Derangements

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Abstract

The Derangements Formula describes the number of fixpoint-free permutations as closed-form formula. This theorem is the 88th theorem of the Top 100 Theorems list.

Contents

1 Derangements

theory *Derangements*
imports
Complex-Main
HOL−Combinatorics.Permutations
begin

1.1 Preliminaries

1.1.1 Additions to *HOL.Finite-Set* Theory

lemma *card-product-dependent*:
assumes *finite* S $\forall x \in S.$ *finite* $(T\ x)$
shows *card* $\{(x, y).\ x \in S \wedge y \in T\ x\} = (\sum x \in S.$ *card* $(T\ x))$
⟨*proof*⟩

1.1.2 Additions to *HOL−Combinatorics.Permutations* Theory

lemma *permutes-imp-bij′*:
assumes *p permutes S*
shows *bij p*
⟨*proof*⟩

lemma *permutesE*:
assumes *p permutes S*
obtains *bij p* $\forall x.\ x \notin S \longrightarrow p\ x = x$
⟨*proof*⟩

lemma *bij-imp-permutes′*:
assumes *bij p* $\forall x.\ x \notin A \longrightarrow p\ x = x$
shows *p permutes A*
⟨*proof*⟩

lemma *permutes-swap*:
assumes *p permutes S*
shows *Fun.swap x y p permutes* (*insert x* (*insert y S*))
⟨*proof*⟩

lemma *bij-extends*:
bij p $\Longrightarrow p\ x = x \Longrightarrow$ *bij* (*p*(*x* := *y*, *inv p y* := *x*))
⟨*proof*⟩

lemma *permutes-add-one*:
assumes *p permutes S* $x \notin S$ $y \in S$
shows *p*(*x* := *y*, *inv p y* := *x*) *permutes* (*insert x S*)
⟨*proof*⟩

lemma *permutations-skip-one*:
assumes *p permutes S x* : *S*
shows *p*(*x* := *x*, *inv p x* := *p x*) *permutes* ($S - \{x\}$)
⟨*proof*⟩

lemma *permutes-drop-cycle-size-two*:
‹*p* ∘ *Transposition.transpose x* (*p x*) *permutes* ($S - \{x, p\ x\}$)›
if ‹*p permutes S*› ‹*p* (*p x*) = *x*›
⟨*proof*⟩

1.2 Fixpoint-Free Permutations

definition *derangements* :: *nat set* ⇒ (*nat* ⇒ *nat*) *set*
where
derangements $S = \{p.\ p$ *permutes* $S \wedge (\forall x \in S.\ p\ x \neq x)\}$

lemma *derangementsI*:
assumes *p permutes S* $\bigwedge x.\ x \in S \Longrightarrow p\ x \neq x$
shows $p \in$ *derangements S*

⟨*proof*⟩

lemma *derangementsE*:
assumes d : *derangements S*
obtains d *permutes* S $\forall x{\in}S.\ d\ x \neq x$
⟨*proof*⟩

1.3 Properties of Derangements

lemma *derangements-inv*:
assumes d: $d \in$ *derangements S*
shows *inv* $d \in$ *derangements S*
⟨*proof*⟩

lemma *derangements-in-image*:
assumes $d \in$ *derangements A* $x \in A$
shows $d\ x \in A$
⟨*proof*⟩

lemma *derangements-in-image-strong*:
assumes $d \in$ *derangements A* $x \in A$
shows $d\ x \in A - \{x\}$
⟨*proof*⟩

lemma *derangements-inverse-in-image*:
assumes $d \in$ *derangements A* $x \in A$
shows *inv* $d\ x \in A$
⟨*proof*⟩

lemma *derangements-fixpoint*:
assumes $d \in$ *derangements A* $x \notin A$
shows $d\ x = x$
⟨*proof*⟩

lemma *derangements-no-fixpoint*:
assumes $d \in$ *derangements A* $x \in A$
shows $d\ x \neq x$
⟨*proof*⟩

lemma *finite-derangements*:
assumes *finite A*
shows *finite* (*derangements A*)
⟨*proof*⟩

1.4 Construction of Derangements

lemma *derangements-empty*[*simp*]:
derangements $\{\} = \{id\}$
⟨*proof*⟩

lemma *derangements-singleton*[*simp*]:
  derangements $\{x\} = \{\}$
⟨*proof*⟩

lemma *derangements-swap*:
  assumes $d \in$ *derangements* S $x \notin S$ $y \notin S$ $x \neq y$
  shows *Fun.swap* x y $d \in$ *derangements* (*insert* x (*insert* y S))
⟨*proof*⟩

lemma *derangements-skip-one*:
  assumes d: $d \in$ *derangements* S **and** $x \in S$ d $(d\ x) \neq x$
  shows $d(x := x,$ *inv* $d\ x := d\ x) \in$ *derangements* $(S - \{x\})$
⟨*proof*⟩

lemma *derangements-add-one*:
  assumes $d \in$ *derangements* S $x \notin S$ $y \in S$
  shows $d(x := y,$ *inv* $d\ y := x) \in$ *derangements* (*insert* x S)
⟨*proof*⟩

lemma *derangements-drop-minimal-cycle*:
  assumes $d \in$ *derangements* S d $(d\ x) = x$
  shows *Fun.swap* x $(d\ x)$ $d \in$ *derangements* $(S - \{x, d\ x\})$
⟨*proof*⟩

1.5 Cardinality of Derangements

1.5.1 Recursive Characterization

fun *count-derangements* :: *nat* ⇒ *nat*
where
  count-derangements *0* = *1*
| *count-derangements* (*Suc* *0*) = *0*
| *count-derangements* (*Suc* (*Suc* *n*)) = (*n* + *1*) ∗ (*count-derangements* (*Suc* *n*) +
count-derangements *n*)

lemma *card-derangements*:
  assumes *finite* S *card* $S = n$
  shows *card* (*derangements* S) = *count-derangements* n
⟨*proof*⟩

1.5.2 Closed-Form Characterization

lemma *count-derangements*:
  real (*count-derangements* *n*) = *fact* *n* ∗ ($\sum k \in \{0..n\}.\ (-1)$ ^ k / *fact* k)
⟨*proof*⟩

1.5.3 Approximation of Cardinality

lemma *two-power-fact-le-fact*:
  assumes $n \geq 1$

shows $2\hat{\ }k * fact\ n \leq (fact\ (n + k) :: 'a :: \{semiring\text{-}char\text{-}0, linordered\text{-}semidom\})$
⟨*proof*⟩

lemma *exp1-approx*:
assumes $n > 0$
shows $exp\ (1::real) - (\sum k<n.\ 1\ /\ fact\ k) \in \{0..2\ /\ fact\ n\}$
⟨*proof*⟩

lemma *exp1-bounds*: $exp\ 1 \in \{8\ /\ 3..11\ /\ 4 :: real\}$
⟨*proof*⟩

lemma *count-derangements-approximation*:
assumes $n \neq 0$
shows $abs(real\ (count\text{-}derangements\ n) - fact\ n\ /\ exp\ 1) < 1\ /\ 2$
⟨*proof*⟩

theorem *derangements-formula*:
assumes $n \neq 0$ *finite S card S* $= n$
shows $int\ (card\ (derangements\ S)) = round\ (fact\ n\ /\ exp\ 1 :: real)$
⟨*proof*⟩

theorem *derangements-formula'*:
assumes $n \neq 0$ *finite S card S* $= n$
shows $card\ (derangements\ S) = nat\ (round\ (fact\ n\ /\ exp\ 1 :: real))$
⟨*proof*⟩

end

References

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