Constructing the Reals as Dedekind Cuts of Rationals

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Abstract

The type of real numbers is constructed from the positive rationals using the method of Dedekind cuts. This development, briefly described in papers by the authors [1, 2], follows the textbook presentation by Gleason [3]. It's notable that the first formalisation of a significant piece of mathematics, by Jutting [4] in 1977, involved a similar construction.

Contents

Remark. This development was part of the Isabelle distribution from about 1999 to 2022. It has been transferred to the AFP, where it may be more useful.

1 The Reals as Dedekind Sections of Positive Rationals

Fundamentals of Abstract Analysis [Gleason, p. 121] provides some of the definitions.

theory *Dedekind-Real*
imports *Complex-Main*
begin

lemma *add-eq-exists*: $\exists\, x.\ a+x = (b::'a::ab\text{-}group\text{-}add)$
⟨*proof*⟩

1.1 Dedekind cuts or sections

definition
$cut :: rat\ set \Rightarrow bool$ **where**
$cut\ A \equiv \{\} \subset A \land A \subset \{0<..\} \land$
$(\forall\, y \in A.\ ((\forall\, z.\ 0<z \land z < y \longrightarrow z \in A) \land (\exists\, u \in A.\ y < u)))$

lemma *cut-of-rat*:
assumes q: $0 < q$ **shows** $cut\ \{r::rat.\ 0 < r \land r < q\}$ (**is** $cut\ ?A$)
⟨*proof*⟩

typedef $preal = Collect\ cut$
⟨*proof*⟩

lemma *Abs-preal-induct* [*induct type*: *preal*]:
$(\bigwedge x.\ cut\ x \Longrightarrow P\ (Abs\text{-}preal\ x)) \Longrightarrow P\ x$
⟨*proof*⟩

lemma *cut-Rep-preal* [*simp*]: $cut\ (Rep\text{-}preal\ x)$
⟨*proof*⟩

definition
$psup :: preal\ set \Rightarrow preal$ **where**
$psup\ P = Abs\text{-}preal\ (\bigcup X \in P.\ Rep\text{-}preal\ X)$

definition
$add\text{-}set :: [rat\ set, rat\ set] \Rightarrow rat\ set$ **where**
$add\text{-}set\ A\ B = \{w.\ \exists\, x \in A.\ \exists\, y \in B.\ w = x + y\}$

definition
$diff\text{-}set :: [rat\ set, rat\ set] \Rightarrow rat\ set$ **where**
$diff\text{-}set\ A\ B = \{w.\ \exists\, x.\ 0 < w \land 0 < x \land x \notin B \land x + w \in A\}$

definition
$mult\text{-}set :: [rat\ set, rat\ set] \Rightarrow rat\ set$ **where**
$mult\text{-}set\ A\ B = \{w.\ \exists\, x \in A.\ \exists\, y \in B.\ w = x * y\}$

definition
inverse-set :: *rat set* ⇒ *rat set* **where**
inverse-set A ≡ {*x*. ∃ *y*. *0* < *x* ∧ *x* < *y* ∧ *inverse y* ∉ *A*}

instantiation *preal* :: {*ord*, *plus*, *minus*, *times*, *inverse*, *one*}
begin

definition
preal-less-def:
$r < s \equiv$ *Rep-preal r* < *Rep-preal s*

definition
preal-le-def:
$r \leq s \equiv$ *Rep-preal r* ⊆ *Rep-preal s*

definition
preal-add-def:
$r + s \equiv$ *Abs-preal* (*add-set* (*Rep-preal r*) (*Rep-preal s*))

definition
preal-diff-def:
$r - s \equiv$ *Abs-preal* (*diff-set* (*Rep-preal r*) (*Rep-preal s*))

definition
preal-mult-def:
$r * s \equiv$ *Abs-preal* (*mult-set* (*Rep-preal r*) (*Rep-preal s*))

definition
preal-inverse-def:
inverse r ≡ *Abs-preal* (*inverse-set* (*Rep-preal r*))

definition *r div s* = *r* ∗ *inverse* (*s*::*preal*)

definition
preal-one-def:
1 ≡ *Abs-preal* {*x*. *0* < *x* ∧ *x* < *1*}

instance ⟨*proof*⟩

end

Reduces equality on abstractions to equality on representatives

declare *Abs-preal-inject* [*simp*]
declare *Abs-preal-inverse* [*simp*]

lemma *rat-mem-preal*: *0* < *q* ⟹ *cut* {*r*::*rat*. *0* < *r* ∧ *r* < *q*}
⟨*proof*⟩

lemma *preal-nonempty*: *cut A* $\Longrightarrow \exists x{\in}A.\ 0 < x$
⟨*proof*⟩

lemma *preal-Ex-mem*: *cut A* $\Longrightarrow \exists x.\ x \in A$
⟨*proof*⟩

lemma *preal-exists-bound*: *cut A* $\Longrightarrow \exists x.\ 0 < x \wedge x \notin A$
⟨*proof*⟩

lemma *preal-exists-greater*: $[\![$*cut A*; $y \in A]\!] \Longrightarrow \exists u \in A.\ y < u$
⟨*proof*⟩

lemma *preal-downwards-closed*: $[\![$*cut A*; $y \in A$; $0 < z$; $z < y]\!] \Longrightarrow z \in A$
⟨*proof*⟩

Relaxing the final premise

lemma *preal-downwards-closed'*: $[\![$*cut A*; $y \in A$; $0 < z$; $z \le y]\!] \Longrightarrow z \in A$
⟨*proof*⟩

A positive fraction not in a positive real is an upper bound. Gleason p. 122 - Remark (1)

lemma *not-in-preal-ub*:
assumes *A*: *cut A*
and *notx*: $x \notin A$
and *y*: $y \in A$
and *pos*: $0 < x$
shows $y < x$
⟨*proof*⟩

preal lemmas instantiated to *Rep-preal X*

lemma *mem-Rep-preal-Ex*: $\exists x.\ x \in$ *Rep-preal X*
thm *preal-Ex-mem*
⟨*proof*⟩

lemma *Rep-preal-exists-bound*: $\exists x{>}0.\ x \notin$ *Rep-preal X*
⟨*proof*⟩

lemmas *not-in-Rep-preal-ub* = *not-in-preal-ub* [*OF cut-Rep-preal*]

1.2 Properties of Ordering

instance *preal* :: *order*
⟨*proof*⟩

lemma *preal-imp-pos*: $[\![$*cut A*; $r \in A]\!] \Longrightarrow 0 < r$
⟨*proof*⟩

instance *preal* :: *linorder*
⟨*proof*⟩

instantiation *preal* :: *distrib-lattice*
begin

definition
  $(inf :: preal \Rightarrow preal \Rightarrow preal) = min$

definition
  $(sup :: preal \Rightarrow preal \Rightarrow preal) = max$

instance
  ⟨*proof*⟩

end

1.3 Properties of Addition

lemma *preal-add-commute*: $(x::preal) + y = y + x$
  ⟨*proof*⟩

Lemmas for proving that addition of two positive reals gives a positive real

lemma *mem-add-set*:
  assumes *cut A cut B*
  shows *cut* (*add-set A B*)
⟨*proof*⟩

lemma *preal-add-assoc*: $((x::preal) + y) + z = x + (y + z)$
  ⟨*proof*⟩

instance *preal* :: *ab-semigroup-add*
⟨*proof*⟩

1.4 Properties of Multiplication

Proofs essentially same as for addition

lemma *preal-mult-commute*: $(x::preal) * y = y * x$
  ⟨*proof*⟩

Multiplication of two positive reals gives a positive real.

lemma *mem-mult-set*:
  assumes *cut A cut B*
  shows *cut* (*mult-set A B*)
⟨*proof*⟩

lemma *preal-mult-assoc*: $((x::preal) * y) * z = x * (y * z)$
  ⟨*proof*⟩

instance *preal* :: *ab-semigroup-mult*

⟨*proof*⟩

Positive real 1 is the multiplicative identity element

lemma *preal-mult-1*: $(1{::}preal) * z = z$
⟨*proof*⟩

instance *preal* :: *comm-monoid-mult*
⟨*proof*⟩

1.5 Distribution of Multiplication across Addition

lemma *mem-Rep-preal-add-iff*:
$(z \in Rep\text{-}preal(r{+}s)) = (\exists\, x \in Rep\text{-}preal\ r.\ \exists\, y \in Rep\text{-}preal\ s.\ z = x + y)$
⟨*proof*⟩

lemma *mem-Rep-preal-mult-iff*:
$(z \in Rep\text{-}preal(r{*}s)) = (\exists\, x \in Rep\text{-}preal\ r.\ \exists\, y \in Rep\text{-}preal\ s.\ z = x * y)$
⟨*proof*⟩

lemma *distrib-subset1*:
$Rep\text{-}preal\ (w * (x + y)) \subseteq Rep\text{-}preal\ (w * x + w * y)$
⟨*proof*⟩

lemma *preal-add-mult-distrib-mean*:
assumes *a*: $a \in Rep\text{-}preal\ w$
and *b*: $b \in Rep\text{-}preal\ w$
and *d*: $d \in Rep\text{-}preal\ x$
and *e*: $e \in Rep\text{-}preal\ y$
shows $\exists\, c \in Rep\text{-}preal\ w.\ a * d + b * e = c * (d + e)$
⟨*proof*⟩

lemma *distrib-subset2*:
$Rep\text{-}preal\ (w * x + w * y) \subseteq Rep\text{-}preal\ (w * (x + y))$
⟨*proof*⟩

lemma *preal-add-mult-distrib2*: $(w * ((x{::}preal) + y)) = (w * x) + (w * y)$
⟨*proof*⟩

lemma *preal-add-mult-distrib*: $(((x{::}preal) + y) * w) = (x * w) + (y * w)$
⟨*proof*⟩

instance *preal* :: *comm-semiring*
⟨*proof*⟩

1.6 Existence of Inverse, a Positive Real

lemma *mem-inverse-set*:
assumes *cut A* **shows** *cut* (*inverse-set A*)
⟨*proof*⟩

1.7 Gleason's Lemma 9-3.4, page 122

lemma *Gleason9-34-exists*:
  assumes A: *cut* A
    and $\forall x{\in}A.\ x + u \in A$
    and $0 \leq z$
  shows $\exists b{\in}A.\ b + (\textit{of-int}\ z) * u \in A$
$\langle proof \rangle$

lemma *Gleason9-34-contra*:
  assumes A: *cut* A
    shows $[\![\forall x{\in}A.\ x + u \in A;\ 0 < u;\ 0 < y;\ y \notin A]\!] \Longrightarrow \textit{False}$
$\langle proof \rangle$

lemma *Gleason9-34*:
  assumes *cut* A $0 < u$
  shows $\exists r \in A.\ r + u \notin A$
  $\langle proof \rangle$

1.8 Gleason's Lemma 9-3.6

lemma *lemma-gleason9-36*:
  assumes A: *cut* A
    and x: $1 < x$
  shows $\exists r \in A.\ r{*}x \notin A$
$\langle proof \rangle$

1.9 Existence of Inverse: Part 2

lemma *mem-Rep-preal-inverse-iff*:
  $(z \in \textit{Rep-preal}(\textit{inverse}\ r)) \longleftrightarrow (0 < z \wedge (\exists y.\ z < y \wedge \textit{inverse}\ y \notin \textit{Rep-preal}\ r))$
  $\langle proof \rangle$

lemma *Rep-preal-one*:
    $\textit{Rep-preal}\ 1 = \{x.\ 0 < x \wedge x < 1\}$
$\langle proof \rangle$

lemma *subset-inverse-mult-lemma*:
  assumes *xpos*: $0 < x$ **and** *xless*: $x < 1$
  shows $\exists v\ u\ y.\ 0 < v \wedge v < y \wedge \textit{inverse}\ y \notin \textit{Rep-preal}\ R\ \wedge$
    $u \in \textit{Rep-preal}\ R \wedge x = v * u$
$\langle proof \rangle$

lemma *subset-inverse-mult*:
    $\textit{Rep-preal}\ 1 \subseteq \textit{Rep-preal}(\textit{inverse}\ r * r)$
  $\langle proof \rangle$

lemma *inverse-mult-subset*: $\textit{Rep-preal}(\textit{inverse}\ r * r) \subseteq \textit{Rep-preal}\ 1$
  $\langle proof \rangle$

lemma *preal-mult-inverse*: *inverse* $r * r = (1::preal)$
⟨*proof*⟩

lemma *preal-mult-inverse-right*: $r *$ *inverse* $r = (1::preal)$
⟨*proof*⟩

Theorems needing *Gleason9-34*

lemma *Rep-preal-self-subset*: *Rep-preal* $(r) \subseteq$ *Rep-preal*$(r + s)$
⟨*proof*⟩

lemma *Rep-preal-sum-not-subset*: ~ *Rep-preal* $(r + s) \subseteq$ *Rep-preal*(r)
⟨*proof*⟩

at last, Gleason prop. 9-3.5(iii) page 123

proposition *preal-self-less-add-left*: $(r::preal) < r + s$
⟨*proof*⟩

1.10 Subtraction for Positive Reals

gleason prop. 9-3.5(iv), page 123: proving $a < b \Longrightarrow \exists\, d.\ a + d = b$. We define the claimed D and show that it is a positive real

lemma *mem-diff-set*:
assumes $r < s$
shows *cut* (*diff-set* (*Rep-preal* s) (*Rep-preal* r))
⟨*proof*⟩

lemma *mem-Rep-preal-diff-iff*:
$r < s \Longrightarrow$
$(z \in$ *Rep-preal* $(s - r)) \longleftrightarrow$
$(\exists\, x.\ 0 < x \wedge 0 < z \wedge x \notin$ *Rep-preal* $r \wedge x + z \in$ *Rep-preal* $s)$
⟨*proof*⟩

proposition *less-add-left*:
fixes $r::preal$
assumes $r < s$
shows $r + (s{-}r) = s$
⟨*proof*⟩

lemma *preal-add-less2-mono1*: $r < (s::preal) \Longrightarrow r + t < s + t$
⟨*proof*⟩

lemma *preal-add-less2-mono2*: $r < (s::preal) \Longrightarrow t + r < t + s$
⟨*proof*⟩

lemma *preal-add-right-less-cancel*: $r + t < s + t \Longrightarrow r < (s::preal)$
⟨*proof*⟩

lemma *preal-add-left-less-cancel*: $t + r < t + s \Longrightarrow r < (s::preal)$

⟨*proof*⟩

lemma *preal-add-less-cancel-left* [*simp*]: $(t + (r::preal) < t + s) \longleftrightarrow (r < s)$
⟨*proof*⟩

lemma *preal-add-less-cancel-right* [*simp*]: $((r::preal) + t < s + t) = (r < s)$
⟨*proof*⟩

lemma *preal-add-le-cancel-left* [*simp*]: $(t + (r::preal) \leq t + s) = (r \leq s)$
⟨*proof*⟩

lemma *preal-add-le-cancel-right* [*simp*]: $((r::preal) + t \leq s + t) = (r \leq s)$
⟨*proof*⟩

lemma *preal-add-right-cancel*: $(r::preal) + t = s + t \Longrightarrow r = s$
⟨*proof*⟩

lemma *preal-add-left-cancel*: $c + a = c + b \Longrightarrow a = (b::preal)$
⟨*proof*⟩

instance *preal* :: *linordered-ab-semigroup-add*
⟨*proof*⟩

1.11 Completeness of type *preal*

Prove that supremum is a cut

Part 1 of Dedekind sections definition

lemma *preal-sup*:
assumes *le*: $\bigwedge X.\ X \in P \Longrightarrow X \leq Y$ **and** $P \neq \{\}$
shows *cut* $(\bigcup X \in P.\ Rep\text{-}preal(X))$
⟨*proof*⟩

lemma *preal-psup-le*:
$[\![\bigwedge X.\ X \in P \Longrightarrow X \leq Y;\ \ x \in P]\!] \Longrightarrow x \leq psup\ P$
⟨*proof*⟩

lemma *psup-le-ub*: $[\![\bigwedge X.\ X \in P \Longrightarrow X \leq Y;\ P \neq \{\}]\!] \Longrightarrow psup\ P \leq Y$
⟨*proof*⟩

Supremum property

proposition *preal-complete*:
assumes *le*: $\bigwedge X.\ X \in P \Longrightarrow X \leq Y$ **and** $P \neq \{\}$
shows $(\exists X \in P.\ Z < X) \longleftrightarrow (Z < psup\ P)$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

1.12 Defining the Reals from the Positive Reals

Here we do quotients the old-fashioned way

definition
realrel :: $((preal * preal) * (preal * preal))$ *set* **where**
$realrel = \{p.\ \exists x1\ y1\ x2\ y2.\ p = ((x1,y1),(x2,y2)) \wedge x1+y2 = x2+y1\}$

definition $Real = UNIV//realrel$

typedef $real = Real$
morphisms *Rep-Real Abs-Real*
⟨*proof*⟩

This doesn't involve the overloaded "real" function: users don't see it

definition
real-of-preal :: $preal \Rightarrow real$ **where**
$\textit{real-of-preal}\ m = \textit{Abs-Real}\ (realrel\ \text{``}\ \{(m + 1,\ 1)\})$

instantiation *real* :: $\{zero,\ one,\ plus,\ minus,\ uminus,\ times,\ inverse,\ ord,\ abs,\ sgn\}$
begin

definition
real-zero-def: $0 = \textit{Abs-Real}(realrel\text{``}\{(1,\ 1)\})$

definition
real-one-def: $1 = \textit{Abs-Real}(realrel\text{``}\{(1 + 1,\ 1)\})$

definition
real-add-def: $z + w =$
$\textit{the-elem}\ (\bigcup (x,y) \in \textit{Rep-Real}\ z.\ \bigcup (u,v) \in \textit{Rep-Real}\ w.$
$\{\ \textit{Abs-Real}(realrel\text{``}\{(x+u,\ y+v)\})\ \})$

definition
real-minus-def: $-\ r = \textit{the-elem}\ (\bigcup (x,y) \in \textit{Rep-Real}\ r.\ \{\ \textit{Abs-Real}(realrel\text{``}\{(y,x)\})\ \})$

definition
real-diff-def: $r - (s::real) = r + -\ s$

definition
real-mult-def:
$z * w =$
$\textit{the-elem}\ (\bigcup (x,y) \in \textit{Rep-Real}\ z.\ \bigcup (u,v) \in \textit{Rep-Real}\ w.$
$\{\ \textit{Abs-Real}(realrel\text{``}\{(x*u + y*v,\ x*v + y*u)\})\ \})$

definition
real-inverse-def: $inverse\ (r::real) \equiv (THE\ s.\ (r = 0 \wedge s = 0) \vee s * r = 1)$

definition
real-divide-def: $r\ div\ (s::real) \equiv r * inverse\ s$

definition
real-le-def: $z \le (w::real) \equiv$
$(\exists x\ y\ u\ v.\ x{+}v \le u{+}y \land (x,y) \in \textit{Rep-Real}\ z \land (u,v) \in \textit{Rep-Real}\ w)$

definition
real-less-def: $x < (y::real) \equiv x \le y \land x \ne y$

definition
real-abs-def: $|r::real| = (\textit{if}\ r < 0\ \textit{then}\ -\ r\ \textit{else}\ r)$

definition
real-sgn-def: $\textit{sgn}\ (x::real) = (\textit{if}\ x{=}0\ \textit{then}\ 0\ \textit{else if}\ 0{<}x\ \textit{then}\ 1\ \textit{else}\ -\ 1)$

instance ⟨*proof*⟩

end

1.13 Equivalence relation over positive reals

lemma *realrel-iff* [*simp*]: $(((x1,y1),(x2,y2)) \in realrel) = (x1 + y2 = x2 + y1)$
⟨*proof*⟩

lemma *preal-trans-lemma*:
assumes $x + y1 = x1 + y$ **and** $x + y2 = x2 + y$
shows $x1 + y2 = x2 + (y1::preal)$
⟨*proof*⟩

lemma *equiv-realrel*: *equiv UNIV realrel*
⟨*proof*⟩

Reduces equality of equivalence classes to the *Dedekind-Real.realrel* relation: $(\textit{Dedekind-Real.realrel}\ ``\ \{x\} = \textit{Dedekind-Real.realrel}\ ``\ \{y\}) = ((x, y) \in \textit{Dedekind-Real.realrel})$

lemmas *equiv-realrel-iff* [*simp*] =
eq-equiv-class-iff [*OF equiv-realrel UNIV-I UNIV-I*]

lemma *realrel-in-real* [*simp*]: $realrel``\{(x,y)\} \in Real$
⟨*proof*⟩

declare *Abs-Real-inject* [*simp*] *Abs-Real-inverse* [*simp*]

Case analysis on the representation of a real number as an equivalence class of pairs of positive reals.

lemma *eq-Abs-Real* [*case-names Abs-Real*, *cases type*: *real*]:
$(\bigwedge x\ y.\ z = \textit{Abs-Real}(realrel``\{(x,y)\}) \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

1.14 Addition and Subtraction

lemma *real-add*:

$Abs\text{-}Real\ (realrel``\{(x,y)\}) + Abs\text{-}Real\ (realrel``\{(u,v)\}) =$
$Abs\text{-}Real\ (realrel``\{(x+u,\ y+v)\})$
⟨*proof*⟩

lemma *real-minus*: $-\ Abs\text{-}Real(realrel``\{(x,y)\}) = Abs\text{-}Real(realrel\ ``\ \{(y,x)\})$
⟨*proof*⟩

instance *real* :: *ab-group-add*
⟨*proof*⟩

1.15 Multiplication

lemma *real-mult-congruent2-lemma*:
$!!(x1::preal).\ [\![x1 + y2 = x2 + y1]\!] \Longrightarrow$
$x * x1 + y * y1 + (x * y2 + y * x2) =$
$x * x2 + y * y2 + (x * y1 + y * x1)$
⟨*proof*⟩

lemma *real-mult-congruent2*:
$(\lambda p1\ p2.$
$(\lambda(x1,y1).\ (\lambda(x2,y2).$
$\{\ Abs\text{-}Real\ (realrel``\{(x1*x2 + y1*y2,\ x1*y2+y1*x2)\})\ \})\ p2)\ p1)$
$respects2\ realrel$
⟨*proof*⟩

lemma *real-mult*:
$Abs\text{-}Real((realrel``\{(x1,y1)\})) * Abs\text{-}Real((realrel``\{(x2,y2)\})) =$
$Abs\text{-}Real(realrel\ ``\ \{(x1*x2+y1*y2,x1*y2+y1*x2)\})$
⟨*proof*⟩

lemma *real-mult-commute*: $(z::real) * w = w * z$
⟨*proof*⟩

lemma *real-mult-assoc*: $((z1::real) * z2) * z3 = z1 * (z2 * z3)$
⟨*proof*⟩

lemma *real-mult-1*: $(1::real) * z = z$
⟨*proof*⟩

lemma *real-add-mult-distrib*: $((z1::real) + z2) * w = (z1 * w) + (z2 * w)$
⟨*proof*⟩

one and zero are distinct

lemma *real-zero-not-eq-one*: $0 \neq (1::real)$
⟨*proof*⟩

instance *real* :: *comm-ring-1*
⟨*proof*⟩

1.16 Inverse and Division

lemma *real-zero-iff*: *Abs-Real* (*realrel* `` {(*x*, *x*)}) = *0*
⟨*proof*⟩

lemma *real-mult-inverse-left-ex*:
assumes $x \neq 0$ **obtains** *y*::*real* **where** $y*x = 1$
⟨*proof*⟩

lemma *real-mult-inverse-left*:
fixes *x* :: *real*
assumes $x \neq 0$ **shows** *inverse* $x * x = 1$
⟨*proof*⟩

1.17 The Real Numbers form a Field

instance *real* :: *field*
⟨*proof*⟩

1.18 The ≤ Ordering

lemma *real-le-refl*: $w \leq (w$::*real*)
⟨*proof*⟩

The arithmetic decision procedure is not set up for type preal. This lemma is currently unused, but it could simplify the proofs of the following two lemmas.

lemma *preal-eq-le-imp-le*:
assumes *eq*: $a+b = c+d$ **and** *le*: $c \leq a$
shows $b \leq (d$::*preal*)
⟨*proof*⟩

lemma *real-le-lemma*:
assumes *l*: $u1 + v2 \leq u2 + v1$
and $x1 + v1 = u1 + y1$
and $x2 + v2 = u2 + y2$
shows $x1 + y2 \leq x2 + (y1$::*preal*)
⟨*proof*⟩

lemma *real-le*:
Abs-Real(*realrel*``{(*x1*,*y1*)}) ≤ *Abs-Real*(*realrel*``{(*x2*,*y2*)}) ⟷ $x1 + y2 \leq x2 + y1$
⟨*proof*⟩

lemma *real-le-antisym*: ⟦$z \leq w$; $w \leq z$⟧ ⟹ $z = (w$::*real*)
⟨*proof*⟩

lemma *real-trans-lemma*:
assumes $x + v \leq u + y$

  and $u + v' \le u' + v$
  and $x2 + v2 = u2 + y2$
shows $x + v' \le u' + (y::preal)$
⟨*proof*⟩

lemma *real-le-trans*: $[\![i \le j; j \le k]\!] \Longrightarrow i \le (k::real)$
⟨*proof*⟩

instance *real* :: *order*
⟨*proof*⟩

instance *real* :: *linorder*
⟨*proof*⟩

instantiation *real* :: *distrib-lattice*
begin

definition
$(inf :: real \Rightarrow real \Rightarrow real) = min$

definition
$(sup :: real \Rightarrow real \Rightarrow real) = max$

instance
⟨*proof*⟩

end

1.19 The Reals Form an Ordered Field

lemma *real-le-eq-diff*: $(x \le y) \longleftrightarrow (x-y \le (0::real))$
⟨*proof*⟩

lemma *real-add-left-mono*:
assumes *le*: $x \le y$ **shows** $z + x \le z + (y::real)$
⟨*proof*⟩

lemma *real-sum-gt-zero-less*: $(0 < s + (-w::real)) \Longrightarrow (w < s)$
⟨*proof*⟩

lemma *real-less-sum-gt-zero*: $(w < s) \Longrightarrow (0 < s + (-w::real))$
⟨*proof*⟩

lemma *real-mult-order*:
fixes $x\ y::real$
assumes $0 < x\ 0 < y$
shows $0 < x * y$
⟨*proof*⟩

lemma *real-mult-less-mono2*: $[\![(0::real) < z; x < y]\!] \Longrightarrow z * x < z * y$
⟨*proof*⟩

instance *real* :: *linordered-field*
⟨*proof*⟩

1.20 Completeness of the reals

The function *real-of-preal* requires many proofs, but it seems to be essential for proving completeness of the reals from that of the positive reals.

lemma *real-of-preal-add*:
real-of-preal $((x::preal) + y) =$ *real-of-preal* x + *real-of-preal* y
⟨*proof*⟩

lemma *real-of-preal-mult*:
real-of-preal $((x::preal) * y) =$ *real-of-preal* x * *real-of-preal* y
⟨*proof*⟩

Gleason prop 9-4.4 p 127

lemma *real-of-preal-trichotomy*:
$\exists m. (x::real) =$ *real-of-preal* $m \lor x = 0 \lor x = -($*real-of-preal* $m)$
⟨*proof*⟩

lemma *real-of-preal-less-iff* [*simp*]:
(*real-of-preal* $m1 <$ *real-of-preal* $m2) = (m1 < m2)$
⟨*proof*⟩

lemma *real-of-preal-le-iff* [*simp*]:
(*real-of-preal* $m1 \leq$ *real-of-preal* $m2) = (m1 \leq m2)$
⟨*proof*⟩

lemma *real-of-preal-zero-less* [*simp*]: $0 <$ *real-of-preal* m
⟨*proof*⟩

1.21 Theorems About the Ordering

lemma *real-gt-zero-preal-Ex*: $(0 < x) \longleftrightarrow (\exists y. x =$ *real-of-preal* $y)$
⟨*proof*⟩

1.22 Completeness of Positive Reals

Supremum property for the set of positive reals

Let P be a non-empty set of positive reals, with an upper bound y. Then P has a least upper bound (written S).

FIXME: Can the premise be weakened to $\forall x \in P.\ x \leq y$?

lemma *posreal-complete*:
assumes *positive-P*: $\forall x \in P.\ (0::real) < x$

and *not-empty-P*: $\exists x.\ x \in P$
and *upper-bound-Ex*: $\exists y.\ \forall x \in P.\ x<y$
shows $\exists s.\ \forall y.\ (\exists x \in P.\ y < x) = (y < s)$
⟨*proof*⟩

1.23 Completeness

lemma *reals-complete*:
fixes S :: *real set*
assumes *notempty-S*: $\exists X.\ X \in S$
and *exists-Ub*: *bdd-above* S
shows $\exists x.\ (\forall s{\in}S.\ s \leq x) \wedge (\forall y.\ (\forall s{\in}S.\ s \leq y) \longrightarrow x \leq y)$
⟨*proof*⟩

1.24 The Archimedean Property of the Reals

theorem *reals-Archimedean*:
fixes x :: *real*
assumes *x-pos*: $0 < x$
shows $\exists n.$ *inverse* (*of-nat* (*Suc* n)) $< x$
⟨*proof*⟩

There must be other proofs, e.g. *Suc* of the largest integer in the cut representing x.

lemma *reals-Archimedean2*: $\exists n.$ (x::*real*) $<$ *of-nat* (n::*nat*)
⟨*proof*⟩

instance *real* :: *archimedean-field*
⟨*proof*⟩

end

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