Deriving class instances for datatypes.*

Christian Sternagel and René Thiemann

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Abstract

We provide a framework for registering automatic methods to derive class instances of datatypes, as it is possible using Haskell's "deriving Ord, Show, ..." feature.

We further implemented such automatic methods to derive comparators, linear orders, parametrizable equality functions, and hash-functions which are required in the Isabelle Collection Framework [1] and the Container Framework [2]. Moreover, for the tactic of Blanchette to show that a datatype is countable, we implemented a wrapper so that this tactic becomes accessible in our framework. All of the generators are based on the infrastructure that is provided by the BNF-based datatype package.

Our formalization was performed as part of the IsaFoR/CeTA project[1] [3]. With our new tactics we could remove several tedious proofs for (conditional) linear orders, and conditional equality operators within IsaFoR and the Container Framework.

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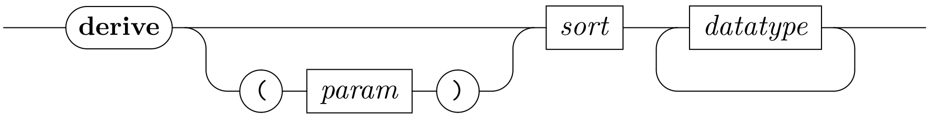
[1] http://cl-informatik.uibk.ac.at/software/ceta

1 Derive Manager

theory *Derive-Manager*
imports *Main*
keywords *print-derives* :: *diag* **and** *derive* :: *thy-decl*
begin

The derive manager allows the user to register various derive-hooks, e.g., for orders, pretty-printers, hash-functions, etc. All registered hooks are accessible via the derive command.



derive (*param*) *sort datatype* calls the hook for deriving *sort* (that may depend on the optional *param*) on *datatype* (if such a hook is registered).

E.g., **derive** *compare-order list* will derive a comparator for datatype *list* which is also used to define a linear order on *list*s.

There is also the diagnostic command **print-derives** that shows the list of currently registered hooks.

ML-file ‹*derive-manager.ML*›

end

2 Shared Utilities for all Generator

In this theory we mainly provide some Isabelle/ML infrastructure that is used by several generators. It consists of a uniform interface to access all the theorems, terms, etc. from the BNF package, and some auxiliary functions which provide recursors on datatypes, common tactics, etc.

theory *Generator-Aux*
imports
  Main
begin

ML-file ‹*bnf-access.ML*›
ML-file ‹*generator-aux.ML*›

lemma *in-set-simps*:
  $x \in set\ (y \# z \# ys) = (x = y \vee x \in set\ (z \# ys))$
  $x \in set\ ([y]) = (x = y)$
  $x \in set\ [] = False$
  $Ball\ (set\ [])\ P = True$

Ball (*set* [*x*]) *P* = *P x*
Ball (*set* (*x* # *y* # *zs*)) *P* = (*P x* ∧ *Ball* (*set* (*y* # *zs*)) *P*)
by *auto*

lemma *conj-weak-cong*: $a = b \Longrightarrow c = d \Longrightarrow (a \wedge c) = (b \wedge d)$ **by** *auto*

lemma *refl-True*: $(x = x) = $ *True* **by** *simp*

end

3 Comparisons

3.1 Comparators and Linear Orders

theory *Comparator*
imports *Main*
begin

Instead of having to define a strict and a weak linear order, $((<), (\leq))$, one can alternative use a comparator to define the linear order, which may deliver three possible outcomes when comparing two values.

datatype *order* = *Eq* | *Lt* | *Gt*

type-synonym *'a comparator* = *'a* ⇒ *'a* ⇒ *order*

In the following, we provide the obvious definitions how to switch between linear orders and comparators.

definition *lt-of-comp* :: *'a comparator* ⇒ *'a* ⇒ *'a* ⇒ *bool* **where**
lt-of-comp acomp x y = (*case acomp x y of Lt* ⇒ *True* | - ⇒ *False*)

definition *le-of-comp* :: *'a comparator* ⇒ *'a* ⇒ *'a* ⇒ *bool* **where**
le-of-comp acomp x y = (*case acomp x y of Gt* ⇒ *False* | - ⇒ *True*)

definition *comp-of-ords* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *'a comparator*
where
comp-of-ords le lt x y = (*if lt x y then Lt else if le x y then Eq else Gt*)

lemma *comp-of-ords-of-le-lt*[*simp*]: *comp-of-ords* (*le-of-comp c*) (*lt-of-comp c*) = *c*
by (*intro ext*, *auto simp*: *comp-of-ords-def le-of-comp-def lt-of-comp-def split*: *order.split*)

lemma *lt-of-comp-of-ords*: *lt-of-comp* (*comp-of-ords le lt*) = *lt*
by (*intro ext*, *auto simp*: *comp-of-ords-def le-of-comp-def lt-of-comp-def split*: *order.split*)

lemma *le-of-comp-of-ords-gen*: (⋀ *x y*. *lt x y* ⟹ *le x y*) ⟹ *le-of-comp* (*comp-of-ords le lt*) = *le*
by (*intro ext*, *auto simp*: *comp-of-ords-def le-of-comp-def lt-of-comp-def split*: *order.split*)

lemma *le-of-comp-of-ords-linorder*: **assumes** *class.linorder le lt*
  shows *le-of-comp* (*comp-of-ords le lt*) = *le*
proof −
  interpret *linorder le lt* **by** *fact*
  show *?thesis* **by** (*rule le-of-comp-of-ords-gen*) *simp*
qed

fun *invert-order*:: *order* ⇒ *order* **where**
  invert-order Lt = *Gt* |
  invert-order Gt = *Lt* |
  invert-order Eq = *Eq*

locale *comparator* =
  fixes *comp* :: *'a comparator*
  assumes *sym*: *invert-order* (*comp x y*) = *comp y x*
    and *weak-eq*: *comp x y* = *Eq* ⟹ *x* = *y*
    and *comp-trans*: *comp x y* = *Lt* ⟹ *comp y z* = *Lt* ⟹ *comp x z* = *Lt*
begin

lemma *eq*: (*comp x y* = *Eq*) = (*x* = *y*)
proof
  assume *x* = *y*
  with *sym* [*of y y*] **show** *comp x y* = *Eq* **by** (*cases comp x y*) *auto*
qed (*rule weak-eq*)

lemma *comp-same* [*simp*]:
  comp x x = *Eq*
  by (*simp add*: *eq*)

abbreviation *lt* ≡ *lt-of-comp comp*
abbreviation *le* ≡ *le-of-comp comp*

sublocale *ordering le lt*
proof
  note [*simp*] = *lt-of-comp-def le-of-comp-def*
  fix *x y z* :: *'a*
  show *le x x* **using** *eq* [*of x x*] **by** (*simp*)
  show *le x y* ⟹ *le y z* ⟹ *le x z*
    by (*cases comp x y comp y z rule*: *order.exhaust* [*case-product order.exhaust*])
      (*auto dest*: *comp-trans simp*: *eq*)
  show *le x y* ⟹ *le y x* ⟹ *x* = *y*
    using *sym* [*of x y*] **by** (*cases comp x y*) (*simp-all add*: *eq*)
  show *lt x y* ⟷ *le x y* ∧ *x* ≠ *y*
    using *eq* [*of x y*] **by** (*cases comp x y*) (*simp-all*)
qed

lemma *linorder*: *class.linorder le lt*
proof (*rule class.linorder.intro*)

interpret *order le lt*
using *ordering-axioms* **by** (*rule ordering-orderI*)
show ‹*class.order le lt*›
by (*fact order-axioms*)
show ‹*class.linorder-axioms le*›
proof
note [*simp*] = *lt-of-comp-def le-of-comp-def*
fix *x y* :: *'a*
show *le x y* ∨ *le y x*
using *sym* [*of x y*] **by** (*cases comp x y*) (*simp-all*)
qed
qed

sublocale *linorder le lt*
by (*rule linorder*)

lemma *Gt-lt-conv*: *comp x y* = *Gt* ⟷ *lt y x*
unfolding *lt-of-comp-def sym*[*of x y*, *symmetric*]
by (*cases comp x y*, *auto*)
lemma *Lt-lt-conv*: *comp x y* = *Lt* ⟷ *lt x y*
unfolding *lt-of-comp-def* **by** (*cases comp x y*, *auto*)
lemma *eq-Eq-conv*: *comp x y* = *Eq* ⟷ *x* = *y*
by (*rule eq*)
lemma *nGt-le-conv*: *comp x y* ≠ *Gt* ⟷ *le x y*
unfolding *le-of-comp-def* **by** (*cases comp x y*, *auto*)
lemma *nLt-le-conv*: *comp x y* ≠ *Lt* ⟷ *le y x*
unfolding *le-of-comp-def sym*[*of x y*, *symmetric*] **by** (*cases comp x y*, *auto*)
lemma *nEq-neq-conv*: *comp x y* ≠ *Eq* ⟷ *x* ≠ *y*
using *eq-Eq-conv*[*of x y*] **by** *simp*

lemmas *le-lt-convs* = *nLt-le-conv nGt-le-conv Gt-lt-conv Lt-lt-conv eq-Eq-conv nEq-neq-conv*

lemma *two-comparisons-into-case-order*:
(*if le x y then* (*if x* = *y then P else Q*) *else R*) = (*case-order P Q R* (*comp x y*))
(*if le x y then* (*if y* = *x then P else Q*) *else R*) = (*case-order P Q R* (*comp x y*))
(*if le x y then* (*if le y x then P else Q*) *else R*) = (*case-order P Q R* (*comp x y*))
(*if le x y then* (*if lt x y then Q else P*) *else R*) = (*case-order P Q R* (*comp x y*))
(*if lt x y then Q else* (*if le x y then P else R*)) = (*case-order P Q R* (*comp x y*))
(*if lt x y then Q else* (*if lt y x then R else P*)) = (*case-order P Q R* (*comp x y*))
(*if lt x y then Q else* (*if x* = *y then P else R*)) = (*case-order P Q R* (*comp x y*))
(*if lt x y then Q else* (*if y* = *x then P else R*)) = (*case-order P Q R* (*comp x y*))
(*if x* = *y then P else* (*if lt y x then R else Q*)) = (*case-order P Q R* (*comp x y*))
(*if x* = *y then P else* (*if lt x y then Q else R*)) = (*case-order P Q R* (*comp x y*))
(*if x* = *y then P else* (*if le y x then R else Q*)) = (*case-order P Q R* (*comp x y*))
(*if x* = *y then P else* (*if le x y then Q else R*)) = (*case-order P Q R* (*comp x y*))
by (*auto simp*: *le-lt-convs split*: *order.splits*)

end

lemma *comp-of-ords*: **assumes** *class.linorder le lt*
  shows *comparator* (*comp-of-ords le lt*)
proof −
  interpret *linorder le lt* **by** *fact*
  show *?thesis*
    by (*unfold-locales*, *auto simp*: *comp-of-ords-def split*: *if-splits*)
qed

definition (**in** *linorder*) *comparator-of* :: *'a comparator* **where**
  comparator-of x y = (*if x* < *y then Lt else if x* = *y then Eq else Gt*)

lemma *comparator-of*: *comparator comparator-of*
  by *unfold-locales* (*auto split*: *if-splits simp*: *comparator-of-def*)

end

3.2 Compare

theory *Compare*
imports *Comparator*
keywords *compare-code* :: *thy-decl*
begin

This introduces a type class for having a proper comparator, similar to *linorder*. Since most of the Isabelle/HOL algorithms work on the latter, we also provide a method which turns linear-order based algorithms into comparator-based algorithms, where two consecutive invocations of linear orders and equality are merged into one comparator invocation. We further define a class which both define a linear order and a comparator, and where the induces orders coincide.

class *compare* =
  fixes *compare* :: *'a comparator*
  assumes *comparator-compare*: *comparator compare*
begin

lemma *compare-Eq-is-eq* [*simp*]:
  compare x y = *Eq* ⟷ *x* = *y*
  by (*rule comparator.eq* [*OF comparator-compare*])

lemma *compare-refl* [*simp*]:
  compare x x = *Eq*
  by *simp*

end

lemma (**in** *linorder*) *le-lt-comparator-of*:
  le-of-comp comparator-of = (≤) *lt-of-comp comparator-of* = (<)
  by (*intro ext*, *auto simp*: *comparator-of-def le-of-comp-def lt-of-comp-def*)+

class *compare-order* = *ord* + *compare* +
assumes *ord-defs*: *le-of-comp compare* = (≤) *lt-of-comp compare* = (<)

compare-order is *compare* and *linorder*, where comparator and orders define the same ordering.

subclass (**in** *compare-order*) *linorder*
by (*unfold ord-defs*[*symmetric*], *rule comparator.linorder*, *rule comparator-compare*)

context *compare-order*
begin

lemma *compare-is-comparator-of*:
compare = *comparator-of*
proof (*intro ext*)
fix *x y* :: *'a*
show *compare x y* = *comparator-of x y*
by (*unfold comparator-of-def*, *unfold ord-defs*[*symmetric*] *lt-of-comp-def*,
cases compare x y, *auto*)
qed

lemmas *two-comparisons-into-compare* =
comparator.two-comparisons-into-case-order[*OF comparator-compare*, *unfolded ord-defs*]

thm *two-comparisons-into-compare*
end

ML-file ‹*compare-code.ML*›

Compare-Code.change-compare-code const ty−vars changes the code equations of some constant such that two consecutive comparisons via (≤), (<)", or (=) are turned into one invocation of *compare*. The difference to a standard *code-unfold* is that here we change the code-equations where an additional sort-constraint on *compare-order* can be added. Otherwise, there would be no *compare*-function.

end

3.3 Example: Modifying the Code-Equations of Red-Black-Trees

theory *RBT-Compare-Order-Impl*
imports
Compare
HOL−Library.RBT-Impl
begin

In the following, we modify all code-equations of the red-black-tree implementation that perform comparisons. As a positive result, they now all require one invocation of comparator, where before two comparisons have been performed. The disadvantage of this simple solution is the additional class constraint on *compare-order*.

compare-code ('*a*) *rbt-ins*
compare-code ('*a*) *rbt-lookup*
compare-code ('*a*) *rbt-del*
compare-code ('*a*) *rbt-map-entry*
compare-code ('*a*) *sunion-with*
compare-code ('*a*) *sinter-with*
compare-code ('*a*) *rbt-split*

export-code *rbt-ins rbt-lookup rbt-del rbt-map-entry rbt-union-with-key rbt-inter-with-key rbt-minus* **in** *Haskell*

end

3.4 A Comparator-Interface to Red-Black-Trees

theory *RBT-Comparator-Impl*
imports
HOL−Library.RBT-Impl
Comparator
begin

For all of the main algorithms of red-black trees, we provide alternatives which are completely based on comparators, and which are provable equivalent. At the time of writing, this interface is used in the Container AFP-entry.

It does not rely on the modifications of code-equations as in the previous subsection.

context
fixes *c* :: '*a comparator*
begin

primrec *rbt-comp-lookup* :: ('*a*, '*b*) *rbt* ⇒ '*a* ⇀ '*b*
where
rbt-comp-lookup RBT-Impl.Empty k = *None*
| *rbt-comp-lookup* (*Branch - l x y r*) *k* =
(*case c k x of Lt* ⇒ *rbt-comp-lookup l k*
| *Gt* ⇒ *rbt-comp-lookup r k*
| *Eq* ⇒ *Some y*)

fun
rbt-comp-ins :: ('*a* ⇒ '*b* ⇒ '*b* ⇒ '*b*) ⇒ '*a* ⇒ '*b* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*
where
rbt-comp-ins f k v RBT-Impl.Empty = *Branch RBT-Impl.R RBT-Impl.Empty k v RBT-Impl.Empty* |
rbt-comp-ins f k v (*Branch RBT-Impl.B l x y r*) = (*case c k x of*
Lt ⇒ *balance* (*rbt-comp-ins f k v l*) *x y r*

| *Gt* ⇒ *balance l x y* (*rbt-comp-ins f k v r*)
| *Eq* ⇒ *Branch RBT-Impl.B l x* (*f k y v*) *r*) |
rbt-comp-ins f k v (*Branch RBT-Impl.R l x y r*) = (*case c k x of*
Lt ⇒ *Branch RBT-Impl.R* (*rbt-comp-ins f k v l*) *x y r*
| *Gt* ⇒ *Branch RBT-Impl.R l x y* (*rbt-comp-ins f k v r*)
| *Eq* ⇒ *Branch RBT-Impl.R l x* (*f k y v*) *r*)

definition *rbt-comp-insert-with-key* :: (*'a* ⇒ *'b* ⇒ *'b* ⇒ *'b*) ⇒ *'a* ⇒ *'b* ⇒ (*'a*,*'b*) *rbt* ⇒ (*'a*,*'b*) *rbt*
where *rbt-comp-insert-with-key f k v t* = *paint RBT-Impl.B* (*rbt-comp-ins f k v t*)

definition *rbt-comp-insert* :: *'a* ⇒ *'b* ⇒ (*'a*, *'b*) *rbt* ⇒ (*'a*, *'b*) *rbt* **where**
rbt-comp-insert = *rbt-comp-insert-with-key* (λ- - *nv*. *nv*)

fun
rbt-comp-del-from-left :: *'a* ⇒ (*'a*,*'b*) *rbt* ⇒ *'a* ⇒ *'b* ⇒ (*'a*,*'b*) *rbt* ⇒ (*'a*,*'b*) *rbt*
and
rbt-comp-del-from-right :: *'a* ⇒ (*'a*,*'b*) *rbt* ⇒ *'a* ⇒ *'b* ⇒ (*'a*,*'b*) *rbt* ⇒ (*'a*,*'b*) *rbt*
and
rbt-comp-del :: *'a*⇒ (*'a*,*'b*) *rbt* ⇒ (*'a*,*'b*) *rbt*
where
rbt-comp-del x RBT-Impl.Empty = *RBT-Impl.Empty* |
rbt-comp-del x (*Branch - a y s b*) =
(*case c x y of*
Lt ⇒ *rbt-comp-del-from-left x a y s b*
| *Gt* ⇒ *rbt-comp-del-from-right x a y s b*
| *Eq* ⇒ *combine a b*) |
rbt-comp-del-from-left x (*Branch RBT-Impl.B lt z v rt*) *y s b* = *balance-left* (*rbt-comp-del x* (*Branch RBT-Impl.B lt z v rt*)) *y s b* |
rbt-comp-del-from-left x a y s b = *Branch RBT-Impl.R* (*rbt-comp-del x a*) *y s b* |
rbt-comp-del-from-right x a y s (*Branch RBT-Impl.B lt z v rt*) = *balance-right a y s* (*rbt-comp-del x* (*Branch RBT-Impl.B lt z v rt*)) |
rbt-comp-del-from-right x a y s b = *Branch RBT-Impl.R a y s* (*rbt-comp-del x b*)

definition *rbt-comp-delete k t* = *paint RBT-Impl.B* (*rbt-comp-del k t*)

definition *rbt-comp-bulkload xs* = *foldr* (λ(*k*, *v*). *rbt-comp-insert k v*) *xs RBT-Impl.Empty*

primrec
rbt-comp-map-entry :: *'a* ⇒ (*'b* ⇒ *'b*) ⇒ (*'a*, *'b*) *rbt* ⇒ (*'a*, *'b*) *rbt*
where
rbt-comp-map-entry k f RBT-Impl.Empty = *RBT-Impl.Empty*
| *rbt-comp-map-entry k f* (*Branch cc lt x v rt*) =
(*case c k x of*
Lt ⇒ *Branch cc* (*rbt-comp-map-entry k f lt*) *x v rt*
| *Gt* ⇒ *Branch cc lt x v* (*rbt-comp-map-entry k f rt*)
| *Eq* ⇒ *Branch cc lt x* (*f v*) *rt*)

function *comp-sunion-with* :: (*'a* ⇒ *'b* ⇒ *'b* ⇒ *'b*) ⇒ (*'a* × *'b*) *list* ⇒ (*'a* × *'b*)

list ⇒ (*'a* × *'b*) *list*
where
  comp-sunion-with f ((*k*, *v*) # *as*) ((*k'*, *v'*) # *bs*) =
   (*case c k' k of*
     Lt ⇒ (*k'*, *v'*) # *comp-sunion-with f* ((*k*, *v*) # *as*) *bs*
    | *Gt* ⇒ (*k*, *v*) # *comp-sunion-with f as* ((*k'*, *v'*) # *bs*)
    | *Eq* ⇒ (*k*, *f k v v'*) # *comp-sunion-with f as bs*)
| *comp-sunion-with f* [] *bs* = *bs*
| *comp-sunion-with f as* [] = *as*
by *pat-completeness auto*
termination by *lexicographic-order*

function *comp-sinter-with* :: (*'a* ⇒ *'b* ⇒ *'b* ⇒ *'b*) ⇒ (*'a* × *'b*) *list* ⇒ (*'a* × *'b*) *list* ⇒ (*'a* × *'b*) *list*
where
  comp-sinter-with f ((*k*, *v*) # *as*) ((*k'*, *v'*) # *bs*) =
  (*case c k' k of*
    Lt ⇒ *comp-sinter-with f* ((*k*, *v*) # *as*) *bs*
   | *Gt* ⇒ *comp-sinter-with f as* ((*k'*, *v'*) # *bs*)
   | *Eq* ⇒ (*k*, *f k v v'*) # *comp-sinter-with f as bs*)
| *comp-sinter-with f* [] - = []
| *comp-sinter-with f* - [] = []
by *pat-completeness auto*
termination by *lexicographic-order*

fun *rbt-split-comp* :: (*'a*, *'b*) *rbt* ⇒ *'a* ⇒ (*'a*, *'b*) *rbt* × *'b option* × (*'a*, *'b*) *rbt*
where
  rbt-split-comp RBT-Impl.Empty k = (*RBT-Impl.Empty*, *None*, *RBT-Impl.Empty*)
| *rbt-split-comp* (*RBT-Impl.Branch - l a b r*) *x* = (*case c x a of*
   Lt ⇒ (*case rbt-split-comp l x of* (*l1*, *β*, *l2*) ⇒ (*l1*, *β*, *rbt-join l2 a b r*))
  | *Gt* ⇒ (*case rbt-split-comp r x of* (*r1*, *β*, *r2*) ⇒ (*rbt-join l a b r1*, *β*, *r2*))
  | *Eq* ⇒ (*l*, *Some b*, *r*))

lemma *rbt-split-comp-size*: (*l2*, *b*, *r2*) = *rbt-split-comp t2 a* ⟹ *size l2* + *size r2* ≤ *size t2*
  by (*induction t2 a arbitrary*: *l2 b r2 rule*: *rbt-split-comp.induct*)
    (*auto split*: *order.splits if-splits prod.splits*)

function *rbt-comp-union-rec* :: (*'a* ⇒ *'b* ⇒ *'b* ⇒ *'b*) ⇒ (*'a*, *'b*) *rbt* ⇒ (*'a*, *'b*) *rbt* ⇒ (*'a*, *'b*) *rbt* **where**
  rbt-comp-union-rec f t1 t2 = (*let* (*f*, *t2*, *t1*) =
   if flip-rbt t2 t1 then (*λk v v'. f k v' v*, *t1*, *t2*) *else* (*f*, *t2*, *t1*) *in*
   if small-rbt t2 then RBT-Impl.fold (*rbt-comp-insert-with-key f*) *t2 t1*
   else (*case t1 of RBT-Impl.Empty* ⇒ *t2*
    | *RBT-Impl.Branch - l1 a b r1* ⇒
     case rbt-split-comp t2 a of (*l2*, *β*, *r2*) ⇒
      rbt-join (*rbt-comp-union-rec f l1 l2*) *a* (*case β of None* ⇒ *b* | *Some b'* ⇒ *f a b b'*) (*rbt-comp-union-rec f r1 r2*)))
  by *pat-completeness auto*

termination
  using *rbt-split-comp-size*
  by (*relation measure* (λ(*f*,*t1*,*t2*). *size t1* + *size t2*)) (*fastforce split*: *if-splits*)+

declare *rbt-comp-union-rec.simps*[*simp del*]

function *rbt-comp-union-swap-rec* :: (′*a* ⇒ ′*b* ⇒ ′*b* ⇒ ′*b*) ⇒ *bool* ⇒ (′*a*, ′*b*) *rbt* ⇒ (′*a*, ′*b*) *rbt* ⇒ (′*a*, ′*b*) *rbt* **where**
  rbt-comp-union-swap-rec f γ t1 t2 = (*let* (γ, *t2*, *t1*) =
    if flip-rbt t2 t1 then (¬γ, *t1*, *t2*) *else* (γ, *t2*, *t1*);
    f′ = (*if* γ *then* (λ*k v v*′. *f k v*′ *v*) *else f*) *in*
    if small-rbt t2 then RBT-Impl.fold (*rbt-comp-insert-with-key f*′) *t2 t1*
    else case t1 of rbt.Empty ⇒ *t2*
      | *Branch x l1 a b r1* ⇒
        case rbt-split-comp t2 a of (*l2*, β, *r2*) ⇒
          rbt-join (*rbt-comp-union-swap-rec f γ l1 l2*) *a* (*case* β *of None* ⇒ *b* | *Some x* ⇒ *f*′ *a b x*) (*rbt-comp-union-swap-rec f γ r1 r2*))
  by *pat-completeness auto*
termination
  using *rbt-split-comp-size*
  by (*relation measure* (λ(*f*,γ,*t1*, *t2*). *size t1* + *size t2*)) (*fastforce split*: *if-splits*)+

declare *rbt-comp-union-swap-rec.simps*[*simp del*]

lemma *rbt-comp-union-swap-rec*: *rbt-comp-union-swap-rec f γ t1 t2* =
  rbt-comp-union-rec (*if* γ *then* (λ*k v v*′. *f k v*′ *v*) *else f*) *t1 t2*
proof (*induction f γ t1 t2 rule*: *rbt-comp-union-swap-rec.induct*)
  case (*1 f γ t1 t2*)
  show *?case*
    using *1*[*OF refl - refl refl - refl - refl*]
   unfolding *rbt-comp-union-swap-rec.simps*[*of - - t1*] *rbt-comp-union-rec.simps*[*of - t1*]
    by (*auto simp*: *Let-def split*: *rbt.splits prod.splits option.splits*)
qed

lemma *rbt-comp-union-swap-rec-code*[*code*]: *rbt-comp-union-swap-rec f γ t1 t2* = (
    let bh1 = *bheight t1*; *bh2* = *bheight t2*; (γ, *t2*, *bh2*, *t1*, *bh1*) =
    if bh1 < *bh2 then* (¬γ, *t1*, *bh1*, *t2*, *bh2*) *else* (γ, *t2*, *bh2*, *t1*, *bh1*);
    f′ = (*if* γ *then* (λ*k v v*′. *f k v*′ *v*) *else f*) *in*
    if bh2 < *4 then RBT-Impl.fold* (*rbt-comp-insert-with-key f*′) *t2 t1*
    else case t1 of rbt.Empty ⇒ *t2*
      | *Branch x l1 a b r1* ⇒
        case rbt-split-comp t2 a of (*l2*, β, *r2*) ⇒
          rbt-join (*rbt-comp-union-swap-rec f γ l1 l2*) *a* (*case* β *of None* ⇒ *b* | *Some x* ⇒ *f*′ *a b x*) (*rbt-comp-union-swap-rec f γ r1 r2*))
  by (*auto simp*: *rbt-comp-union-swap-rec.simps flip-rbt-def small-rbt-def*)

definition *rbt-comp-union-with-key f t1 t2* = *paint RBT-Impl.B* (*rbt-comp-union-swap-rec* *f False t1 t2*)

definition *map-filter-comp-inter f t1 t2* = *List.map-filter* (λ(*k*, *v*).
case rbt-comp-lookup t1 k of None ⇒ *None*
| *Some v′* ⇒ *Some* (*k*, *f k v′ v*)) (*RBT-Impl.entries t2*)

function *rbt-comp-inter-rec* :: (′*a* ⇒ ′*b* ⇒ ′*b* ⇒ ′*b*) ⇒ (′*a*, ′*b*) *rbt* ⇒ (′*a*, ′*b*) *rbt* ⇒ (′*a*, ′*b*) *rbt* **where**
rbt-comp-inter-rec f t1 t2 = (*let* (*f*, *t2*, *t1*) =
if flip-rbt t2 t1 then (λ*k v v′*. *f k v′ v*, *t1*, *t2*) *else* (*f*, *t2*, *t1*) *in*
if small-rbt t2 then rbtreeify (*map-filter-comp-inter f t1 t2*)
else case t1 of RBT-Impl.Empty ⇒ *RBT-Impl.Empty*
| *RBT-Impl.Branch - l1 a b r1* ⇒
case rbt-split-comp t2 a of (*l2*, β, *r2*) ⇒ *let l′* = *rbt-comp-inter-rec f l1 l2*; *r′* = *rbt-comp-inter-rec f r1 r2 in*
(*case* β *of None* ⇒ *rbt-join2 l′ r′* | *Some b′* ⇒ *rbt-join l′ a* (*f a b b′*) *r′*))
by *pat-completeness auto*
termination
using *rbt-split-comp-size*
by (*relation measure* (λ(*f*,*t1*,*t2*). *size t1* + *size t2*)) (*fastforce split*: *if-splits*)+

declare *rbt-comp-inter-rec.simps*[*simp del*]

function *rbt-comp-inter-swap-rec* :: (′*a* ⇒ ′*b* ⇒ ′*b* ⇒ ′*b*) ⇒ *bool* ⇒ (′*a*, ′*b*) *rbt* ⇒ (′*a*, ′*b*) *rbt* ⇒ (′*a*, ′*b*) *rbt* **where**
rbt-comp-inter-swap-rec f γ *t1 t2* = (*let* (γ, *t2*, *t1*) =
if flip-rbt t2 t1 then (¬γ, *t1*, *t2*) *else* (γ, *t2*, *t1*);
f′ = *if* γ *then* (λ*k v v′*. *f k v′ v*) *else f in*
if small-rbt t2 then rbtreeify (*map-filter-comp-inter f′ t1 t2*)
else case t1 of rbt.Empty ⇒ *rbt.Empty*
| *Branch x l1 a b r1* ⇒
(*case rbt-split-comp t2 a of* (*l2*, β, *r2*) ⇒ *let l′* = *rbt-comp-inter-swap-rec f* γ *l1 l2*; *r′* = *rbt-comp-inter-swap-rec f* γ *r1 r2 in*
(*case* β *of None* ⇒ *rbt-join2 l′ r′* | *Some b′* ⇒ *rbt-join l′ a* (*f′ a b b′*) *r′*)))
by *pat-completeness auto*
termination
using *rbt-split-comp-size*
by (*relation measure* (λ(*f*,γ,*t1*,*t2*). *size t1* + *size t2*)) (*fastforce split*: *if-splits*)+

declare *rbt-comp-inter-swap-rec.simps*[*simp del*]

lemma *rbt-comp-inter-swap-rec*: *rbt-comp-inter-swap-rec f* γ *t1 t2* =
rbt-comp-inter-rec (*if* γ *then* (λ*k v v′*. *f k v′ v*) *else f*) *t1 t2*
proof (*induction f* γ *t1 t2 rule*: *rbt-comp-inter-swap-rec.induct*)
case (*1 f* γ *t1 t2*)
show *?case*
using *1*[*OF refl - refl refl - refl - refl*]
unfolding *rbt-comp-inter-swap-rec.simps*[*of - - t1*] *rbt-comp-inter-rec.simps*[*of - t1*]

by (*auto simp*: *Let-def split*: *rbt.splits prod.splits option.splits*)
qed

lemma *comp-inter-with-key-code*[*code*]: *rbt-comp-inter-swap-rec f* γ *t1 t2* = (
let bh1 = *bheight t1*; *bh2* = *bheight t2*; (γ, *t2*, *bh2*, *t1*, *bh1*) =
if bh1 < *bh2 then* ($\neg\gamma$, *t1*, *bh1*, *t2*, *bh2*) *else* (γ, *t2*, *bh2*, *t1*, *bh1*);
f' = (*if* γ *then* (λ*k v v'*. *f k v' v*) *else f*) *in*
if bh2 < *4 then rbtreeify* (*map-filter-comp-inter f' t1 t2*)
else case t1 of rbt.Empty $\Rightarrow$ *rbt.Empty*
| *Branch x l1 a b r1* $\Rightarrow$
(*case rbt-split-comp t2 a of* (*l2*, β, *r2*) $\Rightarrow$ *let l'* = *rbt-comp-inter-swap-rec f* γ *l1 l2*; *r'* = *rbt-comp-inter-swap-rec f* γ *r1 r2 in*
(*case* β *of None* $\Rightarrow$ *rbt-join2 l' r'* | *Some b'* $\Rightarrow$ *rbt-join l' a* (*f' a b b'*) *r'*)))
by (*auto simp*: *rbt-comp-inter-swap-rec.simps flip-rbt-def small-rbt-def*)

definition *rbt-comp-inter-with-key f t1 t2* = *paint RBT-Impl.B* (*rbt-comp-inter-swap-rec f False t1 t2*)

definition *filter-comp-minus t1 t2* =
filter (λ(*k*, -). *rbt-comp-lookup t2 k* = *None*) (*RBT-Impl.entries t1*)

fun *comp-minus* :: (*'a*, *'b*) *rbt* $\Rightarrow$ (*'a*, *'b*) *rbt* $\Rightarrow$ (*'a*, *'b*) *rbt* **where**
comp-minus t1 t2 = (*if small-rbt t2 then RBT-Impl.fold* (λ*k - t*. *rbt-comp-delete k t*) *t2 t1*
else if small-rbt t1 then rbtreeify (*filter-comp-minus t1 t2*)
else case t2 of RBT-Impl.Empty $\Rightarrow$ *t1*
| *RBT-Impl.Branch - l2 a b r2* $\Rightarrow$
case rbt-split-comp t1 a of (*l1*, -, *r1*) $\Rightarrow$ *rbt-join2* (*comp-minus l1 l2*) (*comp-minus r1 r2*))

declare *comp-minus.simps*[*simp del*]

definition *rbt-comp-minus t1 t2* = *paint RBT-Impl.B* (*comp-minus t1 t2*)

context
assumes *c*: *comparator c*
begin

lemma *rbt-comp-lookup*: *rbt-comp-lookup* = *ord.rbt-lookup* (*lt-of-comp c*)
proof (*intro ext*)
fix *k* **and** *t* :: (*'a*,*'b*)*rbt*
show *rbt-comp-lookup t k* = *ord.rbt-lookup* (*lt-of-comp c*) *t k*
by (*induct t*, *unfold rbt-comp-lookup.simps ord.rbt-lookup.simps comparator.two-comparisons-into-case-order*[*OF c*])
(*auto split*: *order.splits*)
qed

lemma *rbt-comp-ins*: *rbt-comp-ins* = *ord.rbt-ins* (*lt-of-comp c*)
proof (*intro ext*)

fix *f k v* **and** *t* :: (*'a*,*'b*)*rbt*
show *rbt-comp-ins f k v t* = *ord.rbt-ins* (*lt-of-comp c*) *f k v t*
by (*induct f k v t rule*: *rbt-comp-ins.induct*, *unfold rbt-comp-ins.simps ord.rbt-ins.simps*
comparator.two-comparisons-into-case-order[*OF c*])
(*auto split*: *order.splits*)
qed

lemma *rbt-comp-insert-with-key*: *rbt-comp-insert-with-key* = *ord.rbt-insert-with-key*
(*lt-of-comp c*)
unfolding *rbt-comp-insert-with-key-def*[*abs-def*] *ord.rbt-insert-with-key-def*[*abs-def*]
unfolding *rbt-comp-ins* **..**

lemma *rbt-comp-insert*: *rbt-comp-insert* = *ord.rbt-insert* (*lt-of-comp c*)
unfolding *rbt-comp-insert-def*[*abs-def*] *ord.rbt-insert-def*[*abs-def*]
unfolding *rbt-comp-insert-with-key* **..**

lemma *rbt-comp-del*: *rbt-comp-del* = *ord.rbt-del* (*lt-of-comp c*)
proof − **{**
fix *k a b* **and** *s t* :: (*'a*,*'b*)*rbt*
have
rbt-comp-del-from-left k t a b s = *ord.rbt-del-from-left* (*lt-of-comp c*) *k t a b s*
rbt-comp-del-from-right k t a b s = *ord.rbt-del-from-right* (*lt-of-comp c*) *k t a b*
s
rbt-comp-del k t = *ord.rbt-del* (*lt-of-comp c*) *k t*
by (*induct k t a b s* **and** *k t a b s* **and** *k t rule*: *rbt-comp-del-from-left-rbt-comp-del-from-right-rbt-comp-del.indu*
unfold
rbt-comp-del.simps ord.rbt-del.simps
rbt-comp-del-from-left.simps ord.rbt-del-from-left.simps
rbt-comp-del-from-right.simps ord.rbt-del-from-right.simps
comparator.two-comparisons-into-case-order[*OF c*],
auto split: *order.split*)
}
thus *?thesis* **by** (*intro ext*)
qed

lemma *rbt-comp-delete*: *rbt-comp-delete* = *ord.rbt-delete* (*lt-of-comp c*)
unfolding *rbt-comp-delete-def*[*abs-def*] *ord.rbt-delete-def*[*abs-def*]
unfolding *rbt-comp-del* **..**

lemma *rbt-comp-bulkload*: *rbt-comp-bulkload* = *ord.rbt-bulkload* (*lt-of-comp c*)
unfolding *rbt-comp-bulkload-def*[*abs-def*] *ord.rbt-bulkload-def*[*abs-def*]
unfolding *rbt-comp-insert* **..**

lemma *rbt-comp-map-entry*: *rbt-comp-map-entry* = *ord.rbt-map-entry* (*lt-of-comp*
c)
proof (*intro ext*)
fix *f k* **and** *t* :: (*'a*,*'b*)*rbt*
show *rbt-comp-map-entry f k t* = *ord.rbt-map-entry* (*lt-of-comp c*) *f k t*
by (*induct t*, *unfold rbt-comp-map-entry.simps ord.rbt-map-entry.simps*

comparator.two-comparisons-into-case-order[*OF c*])
(*auto split*: *order.splits*)
qed

lemma *comp-sunion-with*: *comp-sunion-with* = *ord.sunion-with* (*lt-of-comp c*)
proof (*intro ext*)
fix *f* **and** *as bs* :: ('*a* × '*b*)*list*
show *comp-sunion-with f as bs* = *ord.sunion-with* (*lt-of-comp c*) *f as bs*
by (*induct f as bs rule*: *comp-sunion-with.induct*,
unfold comp-sunion-with.simps ord.sunion-with.simps
comparator.two-comparisons-into-case-order[*OF c*])
(*auto split*: *order.splits*)
qed

lemma *anti-sym*: *lt-of-comp c a x* ⟹ *lt-of-comp c x a* ⟹ *False*
by (*metis c comparator.Gt-lt-conv comparator.Lt-lt-conv order.distinct*(*5*))

lemma *rbt-split-comp*: *rbt-split-comp t x* = *ord.rbt-split* (*lt-of-comp c*) *t x*
by (*induction t x rule*: *rbt-split-comp.induct*)
(*auto simp*: *ord.rbt-split.simps comparator.le-lt-convs*[*OF c*]
split: *order.splits prod.splits dest*: *anti-sym*)

lemma *comp-union-with-key*: *rbt-comp-union-rec f t1 t2* = *ord.rbt-union-rec* (*lt-of-comp c*) *f t1 t2*
proof (*induction f t1 t2 rule*: *rbt-comp-union-rec.induct*)
case (*1 f t1 t2*)
obtain *f' t1' t2'* **where** *flip*: (*f'*, *t2'*, *t1'*) =
(*if flip-rbt t2 t1 then* (λ*k v v'*. *f k v' v*, *t1*, *t2*) *else* (*f*, *t2*, *t1*))
by *fastforce*
show *?case*
proof (*cases t1'*)
case (*Branch - l1 a b r1*)
have *t1-not-Empty*: *t1'* ≠ *RBT-Impl.Empty*
by (*auto simp*: *Branch*)
obtain *l2 β r2* **where** *split*: *rbt-split-comp t2' a* = (*l2*, *β*, *r2*)
by (*cases rbt-split-comp t2' a*) *auto*
show *?thesis*
using *1*[*OF flip refl - - Branch*]
unfolding *rbt-comp-union-rec.simps*[*of - t1*] *ord.rbt-union-rec.simps*[*of - - t1*] *flip*[*symmetric*]
by (*auto simp*: *Branch split rbt-split-comp*[*symmetric*] *rbt-comp-insert-with-key*
split: *prod.splits*)
qed (*auto simp*: *rbt-comp-union-rec.simps*[*of - t1*] *ord.rbt-union-rec.simps*[*of - - t1*] *flip*[*symmetric*]
rbt-comp-insert-with-key rbt-split-comp[*symmetric*])
qed

lemma *comp-sinter-with*: *comp-sinter-with* = *ord.sinter-with* (*lt-of-comp c*)
proof (*intro ext*)

fix *f* **and** *as bs* :: (*'a* × *'b*)*list*
show *comp-sinter-with f as bs* = *ord.sinter-with* (*lt-of-comp c*) *f as bs*
by (*induct f as bs rule*: *comp-sinter-with.induct*,
unfold comp-sinter-with.simps ord.sinter-with.simps
comparator.two-comparisons-into-case-order[*OF c*])
(*auto split*: *order.splits*)
qed

lemma *rbt-comp-union-with-key*: *rbt-comp-union-with-key* = *ord.rbt-union-with-key* (*lt-of-comp c*)
by (*rule ext*)+
(*auto simp*: *rbt-comp-union-with-key-def rbt-comp-union-swap-rec ord.rbt-union-with-key-def*
ord.rbt-union-swap-rec comp-union-with-key)

lemma *comp-inter-with-key*: *rbt-comp-inter-rec f t1 t2* = *ord.rbt-inter-rec* (*lt-of-comp c*) *f t1 t2*
proof (*induction f t1 t2 rule*: *rbt-comp-inter-rec.induct*)
case (*1 f t1 t2*)
obtain *f' t1' t2'* **where** *flip*: (*f'*, *t2'*, *t1'*) =
(*if flip-rbt t2 t1 then* (λ*k v v'*. *f k v' v*, *t1*, *t2*) *else* (*f*, *t2*, *t1*))
by *fastforce*
show *?case*
proof (*cases t1'*)
case (*Branch - l1 a b r1*)
have *t1-not-Empty*: *t1'* ≠ *RBT-Impl.Empty*
by (*auto simp*: *Branch*)
obtain *l2 β r2* **where** *split*: *rbt-split-comp t2' a* = (*l2*, *β*, *r2*)
by (*cases rbt-split-comp t2' a*) *auto*
show *?thesis*
using *1*[*OF flip refl - - Branch*]
unfolding *rbt-comp-inter-rec.simps*[*of - t1*] *ord.rbt-inter-rec.simps*[*of - - t1*] *flip*[*symmetric*]
by (*auto simp*: *Branch split rbt-split-comp*[*symmetric*] *rbt-comp-lookup*
ord.map-filter-inter-def map-filter-comp-inter-def split: *prod.splits*)
qed (*auto simp*: *rbt-comp-inter-rec.simps*[*of - t1*] *ord.rbt-inter-rec.simps*[*of - - t1*] *flip*[*symmetric*]
ord.map-filter-inter-def map-filter-comp-inter-def rbt-comp-lookup rbt-split-comp[*symmetric*])
qed

lemma *rbt-comp-inter-with-key*: *rbt-comp-inter-with-key* = *ord.rbt-inter-with-key* (*lt-of-comp c*)
by (*rule ext*)+
(*auto simp*: *rbt-comp-inter-with-key-def rbt-comp-inter-swap-rec*
ord.rbt-inter-with-key-def ord.rbt-inter-swap-rec comp-inter-with-key)

lemma *comp-minus*: *comp-minus t1 t2* = *ord.rbt-minus-rec* (*lt-of-comp c*) *t1 t2*
proof (*induction t1 t2 rule*: *comp-minus.induct*)
case (*1 t1 t2*)
show *?case*

```
  proof (cases t2)
    case (Branch - l2 a u r2)
    have t2-not-Empty: t2 ≠ RBT-Impl.Empty
      by (auto simp: Branch)
    obtain l1 β r1 where split: rbt-split-comp t1 a = (l1, β, r1)
      by (cases rbt-split-comp t1 a) auto
    show ?thesis
      using 1[OF - - Branch]
      unfolding comp-minus.simps[of t1 t2] ord.rbt-minus-rec.simps[of - t1 t2]
     by (auto simp: Branch split rbt-split-comp[symmetric] rbt-comp-delete rbt-comp-lookup
          filter-comp-minus-def ord.filter-minus-def split: prod.splits)
  qed (auto simp: comp-minus.simps[of t1] ord.rbt-minus-rec.simps[of - t1]
      filter-comp-minus-def ord.filter-minus-def
      rbt-comp-delete rbt-comp-lookup rbt-split-comp[symmetric])
qed

lemma rbt-comp-minus: rbt-comp-minus = ord.rbt-minus (lt-of-comp c)
  by (rule ext)+ (auto simp: rbt-comp-minus-def ord.rbt-minus-def comp-minus)

lemmas rbt-comp-simps =
  rbt-comp-insert
  rbt-comp-lookup
  rbt-comp-delete
  rbt-comp-bulkload
  rbt-comp-map-entry
  rbt-comp-union-with-key
  rbt-comp-inter-with-key
  rbt-comp-minus
end
end

end
```

4 Generating Comparators

```
theory Comparator-Generator
imports
  ../Generator-Aux
  ../Derive-Manager
  Comparator
begin

typedecl ('a,'b,'c,'z)type
```

In the following, we define a generator which for a given datatype *('a, 'b, 'c, 'z) Comparator-Generator.type* constructs a comparator of type *'a comparator ⇒ 'b comparator ⇒ 'c comparator ⇒ 'z comparator ⇒ ('a, 'b, 'c, 'z) Comparator-Generator.type*. To this end, we first compare the index of the constructors, then for equal constructors, we compare the arguments recursively and combine the results lexicographically.

hide-type *type*

4.1 Lexicographic combination of *order*

fun *comp-lex* :: *order list* ⇒ *order*
where
comp-lex (*c* # *cs*) = (*case c of Eq* ⇒ *comp-lex cs* | - ⇒ *c*) |
comp-lex [] = *Eq*

4.2 Improved code for non-lazy languages

The following equations will eliminate all occurrences of *comp-lex* in the generated code of the comparators.

lemma *comp-lex-unfolds*:
comp-lex [] = *Eq*
comp-lex [*c*] = *c*
comp-lex (*c* # *d* # *cs*) = (*case c of Eq* ⇒ *comp-lex* (*d* # *cs*) | *z* ⇒ *z*)
by (*cases c*, *auto*)+

4.3 Pointwise properties for equality, symmetry, and transitivity

The pointwise properties are important during inductive proofs of soundness of comparators. They are defined in a way that are combinable with *comp-lex*.

lemma *comp-lex-eq*: *comp-lex os* = *Eq* ⟷ (∀ *ord* ∈ *set os*. *ord* = *Eq*)
by (*induct os*) (*auto split*: *order.splits*)

definition *trans-order* :: *order* ⇒ *order* ⇒ *order* ⇒ *bool* **where**
trans-order x y z ⟷ *x* ≠ *Gt* ⟶ *y* ≠ *Gt* ⟶ *z* ≠ *Gt* ∧ ((*x* = *Lt* ∨ *y* = *Lt*) ⟶ *z* = *Lt*)

lemma *trans-orderI*:
(*x* ≠ *Gt* ⟹ *y* ≠ *Gt* ⟹ *z* ≠ *Gt* ∧ ((*x* = *Lt* ∨ *y* = *Lt*) ⟶ *z* = *Lt*)) ⟹ *trans-order x y z*
by (*simp add*: *trans-order-def*)

lemma *trans-orderD*:
assumes *trans-order x y z* **and** *x* ≠ *Gt* **and** *y* ≠ *Gt*
shows *z* ≠ *Gt* **and** *x* = *Lt* ∨ *y* = *Lt* ⟹ *z* = *Lt*
using *assms* **by** (*auto simp*: *trans-order-def*)

lemma *All-less-Suc*:
(∀ *i* < *Suc x*. *P i*) ⟷ *P 0* ∧ (∀ *i* < *x*. *P* (*Suc i*))
using *less-Suc-eq-0-disj* **by** *force*

lemma *comp-lex-trans*:

assumes *length xs* = *length ys*
and ∀ *i* < *length zs*. *trans-order* (*xs* ! *i*) (*ys* ! *i*) (*zs* ! *i*)
shows *trans-order* (*comp-lex xs*) (*comp-lex ys*) (*comp-lex zs*)
using *assms*
proof (*induct xs ys zs rule*: *list-induct3*)
case (*Cons x xs y ys z zs*)
then show *?case*
by (*intro trans-orderI*)
(*cases x y z rule*: *order.exhaust* [*case-product order.exhaust order.exhaust*],
auto simp: *All-less-Suc dest*: *trans-orderD*)
qed (*simp add*: *trans-order-def*)

lemma *comp-lex-sym*:
assumes *length xs* = *length ys*
and ∀ *i* < *length ys*. *invert-order* (*xs* ! *i*) = *ys* ! *i*
shows *invert-order* (*comp-lex xs*) = *comp-lex ys*
using *assms* **by** (*induct xs ys rule*: *list-induct2*, *simp*, *case-tac x*) *fastforce+*

declare *comp-lex.simps* [*simp del*]

definition *peq-comp* :: ′*a comparator* ⇒ ′*a* ⇒ *bool*
where
peq-comp acomp x ⟷ (∀ *y*. *acomp x y* = *Eq* ⟷ *x* = *y*)

lemma *peq-compD*: *peq-comp acomp x* ⟹ *acomp x y* = *Eq* ⟷ *x* = *y*
unfolding *peq-comp-def* **by** *auto*

lemma *peq-compI*: (⋀ *y*. *acomp x y* = *Eq* ⟷ *x* = *y*) ⟹ *peq-comp acomp x*
unfolding *peq-comp-def* **by** *auto*

definition *psym-comp* :: ′*a comparator* ⇒ ′*a* ⇒ *bool* **where**
psym-comp acomp x ⟷ (∀ *y*. *invert-order* (*acomp x y*) = (*acomp y x*))

lemma *psym-compD*:
assumes *psym-comp acomp x*
shows *invert-order* (*acomp x y*) = (*acomp y x*)
using *assms* **unfolding** *psym-comp-def* **by** *blast+*

lemma *psym-compI*:
assumes ⋀ *y*. *invert-order* (*acomp x y*) = (*acomp y x*)
shows *psym-comp acomp x*
using *assms* **unfolding** *psym-comp-def* **by** *blast*

definition *ptrans-comp* :: ′*a comparator* ⇒ ′*a* ⇒ *bool* **where**
ptrans-comp acomp x ⟷ (∀ *y z*. *trans-order* (*acomp x y*) (*acomp y z*) (*acomp x z*))

lemma *ptrans-compD*:
assumes *ptrans-comp acomp x*
shows *trans-order* (*acomp x y*) (*acomp y z*) (*acomp x z*)
using *assms* **unfolding** *ptrans-comp-def* **by** *blast+*

lemma *ptrans-compI*:
assumes $\bigwedge$ *y z. trans-order* (*acomp x y*) (*acomp y z*) (*acomp x z*)
shows *ptrans-comp acomp x*
using *assms* **unfolding** *ptrans-comp-def* **by** *blast*

4.4 Separate properties of comparators

definition *eq-comp* :: *'a comparator* ⇒ *bool* **where**
eq-comp acomp ⟷ (∀ *x. peq-comp acomp x*)

lemma *eq-compD2*: *eq-comp acomp* ⟹ *peq-comp acomp x*
unfolding *eq-comp-def* **by** *blast*

lemma *eq-compI2*: ($\bigwedge$ *x. peq-comp acomp x*) ⟹ *eq-comp acomp*
unfolding *eq-comp-def* **by** *blast*

definition *trans-comp* :: *'a comparator* ⇒ *bool* **where**
trans-comp acomp ⟷ (∀ *x. ptrans-comp acomp x*)

lemma *trans-compD2*: *trans-comp acomp* ⟹ *ptrans-comp acomp x*
unfolding *trans-comp-def* **by** *blast*

lemma *trans-compI2*: ($\bigwedge$ *x. ptrans-comp acomp x*) ⟹ *trans-comp acomp*
unfolding *trans-comp-def* **by** *blast*

definition *sym-comp* :: *'a comparator* ⇒ *bool* **where**
sym-comp acomp ⟷ (∀ *x. psym-comp acomp x*)

lemma *sym-compD2*:
sym-comp acomp ⟹ *psym-comp acomp x*
unfolding *sym-comp-def* **by** *blast*

lemma *sym-compI2*: ($\bigwedge$ *x. psym-comp acomp x*) ⟹ *sym-comp acomp*
unfolding *sym-comp-def* **by** *blast*

lemma *eq-compD*: *eq-comp acomp* ⟹ *acomp x y* = *Eq* ⟷ *x* = *y*
by (*rule peq-compD*[*OF eq-compD2*])

lemma *eq-compI*: ($\bigwedge$ *x y. acomp x y* = *Eq* ⟷ *x* = *y*) ⟹ *eq-comp acomp*
by (*intro eq-compI2 peq-compI*)

lemma *trans-compD*: *trans-comp acomp* ⟹ *trans-order* (*acomp x y*) (*acomp y z*) (*acomp x z*)

by (*rule ptrans-compD*[*OF trans-compD2*])

lemma *trans-compI*: ($\bigwedge$ *x y z*. *trans-order* (*acomp x y*) (*acomp y z*) (*acomp x z*)) $\Longrightarrow$ *trans-comp acomp*
by (*intro trans-compI2 ptrans-compI*)

lemma *sym-compD*:
sym-comp acomp $\Longrightarrow$ *invert-order* (*acomp x y*) = (*acomp y x*)
by (*rule psym-compD*[*OF sym-compD2*])

lemma *sym-compI*: ($\bigwedge$ *x y*. *invert-order* (*acomp x y*) = (*acomp y x*)) $\Longrightarrow$ *sym-comp acomp*
by (*intro sym-compI2 psym-compI*)

lemma *eq-sym-trans-imp-comparator*:
assumes *eq-comp acomp* **and** *sym-comp acomp* **and** *trans-comp acomp*
shows *comparator acomp*
proof
fix *x y z*
show *invert-order* (*acomp x y*) = *acomp y x*
using *sym-compD* [*OF* ‹*sym-comp acomp*›] **.**
{
assume *acomp x y* = *Eq*
with *eq-compD* [*OF* ‹*eq-comp acomp*›]
show *x* = *y* **by** *blast*
}
{
assume *acomp x y* = *Lt* **and** *acomp y z* = *Lt*
with *trans-orderD* [*OF trans-compD* [*OF* ‹*trans-comp acomp*›], *of x y z*]
show *acomp x z* = *Lt* **by** *auto*
}
qed

lemma *comparator-imp-eq-sym-trans*:
assumes *comparator acomp*
shows *eq-comp acomp sym-comp acomp trans-comp acomp*
proof −
interpret *comparator acomp* **by** *fact*
show *eq-comp acomp* **using** *eq* **by** (*intro eq-compI*, *auto*)
show *sym-comp acomp* **using** *sym* **by** (*intro sym-compI*, *auto*)
show *trans-comp acomp*
proof (*intro trans-compI trans-orderI*)
fix *x y z*
assume *acomp x y* ≠ *Gt acomp y z* ≠ *Gt*
thus *acomp x z* ≠ *Gt* ∧ (*acomp x y* = *Lt* ∨ *acomp y z* = *Lt* ⟶ *acomp x z* = *Lt*)
using *comp-trans* [*of x y z*] **and** *eq* [*of x y*] **and** *eq* [*of y z*]
by (*cases acomp x y acomp y z rule*: *order.exhaust* [*case-product order.exhaust*]) *auto*

qed
qed

context
fixes *acomp* :: *'a comparator*
assumes *c*: *comparator acomp*
begin
lemma *comp-to-psym-comp*: *psym-comp acomp x*
using *comparator-imp-eq-sym-trans*[*OF c*]
by (*intro sym-compD2*)

lemma *comp-to-peq-comp*: *peq-comp acomp x*
using *comparator-imp-eq-sym-trans* [*OF c*]
by (*intro eq-compD2*)

lemma *comp-to-ptrans-comp*: *ptrans-comp acomp x*
using *comparator-imp-eq-sym-trans* [*OF c*]
by (*intro trans-compD2*)
end

4.5 Auxiliary Lemmas for Comparator Generator

lemma *forall-finite*: $(\forall\ i < (0 :: nat).\ P\ i) = True$
$(\forall\ i < Suc\ 0.\ P\ i) = P\ 0$
$(\forall\ i < Suc\ (Suc\ x).\ P\ i) = (P\ 0 \wedge (\forall\ i < Suc\ x.\ P\ (Suc\ i)))$
by (*auto*, *case-tac i*, *auto*)

lemma *trans-order-different*:
trans-order a b Lt
trans-order Gt b c
trans-order a Gt c
by (*intro trans-orderI*, *auto*)+

lemma *length-nth-simps*:
length [] = *0* *length* (*x* # *xs*) = *Suc* (*length xs*)
(*x* # *xs*) ! *0* = *x* (*x* # *xs*) ! (*Suc n*) = *xs* ! *n* **by** *auto*

4.6 The Comparator Generator

ML-file ‹*comparator-generator.ML*›

end

4.7 Compare Generator

theory *Compare-Generator*
imports
Comparator-Generator
Compare
begin

We provide a generator which takes the comparators of the comparator generator to synthesize suitable *compare*-functions from the *compare*-class.

One can further also use these comparison functions to derive an instance of the *compare-order*-class, and therefore also for *linorder*. In total, we provide the three *derive*-methods where the example type *prod* can be replaced by any other datatype.

- *derive compare prod* creates an instance *prod* :: (*compare*, *compare*) *compare*.
- *derive compare-order prod* creates an instance *prod* :: (*compare*, *compare*) *compare-order*.
- *derive linorder prod* creates an instance *prod* :: (*linorder*, *linorder*) *linorder*.

Usually, the use of *derive linorder* is not recommended if there are comparators available: Internally, the linear orders will directly be converted into comparators, so a direct use of the comparators will result in more efficient generated code. This command is mainly provided as a convenience method where comparators are not yet present. For example, at the time of writing, the Container Framework has partly been adapted to internally use comparators, whereas in other AFP-entries, we did not integrate comparators.

lemma *linorder-axiomsD*: **assumes** *class.linorder le lt*
  shows
  lt x y = (*le x y* ∧ ¬ *le y x*) (**is** *?a*)
  le x x (**is** *?b*)
  le x y ⟹ *le y z* ⟹ *le x z* (**is** *?c1* ⟹ *?c2* ⟹ *?c3*)
  le x y ⟹ *le y x* ⟹ *x* = *y* (**is** *?d1* ⟹ *?d2* ⟹ *?d3*)
  le x y ∨ *le y x* (**is** *?e*)
proof −
  interpret *linorder le lt* **by** *fact*
  show *?a ?b ?c1* ⟹ *?c2* ⟹ *?c3 ?d1* ⟹ *?d2* ⟹ *?d3 ?e* **by** *auto*
qed

named-theorems *compare-simps simp theorems to derive compare = comparator-of*

ML-file ‹*compare-generator.ML*›

end

4.8 Defining Comparators and Compare-Instances for Common Types

theory *Compare-Instances*
imports

  Compare-Generator
  HOL−Library.Char-ord
begin

For all of the following types, we define comparators and register them in the class *compare*: *int*, *integer*, *nat*, *char*, *bool*, *unit*, *sum*, *option*, *list*, and *prod*. We do not register those classes in *compare-order* where so far no linear order is defined, in particular if there are conflicting orders, like pair-wise or lexicographic comparison on pairs.

For *int*, *nat*, *integer* and *char* we just use their linear orders as comparators.

derive (*linorder*) *compare-order int integer nat char*

For *sum*, *list*, *prod*, and *option* we generate comparators which are however are not used to instantiate *linorder*.

derive *compare sum list prod option*

We do not use the linear order to define the comparator for *bool* and *unit*, but implement more efficient ones.

```
fun comparator-unit :: unit comparator where
  comparator-unit x y = Eq

fun comparator-bool :: bool comparator where
  comparator-bool False False = Eq
| comparator-bool False True = Lt
| comparator-bool True True = Eq
| comparator-bool True False = Gt

lemma comparator-unit: comparator comparator-unit
  by (unfold-locales, auto)

lemma comparator-bool: comparator comparator-bool
proof
  fix x y z :: bool
   show invert-order (comparator-bool x y) = comparator-bool y x by (cases x,
(cases y, auto)+)
  show comparator-bool x y = Eq ⟹ x = y by (cases x, (cases y, auto)+)
 show comparator-bool x y = Lt ⟹ comparator-bool y z = Lt ⟹ comparator-bool
x z = Lt
    by (cases x, (cases y, auto), cases y, (cases z, auto)+)
qed


local-setup ‹
  Comparator-Generator.register-foreign-comparator @{typ unit}
    @{term comparator-unit}
    @{thm comparator-unit}
›
```

local-setup ‹
  Comparator-Generator.register-foreign-comparator @{*typ bool*}
    @{*term comparator-bool*}
    @{*thm comparator-bool*}
›

derive *compare bool unit*

It is not directly possible to *derive* (*linorder*) *bool unit*, since *compare* was not defined as *comparator-of*, but as *comparator-bool*. However, we can manually prove this equivalence and then use this knowledge to prove the instance of *compare-order*.

lemma *comparator-bool-comparator-of* [*compare-simps*]:
  comparator-bool = *comparator-of*
proof (*intro ext*)
  fix *a b*
  show *comparator-bool a b* = *comparator-of a b*
    unfolding *comparator-of-def*
    by (*cases a*, (*cases b*, *auto*))
qed

lemma *comparator-unit-comparator-of* [*compare-simps*]:
  comparator-unit = *comparator-of*
proof (*intro ext*)
  fix *a b*
  show *comparator-unit a b* = *comparator-of a b*
    unfolding *comparator-of-def* **by** *auto*
qed

derive (*linorder*) *compare-order bool unit*
end

4.9 Defining Compare-Order-Instances for Common Types

theory *Compare-Order-Instances*
imports
  Compare-Instances
  HOL−Library.List-Lexorder
  HOL−Library.Product-Lexorder
  HOL−Library.Option-ord
begin

We now also instantiate class *compare-order* and not only *compare*. Here, we also prove that our definitions do not clash with existing orders on *list*, *option*, and *prod*.

For *sum* we just define the linear orders via their comparator.

derive *compare-order sum*

```
instance list :: (compare-order)compare-order
proof
  note [simp] = le-of-comp-def lt-of-comp-def comparator-of-def
  show le-of-comp (compare :: 'a list comparator) = (≤)
    unfolding compare-list-def compare-is-comparator-of
  proof (intro ext)
    fix xs ys :: 'a list
    show le-of-comp (comparator-list comparator-of) xs ys = (xs ≤ ys)
    proof (induct xs arbitrary: ys)
      case (Nil ys)
      show ?case
        by (cases ys, simp-all)
    next
      case (Cons x xs yys) note IH = this
      thus ?case
      proof (cases yys)
        case Nil
        thus ?thesis by auto
      next
        case (Cons y ys)
        show ?thesis unfolding Cons
          using IH[of ys]
          by (cases x y rule: linorder-cases, auto)
      qed
    qed
  qed
  show lt-of-comp (compare :: 'a list comparator) = (<)
    unfolding compare-list-def compare-is-comparator-of
  proof (intro ext)
    fix xs ys :: 'a list
    show lt-of-comp (comparator-list comparator-of) xs ys = (xs < ys)
    proof (induct xs arbitrary: ys)
      case (Nil ys)
      show ?case
        by (cases ys, simp-all)
    next
      case (Cons x xs yys) note IH = this
      thus ?case
      proof (cases yys)
        case Nil
        thus ?thesis by auto
      next
        case (Cons y ys)
        show ?thesis unfolding Cons
          using IH[of ys]
          by (cases x y rule: linorder-cases, auto)
      qed
    qed
  qed
```

```
qed

instance prod :: (compare-order, compare-order)compare-order
proof
  note [simp] = le-of-comp-def lt-of-comp-def comparator-of-def
  show le-of-comp (compare :: ('a,'b)prod comparator) = (≤)
    unfolding compare-prod-def compare-is-comparator-of
  proof (intro ext)
    fix xy1 xy2 :: ('a,'b)prod
     show le-of-comp (comparator-prod comparator-of comparator-of) xy1 xy2 =
(xy1 ≤ xy2)
      by (cases xy1, cases xy2, auto)
  qed
  show lt-of-comp (compare :: ('a,'b)prod comparator) = (<)
    unfolding compare-prod-def compare-is-comparator-of
  proof (intro ext)
    fix xy1 xy2 :: ('a,'b)prod
   show lt-of-comp (comparator-prod comparator-of comparator-of) xy1 xy2 = (xy1
< xy2)
      by (cases xy1, cases xy2, auto)
  qed
qed

instance option :: (compare-order)compare-order
proof
  note [simp] = le-of-comp-def lt-of-comp-def comparator-of-def
  show le-of-comp (compare :: 'a option comparator) = (≤)
    unfolding compare-option-def compare-is-comparator-of
  proof (intro ext)
    fix xy1 xy2 :: 'a option
    show le-of-comp (comparator-option comparator-of) xy1 xy2 = (xy1 ≤ xy2)
      by (cases xy1, (cases xy2, auto split: if-splits)+)
  qed
  show lt-of-comp (compare :: 'a option comparator) = (<)
    unfolding compare-option-def compare-is-comparator-of
  proof (intro ext)
    fix xy1 xy2 :: 'a option
    show lt-of-comp (comparator-option comparator-of) xy1 xy2 = (xy1 < xy2)
      by (cases xy1, (cases xy2, auto split: if-splits)+)
  qed
qed

end
```

4.10 Compare Instance for Rational Numbers

```
theory Compare-Rat
imports
  Compare-Generator
```

HOL.Rat
begin

derive (*linorder*) *compare-order rat*

end

4.11 Compare Instance for Real Numbers

theory *Compare-Real*
imports
Compare-Generator
HOL.Real
begin

derive (*linorder*) *compare-order real*

lemma *invert-order-compare-real*[*simp*]: $\bigwedge$ *x y* :: *real*. *invert-order* (*compare x y*) = *compare y x*
by (*simp add*: *comparator-of-def compare-is-comparator-of*)

end

5 Checking Equality Without "="

theory *Equality-Generator*
imports
../*Generator-Aux*
../*Derive-Manager*
begin

typedecl ('*a*,'*b*,'*c*,'*z*)*type*

In the following, we define a generator which for a given datatype ('*a*, '*b*, '*c*, '*z*) *Equality-Generator.type* constructs an equality-test function of type ('*a* ⇒ '*a* ⇒ *bool*) ⇒ ('*b* ⇒ '*b* ⇒ *bool*) ⇒ ('*c* ⇒ '*c* ⇒ *bool*) ⇒ ('*z* ⇒ '*z* ⇒ *bool*) ⇒ ('*a*, '*b*, '*c*, '*z*) *Equality-Generator.type* ⇒ ('*a*, '*b*, '*c*, '*z*) *Equality-Generator.type* ⇒ *bool*. These functions are essential to synthesize conditional equality functions in the container framework, where a strict membership in the *equal*-class must not be enforced.

hide-type *type*

Just a constant to define conjunction on lists of booleans, which will be used to merge the results when having compared the arguments of identical constructors.

definition *list-all-eq* :: *bool list* ⇒ *bool* **where**
list-all-eq = *list-all id*

5.1 Improved Code for Non-Lazy Languages

The following equations will eliminate all occurrences of *list-all-eq* in the generated code of the equality functions.

lemma *list-all-eq-unfold*:
  list-all-eq [] = *True*
  list-all-eq [*b*] = *b*
  list-all-eq (*b1* # *b2* # *bs*) = (*b1* ∧ *list-all-eq* (*b2* # *bs*))
  unfolding *list-all-eq-def*
  by *auto*

lemma *list-all-eq*: *list-all-eq bs* ⟷ (∀ *b* ∈ *set bs*. *b*)
  unfolding *list-all-eq-def list-all-iff* **by** *auto*

5.2 Partial Equality Property

We require a partial property which can be used in inductive proofs.

type-synonym *'a equality* = *'a* ⇒ *'a* ⇒ *bool*

definition *pequality* :: *'a equality* ⇒ *'a* ⇒ *bool*
where
  pequality aeq x ⟷ (∀ *y*. *aeq x y* ⟷ *x* = *y*)

lemma *pequalityD*: *pequality aeq x* ⟹ *aeq x y* ⟷ *x* = *y*
  unfolding *pequality-def* **by** *auto*

lemma *pequalityI*: (⋀ *y*. *aeq x y* ⟷ *x* = *y*) ⟹ *pequality aeq x*
  unfolding *pequality-def* **by** *auto*

5.3 Global equality property

definition *equality* :: *'a equality* ⇒ *bool* **where**
  equality aeq ⟷ (∀ *x*. *pequality aeq x*)

lemma *equalityD2*: *equality aeq* ⟹ *pequality aeq x*
  unfolding *equality-def* **by** *blast*

lemma *equalityI2*: (⋀ *x*. *pequality aeq x*) ⟹ *equality aeq*
  unfolding *equality-def* **by** *blast*

lemma *equalityD*: *equality aeq* ⟹ *aeq x y* ⟷ *x* = *y*
  by (*rule pequalityD*[*OF equalityD2*])

lemma *equalityI*: (⋀ *x y*. *aeq x y* ⟷ *x* = *y*) ⟹ *equality aeq*
  by (*intro equalityI2 pequalityI*)

lemma *equality-imp-eq*:
  equality aeq ⟹ *aeq* = (=)
  by (*intro ext*, *auto dest*: *equalityD*)

lemma *eq-equality*: *equality* (=)
  by (*rule equalityI*, *simp*)

lemma *equality-def'*: *equality f* = (*f* = (=))
  using *equality-imp-eq eq-equality* **by** *blast*

5.4 The Generator

ML-file ‹*equality-generator.ML*›

hide-fact (**open**) *equalityI*

end

5.5 Defining Equality-Functions for Common Types

theory *Equality-Instances*
imports
  Equality-Generator
begin

For all of the following types, we register equality-functions. *int*, *integer*, *nat*, *char*, *bool*, *unit*, *sum*, *option*, *list*, and *prod*. For types without type parameters, we use plain (=), and for the others we use generated ones. These functions will be essential, when the generator is later on invoked on types, which in their definition use one these types.

derive (*eq*) *equality int integer nat char bool unit*
derive *equality sum list prod option*

end

6 Generating Hash-Functions

theory *Hash-Generator*
imports
  ../Generator-Aux
  ../Derive-Manager
  Collections.HashCode
begin

As usual, in the generator we use a dedicated function to combine the results from evaluating the hash-function of the arguments of a constructor, to deliver the global hash-value.

fun *hash-combine* :: *hashcode list* ⇒ *hashcode list* ⇒ *hashcode* **where**
  hash-combine [] [*x*] = *x*
| *hash-combine* (*y* # *ys*) (*z* # *zs*) = *y* ∗ *z* + *hash-combine ys zs*
| *hash-combine* - - = *0*

The first argument of *hash-combine* originates from evaluating the hash-function on the arguments of a constructor, and the second argument of *hash-combine* will be static *magic* numbers which are generated within the generator.

6.1 Improved Code for Non-Lazy Languages

lemma *hash-combine-unfold*:
hash-combine [] [x] = x
hash-combine (y # ys) (z # zs) = $y * z$ + *hash-combine ys zs*
by *auto*

6.2 The Generator

ML-file ‹*hash-generator.ML*›

end

6.3 Defining Hash-Functions for Common Types

theory *Hash-Instances*
imports
Hash-Generator
begin

For all of the following types, we register hashcode-functions. *int*, *integer*, *nat*, *char*, *bool*, *unit*, *sum*, *option*, *list*, and *prod*. For types without type parameters, we use plain *hashcode*, and for the others we use generated ones.

derive (*hashcode*) *hash-code int integer bool char unit nat*

derive *hash-code prod sum option list*

There is no need to *derive hashable prod sum option list* since all of these types are already instances of class *hashable*. Still the above command is necessary to register these types in the generator.

end

7 Countable Datatypes

theory *Countable-Generator*
imports
HOL−Library.Countable
../Derive-Manager
begin

Brian Huffman and Alexander Krauss (old datatype), and Jasmin Blanchette (BNF datatype) have developed tactics which automatically can prove that a datatype is countable. We just make this tactic available in the derive-manager so that one can conveniently write `derive countable some-datatype`.

7.1 Installing the tactic

There is nothing more to do, then to write some boiler-plate ML-code for class-instantiation.

```
setup ‹
  let
    fun derive dtyp-name - thy =
      let
        val base-name = Long-Name.base-name dtyp-name
        val - = writeln (proving that datatype  ^ base-name ^  is countable)
        val sort = @{sort countable}
        val vs =
          let val i = BNF-LFP-Compat.the-spec thy dtyp-name |> #1
          in map (fn (n,-) => (n, sort)) i end
        val thy' = Class.instantiation ([dtyp-name],vs,sort) thy
          |> Class.prove-instantiation-exit (fn ctxt => countable-tac ctxt 1)
        val - = writeln (registered  ^ base-name ^  in class countable)
      in thy' end
  in
    Derive-Manager.register-derive countable register datatypes is class countable
derive
  end
›
```

end

8 Loading Existing Derive-Commands

theory *Derive*
imports
Comparator-Generator/Compare-Instances
Equality-Generator/Equality-Instances
Hash-Generator/Hash-Instances
Countable-Generator/Countable-Generator
begin

We just load the commands to derive comparators, equality-functions, hash-functions, and the command to show that a datatype is countable, so that now all of them are available. There are further generators available in the AFP entries Containers and Show.

print-derives

end

9 Examples

theory *Derive-Examples*
imports
  Derive
  Comparator-Generator/Compare-Order-Instances
  Equality-Generator/Equality-Instances
  HOL.Rat
begin

9.1 Rational Numbers

The rational numbers are not a datatype, so it will not be possible to derive corresponding instances of comparators, hashcodes, etc. via the generators. But we can and should still register the existing instances, so that later datatypes are supported which use rational numbers.

Use the linear order on rationals to define the *compare-order*-instance.

derive (*linorder*) *compare-order rat*

Use (=) as equality function.

derive (*eq*) *equality rat*

First manually define a hashcode function.

instantiation *rat* :: *hashable*
begin
definition *def-hashmap-size* = (λ- :: *rat itself*. *10*)
definition *hashcode* (*r* :: *rat*) = *hashcode* (*quotient-of r*)
instance
  by (*intro-classes*)(*simp-all add*: *def-hashmap-size-rat-def*)
end

And then register it at the generator.

derive (*hashcode*) *hash-code rat*

9.2 A Datatype Without Nested Recursion

datatype *'a bintree* = *BEmpty* | *BNode 'a bintree 'a 'a bintree*

derive *compare-order bintree*
derive *countable bintree*
derive *equality bintree*
derive *hashable bintree*

9.3 Using Other datatypes

datatype *nat-list-list* = *NNil* | *CCons nat list* × *rat option nat-list-list*

derive *compare-order nat-list-list*

derive *countable nat-list-list*
derive (*eq*) *equality nat-list-list*
derive *hashable nat-list-list*

9.4 Mutual Recursion

datatype
'a mtree = MEmpty | MNode 'a 'a mtree-list **and**
'a mtree-list = MNil | MCons 'a mtree 'a mtree-list

derive *compare-order mtree mtree-list*
derive *countable mtree mtree-list*
derive *hashable mtree mtree-list*

For *derive* (*equality*|*comparator*|*hash-code*) *mutual-recursive-type* there is the speciality that only one of the mutual recursive types has to be mentioned in order to register all of them. So one of *mtree* and *mtree-list* suffices.

derive *equality mtree*

9.5 Nested recursion

datatype *'a tree = Empty | Node 'a 'a tree list*
datatype *'a ttree = TEmpty | TNode 'a 'a ttree list tree*

derive *compare-order tree ttree*
derive *countable tree ttree*
derive *equality tree ttree*
derive *hashable tree ttree*

9.6 Examples from IsaFoR

datatype (*'f,'v*) *term = Var 'v | Fun 'f* (*'f,'v*) *term list*
datatype (*'f, 'l*) *lab =*
Lab (*'f, 'l*) *lab 'l*
| *FunLab* (*'f, 'l*) *lab* (*'f, 'l*) *lab list*
| *UnLab 'f*
| *Sharp* (*'f, 'l*) *lab*

derive *compare-order term lab*
derive *countable term lab*
derive *equality term lab*
derive *hashable term lab*

9.7 A Complex Datatype

The following datatype has nested and mutual recursion, and uses other datatypes.

datatype (*'a, 'b*) *complex =*
C1 nat 'a ttree × *rat* + (*'a,'b*) *complex list* |

C2 (*'a*, *'b*) *complex list tree tree 'b* (*'a*, *'b*) *complex* (*'a*, *'b*) *complex2 ttree list*
and (*'a*, *'b*) *complex2* = *D1* (*'a*, *'b*) *complex ttree*

On this last example type we illustrate the difference of the various comparator- and order-generators.

For *complex* we create an instance of *compare-order* which also defines a linear order. Note however that the instance will be *complex* :: (*compare*, *compare*) *compare-order*, i.e., the argument types have to be in class *compare*.

For *complex2* we only derive *compare* which is not a subclass of *linorder*. The instance will be *complex2* :: (*compare*, *compare*) *compare*, i.e., again the argument types have to be in class *compare*.

To avoid the dependence on *compare*, we can also instruct *derive* to be based on *linorder*. Here, the command *derive linorder complex2* will create the instance *complex2* :: (*linorder*, *linorder*) *linorder*, i.e., here the argument types have to be in class *linorder*.

derive *compare-order complex*
derive *compare complex2*
derive *linorder complex2*
derive *countable complex complex2*
derive *equality complex*
derive *hashable complex complex2*

end

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References

[1] P. Lammich and A. Lochbihler. The Isabelle collections framework. In *Proc. ITP'10*, volume 6172 of *LNCS*, pages 339–354, 2010.

[2] A. Lochbihler. Light-weight containers for isabelle: Efficient, extensible, nestable. In *Proc. ITP'13*, volume 7998 of *LNCS*, pages 116–132, 2013.

[3] R. Thiemann and C. Sternagel. Certification of termination proofs using CeTA. In *Proc. TPHOLs'09*, volume 5674 of *LNCS*, pages 452–468, 2009.