

Derangements

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Abstract

The Derangements Formula describes the number of fixpoint-free permutations as closed-form formula. This theorem is the 88th theorem of the Top 100 Theorems list.

Contents

1 Derangements	1
1.1 Preliminaries	1
1.1.1 Additions to <i>HOL.Finite-Set Theory</i>	1
1.1.2 Additions to <i>HOL-Combinatorics.Permutations Theory</i>	2
1.2 Fixpoint-Free Permutations	2
1.3 Properties of Derangements	3
1.4 Construction of Derangements	3
1.5 Cardinality of Derangements	4
1.5.1 Recursive Characterization	4
1.5.2 Closed-Form Characterization	4
1.5.3 Approximation of Cardinality	4

1 Derangements

theory *Derangements*

imports

Complex-Main

HOL-Combinatorics.Permutations

begin

1.1 Preliminaries

1.1.1 Additions to *HOL.Finite-Set Theory*

lemma *card-product-dependent*:

assumes *finite S* $\forall x \in S. \text{finite } (T x)$

shows $\text{card } \{(x, y). x \in S \wedge y \in T x\} = (\sum x \in S. \text{card } (T x))$

<proof>

1.1.2 Additions to *HOL-Combinatorics.Permutations* Theory

lemma *permutes-imp-bij'*:

assumes p permutes S

shows $\text{bij } p$

<proof>

lemma *permutesE*:

assumes p permutes S

obtains $\text{bij } p \ \forall x. x \notin S \longrightarrow p\ x = x$

<proof>

lemma *bij-imp-permutes'*:

assumes $\text{bij } p \ \forall x. x \notin A \longrightarrow p\ x = x$

shows p permutes A

<proof>

lemma *permutes-swap*:

assumes p permutes S

shows $\text{Fun.swap } x\ y\ p$ permutes $(\text{insert } x\ (\text{insert } y\ S))$

<proof>

lemma *bij-extends*:

$\text{bij } p \implies p\ x = x \implies \text{bij } (p(x := y, \text{inv } p\ y := x))$

<proof>

lemma *permutes-add-one*:

assumes p permutes S $x \notin S$ $y \in S$

shows $p(x := y, \text{inv } p\ y := x)$ permutes $(\text{insert } x\ S)$

<proof>

lemma *permutations-skip-one*:

assumes p permutes S $x : S$

shows $p(x := x, \text{inv } p\ x := p\ x)$ permutes $(S - \{x\})$

<proof>

lemma *permutes-drop-cycle-size-two*:

$\langle p \circ \text{Transposition.transpose } x\ (p\ x) \text{ permutes } (S - \{x, p\ x\}) \rangle$

if $\langle p \text{ permutes } S \rangle \ \langle p\ (p\ x) = x \rangle$

<proof>

1.2 Fixpoint-Free Permutations

definition *derangements* :: $\text{nat set} \Rightarrow (\text{nat} \Rightarrow \text{nat}) \text{ set}$

where

$\text{derangements } S = \{p. p \text{ permutes } S \wedge (\forall x \in S. p\ x \neq x)\}$

lemma *derangementsI*:

assumes p permutes $S \ \bigwedge x. x \in S \implies p\ x \neq x$

shows $p \in \text{derangements } S$

<proof>

lemma *derangementsE*:

assumes $d : \text{derangements } S$

obtains $d \text{ permutes } S \ \forall x \in S. d \ x \neq x$

<proof>

1.3 Properties of Derangements

lemma *derangements-inv*:

assumes $d : d \in \text{derangements } S$

shows $\text{inv } d \in \text{derangements } S$

<proof>

lemma *derangements-in-image*:

assumes $d \in \text{derangements } A \ x \in A$

shows $d \ x \in A$

<proof>

lemma *derangements-in-image-strong*:

assumes $d \in \text{derangements } A \ x \in A$

shows $d \ x \in A - \{x\}$

<proof>

lemma *derangements-inverse-in-image*:

assumes $d \in \text{derangements } A \ x \in A$

shows $\text{inv } d \ x \in A$

<proof>

lemma *derangements-fixpoint*:

assumes $d \in \text{derangements } A \ x \notin A$

shows $d \ x = x$

<proof>

lemma *derangements-no-fixpoint*:

assumes $d \in \text{derangements } A \ x \in A$

shows $d \ x \neq x$

<proof>

lemma *finite-derangements*:

assumes *finite* A

shows *finite* ($\text{derangements } A$)

<proof>

1.4 Construction of Derangements

lemma *derangements-empty[simp]*:

$\text{derangements } \{\} = \{\text{id}\}$

<proof>

lemma *derangements-singleton[simp]*:

derangements $\{x\} = \{\}$

<proof>

lemma *derangements-swap*:

assumes $d \in \text{derangements } S \ x \notin S \ y \notin S \ x \neq y$

shows $\text{Fun.swap } x \ y \ d \in \text{derangements } (\text{insert } x \ (\text{insert } y \ S))$

<proof>

lemma *derangements-skip-one*:

assumes $d: d \in \text{derangements } S \ \text{and } x \in S \ d \ (d \ x) \neq x$

shows $d(x := x, \text{inv } d \ x := d \ x) \in \text{derangements } (S - \{x\})$

<proof>

lemma *derangements-add-one*:

assumes $d \in \text{derangements } S \ x \notin S \ y \in S$

shows $d(x := y, \text{inv } d \ y := x) \in \text{derangements } (\text{insert } x \ S)$

<proof>

lemma *derangements-drop-minimal-cycle*:

assumes $d \in \text{derangements } S \ d \ (d \ x) = x$

shows $\text{Fun.swap } x \ (d \ x) \ d \in \text{derangements } (S - \{x, d \ x\})$

<proof>

1.5 Cardinality of Derangements

1.5.1 Recursive Characterization

fun *count-derangements* :: $\text{nat} \Rightarrow \text{nat}$

where

count-derangements $0 = 1$

| *count-derangements* $(\text{Suc } 0) = 0$

| *count-derangements* $(\text{Suc } (\text{Suc } n)) = (n + 1) * (\text{count-derangements } (\text{Suc } n) + \text{count-derangements } n)$

lemma *card-derangements*:

assumes *finite* $S \ \text{card } S = n$

shows $\text{card } (\text{derangements } S) = \text{count-derangements } n$

<proof>

1.5.2 Closed-Form Characterization

lemma *count-derangements*:

$\text{real } (\text{count-derangements } n) = \text{fact } n * (\sum k \in \{0..n\}. (-1) ^ k / \text{fact } k)$

<proof>

1.5.3 Approximation of Cardinality

lemma *two-power-fact-le-fact*:

assumes $n \geq 1$

shows $2^{\wedge}k * \text{fact } n \leq (\text{fact } (n + k) :: 'a :: \{\text{semiring-char-0, linordered-semidom}\})$
<proof>

lemma *exp1-approx*:

assumes $n > 0$

shows $\text{exp } (1 :: \text{real}) - (\sum k < n. 1 / \text{fact } k) \in \{0..2 / \text{fact } n\}$
<proof>

lemma *exp1-bounds*: $\text{exp } 1 \in \{8 / 3..11 / 4 :: \text{real}\}$

<proof>

lemma *count-derangements-approximation*:

assumes $n \neq 0$

shows $\text{abs}(\text{real } (\text{count-derangements } n) - \text{fact } n / \text{exp } 1) < 1 / 2$
<proof>

theorem *derangements-formula*:

assumes $n \neq 0$ *finite* S $\text{card } S = n$

shows $\text{int } (\text{card } (\text{derangements } S)) = \text{round } (\text{fact } n / \text{exp } 1 :: \text{real})$
<proof>

theorem *derangements-formula'*:

assumes $n \neq 0$ *finite* S $\text{card } S = n$

shows $\text{card } (\text{derangements } S) = \text{nat } (\text{round } (\text{fact } n / \text{exp } 1 :: \text{real}))$
<proof>

end

References

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