≠ usubstappt σ V (trm.Func f ϑ)*
moreover
{ assume *(case SFuncs σ f of None ⇒ Aterm (trm.Func f (the (usubstappt σ V ϑ))) | Some t ⇒ if FVT t ∩ V = {} then usubstappt (dotsubstt (the (usubstappt σ V ϑ))) {} t else undeft) ≠ Aterm (trm.Func f (the (usubstappt σ V ϑ)))*
then have *SFuncs σ f ≠ undeft*
using *f2* **by** *meson* **}**
ultimately have *SFuncs σ f ≠ undeft*
using *f5 f3* **by** *fastforce*
then have *?thesis*
using *f9 f8 f7 f2* **by** *(simp add: disjoint-eq-subset-Compl inf.commute)* **}**
then show *?thesis*
by *blast*
qed
next
case *(Plus ϑ1 ϑ2)*
then show *?case* **using** *Pluso-undef* **by** *auto*
next
case *(Times ϑ1 ϑ2)*
then show *?case* **using** *Timeso-undef* **by** *auto*
next
case *(Differential ϑ)*
then show *?case* **using** *Differentialo-undef* **by** *auto*
qed

Uniform Substitutions of Games have monotone taboo output

lemma *usubstappp-fst-mon*: $U \subseteq V \Longrightarrow fst(usubstappp\ \sigma\ U\ \alpha) \subseteq fst(usubstappp\ \sigma\ V\ \alpha)$
proof (*induction* α *arbitrary*: $U\ V$ *rule*: *game-induct*)
case (*Game a*)
then show *?case* **by** (*cases SGames* σ *a*) (*auto*)
next
case (*Assign x* ϑ)
then show *?case* **by** *auto*
next
case (*ODE x* ϑ)
then show *?case* **by** *auto*
next
case (*Test* φ)
then show *?case* **by** *auto*
next
case (*Choice* α β)
then show *?case* **by** (*metis Un-mono fst-pair usubstappp-choice*)
next
case (*Compose* α β)
then show *?case* **by** (*metis fst-pair usubstappp-compose*)
next
case (*Loop* α)
then show *?case* **by** (*metis fst-pair usubstappp-loop*)
next
case (*Dual* α)
then show *?case* **by** (*metis fst-pair usubstappp-dual*)
qed

lemma *usubstappf-and-usubstappp-antimon*:
shows $V \subseteq U \Longrightarrow usubstappf\ \sigma\ U\ \varphi \neq undeff \Longrightarrow usubstappf\ \sigma\ U\ \varphi = usubstappf\ \sigma\ V\ \varphi$
and $V \subseteq U \Longrightarrow snd(usubstappp\ \sigma\ U\ \alpha) \neq undefg \Longrightarrow snd(usubstappp\ \sigma\ U\ \alpha) = snd(usubstappp\ \sigma\ V\ \alpha)$
proof−
have $V \subseteq U \Longrightarrow usubstappf\ \sigma\ U\ \varphi \neq undeff \Longrightarrow usubstappf\ \sigma\ V\ \varphi \neq undeff$
and $V \subseteq U \Longrightarrow snd(usubstappp\ \sigma\ U\ \alpha) \neq undefg \Longrightarrow snd(usubstappp\ \sigma\ V\ \alpha) \neq undefg$
proof (*induction* φ **and** α *arbitrary*: $U\ V$ **and** $U\ V$)
case (*Pred p* ϑ)
then show *?case* **using** *usubstappt-antimon usubstappf-pred*

proof −
have *f1*: $\forall v.\ v \notin V \vee v \in U$
using *Pred.prems*(*1*) **by** *auto*
have *f2*: $\forall z\ f\ za.$ *if* $za = undeff$ *then* (*case za of None* $\Rightarrow$ *z*::*fml option* | *Some* $x \Rightarrow f\ x$) $= z$ *else* (*case za of None* $\Rightarrow z \mid$ *Some* $x \Rightarrow f\ x$) $= f$ (*the za*)
by (*simp add*: *option.case-eq-if*)
have *f3*: (*case usubstappt* $\sigma\ U\ \vartheta$ *of None* $\Rightarrow$ *undeff* | *Some t* $\Rightarrow$ (*case SPreds*

σ p *of None* $\Rightarrow$ *Afml* (*Pred* p t) | *Some* f $\Rightarrow$ *if FVF* f $\cap$ U = {} *then usubstappf* (*dotsubstt* t) {} f *else undeff*)) $\neq$ *undeff*

using *Pred.prems*(*2*) **by** *auto*

have *f4*: $\forall$ z f *za*. *if za* = *undeft then* (*case za of None* $\Rightarrow$ *z*::*fml option* | *Some* x $\Rightarrow$ f x) = z *else* (*case za of None* $\Rightarrow$ z | *Some* x $\Rightarrow$ f x) = f (*the za*)

by (*simp add*: *option.case-eq-if*)

then have *f5*: *usubstappt* σ U ϑ $\neq$ *undeft*

using *f3* **by** *meson*

then have *f6*: (*case usubstappt* σ U ϑ *of None* $\Rightarrow$ *undeff* | *Some* t $\Rightarrow$ (*case SPreds* σ p *of None* $\Rightarrow$ *Afml* (*Pred* p t) | *Some* f $\Rightarrow$ *if FVF* f $\cap$ U = {} *then usubstappf* (*dotsubstt* t) {} f *else undeff*)) = (*case SPreds* σ p *of None* $\Rightarrow$ *Afml* (*Pred p* (*the* (*usubstappt* σ U ϑ))) | *Some* f $\Rightarrow$ *if FVF* f $\cap$ U = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ U ϑ))) {} f *else undeff*)

using *f4* **by** *presburger*

have *f7*: *usubstappt* σ U ϑ = *usubstappt* σ V ϑ

using *f5* **by** (*meson Pred.prems*(*1*) *usubstappt-antimon*)

then have *f8*: *SPreds* σ p = *undeff* $\longrightarrow$ *usubstappf* σ U (*Pred* p ϑ) = *usubstappf* σ V (*Pred* p ϑ)

using *f2 usubstappf.simps*(*1*) **by** *presburger*

obtain *vv* :: *variable set* $\Rightarrow$ *variable set* $\Rightarrow$ *variable* **where**

$\forall$ *x0 x1*. ($\exists$ *v2*. *v2* $\in$ *x1* $\wedge$ ($\exists$ *v3*. *v3* $\in$ *x0* $\wedge$ *v2* = *v3*)) = (*vv x0 x1* $\in$ *x1* $\wedge$ ($\exists$ *v3*. *v3* $\in$ *x0* $\wedge$ *vv x0 x1* = *v3*))

by *moura*

then obtain *vva* :: *variable set* $\Rightarrow$ *variable set* $\Rightarrow$ *variable* **where**

f9: $\forall$ V *Va*. (V $\cap$ *Va* $\neq$ {} $\vee$ ($\forall$ *v*. *v* $\notin$ V $\vee$ ($\forall$ *va*. *va* $\notin$ *Va* $\vee$ *v* $\neq$ *va*))) $\wedge$ (V $\cap$ *Va* = {} $\vee$ *vv Va V* $\in$ V $\wedge$ *vva Va V* $\in$ *Va* $\wedge$ *vv Va V* = *vva Va V*)

by *moura*

then have *f10*: (*FVF* (*the* (*SPreds* σ p)) $\cap$ V $\neq$ {} $\vee$ ($\forall$ *v*. *v* $\notin$ *FVF* (*the* (*SPreds* σ p)) $\vee$ ($\forall$ *va*. *va* $\notin$ V $\vee$ *v* $\neq$ *va*))) $\wedge$ (*FVF* (*the* (*SPreds* σ p)) $\cap$ V = {} $\vee$ *vv V* (*FVF* (*the* (*SPreds* σ p))) $\in$ *FVF* (*the* (*SPreds* σ p)) $\wedge$ *vva V* (*FVF* (*the* (*SPreds* σ p))) $\in$ V $\wedge$ *vv V* (*FVF* (*the* (*SPreds* σ p))) = *vva V* (*FVF* (*the* (*SPreds* σ p))))

by *presburger*

{ assume *vv V* (*FVF* (*the* (*SPreds* σ p))) $\notin$ *FVF* (*the* (*SPreds* σ p)) $\vee$ *vva V* (*FVF* (*the* (*SPreds* σ p))) $\notin$ V $\vee$ *vv V* (*FVF* (*the* (*SPreds* σ p))) $\neq$ *vva V* (*FVF* (*the* (*SPreds* σ p)))

moreover

{ assume (*if FVF* (*the* (*SPreds* σ p)) $\cap$ V = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ V ϑ))) {} (*the* (*SPreds* σ p)) *else undeff*) $\neq$ *undeff*

moreover

{ assume (*if FVF* (*the* (*SPreds* σ p)) $\cap$ V = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ V ϑ))) {} (*the* (*SPreds* σ p)) *else undeff*) $\neq$ *usubstappf* σ V (*Pred* p ϑ)

then have (*case SPreds* σ p *of None* $\Rightarrow$ *Afml* (*Pred p* (*the* (*usubstappt* σ V ϑ))) | *Some* f $\Rightarrow$ *if FVF* f $\cap$ V = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ V ϑ))) {} f *else undeff*) $\neq$ (*if FVF* (*the* (*SPreds* σ p)) $\cap$ V = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ V ϑ))) {} (*the* (*SPreds* σ p)) *else undeff*)

using *f7 f5 f4* **by** *simp*

then have *SPreds* σ p = *undeff*

using *f2* **by** (*metis* (*no-types*, *lifting*) ‹(*if FVF* (*the* (*SPreds* σ *p*)) $\cap$ *V* = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *V* ϑ))) {} (*the* (*SPreds* σ *p*)) *else undeff*) $\neq$ *usubstappf* σ *V* (*Pred p* ϑ)› *calculation f5 f7 option.collapse usubstappf-pred*)**}**

ultimately have *usubstappf* σ *V* (*Pred p* ϑ) = *undeff* $\longrightarrow$ *SPreds* σ *p* = *undeff*

by *fastforce* **}**

moreover

{ assume *usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *V* ϑ))) {} (*the* (*SPreds* σ *p*)) $\neq$ *usubstappf* σ *U* (*Pred p* ϑ)

then have *usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *V* ϑ))) {} (*the* (*SPreds* σ *p*)) $\neq$ (*case SPreds* σ *p of None* $\Rightarrow$ *Afml* (*Pred p* (*the* (*usubstappt* σ *U* ϑ))) | *Some f* $\Rightarrow$ *if FVF f* $\cap$ *U* = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *U* ϑ))) {} *f else undeff*)

using *f6* **by** *simp*

then have *usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *V* ϑ))) {} (*the* (*SPreds* σ *p*)) $\neq$ (*case SPreds* σ *p of None* $\Rightarrow$ *Afml* (*Pred p* (*the* (*usubstappt* σ *V* ϑ))) | *Some f* $\Rightarrow$ *if FVF f* $\cap$ *U* = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *V* ϑ))) {} *f else undeff*)

using *f7* **by** (*metis f7*)

moreover

{ assume (*case SPreds* σ *p of None* $\Rightarrow$ *Afml* (*Pred p* (*the* (*usubstappt* σ *V* ϑ))) | *Some f* $\Rightarrow$ *if FVF f* $\cap$ *U* = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *V* ϑ))) {} *f else undeff*) $\neq$ (*if FVF* (*the* (*SPreds* σ *p*)) $\cap$ *U* = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *V* ϑ))) {} (*the* (*SPreds* σ *p*)) *else undeff*)

then have *SPreds* σ *p* = *undeff*

using *f2* **by** (*metis* (*no-types*, *lifting*) *Pred.prems*(*2*) ‹*usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *V* ϑ))) {} (*the* (*SPreds* σ *p*)) $\neq$ *usubstappf* σ *U* (*Pred p* ϑ)› *f5 f7 option.collapse usubstappf-pred usubstappf-pred2*)**}**

ultimately have *FVF* (*the* (*SPreds* σ *p*)) $\cap$ *U* = {} $\longrightarrow$ *SPreds* σ *p* = *undeff*

by *force* **}**

ultimately have *FVF* (*the* (*SPreds* σ *p*)) $\cap$ *U* = {} $\wedge$ *usubstappf* σ *V* (*Pred p* ϑ) = *undeff* $\longrightarrow$ *SPreds* σ *p* = *undeff*

using *f10* **by** (*metis Pred.prems*(*2*)) **}**

moreover

{ assume *FVF* (*the* (*SPreds* σ *p*)) $\cap$ *U* $\neq$ {}

then have (*if FVF* (*the* (*SPreds* σ *p*)) $\cap$ *U* = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *V* ϑ))) {} (*the* (*SPreds* σ *p*)) *else undeff*) $\neq$ (*case SPreds* σ *p of None* $\Rightarrow$ *Afml* (*Pred p* (*the* (*usubstappt* σ *U* ϑ))) | *Some f* $\Rightarrow$ *if FVF f* $\cap$ *U* = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *U* ϑ))) {} *f else undeff*)

using *f6 Pred.prems*(*2*) **by** *auto*

then have (*if FVF* (*the* (*SPreds* σ *p*)) $\cap$ *U* = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *V* ϑ))) {} (*the* (*SPreds* σ *p*)) *else undeff*) $\neq$ (*case SPreds* σ *p of None* $\Rightarrow$ *Afml* (*Pred p* (*the* (*usubstappt* σ *V* ϑ))) | *Some f* $\Rightarrow$ *if FVF f* $\cap$ *U* = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *V* ϑ))) {} *f else undeff*)

using *f7* **by** (*metis* ‹*FVF* (*the* (*SPreds* σ *p*)) $\cap$ *U* $\neq$ {}›)

then have (*case SPreds* σ *p of None* $\Rightarrow$ *Afml* (*Pred p* (*the* (*usubstappt* σ *V* ϑ))) | *Some f* $\Rightarrow$ *if FVF f* $\cap$ *U* = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* σ

V ϑ))) {} *f else undeff*) ≠ (*if FVF* (*the* (*SPreds σ p*)) ∩ *U* = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt σ V ϑ*))) {} (*the* (*SPreds σ p*)) *else undeff*)
by *simp*
then have *SPreds σ p* = *undeff*
using *f2*
proof −
show *?thesis*
by (*metis* (*no-types*) *Pred.prems*(*2*) ‹*FVF* (*the* (*SPreds σ p*)) ∩ *U* ≠ {}› *option.discI option.expand option.sel usubstappf-pred2*)
qed }
ultimately have *usubstappf σ V* (*Pred p ϑ*) = *undeff* ⟶ *SPreds σ p* = *undeff*
using *f9 f1* **by** *meson*
then show *?thesis*
using *f8* **by** (*metis* (*full-types*) *Pred.prems*(*2*))
qed

next
case (*Geq ϑ η*)
then show *?case* **using** *usubstappt-antimon* **using** *Geqo-undef* **by** *auto*
next
case (*Not x*)
then show *?case* **using** *Noto-undef* **by** *auto*
next
case (*And x1 x2*)
then show *?case* **using** *Ando-undef* **by** *auto*
next
case (*Exists x1 x2*)
then show *?case* **using** *Existso-undef*
by (*metis* (*no-types*, *lifting*) *Un-mono subsetI usubstappf.simps*(*5*))
next
case (*Diamond x1 x2*)
then show *?case* **using** *Diamondo-undef usubstappf.simps*(*6*) *usubstappp-fst-mon* **by** *metis*
next
case (*Game a*)
then show *?case* **by** (*cases SGames σ a*) (*auto*)
next
case (*Assign x ϑ*)
then show *?case* **using** *usubstappt-antimon* **by** (*metis Assigno-undef snd-conv usubstappp.simps*(*2*))
next
case (*ODE x ϑ*)
then show *?case* **using** *usubstappt-antimon ODEo-undef*
by (*metis* (*no-types*, *hide-lams*) *Un-mono order-refl snd-conv usubstappp.simps*(*8*))
next
case (*Test φ*)
then show *?case* **by** (*metis Testo-undef snd-conv usubstappp.simps*(*3*))
next

case (*Choice* α β)
then show *?case* **using** *Choiceo-undef* **by** *auto*
next
case (*Compose* α β)
then show *?case*
using *usubstappp-compose*[**where** σ=σ **and** U=U **and** α=α **and** β=β] *usubstappp-compose*[**where** σ=σ **and** U=V **and** α=α **and** β=β] *Composeo-undef*[**where** α=‹*snd* (*usubstappp* σ U α)› **and** β=‹*snd* (*usubstappp* σ (*fst* (*usubstappp* σ U α)) β)›] *Composeo-undef*[**where** α=‹*snd* (*usubstappp* σ V α)› **and** β=‹*snd* (*usubstappp* σ (*fst* (*usubstappp* σ V α)) β)›] *snd-conv usubstappp-fst-mon* **by** *metis*

next
case (*Loop* α)
then show *?case* **using** *Loopo-undef snd-conv usubstappp.simps*(6) *usubstappp-fst-mon* **by** *metis*
next
case (*Dual* α)
then show *?case* **by** (*metis Dualo-undef snd-conv usubstappp.simps*(7))
qed
from *this* **show** $V{\subseteq}U \Longrightarrow$ *usubstappf* σ U $\varphi \neq$ *undeff* $\Longrightarrow$ *usubstappf* σ U φ = *usubstappf* σ V φ
and $V{\subseteq}U \Longrightarrow$ *snd*(*usubstappp* σ U α) $\neq$ *undefg* $\Longrightarrow$ *snd*(*usubstappp* σ U α) = *snd*(*usubstappp* σ V α) **using** *usubstappf-and-usubstappp-det* **by** *auto*
qed

lemma *usubstappf-antimon*: $V{\subseteq}U \Longrightarrow$ *usubstappf* σ U $\varphi \neq$ *undeff* $\Longrightarrow$ *usubstappf* σ U φ = *usubstappf* σ V φ
using *usubstappf-and-usubstappp-antimon* **by** *simp*

lemma *usubstappp-antimon*: $V{\subseteq}U \Longrightarrow$ *snd*(*usubstappp* σ U α) $\neq$ *undefg* $\Longrightarrow$ *snd*(*usubstappp* σ U α) = *snd*(*usubstappp* σ V α)
using *usubstappf-and-usubstappp-antimon* **by** *simp*

7.3 Taboo Lemmas

lemma *usubstappp-loop-conv*: *snd* (*usubstappp* σ U (*Loop* α)) $\neq$ *undefg* $\Longrightarrow$
snd(*usubstappp* σ U α) $\neq$ *undefg* $\wedge$
snd(*usubstappp* σ (*fst*(*usubstappp* σ U α)) α) $\neq$ *undefg*

proof
assume *a*: *snd* (*usubstappp* σ U (*Loop* α)) $\neq$ *undefg*
have *fact*: *fst*(*usubstappp* σ U α) $\supseteq$ U **using** *usubst-taboos-mon* **by** *simp*
show *snd*(*usubstappp* σ (*fst*(*usubstappp* σ U α)) α) $\neq$ *undefg* **using** *a usubstappp-loop Loopo-undef* **by** *simp*
then show *snd*(*usubstappp* σ U α) $\neq$ *undefg* **using** *a usubstappp-loop Loopo-undef fact usubstappp-antimon* **by** *auto*
qed

Lemma 13 of http://arxiv.org/abs/1902.07230

lemma *usubst-taboos*: *snd*(*usubstappp* σ *U* α)≠*undefg* ⟹ *fst*(*usubstappp* σ *U* α) ⊇ *U* ∪ *BVG*(*the* (*snd*(*usubstappp* σ *U* α)))
proof (*induction* α *arbitrary*: *U rule*: *game-induct*)
case (*Game a*)
then show *?case* **by** (*cases SGames* σ *a*) (*auto*)
next
case (*Assign x* ϑ)
then show *?case*
using *BVG-assign Assigno-undef*
by (*metis* (*no-types*, *lifting*) *Assigno.elims BVG-assign-other fst-pair option.sel singletonI snd-pair subsetI union-or usubstappp.simps*(*2*))
next
case (*ODE x* ϑ)
then show *?case*
using *BVG-ODE ODEo-undef*
by (*metis* (*no-types*, *lifting*) *ODEo.elims Un-least fst-pair option.sel snd-conv sup.coboundedI2 usubst-taboos-mon usubstappp.simps*(*8*))
next
case (*Test p*)
then show *?case*
using *BVG-test Testo-undef usubst-taboos-mon* **by** *auto*
next
case (*Choice* α β)
then show *?case*
proof–
from *Choice* **have** *IHa*: *fst*(*usubstappp* σ *U* α) ⊇ *U* ∪ *BVG*(*the* (*snd*(*usubstappp* σ *U* α))) **by** (*simp add*: *Choiceo-undef*)
from *Choice* **have** *IHb*: *fst*(*usubstappp* σ *U* β) ⊇ *U* ∪ *BVG*(*the* (*snd*(*usubstappp* σ *U* β))) **by** (*simp add*: *Choiceo-undef*)
have *fact*: *BVG*(*the* (*snd*(*usubstappp* σ *U* α))) ∪ *BVG*(*the* (*snd*(*usubstappp* σ *U* β))) ⊇ *BVG*(*the* (*snd*(*usubstappp* σ *U* (*Choice* α β)))) **using** *BVG-choice*

proof −
have *Agame* (*the* (*snd* (*usubstappp* σ *U* α)) ∪∪ *the* (*snd* (*usubstappp* σ *U* β))) = *Choiceo* (*snd* (*usubstappp* σ *U* α)) (*snd* (*usubstappp* σ *U* β))
by (*metis* (*no-types*) *Choice.prems Choiceo.simps*(*1*) *option.collapse usubstappp-choice-conv*)
then show *?thesis*
by (*metis* (*no-types*) *BVG-choice Choice.prems Pair-inject option.collapse option.inject surjective-pairing usubstappp.simps*(*4*))
qed
from *IHa* **and** *IHb* **have** *fst*(*usubstappp* σ *U* α) ∪ *fst*(*usubstappp* σ *U* β) ⊇ *U* ∪ *BVG*(*the* (*snd*(*usubstappp* σ *U* α))) ∪ *BVG*(*the* (*snd*(*usubstappp* σ *U* β))) **by** *auto*
then have *fst*(*usubstappp* σ *U* (*Choice* α β)) ⊇ *U* ∪ *BVG*(*the* (*snd*(*usubstappp* σ *U* α))) ∪ *BVG*(*the* (*snd*(*usubstappp* σ *U* β))) **using** *usubstappp.simps Let-def* **by** *auto*
then show *fst*(*usubstappp* σ *U* (*Choice* α β)) ⊇ *U* ∪ *BVG*(*the* (*snd*(*usubstappp*

σ U (*Choice* α β)))) **using** *usubstappp.simps fact* **by** *auto*
qed
next
case (*Compose* α β)
then show *?case*
proof−
let *?V* = *fst*(*usubstappp* σ U α)
let *?W* = *fst*(*usubstappp* σ *?V* β)
from *Compose* **have** *IHa*: *?V* $\supseteq$ $U \cup$ *BVG*(*the* (*snd*(*usubstappp* σ U α))) **by** (*simp add*: *Composeo-undef*)
from *Compose* **have** *IHb*: *?W* $\supseteq$ *?V* $\cup$ *BVG*(*the* (*snd*(*usubstappp* σ *?V* β))) **by** (*simp add*: *Composeo-undef*)
have *fact*: *BVG*(*the* (*snd*(*usubstappp* σ U α))) $\cup$ *BVG*(*the* (*snd*(*usubstappp* σ *?V* β))) $\supseteq$ *BVG*(*the* (*snd*(*usubstappp* σ U (*Compose* α β)))) **using** *usubstappp.simps BVG-compose*

proof −
have *f1*: $\forall z.\ z$ = *undefg* $\vee$ *Agame* (*the* z) = z
using *option.collapse* **by** *blast*
then have *Agame* (*the* (*snd* (*usubstappp* σ U α)) ;; *the* (*snd* (*usubstappp* σ (*fst* (*usubstappp* σ U α)) β))) = *snd* (*usubstappp* σ U (α ;; β))
using *Compose.prems Composeo-undef* **by** *auto*
then show *?thesis*
using *f1* **by** (*metis* (*no-types*) *BVG-compose Compose.prems option.inject*)
qed
have *?W* $\supseteq$ $U \cup$ *BVG*(*the* (*snd*(*usubstappp* σ U α))) $\cup$ *BVG*(*the* (*snd*(*usubstappp* σ *?V* β))) **using** *usubstappp.simps Let-def IHa IHb* **by** *auto*
then have *?W* $\supseteq$ $U \cup$ *BVG*(*the* (*snd*(*usubstappp* σ U (*Compose* α β)))) **using** *fact* **by** *auto*
then show *fst*(*usubstappp* σ U (*Compose* α β)) $\supseteq$ $U \cup$ *BVG*(*the* (*snd*(*usubstappp* σ U (*Compose* α β)))) **using** *usubstappp.simps Let-def* **by** *simp*
qed
next
case (*Loop* α)
then show *?case*
proof−
let *?V* = *fst*(*usubstappp* σ U α)
let *?W* = *fst*(*usubstappp* σ *?V* α)
from *Loop* **have** *def*α: *snd*(*usubstappp* σ U α) $\neq$ *undefg* **using** *usubstappp-loop-conv* **by** *simp*
from *Loop* **have** *IHdef*: *?V* $\supseteq$ $U \cup$ *BVG*(*the* (*snd*(*usubstappp* σ U α)))
using *def*α *usubstappp-loop*[**where** σ=σ **and** U=U **and** α=α] *Loopo-undef*[**where** α=‹*snd* (*usubstappp* σ (*fst* (*usubstappp* σ U α)) α)›] **by** *auto*
from *Loop* **have** *IH*: *?W* $\supseteq$ *?V* $\cup$ *BVG*(*the* (*snd*(*usubstappp* σ *?V* α))) **by** (*simp add*: *Loopo-undef*)
then have *Vfix*: *?V* $\supseteq$ *BVG*(*the* (*snd*(*usubstappp* σ *?V* α)))
using *usubstappp-det* **by** (*metis IHdef Loop.prems le-sup-iff usubstappp-loop-conv*)
then have *?V* $\supseteq$ $U \cup$ *BVG*(*the* (*snd*(*usubstappp* σ U (*Loop* α))))
using *usubstappp.simps Vfix IHdef BVG-loop usubst-taboos-mon usubstappp-loop-conv*

proof −
have *f1*: $\forall z.\ z = undefg \vee Agame\ (the\ z) = z$
using *option.collapse* **by** *blast*
have $snd\ (usubstappp\ \sigma\ U\ \alpha) \neq undefg \wedge snd\ (usubstappp\ \sigma\ (fst\ (usubstappp\ \sigma\ U\ \alpha))\ \alpha) \neq undefg$
using *Loop.prems usubstappp-loop-conv* **by** *blast*
then have $Agame\ (Loop\ (the\ (snd\ (usubstappp\ \sigma\ (fst\ (usubstappp\ \sigma\ U\ \alpha))\ \alpha)))) = snd\ (usubstappp\ \sigma\ U\ (Loop\ \alpha))$
by *force*
then show *?thesis*
using *f1* **by** (*metis* (*no-types*) *BVG-loop Loop.prems Vfix option.inject sup.absorb-iff1 sup.mono usubst-taboos-mon*)
qed
then show $fst(usubstappp\ \sigma\ U\ (Loop\ \alpha)) \supseteq U \cup BVG(the\ (snd(usubstappp\ \sigma\ U\ (Loop\ \alpha))))$ **using** *usubstappp.simps Let-def* **by** *simp*
qed
next
case (*Dual* α)
then show *?case*

proof−
let $?V = fst(usubstappp\ \sigma\ U\ \alpha)$
from *Dual* **have** *IH*: $?V \supseteq U \cup BVG(the\ (snd(usubstappp\ \sigma\ U\ \alpha)))$ **by** (*simp add*: *Dualo-undef*)
have *fact*: $BVG(the\ (snd(usubstappp\ \sigma\ U\ \alpha))) \supseteq BVG(the\ (snd(usubstappp\ \sigma\ U\ (Dual\ \alpha))))$ **using** *usubstappp.simps BVG-dual*
by (*metis* (*no-types*, *lifting*) *Dual.prems Dualo.simps*(*1*) *Dualo.simps*(*2*) *option.collapse option.sel snd-pair*)
then have $?V \supseteq U \cup BVG(the\ (snd(usubstappp\ \sigma\ U\ (Dual\ \alpha))))$ **using** *IH fact* **by** *auto*
then show $fst(usubstappp\ \sigma\ U\ (Dual\ \alpha)) \supseteq U \cup BVG(the\ (snd(usubstappp\ \sigma\ U\ (Dual\ \alpha))))$ **using** *usubstappp.simps Let-def* **by** *simp*
qed
qed

7.4 Substitution Adjoints

Modified interpretation *repI I f d* replaces the interpretation of constant function f in the interpretation I with d

definition *repc* :: *interp* ⇒ *ident* ⇒ *real* ⇒ *interp*
where *repc I f d* ≡ *mkinterp*((λ*c*. *if c* = *f then d else Consts I c*), *Funcs I*, *Preds I*, *Games I*)

lemma *repc-consts* [*simp*]: *Consts* (*repc I f d*) *c* = (*if* (*c*=*f*) *then d else Consts I c*)
unfolding *repc-def* **by** *auto*
lemma *repc-funcs* [*simp*]: *Funcs* (*repc I f d*) = *Funcs I*
unfolding *repc-def* **by** *simp*

lemma *repc-preds* [*simp*]: *Preds* (*repc I f d*) = *Preds I*
unfolding *repc-def* **by** *simp*
lemma *repc-games* [*simp*]: *Games* (*repc I f d*) = *Games I*
unfolding *repc-def* **by** (*simp add*: *mon-mono*)

lemma *adjoint-stays-mon*: *mono* (*case SGames σ a of None ⇒ Games I a | Some r ⇒ λX. game-sem I r X*)
using *game-sem-mono game-sem.simps*(*1*)
by (*metis disjE-realizer2 option.case-distrib*)

adjoint interpretation *adjoint σ I ω* to *σ* of interpretation *I* in state *ω*

definition *adjoint*:: *usubst ⇒* (*interp ⇒ state ⇒ interp*)
where *adjoint σ I ω* = *mkinterp*(
(*λf*. (*case SConst σ f of None ⇒ Consts I f| Some r ⇒ term-sem I r ω*)),
(*λf*. (*case SFuncs σ f of None ⇒ Funcs I f | Some r ⇒ λd. term-sem* (*repc I dotid d*) *r ω*)),
(*λp*. (*case SPreds σ p of None ⇒ Preds I p | Some r ⇒ λd. ω∈fml-sem* (*repc I dotid d*) *r*)),
(*λa*. (*case SGames σ a of None ⇒ Games I a | Some r ⇒ λX. game-sem I r X*))
)

Simple Observations about Adjoints **lemma** *adjoint-consts*: *Consts* (*adjoint σ I ω*) *f* = *term-sem I* (*case SConst σ f of Some r ⇒ r | None ⇒ Const f*) *ω*
unfolding *adjoint-def* **by** (*cases SConst σ f*=*None*) (*auto*)

lemma *adjoint-funcs*: *Funcs* (*adjoint σ I ω*) *f* = (*case SFuncs σ f of None ⇒ Funcs I f | Some r ⇒ λd. term-sem* (*repc I dotid d*) *r ω*)
unfolding *adjoint-def* **by** *auto*

lemma *adjoint-funcs-match*: *SFuncs σ f*=*Some r ⟹ Funcs* (*adjoint σ I ω*) *f* = (*λd. term-sem* (*repc I dotid d*) *r ω*)
using *adjoint-funcs* **by** *auto*

lemma *adjoint-funcs-skip*: *SFuncs σ f*=*None ⟹ Funcs* (*adjoint σ I ω*) *f* = *Funcs I f*
using *adjoint-funcs* **by** *auto*

lemma *adjoint-preds*: *Preds* (*adjoint σ I ω*) *p* = (*case SPreds σ p of None ⇒ Preds I p | Some r ⇒ λd. ω∈fml-sem* (*repc I dotid d*) *r*)
unfolding *adjoint-def* **by** *auto*

lemma *adjoint-preds-skip*: *SPreds σ p*=*None ⟹ Preds* (*adjoint σ I ω*) *p* = *Preds I p*
using *adjoint-preds* **by** *simp*

lemma *adjoint-preds-match*: *SPreds σ p*=*Some r ⟹ Preds* (*adjoint σ I ω*) *p* = (*λd. ω∈fml-sem* (*repc I dotid d*) *r*)
using *adjoint-preds* **by** *simp*

lemma *adjoint-games* [*simp*]: *Games* (*adjoint* σ *I* ω) *a* = (*case SGames* σ *a of None* ⇒ *Games I a* | *Some r* ⇒ λX. *game-sem I r X*)
unfolding *adjoint-def* **using** *adjoint-stays-mon Games-mkinterp* **by** *simp*

lemma *adjoint-dotsubstt*: *adjoint* (*dotsubstt* ϑ) *I* ω = *repc I dotid* (*term-sem I* ϑ ω)

proof–
let *?lhs* = *adjoint* (*dotsubstt* ϑ) *I* ω
let *?rhs* = *repc I dotid* (*term-sem I* ϑ ω)
have *feq*: *Funcs ?lhs* = *Funcs ?rhs* **using** *repc-funcs adjoint-funcs dotsubstt-def* **by** *auto*
moreover have *peq*: *Preds ?lhs* = *Preds ?rhs* **using** *repc-preds adjoint-preds dotsubstt-def* **by** *auto*
moreover have *geq*: *Games ?lhs* = *Games ?rhs* **using** *repc-games adjoint-games dotsubstt-def* **by** *auto*
moreover have *ceq*: *Consts ?lhs* = *Consts ?rhs* **using** *repc-consts adjoint-consts dotsubstt-def* **by** *auto*
show *?thesis* **using** *mkinterp-eq*[**where** *I*=‹*?lhs*› **and** *J*=‹*?rhs*›] *feq peq geq ceq* **by** *simp*
qed

7.5 Uniform Substitution for Terms

Lemma 15 of http://arxiv.org/abs/1902.07230

theorem *usubst-term*: *Uvariation* ν ω *U* ⟹ *usubstappt* σ *U* ϑ≠*undeft* ⟹
term-sem I (*the* (*usubstappt* σ *U* ϑ)) ν = *term-sem* (*adjoint* σ *I* ω) ϑ ν
proof–
assume *vaouter*: *Uvariation* ν ω *U*
assume *defouter*: *usubstappt* σ *U* ϑ≠*undeft*
show *usubstappt* σ *U* ϑ≠*undeft* ⟹ *term-sem I* (*the* (*usubstappt* σ *U* ϑ)) ν = *term-sem* (*adjoint* σ *I* ω) ϑ ν **for** σ ϑ
using *vaouter* **proof** (*induction arbitrary*: ν ω *rule*: *usubstappt-induct*)
case (*Var* σ *U x*)
then show *?case* **by** *simp*
next
case (*Number* σ *U r*)
then show *?case* **by** *simp*
next
case (*Const* σ *U f*)
then show *?case*
proof (*cases SConst* σ *f*)
case *None*
then show *?thesis* **using** *adjoint-consts* **by** (*simp add*: *usappconst-def*)
next
case (*Some r*)
then have *varcond*: *FVT*(*r*)∩*U*={} **using** *Const usubstappt-const usubstappt-const-conv* **by** (*metis option.inject option.simps*(*3*))

from *Some* **have** *term-sem I* (*the* (*usubstappt σ U* (*Const f*))) *ν* = *term-sem I r ν* **by** (*simp add*: *varcond*)
also have ... = *term-sem I r ω* **using** *Const coincidence-term-cor*[*of ν ω U r*] *varcond* **by** *simp*
also have ... = *Consts* (*adjoint σ I ω*) *f* **using** *Some adjoint-consts* **by** *simp*
also have ... = *term-sem* (*adjoint σ I ω*) (*Const f*) *ν* **by** *auto*
finally show *term-sem I* (*the* (*usubstappt σ U* (*Const f*))) *ν* = *term-sem* (*adjoint σ I ω*) (*Const f*) *ν* **.**
qed
next
case (*FuncMatch σ U f ϑ*)
then have *va*: *Uvariation ν ω U* **by** *simp*
then show *?case*
proof (*cases SFuncs σ f*)
case *None*
from *FuncMatch* **and** *None* **have** *IHsubterm*: *term-sem I* (*the* (*usubstappt σ U ϑ*)) *ν* = *term-sem* (*adjoint σ I ω*) *ϑ ν* **using** *va*
by (*simp add*: *FuncMatch.IH*(*1*) *usubstappt-func-conv*)
from *None* **show** *?thesis* **using** *usubstappt-func IHsubterm adjoint-funcs*
by (*metis* (*no-types*, *lifting*) *FuncMatch.prems*(*1*) *option.case-eq-if option.sel term-sem.simps*(*4*) *usubstappt.simps*(*4*))
next
case (*Some r*)
then have *varcond*: *FVT*(*r*)∩*U*={} **using** *FuncMatch usubstappt-func usubstappt-func2 usubstappt-func-conv* **by** *meson*
from *FuncMatch* **have** *subdef*: *usubstappt σ U ϑ* ≠ *undeft* **using** *usubstappt-func-conv* **by** *auto*
from *FuncMatch* **and** *Some*
have *IHsubterm*: *term-sem I* (*the* (*usubstappt σ U ϑ*)) *ν* = *term-sem* (*adjoint σ I ω*) *ϑ ν* **using** *va subdef* **by** *blast*
from *FuncMatch* **and** *Some*
have *IHsubsubst*: ⋀*ν ω*. *Uvariation ν ω* {} ⟹ *term-sem I* (*the* (*usubstappt* (*dotsubstt* (*the* (*usubstappt σ U ϑ*))) {} *r*)) *ν* = *term-sem* (*adjoint* (*dotsubstt* (*the* (*usubstappt σ U ϑ*))) *I ω*) *r ν*
using *subdef varcond* **by** *fastforce*

let *?d* = *term-sem I* (*the* (*usubstappt σ U ϑ*)) *ν*
have *deq*: *?d* = *term-sem* (*adjoint σ I ω*) *ϑ ν* **by** (*rule IHsubterm*)
let *?dotIa* = *adjoint* (*dotsubstt* (*the* (*usubstappt σ U ϑ*))) *I ν*

from *Some*
have *term-sem I* (*the* (*usubstappt σ U* (*Func f ϑ*))) *ν* = *term-sem I* (*the* (*usubstappt* (*dotsubstt* (*the* (*usubstappt σ U ϑ*))) {} *r*)) *ν* **using** *subdef varcond* **by** *force*
also have ... = *term-sem ?dotIa r ν* **using** *IHsubsubst*[**where** *ν*=*ν* **and** *ω*=*ν*] *Uvariation-empty* **by** *auto*
also have ... = *term-sem* (*repc I dotid ?d*) *r ν* **using** *adjoint-dotsubstt*[**where** *ϑ*=‹*the* (*usubstappt σ U ϑ*)› **and** *I*=*I* **and** *ω*=*ν*] **by** *simp*
also have ... = *term-sem* (*repc I dotid ?d*) *r ω* **using** *coincidence-term-cor*[**where**

$\omega{=}\nu$ **and** $\omega'{=}\omega$ **and** $U{=}U$ **and** $\vartheta{=}r$ **and** $I{=}\langle repc\ I\ dotid\ ?d\rangle]$ *va varcond* **by** *simp*

also have ... = (*Funcs* (*adjoint* σ I ω) f)(*?d*) **using** *adjoint-funcs-match Some* **by** *simp*
also have ... = (*Funcs* (*adjoint* σ I ω) f)(*term-sem* (*adjoint* σ I ω) ϑ ν) **using** *deq* **by** *simp*
also have ... = *term-sem* (*adjoint* σ I ω) (*Func* f ϑ) ν **by** *simp*
finally show *term-sem I* (*the* (*usubstappt* σ U (*Func* f ϑ))) ν = *term-sem* (*adjoint* σ I ω) (*Func* f ϑ) ν **.**
qed
next
case (*Plus* σ U ϑ η)
then show *?case* **using** *Pluso-undef* **by** *auto*
next
case (*Times* σ U ϑ η)
then show *?case* **using** *Timeso-undef* **by** *auto*
next
case (*Differential* σ U ϑ)
from *Differential* **have** *subdef*: *usubstappt* σ *allvars* $\vartheta \neq$ *undeft* **using** *usubstappt-differential-conv* **by** *simp*
from *Differential* **have** *IH*: $\bigwedge\nu$. *term-sem I* (*the* (*usubstappt* σ *allvars* ϑ)) ν = *term-sem* (*adjoint* σ I ω) ϑ ν **using** *subdef Uvariation-univ* **by** *blast*

have *term-sem I* (*the* (*usubstappt* σ U (*Differential* ϑ))) ν = *term-sem I* (*Differential* (*the* (*usubstappt* σ *allvars* ϑ))) ν **using** *subdef* **by** *force*
also have ... = *sum*(λx. ν(*DVar x*)∗*deriv*(λX. *term-sem I* (*the* (*usubstappt* σ *allvars* ϑ)) (*repv* ν (*RVar x*) *X*))(ν(*RVar x*)))(*allidents*) **by** *simp*
also have ... = *sum*(λx. ν(*DVar x*)∗*deriv*(λX. *term-sem* (*adjoint* σ I ω) ϑ (*repv* ν (*RVar x*) *X*))(ν(*RVar x*)))(*allidents*) **using** *IH* **by** *auto*
also have ... = *term-sem* (*adjoint* σ I ω) (*Differential* ϑ) ν **by** *simp*
finally show *term-sem I* (*the* (*usubstappt* σ U (*Differential* ϑ))) ν = *term-sem* (*adjoint* σ I ω) (*Differential* ϑ) ν **.**
qed
qed

7.6 Uniform Substitution for Formulas and Games

Separately Prove Crucial Ingredient for the ODE Case of *usubst-fml-game*
lemma *same-ODE-same-sol*:
($\bigwedge\nu$. *Uvariation* ν (*F*(0)) {*RVar x*,*DVar x*} $\Longrightarrow$ *term-sem I* ϑ ν = *term-sem J* η ν)
$\Longrightarrow$ *solves-ODE I F x* ϑ = *solves-ODE J F x* η
using *Uvariation-Vagree Vagree-def solves-ODE-def*

proof–
assume *va*: $\bigwedge\nu$. *Uvariation* ν (*F*(0)) {*RVar x*,*DVar x*} $\Longrightarrow$ *term-sem I* ϑ ν = *term-sem J* η ν
then have *va2*: $\bigwedge\nu$. *Uvariation* ν (*F*(0)) {*RVar x*,*DVar x*} $\Longrightarrow$ *term-sem J* η ν = *term-sem I* ϑ ν **by** *simp*

have *one*: $\bigwedge I\ J\ \vartheta\ \eta.\ (\bigwedge \nu.\ Uvariation\ \nu\ (F(0))\ \{RVar\ x, DVar\ x\} \Longrightarrow term\text{-}sem\ I\ \vartheta\ \nu = term\text{-}sem\ J\ \eta\ \nu)$
$\Longrightarrow solves\text{-}ODE\ I\ F\ x\ \vartheta \Longrightarrow solves\text{-}ODE\ J\ F\ x\ \eta$
proof–
fix $I\ J\ \vartheta\ \eta$
assume *vaflow*: $\bigwedge \nu.\ Uvariation\ \nu\ (F(0))\ \{RVar\ x, DVar\ x\} \Longrightarrow term\text{-}sem\ I\ \vartheta\ \nu = term\text{-}sem\ J\ \eta\ \nu$
assume *sol*: *solves-ODE I F x ϑ*
from *vaflow* **and** *sol* **show** *solves-ODE J F x η* **unfolding** *solves-ODE-def* **using** *Uvariation-Vagree coincidence-term*
by (*metis double-complement solves-Vagree sol*)
qed
show *solves-ODE I F x ϑ = solves-ODE J F x η* **using** *one*[**where** *ϑ=ϑ* **and** *η=η*, *OF va*] *one*[**where** *ϑ=η* **and** *η=ϑ*, *OF va2*]
by *force*
qed

lemma *usubst-ode*:
assumes *subdef*: *usubstappt σ {RVar x,DVar x} ϑ ≠ undeft*
shows *solves-ODE I F x* (*the* (*usubstappt σ {RVar x,DVar x} ϑ*)) = *solves-ODE* (*adjoint σ I* (*F*(*0*))) *F x ϑ*
proof–
have *vaflow*: $\bigwedge F\ \vartheta\ \zeta.\ solves\text{-}ODE\ I\ F\ x\ \vartheta \Longrightarrow Uvariation\ (F(\zeta))\ (F(0))\ \{RVar\ x, DVar\ x\}$ **using** *solves-Vagree-trans* **by** *simp*
from *subdef* **have** *IH*: $\bigwedge \nu.\ Uvariation\ \nu\ (F(0))\ \{RVar\ x, DVar\ x\} \Longrightarrow term\text{-}sem\ I\ (the\ (usubstappt\ \sigma\ \{RVar\ x, DVar\ x\}\ \vartheta))\ \nu = term\text{-}sem\ (adjoint\ \sigma\ I\ (F(0)))\ \vartheta\ \nu$ **by** (*simp add*: *usubst-term*)
then show *?thesis* **using** *IH vaflow solves-ODE-def Uvariation-Vagree same-ODE-same-sol* **by** *blast*
qed

lemma *usubst-ode-ext*:
assumes *uv*: *Uvariation* (*F*(*0*)) *ω* (*U*∪{*RVar x,DVar x*})
assumes *subdef*: *usubstappt σ* (*U*∪{*RVar x,DVar x*}) *ϑ ≠ undeft*
shows *solves-ODE I F x* (*the* (*usubstappt σ* (*U*∪{*RVar x,DVar x*}) *ϑ*)) = *solves-ODE* (*adjoint σ I ω*) *F x ϑ*

proof–
have *vaflow1*: $\bigwedge F\ \vartheta\ \zeta.\ solves\text{-}ODE\ I\ F\ x\ (the\ (usubstappt\ \sigma\ (U \cup \{RVar\ x, DVar\ x\})\ \vartheta)) \Longrightarrow Uvariation\ (F(\zeta))\ (F(0))\ \{RVar\ x, DVar\ x\}$ **using** *solves-Vagree-trans* **by** *simp*
have *vaflow2*: $\bigwedge F\ \vartheta\ \zeta.\ solves\text{-}ODE\ (adjoint\ \sigma\ I\ \omega)\ F\ x\ \vartheta \Longrightarrow Uvariation\ (F(\zeta))\ (F(0))\ \{RVar\ x, DVar\ x\}$ **using** *solves-Vagree-trans* **by** *simp*
from *subdef* **have** *IH*: $\bigwedge \nu.\ Uvariation\ \nu\ (F(0))\ (U \cup \{RVar\ x, DVar\ x\}) \Longrightarrow term\text{-}sem\ I\ (the\ (usubstappt\ \sigma\ (U \cup \{RVar\ x, DVar\ x\})\ \vartheta))\ \nu = term\text{-}sem\ (adjoint\ \sigma\ I\ (F(0)))\ \vartheta\ \nu$ **using** *Uvariation-refl Uvariation-trans usubst-term* **by** *blast*
have *l2r*: *solves-ODE I F x* (*the* (*usubstappt σ* (*U*∪{*RVar x,DVar x*}) *ϑ*)) ⟹ *solves-ODE* (*adjoint σ I ω*) *F x ϑ*

using *vaflow1 subdef same-ODE-same-sol Uvariation-trans usubst-term uv*

proof −
assume *a1*: *solves-ODE I F x* (*the* (*usubstappt σ* (*U* ∪ {*RVar x*, *DVar x*}) *ϑ*))
obtain *rr* :: *trm* ⇒ *interp* ⇒ *trm* ⇒ *interp* ⇒ *char* ⇒ (*real* ⇒ *variable* ⇒ *real*) ⇒ *variable* ⇒ *real* **where**
f2: ∀ *x0 x1 x2 x3 x4 x5*. (∃ *v6*. *Uvariation v6* (*x5 0*) {*RVar x4*, *DVar x4*} ∧ *term-sem x3 x2 v6* ≠ *term-sem x1 x0 v6*) = (*Uvariation* (*rr x0 x1 x2 x3 x4 x5*) (*x5 0*) {*RVar x4*, *DVar x4*} ∧ *term-sem x3 x2* (*rr x0 x1 x2 x3 x4 x5*) ≠ *term-sem x1 x0* (*rr x0 x1 x2 x3 x4 x5*))
by *moura*
have *f3*: *Uvariation* (*F 0*) *ω* (*insert* (*RVar x*) (*U* ∪ {*DVar x*}))
using *uv* **by** *auto*
have *f4*: {*DVar x*} ∪ {} ∪ {*DVar x*} = *insert* (*DVar x*) ({*DVar x*} ∪ {} ∪ {}) ⟶ {*RVar x*} ∪ {*DVar x*} ∪ *insert* (*RVar x*) (*U* ∪ {*DVar x*}) = *insert* (*RVar x*) (*U* ∪ {*DVar x*})
by *fastforce*
{ assume {*DVar x*} ∪ {} ∪ {*DVar x*} = *insert* (*DVar x*) ({*DVar x*} ∪ {} ∪ {}) ∧ *Uvariation* (*rr* (*the* (*usubstappt σ* (*U* ∪ {*RVar x*, *DVar x*}) *ϑ*)) *I ϑ* (*USubst.adjoint σ I ω*) *x F*) *ω* ({*RVar x*} ∪ {*DVar x*} ∪ *insert* (*RVar x*) (*U* ∪ {*DVar x*}))
then have ¬ *Uvariation* (*rr* (*the* (*usubstappt σ* (*U* ∪ {*RVar x*, *DVar x*}) *ϑ*)) *I ϑ* (*USubst.adjoint σ I ω*) *x F*) (*F 0*) {*RVar x*, *DVar x*} ∨ *term-sem* (*USubst.adjoint σ I ω*) *ϑ* (*rr* (*the* (*usubstappt σ* (*U* ∪ {*RVar x*, *DVar x*}) *ϑ*)) *I ϑ* (*USubst.adjoint σ I ω*) *x F*) = *term-sem I* (*the* (*usubstappt σ* (*U* ∪ {*RVar x*, *DVar x*}) *ϑ*)) (*rr* (*the* (*usubstappt σ* (*U* ∪ {*RVar x*, *DVar x*}) *ϑ*)) *I ϑ* (*USubst.adjoint σ I ω*) *x F*)
using *f4 subdef usubst-term* **by** *auto* **}**
then have ¬ *Uvariation* (*rr* (*the* (*usubstappt σ* (*U* ∪ {*RVar x*, *DVar x*}) *ϑ*)) *I ϑ* (*USubst.adjoint σ I ω*) *x F*) (*F 0*) {*RVar x*, *DVar x*} ∨ *term-sem* (*USubst.adjoint σ I ω*) *ϑ* (*rr* (*the* (*usubstappt σ* (*U* ∪ {*RVar x*, *DVar x*}) *ϑ*)) *I ϑ* (*USubst.adjoint σ I ω*) *x F*) = *term-sem I* (*the* (*usubstappt σ* (*U* ∪ {*RVar x*, *DVar x*}) *ϑ*)) (*rr* (*the* (*usubstappt σ* (*U* ∪ {*RVar x*, *DVar x*}) *ϑ*)) *I ϑ* (*USubst.adjoint σ I ω*) *x F*)
using *f3* **by** (*metis* (*no-types*) *Uvariation-trans insert-is-Un*)
then show *?thesis*
using *f2 a1* **by** (*meson same-ODE-same-sol*)
qed
have *r2l*: *solves-ODE* (*adjoint σ I ω*) *F x ϑ* ⟹ *solves-ODE I F x* (*the* (*usubstappt σ* (*U*∪{*RVar x*,*DVar x*}) *ϑ*))
using *vaflow2 subdef same-ODE-same-sol Uvariation-trans usubst-term uv*

proof −
assume *a1*: *solves-ODE* (*USubst.adjoint σ I ω*) *F x ϑ*
obtain *rr* :: *trm* ⇒ *interp* ⇒ *trm* ⇒ *interp* ⇒ *char* ⇒ (*real* ⇒ *variable* ⇒ *real*) ⇒ *variable* ⇒ *real* **where**
∀ *x0 x1 x2 x3 x4 x5*. (∃ *v6*. *Uvariation v6* (*x5 0*) {*RVar x4*, *DVar x4*} ∧ *term-sem x3 x2 v6* ≠ *term-sem x1 x0 v6*) = (*Uvariation* (*rr x0 x1 x2 x3 x4 x5*) (*x5 0*) {*RVar x4*, *DVar x4*} ∧ *term-sem x3 x2* (*rr x0 x1 x2 x3 x4 x5*) ≠ *term-sem x1 x0* (*rr x0 x1 x2 x3 x4 x5*))
by *moura*
then have *f2*: ∀ *f c i t ia ta*. *Uvariation* (*rr ta ia t i c f*) (*f 0*) {*RVar c*, *DVar*

$c\} \wedge$ *term-sem i t* (*rr ta ia t i c f*) $\neq$ *term-sem ia ta* (*rr ta ia t i c f*) $\vee$ *solves-ODE i f c t* = *solves-ODE ia f c ta*
by (*meson same-ODE-same-sol*)
have *f3*: *Uvariation* (*F 0*) ω ($\{$*RVar x*, *DVar x*$\}$ $\cup$ *U*)
using *uv* **by** *force*
have *f4*: *usubstappt* σ ($\{$*RVar x*, *DVar x*$\}$ $\cup$ *U*) $\vartheta \neq$ *undeft*
using *subdef* **by** *auto*
{ assume *Uvariation* (*rr* ϑ (*USubst.adjoint* σ *I* ω) (*the* (*usubstappt* σ (*U* $\cup$ $\{$*RVar x*, *DVar x*$\}$) ϑ)) *I x F*) ω ($\{$*RVar x*, *DVar x*$\}$ $\cup$ ($\{$*RVar x*, *DVar x*$\}$ $\cup$ *U*))
then have $\neg$ *Uvariation* (*rr* ϑ (*USubst.adjoint* σ *I* ω) (*the* (*usubstappt* σ (*U* $\cup$ $\{$*RVar x*, *DVar x*$\}$) ϑ)) *I x F*) (*F 0*) $\{$*RVar x*, *DVar x*$\}$ $\vee$ *term-sem I* (*the* (*usubstappt* σ (*U* $\cup$ $\{$*RVar x*, *DVar x*$\}$) ϑ)) (*rr* ϑ (*USubst.adjoint* σ *I* ω) (*the* (*usubstappt* σ (*U* $\cup$ $\{$*RVar x*, *DVar x*$\}$) ϑ)) *I x F*) = *term-sem* (*USubst.adjoint* σ *I* ω) ϑ (*rr* ϑ (*USubst.adjoint* σ *I* ω) (*the* (*usubstappt* σ (*U* $\cup$ $\{$*RVar x*, *DVar x*$\}$) ϑ)) *I x F*)
using *f4* **by** (*simp add*: *Un-commute usubst-term*) **}**
then show *?thesis*
using *f3 f2 a1* **by** (*meson Uvariation-trans*)
qed
from *l2r* **and** *r2l* **show** *?thesis* **by** *auto*
qed

lemma *usubst-ode-ext2*:
assumes *subdef*: *usubstappt* σ (*U*$\cup\{$*RVar x*,*DVar x*$\}$) $\vartheta \neq$ *undeft*
assumes *uv*: *Uvariation* (*F*(*0*)) ω (*U*$\cup\{$*RVar x*,*DVar x*$\}$)
shows *solves-ODE I F x* (*the* (*usubstappt* σ (*U*$\cup\{$*RVar x*,*DVar x*$\}$) ϑ)) = *solves-ODE* (*adjoint* σ *I* ω) *F x* ϑ
using *usubst-ode-ext subdef uv* **by** *blast*

Separately Prove the Loop Case of *usubst-fml-game* **lemma** *union-comm*: *A*$\cup$*B*=*B*$\cup$*A*
by *auto*

definition *loopfp*τ:: *game* $\Rightarrow$ *interp* $\Rightarrow$ (*state set* $\Rightarrow$ *state set*)
where *loopfp*τ α *I X* = *lfp*(λ*Z*. *X* $\cup$ *game-sem I* α *Z*)

lemma *usubst-game-loop*:
assumes *uv*: *Uvariation* ν ω *U*
and *IH*α*rec*: $\bigwedge\nu$ ω *X*. *Uvariation* ν ω (*fst*(*usubstappp* σ *U* α)) $\Longrightarrow$ *snd* (*usubstappp* σ (*fst*(*usubstappp* σ *U* α)) α)$\neq$*undefg* $\Longrightarrow$
($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp* σ (*fst*(*usubstappp* σ *U* α)) α))) *X*) = ($\nu \in$ *game-sem* (*adjoint* σ *I* ω) α *X*)
shows *snd* (*usubstappp* σ *U* (*Loop* α))$\neq$*undefg* $\Longrightarrow$ ($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp* σ *U* (*Loop* α)))) *X*) = ($\nu \in$ *game-sem* (*adjoint* σ *I* ω) (*Loop* α) *X*)
proof–
assume *def*: *snd* (*usubstappp* σ *U* (*Loop* α))$\neq$*undefg*
have *loopfix*: $\bigwedge\alpha$ *I X*. *loopfp*τ α *I X* = *game-sem I* (*Loop* α) *X* **unfolding** *loopfp*τ*-def* **using** *game-sem-loop* **by** *metis*

let *?σαloopoff* = *the* (*snd* (*usubstappp σ U* (*Loop α*)))
let *?σα* = *the* (*snd*(*usubstappp σ* (*fst*(*usubstappp σ U α*)) *α*))
let *?σαloop* = *Loop ?σα*
have *loopform*: *?σαloopoff* = *?σαloop* **using** *usubstappp-loop*
by (*metis* (*mono-tags*, *lifting*) *Loopo.simps*(*1*) *def option.exhaust-sel option.inject snd-conv usubstappp-loop-conv*)
let *?τ* = *loopfpτ ?σαloop I*
let *?ϱ* = *loopfpτ α* (*adjoint σ I ω*)
let *?V* = *fst*(*usubstappp σ U α*)
have *fact1*: ⋀*V*. *snd*(*usubstappp σ V α*)≠*undefg* ⟹ *fst*(*usubstappp σ V α*) ⊇ *BVG*(*the* (*snd*(*usubstappp σ V α*))) **using** *usubst-taboos* **by** *simp*
have *fact2*: ⋀*V W*. *snd*(*usubstappp σ V α*)≠*undefg* ⟹ *snd*(*usubstappp σ W α*)≠*undefg* ⟹ (*fst*(*usubstappp σ V α*) ⊇ *BVG*(*the* (*snd*(*usubstappp σ W α*)))) **using** *fact1 usubst-taboos usubstappp-det* **by** *metis*
have *VgeqBV*: *?V* ⊇ *BVG*(*?σα*) **using** *usubst-taboos fact2 def usubstappp-loop-conv* **by** *simp*
have *uvV*: *Vagree ν ω* (−*?V*) **using** *uv*
by (*metis Uvariation-Vagree Uvariation-mon double-compl usubst-taboos-mon*)

have *τeq*: *?τ*(*X*) = *game-sem I ?σαloop X* **using** *loopfix* **by** *auto*
have *ϱeq*: *?ϱ*(*X*) = *game-sem* (*adjoint σ I ω*) (*Loop α*) *X* **using** *loopfix* **by** *auto*
have *τisϱ*: *selectlike* (*?τ*(*X*)) *ω* (−*?V*)= *selectlike* (*?ϱ*(*X*)) *ω* (−*?V*)
proof−
let *?f* = *λZ*. *X* ∪ *game-sem I ?σα Z*
let *?g* = *λY*. *X* ∪ *game-sem* (*adjoint σ I ω*) *α Y*
let *?R* = *λZ Y*. *selectlike Z ω* (−*?V*) = *selectlike Y ω* (−*?V*)
have *?R* (*lfp ?f*) (*lfp ?g*)
proof (*induction rule*: *lfp-lockstep-induct*[**where** *f*=‹*?f*› **and** *g*=‹*?g*› **and** *R*=‹*?R*›])
case *monof*
then show *?case* **using** *game-sem-loop-fixpoint-mono* **by** *simp*
next
case *monog*
then show *?case* **using** *game-sem-loop-fixpoint-mono* **by** *simp*
next
case (*step A B*)
then have *IHfp*: *selectlike A ω* (−*?V*) = *selectlike B ω* (−*?V*) **by** *simp*
show *selectlike* (*X* ∪ *game-sem I ?σα A*) *ω* (−*?V*) = *selectlike* (*X* ∪ *game-sem* (*adjoint σ I ω*) *α B*) *ω* (−*?V*)
proof (*rule selectlike-equal-cocond-corule*
)
fix *μ*
assume *muvar*: *Uvariation μ ω ?V*
have *forw*: (*μ* ∈ *X* ∪ *game-sem I ?σα A*) = (*μ* ∈ *X* ∪ *game-sem I ?σα* (*selectlike A μ* (−*BVG*(*?σα*)))) **using** *boundeffect* **by** *auto*

have (*μ* ∈ *X* ∪ *game-sem* (*adjoint σ I ω*) *α B*) = (*μ* ∈ *X* ∪ *game-sem I ?σα B*) **using** *IHαrec*[*OF muvar*]
using *def usubstappp-loop-conv* **by** *auto*

also have ... = ($\mu \in X \cup$ *game-sem I ?σα* (*selectlike B μ* ($-BVG(?\sigma\alpha)$))) **using** *boundeffect* **by** *simp*
finally have *backw*: ($\mu \in X \cup$ *game-sem* (*adjoint σ I ω*) *α B*) = ($\mu \in X \cup$ *game-sem I ?σα* (*selectlike B μ* ($-BVG(?\sigma\alpha)$))) **.**

have *samewin*: *selectlike A μ* ($-BVG(?\sigma\alpha)$) = *selectlike B μ* ($-BVG(?\sigma\alpha)$) **using** *IHfp selectlike-antimon VgeqBV muvar Uvariation-trans selectlike-equal-cocond*

proof −
have *Vagree μ ω* (− *fst* (*usubstappp σ U α*))
by (*metis Uvariation-Vagree double-complement muvar*)
then have *selectlike A μ* (− *fst* (*usubstappp σ U α*)) = *selectlike B μ* (− *fst* (*usubstappp σ U α*))
using *IHfp selectlike-Vagree* **by** *presburger*
then show *?thesis*
by (*metis* (*no-types*) *Compl-subset-Compl-iff VgeqBV selectlike-compose sup.absorb-iff2*)
qed
from *forw* **and** *backw* **show** ($\mu \in X \cup$ *game-sem I ?σα A*) = ($\mu \in X \cup$ *game-sem* (*adjoint σ I ω*) *α B*) **using** *samewin* **by** *auto*
qed

next
case (*union M*)
then show *?case* **using** *selectlike-Sup*[**where** ν=ω **and** V=‹− *?V*›] *fst-proj-def snd-proj-def* **by** (*simp*) (*blast*)
qed
then show *?thesis*
using τ*eq loopfix loopfpτ-def* **by** *auto*
qed
show *?thesis* **using** τ*eq ϱeq τisϱ similar-selectlike-mem*[*OF uvV*] **by** (*metis loopform*)
qed

lemma *usubst-fml-game*:
assumes *vaouter*: *Uvariation ν ω U*
shows *usubstappf σ U φ*≠*undeff* ⟹ ($\nu \in$ *fml-sem I* (*the* (*usubstappf σ U φ*))) = ($\nu \in$ *fml-sem* (*adjoint σ I ω*) *φ*)
and *snd* (*usubstappp σ U α*)≠*undefg* ⟹ ($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp σ U α*))) *X*) = ($\nu \in$ *game-sem* (*adjoint σ I ω*) *α X*)
proof−
show *usubstappf σ U φ*≠*undeff* ⟹ ($\nu \in$ *fml-sem I* (*the* (*usubstappf σ U φ*))) = ($\nu \in$ *fml-sem* (*adjoint σ I ω*) *φ*)
and *snd* (*usubstappp σ U α*)≠*undefg* ⟹ ($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp σ U α*))) *X*) = ($\nu \in$ *game-sem* (*adjoint σ I ω*) *α X*) **for** *σ φ α*
using *vaouter* **proof** (*induction φ* **and** *α arbitrary*: *ν ω* **and** *ν ω X rule*: *usubstappfp-induct*)

case (*Pred* σ U p ϑ)
then have *va*: *Uvariation* ν ω U **by** *simp*
then show *?case*
proof (*cases SPreds* σ p)
case *None*
then show *?thesis* **using** *usubst-term*[*OF va*] *adjoint-preds-skip*

proof −
have $\forall p\ V\ c\ t$. *usubstappf* p V (*Pred* c t) = (*if usubstappt* p V t = *undeft* *then undeff else case SPreds* p c *of None* ⇒ *Afml* (*Pred* c (*the* (*usubstappt* p V t))) | *Some* f ⇒ *if FVF* f ∩ V = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* p V t))) {} f *else undeff*)
by (*simp add*: *option.case-eq-if*)
then have *f1*: $\forall p\ V\ c\ t$. *if usubstappt* p V t = *undeft then usubstappf* p V (*Pred* c t) = *undeff else usubstappf* p V (*Pred* c t) = (*case SPreds* p c *of None* ⇒ *Afml* (*Pred* c (*the* (*usubstappt* p V t))) | *Some* f ⇒ *if FVF* f ∩ V = {} *then usubstappf* (*dotsubstt* (*the* (*usubstappt* p V t))) {} f *else undeff*)
by *presburger*
then have *usubstappt* σ U ϑ ≠ *undeft*
by (*meson Pred.prems*(*1*))
then have *f2*: *usubstappf* σ U (*Pred* p ϑ) = *Afml* (*Pred* p (*the* (*usubstappt* σ U ϑ)))
using *None* **by** *force*
have *usubstappt* σ U ϑ ≠ *undeft*
using *f1* **by** (*meson Pred.prems*(*1*))
then show *?thesis*
using *f2* **by** (*simp add*: *None* ‹⋀ϑ σ I. *usubstappt* σ U ϑ ≠ *undeft* ⟹ *term-sem* I (*the* (*usubstappt* σ U ϑ)) ν = *term-sem* (*USubst.adjoint* σ I ω) ϑ ν› *adjoint-preds-skip*)
qed

next
case (*Some r*)
then have *varcond*: *FVF*(*r*)∩*U*={} **using** *Pred usubstappf-pred usubstappf-pred2 usubstappf-pred-conv* **by** *meson*
from *Pred* **have** *subdef*: *usubstappt* σ U ϑ ≠ *undeft* **using** *usubstappf-pred-conv* **by** *auto*
from *Pred* **and** *Some*
have *IHsubsubst*: ⋀ν ω. *Uvariation* ν ω {} ⟹ (ν ∈ *fml-sem* I (*the* (*usubstappf* (*dotsubstt* (*the* (*usubstappt* σ U ϑ))) {} *r*))) = (ν ∈ *fml-sem* (*adjoint* (*dotsubstt* (*the* (*usubstappt* σ U ϑ))) I ω) *r*)
using *subdef varcond* **by** *fastforce*

let *?d* = *term-sem* I (*the* (*usubstappt* σ U ϑ)) ν
have *deq*: *?d* = *term-sem* (*adjoint* σ I ω) ϑ ν **by** (*rule usubst-term*[*OF va subdef*])
let *?dotIa* = *adjoint* (*dotsubstt* (*the* (*usubstappt* σ U ϑ))) I ν

from *Some*

have (ν∈*fml-sem I* (*the* (*usubstappf* σ *U* (*Pred p* ϑ)))) = (ν∈*fml-sem I* (*the* (*usubstappf* (*dotsubstt* (*the* (*usubstappt* σ *U* ϑ))) {} *r*)))
using *subdef varcond* **by** *force*
also have ... = (ν∈*fml-sem ?dotIa r*) **using** *IHsubsubst*[**where** ν=ν **and** ω=ν] *Uvariation-empty* **by** *auto*
also have ... = (ν∈*fml-sem* (*repc I dotid ?d*) *r*) **using** *adjoint-dotsubstt*[**where** ϑ=‹*the* (*usubstappt* σ *U* ϑ)› **and** *I*=*I* **and** ω=ν] **by** *simp*
also have ... = (ω∈*fml-sem* (*repc I dotid ?d*) *r*) **using** *coincidence-formula-cor*[**where** ω=ν **and** ω'=ω **and** *U*=*U* **and** φ=*r* **and** *I*=‹*repc I dotid ?d*›] *va varcond* **by** *simp*

also have ... = (*Preds* (*adjoint* σ *I* ω) *p*)(*?d*) **using** *adjoint-preds-match Some* **by** *simp*
also have ... = (*Preds* (*adjoint* σ *I* ω) *p*)(*term-sem* (*adjoint* σ *I* ω) ϑ ν) **using** *deq* **by** *simp*
also have ... = (ν∈*fml-sem* (*adjoint* σ *I* ω) (*Pred p* ϑ)) **by** *simp*
finally show (ν∈*fml-sem I* (*the* (*usubstappf* σ *U* (*Pred p* ϑ)))) = (ν∈*fml-sem* (*adjoint* σ *I* ω) (*Pred p* ϑ)) **.**
qed

next
case (*Geq* σ *U* ϑ η)

then have *def1*: *usubstappt* σ *U* ϑ ≠ *undeft* **using** *usubstappf-geq-conv* **by** *simp*

moreover have *def2*: *usubstappt* σ *U* η ≠ *undeft* **using** *usubstappf-geq-conv Geq.prems*(*1*) **by** *blast*
show *?case*
using *usubst-term*[*OF* ‹*Uvariation* ν ω *U*›, *OF def1*] *usubst-term*[*OF* ‹*Uvariation* ν ω *U*›,*OF def2*] *usubstappf-geqr*[*OF* ‹*usubstappf* σ *U* (*Geq* ϑ η) ≠ *undeff*›] **by** *force*
next
case (*Not* σ *U* φ)
then show *?case* **using** *Noto-undef* **by** *auto*
next
case (*And* σ *U* φ ψ)
then show *?case* **using** *Ando-undef* **by** *auto*

next
case (*Exists* σ *U x* φ)
then have *IH*: $\bigwedge\nu$ ω. *Uvariation* ν ω (*U*∪{*x*}) ⟹ (ν ∈ *fml-sem I* (*the* (*usubstappf* σ (*U*∪{*x*}) φ))) = (ν ∈ *fml-sem* (*adjoint* σ *I* ω) φ) **by** *force*
from *Exists* **have** *Uvariation* ν ω *U* **by** *simp*

then have *Uvar*: $\bigwedge$*d*. *Uvariation* (*repv* ν *x d*) ω (*U*∪{*x*}) **using** *Uvariation-repv Uvariation-trans Uvariation-sym*
using *Exists.prems*(*2*) *Uvariation-def* **by** *auto*
have (ν∈*fml-sem I* (*the* (*usubstappf* σ *U* (*Exists x* φ)))) = (ν∈*fml-sem I* (*Exists x* (*the* (*usubstappf* σ (*U*∪{*x*}) φ))))

using *usubstappf-exists Exists.prems(1)* **by** *fastforce*
also have ... = $(\exists\, d.\ (repv\ \nu\ x\ d) \in fml\text{-}sem\ I\ (the\ (usubstappf\ \sigma\ (U \cup \{x\})\ \varphi)))$ **by** *simp*
also have ... = $(\exists\, d.\ (repv\ \nu\ x\ d) \in fml\text{-}sem\ (adjoint\ \sigma\ I\ \omega)\ \varphi)$ **using** *IH Uvar* **by** *auto*
also have ... = $(\nu \in fml\text{-}sem\ (adjoint\ \sigma\ I\ \omega)\ (Exists\ x\ \varphi))$ **by** *auto*
finally have $(\nu \in fml\text{-}sem\ I\ (the\ (usubstappf\ \sigma\ U\ (Exists\ x\ \varphi)))) = (\nu \in fml\text{-}sem\ (adjoint\ \sigma\ I\ \omega)\ (Exists\ x\ \varphi))$ **.**
from *this* **show** *?case* **by** *simp*

next
case $(Diamond\ \sigma\ U\ \alpha\ \varphi)$
let $?V = fst(usubstappp\ \sigma\ U\ \alpha)$
from *Diamond* **have** *IHα*: $\bigwedge X.\ Uvariation\ \nu\ \omega\ U \Longrightarrow (\nu \in game\text{-}sem\ I\ (the\ (snd\ (usubstappp\ \sigma\ U\ \alpha)))\ X) = (\nu \in game\text{-}sem\ (adjoint\ \sigma\ I\ \omega)\ \alpha\ X)$ **by** *fastforce*

from *Diamond* **have** *IHφ*: $\bigwedge \nu\ \omega.\ Uvariation\ \nu\ \omega\ (fst(usubstappp\ \sigma\ U\ \alpha)) \Longrightarrow (\nu \in fml\text{-}sem\ I\ (the\ (usubstappf\ \sigma\ (fst(usubstappp\ \sigma\ U\ \alpha))\ \varphi))) = (\nu \in fml\text{-}sem\ (adjoint\ \sigma\ I\ \omega)\ \varphi)$
by (*simp add*: *Diamondo-undef*)
from *Diamond* **have** *uv*: $Uvariation\ \nu\ \omega\ U$ **by** *simp*
have $(\nu \in fml\text{-}sem\ I\ (the\ (usubstappf\ \sigma\ U\ (Diamond\ \alpha\ \varphi)))) = (\nu \in fml\text{-}sem\ I\ (let\ V\alpha = usubstappp\ \sigma\ U\ \alpha\ in\ Diamond\ (the\ (snd\ V\alpha))\ (the\ (usubstappf\ \sigma\ (fst\ V\alpha)\ \varphi))))$
by (*metis Diamond.prems(1) Diamondo.elims option.sel usubstappf.simps(6)*)

also have ... = $(\nu \in fml\text{-}sem\ I\ (Diamond\ (the\ (snd(usubstappp\ \sigma\ U\ \alpha)))\ (the\ (usubstappf\ \sigma\ (fst(usubstappp\ \sigma\ U\ \alpha))\ \varphi))))$ **by** *simp*
also have ... = $(\nu \in game\text{-}sem\ I\ (the\ (snd(usubstappp\ \sigma\ U\ \alpha)))\ (fml\text{-}sem\ I\ (the\ (usubstappf\ \sigma\ (fst(usubstappp\ \sigma\ U\ \alpha))\ \varphi))))$ **by** *simp*
also have ... = $(\nu \in game\text{-}sem\ I\ (the\ (snd(usubstappp\ \sigma\ U\ \alpha)))\ (selectlike\ (fml\text{-}sem\ I\ (the\ (usubstappf\ \sigma\ (fst(usubstappp\ \sigma\ U\ \alpha))\ \varphi)))\ \nu\ (-BVG(the\ (snd(usubstappp\ \sigma\ U\ \alpha))))))$ **using** *boundeffect* **by** *auto*
finally have *forw*: $(\nu \in fml\text{-}sem\ I\ (the\ (usubstappf\ \sigma\ U\ (Diamond\ \alpha\ \varphi)))) = (\nu \in game\text{-}sem\ I\ (the\ (snd(usubstappp\ \sigma\ U\ \alpha)))\ (selectlike\ (fml\text{-}sem\ I\ (the\ (usubstappf\ \sigma\ (fst(usubstappp\ \sigma\ U\ \alpha))\ \varphi)))\ \nu\ (-BVG(the\ (snd(usubstappp\ \sigma\ U\ \alpha))))))$ **.**

have $(\nu \in fml\text{-}sem\ (adjoint\ \sigma\ I\ \omega)\ (Diamond\ \alpha\ \varphi)) = (\nu \in game\text{-}sem\ (adjoint\ \sigma\ I\ \omega)\ \alpha\ (fml\text{-}sem\ (adjoint\ \sigma\ I\ \omega)\ \varphi))$ **by** *simp*
also have ... = $(\nu \in game\text{-}sem\ I\ (the\ (snd(usubstappp\ \sigma\ U\ \alpha)))\ (fml\text{-}sem\ (adjoint\ \sigma\ I\ \omega)\ \varphi))$ **using** *IHα uv* **by** *simp*
also have ... = $(\nu \in game\text{-}sem\ I\ (the\ (snd(usubstappp\ \sigma\ U\ \alpha)))\ (selectlike\ (fml\text{-}sem\ (adjoint\ \sigma\ I\ \omega)\ \varphi)\ \nu\ (-BVG(the\ (snd(usubstappp\ \sigma\ U\ \alpha))))))$ **using** *boundeffect* **by** *auto*
finally have *backw*: $(\nu \in fml\text{-}sem\ (adjoint\ \sigma\ I\ \omega)\ (Diamond\ \alpha\ \varphi)) = (\nu \in game\text{-}sem\ I\ (the\ (snd(usubstappp\ \sigma\ U\ \alpha)))\ (selectlike\ (fml\text{-}sem\ (adjoint\ \sigma\ I\ \omega)\ \varphi)\ \nu\ (-BVG(the\ (snd(usubstappp\ \sigma\ U\ \alpha))))))$ **.**

have *samewin*: $selectlike\ (fml\text{-}sem\ I\ (the\ (usubstappf\ \sigma\ (fst(usubstappp\ \sigma\ U$

$\alpha)$ $\varphi)))$ ν $(-BVG(the\ (snd(usubstappp\ \sigma\ U\ \alpha))))$ = *selectlike* (*fml-sem* (*adjoint* σ I ω) φ) ν $(-BVG(the\ (snd(usubstappp\ \sigma\ U\ \alpha))))$

proof (*rule selectlike-equal-cocond-corule*)

fix μ

assume *muvar*: *Uvariation* μ ν $(BVG(the\ (snd(usubstappp\ \sigma\ U\ \alpha))))$

have $U\mu\omega$: *Uvariation* μ ω *?V* **using** *muvar uv Uvariation-trans union-comm usubst-taboos Uvariation-mon*

proof−

have $U\mu\nu$: *Uvariation* μ ν $(BVG(the\ (snd(usubstappp\ \sigma\ U\ \alpha))))$ **by** (*rule muvar*)

have $U\nu\omega$: *Uvariation* ν ω U **by** (*rule uv*)

have *Uvariation* μ ω $(U \cup BVG(the\ (snd(usubstappp\ \sigma\ U\ \alpha))))$ **using** *Uvariation-trans*[*OF* $U\mu\nu$ $U\nu\omega$] *union-comm* **by** (*rule HOL.back-subst*)

thus *?thesis* **using** *usubst-taboos Uvariation-mon* **by** (*metis* (*mono-tags*, *lifting*) *Diamond.prems*(*1*) *Diamondo-undef Uvariation-mon usubst-taboos usubstappf.simps*(*6*))

qed

have $(\mu \in$ *fml-sem* (*adjoint* σ I ω) $\varphi) = (\mu \in$ *fml-sem* I (*the* (*usubstappf* σ (*fst*(*usubstappp* σ U α)) $\varphi)))$

using *muvar Uvariation-trans uv IH*φ *boundeffect Uvariation-mon usubst-taboos* $U\mu\omega$ **by** *auto*

then show $(\mu \in$ *fml-sem* I (*the* (*usubstappf* σ (*fst* (*usubstappp* σ U $\alpha)$) $\varphi)))$ = $(\mu \in$ *fml-sem* (*adjoint* σ I ω) $\varphi)$ **by** *simp*

qed

from *forw* **and** *backw* **show** $(\nu \in$ *fml-sem* I (*the* (*usubstappf* σ U (*Diamond* α $\varphi))))) = (\nu \in$ *fml-sem* (*adjoint* σ I ω) (*Diamond* α $\varphi))$ **using** *samewin* **by** *auto*

next

case (*Game* σ U a)

then show *?case* **using** *adjoint-games usubstappp-game* **by** (*cases SGames* σ a) *auto*

next

case (*Assign* σ U x ϑ)

then show *?case* **using** *usubst-term Assigno-undef*

proof −

have *f1*: *usubstappt* σ U $\vartheta \neq$ *undeft*

using *Assign.prems*(*1*) *Assigno-undef* **by** *auto*

{ assume *repv* ν x (*term-sem* (*USubst.adjoint* σ I ω) ϑ $\nu) \in X$

then have *repv* ν x (*term-sem* I (*the* (*usubstappt* σ U $\vartheta)$) $\nu) \in X$

using *f1 Assign.prems*(*2*) *usubst-term* **by** *auto*

then have *repv* ν x (*term-sem* (*USubst.adjoint* σ I ω) ϑ $\nu) \in X \longrightarrow$ $(\nu \in$ *game-sem* I (*the* (*snd* (*usubstappp* σ U $(x := \vartheta))))$ $X) = (\nu \in$ *game-sem* (*USubst.adjoint* σ I ω) $(x := \vartheta)$ $X)$

using *f1* **by** *force* **}**

moreover

```
    { assume repv ν x (term-sem (USubst.adjoint σ I ω) ϑ ν) ∉ X
      then have repv ν x (term-sem I (the (usubstappt σ U ϑ)) ν) ∉ X ∧ repv
ν x (term-sem (USubst.adjoint σ I ω) ϑ ν) ∉ X
        using f1 Assign.prems(2) usubst-term by presburger
      then have ?thesis
        using f1 by force }
    ultimately show ?thesis
      by fastforce
  qed

 next
  case (Test σ U φ)
  then show ?case using Testo-undef by auto

 next
  case (Choice σ U α β)
  from Choice have IHα: ⋀X. Uvariation ν ω U ⟹ (ν ∈ game-sem I (the (snd
(usubstappp σ U α))) X) = (ν ∈ game-sem (adjoint σ I ω) α X) by (simp add:
Choiceo-undef)
  from Choice have IHβ: ⋀X. Uvariation ν ω U ⟹ (ν ∈ game-sem I (the (snd
(usubstappp σ U β))) X) = (ν ∈ game-sem (adjoint σ I ω) β X) by (simp add:
Choiceo-undef)
  from Choice have uv: Uvariation ν ω U by simp
  show ?case using IHα IHβ uv

  proof −
    have f1: Agame (the (snd (usubstappp σ U α))) = snd (usubstappp σ U α)
      by (meson Choice(3) option.collapse usubstappp-choice-conv)
    have Agame (the (snd (usubstappp σ U β))) = snd (usubstappp σ U β)
      by (meson Choice(3) option.collapse usubstappp-choice-conv)
    then have snd (usubstappp σ U (α ∪∪ β)) = Agame (the (snd (usubstappp
σ U α)) ∪∪ the (snd (usubstappp σ U β)))
      using f1 by (metis Choiceo.simps(1) snd-conv usubstappp.simps(4))
     then have f2: game-sem I (the (snd (usubstappp σ U α))) X ∪ game-sem
I (the (snd (usubstappp σ U β))) X = game-sem I (the (snd (usubstappp σ U (α
∪∪ β)))) X
      by simp
    moreover
    { assume ν ∉ game-sem I (the (snd (usubstappp σ U β))) X
      have (ν ∉ game-sem I (the (snd (usubstappp σ U (α ∪∪ β)))) X) = (ν ∈
game-sem (adjoint σ I ω) (α ∪∪ β) X) ⟶ ν ∉ game-sem I (the (snd (usubstappp
σ U β))) X ∧ ν ∉ game-sem (adjoint σ I ω) (α ∪∪ β) X
        using f2 Choice(4) IHα IHβ by auto
      then have (ν ∉ game-sem I (the (snd (usubstappp σ U (α ∪∪ β)))) X) ≠
(ν ∈ game-sem (adjoint σ I ω) (α ∪∪ β) X)
        using f2 Choice(4) IHα by auto }
    ultimately show ?thesis
      using Choice(4) IHβ by auto
  qed
```

next
case (*Compose* σ U α β)
let *?V* = *fst*(*usubstappp* σ U α)
from *Compose* **have** *IH*α: $\bigwedge X$. *Uvariation* ν ω $U \Longrightarrow$ ($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp* σ U α))) X) = ($\nu \in$ *game-sem* (*adjoint* σ I ω) α X) **by** (*simp add*: *Composeo-undef*)
from *Compose* **have** *IH*β: $\bigwedge \nu$ ω X. *Uvariation* ν ω *?V* $\Longrightarrow$ ($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp* σ *?V* β))) X) = ($\nu \in$ *game-sem* (*adjoint* σ I ω) β X) **by** (*simp add*: *Composeo-undef*)
from *Compose* **have** *uv*: *Uvariation* ν ω U **by** *simp*
have ($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp* σ U (*Compose* α β)))) X) = ($\nu \in$ *game-sem I* (*Compose* (*the* (*snd*(*usubstappp* σ U α))) (*the* (*snd*(*usubstappp* σ *?V* β)))) X)
by (*metis* (*no-types*, *lifting*) *Compose.prems*(*1*) *Composeo.elims option.sel snd-pair usubstappp.simps*(*5*))
also have ... = ($\nu \in$ *game-sem I* (*the* (*snd*(*usubstappp* σ U α))) (*game-sem I* (*the* (*snd*(*usubstappp* σ *?V* β))) X)) **by** *simp*
also have ... = ($\nu \in$ *game-sem I* (*the* (*snd*(*usubstappp* σ U α))) (*selectlike* (*game-sem I* (*the* (*snd*(*usubstappp* σ *?V* β))) X) ν ($-$*BVG*(*the*(*snd*(*usubstappp* σ U α)))))) **using** *boundeffect* **by** *auto*
finally have *forw*: ($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp* σ U (*Compose* α β)))) X) = ($\nu \in$ *game-sem I* (*the* (*snd*(*usubstappp* σ U α))) (*selectlike* (*game-sem I* (*the* (*snd*(*usubstappp* σ *?V* β))) X) ν ($-$*BVG*(*the*(*snd*(*usubstappp* σ U α)))))) .

have ($\nu \in$ *game-sem* (*adjoint* σ I ω) (*Compose* α β) X) = ($\nu \in$ *game-sem* (*adjoint* σ I ω) α ((*game-sem* (*adjoint* σ I ω) β) X)) **by** *simp*
also have ... = ($\nu \in$ *game-sem I* (*the* (*snd*(*usubstappp* σ U α))) ((*game-sem* (*adjoint* σ I ω) β) X)) **using** *IH*α[*OF uv*] **by** *auto*
also have ... = ($\nu \in$ *game-sem I* (*the* (*snd*(*usubstappp* σ U α))) (*selectlike* ((*game-sem* (*adjoint* σ I ω) β) X) ν ($-$*BVG*(*the*(*snd*(*usubstappp* σ U α)))))) **using** *boundeffect* **by** *auto*
finally have *backw*: ($\nu \in$ *game-sem* (*adjoint* σ I ω) (*Compose* α β) X) = ($\nu \in$ *game-sem I* (*the* (*snd*(*usubstappp* σ U α))) (*selectlike* ((*game-sem* (*adjoint* σ I ω) β) X) ν ($-$*BVG*(*the*(*snd*(*usubstappp* σ U α)))))) .

have *samewin*: *selectlike* (*game-sem I* (*the* (*snd*(*usubstappp* σ *?V* β))) X) ν ($-$*BVG*(*the*(*snd*(*usubstappp* σ U α)))) = *selectlike* ((*game-sem* (*adjoint* σ I ω) β) X) ν ($-$*BVG*(*the*(*snd*(*usubstappp* σ U α))))
proof (*rule selectlike-equal-cocond-corule*)
fix μ
assume *muvar*: *Uvariation* μ ν (*BVG*(*the*(*snd*(*usubstappp* σ U α))))
have *U*$\mu\omega$: *Uvariation* μ ω *?V* **using** *muvar uv Uvariation-trans union-comm usubst-taboos Uvariation-mon*

proof $-$
have *Uvariation* μ ω (*BVG* (*the* (*snd* (*usubstappp* σ U α))) $\cup$ U) **by** (*meson Uvariation-trans muvar uv*)
then show *?thesis* **using** *Uvariation-mon union-comm usubst-taboos*

by (*metis* (*no-types*, *lifting*) *Compose.prems*(*1*) *Composeo-undef Pair-inject prod.collapse usubstappp-compose*)

qed
have ($\mu \in$ *game-sem I* (*the*(*snd*(*usubstappp* σ *?V* β))) *X*) = ($\mu \in$ *game-sem* (*adjoint* σ *I* ω) β *X*)
using *muvar Uvariation-trans uv IH*β *boundeffect Uvariation-mon usubst-taboos U*$\mu\omega$ **by** *auto*
then show ($\mu \in$ *game-sem I* (*the*(*snd*(*usubstappp* σ *?V* β))) *X*) = ($\mu \in$ *game-sem* (*adjoint* σ *I* ω) β *X*) **by** *simp*
qed
from *forw* **and** *backw* **show** ($\nu \in$ *game-sem I* (*the*(*snd* (*usubstappp* σ *U* (*Compose* α β)))) *X*) = ($\nu \in$ *game-sem* (*adjoint* σ *I* ω) (*Compose* α β) *X*) **using** *samewin* **by** *auto*

next
case (*Loop* σ *U* α)
let *?V* = *fst*(*usubstappp* σ *U* α)
from *Loop* **have** *selfdef*: *snd* (*usubstappp* σ *U* (*Loop* α)) $\neq$ *undefg* **by** *auto*
from *Loop* **have** *IH*α*rec*: $\bigwedge\nu\ \omega\ X$. *Uvariation* ν ω *?V* $\Longrightarrow$ ($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp* σ *?V* α))) *X*) = ($\nu \in$ *game-sem* (*adjoint* σ *I* ω) α *X*) **by** *fastforce*
from *Loop* **have** *uv*: *Uvariation* ν ω *U* **by** *simp*
show ($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp* σ *U* (*Loop* α)))) *X*) = ($\nu \in$ *game-sem* (*adjoint* σ *I* ω) (*Loop* α) *X*)
using *usubst-game-loop IH*α*rec Loop.prems*(*2*) *selfdef* **by** *blast*

next
case (*Dual* σ *U* α)
from *Dual* **have** *IH*α: $\bigwedge X$. *Uvariation* ν ω *U* $\Longrightarrow$ ($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp* σ *U* α))) *X*) = ($\nu \in$ *game-sem* (*adjoint* σ *I* ω) α *X*) **by** *force*
from *Dual* **have** *uv*: *Uvariation* ν ω *U* **by** *simp*
from *Dual* **have** *def*: *snd* (*usubstappp* σ *U* ($\alpha\hat{}$*d*)) $\neq$ *undefg* **by** *simp*

have ($\nu \in -$*game-sem I* (*the* (*snd* (*usubstappp* σ *U* α))) ($-$*X*)) = ($\nu \in -$*game-sem* (*adjoint* σ *I* ω) α ($-$*X*)) **using** *IH*α[*OF uv*] **by** *simp*
then have ($\nu \in$ *game-sem I* ((*the* (*snd* (*usubstappp* σ *U* α)))$\hat{}$*d*) *X*) = ($\nu \in$ *game-sem* (*adjoint* σ *I* ω) ($\alpha\hat{}$*d*) *X*) **using** *game-sem.simps*(*7*) **by** *auto*
then show *?case* **using** *usubstappp-dual Dualo-undef*
proof $-$
have $\bigwedge\sigma\ V\ \alpha$. *snd* (*usubstappp* σ *U* ($\alpha\hat{}$*d*)) = *Dualo* (*snd* (*usubstappp* σ *U* α)) **by** *simp*
then have *snd* (*usubstappp* σ *U* α) $\neq$ *undefg* **using** *Dualo-undef def* **by** *presburger*
then show *?thesis*
using ‹($\nu \in$ *game-sem I* ((*the* (*snd* (*usubstappp* σ *U* α)))$\hat{}$*d*) *X*) = ($\nu \in$ *game-sem* (*USubst.adjoint* σ *I* ω) ($\alpha\hat{}$*d*) *X*)› **by** *force*
qed

next
case (*ODE* σ *U* x ϑ)
then have *va*: *Uvariation* ν ω *U* **by** *simp*
from *ODE* **have** *subdef*: *usubstappt* σ (*U*∪{*RVar x*,*DVar x*}) $\vartheta \neq$ *undeft* **by** (*simp add*: *ODEo-undef*)
from *ODE* **have** *IH*: *term-sem I* (*the* (*usubstappt* σ (*U*∪{*RVar x*,*DVar x*}) ϑ)) ν = *term-sem* (*adjoint* σ *I* ω) ϑ ν **using** *va*
by (*metis ODEo-undef fst-pair snd-conv usubst-taboos-mon usubst-term usubstappp.simps*(8) *usubstappt-antimon*)
have ($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp* σ *U* (*ODE x* ϑ)))) *X*) = ($\nu \in$ *game-sem I* (*the* (*ODEo x* (*usubstappt* σ (*U*∪{*RVar x*,*DVar x*}) ϑ))) *X*) **by** *simp*
also have ... = ($\nu \in$ *game-sem I* (*ODE x* (*the* (*usubstappt* σ (*U*∪{*RVar x*,*DVar x*}) ϑ))) *X*) **using** *subdef* **by** *force*
also have ... = ($\exists$ *F T*. *Vagree* ν (*F*(0)) (−{*DVar x*}) ∧ *F*(*T*) ∈ *X* ∧ *solves-ODE I F x* (*the* (*usubstappt* σ (*U*∪{*RVar x*,*DVar x*}) ϑ))) **by** *simp*
also have ... = ($\exists$ *F T*. *Uvariation* ν (*F*(0)) {*DVar x*} ∧ *F*(*T*) ∈ *X* ∧ *solves-ODE I F x* (*the* (*usubstappt* σ (*U*∪{*RVar x*,*DVar x*}) ϑ))) **using** *Uvariation-Vagree* **by** (*metis double-compl*)
also have ... = ($\exists$ *F T*. *Uvariation* ν (*F*(0)) {*DVar x*} ∧ *F*(*T*) ∈ *X* ∧ *solves-ODE* (*adjoint* σ *I* ω) *F x* ϑ)
using *usubst-ode-ext2*[*OF subdef*] *va solves-Vagree-trans Uvariation-trans Uvariation-sym-rel Uvariation-mon* **by** (*meson subset-insertI*)
also have ... = ($\exists$ *F T*. *Vagree* ν (*F*(0)) (−{*DVar x*}) ∧ *F*(*T*) ∈ *X* ∧ *solves-ODE* (*adjoint* σ *I* ω) *F x* ϑ) **using** *Uvariation-Vagree* **by** (*metis double-compl*)
also have ... = ($\nu \in$ *game-sem* (*adjoint* σ *I* ω) (*ODE x* ϑ) *X*) **using** *solves-ODE-def* **by** *simp*
finally show ($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp* σ *U* (*ODE x* ϑ)))) *X*) = ($\nu \in$ *game-sem* (*adjoint* σ *I* ω) (*ODE x* ϑ) *X*) .
qed
qed

Lemma 16 of http://arxiv.org/abs/1902.07230

theorem *usubst-fml*: *Uvariation* ν ω *U* $\Longrightarrow$ *usubstappf* σ *U* $\varphi \neq$ *undeff* $\Longrightarrow$
($\nu \in$ *fml-sem I* (*the* (*usubstappf* σ *U* φ))) = ($\nu \in$ *fml-sem* (*adjoint* σ *I* ω) φ)
using *usubst-fml-game* **by** *simp*

Lemma 17 of http://arxiv.org/abs/1902.07230

theorem *usubst-game*: *Uvariation* ν ω *U* $\Longrightarrow$ *snd* (*usubstappp* σ *U* α) $\neq$ *undefg* $\Longrightarrow$
($\nu \in$ *game-sem I* (*the* (*snd* (*usubstappp* σ *U* α))) *X*) = ($\nu \in$ *game-sem* (*adjoint* σ *I* ω) α *X*)
using *usubst-fml-game* **by** *simp*

7.7 Soundness of Uniform Substitution of Formulas

abbreviation *usubsta*:: *usubst* ⇒ *fml* ⇒ *fmlo*
where *usubsta* σ $\varphi \equiv$ *usubstappf* σ {} φ

Theorem 18 of http://arxiv.org/abs/1902.07230

theorem *usubst-sound*: *usubsta* σ $\varphi \neq$ *undefF* $\Longrightarrow$ *valid* $\varphi \Longrightarrow$ *valid* (*the* (*usubsta* σ φ))
proof–
assume *def*: *usubsta* σ $\varphi \neq$ *undefF*
assume *prem*: *valid* φ
from *prem* **have** *premc*: $\bigwedge I$ ω. $\omega \in$ *fml-sem* I φ **using** *valid-def* **by** *auto*
show *valid* (*the* (*usubsta* σ φ)) **unfolding** *valid-def*
proof (*clarify*)
fix I ω
have ($\omega \in$ *fml-sem* I (*the* (*usubsta* σ φ))) = ($\omega \in$ *fml-sem* (*adjoint* σ I ω) φ)
using *usubst-fml* **by** (*simp add*: *usubst-fml def*)
also have ... = *True* **using** *premc* **by** *simp*
finally have ($\omega \in$ *fml-sem* I (*the* (*usubsta* σ φ))) = *True* **.**
from *this* **show** $\omega \in$ *fml-sem* I (*the* (*usubstappf* σ {} φ)) **by** *simp*
qed
qed

7.8 Soundness of Uniform Substitution of Rules

Uniform Substitution applied to a rule or inference

definition *usubstr*:: *usubst* $\Rightarrow$ *inference* $\Rightarrow$ *inference option*
where *usubstr* σ $R \equiv$ *if* (*usubstappf* σ *allvars* (*snd* R) $\neq$ *undefF* $\wedge$ ($\forall \varphi \in$ *set* (*fst* R). *usubstappf* σ *allvars* $\varphi \neq$ *undefF*)) *then*
Some(*map*(*the o* (*usubstappf* σ *allvars*))(*fst* R), *the* (*usubstappf* σ *allvars* (*snd* R)))
else
None

Simple observations about applying uniform substitutions to a rule

lemma *usubstr-conv*: *usubstr* σ $R \neq$ *None* $\Longrightarrow$
usubstappf σ *allvars* (*snd* R) $\neq$ *undefF* $\wedge$
($\forall \varphi \in$ *set* (*fst* R). *usubstappf* σ *allvars* $\varphi \neq$ *undefF*)
by (*metis usubstr-def*)

lemma *usubstr-union-undef*: (*usubstr* σ ((*append* A B), C) $\neq$ *None*) = (*usubstr* σ (A, C) $\neq$ *None* $\wedge$ *usubstr* σ (B, C) $\neq$ *None*)
using *usubstr-def* **by** *auto*
lemma *usubstr-union-undef2*: (*usubstr* σ ((*append* A B), C) $\neq$ *None*) $\Longrightarrow$ (*usubstr* σ (A, C) $\neq$ *None* $\wedge$ *usubstr* σ (B, C) $\neq$ *None*)
using *usubstr-union-undef* **by** *blast*

lemma *usubstr-cons-undef*: (*usubstr* σ ((*Cons* A B), C) $\neq$ *None*) = (*usubstr* σ ([A], C) $\neq$ *None* $\wedge$ *usubstr* σ (B, C) $\neq$ *None*)
using *usubstr-def* **by** *auto*
lemma *usubstr-cons-undef2*: (*usubstr* σ ((*Cons* A B), C) $\neq$ *None*) $\Longrightarrow$ (*usubstr* σ ([A], C) $\neq$ *None* $\wedge$ *usubstr* σ (B, C) $\neq$ *None*)
using *usubstr-cons-undef* **by** *blast*

lemma *usubstr-cons*: (*usubstr* σ ((*Cons* A B), C) $\neq$ *None*) $\Longrightarrow$

the (*usubstr* σ ((*Cons A B*), *C*)) = (*Cons* (*the* (*usubstappf* σ *allvars A*)) (*fst* (*the* (*usubstr* σ (*B*, *C*)))), *snd* (*the* (*usubstr* σ ([*A*], *C*))))
using *usubstr-union-undef map-cons usubstr-def*
proof–
assume *def*: (*usubstr* σ ((*Cons A B*), *C*) $\neq$ *None*)
let *?R* = ((*Cons A B*), *C*)
have *the* (*usubstr* σ *?R*) = (*map*(*the o* (*usubstappf* σ *allvars*))(*fst ?R*) , *the* (*usubstappf* σ *allvars* (*snd ?R*))) **using** *def usubstr-def* **by** (*metis option.sel*)
also have ... = (*Cons* (*the* (*usubstappf* σ *allvars A*)) (*map*(*the o* (*usubstappf* σ *allvars*))(*B*)) , *the* (*usubstappf* σ *allvars* (*snd ?R*))) **using** *map-cons* **by** *auto*
also have ... = (*Cons* (*the* (*usubstappf* σ *allvars A*)) (*fst* (*the* (*usubstr* σ (*B*, *C*)))) , *the* (*usubstappf* σ *allvars* (*snd ?R*))) **using** *usubstr-cons-undef2*[*OF def*] *usubstr-def* **by** (*metis* (*no-types*, *lifting*) *fst-conv option.sel*)
also have ... = (*Cons* (*the* (*usubstappf* σ *allvars A*)) (*fst* (*the* (*usubstr* σ (*B*, *C*)))) , *snd* (*the* (*usubstr* σ ([*A*], *C*)))) **using** *def usubstr-def* **by** *auto*
ultimately show *the* (*usubstr* σ ((*Cons A B*), *C*)) = (*Cons* (*the* (*usubstappf* σ *allvars A*)) (*fst* (*the* (*usubstr* σ (*B*, *C*)))) , *snd* (*the* (*usubstr* σ ([*A*], *C*)))) **by** *simp*
qed

lemma *usubstr-union*: (*usubstr* σ ((*append A B*), *C*) $\neq$ *None*) $\Longrightarrow$
the (*usubstr* σ ((*append A B*), *C*)) = (*append* (*fst* (*the* (*usubstr* σ (*A*, *C*)))) (*fst* (*the* (*usubstr* σ (*B*, *C*)))), *snd* (*the* (*usubstr* σ (*A*, *C*))))
using *usubstr-union-undef2*

proof–
assume *def*: (*usubstr* σ ((*append A B*), *C*) $\neq$ *None*)
let *?R* = ((*append A B*), *C*)
have *the* (*usubstr* σ *?R*) = (*map*(*the o* (*usubstappf* σ *allvars*))(*fst ?R*) , *the* (*usubstappf* σ *allvars* (*snd ?R*))) **using** *def usubstr-def* **by** (*metis option.sel*)
also have ... = (*map*(*the o* (*usubstappf* σ *allvars*))(*fst ?R*) , *snd* (*the* (*usubstr* σ (*A*, *C*)))) **using** *usubstr-union-undef2*[*OF def*] *usubstr-def* **by** (*metis option.sel sndI*)
also have ... = (*append* (*map*(*the o* (*usubstappf* σ *allvars*))(*A*)) (*map*(*the o* (*usubstappf* σ *allvars*))(*B*)) , *snd* (*the* (*usubstr* σ (*A*, *C*)))) **using** *usubstr-union-undef2*[*OF def*] *map-append* **by** *simp*
also have ... = (*append* (*fst* (*the* (*usubstr* σ (*A*, *C*)))) (*fst* (*the* (*usubstr* σ (*B*, *C*)))), *snd* (*the* (*usubstr* σ (*A*, *C*)))) **using** *usubstr-union-undef2*[*OF def*] *usubstr-def* **by** (*metis* (*no-types*, *lifting*) *fst-conv option.sel*)
ultimately show *the* (*usubstr* σ ((*append A B*), *C*)) = (*append* (*fst* (*the* (*usubstr* σ (*A*, *C*)))) (*fst* (*the* (*usubstr* σ (*B*, *C*)))), *snd* (*the* (*usubstr* σ (*A*, *C*)))) **by** *simp*
qed

lemma *usubstr-length*: *usubstr* σ *R* $\neq$ *None* $\Longrightarrow$ *length* (*fst* (*the* (*usubstr* σ *R*))) = *length* (*fst R*)
by (*metis fst-pair length-map option.sel usubstr-def*)

lemma *usubstr-nth*: *usubstr* σ *R* $\neq$ *None* $\Longrightarrow$ *0*$\leq$*k* $\Longrightarrow$ *k*<*length* (*fst R*) $\Longrightarrow$
nth (*fst* (*the* (*usubstr* σ *R*))) *k* = *the* (*usubstappf* σ *allvars* (*nth* (*fst R*) *k*))

proof–
assume *a1*: *usubstr σ R ≠ None*
assume *a2*: *0≤k*
assume *a3*: *k<length (fst R)*
show *nth (fst (the (usubstr σ R))) k = the (usubstappf σ allvars (nth (fst R) k))*
using *a1 a2 a3* **proof** (*induction R arbitrary*: *k*)
case (*Pair A C*)
then show *?case*
proof (*induction A arbitrary*: *k*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons D E*)
then have *IH*: ⋀*k*. *usubstr σ (E, C) ≠ None ⟹ 0 ≤ k ⟹ k < length E ⟹ nth (fst (the (usubstr σ (E, C)))) k = the (usubstappf σ allvars (nth E k))* **by** *simp*
then show *?case*
proof (*cases k*)
case *0*
then show *?thesis* **using** *Cons usubstr-cons* **by** *simp*
next
case (*Suc n*)
then have *smaller*: *n<length E* **using** *Cons.prems(3)* **by** *auto*
have *nati*: *0≤n* **by** *simp*
have *def*: *usubstr σ (E, C) ≠ None* **using** *usubstr-cons-undef2*[*OF Cons.prems(1)*] **by** *blast*
have *nth (fst (the (usubstr σ (E, C)))) n = the (usubstappf σ allvars (nth (fst (E,C)) n))* **using** *IH*[*OF def*, *OF nati*, *OF smaller*] **by** *simp*
then show *?thesis* **using** *Cons usubstr-cons* **by** (*simp add*: *Suc*)
qed
qed
qed
qed

Theorem 19 of http://arxiv.org/abs/1902.07230

theorem *usubst-rule-sound*: *usubstr σ R ≠ None ⟹ locally-sound R ⟹ locally-sound (the (usubstr σ R))*
proof–
assume *def*: *usubstr σ R ≠ None*
assume *prem*: *locally-sound R*
let *?σD = usubstr σ R*
fix *ω*
from *usubst-fml* **have** *substeq*: ⋀*I ν φ*. *usubstappf σ allvars φ ≠ undefF ⟹ (ν ∈ fml-sem I (the (usubstappf σ allvars φ))) = (ν ∈ fml-sem (adjoint σ I ω) φ)* **using** *Uvariation-univ* **by** *blast*
then have *substval*: ⋀*I*. *usubstappf σ allvars φ ≠ undefF ⟹ valid-in I (the (usubstappf σ allvars φ)) = valid-in (adjoint σ I ω) φ* **unfolding** *valid-in-def* **by** *auto*
show *locally-sound (the (usubstr σ R))* **unfolding** *locally-sound-def*

proof (*clarify*)
fix *I*
assume $\forall k{\geq}0.\ k < length\ (fst\ (the\ (usubstr\ \sigma\ R))) \longrightarrow valid\text{-}in\ I\ (nth\ (fst\ (the\ (usubstr\ \sigma\ R)))\ k)$
then have $\forall k{\geq}0.\ k < length\ (fst\ R) \longrightarrow valid\text{-}in\ (adjoint\ \sigma\ I\ \omega)\ (nth\ (fst\ R)\ k)$ **using** *substval usubstr-nth usubstr-length substeq valid-in-def* **by** (*metis def nth-mem usubstr-def*)
then have *valid-in* (*adjoint* σ *I* ω) (*snd R*) **using** *prem* **unfolding** *locally-sound-def* **by** *simp*
from *this* **show** *valid-in I* (*snd* (*the* (*usubstr* σ *R*))) **using** *usubst-fml substeq usubstr-def valid-in-def* **by** (*metis def option.sel snd-conv*)
qed
qed

end
theory *Ids*
imports *Complex-Main*
Syntax
begin

Some specific identifiers used in Axioms

abbreviation *hgid1*::*ident* **where** *hgid1* ≡ *CHR ''a''*
abbreviation *hgid2*::*ident* **where** *hgid2* ≡ *CHR ''b''*
abbreviation *hgidc*::*ident* **where** *hgidc* ≡ *CHR ''c''*
abbreviation *hgidd*::*ident* **where** *hgidd* ≡ *CHR ''d''*
abbreviation *pid1*::*ident* **where** *pid1* ≡ *CHR ''p''*
abbreviation *pid2*::*ident* **where** *pid2* ≡ *CHR ''q''*
abbreviation *fid1*::*ident* **where** *fid1* ≡ *CHR ''f''*
abbreviation *xid1*::*variable* **where** *xid1* ≡ *RVar* (*CHR ''x''*)
end
theory *Axioms*
imports
Syntax
Denotational-Semantics
Ids
begin

8 Axioms and Axiomatic Proof Rules of Differential Game Logic

8.1 Axioms

abbreviation *pusall*:: *fml*
where *pusall* ≡ ⟨*Game hgidc*⟩ *TT*

abbreviation *nothing*:: *trm*
where *nothing* ≡ *Number 0*

named-theorems *axiom-defs Axiom definitions*

definition *box-axiom* :: *fml*
  where [*axiom-defs*]:
box-axiom ≡ (*Box* (*Game hgid1*) *pusall*) ↔ *Not*(*Diamond* (*Game hgid1*) (*Not*(*pusall*)))

definition *assigneq-axiom* :: *fml*
  where [*axiom-defs*]:
assigneq-axiom ≡ (*Diamond* (*Assign xid1* (*Const fid1*)) *pusall*) ↔ *Exists xid1* (*Equals* (*Var xid1*) (*Const fid1*) && *pusall*)

definition *stutterd-axiom* :: *fml*
  where [*axiom-defs*]:
stutterd-axiom ≡ (*Diamond* (*Assign xid1* (*Var xid1*)) *pusall*) ↔ *pusall*

definition *test-axiom* :: *fml*
  where [*axiom-defs*]:
test-axiom ≡ *Diamond* (*Test* (*Pred pid2 nothing*)) (*Pred pid1 nothing*) ↔ (*Pred pid2 nothing* && *Pred pid1 nothing*)

definition *choice-axiom* :: *fml*
  where [*axiom-defs*]:
choice-axiom ≡ *Diamond* (*Choice* (*Game hgid1*) (*Game hgid2*)) *pusall* ↔ (*Diamond* (*Game hgid1*) *pusall* || *Diamond* (*Game hgid2*) *pusall*)

definition *compose-axiom* :: *fml*
  where [*axiom-defs*]:
compose-axiom ≡ *Diamond* (*Compose* (*Game hgid1*) (*Game hgid2*)) *pusall* ↔ *Diamond* (*Game hgid1*) (*Diamond* (*Game hgid2*) *pusall*)

definition *iterate-axiom* :: *fml*
  where [*axiom-defs*]:
iterate-axiom ≡ *Diamond* (*Loop* (*Game hgid1*)) *pusall* ↔ (*pusall* || *Diamond* (*Game hgid1*) (*Diamond* (*Loop* (*Game hgid1*)) *pusall*))

definition *dual-axiom* :: *fml*
  where [*axiom-defs*]:
dual-axiom ≡ *Diamond* (*Dual* (*Game hgid1*)) *pusall* ↔ !(*Diamond* (*Game hgid1*) (!*pusall*))

8.2 Axiomatic Rules

named-theorems *rule-defs Rule definitions*

definition *mon-rule* :: *inference*
  where [*rule-defs*]:
mon-rule ≡ ([(⟨*Game hgidc*⟩ *TT*) → (⟨*Game hgidd*⟩ *TT*)], (⟨*Game hgid1*⟩(⟨*Game hgidc*⟩ *TT*)) → (⟨*Game hgid1*⟩(⟨*Game hgidd*⟩ *TT*)))

definition *FP-rule* :: *inference*
where [*rule-defs*]:
FP-rule ≡ ([((⟨*Game hgidc*⟩*TT*) || ⟨*Game hgid1*⟩⟨*Game hgidd*⟩*TT*) → ⟨*Game hgidd*⟩*TT*], (⟨*Loop* (*Game hgid1*)⟩⟨*Game hgidc*⟩*TT*) → (⟨*Game hgidd*⟩*TT*))

definition *MP-rule* :: *inference*
where [*rule-defs*]:
MP-rule ≡ ([*Pred pid1 nothing* , *Pred pid1 nothing* → *Pred pid2 nothing*], *Pred pid2 nothing*)

definition *gena-rule* :: *inference*
where [*rule-defs*]:
gena-rule ≡ ([*pusall*], *Exists xid1 pusall*)

8.3 Soundness / Validity Proofs for Axioms

Because an axiom in a uniform substitution calculus is an individual formula, proving the validity of that formula suffices to prove soundness

lemma *box-valid*: *valid box-axiom*
unfolding *box-axiom-def Box-def Or-def* **by** *simp*

lemma *assigneq-valid*: *valid assigneq-axiom*
unfolding *assigneq-axiom-def* **by** (*auto simp add*: *valid-equiv*)

lemma *stutterd-valid*: *valid stutterd-axiom*
unfolding *stutterd-axiom-def* **by** (*auto simp add*: *valid-equiv*)

lemma *test-valid*: *valid test-axiom*
unfolding *test-axiom-def Or-def* **using** *valid-equiv* **by** *fastforce*

lemma *choice-valid*: *valid choice-axiom*
unfolding *choice-axiom-def Or-def* **by** (*auto simp add*: *valid-equiv*)

lemma *compose-valid*: *valid compose-axiom*
unfolding *compose-axiom-def Or-def* **by** (*simp add*: *valid-equiv*)

lemma *dual-valid*: *valid dual-axiom*
unfolding *dual-axiom-def* **using** *valid-equiv fml-sem-not* **using** *fml-sem.simps*(*6*) *game-sem.simps*(*7*) **by** *presburger*

lemma *iterate-valid*: *valid iterate-axiom*

proof−
have ∀ *I*. *fml-sem I* (*Diamond* (*Loop* (*Game hgid1*)) *pusall*) = *fml-sem I* (*pusall*

|| *Diamond* (*Game hgid1*) (*Diamond* (*Loop* (*Game hgid1*)) *pusall*))
proof
fix *I*
have *fml-sem I* (*Diamond* (*Loop* (*Game hgid1*)) *pusall*) = *game-sem I* (*Loop* (*Game hgid1*)) (*fml-sem I pusall*) **by** (*rule fml-sem.simps*(*6*))
also have ... = *game-sem I* (*Choice Skip* (*Compose* (*Game hgid1*) (*Loop* (*Game hgid1*)))) (*fml-sem I pusall*)
using *game-equiv-subst*[**where** *I*=*I* **and** *X*=‹*fml-sem I pusall*›, *OF loop-iterate-equiv*[**where** *α*=‹*Game hgid1*›]] **by** *blast*
also have ... = *fml-sem I* (*Diamond* (*Choice Skip* (*Compose* (*Game hgid1*) (*Loop* (*Game hgid1*)))) *pusall*) **by** *simp*
also have ... = *fml-sem I* (*Diamond Skip pusall* || *Diamond* (*Compose* (*Game hgid1*) (*Loop* (*Game hgid1*))) *pusall*) **by** *simp*
also have ... = *fml-sem I* (*pusall* || *Diamond* (*Compose* (*Game hgid1*) (*Loop* (*Game hgid1*))) *pusall*) **by** *simp*
also have ... = *fml-sem I* (*pusall* || *Diamond* (*Game hgid1*) (*Diamond* (*Loop* (*Game hgid1*)) *pusall*)) **by** *simp*
finally show *fml-sem I* (*Diamond* (*Loop* (*Game hgid1*)) *pusall*) = *fml-sem I* (*pusall* || *Diamond* (*Game hgid1*) (*Diamond* (*Loop* (*Game hgid1*)) *pusall*)) **.**
qed
then have *valid* ((*Diamond* (*Loop* (*Game hgid1*)) *pusall*) ↔ (*pusall* || *Diamond* (*Game hgid1*) (*Diamond* (*Loop* (*Game hgid1*)) *pusall*))) **using** *valid-equiv* **by** (*rule rev-iffD2*)
then show *valid iterate-axiom* **unfolding** *iterate-axiom-def* **by** *auto*
qed

8.4 Local Soundness Proofs for Axiomatic Rules

lemma *mon-locsound*: *locally-sound mon-rule*
unfolding *mon-rule-def locally-sound-def* **using** *valid-in-impl monotone* **by** *simp*

lemma *FP-locsound*: *locally-sound FP-rule*
unfolding *FP-rule-def locally-sound-def* **using** *valid-in-impl game-sem-loop* **by** *auto*

lemma *MP-locsound*: *locally-sound MP-rule*
unfolding *MP-rule-def locally-sound-def valid-in-def* **using** *fml-sem-implies less-Suc-eq* **by** *auto*

lemma *gena-locsound*: *locally-sound gena-rule*
unfolding *gena-rule-def locally-sound-def valid-in-def* **using** *fml-sem-implies less-Suc-eq* **by** *auto*

end

9 dGL Formalization

theory *Differential-Game-Logic*
imports

Complex-Main
Lib
Identifiers
Syntax
Denotational-Semantics
Static-Semantics
Coincidence
USubst
Axioms
begin

This formalization of Differential Game Logic http://arxiv.org/abs/1902.07230 [4] consists of the syntax, denotational semantics, static semantics, uniform substitution lemmas, uniform substitution soundness proofs, and soundness proofs for axioms.

end

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