

Derangements

Lukas Bulwahn

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Abstract

The Derangements Formula describes the number of fixpoint-free permutations as closed-form formula. This theorem is the 88th theorem of the Top 100 Theorems list.

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1 Derangements

```
theory Derangements
imports
  Complex-Main
  HOL-Library.Permutations
begin
```

1.1 Preliminaries

1.1.1 Additions to *Finite-Set* Theory

```
lemma card-product-dependent:
  assumes finite S  $\forall x \in S. \text{finite } (T x)$ 
  shows card  $\{(x, y). x \in S \wedge y \in T x\} = (\sum x \in S. \text{card } (T x))$ 
<proof>
```

1.1.2 Additions to *Permutations Theory*

lemma *permutes-imp-bij'*:

assumes p permutes S

shows $\text{bij } p$

<proof>

lemma *permutesE*:

assumes p permutes S

obtains $\text{bij } p \ \forall x. x \notin S \longrightarrow p\ x = x$

<proof>

lemma *bij-imp-permutes'*:

assumes $\text{bij } p \ \forall x. x \notin A \longrightarrow p\ x = x$

shows p permutes A

<proof>

lemma *permutes-swap*:

assumes p permutes S

shows $\text{Fun.swap } x\ y\ p$ permutes $(\text{insert } x\ (\text{insert } y\ S))$

<proof>

lemma *bij-extends*:

$\text{bij } p \implies p\ x = x \implies \text{bij } (p(x := y, \text{inv } p\ y := x))$

<proof>

lemma *permutes-add-one*:

assumes p permutes S $x \notin S$ $y \in S$

shows $p(x := y, \text{inv } p\ y := x)$ permutes $(\text{insert } x\ S)$

<proof>

lemma *permutations-skip-one*:

assumes p permutes S $x : S$

shows $p(x := x, \text{inv } p\ x := p\ x)$ permutes $(S - \{x\})$

<proof>

lemma *permutes-drop-cycle-size-two*:

assumes p permutes S $p\ (p\ x) = x$

shows $\text{Fun.swap } x\ (p\ x)\ p$ permutes $(S - \{x, p\ x\})$

<proof>

1.2 Fixpoint-Free Permutations

definition *derangements* :: $\text{nat set} \Rightarrow (\text{nat} \Rightarrow \text{nat}) \text{ set}$

where

$\text{derangements } S = \{p. p \text{ permutes } S \wedge (\forall x \in S. p\ x \neq x)\}$

lemma *derangementsI*:

assumes p permutes $S \ \bigwedge x. x \in S \implies p\ x \neq x$

shows $p \in \text{derangements } S$

<proof>

lemma *derangementsE*:

assumes $d : \text{derangements } S$

obtains $d \text{ permutes } S \ \forall x \in S. d\ x \neq x$

<proof>

1.3 Properties of Derangements

lemma *derangements-inv*:

assumes $d: d \in \text{derangements } S$

shows $\text{inv } d \in \text{derangements } S$

<proof>

lemma *derangements-in-image*:

assumes $d \in \text{derangements } A \ x \in A$

shows $d\ x \in A$

<proof>

lemma *derangements-in-image-strong*:

assumes $d \in \text{derangements } A \ x \in A$

shows $d\ x \in A - \{x\}$

<proof>

lemma *derangements-inverse-in-image*:

assumes $d \in \text{derangements } A \ x \in A$

shows $\text{inv } d\ x \in A$

<proof>

lemma *derangements-fixpoint*:

assumes $d \in \text{derangements } A \ x \notin A$

shows $d\ x = x$

<proof>

lemma *derangements-no-fixpoint*:

assumes $d \in \text{derangements } A \ x \in A$

shows $d\ x \neq x$

<proof>

lemma *finite-derangements*:

assumes *finite* A

shows *finite* $(\text{derangements } A)$

<proof>

1.4 Construction of Derangements

lemma *derangements-empty[simp]*:

derangements $\{\} = \{\text{id}\}$

<proof>

lemma *derangements-singleton*[simp]:

derangements { x } = {}
<proof>

lemma *derangements-swap*:

assumes $d \in \text{derangements } S \ x \notin S \ y \notin S \ x \neq y$
shows $\text{Fun.swap } x \ y \ d \in \text{derangements } (\text{insert } x \ (\text{insert } y \ S))$
<proof>

lemma *derangements-skip-one*:

assumes $d: d \in \text{derangements } S \ \text{and } x \in S \ d \ (d \ x) \neq x$
shows $d(x := x, \text{inv } d \ x := d \ x) \in \text{derangements } (S - \{x\})$
<proof>

lemma *derangements-add-one*:

assumes $d \in \text{derangements } S \ x \notin S \ y \in S$
shows $d(x := y, \text{inv } d \ y := x) \in \text{derangements } (\text{insert } x \ S)$
<proof>

lemma *derangements-drop-minimal-cycle*:

assumes $d \in \text{derangements } S \ d \ (d \ x) = x$
shows $\text{Fun.swap } x \ (d \ x) \ d \in \text{derangements } (S - \{x, d \ x\})$
<proof>

1.5 Cardinality of Derangements

1.5.1 Recursive Characterization

fun *count-derangements* :: $\text{nat} \Rightarrow \text{nat}$

where

count-derangements 0 = 1
| *count-derangements* (Suc 0) = 0
| *count-derangements* (Suc (Suc n)) = (n + 1) * (*count-derangements* (Suc n) +
count-derangements n)

lemma *card-derangements*:

assumes *finite* $S \ \text{card } S = n$
shows $\text{card } (\text{derangements } S) = \text{count-derangements } n$
<proof>

1.5.2 Closed-Form Characterization

lemma *count-derangements*:

$\text{real } (\text{count-derangements } n) = \text{fact } n * (\sum k \in \{0..n\}. (-1) ^ k / \text{fact } k)$
<proof>

1.5.3 Approximation of Cardinality

lemma *two-power-fact-le-fact*:

assumes $n \geq 1$

shows $2^k * \text{fact } n \leq (\text{fact } (n + k) :: 'a :: \{\text{semiring-char-0, linordered-semidom}\})$
<proof>

lemma *exp1-approx*:

assumes $n > 0$

shows $\text{exp } (1 :: \text{real}) - (\sum_{k < n}. 1 / \text{fact } k) \in \{0..2 / \text{fact } n\}$
<proof>

lemma *exp1-bounds*: $\text{exp } 1 \in \{8 / 3..11 / 4 :: \text{real}\}$

<proof>

lemma *count-derangements-approximation*:

assumes $n \neq 0$

shows $\text{abs}(\text{real } (\text{count-derangements } n) - \text{fact } n / \text{exp } 1) < 1 / 2$
<proof>

theorem *derangements-formula*:

assumes $n \neq 0$ *finite* S $\text{card } S = n$

shows $\text{int } (\text{card } (\text{derangements } S)) = \text{round } (\text{fact } n / \text{exp } 1 :: \text{real})$
<proof>

theorem *derangements-formula'*:

assumes $n \neq 0$ *finite* S $\text{card } S = n$

shows $\text{card } (\text{derangements } S) = \text{nat } (\text{round } (\text{fact } n / \text{exp } 1 :: \text{real}))$
<proof>

end

References

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