

# Solving Cubic and Quartic Equations\*

René Thiemann

December 14, 2021

## Abstract

We formalize Cardano's formula to solve a cubic equation

$$ax^3 + bx^2 + cx + d = 0,$$

as well as Ferrari's formula to solve a quartic equation [1]. We further turn both formulas into executable algorithms based on the algebraic number implementation in the AFP [2]. To this end we also slightly extended this library, namely by making the minimal polynomial of an algebraic number executable, and by defining and implementing  $n$ -th roots of complex numbers.

## Contents

<b>1</b>	<b>Ferrari's formula for solving quartic equations</b>	<b>2</b>
1.1	Translation to depressed case . . . . .	2
1.2	Solving the depressed case via Ferrari's formula . . . . .	2
<b>2</b>	<b>Cardano's formula for solving cubic equations</b>	<b>4</b>
2.1	Translation to depressed case . . . . .	4
2.2	Solving the depressed case in arbitrary fields . . . . .	4
2.3	Solving the depressed case for complex numbers . . . . .	5
2.4	Solving the depressed case for real numbers . . . . .	7
<b>3</b>	<b><math>n</math>-th roots of complex numbers</b>	<b>11</b>
3.1	An algorithm to compute all complex roots of (algebraic) complex numbers . . . . .	11
3.2	A definition of <i>the</i> complex root of a complex number . . . . .	13
<b>4</b>	<b>Algorithms to compute all complex and real roots of a cubic polynomial</b>	<b>22</b>

---

\*Supported by FWF (Austrian Science Fund) project Y757.

## 5 Algorithms to compute all complex and real roots of a quartic polynomial

27

### 1 Ferrari's formula for solving quartic equations

**theory** *Ferraris-Formula*

**imports**

*Polynomial-Factorization.Explicit-Roots*  
*Polynomial-Interpolation.Ring-Hom-Poly*  
*Complex-Geometry.More-Complex*

**begin**

#### 1.1 Translation to depressed case

Solving an arbitrary quartic equation can easily be turned into the depressed case, i.e., where there is no cubic part.

**lemma** *to-depressed-quartic*: **fixes**  $a_4 :: 'a :: \text{field-char-0}$

**assumes**  $a_4: a_4 \neq 0$

**and**  $b: b = a_3 / a_4$

**and**  $c: c = a_2 / a_4$

**and**  $d: d = a_1 / a_4$

**and**  $e: e = a_0 / a_4$

**and**  $p: p = c - (3/8) * b^2$

**and**  $q: q = (b^3 - 4*b*c + 8 * d) / 8$

**and**  $r: r = (-3 * b^4 + 256 * e - 64 * b * d + 16 * b^2 * c) / 256$

**and**  $x: x = y - b/4$

**shows**  $a_4 * x^4 + a_3 * x^3 + a_2 * x^2 + a_1 * x + a_0 = 0$

$\longleftrightarrow y^4 + p * y^2 + q * y + r = 0$

**proof** -

**have**  $a_4 * x^4 + a_3 * x^3 + a_2 * x^2 + a_1 * x + a_0 = 0 \longleftrightarrow$

$(a_4 * x^4 + a_3 * x^3 + a_2 * x^2 + a_1 * x + a_0) / a_4 = 0$  **using**  $a_4$  **by** *auto*

**also have**  $(a_4 * x^4 + a_3 * x^3 + a_2 * x^2 + a_1 * x + a_0) / a_4$

$= x^4 + b * x^3 + c * x^2 + d * x + e$

**unfolding**  $b c d e$  **using**  $a_4$  **by** (*simp add: field-simps*)

**also have**  $\dots = y^4 + p * y^2 + q * y + r$

**unfolding**  $x p q r$

**by** (*simp add: field-simps power4-eq-xxxx power3-eq-cube power2-eq-square*)

**finally show** *?thesis* .

**qed**

**lemma** *biquadratic-solution*: **fixes**  $p q :: 'a :: \text{field-char-0}$

**shows**  $y^4 + p * y^2 + q = 0 \longleftrightarrow (\exists z. z^2 + p * z + q = 0 \wedge z = y^2)$

**by** (*auto simp: field-simps power4-eq-xxxx power2-eq-square*)

#### 1.2 Solving the depressed case via Ferrari's formula

**lemma** *depressed-quartic-Ferrari*: **fixes**  $p q r :: 'a :: \text{field-char-0}$

```

assumes cubic-root:  $8 * m^3 + (8 * p) * m^2 + (2 * p^2 - 8 * r) * m - q^2 = 0$ 
and q0:  $q \neq 0$  — otherwise m might be zero, so a is zero and then there is a
division by zero in b1 and b2
and sqrt:  $a * a = 2 * m$ 
and b1:  $b1 = p / 2 + m - q / (2 * a)$ 
and b2:  $b2 = p / 2 + m + q / (2 * a)$ 
shows  $y^4 + p * y^2 + q * y + r = 0 \iff poly [ :b1, a, 1:] y = 0 \vee poly [ :b2, -a, 1:] y = 0$ 
proof –
let ?N =  $y^2 + p / 2 + m$ 
let ?M =  $a * y - q / (2 * a)$ 
from cubic-root q0 have m0:  $m \neq 0$  by auto
from sqrt m0 have a0:  $a \neq 0$  by auto
define N where  $N = ?N$ 
define M where  $M = ?M$ 
note powers = field-simps power4-eq-xxxx power3-eq-cube power2-eq-square
from cubic-root have  $8 * m^3 = - (8 * p) * m^2 - (2 * p^2 - 8 * r) * m + q^2$ 
by (simp add: powers)
from arg-cong[OF this, of (*) 4]
have id:  $32 * m^3 = 4 * (- (8 * p) * m^2 - (2 * p^2 - 8 * r) * m + q^2)$  by
simp
let ?add =  $2 * y^2 * m + p * m + m^2$ 
have  $y^4 + p * y^2 + q * y + r = 0 \iff$ 
 $(y^2 + p / 2)^2 = -q * y - r + p^2 / 4$ 
by (simp add: powers, algebra)
also have ...  $\iff (y^2 + p / 2)^2 + ?add = -q * y - r + p^2 / 4 + ?add$ 
by simp
also have ...  $\iff ?N^2 = 2 * m * y^2 - q * y + m^2 + m * p + p^2 / 4 - r$ 
by (simp add: powers)
also have  $2 * m * y^2 - q * y + m^2 + m * p + p^2 / 4 - r =$ 
 $?M^2$  using m0 id a0 sqrt by (simp add: powers, algebra)
also have  $?N^2 = ?M^2 \iff (?N + ?M) * (?N - ?M) = 0$ 
unfolding N-def[symmetric] M-def[symmetric] by algebra
also have ...  $\iff ?N + ?M = 0 \vee ?N - ?M = 0$  by simp
also have  $?N + ?M = y^2 + a * y + b1$ 
by (simp add: b1)
also have  $?N - ?M = y^2 - a * y + b2$ 
by (simp add: b2)
also have  $y^2 + a * y + b1 = 0 \iff poly [ :b1, a, 1:] y = 0$ 
by (simp add: powers)
also have  $y^2 - a * y + b2 = 0 \iff poly [ :b2, -a, 1:] y = 0$ 
by (simp add: powers)
finally show ?thesis .
qed

```

end

## 2 Cardano's formula for solving cubic equations

```

theory Cardanos-Formula
  imports
    Polynomial-Factorization.Explicit-Roots
    Polynomial-Interpolation.Ring-Hom-Poly
    Complex-Geometry.More-Complex
    Algebraic-Numbers.Complex-Roots-Real-Poly
begin

```

### 2.1 Translation to depressed case

Solving an arbitrary cubic equation can easily be turned into the depressed case, i.e., where there is no quadratic part.

**lemma** *to-depressed-cubic*: **fixes**  $a :: 'a :: \text{field-char-0}$

**assumes**  $a \neq 0$

**and**  $xy$ :  $x = y - b / (3 * a)$

**and**  $e$ :  $e = (c - b^2 / (3 * a)) / a$

**and**  $f$ :  $f = (d + 2 * b^3 / (27 * a^2) - b * c / (3 * a)) / a$

**shows**  $(a * x^3 + b * x^2 + c * x + d = 0) \longleftrightarrow y^3 + e * y + f = 0$

**proof** –

**let**  $?yexp = y^3 + e * y + f$

**have**  $a * x^3 + b * x^2 + c * x + d = 0 \longleftrightarrow (a * x^3 + b * x^2 + c * x + d) / a = 0$

**using**  $a$  **by** *auto*

**also have**  $(a * x^3 + b * x^2 + c * x + d) / a = ?yexp$  **unfolding**  $xy$   $e$   $f$  *power3-eq-cube power2-eq-square* **using**  $a$

**by** (*simp add: field-simps*)

**finally show** *?thesis* .

**qed**

### 2.2 Solving the depressed case in arbitrary fields

**lemma** *cubic-depressed*: **fixes**  $e :: 'a :: \text{field-char-0}$

**assumes**  $yz$ :  $e \neq 0 \implies z^2 - y * z - e / 3 = 0$

**and**  $u$ :  $e \neq 0 \implies u = z^3$

**and**  $v$ :  $v = -(e^3 / 27)$

**shows**  $y^3 + e * y + f = 0 \longleftrightarrow (\text{if } e = 0 \text{ then } y^3 = -f \text{ else } u^2 + f * u + v = 0)$

**proof** –

**let**  $?yexp = y^3 + e * y + f$

**show** *?thesis*

**proof** (*cases e = 0*)

**case** *False*

**note**  $yz = yz[OF \text{ False}]$

**from**  $yz$  **have**  $eyz$ :  $e = 3 * (z^2 - y * z)$  **by** *auto*

**from**  $yz$  *False* **have**  $z0$ :  $z \neq 0$  **by** *auto*

**have**  $?yexp = 0 \longleftrightarrow z^3 * ?yexp = 0$  **using**  $z0$  **by** *simp*

**also have**  $z^3 * ?yexp = z^6 + f * z^3 - e^3 / 27$  **unfolding**  $eyz$  **by** *algebra*

```

    also have ... = u^2 + f * u + v unfolding u[OF False] v by algebra
    finally show ?thesis using False by auto
next
  case True
  show ?thesis unfolding True by (auto, algebra)
qed
qed

```

## 2.3 Solving the depressed case for complex numbers

In the complex-numbers-case, the quadratic equation for  $u$  is always solvable, and the main challenge here is prove that it does not matter which solution of the quadratic equation is considered (this is the `diff:False` case in the proof below.)

```

lemma solve-cubic-depressed-Cardano-complex: fixes e :: complex
  assumes e0: e ≠ 0
  and v: v = - (e ^ 3 / 27)
  and u: u^2 + f * u + v = 0
shows y^3 + e * y + f = 0 ↔ (∃ z. z^3 = u ∧ y = z - e / (3 * z))
proof -
  from v e0 have v0: v ≠ 0 by auto
  from e0 have (if e = 0 then x else y) = y for x y :: bool by auto
  note main = cubic-depressed[OF - - v, unfolded this]
  show ?thesis (is ?l = ?r)
proof
  assume ?r
  then obtain z where z: z^3 = u and y: y = z - e / (3 * z) by auto
  from u v0 have u0: u ≠ 0 by auto
  from z u0 have z0: z ≠ 0 by auto
  show ?l
  proof (subst main)
    show u^2 + f * u + v = 0 by fact
    show u = z^3 unfolding z by simp
    show z^2 - y * z - e / 3 = 0 unfolding y using z0
    by (auto simp: field-simps power2-eq-square)
  qed
next
  assume ?l
  let ?yexp = y^3 + e * y + f
  have y0: ?yexp = 0 using ‹?l› by auto
  define p where p = [-e/3, -y, 1:]
  have deg: degree p = 2 unfolding p-def by auto
  define z where z = hd (roots2 p)
  have z ∈ set (roots2 p) unfolding roots2-def Let-def z-def by auto
  with roots2[OF deg] have pz: poly p z = 0 by auto
  from pz e0 have z0: z ≠ 0 unfolding p-def by auto
  from pz have yz: y * z = z * z - e / 3 unfolding p-def by (auto simp:
field-simps)

```

```

from arg-cong[OF this, of  $\lambda x. x / z$ ] z0 have  $y = z - e / (3 * z)$ 
  by (auto simp: field-simps)
have  $\exists u z. u^2 + f * u + v = 0 \wedge z^3 = u \wedge y = z - e / (3 * z)$ 
proof (intro exI conjI)
  show  $y = z - e / (3 * z)$  by fact
  from y0 have  $0 = ?yexp * z^3$  by auto
  also have  $\dots = (y * z)^3 + e * (y * z) * z^2 + f * z^3$  by algebra
  also have  $\dots = (z^3)^2 + f * (z^3) + v$  unfolding yz v by algebra
  finally show  $(z^3)^2 + f * (z^3) + v = 0$  by simp
qed simp
then obtain uu z where
   $*: uu^2 + f * uu + v = 0 \wedge z^3 = uu \wedge y = z - e / (3 * z)$  by blast
  show ?r
proof (cases uu = u)
  case True
    thus ?thesis using  $*$  by auto
  next
    case diff: False
      define p where  $p = [:v, f, 1:]$ 
      have p2: degree p = 2 unfolding p-def by auto
      have poly: poly p u = 0 poly p uu = 0 using  $u *(1)$  unfolding p-def
        by (auto simp: field-simps power2-eq-square)
      have u0:  $u \neq 0 \wedge uu \neq 0$  using poly v0 unfolding p-def by auto
      {
        from poly(1) have  $[: -u, 1:] \text{ dvd } p$  by (meson poly-eq-0-iff-dvd)
        then obtain q where  $pq: p = q * [: -u, 1:]$  by auto
        from poly(2)[unfolded pq poly-mult] diff have poly q uu = 0 by auto
        hence  $[: -uu, 1:] \text{ dvd } q$  by (meson poly-eq-0-iff-dvd)
        then obtain q' where  $qq': q = q' * [: -uu, 1:]$  by auto
        with pq have  $pq: p = q' * [: -uu, 1:] * [: -u, 1:]$  by auto
        from pq[unfolded p-def] have  $q': q' \neq 0$  by auto
        from arg-cong[OF pq, of degree, unfolded p2]
        have  $2 = \text{degree } (q' * [: -uu, 1:] * [: -u, 1:])$  .
        also have  $\dots = \text{degree } q' + \text{degree } [: -uu, 1:] + \text{degree } [: -u, 1:]$ 
          apply (subst degree-mult-eq)
          subgoal using  $q'$  by (metis mult-eq-0-iff pCons-eq-0-iff zero-neq-one)
          subgoal by force
          by (subst degree-mult-eq[OF q'], auto)
        also have  $\dots = \text{degree } q' + 2$  by simp
        finally have dq: degree q' = 0 by simp
        from dq obtain c where  $q': q' = [: c:]$  by (metis degree-eq-zeroE)
        from pq[unfolded q' p-def] have  $c = 1$  by auto
        with  $q'$  have  $q' = 1$  by simp
        with pq have  $[: -u, 1:] * [: -uu, 1:] = p$  by simp
      }
    }
from this[unfolded p-def, simplified] have prod:  $uu * u = v$  by simp
hence uu:  $u = v / uu$  using u0 by (simp add: field-simps)
define zz where  $zz = - e / (3 * z)$ 
show ?r using  $*(2-)$  uu unfolding v using u0

```

by (intro exI[of - zz], auto simp: zz-def field-simps)  
qed  
qed  
qed

## 2.4 Solving the depressed case for real numbers

**definition** *discriminant-cubic-depressed* :: 'a :: comm-ring-1  $\Rightarrow$  'a  $\Rightarrow$  'a where  
*discriminant-cubic-depressed* e f = - (4 \* e<sup>3</sup> + 27 \* f<sup>2</sup>)

**lemma** *discriminant-cubic-depressed*: **assumes** [-x,1:] \* [-y,1:] \* [-z,1:] = [-f,e,0,1:]

**shows** *discriminant-cubic-depressed* e f = (x-y)<sup>2</sup> \* (x-z)<sup>2</sup> \* (y-z)<sup>2</sup>

**proof** -

**from** *assms* **have** f: f = - (z \* (y \* x)) **and** e: e = y \* x - z \* (-y - x) **and**

z: z = -y - x **by** *auto*

**show** *thesis* **unfolding** *discriminant-cubic-depressed-def* e f z

**by** (*simp add: power2-eq-square power3-eq-cube field-simps*)

**qed**

If the discriminant is negative, then there is exactly one real root

**lemma** *solve-cubic-depressed-Cardano-real*: **fixes** e f v u :: real

**defines** y1  $\equiv$  root 3 u - e / (3 \* root 3 u)

**and**  $\Delta \equiv$  *discriminant-cubic-depressed* e f

**assumes** e0: e  $\neq$  0

**and** v: v = - (e<sup>3</sup> / 27)

**and** u: u<sup>2</sup> + f \* u + v = 0

**shows** y1<sup>3</sup> + e \* y1 + f = 0

$\Delta \neq 0 \implies y^3 + e * y + f = 0 \implies y = y1$

**proof** -

**let** ?c = *complex-of-real*

**let** ?y = ?c y

**let** ?e = ?c e

**let** ?u = ?c u

**let** ?v = ?c v

**let** ?f = ?c f

{

**fix** y :: real

**let** ?y = ?c y

**have** y<sup>3</sup> + e \* y + f = 0  $\longleftrightarrow$  ?c (y<sup>3</sup> + e \* y + f) = ?c 0

**using** *of-real-eq-iff* **by** *blast*

**also have** ...  $\longleftrightarrow$  ?y<sup>3</sup> + ?e \* ?y + ?f = 0 **by** *simp*

**also have** ...  $\longleftrightarrow$  ( $\exists z. z^3 = ?u \wedge ?y = z - ?e / (3 * z)$ )

**proof** (*rule solve-cubic-depressed-Cardano-complex*)

**show** ?e  $\neq$  0 **using** e0 **by** *auto*

**show** ?v = - (?e<sup>3</sup> / 27) **unfolding** v **by** *simp*

**show** ?u<sup>2</sup> + ?f \* ?u + ?v = 0 **using** *arg-cong[OF u, of ?c]* **by** *simp*

**qed**

**finally have** y<sup>3</sup> + e \* y + f = 0  $\longleftrightarrow$  ( $\exists z. z^3 = ?u \wedge ?y = z - ?e / (3 *$

```

z)) .
} note pre = this
show y1:  $y1^3 + e * y1 + f = 0$  unfolding pre y1-def
  by (intro exI[of - ?c (root 3 u)], simp only: of-real-power[symmetric],
    simp del: of-real-power add: odd-real-root-pow)
from u have {z. poly [:v,f,1:] z = 0} ≠ {}
  by (auto simp add: field-simps power2-eq-square)
hence set (rroots2 [:v,f,1:]) ≠ {}
  by (subst rroots2[symmetric], auto)
hence rroots2 [:v,f,1:] ≠ [] by simp
from this[unfolded rroots2-def Let-def, simplified]
have  $f^2 - 4 * v \geq 0$ 
  by (auto split: if-splits simp: numeral-2-eq-2 field-simps power2-eq-square)
hence delta-le-0:  $\Delta \leq 0$  unfolding  $\Delta$ -def discriminant-cubic-depressed-def v by
auto

assume Delta-non-0:  $\Delta \neq 0$ 
with delta-le-0 have delta-neg:  $\Delta < 0$  by simp
let ?p = [:f,e,0,1:]
have poly: poly ?p y = 0  $\longleftrightarrow$   $y^3 + e * y + f = 0$  for y
  by (simp add: field-simps power2-eq-square power3-eq-cube)
from y1 have poly ?p y1 = 0 unfolding poly .
hence [: -y1, 1:] dvd ?p using poly-eq-0-iff-dvd by blast
then obtain q where pq: ?p = [: -y1, 1:] * q by blast
{
  fix y2
  assume poly ?p y2 = 0  $y2 \neq y1$ 
  from this[unfolded pq] poly-mult have poly q y2 = 0 by auto
  from this[unfolded poly-eq-0-iff-dvd] obtain r where qr: q = [: -y2, 1:] * r by
blast
{
  have r0: r ≠ 0 using pq unfolding qr poly-mult by auto
  have 3 = degree ?p by simp
  also have ... = 2 + degree r unfolding pq qr
    apply (subst degree-mult-eq, force)
  subgoal using r0 pq qr by force
  by (subst degree-mult-eq[OF - r0], auto)
  finally have degree r = 1 by simp
  from degree1-coeffs[OF this] obtain yy a where r: r = [:yy,a:] by auto
  define y3 where y3 = -yy
  with r have r: r = [: -y3,a:] by auto
  from pq[unfolded qr r] have a = 1 by auto
  with r have  $\exists y3. r = [: -y3, 1:]$  by auto
}
then obtain y3 where r: r = [: -y3, 1:] by auto
have py: ?p = [: -y1, 1:] * [: -y2, 1:] * [: -y3, 1:] unfolding pq qr r by algebra
from discriminant-cubic-depressed[OF this[symmetric], folded  $\Delta$ -def]
have delta:  $\Delta = (y1 - y2)^2 * (y1 - y3)^2 * (y2 - y3)^2$  .
have d0:  $\Delta \geq 0$  unfolding delta by auto

```



```

    with delta-neg have False by auto
  }
  with y1 show  $y^3 + e * y + f = 0 \implies y = y1$  unfolding poly by auto
qed

```

If the discriminant is non-negative, then all roots are real

**lemma** *solve-cubic-depressed-Cardano-all-real-roots*: fixes  $e f v :: \text{real}$  and  $y :: \text{complex}$

```

  defines  $\Delta \equiv \text{discriminant-cubic-depressed } e f$ 
  assumes Delta:  $\Delta \geq 0$ 
  and rt:  $y^3 + e * y + f = 0$ 
shows  $y \in \mathbb{R}$ 
proof -
  note powers = field-simps power3-eq-cube power2-eq-square
  let ?c = complex-of-real
  let ?e = ?c e
  let ?f = ?c f
  let ?cp = [?:?f, ?e, 0, 1:]
  let ?p = [?:f, e, 0, 1:]
  from odd-degree-imp-real-root[of ?p] obtain x1 where poly ?p  $x1 = 0$  by auto
  hence [:-x1, 1:] dvd ?p using poly-eq-0-iff-dvd by blast
  then obtain q where pq: ?p = [:-x1, 1:] * q by auto
  from arg-cong[OF pq, of degree]
  have  $\mathfrak{3} = \text{degree } ([:-x1, 1:] * q)$  by simp
  also have  $\dots = 1 + \text{degree } q$ 
    by (subst degree-mult-eq, insert pq, auto)
  finally have dq:  $\text{degree } q = 2$  by auto
  let ?cc = map-poly ?c
  let ?q = ?cc q
  have cpq: ?cc ?p = ?cc [:-x1, 1:] * ?q unfolding pq hom-distrib by simp
  let ?x1 = ?c x1
  have dq':  $\text{degree } ?q = 2$  using dq by simp
  have  $\neg \text{constant } (\text{poly } ?q)$  using dq by (simp add: constant-degree)
  from fundamental-theorem-of-algebra[OF this] obtain x2 where x2: poly ?q  $x2 = 0$  by blast
  have  $x2 \in \mathbb{R}$ 
  proof (rule ccontr)
    assume x2r:  $x2 \notin \mathbb{R}$ 
    define x3 where  $x3 = \text{cnj } x2$ 
    from x2r have x23:  $x2 \neq x3$  unfolding x3-def using Reals-cnj-iff by force
    have x3: poly ?q  $x3 = 0$  unfolding x3-def
      by (rule complex-conjugate-root[OF - x2], auto)
    from x2[unfolded poly-eq-0-iff-dvd] obtain r where qr: ?q = [:-x2, 1:] * r by auto
    from arg-cong[OF this[symmetric], of  $\lambda x. \text{poly } x x3$ , unfolded poly-mult x3 mult-eq-0-iff] x23
    have x3: poly r  $x3 = 0$  by auto
    from arg-cong[OF qr, of degree]
    have  $2 = \text{degree } ([:-x2, 1:] * r)$  using dq' by simp

```

**also have**  $\dots = 1 + \text{degree } r$  **by** (*subst degree-mult-eq, insert pq qr, auto*)  
**finally have**  $\text{degree } r = 1$  **by** *simp*  
**from** *degree1-coeffs[OF this]* **obtain**  $a \ b$  **where**  $r: r = [a, b]$  **by** *auto*  
**from** *cpq[unfolded qr r]* **have**  $b1: b = 1$  **by** *simp*  
**with**  $x3 \ r$  **have**  $a + x3 = 0$  **by** *simp*  
**hence**  $a = -x3$  **by** *algebra*  
**with**  $b1 \ r$  **have**  $r: r = [-x3, 1]$  **by** *auto*  
**have**  $?cc \ ?p = ?cc \ [-x1, 1] * [-x2, 1] * [-x3, 1]$  **unfolding** *cpq qr r* **by**  
*algebra*  
**also have**  $?cc \ [-x1, 1] = [-?x1, 1]$  **by** *simp*  
**also have**  $?cc \ ?p = ?cp$  **by** *simp*  
**finally have**  $id: [-?x1, 1] * [-x2, 1] * [-x3, 1] = ?cp$  **by** *simp*  
**define**  $x23$  **where**  $x23 = -4 * (Im \ x2)^2$   
**define**  $x12c$  **where**  $x12c = ?x1 - x2$   
**define**  $x12$  **where**  $x12 = (Re \ x12c)^2 + (Im \ x12c)^2$   
**have**  $x23-0: x23 < 0$  **unfolding** *x23-def* **using** *x2r* **using** *complex-is-Real-iff*  
**by** *force*  
**have**  $Im \ x12c \neq 0$  **unfolding** *x12c-def* **using** *x2r* **using** *complex-is-Real-iff* **by**  
*force*  
**hence**  $(Im \ x12c)^2 > 0$  **by** *simp*  
**hence**  $x12: x12 > 0$  **unfolding** *x12-def* **using** *sum-power2-gt-zero-iff* **by** *auto*  
**from** *discriminant-cubic-depressed[OF id]*  
**have**  $?c \ \Delta = ((?x1 - x2)^2 * (?x1 - x3)^2) * (x2 - x3)^2$   
**unfolding** *\Delta-def* *discriminant-cubic-depressed-def* **by** *simp*  
**also have**  $(x2 - x3)^2 = ?c \ x23$  **unfolding** *x3-def* *x23-def* **by** (*simp add:*  
*complex-eq-iff* *power2-eq-square*)  
**also have**  $(?x1 - x2)^2 * (?x1 - x3)^2 = ((?x1 - x2) * (?x1 - x3))^2$   
**by** (*simp add: power2-eq-square*)  
**also have**  $?x1 - x3 = cnj \ (?x1 - x2)$  **unfolding** *x3-def* **by** *simp*  
**also have**  $(?x1 - x2) = x12c$  **unfolding** *x12c-def* **..**  
**also have**  $x12c * cnj \ x12c = ?c \ x12$  **by** (*simp add: x12-def* *complex-eq-iff*  
*power2-eq-square*)  
**finally have**  $?c \ \Delta = ?c \ (x12^2 * x23)$  **by** *simp*  
**hence**  $\Delta = x12^2 * x23$  **by** (*rule of-real-hom.injectivity*)  
**also have**  $\dots < 0$  **using** *x12* *x23-0* **by** (*meson* *mult-pos-neg* *zero-less-power*)  
**finally show** *False* **using** *Delta* **by** *simp*  
**qed**  
**with**  $x2$  **obtain**  $x2$  **where**  $poly \ ?q \ (?c \ x2) = 0$  **unfolding** *Reals-def* **by** *auto*  
**hence**  $x2: poly \ q \ x2 = 0$  **by** *simp*  
**from** *x2[unfolded poly-eq-0-iff-dvd]* **obtain**  $r$  **where**  $qr: q = [-x2, 1] * r$  **by**  
*auto*  
**from** *arg-cong[OF qr, of degree]*  
**have**  $2 = \text{degree} \ ([-x2, 1] * r)$  **using** *dq'* **by** *simp*  
**also have**  $\dots = 1 + \text{degree } r$  **by** (*subst degree-mult-eq, insert pq qr, auto*)  
**finally have**  $\text{degree } r = 1$  **by** *simp*  
**from** *degree1-coeffs[OF this]* **obtain**  $a \ b$  **where**  $r: r = [a, b]$  **by** *auto*  
**from** *pq[unfolded qr r]* **have**  $b1: b = 1$  **by** *simp*  
**define**  $x3$  **where**  $x3 = -a$   
**have**  $r: r = [-x3, 1]$  **unfolding** *r* *b1* *x3-def* **by** *simp*

```

let ?pp = [:-x1,1:] * [:-x2,1:] * [:-x3,1:]
have id: ?p = ?pp unfolding pq qr r by linarith
have True  $\longleftrightarrow$   $y^3 + e * y + f = 0$  using rt by auto
also have  $y^3 + e * y + f = \text{poly } (?cc \ ?p) \ y$  by (simp add: powers)
also have ... = poly (?cc ?pp) y unfolding id by simp
also have ?cc ?pp = [:-?c x1, 1:] * [:-?c x2, 1:] * [:- ?c x3, 1:]
  by simp
also have poly ... y = 0  $\longleftrightarrow$  y = ?c x1  $\vee$  y = ?c x2  $\vee$  y = ?c x3
  unfolding poly-mult mult-eq-0-iff by auto
finally show y  $\in$   $\mathbb{R}$  by auto
qed

end

```

### 3 $n$ -th roots of complex numbers

**theory** Complex-Roots

**imports**

Complex-Geometry.More-Complex  
 Algebraic-Numbers.Complex-Algebraic-Numbers  
 Factor-Algebraic-Polynomial.Roots-via-IA  
 HOL-Library.Product-Lexorder

**begin**

#### 3.1 An algorithm to compute all complex roots of (algebraic) complex numbers

**definition** all-roots :: nat  $\Rightarrow$  complex  $\Rightarrow$  complex list **where**

all-roots n x = (if n = 0 then [] else  
 if algebraic x then  
 (let p = min-int-poly x;  
 q = poly-nth-root n p;  
 xs = complex-roots-of-int-poly q  
 in filter ( $\lambda y. y^n = x$ ) xs)  
 else (SOME ys. set ys = {y.  $y^n = x$ }))

**lemma** all-roots: **assumes** n0:  $n \neq 0$  **shows** set (all-roots n x) = {y.  $y^n = x$ }

**proof** (cases algebraic x)

**case** True

**hence** id: (if n = 0 then y else if algebraic x then z else u) = z

**for** y z u :: complex list **using** n0 **by** auto

**define** p **where** p = poly-nth-root n (min-int-poly x)

**show** ?thesis **unfolding** Let-def p-def[symmetric] all-roots-def id

**proof** (standard, force, standard, simp)

**fix** y

**assume** y:  $y^n = x$

**have** min-int-poly x represents x **using** True **by** auto

**from** represents-nth-root[OF n0 y this]

```

    have p represents y unfolding p-def by auto
    thus  $y \in \text{set } (\text{complex-roots-of-int-poly } p)$ 
    by (subst complex-roots-of-int-poly, auto)
  qed
next
case False
  hence id: (if  $n = 0$  then y else if algebraic x then z else u) = u
    for y z u :: complex list using n0 by auto
  show ?thesis unfolding Let-def all-roots-def id
    by (rule someI-ex, rule finite-list, insert n0, blast)
  qed

```

TODO: One might change *complex-roots-of-int-poly* to *complex-roots-of-int-poly3* in order to avoid an unnecessary factorization of an integer polynomial. However, then this change already needs to be performed within the definition of *all-roots*.

```

lift-definition all-roots-part1 :: nat  $\Rightarrow$  complex  $\Rightarrow$  complex genuine-roots-aux is
   $\lambda n x.$  if  $n = 0 \vee x = 0 \vee \neg$  algebraic x then (1,[],0, filter-fun-complex 1)
    else let p = min-int-poly x;
          q = poly-nth-root n p;
          zeros = complex-roots-of-int-poly q;
          r = Polynomial.monom 1 n - [:x:]
          in (r,zeros, n, filter-fun-complex r)
subgoal for n x
proof (cases  $n = 0 \vee x = 0 \vee \neg$  algebraic x)
  case True
    thus ?thesis by (simp add: filter-fun-complex)
  next
  case False
    hence *: algebraic x  $n \neq 0$   $x \neq 0$  by auto
    {
      fix z
      assume zn:  $z^n = x$ 
      from *(1) have repr: min-int-poly x represents x by auto
      from represents-nth-root[OF *(2) zn repr]
      have poly-nth-root n (min-int-poly x) represents z .
    }
    moreover have card {z.  $z^n = x$ } = n
      by (rule card-nth-roots) (use * in auto)
    ultimately show ?thesis using *
      by (auto simp: Let-def complex-roots-of-int-poly filter-fun-complex poly-monom)
    qed
  done

```

```

lemma all-roots-code[code]:
  all-roots n x = (if  $n = 0$  then [] else if  $x = 0$  then [0]
    else if algebraic x then genuine-roots-impl (all-roots-part1 n x)
    else Code.abort (STR "all-roots invoked on non-algebraic number") ( $\lambda \_.$ 
all-roots n x))

```

```

proof (cases n = 0)
  case True
  thus ?thesis unfolding all-roots-def by simp
next
  case n: False
  show ?thesis
  proof (cases x = 0)
    case x: False
    show ?thesis
    proof (cases algebraic x)
      case False
      with n x show ?thesis by simp
    next
    case True
    define t where t = ?thesis
    have t  $\longleftrightarrow$  filter ( $\lambda y. y \wedge n = x$ )
      (complex-roots-of-int-poly (poly-nth-root n (min-int-poly x)))
      = genuine-roots-impl (all-roots-part1 n x)
      unfolding t-def
      by (subst all-roots-def[of n x], unfold Let-def, insert n x True, auto)
    also have ... using n x True unfolding genuine-roots-impl-def
      by (transfer, simp add: Let-def genuine-roots-def poly-monom)
    finally show ?thesis unfolding t-def by simp
  qed
next
  case x: True
  have set (all-roots n 0) = {0} unfolding all-roots[OF n] using n by simp
  moreover have distinct (all-roots n 0) unfolding all-roots-def using n
    by (auto intro!: distinct-filter complex-roots-of-int-poly)
  ultimately have all-roots n 0 = [0]
    by (smt (verit, del-insts) distinct.simps(2) distinct-singleton insert-ident
list.set-cases list.set-intros(1) list.simps(15) mem-Collect-eq set-empty singleton-conv)
  moreover have ?thesis  $\longleftrightarrow$  all-roots n 0 = [0] using n x by simp
  ultimately show ?thesis by auto
  qed
qed

```

### 3.2 A definition of *the* complex root of a complex number

While the definition of the complex root is quite natural and easy, the main task is a criterion to determine which of all possible roots of a complex number is the chosen one.

**definition** *croot* :: nat  $\Rightarrow$  complex  $\Rightarrow$  complex **where**  
*croot* n x = (rcis (root n (cmod x)) (Arg x / of-nat n))

**lemma** *croot-0[simp]*: *croot* n 0 = 0 *croot* 0 x = 0  
**unfolding** *croot-def* **by** auto

**lemma** *croot-power*: **assumes** n: n  $\neq$  0

**shows**  $(\text{croot } n \ x) \wedge n = x$   
**unfolding** *croot-def DeMoiivre2*  
**by** (*subst real-root-pow-pos2, insert n, auto simp: rcis-cmod-Arg*)

**lemma** *Arg-of-real*:  $\text{Arg} (\text{of-real } x) =$   
*(if  $x < 0$  then  $\pi$  else  $0$ )*  
**proof** (*cases  $x = 0$* )  
**case** *False*  
**hence**  $x < 0 \vee x > 0$  **by** *auto*  
**thus** *?thesis* **by** (*intro cis-Arg-unique, auto*  
*simp: complex-sgn-def scaleR-complex.ctr complex-eq-iff*)  
**qed** (*auto simp: Arg-def*)

**lemma** *Arg-rcis-cis[simp]*: **assumes**  $x > 0$   
**shows**  $\text{Arg} (\text{rcis } x \ y) = \text{Arg} (\text{cis } y)$   
**using** *assms unfolding rcis-def* **by** *simp*

**lemma** *cis-Arg-1[simp]*:  $\text{cis} (\text{Arg } 1) = 1$   
**using** *Arg-of-real[of 1]* **by** *simp*

**lemma** *cis-Arg-power[simp]*: **assumes**  $x \neq 0$   
**shows**  $\text{cis} (\text{Arg} (x \wedge n)) = \text{cis} (\text{Arg } x * \text{real } n)$   
**proof** (*induct n*)  
**case** (*Suc n*)  
**show** *?case* **unfolding** *power.simps*  
**proof** (*subst cis-arg-mult*)  
**show**  $\text{cis} (\text{Arg } x + \text{Arg} (x \wedge n)) = \text{cis} (\text{Arg } x * \text{real} (\text{Suc } n))$   
**unfolding** *mult.commute[of Arg x] DeMoiivre[symmetric]*  
**unfolding** *power.simps* **using** *Suc*  
**by** (*metis DeMoiivre cis-mult mult.commute*)  
**show**  $x * x \wedge n \neq 0$  **using** *assms* **by** *auto*  
**qed**  
**qed** *simp*

**lemma** *Arg-croot[simp]*:  $\text{Arg} (\text{croot } n \ x) = \text{Arg } x / \text{real } n$   
**proof** (*cases  $n = 0 \vee x = 0$* )  
**case** *True*  
**thus** *?thesis* **by** (*auto simp: Arg-def*)  
**next**  
**case** *False*  
**hence**  $n: n \neq 0$  **and**  $x: x \neq 0$  **by** *auto*  
**let** *?root = croot n x*  
**from**  $n$  **have**  $n1: \text{real } n \geq 1 \ \text{real } n > 0 \ \text{real } n \neq 0$  **by** *auto*  
**have** *bounded*:  $-\pi < \text{Arg } x / \text{real } n \wedge \text{Arg } x / \text{real } n \leq \pi$   
**proof** (*cases  $\text{Arg } x < 0$* )  
**case** *True*  
**from** *Arg-bounded[of x]* **have**  $-\pi < \text{Arg } x$  **by** *auto*  
**also** **have**  $\dots \leq \text{Arg } x / \text{real } n$  **using**  $n1$  *True*

by (smt (z3) div-by-1 divide-minus-left frac-le)  
 finally have one:  $-pi < Arg\ x / real\ n$  .  
 have  $Arg\ x / real\ n \leq 0$  using True n1  
 by (smt (verit) divide-less-0-iff)  
 also have  $\dots \leq pi$  by simp  
 finally show ?thesis using one by auto  
 next  
 case False  
 hence ax:  $Arg\ x \geq 0$  by auto  
 have  $Arg\ x / real\ n \leq Arg\ x$  using n1 ax  
 by (smt (verit) div-by-1 frac-le)  
 also have  $\dots \leq pi$  using Arg-bounded[of x] by simp  
 finally have one:  $Arg\ x / real\ n \leq pi$  .  
 have  $-pi < 0$  by simp  
 also have  $\dots \leq Arg\ x / real\ n$  using ax n1 by simp  
 finally show ?thesis using one by auto  
 qed  
 have  $Arg\ ?root = Arg\ (cis\ (Arg\ x / real\ n))$   
 unfolding croot-def using x n by simp  
 also have  $\dots = Arg\ x / real\ n$   
 by (rule cis-Arg-unique, force, insert bounded, auto)  
 finally show ?thesis .  
 qed

lemma cos-abs[simp]:  $cos\ (abs\ x :: real) = cos\ x$   
 proof (cases  $x < 0$ )  
 case True  
 hence abs:  $abs\ x = -x$  by simp  
 show ?thesis unfolding abs by simp  
 qed simp

lemma cos-mono-le: assumes  $abs\ x \leq pi$   
 and  $abs\ y \leq pi$   
 shows  $cos\ x \leq cos\ y \iff abs\ y \leq abs\ x$   
 proof -  
 have  $cos\ x \leq cos\ y \iff cos\ (abs\ x) \leq cos\ (abs\ y)$  by simp  
 also have  $\dots \iff abs\ y \leq abs\ x$   
 by (subst cos-mono-le-eq, insert assms, auto)  
 finally show ?thesis .  
 qed

lemma abs-add-2-mult-bound: fixes  $x :: 'a :: linordered-idom$   
 assumes  $xy: |x| \leq y$   
 shows  $|x| \leq |x + 2 * of-int\ i * y|$   
 proof (cases  $i = 0$ )  
 case i: False  
 let ?oi = of-int ::  $int \Rightarrow 'a$   
 from xy have  $y \geq 0$  by auto  
 consider (pp)  $x \geq 0\ i \geq 0$

```

| (nn)  $x \leq 0 \wedge i \leq 0$ 
| (pn)  $x \geq 0 \wedge i \leq 0$ 
| (np)  $x \leq 0 \wedge i \geq 0$ 
by linarith
thus ?thesis
proof cases
  case pp
  thus ?thesis using y by simp
next
  case nn
  have  $x \geq x + 2 * ?oi \ i * y$ 
  using nn y by (simp add: mult-nonneg-nonpos2)
  with nn show ?thesis by linarith
next
  case pn
  with i have  $0 \leq x \wedge i < 0$  by auto
  define j where  $j = \text{nat } (-i) - 1$ 
  define z where  $z = x - 2 * y$ 
  define u where  $u = 2 * ?oi \ (\text{nat } j) * y$ 
  have u:  $u \geq 0$  unfolding u-def using y by auto
  have i:  $i = - \text{int } (\text{Suc } j)$ 
  using <i < 0> unfolding j-def by simp
  have id:  $x + 2 * ?oi \ i * y = z - u$ 
  unfolding i z-def u-def by (simp add: field-simps)
  have z:  $z \leq 0 \wedge \text{abs } z \geq x$  using xy y pn(1)
  unfolding z-def by auto
  show ?thesis unfolding id using pn(1) z u by simp
next
  case np
  with i have  $0 \geq x \wedge i > 0$  by auto
  define j where  $j = \text{nat } i - 1$ 
  have i:  $i = \text{int } (\text{Suc } j)$ 
  using <i > 0> unfolding j-def by simp
  define u where  $u = 2 * ?oi \ (\text{nat } j) * y$ 
  have u:  $u \geq 0$  unfolding u-def using y by auto
  define z where  $z = -x - 2 * y$ 
  have id:  $x + 2 * ?oi \ i * y = -z + u$ 
  unfolding i z-def u-def by (simp add: field-simps)
  have z:  $z \leq 0 \wedge \text{abs } z \geq -x$  using xy y np(1)
  unfolding z-def by auto
  show ?thesis unfolding id using np(1) z u by simp
qed
qed simp

```

```

lemma abs-eq-add-2-mult: fixes y :: 'a :: linordered-idom
  assumes abs-id:  $|x| = |x + 2 * \text{of-int } i * y|$ 
  and xy:  $-y < x \wedge x \leq y$ 
  and i:  $i \neq 0$ 
  shows  $x = y \wedge i = -1$ 

```



```

proof –
  let ?oi = of-int :: int ⇒ 'a
  from xy have y: y > 0 by auto
  consider (pp) x ≥ 0 i ≥ 0
    | (nn) x < 0 i ≤ 0
    | (pn) x ≥ 0 i ≤ 0
    | (np) x < 0 i ≥ 0
  by linarith
  hence ?thesis ∨ x = ?oi (- i) * y
proof cases
  case pp
    thus ?thesis using y abs-id xy i by simp
  next
    case nn
      hence |x + 2 * ?oi i * y| =
        - (x + 2 * ?oi i * y)
      using y nn
      by (intro abs-of-nonpos add-nonpos-nonpos,
        force, simp, intro mult-nonneg-nonpos, auto)
      thus ?thesis using y abs-id xy i nn
      by auto
    next
      case pn
        with i have 0 ≤ x i < 0 by auto
        define j where j = nat (-i) - 1
        define z where z = x - 2 * y
        define u where u = 2 * ?oi (nat j) * y
        have u: u ≥ 0 unfolding u-def using y by auto
        have i: i = - int (Suc j)
          using ⟨i < 0⟩ unfolding j-def by simp
        have id: x + 2 * ?oi i * y = z - u
          unfolding i z-def u-def by (simp add: field-simps)
        have z: z ≤ 0 abs z ≥ x using xy y pn(1)
          unfolding z-def by auto
        from abs-id[unfolded id] have z - u = -x
          using z u pn by auto
        from this[folded id] have x = of-int (-i) * y
          by auto
        thus ?thesis by auto
      next
        case np
          with i have 0 ≥ x i > 0 by auto
          define j where j = nat i - 1
          have i: i = int (Suc j)
            using ⟨i > 0⟩ unfolding j-def by simp
          define u where u = 2 * ?oi (nat j) * y
          have u: u ≥ 0 unfolding u-def using y by auto
          define z where z = - x - 2 * y
          have id: x + 2 * ?oi i * y = - z + u

```

```

    unfolding i z-def u-def by (simp add: field-simps)
  have z: z ≤ 0
    using xy y np(1) unfolding z-def by auto
  from abs-id[unfolded id] have - z + u = - x
    using u z np by auto
  from this[folded id] have x = of-int (- i) * y
    by auto
  thus ?thesis by auto
qed
thus ?thesis
proof
  assume x = ?oi (- i) * y
  with xy i y
  show ?thesis
  by (smt (verit, ccfv-SIG) less-le minus-less-iff mult-le-cancel-right2 mult-minus1-right
  mult-minus-left mult-of-int-commute of-int-hom.hom-one of-int-le-1-iff of-int-minus)
qed
qed

```

This is the core lemma. It tells us that *croot* will choose the principal root, i.e. the root with largest real part and if there are two roots with identical real part, then the largest imaginary part. This criterion will be crucial for implementing *croot*.

```

lemma croot-principal: assumes n: n ≠ 0
  and y: y ^ n = x
  and neq: y ≠ croot n x
shows Re y < Re (croot n x) ∨ Re y = Re (croot n x) ∧ Im y < Im (croot n x)
proof (cases x = 0)
  case True
  with neq y have False by auto
  thus ?thesis ..
next
  case x: False
  let ?root = croot n x
  from n have n1: real n ≥ 1 real n > 0 real n ≠ 0 by auto
  from x y n have y0: y ≠ 0 by auto
  from croot-power[OF n, of x] y
  have id: ?root ^ n = y ^ n by simp
  hence cmod (?root ^ n) = cmod (y ^ n) by simp
  hence norm-eq: cmod ?root = cmod y using n unfolding norm-power
  by (meson gr-zeroI norm-ge-zero power-eq-imp-eq-base)
  have cis (Arg y * real n) = cis (Arg (y ^ n)) by (subst cis-Arg-power[OF y0],
simp)
  also have ... = cis (Arg x) using y by simp
  finally have ciseq: cis (Arg y * real n) = cis (Arg x) by simp
  from cis-eq[OF ciseq] obtain i where
    Arg y * real n - Arg x = 2 * real-of-int i * pi
  by auto
  hence Arg y * real n = Arg x + 2 * real-of-int i * pi by auto

```

```

from arg-cong[OF this, of  $\lambda x. x / \text{real } n$ ] n1
have Argy: Arg y = Arg ?root + 2 * real-of-int i * pi / real n
  by (auto simp: field-simps)
have i0: i  $\neq$  0
proof
  assume i = 0
  hence Arg y = Arg ?root unfolding Argy by simp
  with norm-eq have ?root = y by (metis rcis-cmod-Arg)
  with neq show False by simp
qed
from y0 have cy0: cmod y > 0 by auto
from Arg-bounded[of x] have abs-pi: abs (Arg x)  $\leq$  pi by auto
have Re y  $\leq$  Re ?root  $\iff$  Re y / cmod y  $\leq$  Re ?root / cmod y
  using cy0 unfolding divide-le-cancel by simp
also have cosy: Re y / cmod y = cos (Arg y) unfolding cos-arg[OF y0] ..
also have cosrt: Re ?root / cmod y = cos (Arg ?root)
  unfolding norm-eq[symmetric] by (subst cos-arg, insert norm-eq cy0, auto)
also have cos (Arg y)  $\leq$  cos (Arg ?root)  $\iff$  abs (Arg ?root)  $\leq$  abs (Arg y)
  by (rule cos-mono-le, insert Arg-bounded[of y] Arg-bounded[of ?root], auto)
also have ...  $\iff$  abs (Arg ?root) * real n  $\leq$  abs (Arg y) * real n
  unfolding mult-le-cancel-right using n1 by simp
also have ...  $\iff$  abs (Arg x)  $\leq$  |Arg x + 2 * real-of-int i * pi|
  unfolding Argy using n1 by (simp add: field-simps)
also have ... using abs-pi
  by (rule abs-add-2-mult-bound)
finally have le: Re y  $\leq$  Re (croot n x) .
show ?thesis
proof (cases Re y = Re (croot n x))
  case False
  with le show ?thesis by auto
next
  case True
  hence Re y / cmod y = Re ?root / cmod y by simp
  hence cos (Arg y) = cos (Arg ?root) unfolding cosy cosrt .
  hence cos (abs (Arg y)) = cos (abs (Arg ?root)) unfolding cos-abs .
  from cos-inj-pi[OF - - - this]
  have abs (Arg y) = abs (Arg ?root)
  using Arg-bounded[of y] Arg-bounded[of ?root] by auto
  hence abs (Arg y) * real n = abs (Arg ?root) * real n by simp
  hence abs (Arg x) = |Arg x + 2 * real-of-int i * pi| unfolding Argy
  using n1 by (simp add: field-simps)
  from abs-eq-add-2-mult[OF this - - <i  $\neq$  0>] Arg-bounded[of x]
  have Argx: Arg x = pi and i: i = -1 by auto
  have Argy: Arg y = -pi / real n
  unfolding Argy Arg-croot i Argx by simp
  have Im ?root > Im y  $\iff$  Im ?root / cmod ?root > Im y / cmod y
  unfolding norm-eq using cy0
  by (meson divide-less-cancel divide-strict-right-mono)
  also have ...  $\iff$  sin (Arg ?root) > sin (Arg y)

```

```

    by (subst (1 2) sin-arg, insert y0 norm-eq, auto)
  also have ...  $\longleftrightarrow$   $\sin(-\pi / \text{real } n) < \sin(\pi / \text{real } n)$ 
    unfolding Argy Arg-croot Argx by simp
  also have ...
  proof -
    have  $\sin(-\pi / \text{real } n) < 0$ 
      using n1 by (smt (verit) Arg-bounded Argy divide-neg-pos sin-gt-zero
        sin-minus)
    also have ...  $< \sin(\pi / \text{real } n)$ 
      using n1 calculation by fastforce
    finally show ?thesis .
  qed
  finally show ?thesis using le by auto
  qed
  qed

```

```

lemma croot-unique: assumes n:  $n \neq 0$ 
  and y:  $y^n = x$ 
  and y-max-Re-Im:  $\bigwedge z. z^n = x \implies \text{Re } z < \text{Re } y \vee \text{Re } z = \text{Re } y \wedge \text{Im } z \leq \text{Im } y$ 
shows croot n x = y
proof (rule ccontr)
  assume croot n x  $\neq$  y
  from croot-principal[OF n y this[symmetric]]
  have  $\text{Re } y < \text{Re } (\text{croot } n \ x) \vee$ 
     $\text{Re } y = \text{Re } (\text{croot } n \ x) \wedge \text{Im } y < \text{Im } (\text{croot } n \ x)$  .
  with y-max-Re-Im[OF croot-power[OF n]]
  show False by auto
  qed

```

```

lemma csqrt-is-croot-2:  $\text{csqrt} = \text{croot } 2$ 
proof
  fix x
  show  $\text{csqrt } x = \text{croot } 2 \ x$ 
  proof (rule sym, rule croot-unique, force, force)
    let ?p =  $[-x, 0, 1:]$ 
    let ?cx =  $\text{csqrt } x$ 
    have p:  $?p = [?:?cx, 1:] * [:-?cx, 1:]$ 
      by (simp add: power2-eq-square[symmetric])
    fix y
    assume  $y^2 = x$ 
    hence  $\text{True} \longleftrightarrow \text{poly } ?p \ y = 0$ 
      by (auto simp: power2-eq-square)
    also have ...  $\longleftrightarrow y = - ?cx \vee y = ?cx$ 
      unfolding p poly-mult mult-eq-0-iff poly-root-factor by auto
    finally have  $y = - ?cx \vee y = ?cx$  by simp
    thus  $\text{Re } y < \text{Re } ?cx \vee \text{Re } y = \text{Re } ?cx \wedge \text{Im } y \leq \text{Im } ?cx$ 
  proof
    assume y:  $y = - ?cx$ 

```

```

show ?thesis
proof (cases Re ?cx = 0)
  case False
    with csqrt-principal[of x] have Re ?cx > 0 by simp
    thus ?thesis unfolding y by simp
  next
    case True
      with csqrt-principal[of x] have Im ?cx ≥ 0 by simp
      thus ?thesis unfolding y using True by auto
    qed
  qed auto
qed
qed

lemma croot-via-root-selection: assumes roots: set ys = { y. yn = x }
  and n: n ≠ 0
shows croot n x = arg-min-list (λ y. (- Re y, - Im y)) ys
  (is - = arg-min-list ?f ys)
proof (rule croot-unique[OF n])
  let ?y = arg-min-list ?f ys
  have rt: croot n xn = x using n by (rule croot-power)
  hence croot n x ∈ set ys unfolding roots by auto
  hence ys: ys ≠ [] by auto
  from arg-min-list-in[OF this] have ?y ∈ set ys by auto
  from this[unfolded roots]
  show ?yn = x by auto
  fix z
  assume zn = x
  hence z: z ∈ set ys unfolding roots by auto
  from f-arg-min-list-f[OF ys, of ?f] z
  have ?f ?y ≤ ?f z by simp
  thus Re z < Re ?y ∨ Re z = Re ?y ∧ Im z ≤ Im ?y by auto
qed

lemma croot-impl[code]: croot n x = (if n = 0 then 0 else
  arg-min-list (λ y. (- Re y, - Im y)) (all-roots n x))
proof (cases n = 0)
  case n0: False
    hence id: (if n = 0 then y else z) = z
    for y z u :: complex by auto
    show ?thesis unfolding id Let-def
    by (rule croot-via-root-selection[OF - n0], rule all-roots[OF n0])
  qed auto

end

```

## 4 Algorithms to compute all complex and real roots of a cubic polynomial

```

theory Cubic-Polynomials
  imports
    Cardanos-Formula
    Complex-Roots
begin

```

The real case where a result is only delivered if the discriminant is negative

```

definition solve-depressed-cubic-Cardano-real :: real  $\Rightarrow$  real  $\Rightarrow$  real option where
  solve-depressed-cubic-Cardano-real e f = (
    if e = 0 then Some (root 3 (-f)) else
      let v = - (e ^ 3 / 27) in
        case rroots2 [:v,f,1:] of
          [u,-]  $\Rightarrow$  let rt = root 3 u in Some (rt - e / (3 * rt))
          | -  $\Rightarrow$  None)

```

**lemma** *solve-depressed-cubic-Cardano-real*:

```

assumes solve-depressed-cubic-Cardano-real e f = Some y
shows {y. y ^ 3 + e * y + f = 0} = {y}

```

**proof** (*cases e = 0*)

**case** *True*

```

have {y. y ^ 3 + e * y + f = 0} = {y. y ^ 3 = -f} unfolding True

```

```

  by (auto simp add: field-simps)

```

```

also have ... = {root 3 (-f)}

```

```

  using odd-real-root-unique[of 3 - -f] odd-real-root-pow[of 3] by auto

```

```

also have root 3 (-f) = y using assms unfolding True solve-depressed-cubic-Cardano-real-def
  by auto

```

```

finally show ?thesis .

```

**next**

**case** *False*

```

define v where v = - (e ^ 3 / 27)

```

```

note * = assms[unfolded solve-depressed-cubic-Cardano-real-def Let-def, folded v-def]

```

```

let ?rr = rroots2 [:v,f,1:]

```

```

from * False obtain u u' where rr: ?rr = [u,u']

```

```

  by (cases ?rr; cases tl ?rr; cases tl (tl ?rr); auto split: if-splits)

```

```

from * [unfolded rr list.simps] False

```

```

have y: y = root 3 u - e / (3 * root 3 u) by auto

```

```

have u  $\in$  set (rroots2 [:v,f,1:]) unfolding rr by auto

```

```

also have set (rroots2 [:v,f,1:]) = {u. poly [:v,f,1:] u = 0}

```

```

  by (subst rroots2, auto)

```

```

finally have u: u ^ 2 + f * u + v = 0 by (simp add: field-simps power2-eq-square)

```

```

note Cardano = solve-cubic-depressed-Cardano-real[OF False v-def u]

```

```

have 2: 2 = Suc (Suc 0) by simp

```

```

from rr have 0: f^2 - 4 * v  $\neq$  0 unfolding rroots2-def Let-def

```

```

  by (auto split: if-splits simp: 2)

```

```

hence 0: discriminant-cubic-depressed e f  $\neq$  0

```

**unfolding** *discriminant-cubic-depressed-def v-def* **by** *auto*  
**show** *?thesis* **using** *Cardano(1) Cardano(2)[OF 0]* **unfolding** *y[symmetric]* **by**  
*blast*  
**qed**

The complex case

**definition** *solve-depressed-cubic-complex* :: *complex*  $\Rightarrow$  *complex*  $\Rightarrow$  *complex list*  
**where**

*solve-depressed-cubic-complex e f* = (let  
   *ys* = (if *e* = 0 then *all-roots* 3 (- *f*) else (let  
     *u* = *hd* (*roots2* [: - (*e* ^ 3 / 27) ,*f*,1:]);  
     *zs* = *all-roots* 3 *u*  
     in *map* ( $\lambda z. z - e / (3 * z)$ ) *zs*))  
 in *remdups ys*)

**lemma** *solve-depressed-cubic-complex-code[code]*:

*solve-depressed-cubic-complex e f* = (let  
   *ys* = (if *e* = 0 then *all-roots* 3 (- *f*) else (let  
     *f2* = *f* / 2;  
     *u* = - *f2* + *csqrt* (*f2*^2 + *e* ^ 3 / 27);  
     *zs* = *all-roots* 3 *u*  
     in *map* ( $\lambda z. z - e / (3 * z)$ ) *zs*))  
 in *remdups ys*)

**unfolding** *solve-depressed-cubic-complex-def* *Let-def roots2-def*  
**by** (*simp add: numeral-2-eq-2*)

**lemma** *solve-depressed-cubic-complex*: *y*  $\in$  *set* (*solve-depressed-cubic-complex e f*)

$\longleftrightarrow$  (*y*^3 + *e* \* *y* + *f* = 0)

**proof** (*cases e = 0*)

**case** *True*

**thus** *?thesis* **by** (*simp add: solve-depressed-cubic-complex-def Let-def all-roots eq-neg-iff-add-eq-0*)

**next**

**case** *e0: False*

**hence** *id*: (if *e* = 0 then *x* else *y*) = *y* **for** *x y* :: *complex list* **by** *simp*

**define** *v* **where** *v* = - (*e* ^ 3 / 27)

**define** *p* **where** *p* = [:*v*, *f*, 1:]

**have** *p2*: *degree p* = 2 **unfolding** *p-def* **by** *auto*

**let** *?u* = *hd* (*roots2 p*)

**define** *u* **where** *u* = *?u*

**have** *u*  $\in$  *set* (*roots2 p*) **unfolding** *roots2-def Let-def u-def* **by** *auto*

**with** *roots2[OF p2]* **have** *poly p u* = 0 **by** *auto*

**hence** *u*: *u*^2 + *f* \* *u* + *v* = 0 **unfolding** *p-def*

**by** (*simp add: field-simps power2-eq-square*)

**note** *cube-roots* = *all-roots[of 3, simplified]*

**show** *?thesis* **unfolding** *solve-depressed-cubic-complex-def Let-def set-remdups set-map id cube-roots*

```

unfolding v-def[symmetric] p-def[symmetric] set-concat set-map
  u-def[symmetric]
proof -
  have p: {x. poly p x = 0} = {u. u2 + f * u + v = 0} unfolding p-def by
(auto simp: field-simps power2-eq-square)
  have cube:  $\bigcup$  (set 'all-roots 3' {x. poly p x = 0}) = {z.  $\exists$  u. u2 + f * u +
v = 0  $\wedge$  z3 = u}
  unfolding p by (auto simp: cube-roots)
  show (y  $\in$  ( $\lambda$ z. z - e / (3 * z)) ' {y. y3 = u}) = (y3 + e * y + f = 0)
  using solve-cubic-depressed-Cardano-complex[OF e0 v-def u] cube by blast
qed
qed

```

For the general real case, we first try Cardano with negative discriminant and only if it is not applicable, then we go for the calculation using complex numbers. Note that for non-negative delta no filter is required to identify the real roots from the list of complex roots, since in that case we already know that all roots are real.

**definition** solve-depressed-cubic-real :: real  $\Rightarrow$  real  $\Rightarrow$  real list **where**  
 solve-depressed-cubic-real e f = (case solve-depressed-cubic-Cardano-real e f  
 of Some y  $\Rightarrow$  [y]  
 | None  $\Rightarrow$  map Re (solve-depressed-cubic-complex (of-real e) (of-real f)))

**lemma** solve-depressed-cubic-real-code[code]: solve-depressed-cubic-real e f =  
 (if e = 0 then [root 3 (-f)] else  
 let v = e<sup>3</sup> / 27;  
 f2 = f / 2;  
 f2v = f2<sup>2</sup> + v in  
 if f2v > 0 then  
 let u = -f2 + sqrt f2v;  
 rt = root 3 u  
 in [rt - e / (3 \* rt)]  
 else  
 let ce3 = of-real e / 3;  
 u = - of-real f2 + csqrt (of-real f2v) in  
 map Re (remdups (map ( $\lambda$ rt. rt - ce3 / rt) (all-roots 3 u))))

```

proof -
have id: rroots2 [:v, f, 1:] = (let
  f2 = f / 2;
  bac = f22 - v in
  if bac = 0 then [- f2] else
  if bac < 0 then [] else let e = sqrt bac in [- f2 + e, - f2 - e]) for v
unfolding rroots2-def Let-def numeral-2-eq-2 by auto
define foo :: real  $\Rightarrow$  real  $\Rightarrow$  real option where
  foo f2v f2 = (case (if f2v = 0 then [- f2] else []) of []  $\Rightarrow$  None | -  $\Rightarrow$  None)
for f2v f2
have solve-depressed-cubic-real e f = (if e = 0 then [root 3 (-f)] else
  let v = e3 / 27;
  f2 = f / 2;

```



```

      f2v = f22 + v in
    if f2v > 0 then
      let u = -f2 + sqrt f2v;
          rt = root 3 u
      in [rt - e / (3 * rt)]
    else
      (case foo f2v f2 of
        None => let u = - cor f2 + csqrt (cor f2v) in
          map Re
            (remdups (map (λz. z - cor e / (3 * z)) (all-roots 3 u)))
        | Some y => [])
      unfolding solve-depressed-cubic-real-def solve-depressed-cubic-Cardano-real-def

      solve-depressed-cubic-complex-code
      Let-def id foo-def
      by (auto split: if-splits)
    also have id: foo f2v f2 = None
      for f2v f2 unfolding foo-def by auto
    ultimately show ?thesis by (auto simp: Let-def)
  qed

lemma solve-depressed-cubic-real: y ∈ set (solve-depressed-cubic-real e f)
  ↔ (y3 + e * y + f = 0)
proof (cases solve-depressed-cubic-Cardano-real e f)
  case (Some x)
  show ?thesis unfolding solve-depressed-cubic-real-def Some option.simps
    using solve-depressed-cubic-Cardano-real[OF Some] by auto
next
  case None
  from this[unfolded solve-depressed-cubic-Cardano-real-def Let-def rroots2-def]
  have disc: 0 ≤ discriminant-cubic-depressed e f unfolding discriminant-cubic-depressed-def
    by (auto split: if-splits simp: numeral-2-eq-2)
  let ?c = complex-of-real
  let ?y = ?c y
  let ?e = ?c e
  let ?f = ?c f
  have sub: set (solve-depressed-cubic-complex ?e ?f) ⊆ ℝ
  proof
    fix y
    assume y: y ∈ set (solve-depressed-cubic-complex ?e ?f)
    show y ∈ ℝ
    by (rule solve-cubic-depressed-Cardano-all-real-roots[OF disc y[unfolded solve-depressed-cubic-complex]])
  qed
  have y3 + e * y + f = 0 ↔ (?c (y3 + e * y + f) = ?c 0) unfolding
of-real-eq-iff by simp
  also have ... ↔ ?y3 + ?e * ?y + ?f = 0 by simp
  also have ... ↔ ?y ∈ set (solve-depressed-cubic-complex ?e ?f)
    unfolding solve-depressed-cubic-complex ..
  also have ... ↔ y ∈ Re ' set (solve-depressed-cubic-complex ?e ?f) using sub

```

by force  
 finally show ?thesis unfolding solve-depressed-cubic-real-def None by auto  
 qed

Combining the various algorithms

**lemma** degree3-coeffs: degree  $p = 3 \implies$   
 $\exists a b c d. p = [: d, c, b, a :] \wedge a \neq 0$   
 by (metis One-nat-def Suc-1 degree2-coeffs degree-pCons-eq-if nat.inject numeral-3-eq-3 pCons-cases zero-neq-numeral)

**definition** roots3-generic :: ('a :: field-char-0  $\Rightarrow$  'a  $\Rightarrow$  'a list)  $\Rightarrow$  'a poly  $\Rightarrow$  'a list  
 where

roots3-generic depressed-solver  $p =$  (let  
 cs = coeffs  $p$ ;  
 $a = cs ! 3$ ;  $b = cs ! 2$ ;  $c = cs ! 1$ ;  $d = cs ! 0$ ;  
 $a3 = 3 * a$ ;  
 $ba3 = b / a3$ ;  
 $b2 = b * b$ ;  
 $b3 = b2 * b$ ;  
 $e = (c - b2 / a3) / a$ ;  
 $f = (d + 2 * b3 / (27 * a^2) - b * c / a3) / a$ ;  
 roots = depressed-solver  $e f$   
 in map ( $\lambda y. y - ba3$ ) roots)

**lemma** roots3-generic: assumes deg: degree  $p = 3$   
 and solver:  $\bigwedge e f y. y \in \text{set } (\text{depressed-solver } e f) \iff y^3 + e * y + f = 0$   
 shows set (roots3-generic depressed-solver  $p$ ) = { $x. \text{poly } p x = 0$ }

proof -

**note** powers = field-simps power3-eq-cube power2-eq-square  
**from** degree3-coeffs[OF deg] **obtain**  $a b c d$  **where**  
 $p: p = [:d,c,b,a:]$  **and**  $a: a \neq 0$  **by** auto  
**have** coeffs: coeffs  $p ! 3 = a$  coeffs  $p ! 2 = b$  coeffs  $p ! 1 = c$  coeffs  $p ! 0 = d$   
**unfolding**  $p$  **using**  $a$  **by** auto  
**define**  $e$  **where**  $e = (c - b^2 / (3 * a)) / a$   
**define**  $f$  **where**  $f = (d + 2 * b^3 / (27 * a^2) - b * c / (3 * a)) / a$   
**note** def = roots3-generic-def[of depressed-solver  $p$ , unfolded Let-def coeffs,  
 folded power3-eq-cube, folded power2-eq-square, folded e-def f-def]  
 {  
 fix  $x :: 'a$   
**define**  $y$  **where**  $y = x + b / (3 * a)$   
**have**  $xy: x = y - b / (3 * a)$  **unfolding**  $y$ -def **by** auto  
**have** poly  $p x = 0 \iff a * x^3 + b * x^2 + c * x + d = 0$  **unfolding**  $p$   
**by** (simp add: powers)  
**also have** ...  $\iff (y^3 + e * y + f = 0)$   
**unfolding** to-depressed-cubic[OF  $a xy e$ -def  $f$ -def] ..  
**also have** ...  $\iff y \in \text{set } (\text{depressed-solver } e f)$   
**unfolding** solver ..  
**also have** ...  $\iff x \in \text{set } (\text{roots3-generic depressed-solver } p)$  **unfolding**  $xy$  def  
**by** auto

**finally have**  $\text{poly } p \ x = 0 \iff x \in \text{set } (\text{roots3-generic depressed-solver } p)$  **by**  
*auto*  
**}**  
**thus** *?thesis* **by** *auto*  
**qed**

**definition** *roots3* :: *complex poly*  $\Rightarrow$  *complex list* **where**  
*roots3* = *roots3-generic solve-depressed-cubic-complex*

**lemma** *roots3*: **assumes** *deg: degree*  $p = 3$   
**shows**  $\text{set } (\text{roots3 } p) = \{ x. \text{poly } p \ x = 0 \}$   
**unfolding** *roots3-def* **by** (*rule roots3-generic*[*OF deg solve-depressed-cubic-complex*])

**definition** *rroots3* :: *real poly*  $\Rightarrow$  *real list* **where**  
*rroots3* = *roots3-generic solve-depressed-cubic-real*

**lemma** *rroots3*: **assumes** *deg: degree*  $p = 3$   
**shows**  $\text{set } (\text{rroots3 } p) = \{ x. \text{poly } p \ x = 0 \}$   
**unfolding** *rroots3-def* **by** (*rule roots3-generic*[*OF deg solve-depressed-cubic-real*])

**end**

## 5 Algorithms to compute all complex and real roots of a quartic polynomial

**theory** *Quartic-Polynomials*

**imports**

*Ferraris-Formula*

*Cubic-Polynomials*

**begin**

The complex case is straight-forward

**definition** *solve-depressed-quartic-complex* :: *complex*  $\Rightarrow$  *complex*  $\Rightarrow$  *complex*  $\Rightarrow$   
*complex list* **where**

*solve-depressed-quartic-complex*  $p \ q \ r = \text{remdups}$  (*if*  $q = 0$  *then*  
*(concat (map*  $(\lambda \ z. \text{let } y = \text{csqrt } z \text{ in } [y, -y]) (\text{roots2 } [:r, p, 1:]))$ ) *else*  
*let* *cubics* = *roots3*  $[: - (q^2), 2 * p^2 - 8 * r, 8 * p, 8:];$   
*m* = *hd cubics*; — *select any root of the cubic polynomial*  
*a* = *csqrt*  $(2 * m);$   
*p2m* =  $p / 2 + m;$   
*q2a* =  $q / (2 * a);$   
*b1* =  $p2m - q2a;$   
*b2* =  $p2m + q2a$   
*in* (*roots2*  $[:b1, a, 1:] @ \text{roots2 } [:b2, -a, 1:]))$

**lemma** *solve-depressed-quartic-complex*:  $x \in \text{set } (\text{solve-depressed-quartic-complex } p \ q \ r)$   
 $\iff (x^4 + p * x^2 + q * x + r = 0)$

```

proof –
  note powers = field-simps power4-eq-xxxx power3-eq-cube power2-eq-square
  show ?thesis
  proof (cases q = 0)
    case True
      have csqrt:  $z = x^{\wedge}2 \iff (x = \text{csqrt } z \vee x = - \text{csqrt } z)$  for z
        by (metis power2-csqrt power2-eq-iff)
      have  $(x^{\wedge}4 + p * x^2 + q * x + r = 0) \iff (x^{\wedge}4 + p * x^2 + r = 0)$ 
        unfolding True by simp
      also have ...  $\iff (\exists z. z^2 + p * z + r = 0 \wedge z = x^2)$  unfolding bi-
        quadratic-solution by simp
      also have ...  $\iff (\exists z. \text{poly } [:r,p,1:] z = 0 \wedge z = x^2)$ 
        by (simp add: powers)
      also have ...  $\iff (\exists z \in \text{set } (\text{roots2 } [:r,p,1:]). z = x^2)$ 
        by (subst roots2[symmetric], auto)
      also have ...  $\iff (\exists z \in \text{set } (\text{roots2 } [:r,p,1:]). x = \text{csqrt } z \vee x = - \text{csqrt } z)$ 
        unfolding csqrt ..
      also have ...  $\iff (x \in \text{set } (\text{solve-depressed-quartic-complex } p \ q \ r))$ 
        unfolding solve-depressed-quartic-complex-def id unfolding True Let-def by
        auto
      finally show ?thesis ..
    next
      case q0: False
        hence id: (if q = 0 then x else y) = y for x y :: complex list by auto
        note powers = field-simps power4-eq-xxxx power3-eq-cube power2-eq-square
        let ?poly = [:- q2, 2 * p2 - 8 * r, 8 * p, 8:]
        from roots3[of ?poly] have croots: set (roots3 ?poly) = {x. poly ?poly x = 0}
        by auto
        from fundamental-theorem-of-algebra-alt[of ?poly]
        have {x. poly ?poly x = 0}  $\neq \{\}$  by auto
        with croots have roots3 ?poly  $\neq []$  by auto
        then obtain m rest where rts: roots3 ?poly = m # rest by (cases roots3
        ?poly, auto)
        hence hd: hd (roots3 ?poly) = m by auto
        from roots[unfolded rts] have poly ?poly m = 0 by auto
        hence mrt:  $8 * m^3 + (8 * p) * m^2 + (2 * p^2 - 8 * r) * m - q^2 = 0$ 
          and m0:  $m \neq 0$  using q0
          by (auto simp: powers)
        define b1 where  $b1 = p / 2 + m - q / (2 * \text{csqrt } (2 * m))$ 
        define b2 where  $b2 = p / 2 + m + q / (2 * \text{csqrt } (2 * m))$ 
        have csqrt:  $\text{csqrt } x * \text{csqrt } x = x$  for x by (metis power2-csqrt power2-eq-square)
        show ?thesis unfolding solve-depressed-quartic-complex-def id Let-def set-remdups
        set-append hd
          unfolding b1-def[symmetric] b2-def[symmetric]
          apply (subst depressed-quartic-Ferrari[OF mrt q0 csqrt b1-def b2-def])
          apply (subst (1 2) roots2[symmetric], auto)
          done
      qed
    qed

```

The main difference in the real case is that a specific cubic root has to be used, namely a positive one. In the soundness proof we show that such a cubic root always exists.

**definition** *solve-depressed-quartic-real* :: *real* ⇒ *real* ⇒ *real* ⇒ *real list* **where**  
*solve-depressed-quartic-real* *p q r* = *remdups* (if *q* = 0 then  
 (*concat* (*map* ( $\lambda z. rroots2 [-z,0,1:]$ ) (*rroots2* [:*r,p,1:*])) *else*  
*let cubics* = *rroots3* [: - (*q*<sup>2</sup>), 2 \* *p*<sup>2</sup> - 8 \* *r*, 8 \* *p*, 8:];  
*m* = *the* (*find* ( $\lambda m. m > 0$ ) *cubics*); — select any positive root of the  
 cubic polynomial  
*a* = *sqrt* (2 \* *m*);  
*p2m* = *p* / 2 + *m*;  
*q2a* = *q* / (2 \* *a*);  
*b1* = *p2m* - *q2a*;  
*b2* = *p2m* + *q2a*  
*in* (*rroots2* [:*b1,a,1:*] @ *rroots2* [:*b2,-a,1:*]))

**lemma** *solve-depressed-quartic-real*:  $x \in set (solve-depressed-quartic-real\ p\ q\ r)$   
 $\longleftrightarrow (x^4 + p * x^2 + q * x + r = 0)$

**proof** —

**note** *powers* = *field-simps* *power4-eq-xxxx* *power3-eq-cube* *power2-eq-square*  
**show** *?thesis*

**proof** (*cases* *q* = 0)

**case** *True*

**have** *sqrt*:  $z = x^2 \longleftrightarrow (x \in set (rroots2 [-z,0,1:]))$  **for** *z*

**by** (*subst* *rroots2[symmetric]*, *auto* *simp*: *powers*)

**have**  $(x^4 + p * x^2 + q * x + r = 0) \longleftrightarrow (x^4 + p * x^2 + r = 0)$

**unfolding** *True* **by** *simp*

**also have** ...  $\longleftrightarrow (\exists z. z^2 + p * z + r = 0 \wedge z = x^2)$  **unfolding** *bi-quadratic-solution* **by** *simp*

**also have** ...  $\longleftrightarrow (\exists z. poly [:r,p,1:] z = 0 \wedge z = x^2)$

**by** (*simp* *add*: *powers*)

**also have** ...  $\longleftrightarrow (\exists z \in set (rroots2 [:r,p,1:]). z = x^2)$

**by** (*subst* *rroots2[symmetric]*, *auto*)

**also have** ...  $\longleftrightarrow (\exists z \in set (rroots2 [:r,p,1:]). x \in set (rroots2 [-z,0,1:])))$

**unfolding** *sqrt* ..

**also have** ...  $\longleftrightarrow (x \in set (solve-depressed-quartic-real\ p\ q\ r))$

**unfolding** *solve-depressed-quartic-real-def* *id* **unfolding** *True* *Let-def* **by** *auto*

**finally show** *?thesis* ..

**next**

**case** *q0*: *False*

**hence** *id*: (if *q* = 0 then *x* else *y*) = *y* **for** *x y* :: *real list* **by** *auto*

**note** *powers* = *field-simps* *power4-eq-xxxx* *power3-eq-cube* *power2-eq-square*

**let** *?poly* = [: - *q*<sup>2</sup>, 2 \* *p*<sup>2</sup> - 8 \* *r*, 8 \* *p*, 8:]

**define** *cubics* **where** *cubics* = *rroots3* *?poly*

**from** *rroots3[of ?poly, folded cubics-def]*

**have** *rroots*: *set* *cubics* = {*x. poly ?poly x* = 0} **by** *auto*

**from** *odd-degree-imp-real-root[of ?poly]*

**have** {*x. poly ?poly x* = 0} ≠ {} **by** *auto*

```

with roots have cubics ≠ [] by auto
have ∃ m. m ∈ set cubics ∧ m > 0
proof (rule ccontr)
  assume ¬ ?thesis
  from this[unfolded roots] have rt: poly ?poly m = 0 ⇒ m ≤ 0 for m by
auto
  have poly ?poly 0 = - (q^2) by simp
  also have ... < 0 using q0 by auto
  finally have lt: poly ?poly 0 ≤ 0 by simp
  from poly-pinfy-gt-lc[of ?poly] obtain n0 where ∧ n. n ≥ n0 ⇒ 8 ≤ poly
?poly n by auto
  from this[of max n0 0] have poly ?poly (max n0 0) ≥ 0 by auto
  from IVT[of poly ?poly, OF lt this] obtain m where m ≥ 0 and poly: poly
?poly m = 0 by auto
  from rt[OF this(2)] this(1) have m = 0 by auto
  thus False using poly q0 by simp
qed
hence find (λ m. m > 0) cubics ≠ None unfolding find-None-iff by auto
then obtain m where find: find (λ m. m > 0) cubics = Some m by auto
from find-Some-D[OF this] have m: m ∈ set cubics and m-0: m > 0 by auto
with roots have poly ?poly m = 0 by auto
hence mrt: 8*m^3 + (8 * p) * m^2 + (2 * p^2 - 8 * r) * m - q^2 = 0
  by (auto simp: powers)
from m-0 have sqrt: sqrt (2 * m) * sqrt (2 * m) = 2 * m by simp
define b1 where b1 = p / 2 + m - q / (2 * sqrt (2 * m))
define b2 where b2 = p / 2 + m + q / (2 * sqrt (2 * m))
show ?thesis unfolding solve-depressed-quartic-real-def id Let-def set-remdups
set-append
  cubics-def[symmetric] find option.sel
  unfolding b1-def[symmetric] b2-def[symmetric]
  apply (subst depressed-quartic-Ferrari[OF mrt q0 sqrt b1-def b2-def])
  apply (subst (1 2) roots2[symmetric], auto)
  done

```

qed

qed

Combining the various algorithms

```

lemma numeral-4-eq-4: 4 = Suc (Suc (Suc (Suc 0)))
  by (simp add: eval-nat-numeral)

```

```

lemma degree4-coeffs: degree p = 4 ⇒
  ∃ a b c d e. p = [: e, d, c, b, a :] ∧ a ≠ 0
  using degree3-coeffs degree-pCons-eq-if nat.inject numeral-3-eq-3 numeral-4-eq-4
pCons-cases zero-neq-numeral
  by metis

```

```

definition roots4-generic :: ('a :: field-char-0 ⇒ 'a ⇒ 'a ⇒ 'a list) ⇒ 'a poly ⇒
'a list where
  roots4-generic depressed-solver p = (let

```

```

cs = coeffs p;
cs = coeffs p;
a4 = cs ! 4; a3 = cs ! 3; a2 = cs ! 2; a1 = cs ! 1; a0 = cs ! 0;
b = a3 / a4;
c = a2 / a4;
d = a1 / a4;
e = a0 / a4;
b2 = b * b;
b3 = b2 * b;
b4 = b3 * b;
b4' = b / 4;
p = c - 3/8 * b2;
q = (b3 - 4*b*c + 8 * d) / 8;
r = (-3 * b4 + 256 * e - 64 * b * d + 16 * b2 * c) / 256;
roots = depressed-solver p q r
in map (λ y. y - b4') roots)

```

**lemma** *roots4-generic*: **assumes** *deg*: degree  $p = 4$   
**and** *solver*:  $\bigwedge p q r y. y \in \text{set } (\text{depressed-solver } p q r) \longleftrightarrow y^4 + p * y^2 + q * y + r = 0$   
**shows**  $\text{set } (\text{roots4-generic depressed-solver } p) = \{x. \text{poly } p x = 0\}$   
**proof** –  
**note** *powers* = *field-simps power4-eq-xxxx power3-eq-cube power2-eq-square*  
**from** *degree4-coeffs*[*OF deg*] **obtain**  $a_4 a_3 a_2 a_1 a_0$  **where**  
 $p: p = [:a_0, a_1, a_2, a_3, a_4:]$  **and**  $a_4: a_4 \neq 0$  **by** *auto*  
**have** *coeffs*:  $\text{coeffs } p ! 4 = a_4 \text{ coeffs } p ! 3 = a_3 \text{ coeffs } p ! 2 = a_2 \text{ coeffs } p ! 1 = a_1 \text{ coeffs } p ! 0 = a_0$   
**unfolding**  $p$  **using**  $a_4$  **by** *auto*  
**define**  $b$  **where**  $b = a_3 / a_4$   
**define**  $c$  **where**  $c = a_2 / a_4$   
**define**  $d$  **where**  $d = a_1 / a_4$   
**define**  $e$  **where**  $e = a_0 / a_4$   
**note** *def* = *roots4-generic-def*[*of depressed-solver p, unfolded Let-def coeffs, folded b-def c-def d-def e-def,*  
*folded power4-eq-xxxx, folded power3-eq-cube, folded power2-eq-square*]  
**let**  $?p = p$   
{  
**fix**  $x$   
**define**  $y$  **where**  $y = x + b / 4$   
**define**  $p$  **where**  $p = c - (3/8) * b^2$   
**define**  $q$  **where**  $q = (b^3 - 4*b*c + 8 * d) / 8$   
**define**  $r$  **where**  $r = (-3 * b^4 + 256 * e - 64 * b * d + 16 * b^2 * c) / 256$   
**note** *def* = *def*[*folded p-def q-def r-def*]  
**have**  $xy: x = y - b / 4$  **unfolding**  $y\text{-def}$  **by** *auto*  
**have**  $\text{poly } ?p x = 0 \longleftrightarrow a_4 * x^4 + a_3 * x^3 + a_2 * x^2 + a_1 * x + a_0 = 0$   
**unfolding**  $p$   
**by** (*simp add: powers*)  
**also have**  $\dots \longleftrightarrow (y^4 + p * y^2 + q * y + r = 0)$   
**unfolding** *to-depressed-quartic*[*OF a4 b-def c-def d-def e-def p-def q-def r-def*

```

xy] ..
  also have ...  $\longleftrightarrow y \in \text{set } (\text{depressed-solver } p \ q \ r)$ 
    unfolding solver ..
  also have ...  $\longleftrightarrow x \in \text{set } (\text{roots4-generic depressed-solver } ?p)$  unfolding xy
def by auto
  finally have poly ?p  $x = 0 \longleftrightarrow x \in \text{set } (\text{roots4-generic depressed-solver } ?p)$ 
by auto
}
thus ?thesis by simp
qed

```

**definition** *roots4* :: *complex poly*  $\Rightarrow$  *complex list* **where**  
*roots4* = *roots4-generic solve-depressed-quartic-complex*

**lemma** *roots4*: **assumes** *deg*: *degree*  $p = 4$   
**shows**  $\text{set } (\text{roots4 } p) = \{ x. \text{poly } p \ x = 0 \}$   
**unfolding** *roots4-def* **by** (*rule* *roots4-generic*[*OF deg solve-depressed-quartic-complex*])

**definition** *rroots4* :: *real poly*  $\Rightarrow$  *real list* **where**  
*rroots4* = *roots4-generic solve-depressed-quartic-real*

**lemma** *rroots4*: **assumes** *deg*: *degree*  $p = 4$   
**shows**  $\text{set } (\text{rroots4 } p) = \{ x. \text{poly } p \ x = 0 \}$   
**unfolding** *rroots4-def* **by** (*rule* *roots4-generic*[*OF deg solve-depressed-quartic-real*])

end

## References

- [1] G. Cardano. *Ars Magna, The Great Art or the Rules of Algebra*. 1545. [https://en.wikipedia.org/wiki/Ars\\_Magna\\_\(Cardano\\_book\)](https://en.wikipedia.org/wiki/Ars_Magna_(Cardano_book)).
- [2] R. Thiemann, A. Yamada, and S. Joosten. Algebraic numbers in Isabelle/HOL. *Archive of Formal Proofs*, Dec. 2015. [https://isa-afp.org/entries/Algebraic\\_Numbers.html](https://isa-afp.org/entries/Algebraic_Numbers.html), Formal proof development.