Deriving class instances for datatypes.*

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Abstract

We provide a framework for registering automatic methods to derive class instances of datatypes, as it is possible using Haskell’s “deriving Ord, Show, . . .” feature.

We further implemented such automatic methods to derive comparators, linear orders, parametrizable equality functions, and hash-functions which are required in the Isabelle Collection Framework [1] and the Container Framework [2]. Moreover, for the tactic of Blanchette to show that a datatype is countable, we implemented a wrapper so that this tactic becomes accessible in our framework. All of the generators are based on the infrastructure that is provided by the BNF-based datatype package.

Our formalization was performed as part of the IsaFoR/CeTA project[3]. With our new tactics we could remove several tedious proofs for (conditional) linear orders, and conditional equality operators within IsaFoR and the Container Framework.

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1http://cl-informatik.uibk.ac.at/software/ceta
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1 Derive Manager

theory Derive-Manager
imports Main
keywords print-derives :: diag and derive :: thy-decl
begin

The derive manager allows the user to register various derive-hooks, e.g., for orders, pretty-printers, hash-functions, etc. All registered hooks are accessible via the derive command.

derive (param) sort datatype calls the hook for deriving sort (that may depend on the optional param) on datatype (if such a hook is registered).

E.g., derive compare-order list will derive a comparator for datatype list which is also used to define a linear order on lists.

There is also the diagnostic command print-derives that shows the list of currently registered hooks.
ML-file ⟨derive-manager.ML⟩

end

2 Shared Utilities for all Generator

In this theory we mainly provide some Isabelle/ML infrastructure that is used by several generators. It consists of a uniform interface to access all the theorems, terms, etc. from the BNF package, and some auxiliary functions which provide recursors on datatypes, common tactics, etc.

theory Generator-Aux
imports Main
begin

ML-file ⟨bnf-access.ML⟩
ML-file ⟨generator-aux.ML⟩

lemma in-set-simps:
  \( x \in \text{set} (y # z # ys) = (x = y \lor x \in \text{set} (z # ys)) \)
  \( x \in \text{set} [y] = (x = y) \)
  \( x \in \text{set} [] = \text{False} \)
  \( \forall x (\text{set} [x]) P = \text{True} \)
Ball (set [x]) P = P x
Ball (set (x ≠ y ≠ zs)) P = (P x ∧ Ball (set (y ≠ zs)) P)
by auto

lemma conj-weak-cong: a = b ⟹ c = d ⟹ (a ∧ c) = (b ∧ d) by auto

lemma refl-True: (x = x) = True by simp

end

3 Comparisons

3.1 Comparators and Linear Orders

theory Comparator
imports Main
begin

Instead of having to define a strict and a weak linear order, ((<), (≤)),
one can alternative use a comparator to define the linear order, which may
deliver three possible outcomes when comparing two values.

datatype order = Eq | Lt | Gt

type-synonym 'a comparator = 'a ⇒ 'a ⇒ 'a ⇒ bool

In the following, we provide the obvious definitions how to switch be-
tween linear orders and comparators.

definition lt-of-comp :: 'a comparator ⇒ 'a ⇒ 'a ⇒ bool where
lt-of-comp acomp x y = (case acomp x y of Lt ⇒ True | - ⇒ False)

definition le-of-comp :: 'a comparator ⇒ 'a ⇒ 'a ⇒ bool where
le-of-comp acomp x y = (case acomp x y of Gt ⇒ False | - ⇒ True)

definition comp-of-ords :: ('a ⇒ 'a ⇒ bool) ⇒ ('a ⇒ 'a ⇒ bool) ⇒ 'a comparator
where
comp-of-ords le lt x y = (if lt x y then Lt else if le x y then Eq else Gt)

lemma comp-of-ords-of-le-lt[simp]: comp-of-ords (le-of-comp c) (lt-of-comp c) = c
by (intro ext, auto simp: comp-of-ords-def le-of-comp-def lt-of-comp-def split: order.split)

lemma lt-of-comp-of-ords: lt-of-comp (comp-of-ords le lt) = lt
by (intro ext, auto simp: comp-of-ords-def le-of-comp-def lt-of-comp-def split: order.split)

lemma le-of-comp-of-ords-gen: (∀ x y. lt x y ⟹ le x y) ⟹ le-of-comp (comp-of-ords le lt) = le
by (intro ext, auto simp: comp-of-ords-def le-of-comp-def lt-of-comp-def split: order.split)
lemma le-of-comp-of-ords-linorder: assumes class.linorder le lt
  shows le-of-comp (comp-of-ords le lt) = le
proof -
  interpret linorder le lt by fact
  show ?thesis by (rule le-of-comp-of-ords-gen) simp
qed

fun invert-order:: order ⇒ order where
  invert-order Lt = Gt |
  invert-order Gt = Lt |
  invert-order Eq = Eq

locale comparator =
  fixes comp :: 'a comparator
  assumes sym: invert-order (comp x y) = comp y x
    and weak-eq: comp x y = Eq ⇒ x = y
    and trans: comp x y = Lt ⇒ comp y z = Lt ⇒ comp x z = Lt
begin

lemma eq: (comp x y = Eq) = (x = y)
proof
  assume x = y
  with sym [of y y] show comp x y = Eq by (cases comp x y) auto
qed (rule weak-eq)

lemma comp-same [simp]:
  comp x x = Eq
by (simp add: eq)

abbreviation lt ≡ lt-of-comp comp
abbreviation le ≡ le-of-comp comp

lemma linorder: class.linorder le lt
proof
  note [simp] = lt-of-comp-def le-of-comp-def
  fix x y z :: 'a
  show lt x y = (le x y ∧ ¬ le y x)
    using sym [of x y] by (cases comp x y) (simp-all)
  show le x y ∨ le y x
    using sym [of x y] by (cases comp x y) (simp-all)
  show le x x using eq [of x x] by (simp)
  show le x y ⇒ le y x ⇒ x = y
    using sym [of x y] by (cases comp x y) (simp-all: eq)
  show le x y ⇒ le y z ⇒ le x z
    by (cases comp x y comp y z rule: order.exhaust [case-product order.exhaust!])
    (auto dest: trans simp: eq)
qed
sublocale linorder le lt 
by (rule linorder)

lemma Gt-lt-conv: comp x y = Gt ↔ lt y x
  unfolding lt-of-comp-def sym[of x y, symmetric]
  by (cases comp x y, auto)

lemma Lt-lt-conv: comp x y = Lt ↔ lt x y
  unfolding lt-of-comp-def by (cases comp x y, auto)

lemma eq-Eq-conv: comp x y = Eq ↔ x = y
  by (rule eq)

lemma nGt-le-conv: comp x y ≠ Gt ↔ le x y
  unfolding le-of-comp-def by (cases comp x y, auto)

lemma nLt-le-conv: comp x y ≠ Lt ↔ le y x
  unfolding le-of-comp-def symmetric[of x y, symmetric] by (cases comp x y, auto)

lemma nEq-neq-conv: comp x y ≠ Eq ↔ x ≠ y
  using eq-Eq-conv[of x y] by simp

lemmas le-lt-convs = nLt-le-conv nGt-le-conv Gt-lt-conv Lt-lt-conv eq-Eq-conv nEq-neq-conv

lemma two-comparisons-into-case-order:
  (if le x y then (if x = y then P else Q) else R) = (case-order P Q R (comp x y))
  (if le x y then (if y = x then P else Q) else R) = (case-order P Q R (comp x y))
  (if le x y then (if le y x then P else Q) else R) = (case-order P Q R (comp x y))
  (if le x y then (if lt y x then Q else P) else R) = (case-order P Q R (comp x y))
  (if le x y then Q else (if le x y then Q else R)) = (case-order P Q R (comp x y))
  (if le x y then Q else (if le y x then Q else P)) = (case-order P Q R (comp x y))
  (if le x y then Q else (if y = x then P else R)) = (case-order P Q R (comp x y))
  (if le x y then Q else (if le y x then R else P)) = (case-order P Q R (comp x y))
  (if lt x y then Q else (if le y x then Q else R)) = (case-order P Q R (comp x y))
  (if lt x y then Q else (if le x y then Q else R)) = (case-order P Q R (comp x y))
  by (auto simp: le-lt-convs split: order.splits)

end

lemma comp-of-ords: assumes class.linorder le lt
  shows comparator (comp-of-ords le lt)
proof -
  interpret linorder le lt by fact
  show ?thesis
    by (unfold-locales, auto simp: comp-of-ords-def split: if-splits)
qed

definition (in linorder) comparator-of :: 'a comparator where
  comparator-of x y = (if x < y then Lt else if x = y then Eq else Gt)

lemma comparator-of: comparator comparator-of
by unfold-locales (auto split: if-splits simp: comparator-of-def)

end

3.2 Compare

theory Compare
imports Comparator
keywords compare-code :: thy-decl
begin

This introduces a type class for having a proper comparator, similar to linorder. Since most of the Isabelle/HOL algorithms work on the latter, we also provide a method which turns linear-order based algorithms into comparator-based algorithms, where two consecutive invocations of linear orders and equality are merged into one comparator invocation. We further define a class which both define a linear order and a comparator, and where the induces orders coincide.

class compare = 
  fixes compare :: 'a comparator
  assumes comparator-compare: comparator compare
begin

lemma compare-Eq-is-eq [simp]:
  compare x y = Eq ←→ x = y
  by (rule comparator.eq [OF comparator-compare])

lemma compare-refl [simp]:
  compare x x = Eq
  by simp

end

lemma (in linorder) le-lt-comparator-of:
  le-of-comp comparator-of = (≤)
  lt-of-comp comparator-of = (<)
  by (intro ext, auto simp: comparator-of-def le-of-comp-def lt-of-comp-def)+

class compare-order = ord + compare +
  assumes ord-defs: le-of-comp compare = (≤) lt-of-comp compare = (<)

  compare-order is compare and linorder, where comparator and orders define the same ordering.

subclass (in compare-order) linorder
  by (unfold ord-defs[symmetric], rule comparator.linorder, rule comparator-compare)

context compare-order
begin

lemma compare-is-comparator-of:

end
\textbf{proof} (intro ext)
fix \(x\ y::'a\)
show \(\text{compare } x\ y = \text{comparator-of } x\ y\)
\hspace{1em}by (unfold comparator-of-def, unfold ord-defs[symmetric] lt-of-comp-def, cases compare \(x\ y\), auto)
\textbf{qed}

\textbf{lemmas} two-comparisons-into-compare =
comparator.two-comparisons-into-case-order[OF comparator-compare, unfolded ord-defs]

\textbf{thm} two-comparisons-into-compare
end

\textbf{ML-file} (compare-code.ML)

\textit{Compare-Code.change-compare-code const ty-vars} changes the code equations of some constant such that two consecutive comparisons via \((\leq), (\prec)\), or \((=)\) are turned into one invocation of \texttt{compare}. The difference to a standard \texttt{code-unfold} is that here we change the code-equations where an additional sort-constraint on \texttt{compare-order} can be added. Otherwise, there would be no \texttt{compare}-function.

\textbf{end}

\subsection{3.3 Example: Modifying the Code-Equations of Red-Black-Trees}

\textbf{theory} RBT-Compare-Order-Impl
\textbf{imports}
\texttt{Compare}
\texttt{HOL-Library.RBT-Impl}
\textbf{begin}

In the following, we modify all code-equations of the red-black-tree implementation that perform comparisons. As a positive result, they now all require one invocation of comparator, where before two comparisons have been performed. The disadvantage of this simple solution is the additional class constraint on \texttt{compare-order}.

\texttt{compare-code ('a) rbt-ins}
\texttt{compare-code ('a) rbt-lookup}
\texttt{compare-code ('a) rbt-del}
\texttt{compare-code ('a) rbt-map-entry}
\texttt{compare-code ('a) sunion-with}
\texttt{compare-code ('a) sinter-with}

\texttt{export-code rbt-ins rbt-lookup rbt-del rbt-map-entry rbt-union-with-key rbt-inter-with-key}
in Haskell
3.4 A Comparator-Interface to Red-Black-Trees

theory RBT-Comparator-Impl
imports
  HOL-Library.RBT-Impl
  Comparator
begin

  For all of the main algorithms of red-black trees, we provide alternatives which are completely based on comparators, and which are provable equivalent. At the time of writing, this interface is used in the Container AFP-entry.

  It does not rely on the modifications of code-equations as in the previous subsection.

context
  fixes c :: 'a comparator
begin

primrec rbt-comp-lookup :: ('a, 'b) rbt ⇒ 'a ⇒ 'b
where
  rbt-comp-lookup RBT-Impl.Empty k = None
| rbt-comp-lookup (Branch - l x y r) k = (case c k x of Lt ⇒ rbt-comp-lookup l k | Gt ⇒ rbt-comp-lookup r k | Eq ⇒ Some y)

fun rbt-comp-ins :: ('a ⇒ 'b ⇒ 'b ⇒ 'b) ⇒ 'a ⇒ 'b ⇒ ('a, 'b) rbt
where
  rbt-comp-ins f k v RBT-Impl.Empty = Branch RBT-Impl.R RBT-Impl.Empty k v
| rbt-comp-ins f k v (Branch RBT-Impl.B l x y r) = (case c k x of Lt ⇒ balance (rbt-comp-ins f k v l) x y r | Gt ⇒ balance l x y (rbt-comp-ins f k v r) | Eq ⇒ Branch RBT-Impl.B l x (f k y v) r)
| rbt-comp-ins f k v (Branch RBT-Impl.R l x y r) = (case c k x of Lt ⇒ Branch RBT-Impl.R (rbt-comp-ins f k v l) x y r | Gt ⇒ Branch RBT-Impl.R l x y (rbt-comp-ins f k v r) | Eq ⇒ Branch RBT-Impl.R l x (f k y v) r)

definition rbt-comp-insert-with-key :: ('a ⇒ 'b ⇒ 'b ⇒ 'b) ⇒ 'a ⇒ 'b ⇒ ('a, 'b) rbt
where rbt-comp-insert-with-key f k v t = paint RBT-Impl.B (rbt-comp-ins f k v t)

definition rbt-comp-insert :: 'a ⇒ 'b ⇒ ('a, 'b) rbt ⇒ ('a, 'b) rbt
where
  rbt-comp-insert = rbt-comp-insert-with-key (λ - - nv. nv)
fun rbt-comp-del-from-left :: 'a ⇒ ('a,'b) rbt ⇒ 'a ⇒ 'b ⇒ ('a,'b) rbt
and rbt-comp-del-from-right :: 'a ⇒ ('a,'b) rbt ⇒ 'a ⇒ 'b ⇒ ('a,'b) rbt
and rbt-comp-del :: 'a⇒ ('a,'b) rbt ⇒ ('a,'b) rbt

where
rbt-comp-del x RBT-Impl.Empty = RBT-Impl.Empty |
rbt-comp-del x (Branch - a y s b) =
  (case c x y of
   Lt ⇒ rbt-comp-del-from-left x a y s b |
   Gt ⇒ rbt-comp-del-from-right x a y s b |
   Eq ⇒ combine a b) |

(rbt-comp-del x (Branch RBT-Impl.B lt z v rt)) y s b = balance-left
(rbt-comp-del-from-left x (Branch RBT-Impl.B lt z v rt)) x a y s b |
(rbt-comp-del-from-left x a y s b = Branch RBT-Impl.R (rbt-comp-del x a) y s b |
(rbt-comp-del-from-right x a y s (Branch RBT-Impl.B lt z v rt)) = balance-right
y s (rbt-comp-del x (Branch RBT-Impl.B lt z v rt)) |
(rbt-comp-del-from-right x a y s b = Branch RBT-Impl.R a y s (rbt-comp-del x b))

definition rbt-comp-delete k t = paint RBT-Impl.B (rbt-comp-del k t)

definition rbt-comp-bulkload xs = foldr (λ(k, v). rbt-comp-insert k v) xs RBT-Impl.Empty

primrec rbt-comp-map-entry :: 'a ⇒ ('b ⇒ 'b) ⇒ ('b ⇒ 'b) ⇒ ('a,'b) rbt ⇒ ('a,'b) rbt

where
rbt-comp-map-entry k f RBT-Impl.Empty = RBT-Impl.Empty |
rbt-comp-map-entry k f (Branch cc lt x v rt) =
  (case c k x of
   Lt ⇒ Branch cc (rbt-comp-map-entry k f lt) x v rt |
   Gt ⇒ Branch cc lt x v (rbt-comp-map-entry k f rt) |
   Eq ⇒ Branch cc lt x (f v) rt)

function comp-sunion-with :: ('a ⇒ 'b ⇒ 'b ⇒ 'b) ⇒ ('a × 'b) list ⇒ ('a × 'b) list

where
comp-sunion-with f ((k, v) # as) ((k', v') # bs) =
  (case k k' of
   Lt ⇒ (k', v') # comp-sunion-with f ((k, v) # as) bs |
   Gt ⇒ (k, v) # comp-sunion-with f as ((k', v') # bs) |
   Eq ⇒ (k, f k v v') # comp-sunion-with f as bs)

| comp-sunion-with f [] bs = bs |
| comp-sunion-with f as [] = as
by pat-completeness auto
termination by lexicographic-order

function comp-sinter-with :: ('a ⇒ 'b ⇒ 'b ⇒ 'b) ⇒ ('a × 'b) list ⇒ ('a × 'b) list
list ⇒ ('a × 'b) list

where

\[ \text{comp-sinter-with } f \ ((k, v) \# as) ((k', v') \# bs) = \]

\(\begin{align*}
& (\text{case } c k' k \text{ of} \\
& \quad \text{Lt} \Rightarrow \text{comp-sinter-with } f \ ((k, v) \# as) bs \\
& \quad \text{Gt} \Rightarrow \text{comp-sinter-with } f \ ( (k', v') \# bs) \\
& \quad \text{Eq} \Rightarrow (k, f k v v') \# \text{comp-sinter-with } f \ as \ bs) \\
\end{align*} \]

\[ \text{comp-sinter-with } f \ [] - = [] \\
\text{comp-sinter-with } f - [] = [] \]

by pat-completeness auto

termination by lexicographic-order

definition rbt-comp-union-with-key :: ('a ⇒ 'b ⇒ 'b ⇒ 'b) ⇒ ('a, 'b) rbt ⇒ ('a, 'b) rbt

where

\[ \text{rbt-comp-union-with-key } f \ t1 t2 = \]

\(\begin{align*}
& (\text{case } \text{RBTL-Impl.compare-height } t1 t1 t2 t2 \\
& \text{of compare.EQ ⇒ rbtreeify (comp-sunion-with } f \ (\text{RBTL-Impl.entries } t1) (\text{RBTL-Impl.entries } t2)) \\
& \quad \text{compare.LT ⇒ RBTL-Impl.fold (rbt-comp-insert-with-key } (\lambda k v w. f k w v) \) t1 t2 \\
& \quad \text{compare.GT ⇒ RBTL-Impl.fold (rbt-comp-insert-with-key } f \) t2 t1) \\
\end{align*} \]

definition rbt-comp-inter-with-key :: ('a ⇒ 'b ⇒ 'b ⇒ 'b) ⇒ ('a, 'b) rbt ⇒ ('a, 'b) rbt

where

\[ \text{rbt-comp-inter-with-key } f \ t1 t2 = \]

\(\begin{align*}
& (\text{case } \text{RBTL-Impl.compare-height } t1 t1 t2 t2 \\
& \text{of compare.EQ ⇒ rbtreeify (comp-sinter-with } f \ (\text{RBTL-Impl.entries } t1) (\text{RBTL-Impl.entries } t2)) \\
& \quad \text{compare.LT ⇒ rbtreeify (List.map-filter (\lambda (k, v). map-option (\lambda w. (k, f k v w)) \) \) t1 \) (RBTL-Impl.entries } t2)) \\
& \quad \text{compare.GT ⇒ rbtreeify (List.map-filter (\lambda (k, v). map-option (\lambda w. (k, f k w v)) \) (rbt-comp-lookup } t1 k) \ (RBTL-Impl.entries t2)) \\
\end{align*} \]

class context

assumes \(c: \text{comparator } c\)

begin

lemma rbt-comp-lookup: rbt-comp-lookup = ord.rbt-lookup (lt-of-comp c)

proof (intro ext)

fix \(k\) and \(t:: ('a,'b)\)

show \(\text{rbt-comp-lookup } t k = \text{ord.rbt-lookup } (\text{lt-of-comp } c) \ t k\)

by (induct \(t\), unfold rbt-comp-lookup.simps ord.rbt-lookup.simps \(c: \text{comparator}\).two-comparisons-into-case-order[OF \(c\)])

(auto split: order.splits)

qed
lemma rbt-comp-ins: rbt-comp-ins = ord.rbt-ins (lt-of-comp c)
proof (intro ext)
  fix f k v and t :: ('a, 'b) rbt
  show rbt-comp-ins f k v t = ord.rbt-ins (lt-of-comp c) f k v t
  by (induct f k v t rule: rbt-comp-ins.induct, unfold rbt-comp-ins.simps ord.rbt-ins.simps
      comparator.two-comparisons-into-case-order[OF c])
      (auto split: order.splits)
qed

lemma rbt-comp-insert-with-key: rbt-comp-insert-with-key = ord.rbt-insert-with-key (lt-of-comp c)
unfolding rbt-comp-insert-with-key-def[abs-def] ord.rbt-insert-with-key-def[abs-def]
unfolding rbt-comp-ins ..

lemma rbt-comp-del: rbt-comp-del = ord.rbt-del (lt-of-comp c)
proof -
  fix k a b and s t :: ('a, 'b) rbt
  have
      rbt-comp-del-from-left k t a b s = ord.rbt-del-from-left (lt-of-comp c) k t a b s
      rbt-comp-del-from-right k t a b s = ord.rbt-del-from-right (lt-of-comp c) k t a b s
  by (induct k t a b s and k t a b s and k t rule: rbt-comp-del-from-left-rbt-comp-del-from-right-rbt-comp-del.induct
      unfold
      rbt-comp-del.simps ord.rbt-del.simps
      rbt-comp-del-from-left.simps ord.rbt-del-from-left.simps
      rbt-comp-del-from-right.simps ord.rbt-del-from-right.simps
      comparator.two-comparisons-into-case-order[OF c],
      auto split: order.split)
  thus ?thesis by (intro ext)
qed

lemma rbt-comp-delete: rbt-comp-delete = ord.rbt-delete (lt-of-comp c)
unfolding rbt-comp-delete-def[abs-def] ord.rbt-delete-def[abs-def]
unfolding rbt-comp-del ..

lemma rbt-comp-bulkload: rbt-comp-bulkload = ord.rbt-bulkload (lt-of-comp c)
unfolding rbt-comp-bulkload-def[abs-def] ord.rbt-bulkload-def[abs-def]
unfolding rbt-comp-insert ..

lemma rbt-comp-map-entry: rbt-comp-map-entry = ord.rbt-map-entry (lt-of-comp c)
proof (intro ext)
  fix f k and t :: ('a, 'b) rbt
show \( \text{rbt-comp-map-entry } f \ k \ t = \text{ord.rbt-map-entry}(\text{lt-of-comp} \ c) \ f \ k \ t \)
by (induct \( t \), unfold \text{rbt-comp-map-entry.simps} \text{ord.rbt-map-entry.simps} \text{comparator.two-comparisons-into-case-order}[\text{OF} \ c])
(auto split: \text{order.splits})
qed

lemma \( \text{comp-sunion-with: comp-sunion-with} = \text{ord.sunion-with}(\text{lt-of-comp} \ c) \)
proof (intro ext)
fix \( f \) and \( as \) :: \(('a \times 'b)\)list
show \( \text{comp-sunion-with} \ f \ as \ bs = \text{ord.sunion-with}(\text{lt-of-comp} \ c) \ f \ as \ bs \)
by (induct \( f \) as \( bs \) rule: \( \text{comp-sunion-with.induct} \),
unfold \text{comp-sunion-with.simps} \text{ord.sunion-with.simps}
\text{comparator.two-comparisons-into-case-order}[\text{OF} \ c])
(auto split: \text{order.splits})
qed

lemma \( \text{comp-sinter-with: comp-sinter-with} = \text{ord.sinter-with}(\text{lt-of-comp} \ c) \)
proof (intro ext)
fix \( f \) and \( as \) :: \(('a \times 'b)\)list
show \( \text{comp-sinter-with} \ f \ as \ bs = \text{ord.sinter-with}(\text{lt-of-comp} \ c) \ f \ as \ bs \)
by (induct \( f \) as \( bs \) rule: \( \text{comp-sinter-with.induct} \),
unfold \text{comp-sinter-with.simps} \text{ord.sinter-with.simps}
\text{comparator.two-comparisons-into-case-order}[\text{OF} \ c])
(auto split: \text{order.splits})
qed

lemma \( \text{rbt-comp-union-with-key: rbt-comp-union-with-key} = \text{ord.rbt-union-with-key}(\text{lt-of-comp} \ c) \)
unfolding \text{rbt-comp-union-with-key-def[abs-def]} \text{ord.rbt-union-with-key-def[abs-def]}
unfolding \text{rbt-comp-insert-with-key comp-sunion-with} ..

lemma \( \text{rbt-comp-inter-with-key: rbt-comp-inter-with-key} = \text{ord.rbt-inter-with-key}(\text{lt-of-comp} \ c) \)
unfolding \text{rbt-comp-inter-with-key-def[abs-def]} \text{ord.rbt-inter-with-key-def[abs-def]}
unfolding \text{rbt-comp-insert-with-key comp-sinter-with rbt-comp-lookup} ..

lemmas \( \text{rbt-comp-simps} = \)
\text{rbt-comp-insert} \text{rbt-comp-lookup} \text{rbt-comp-delete} \text{rbt-comp-bulkload} \text{rbt-comp-map-entry} \text{rbt-comp-union-with-key} \text{rbt-comp-inter-with-key}
end end end

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4 Generating Comparators

theory Comparator-Generator
imports
  ../Generator-Aux
  ../Derive-Manager
Comparator
begin

typedecl ('a,'b,'c,'z)type

In the following, we define a generator which for a given datatype ('a,
'b, 'c, 'z) Comparator-Generator.type constructs a comparator of type 'a
comparator ⇒ 'b comparator ⇒ 'c comparator ⇒ 'z comparator ⇒ ('a, 'b,
'c, 'z) Comparator-Generator.type. To this end, we first compare the index
of the constructors, then for equal constructors, we compare the arguments
recursively and combine the results lexicographically.

hide-type type

4.1 Lexicographic combination of order

fun comp-lex :: order list ⇒ order
where
  comp-lex (c # cs) = (case c of Eq ⇒ comp-lex cs | _ ⇒ c) |
  comp-lex [] = Eq

4.2 Improved code for non-lazy languages

The following equations will eliminate all occurrences of comp-lex in the
generated code of the comparators.

lemma comp-lex-unfolds:
  comp-lex [] = Eq
  comp-lex [c] = c
  comp-lex (c # d # cs) = (case c of Eq ⇒ comp-lex (d # cs) | z ⇒ z)
by (cases c, auto)+

4.3 Pointwise properties for equality, symmetry, and transi-
tivity

The pointwise properties are important during inductive proofs of sound-
ness of comparators. They are defined in a way that are combinable with
cmp-lex.

lemma comp-lex-eq: comp-lex os = Eq ⇐⇒ (∀ ord ∈ set os. ord = Eq)
by (induct os) (auto split: order.splits)

definition trans-order :: order ⇒ order ⇒ order ⇒ bool where
trans-order x y z ⇐⇒ x ≠ Gt ⇒ y ≠ Gt ⇒ z ≠ Gt ∧ ((x = Lt ∨ y = Lt)
→ z = Lt)
lemma \textit{trans-orderI}:
\[(x \neq Gt \implies y \neq Gt \implies z \neq Gt \land ((x = Lt \lor y = Lt) \implies z = Lt)) \implies\]
\textit{trans-order} \(x \ y \ z\)
\hspace{1em} by \((\text{simp add: trans-order-def})\)

lemma \textit{trans-orderD}:
\begin{itemize}
  \item \textbf{assumes} \textit{trans-order} \(x \ y \ z\) \textbf{and} \(x \neq Gt\) \textbf{and} \(y \neq Gt\)
  \item \textbf{shows} \(z \neq Gt\) \textbf{and} \(x = Lt \lor y = Lt \implies z = Lt\)
\end{itemize}
\hspace{1em} using \textit{assms} \hspace{1em} by \((\text{auto simp: trans-order-def})\)

lemma \textit{All-less-Suc}:
\[(\forall i < Suc \ x. \ P \ i) \longleftrightarrow P \ 0 \land (\forall i < x. \ P \ (Suc \ i))\]
\hspace{1em} using \textit{less-Suc-eq-0-disj} \hspace{1em} by \textit{force}

lemma \textit{comp-lex-trans}:
\begin{itemize}
  \item \textbf{assumes} \textit{length} \(xs = length \ ys\) \textbf{and} \textit{length} \(ys = length \ zs\)
  \item \(\forall \ i < length \ zs. \ \textit{trans-order} \ (xs ! i) \ (ys ! i) \ (zs ! i)\)
  \item \textbf{shows} \textit{trans-order} \((\text{comp-lex} \ xs) \ (\text{comp-lex} \ ys) \ (\text{comp-lex} \ zs)\)
\end{itemize}
\hspace{1em} using \textit{assms} \hspace{1em} proof \((\text{induct} \ xs \ ys \ zs \ \text{rule: list-induct3})\)
\hspace{1em} case \((\text{Cons} \ x \ xs \ y \ ys \ z \ zs)\)
\hspace{2em} then show ?case \hspace{1em} by \((\text{intro \ trans-orderI})\)
\hspace{4em} (cases \(x \ y \ z \ \text{rule: order.exhaust \ [case-product \ order.exhaust \ order.exhaust \ order.exhaust]})\)
\hspace{5em} auto simp: \textit{All-less-Suc dest: trans-orderD}\)
\hspace{1em} qed \((\text{simp add: trans-order-def})\)

lemma \textit{comp-lex-sym}:
\begin{itemize}
  \item \textbf{assumes} \textit{length} \(xs = length \ ys\)
  \item \(\forall \ i < length \ ys. \ \textit{invert-order} \ (xs ! i) = ys ! i\)
  \item \textbf{shows} \textit{invert-order} \((\text{comp-lex} \ xs) = \text{comp-lex} \ ys\)
\end{itemize}
\hspace{1em} using \textit{assms} \hspace{1em} by \((\text{induct} \ xs \ ys \ zs \ \text{rule: list-induct2, simp, case-tac \(x \)})\) \textit{fastforce+}

declare \textit{comp-lex.simps} \[\text{simp del}]\]

definition \textit{peq-comp} :: \('a \ comparator \Rightarrow 'a \Rightarrow bool\)
\begin{itemize}
  \item \textbf{where} \textit{peq-comp} \(a \text{comp} \ x \leftarrow \ (\forall \ y. \ a \text{comp} \ x \ y = Eq \leftarrow x = y)\)
\end{itemize}

lemma \textit{peq-compD}: \textit{peq-comp} \(a \text{comp} \ x \ \Longrightarrow \ a \text{comp} \ x \ y = Eq \leftarrow x = y\)
\hspace{1em} unfolding \textit{peq-comp-def} \hspace{1em} by \textit{auto}

lemma \textit{peq-compI}:
\[(\forall y. \ a \text{comp} \ x \ y = Eq \leftarrow x = y) \Longrightarrow \textit{peq-comp} \ a \text{comp} \ x\]
\hspace{1em} unfolding \textit{peq-comp-def} \hspace{1em} by \textit{auto}

definition \textit{psym-comp} :: \('a \ comparator \Rightarrow 'a \Rightarrow bool\)
\begin{itemize}
  \item \textbf{where} \textit{psym-comp} \(a \text{comp} \ x \leftarrow \ (\forall \ y. \ \text{invert-order} \ (a \text{comp} \ x \ y) = (a \text{comp} \ y \ x))\)
\end{itemize}
lemma \textit{psym-compD}:
assumes \textit{psym-comp} acomp \(x\)
shows invert-order \((\textit{acomp} \(x\) \(y\)) = (\textit{acomp} \(y\) \(x\))\)
using \textit{assms unfolding psym-comp-def by blast+}

lemma \textit{psym-compI}:
assumes \(\forall y.\) invert-order \((\textit{acomp} \(x\) \(y\)) = (\textit{acomp} \(y\) \(x\))\)
shows \textit{psym-comp} acomp \(x\)
using \textit{assms unfolding psym-comp-def by blast}

definition \textit{ptrans-comp} :: \(\forall a.\) comparator \(\Rightarrow\) bool where
\textit{ptrans-comp} acomp \(x\) \(\iff\) \((\forall y z.\) trans-order \((\textit{acomp} \(x\) \(y\)) (\textit{acomp} \(y\) \(z\)) (\textit{acomp} \(x\) \(z\)))\)

lemma \textit{ptrans-compD}:
assumes \textit{ptrans-comp} acomp \(x\)
shows trans-order \((\textit{acomp} \(x\) \(y\)) (\textit{acomp} \(y\) \(z\)) (\textit{acomp} \(x\) \(z\))\)
using \textit{assms unfolding ptrans-comp-def by blast+}

lemma \textit{ptrans-compI}:
assumes \(\forall y z.\) trans-order \((\textit{acomp} \(x\) \(y\)) (\textit{acomp} \(y\) \(z\)) (\textit{acomp} \(x\) \(z\))\)
shows \textit{ptrans-comp} acomp \(x\)
using \textit{assms unfolding ptrans-comp-def by blast}

4.4 Separate properties of comparators
definition \textit{eq-comp} :: \(\forall a.\) comparator \(\Rightarrow\) bool where
\textit{eq-comp} acomp \(\iff\) \((\forall x.\) peq-comp acomp \(x\))

lemma \textit{eq-compD2}: \textit{eq-comp} acomp \(\Rightarrow\) peq-comp acomp \(x\)
unfolding \textit{eq-comp-def} by blast

lemma \textit{eq-compI2}: \((\forall x.\) peq-comp acomp \(x\)) \(\Rightarrow\) eq-comp acomp
unfolding \textit{eq-comp-def} by blast

definition \textit{trans-comp} :: \(\forall a.\) comparator \(\Rightarrow\) bool where
\textit{trans-comp} acomp \(\iff\) \((\forall x.\) ptrans-comp acomp \(x\))

lemma \textit{trans-compD2}: \textit{trans-comp} acomp \(\Rightarrow\) ptrans-comp acomp \(x\)
unfolding \textit{trans-comp-def} by blast

lemma \textit{trans-compI2}: \((\forall x.\) ptrans-comp acomp \(x\)) \(\Rightarrow\) trans-comp acomp
unfolding \textit{trans-comp-def} by blast

definition \textit{sym-comp} :: \(\forall a.\) comparator \(\Rightarrow\) bool where
\textit{sym-comp} acomp \(\iff\) \((\forall x.\) psym-comp acomp \(x\))
lemma sym-compD2:
    sym-comp acomp ⇒ psym-comp acomp x
    unfolding sym-comp-def by blast

lemma sym-compI2: (∀ x. psym-comp acomp x) ⇒ sym-comp acomp
    unfolding sym-comp-def by blast

lemma eq-compD: eq-comp acomp ⇒ acomp x y = Eq ↔ x = y
    by (rule peq-compD[OF eq-compD2])

lemma eq-compI: (∀ x y. acomp x y = Eq ↔ x = y) ⇒ eq-comp acomp
    by (intro eq-compI2 peq-compI)

lemma trans-compD: trans-comp acomp ⇒ trans-order (acomp x y) (acomp y z) (acomp x z)
    by (rule ptrans-compD[OF trans-compD2])

lemma trans-compI: (∀ x y z. trans-order (acomp x y) (acomp y z) (acomp x z)) ⇒ trans-comp acomp
    by (intro trans-compI2 ptrans-compI)

lemma sym-compD:
    sym-comp acomp ⇒ invert-order (acomp x y) = (acomp y x)
    by (rule psym-compD[OF sym-compD2])

lemma sym-compI: (∀ x y. invert-order (acomp x y) = (acomp y x)) ⇒ sym-comp acomp
    by (intro sym-compI2 psym-compI)

lemma eq-sym-trans-imp-comparator:
    assumes eq-comp acomp and sym-comp acomp and trans-comp acomp
    shows comparator acomp
proof
    fix x y z
    show invert-order (acomp x y) = acomp y x
        using sym-compD[OF ⟨sym-comp acomp⟩] .
    
    { assume acomp x y = Eq
      with eq-compD[OF ⟨eq-comp acomp⟩]
      show x = y by blast
    }

    { assume acomp x y = Lt and acomp y z = Lt
      with trans-orderD[OF trans-compD[OF ⟨trans-comp acomp⟩], of x y z]
      show acomp x z = Lt by auto
    }
qed
lemma comparator-imp-eq-sym-trans:
  assumes comparator acomp
  shows eq-comp acomp sym-comp acomp trans-comp acomp
proof –
  interpret comparator acomp by fact
  show eq-comp acomp using eq by (intro eq-compI, auto)
  show sym-comp acomp using sym by (intro sym-compI, auto)
  show trans-comp acomp
proof (intro trans-compI trans-orderI)
  fix x y z
  assume acomp x y \neq Gt acomp y z \neq Gt
  thus acomp x z \neq Gt \land (acomp x y = Lt \lor acomp y z = Lt \rightarrow acomp x z = Lt)
  using trans \[of x y z\] and eq \[of x y\] and eq \[of y z\]
  by (cases acomp x y acomp y z rule: order.exhaust [case-product order.exhaust])
  auto
  qed
  qed

category
  fixes acomp :: 'a comparator
  assumes c: comparator acomp
begin
lemma comp-to-psym-comp: psym-comp acomp x
  using comparator-imp-eq-sym-trans[OF c]
  by (intro sym-compD2)

lemma comp-to-peq-comp: peq-comp acomp x
  using comparator-imp-eq-sym-trans [OF c]
  by (intro eq-compD2)

lemma comp-to-ptrans-comp: ptrans-comp acomp x
  using comparator-imp-eq-sym-trans [OF c]
  by (intro trans-compD2)
end

4.5 Auxiliary Lemmas for Comparator Generator

lemma forall-finite: (\forall i \in \{0 :: nat\}. P i) = True
  (\forall i < Suc 0. P i) = P 0
  (\forall i < Suc (Suc x). P i) = (P 0 \land (\forall i < Suc x. P (Suc i)))
  by (auto, case_tac i, auto)

lemma trans-order-different:
  trans-order a b Lt
  trans-order Gt b c
  trans-order a Gt c
  by (intro trans-orderI, auto)
lemma length-nth-simps:
length [] = 0 length (x # xs) = Suc (length xs)
(x # xs) ! 0 = x (x # xs) ! (Suc n) = xs ! n by auto

4.6 The Comparator Generator

ML-file ⟨comparator-generator.ML⟩ end

4.7 Compare Generator

theory Compare-Generator
imports Comparator-Generator Compare begin

We provide a generator which takes the comparators of the comparator generator to synthesize suitable compare-functions from the compare-class.

One can further also use these comparison functions to derive an instance of the compare-order-class, and therefore also for linorder. In total, we provide the three derive-methods where the example type prod can be replaced by any other datatype.

- *derive compare prod* creates an instance prod :: (compare, compare) compare.
- *derive compare-order prod* creates an instance prod :: (compare, compare) compare-order.
- *derive linorder prod* creates an instance prod :: (linorder, linorder) linorder.

Usually, the use of derive linorder is not recommended if there are comparators available: Internally, the linear orders will directly be converted into comparators, so a direct use of the comparators will result in more efficient generated code. This command is mainly provided as a convenience method where comparators are not yet present. For example, at the time of writing, the Container Framework has partly been adapted to internally use comparators, whereas in other AFP-entries, we did not integrate comparators.

lemma linorder-axiomsD: assumes class.linorder le lt shows
  lt x y = (le x y ∧ ¬ le y x) (is ?a)
  le x x (is ?b)
  le x y → le y z → le x z (is ?c1 → ?c2 → ?c3)
  le x y → le y x → x = y (is ?d1 → ?d2 → ?d3)
  le x y ∨ le y x (is ?e)
proof
  -
  interpret linorder le lt by fact
qed
	named-theorems compare-simps simp theorems to derive compare = comparator-of

ML-file (compare-generator.ML)
end

4.8 Defining Comparators and Compare-Instances for Common Types

theory Compare-Instances
imports
  Compare-Generator
  HOL-Library.Char-ord
begin
  For all of the following types, we define comparators and register them in the class compare: int, integer, nat, char, bool, unit, sum, option, list, and prod. We do not register those classes in compare-order where so far no linear order is defined, in particular if there are conflicting orders, like pair-wise or lexicographic comparison on pairs.

  For int, nat, integer and char we just use their linear orders as comparators.

derive (linorder) compare-order int integer nat char

  For sum, list, prod, and option we generate comparators which are however are not used to instantiate linorder.

derive compare sum list prod option

  We do not use the linear order to define the comparator for bool and unit, but implement more efficient ones.

fun comparator-unit :: unit comparator where
  comparator-unit x y = Eq

fun comparator-bool :: bool comparator where
  comparator-bool False False = Eq
  | comparator-bool False True = Lt
  | comparator-bool True True = Eq
  | comparator-bool True False = Gt

lemma comparator-unit: comparator comparator-unit
  by (unfold-locales, auto)

lemma comparator-bool: comparator comparator-bool
proof
  fix x y z :: bool
  show invert-order (comparator_bool x y) = comparator_bool y x by (cases x, (cases y, auto)+)
  show comparator_bool x y = Eq \implies x = y by (cases x, (cases y, auto)+)
  show comparator_bool x y = Lt \implies comparator_bool y z = Lt \implies comparator_bool x z = Lt
    by (cases x, (cases y, auto), cases y, (cases z, auto)+)
qed

local-setup ⟨
  Comparator-Generator.register-foreign-comparator @{typ unit}
    @{term comparator-unit}
    @{thm comparator-unit}
⟩

local-setup ⟨
  Comparator-Generator.register-foreign-comparator @{typ bool}
    @{term comparator_bool}
    @{thm comparator_bool}
⟩

derive compare_bool unit

It is not directly possible to derive (linorder) bool unit, since compare was not defined as comparator_of, but as comparator_bool. However, we can manually prove this equivalence and then use this knowledge to prove the instance of compare-order.

lemma comparator_bool-comparator-of [compare-simps]:
  comparator_bool = comparator_of
proof (intro ext)
  fix a b
  show comparator_bool a b = comparator_of a b
    unfolding comparator_of_def
    by (cases a, (cases b, auto))
qed

lemma comparator-unit-comparator-of [compare-simps]:
  comparator_unit = comparator_of
proof (intro ext)
  fix a b
  show comparator_unit a b = comparator_of a b
    unfolding comparator_of_def by auto
qed

derive (linorder) compare-order bool unit
end
4.9 Defining Compare-Order-Instances for Common Types

theory Compare-Order-Instances
imports
  Compare-Instances
  HOL-Library.List-Lexorder
  HOL-Library.Product-Lexorder
  HOL-Library.Option-ord
begin

We now also instantiate class \textit{compare-order} and not only \textit{compare}. Here, we also prove that our definitions do not clash with existing orders on \textit{list}, \textit{option}, and \textit{prod}.

For \textit{sum} we just define the linear orders via their comparator.

derive compare-order sum

instance list :: (compare-order)compare-order
proof

note [simp] = le-of-comp-def lt-of-comp-def comparator-of-def
show le-of-comp (compare :: 'a list comparator) = (\leq)
  unfolding compare-list-def compare-is-comparator-of
proof (intro ext)
  fix xs ys :: 'a list
  show le-of-comp (comparator-list comparator-of) xs ys = (xs \leq ys)
  proof (induct xs arbitrary: ys)
    case (Nil ys)
    thus ?case
    by (cases ys, simp-all)
  next
    case (Cons x xs yys)
    note IH = this
    thus ?thesis by auto
  next
    case (Cons y ys)
    show ?thesis unfolding Cons
      using IH[of ys]
      by (cases x y rule: linorder-cases, auto)
  qed
  qed

show lt-of-comp (compare :: 'a list comparator) = (<)
  unfolding compare-list-def compare-is-comparator-of
proof (intro ext)
  fix xs ys :: 'a list
  show lt-of-comp (comparator-list comparator-of) xs ys = (xs < ys)
  proof (induct xs arbitrary: ys)
    case (Nil ys)

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show \(?case \)
  by \((cases \, ys, \, simp-all)\)
next
  case \((Cons \, x \, xs \, yys)\) note \(IH = \) this
  thus \(?case \)
  proof \((cases \, yys)\)
    case Nil
    thus \(?thesis \) by auto
  next
    case \((Cons \, y \, ys)\)
    show \(?thesis \) unfolding \(Cons\)
      using \(IH[of ys]\)
      by \((cases \, x \, y \, rule: \, linorder-cases, \, auto)\)
  qed
  qed
  qed

instance \(prod :: \) (compare-order, compare-order) compare-order
  proof
    note \([simp] = le-of-comp-def lt-of-comp-def comparator-of-def\)
    show \(le-of-comp \) (compare \:: \('a, 'b) prod comparator) \((\le)\)
      unfolding \(compare-prod-def \) compare-is-comparator-of
    proof \((intro \, ext)\)
      fix \(xy1 \, xy2 :: ('a, 'b) prod\)
      show \(le-of-comp \) (comparator-prod comparator-of \) \(xy1 \, xy2 = (xy1 \, \le \, xy2)\)
        by \((cases \, xy1, \, cases \, xy2, \, auto)\)
    qed
    show \(lt-of-comp \) (compare \:: \('a, 'b) prod comparator) \((<)\)
      unfolding \(compare-prod-def \) compare-is-comparator-of
    proof \((intro \, ext)\)
      fix \(xy1 \, xy2 :: ('a, 'b) prod\)
      show \(lt-of-comp \) (comparator-prod comparator-of \) \(xy1 \, xy2 = (xy1 \, < \, xy2)\)
        by \((cases \, xy1, \, cases \, xy2, \, auto)\)
    qed
    qed

instance \(option :: \) (compare-order) compare-order
  proof
    note \([simp] = le-of-comp-def lt-of-comp-def comparator-of-def\)
    show \(le-of-comp \) (compare \:: \('a \, option \, comparator) \((\le)\)
      unfolding \(compare-option-def \) compare-is-comparator-of
    proof \((intro \, ext)\)
      fix \(xy1 \, xy2 :: 'a \, option\)
      show \(le-of-comp \) (comparator-option comparator-of \) \(xy1 \, xy2 = (xy1 \, \le \, xy2)\)
        by \((cases \, xy1, \, cases \, xy2, \, auto \, split: \, if-splits)+)\)
  qed
show lt-of-comp (compare :: 'a option comparator) = (<)
unfolding compare-option-def compare-is-comparator-of
proof (intro ext)
  fix xy1 xy2 :: 'a option
  show lt-of-comp (comparator-option comparator-of) xy1 xy2 = (xy1 < xy2)
    by (cases xy1, (cases xy2, auto split: if-splits)+)
qed
qed

4.10 Compare Instance for Rational Numbers

theory Compare-Rat
import Compare-Generator
  HOL.Rat
begin
derive (linorder) compare-order rat
end

4.11 Compare Instance for Real Numbers

theory Compare-Real
import Compare-Generator
  HOL.Real
begin
derive (linorder) compare-order real

lemma invert-order-compare-real[simp]: ∀ x y :: real. invert-order (compare x y) = compare y x
  by (simp add: comparator-of-def compare-is-comparator-of)
end

5 Checking Equality Without "="

theory Equality-Generator
import ..Generator-Aux ..Derive-Manager
begin
  typedef ('a,'b,'c,'z)type
In the following, we define a generator which for a given datatype \((a, b, c, z)\) \textit{Equality-Generator.type} constructs an equality-test function of type \((a \Rightarrow a \Rightarrow \text{bool}) \Rightarrow (b \Rightarrow b \Rightarrow \text{bool}) \Rightarrow (c \Rightarrow c \Rightarrow \text{bool}) \Rightarrow (z \Rightarrow z \Rightarrow \text{bool}) \Rightarrow (a, b, c, z)\) \textit{Equality-Generator.type} \Rightarrow \text{bool}. These functions are essential to synthesize conditional equality functions in the container framework, where a strict membership in the \textit{equal}-class must not be enforced.

\textbf{hide-type type}

Just a constant to define conjunction on lists of booleans, which will be used to merge the results when having compared the arguments of identical constructors.

\textbf{definition list-all-eq :: bool list \Rightarrow bool where}
\textit{list-all-eq} = \textit{list-all id}

\textbf{5.1 Improved Code for Non-Lazy Languages}

The following equations will eliminate all occurrences of \textit{list-all-eq} in the generated code of the equality functions.

\textbf{lemma list-all-eq-unfold:}
\textit{list-all-eq} \[\] = \textit{True}
\textit{list-all-eq} \[b\] = \textit{b}
\textit{list-all-eq} \((b1 \# b2 \# bs)\) = \((b1 \wedge \textit{list-all-eq} (b2 \# bs))\)
\textit{unfolding} \textit{list-all-eq-def} \textit{by auto}

\textbf{lemma list-all-eq: list-all-eq bs \[\leftrightarrow (\forall b \in \text{set bs}. b)\]}
\textit{unfolding} \textit{list-all-eq-def list-all-iff} \textit{by auto}

\textbf{5.2 Partial Equality Property}

We require a partial property which can be used in inductive proofs.

\textbf{type-synonym 'a equality = 'a \Rightarrow 'a \Rightarrow bool}

\textbf{definition pequality :: 'a equality \Rightarrow 'a \Rightarrow bool where}
\textit{pequality} \textit{aeq x} \[\leftrightarrow (\forall y. \textit{aeq x y} \leftrightarrow x = y)\]

\textbf{lemma pequalityD: pequality aeq x \Rightarrow aeq x y \leftrightarrow x = y}
\textit{unfolding} \textit{pequality-def} \textit{by auto}

\textbf{lemma pequalityI: (\forall y. \textit{aeq x y} \leftrightarrow x = y) \Rightarrow \textit{pequality} aeq x}
\textit{unfolding} \textit{pequality-def} \textit{by auto}

\textbf{5.3 Global equality property}

\textbf{definition equality :: 'a equality \Rightarrow bool where}

\[25\]
equality aeq \iff (\forall x. \text{pequality aeq } x)

\text{lemma equalityD2: equality aeq \implies pequality aeq x}
\text{unfolding equality-def by blast}

\text{lemma equalityI2: (\land x. \text{pequality aeq } x) \implies equality aeq}
\text{unfolding equality-def by blast}

\text{lemma equalityD: equality aeq \implies aeq x y \longleftrightarrow x = y}
\text{by (rule pequalityD[OF equalityD2])}

\text{lemma equalityI: (\land x y. aeq x y \longleftrightarrow x = y) \implies equality aeq}
\text{by (intro equalityI2 pequalityI)}

\text{lemma equality-imp-eq;}
\text{equality aeq \implies aeq = (=)}
\text{by (intro ext, auto dest: equalityD)}

\text{lemma eq-equality: equality (=)}
\text{by (rule equalityI, simp)}

\text{lemma equality-def': equality f = (f = (=))}
\text{using equality-imp-eq eq-equality by blast}

\textbf{5.4 The Generator}
\textbf{ML-file} (equality-generator.ML)
\textbf{hide-fact} (open) equalityI
\textbf{end}

\textbf{5.5 Defining Equality-Functions for Common Types}
\textbf{theory} Equality-Instances
\textbf{imports}
  Equality-Generator
\textbf{begin}
  For all of the following types, we register equality-functions. \textit{int, integer, nat, char, bool, unit, sum, option, list, and prod.} For types without type parameters, we use plain (=), and for the others we use generated ones. These functions will be essential, when the generator is later on invoked on types, which in their definition use one these types.
\textbf{derive} (eq) equality int integer nat char bool unit
\textbf{derive} equality sum list prod option
\textbf{end}
6 Generating Hash-Functions

theory Hash-Generator
imports
../Generator-Aux
../Derive-Manager
Collections.HashCode
begin

As usual, in the generator we use a dedicated function to combine the
results from evaluating the hash-function of the arguments of a constructor,
to deliver the global hash-value.

fun hash-combine :: hashcode list ⇒ hashcode list ⇒ hashcode where
  hash-combine [] [x] = x
  | hash-combine (y # ys) (z # zs) = y * z + hash-combine ys zs
  | hash-combine _ _ = 0

The first argument of hash-combine originates from evaluating the hash-
function on the arguments of a constructor, and the second argument of
hash-combine will be static magic numbers which are generated within the
generator.

6.1 Improved Code for Non-Lazy Languages

lemma hash-combine-unfold:
  hash-combine [] [x] = x
  hash-combine (y # ys) (z # zs) = y * z + hash-combine ys zs
by auto

6.2 The Generator

ML-file ⟨hash-generator.ML⟩

end

6.3 Defining Hash-Functions for Common Types

theory Hash-Instances
imports
    Hash-Generator
begin

For all of the following types, we register hashcode-functions. int, integer,
nat, char, bool, unit, sum, option, list, and prod. For types without type
parameters, we use plain hashcode, and for the others we use generated
ones.

derive (hashcode) hash-code int integer bool char unit nat

derive hash-code prod sum option list
There is no need to derive hashable prod sum option list since all of these types are already instances of class hashable. Still the above command is necessary to register these types in the generator.

7 Countable Datatypes

theory Countable-Generator
imports HOL-Library.Countable
../Derive-Manager
begin
  Brian Huffman and Alexander Krauss (old datatype), and Jasmin Blanchette (BNF datatype) have developed tactics which automatically can prove that a datatype is countable. We just make this tactic available in the derive-manager so that one can conveniently write derive countable some-datatype.

7.1 Installing the tactic

There is nothing more to do, then to write some boiler-plate ML-code for class-instantiation.

setup ⟨
  let
  fan derive dtyp-name - thy =
    let
      fun derive dtyp-name - thy =
        let
          val base-name = Long-Name.base-name dtyp-name
          val - = writeln (proving that datatype `base-name` is countable)
          val sort = @{sort countable}
          val vs =
            let val i = BNF-LFP-Compat.the-spec thy dtyp-name |> #1
              in map (fn (n,sort) => (n,sort)) i end
          val thy' = Class.instantiation ([|dtyp-name|],vs,sort) thy
            |> Class.prove-instantiation-exit (fn ctx => countable-tac ctx 1)
          val - = writeln (registered `base-name` in class countable)
        in thy' end
      in
      Derive-Manager.register-derive countable register datatypes is class countable
      derive
    end
  in
  Derive-Manager.register-derive countable register datatypes is class countable
  derive
  end
⟩

8 Loading Existing Derive-Commands

theory Derive
We just load the commands to derive comparators, equality-functions, hash-functions, and the command to show that a datatype is countable, so that now all of them are available. There are further generators available in the AFP entries Containers and Show.

print-derives

end

9 Examples

theory Derive-Examples
imports Derive Comparator-Generator/Compare-Order-Instances Equality-Generator/Equality-Instances HOL.Rat
begin

9.1 Rational Numbers

The rational numbers are not a datatype, so it will not be possible to derive corresponding instances of comparators, hashcodes, etc. via the generators. But we can and should still register the existing instances, so that later datatypes are supported which use rational numbers.

Use the linear order on rationals to define the compare-order-instance.

derive (linorder) compare-order rat

Use (=) as equality function.

derive (eq) equality rat

First manually define a hashcode function.

instantiation rat :: hashable
begin
definition def-hashmap-size = (λ- :: rat itself. 10)
definition hashcode (r :: rat) = hashcode (quotient-of r)
instance
  by (intro-classes)(simp-all add: def-hashmap-size-rat-def)
end

And then register it at the generator.

derive (hashcode) hash-code rat
9.2 A Datatype Without Nested Recursion

\texttt{datatype \texttt{'}a bintree = BEmpty | BNode \texttt{'}a bintree \texttt{'}a bintree}

\texttt{derive compare-order bintree}
\texttt{derive countable bintree}
\texttt{derive equality bintree}
\texttt{derive hashable bintree}

9.3 Using Other datatypes

\texttt{datatype nat-list-list = NNil | CCons nat list \times rat option nat-list-list}

\texttt{derive compare-order nat-list-list}
\texttt{derive countable nat-list-list}
\texttt{derive (eq) equality nat-list-list}
\texttt{derive hashable nat-list-list}

9.4 Mutual Recursion

\texttt{datatype \texttt{'}a mtree = MEmpty | MNode \texttt{'}a \texttt{'}a mtree-list and}
\texttt{\texttt{'}a mtree-list = MNil | MCons \texttt{'}a mtree \texttt{'}a mtree-list}

\texttt{derive compare-order mtree mtree-list}
\texttt{derive countable mtree mtree-list}
\texttt{derive hashable mtree mtree-list}

For \texttt{derive (equality|comparator|hash-code) mutual-recursive-type} there is the speciality that only one of the mutual recursive types has to be mentioned in order to register all of them. So one of \texttt{mtree} and \texttt{mtree-list} suffices.

\texttt{derive equality mtree}

9.5 Nested recursion

\texttt{datatype \texttt{'}a tree = Empty | Node \texttt{'}a \texttt{'}a tree list}
\texttt{datatype \texttt{'}a ttree = TEmpty | TNode \texttt{'}a \texttt{'}a ttree list tree}

\texttt{derive compare-order tree ttree}
\texttt{derive countable tree ttree}
\texttt{derive equality tree ttree}
\texttt{derive hashable tree ttree}

9.6 Examples from IsaFoR

\texttt{datatype (f,\texttt{'}v) term = Var \texttt{'}v | Fun f (f,\texttt{'}v) term list}
\texttt{datatype (f, \texttt{'}l) lab =}
\texttt{\texttt{Lab (f, \texttt{'}l) lab \texttt{'}l}}
\texttt{\texttt{| FunLab (f, \texttt{'}l) lab (f, \texttt{'}l) lab list}}
\texttt{\texttt{| UnLab f}}
9.7 A Complex Datatype

The following datatype has nested and mutual recursion, and uses other datatypes.

```plaintext
datatype ('a, 'b) complex =
  C1 nat 'a tree × 'rat + ('a, 'b) complex list |
  C2 ('a, 'b) complex list tree tree 'b ('a, 'b) complex ('a, 'b) complex2 tree list
and ('a, 'b) complex2 = D1 ('a, 'b) complex tree
```

On this last example type we illustrate the difference of the various comparator- and order-generators.

For complex we create an instance of `compare-order` which also defines a linear order. Note however that the instance will be `complex :: (compare, compare) compare-order`, i.e., the argument types have to be in class `compare`.

For complex2 we only derive `compare` which is not a subclass of `linorder`. The instance will be `complex2 :: (compare, compare) compare`, i.e., again the argument types have to be in class `compare`.

To avoid the dependence on `compare`, we can also instruct `derive` to be based on `linorder`. Here, the command `derive linorder complex2` will create the instance `complex2 :: (linorder, linorder) linorder`, i.e., here the argument types have to be in class `linorder`.

```plaintext
derive compare-order complex
derive compare complex2
derive linorder complex2
derive countable complex complex2
derive equality complex
derive hashable complex complex2
```

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References

