Countable Sums and Discrete (Sub)Distributions

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March 19, 2025

Abstract

We provide elementary formalizations of countable sums over positive real numbers, and of discrete probabilistic subdistributions and distributions. This is intended as a lightweight alternative to the corresponding concepts from the Isabelle distribution, which are defined using their continuous counterparts (namely Lebesgue integral and general probability distributions) and therefore have significant dependencies.

Contents

1 Infinite Sums of Positive Reals

This is a theory of infinite sums of positive reals defined as limits of finite sums. The goal is to make reasoning about these infinite sums almost as easy as that about finite sums.

theory *Infinite-Sums-of-Positive-Reals*
imports *Complex-Main HOL−Library.Countable-Set*
begin

1.1 Preliminaries

lemma *real-pm-iff*:
$\bigwedge a\ b\ c.\ (a{::}real) + b \leq c \longleftrightarrow a \leq c - b$
$\bigwedge a\ b\ c.\ (a{::}real) + b \leq c \longleftrightarrow b \leq c - a$

$\bigwedge a\ b\ c.\ (a{::}real) \le b - c \longleftrightarrow c \le b - a$
⟨*proof*⟩

lemma *real-md-iff*:
$\bigwedge a\ b\ c.\ a \ge 0 \Longrightarrow b > 0 \Longrightarrow c \ge 0 \Longrightarrow (a{::}real) * b \le c \longleftrightarrow a \le c\ /\ b$
$\bigwedge a\ b\ c.\ a > 0 \Longrightarrow b \ge 0 \Longrightarrow c \ge 0 \Longrightarrow (a{::}real) * b \le c \longleftrightarrow b \le c\ /\ a$
$\bigwedge a\ b\ c.\ a > 0 \Longrightarrow b \ge 0 \Longrightarrow c > 0 \Longrightarrow (a{::}real) \le b\ /\ c \longleftrightarrow c \le b\ /\ a$
⟨*proof*⟩

lemma *disjoint-finite-aux*:
$\forall i{\in}I.\ \forall j{\in}I.\ i \ne j \longrightarrow A\ i \cap A\ j = \{\} \Longrightarrow B \subseteq \bigcup\ (A\ `\ I) \Longrightarrow finite\ B \Longrightarrow finite\ \{i \in I.\ B \cap A\ i \ne \{\}\}$
⟨*proof*⟩

lemma *incl-UNION-aux*: $B \subseteq \bigcup\ (A\ `\ I) \Longrightarrow B = \bigcup\ ((\lambda i.\ (B \cap A\ i))\ `\ \{i \in I.\ B \cap A\ i \ne \{\}\})$
⟨*proof*⟩

lemma *incl-UNION-aux2*: $B \subseteq \bigcup\ (A\ `\ I) \longleftrightarrow B = \bigcup\ ((\lambda i.\ (B \cap A\ i))\ `\ I)$
⟨*proof*⟩

lemma *sum-singl*[*simp*]: $sum\ f\ \{a\} = f\ a$
⟨*proof*⟩

lemma *sum-two*[*simp*]: $a1 \ne a2 \Longrightarrow sum\ f\ \{a1,a2\} = f\ a1 + f\ a2$
⟨*proof*⟩

lemma *sum-three*[*simp*]: $a1 \ne a2 \Longrightarrow a1 \ne a3 \Longrightarrow a2 \ne a3 \Longrightarrow sum\ f\ \{a1,a2,a3\} = f\ a1 + f\ a2 + f\ a3$
⟨*proof*⟩

lemma *Sup-leq*:
$A \ne \{\} \Longrightarrow \forall a{\in}A.\ \exists b{\in}B.\ (a{::}real) \le b \Longrightarrow bdd\text{-}above\ B \Longrightarrow Sup\ A \le Sup\ B$
⟨*proof*⟩

lemma *Sup-image-leq*:
$A \ne \{\} \Longrightarrow \forall a{\in}A.\ \exists b{\in}B.\ (f\ a{::}real) \le g\ b \Longrightarrow bdd\text{-}above\ (g\ `\ B) \Longrightarrow Sup\ (f\ `\ A) \le Sup\ (g\ `\ B)$
⟨*proof*⟩

lemma *Sup-cong*:
assumes $A \ne \{\} \lor B \ne \{\}$ $\forall a{\in}A.\ \exists b{\in}B.\ (a{::}real) \le b$ $\forall b{\in}B.\ \exists a{\in}A.\ (b{::}real) \le a$
$bdd\text{-}above\ A \lor bdd\text{-}above\ B$
shows $Sup\ A = Sup\ B$
⟨*proof*⟩

lemma *Sup-image-cong*:
 A ≠ {} ∨ B ≠ {} ⟹ ∀ a∈A. ∃ b∈B. (f a::real) ≤ g b ⟹ ∀ b∈B. ∃ a∈A. (g b::real) ≤ f a ⟹
 bdd-above (f ‘ A) ∨ bdd-above (g ‘ B) ⟹
 Sup (f ‘ A) = Sup (g ‘ B)
 ⟨proof⟩

lemma *Sup-congL*:
 A ≠ {} ⟹ ∀ a∈A. ∃ b∈B. (a::real) ≤ b ⟹ ∀ b∈B. b ≤ Sup A ⟹ Sup A = Sup B
 ⟨proof⟩

lemma *Sup-image-congL*:
 A ≠ {} ⟹ ∀ a∈A. ∃ b∈B. (f a::real) ≤ g b ⟹ ∀ b∈B. g b ≤ Sup (f ‘ A) ⟹ Sup (f ‘ A) = Sup (g ‘ B)
 ⟨proof⟩

lemma *Sup-congR*:
 B ≠ {} ⟹ ∀ a∈A. a ≤ Sup B ⟹ ∀ b∈B. ∃ a∈A. (b::real) ≤ a ⟹ Sup A = Sup B
 ⟨proof⟩

lemma *Sup-image-congR*:
 B ≠ {} ⟹ ∀ a∈A. f a ≤ Sup (g ‘ B) ⟹ ∀ b∈B. ∃ a∈A. (g b::real) ≤ f a ⟹ Sup (f ‘ A) = Sup (g ‘ B)
 ⟨proof⟩

lemma *Sup-eq-0-iff*:
 assumes A ≠ {} bdd-above A (∀ a∈A. (a::real) ≥ 0)
 shows Sup A = 0 ⟷ (∀ a∈A. a = 0)
 ⟨proof⟩

lemma *plus-Sup-commute*:
 assumes f1: {f1 b1 | b1. φ1 b1} ≠ {} bdd-above {f1 b1 | b1. φ1 b1} **and**
 f2: {f2 b2 | b2. φ2 b2} ≠ {} bdd-above {f2 b2 | b2. φ2 b2}
 shows
 Sup {(f1 b1::real) | b1 . φ1 b1} + Sup {f2 b2 | b2 . φ2 b2} =
 Sup {f1 b1 + f2 b2 | b1 b2. φ1 b1 ∧ φ2 b2} (**is** ?L1 + ?L2 = ?R)
⟨proof⟩

lemma *plus-Sup-commute′*:
 assumes f1: A1 ≠ {} bdd-above A1 **and**
 f2: A2 ≠ {} bdd-above A2
 shows Sup A1 + Sup A2 = Sup {(a1::real) + a2 | a1 a2. a1 ∈ A1 ∧ a2 ∈ A2}

 ⟨proof⟩

lemma *plus-SupR*: $A \neq \{\} \Longrightarrow \textit{bdd-above}\ A \Longrightarrow \textit{Sup}\ A + (b{::}\textit{real}) = \textit{Sup}\ \{a + b \mid a.\ a \in A\}$
⟨*proof*⟩

lemma *plus-SupL*: $A \neq \{\} \Longrightarrow \textit{bdd-above}\ A \Longrightarrow (b{::}\textit{real}) + \textit{Sup}\ A = \textit{Sup}\ \{b + a \mid a.\ a \in A\}$
⟨*proof*⟩

lemma *mult-Sup-commute*:
assumes *f1*: $\{f1\ b1 \mid b1.\ \varphi 1\ b1\} \neq \{\}$ $\textit{bdd-above}\ \{f1\ b1 \mid b1.\ \varphi 1\ b1\}$ $\forall b1.\ \varphi 1\ b1 \longrightarrow f1\ b1 \geq 0$ **and**
f2: $\{f2\ b2 \mid b2.\ \varphi 2\ b2\} \neq \{\}$ $\textit{bdd-above}\ \{f2\ b2 \mid b2.\ \varphi 2\ b2\}$ $\forall b2.\ \varphi 2\ b2 \longrightarrow f2\ b2 \geq 0$
shows
$\textit{Sup}\ \{(f1\ b1{::}\textit{real}) \mid b1\ .\ \varphi 1\ b1\} * \textit{Sup}\ \{f2\ b2 \mid b2\ .\ \varphi 2\ b2\} =$
$\textit{Sup}\ \{f1\ b1 * f2\ b2 \mid b1\ b2.\ \varphi 1\ b1 \wedge \varphi 2\ b2\}$ (**is** $?L1 * ?L2 = ?R$)
⟨*proof*⟩

lemma *mult-Sup-commute′*:
assumes $A1 \neq \{\}$ $\textit{bdd-above}\ A1$ $\forall a1 \in A1.\ a1 \geq 0$ **and**
$A2 \neq \{\}$ $\textit{bdd-above}\ A2$ $\forall a2 \in A2.\ a2 \geq 0$
shows $\textit{Sup}\ A1 * \textit{Sup}\ A2 = \textit{Sup}\ \{(a1{::}\textit{real}) * a2 \mid a1\ a2.\ a1 \in A1 \wedge a2 \in A2\}$

⟨*proof*⟩

lemma *mult-SupR*: $A \neq \{\} \Longrightarrow \textit{bdd-above}\ A \Longrightarrow \forall a \in A.\ a \geq 0 \Longrightarrow b \geq 0 \Longrightarrow$
$\textit{Sup}\ A * (b{::}\textit{real}) = \textit{Sup}\ \{a * b \mid a.\ a \in A\}$
⟨*proof*⟩

lemma *mult-SupL*: $A \neq \{\} \Longrightarrow \textit{bdd-above}\ A \Longrightarrow \forall a \in A.\ a \geq 0 \Longrightarrow b \geq 0 \Longrightarrow$
$(b{::}\textit{real}) * \textit{Sup}\ A = \textit{Sup}\ \{b * a \mid a.\ a \in A\}$
⟨*proof*⟩

lemma *sum-mono3*:
$\textit{finite}\ B \Longrightarrow A \subseteq B \Longrightarrow (\bigwedge b.\ b \in B - A \Longrightarrow 0 \leq g\ b) \Longrightarrow (\bigwedge a.\ a \in A \Longrightarrow (f\ a{::}\textit{real}) \leq g\ a) \Longrightarrow$
$\textit{sum}\ f\ A \leq \textit{sum}\ g\ B$
⟨*proof*⟩

lemma *sum-Sup-commute*:
fixes $h :: 'a \Rightarrow \textit{real}$
assumes $\textit{finite}\ J$ **and** $\forall i \in J.\ \{h\ b \mid b.\ \varphi\ i\ b\} \neq \{\} \wedge \textit{bdd-above}\ \{h\ b \mid b.\ \varphi\ i\ b\}$
shows $\textit{sum}\ (\lambda i.\ \textit{Sup}\ \{h\ b \mid b.\ \varphi\ i\ b\})\ J =$
$\textit{Sup}\ \{\textit{sum}\ (\lambda i.\ h\ (b\ i))\ J \mid b\ .\ \forall i \in J.\ \varphi\ i\ (b\ i)\}$

⟨*proof*⟩

1.2 Positivity, boundedness and infinite summation

definition *positive* :: $('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow bool$ **where**
positive f A $\equiv \forall a{\in}A.\ f\ a \geq 0$

definition *sbounded* :: $('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow bool$ **where**
sbounded f A $\equiv \exists r.\ \forall B.\ B \subseteq A \wedge finite\ B \longrightarrow sum\ f\ B \leq r$

definition *isum* :: $('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow real$ **where**
isum f A $\equiv Sup\ (sum\ f\ `\ \{B \mid B\ .\ B \subseteq A \wedge finite\ B\})$

lemma *positive-mono*: *positive p A* $\Longrightarrow B \subseteq A \Longrightarrow$ *positive p B*
⟨*proof*⟩

lemma *positive-eq*:
assumes *positive f A* **and** $\forall a{\in}A.\ f1\ a = f\ a$
shows *positive f1 A*
⟨*proof*⟩

lemma *sbounded-eq*:
assumes *sbounded f A* **and** $\forall a{\in}A.\ f1\ a = f\ a$
shows *sbounded f1 A*
⟨*proof*⟩

lemma *finite-imp-sbounded*: *positive f A* $\Longrightarrow$ *finite A* $\Longrightarrow$ *sbounded f A*
⟨*proof*⟩

lemma *sbounded-empty*[*simp*,*intro*!]: *sbounded f* {}
⟨*proof*⟩

lemma *sbounded-insert*[*simp*]: *sbounded f* (*insert a A*) $\longleftrightarrow$ *sbounded f A*
⟨*proof*⟩

lemma *sbounded-Un*[*simp*]: *sbounded f* $(A1 \cup A2) \longleftrightarrow$ *sbounded f A1* $\wedge$ *sbounded f A2*
⟨*proof*⟩

lemma *sbounded-UNION*:
assumes *finite I* **shows** *sbounded f* $(\bigcup i{\in}I.\ A\ i) \longleftrightarrow (\forall i{\in}I.\ sbounded\ f\ (A\ i))$
⟨*proof*⟩

lemma *sbounded-mono*: $A \subseteq B \Longrightarrow$ *sbounded f B* $\Longrightarrow$ *sbounded f A*
⟨*proof*⟩

lemma *sbounded-reindex*: *sbounded* (*f o u*) *A* $\Longrightarrow$ *sbounded f* (*u* ‘ *A*)
⟨*proof*⟩

lemma *sbounded-reindex-inj-on*: *inj-on u A* $\Longrightarrow$ *sbounded f* (*u* ‘ *A*) $\longleftrightarrow$ *sbounded* (*f o u*) *A*
⟨*proof*⟩

lemma *sbounded-swap*:
sbounded (λ(*a*,*b*). *f a b*) ($A \times B$) $\longleftrightarrow$ *sbounded* (λ(*b*,*a*). *f a b*) ($B \times A$)
⟨*proof*⟩

lemma *sbounded-constant-0*:
assumes $\forall\, a{\in}A.\ f\ a = (0{::}real)$
shows *sbounded f A*
⟨*proof*⟩

lemma *sbounded-setminus*:
assumes *sbounded f A* **and** $\forall\, b{\in}B{-}A.\ f\ b = 0$
shows *sbounded f B*
⟨*proof*⟩

lemma *isum-eq-sum*:
positive f A $\Longrightarrow$ *finite A* $\Longrightarrow$ *isum f A* = *sum f A*
⟨*proof*⟩

lemma *isum-cong*:
assumes $A = B$ **and** $\bigwedge x.\ x \in B \Longrightarrow g\ x = h\ x$
shows *isum g A* = *isum h B*
⟨*proof*⟩

lemma *isum-mono*:
assumes *sbounded h A* **and** $\bigwedge x.\ x \in A \Longrightarrow g\ x \le h\ x$
shows *isum g A* $\le$ *isum h A*
⟨*proof*⟩

lemma *isum-mono'*:
assumes *sbounded g B* **and** $A \subseteq B$
shows *isum g A* $\le$ *isum g B*
⟨*proof*⟩

lemma *isum-empty*[*simp*]: *isum g* {} = *0*

⟨*proof*⟩

lemma *isum-const-zero*[*simp*]: *isum* (λx. *0*) *A* = *0*
⟨*proof*⟩

lemma *isum-const-zero'*: $\forall x \in A$. *g x* = *0* $\Longrightarrow$ *isum g A* = *0*
⟨*proof*⟩

lemma *isum-eq-0-iff*: *positive f A* $\Longrightarrow$ *sbounded f A* $\Longrightarrow$ *isum f A* = *0* $\longleftrightarrow$ ($\forall a \in A$. *f a* = *0*)
⟨*proof*⟩

lemma *isum-reindex*: *inj-on h A* $\Longrightarrow$ *isum g* (*h* ‘ *A*) = *isum* ($g \circ h$) *A*
⟨*proof*⟩

lemma *isum-reindex-cong*: *inj-on l B* $\Longrightarrow$ *A* = *l* ‘ *B* $\Longrightarrow$
($\bigwedge x$. $x \in B \Longrightarrow g$ (*l x*) = *h x*) $\Longrightarrow$ *isum g A* = *isum h B*
⟨*proof*⟩

lemma *isum-reindex-cong'*:
($\bigwedge x\ y$. $x \in A \Longrightarrow y \in A \Longrightarrow x \neq y \Longrightarrow h\ x = h\ y \Longrightarrow g\ (h\ x) = 0$) $\Longrightarrow$ *isum g* (*h* ‘ *A*) = *isum* ($g \circ h$) *A*
⟨*proof*⟩

lemma *isum-zeros-cong*:
assumes *sbounded g* ($S \cap T$) $\vee$ *sbounded h* ($S \cap T$)
and ($\bigwedge i$. $i \in T - S \Longrightarrow h\ i = 0$) **and** ($\bigwedge i$. $i \in S - T \Longrightarrow g\ i = 0$)
and ($\bigwedge x$. $x \in S \cap T \Longrightarrow g\ x = h\ x$)
shows *isum g S* = *isum h T*
⟨*proof*⟩

lemma *isum-zeros-congL*:
sbounded g S $\Longrightarrow S \subseteq T \Longrightarrow \forall i \in T - S$. *g i* = *0* $\Longrightarrow$ *isum g S* = *isum g T*
⟨*proof*⟩

lemma *isum-zeros-congR*:
sbounded g S $\Longrightarrow S \subseteq T \Longrightarrow \forall i \in T - S$. *g i* = *0* $\Longrightarrow$ *isum g T* = *isum g S*
⟨*proof*⟩

lemma *isum-singl*[*simp*]: $f\ a \geq (0{::}real) \Longrightarrow isum\ f\ \{a\} = f\ a$
  ⟨*proof*⟩

lemma *isum-two*[*simp*]: $a1 \neq a2 \Longrightarrow f\ a1 \geq (0{::}real) \Longrightarrow f\ a2 \geq 0 \Longrightarrow isum\ f\ \{a1,a2\} = f\ a1 + f\ a2$
  ⟨*proof*⟩

lemma *isum-three*[*simp*]: $a1 \neq a2 \Longrightarrow a1 \neq a3 \Longrightarrow a2 \neq a3 \Longrightarrow f\ a1 \geq 0 \Longrightarrow f\ a2 \geq (0{::}real) \Longrightarrow f\ a3 \geq 0 \Longrightarrow$
 $isum\ f\ \{a1,a2,a3\} = f\ a1 + f\ a2 + f\ a3$
  ⟨*proof*⟩

lemma *isum-ge-0*: $positive\ f\ A \Longrightarrow sbounded\ f\ A \Longrightarrow isum\ f\ A \geq 0$
  ⟨*proof*⟩

lemma *in-le-isum*: $positive\ f\ A \Longrightarrow sbounded\ f\ A \Longrightarrow a \in A \Longrightarrow f\ a \leq isum\ f\ A$
  ⟨*proof*⟩

lemma *isum-eq-singl*:
  assumes *fx*: $f\ a = x$ **and** *f*: $\forall a'.\ a' \neq a \longrightarrow f\ a' = 0$ **and** *x*: $x \geq 0$ **and** *a*: $a \in A$
  shows $isum\ f\ A = x$
  ⟨*proof*⟩

lemma *isum-le-singl*:
  assumes *fx*: $f\ a \leq x$ **and** *f*: $\forall a'.\ a' \neq a \longrightarrow f\ a' = 0$ **and** *x*: $f\ a \geq 0$ **and** *a*: $a \in A$
  shows $isum\ f\ A \leq x$
  ⟨*proof*⟩

lemma *isum-insert*[*simp*]: $a \notin A \Longrightarrow sbounded\ f\ A \Longrightarrow f\ a \geq 0 \Longrightarrow isum\ f\ (insert\ a\ A) = isum\ f\ A + f\ a$
  ⟨*proof*⟩

lemma *isum-UNION*:
  assumes *dsj*: $\forall i{\in}I.\ \forall j{\in}I.\ i \neq j \longrightarrow A\ i \cap A\ j = \{\}$ **and** *sb*: $sbounded\ g\ (\bigcup\ (A\ `\ I))$
  shows $isum\ g\ (\bigcup\ (A\ `\ I)) = isum\ (\lambda i.\ isum\ g\ (A\ i))\ I$
⟨*proof*⟩

lemma *isum-Un*[*simp*]:
  assumes $positive\ f\ A1$ $sbounded\ f\ A1$ $positive\ f\ A2$ $sbounded\ f\ A2$ $A1 \cap A2 = \{\}$
  shows $isum\ f\ (A1 \cup A2) = isum\ f\ A1 + isum\ f\ A2$
⟨*proof*⟩

lemma *isum-Sigma*:
  assumes *sbd*: $sbounded\ (\lambda(a,b).\ f\ a\ b)\ (Sigma\ A\ Bs)$

shows $isum\ (\lambda(a,b).\ f\ a\ b)\ (Sigma\ A\ Bs) = isum\ (\lambda a.\ isum\ (f\ a)\ (Bs\ a))\ A$
⟨proof⟩

lemma *isum-Times*:
assumes $sbounded\ (\lambda(a,b).\ f\ a\ b)\ (A \times B)$
shows $isum\ (\lambda(a,b).\ f\ a\ b)\ (A \times B) = isum\ (\lambda a.\ isum\ (f\ a)\ B)\ A$
⟨proof⟩

lemma *isum-swap*:
assumes $sbounded\ (\lambda(a,b).\ f\ a\ b)\ (A \times B)$
shows $isum\ (\lambda a.\ isum\ (f\ a)\ B)\ A = isum\ (\lambda b.\ isum\ (\lambda a.\ f\ a\ b)\ A)\ B$ (**is** $?L = ?R$)
⟨proof⟩

lemma *isum-plus*:
assumes $f1$: $positive\ f1\ A\ sbounded\ f1\ A$
and $f2$: $positive\ f2\ A\ sbounded\ f2\ A$
shows $isum\ (\lambda a.\ f1\ a + f2\ a)\ A = isum\ f1\ A + isum\ f2\ A$
⟨proof⟩

lemma *sbounded-product*:
assumes f: $positive\ f\ A\ sbounded\ f\ A$ **and** g: $positive\ g\ B\ sbounded\ g\ B$
shows $sbounded\ (\lambda(a,b).\ f\ a * g\ b)\ (A \times B)$
⟨proof⟩

lemma *sbounded-multL*: $x \geq 0 \Longrightarrow sbounded\ f\ A \Longrightarrow sbounded\ (\lambda a.\ x * f\ a)\ A$
⟨proof⟩

lemma *sbounded-multL-strict*[*simp*]:
assumes x: $x > 0$
shows $sbounded\ (\lambda a.\ x * f\ a)\ A \longleftrightarrow sbounded\ f\ A$
⟨proof⟩

lemma *sbounded-multR*: $x \geq 0 \Longrightarrow sbounded\ f\ A \Longrightarrow sbounded\ (\lambda a.\ f\ a * x)\ A$
⟨proof⟩

lemma *sbounded-multR-strict*[*simp*]:
assumes x: $x > 0$
shows $sbounded\ (\lambda a.\ f\ a * x)\ A \longleftrightarrow sbounded\ f\ A$
⟨proof⟩

lemma *positive-sbounded-multL*:
assumes f: $positive\ f\ A\ sbounded\ f\ A$ **and** g: $\forall a \in A.\ g\ a \leq x$

shows *sbounded* $(\lambda a.\ f\ a * g\ a)\ A$
⟨*proof*⟩

lemma *positive-sbounded-multR*:
assumes *f*: *positive f A sbounded f A* **and** *g*: $\forall\ a{\in}A.\ g\ a \leq x$
shows *sbounded* $(\lambda a.\ g\ a * f\ a)\ A$
⟨*proof*⟩

lemma *isum-product-Times*:
assumes *f*: *positive f A sbounded f A* **and** *g*: *positive g B sbounded g B*
shows *isum f A* $*$ *isum g B* $=$ *isum* $(\lambda(a,b).\ f\ a * g\ b)\ (A \times B)$
⟨*proof*⟩

lemma *isum-product*:
assumes *f*: *positive f A sbounded f A* **and** *g*: *positive g B sbounded g B*
shows *isum f A* $*$ *isum g B* $=$ *isum* $(\lambda a.$ *isum* $(\lambda b.\ f\ a * g\ b)\ B)\ A$
⟨*proof*⟩

lemma *isum-distribR*:
assumes *f*: *positive f* $(A{::}'a\ set)$ *sbounded f A* **and** *r*: $r \geq 0$
shows *isum f A* $* r =$ *isum* $(\lambda a.\ f\ a * r)\ A$
⟨*proof*⟩

lemma *isum-distribL*:
assumes *f*: *positive f* $(A{::}'a\ set)$ *sbounded f A* **and** *r*: $r \geq 0$
shows $r *$ *isum f A* $=$ *isum* $(\lambda a.\ r * f\ a)\ A$
⟨*proof*⟩

end

2 Discrete Subdistributions and Distributions

This theory defines countably discrete probability (sub)distributions and their monadic operators, namely:

- Kleisli extension, "ext"
- functorial action, the lifting operator "lift"
- monad unit, the indicator function "ind"
- monad counit, the flattening operators "flat" for subdistributions and "dflat" for subdistributions

Basic facts about them are proved, including the monadic laws.

In all operators except the monad counit (flattening/averaging), the operators for distributions are restrictions of those for subdistributions. For flattening, as explained later we must use two distinct operators "flat" and "dflat".

We also define the expectation operator, "expd", which is the Lebesgue integral for the discrete case.

theory *Discrete-Subdistributions-and-Distributions*
  imports *Infinite-Sums-of-Positive-Reals*
begin

2.1 Definitions and Basic Properties

definition *Subdis* :: $'a\ set \Rightarrow ('a \Rightarrow real)\ set$ **where**
  $Subdis\ A \equiv \{p.\ positive\ p\ A \wedge sbounded\ p\ A \wedge isum\ p\ A \leq 1\}$

definition *Dis* :: $'a\ set \Rightarrow ('a \Rightarrow real)\ set$ **where**
  $Dis\ A \equiv \{p.\ p \in Subdis\ A \wedge isum\ p\ A \geq 1\}$

lemma *Dis-incl-Subdis*: $Dis\ A \subseteq Subdis\ A$ ⟨*proof*⟩

lemma *Subdis-mono*: $p \in Subdis\ A \Longrightarrow B \subseteq A \Longrightarrow p \in Subdis\ B$
  ⟨*proof*⟩

lemma *Subdis-Dis2*: $Subdis\ (Subdis\ A) \subseteq Subdis\ (Dis\ A)$
  ⟨*proof*⟩

lemma *Subdis-ge-0*: $p \in Subdis\ A \Longrightarrow a \in A \Longrightarrow p\ a \geq 0$
  ⟨*proof*⟩

lemma *Subdis-le-1*: $p \in Subdis\ A \Longrightarrow a \in A \Longrightarrow p\ a \leq 1$
  ⟨*proof*⟩

lemma *Subdis-eq*:
  assumes $p \in Subdis\ A$ **and** $\forall a{\in}A.\ p1\ a = p\ a$
  shows $p1 \in Subdis\ A$
  ⟨*proof*⟩

lemma *Dis-Subdis-mono*: $p \in Dis\ A \Longrightarrow B \subseteq A \Longrightarrow p \in Subdis\ B$
  ⟨*proof*⟩

lemma *Dis-zeros-mono*: $p \in Dis\ A \Longrightarrow B \subseteq A \Longrightarrow \forall a{\in}A{-}B.\ p\ a = 0 \Longrightarrow p \in Dis\ B$
  ⟨*proof*⟩

lemma *Dis-ge-0*: $p \in Dis\ A \Longrightarrow a \in A \Longrightarrow p\ a \geq 0$

⟨proof⟩

lemma *Dis-le-1*: $p \in Dis\ A \Longrightarrow a \in A \Longrightarrow p\ a \le 1$
⟨proof⟩

lemma *Dis-isum-1*: $p \in Dis\ A \Longrightarrow isum\ p\ A = 1$
⟨proof⟩

lemma *Dis-sum-1*: $p \in Dis\ A \Longrightarrow finite\ A \Longrightarrow sum\ p\ A = 1$
⟨proof⟩

lemma *Dis-eq*:
assumes $p \in Dis\ A$ **and** $\forall a \in A.\ p1\ a = p\ a$
shows $p1 \in Dis\ A$
⟨proof⟩

lemma *Subdis-le-1-eq-1*: $p \in Subdis\ A \Longrightarrow 1 \le isum\ p\ A \Longrightarrow isum\ p\ A = 1$
⟨proof⟩

lemma *Subdis-sum-le-1*: $p \in Subdis\ A \Longrightarrow finite\ A \Longrightarrow sum\ p\ A \le 1$
⟨proof⟩

lemma *Subdis-sum-ge-0*: $p \in Subdis\ A \Longrightarrow finite\ A \Longrightarrow sum\ p\ A \ge 0$
⟨proof⟩

lemma *Subdis-sum-ge-0-sub*: $p \in Subdis\ A \Longrightarrow B \subseteq A \Longrightarrow finite\ B \Longrightarrow sum\ p\ B \ge 0$
⟨proof⟩

lemma *Subdis-sum-le-1-sub*: $p \in Subdis\ A \Longrightarrow B \subseteq A \Longrightarrow finite\ B \Longrightarrow sum\ p\ B \le 1$
⟨proof⟩

lemma *Subdis-sboundedL*:
assumes $p \in Subdis\ A\ \forall a \in A.\ g\ a \le x$
shows $sbounded\ (\lambda a.\ p\ a * g\ a)\ A$
⟨proof⟩

lemma *Subdis-sboundedR*:
assumes $p \in Subdis\ A\ \forall a \in A.\ g\ a \le x$
shows $sbounded\ (\lambda a.\ g\ a * p\ a)\ A$
⟨proof⟩

lemma *Subdis-isum-leL*:
assumes p: $p \in Subdis\ A$ **and** g: $positive\ g\ A\ \forall a \in A.\ g\ a \le x$ **and** x: $x \ge 0$
shows $isum\ (\lambda a.\ p\ a * g\ a)\ A \le x$
⟨proof⟩

lemma *Subdis-isum-leR*:
  assumes p: $p \in \mathit{Subdis}\ A$ **and** g: $\mathit{positive}\ g\ A\ \forall\, a{\in}A.\ g\ a \leq x$ **and** x: $x \geq 0$
  shows $\mathit{isum}\ (\lambda a.\ g\ a * p\ a)\ A \leq x$
$\langle \mathit{proof} \rangle$

lemma *Subdis-sum-le-Max*:
  assumes $\mathit{finite}\ A\ p \in \mathit{Subdis}\ A\ \mathit{positive}\ g\ A\ A \neq \{\}$
  shows $(\sum a{\in}A.\ p\ a * g\ a) \leq \mathit{Max}\ (g\ `\ A)$
$\langle \mathit{proof} \rangle$

lemma *Subdis-sum-le*:
  assumes $\mathit{finite}\ A\ p \in \mathit{Subdis}\ A\ \mathit{positive}\ g\ A\ A \neq \{\}\ \forall\, a{\in}A.\ g\ a \leq x$
  shows $(\sum a{\in}A.\ p\ a * g\ a) \leq x$
  $\langle \mathit{proof} \rangle$

2.2 Monadic structure

definition $\mathit{ind} :: {}'a \Rightarrow ({}'a \Rightarrow \mathit{real})$ **where**
  $\mathit{ind}\ a \equiv \lambda a'.\ \mathit{if}\ a' = a\ \mathit{then}\ 1\ \mathit{else}\ 0$

lemma *ind-simps*[*simp*]: $\bigwedge a.\ \mathit{ind}\ a\ a = 1$
  $\bigwedge a\ a'.\ a' \neq a \Longrightarrow \mathit{ind}\ a'\ a = 0$
  $\langle \mathit{proof} \rangle$

lemma *ind-eq-0-iff*[*simp*]: $\mathit{ind}\ a\ a' = 0 \longleftrightarrow a \neq a'$
  $\langle \mathit{proof} \rangle$

lemma *ind-eq-1-iff*[*simp*]: $\mathit{ind}\ a\ a' = 1 \longleftrightarrow a = a'$
  $\langle \mathit{proof} \rangle$

lemma *ind-ge-0*: $\mathit{ind}\ a\ a' \geq 0$
  $\langle \mathit{proof} \rangle$

lemma *ind-le-1*: $\mathit{ind}\ a\ a' \leq 1$
  $\langle \mathit{proof} \rangle$

lemma *positive-ind*[*simp*]: $\mathit{positive}\ (\mathit{ind}\ a)\ A$
  $\langle \mathit{proof} \rangle$

lemma *sbounded-ind*[*simp*]: $\mathit{sbounded}\ (\mathit{ind}\ a)\ A$
  $\langle \mathit{proof} \rangle$

lemma *sum-ind*[*simp*]: $\bigwedge a\ B.\ \mathit{finite}\ B \Longrightarrow a \in B \Longrightarrow \mathit{sum}\ (\mathit{ind}\ a)\ B = 1$
  $\bigwedge a\ B.\ \mathit{finite}\ B \Longrightarrow a \notin B \Longrightarrow \mathit{sum}\ (\mathit{ind}\ a)\ B = 0$
  $\langle \mathit{proof} \rangle$

lemma *isum-ind*[*simp*]: $\bigwedge a\ A.\ a \in A \Longrightarrow \mathit{isum}\ (\mathit{ind}\ a)\ A = 1$
  $\bigwedge a\ A.\ a \notin A \Longrightarrow \mathit{isum}\ (\mathit{ind}\ a)\ A = 0$
  $\langle \mathit{proof} \rangle$

lemma *ind-Subdis*[*simp*, *intro*!]: *ind a* ∈ *Subdis A*
⟨*proof*⟩

lemma *Dis-ind*[*simp*, *intro*!]: *a* ∈ *A* ⟹ *ind a* ∈ *Dis A*
⟨*proof*⟩

lemma *ind-mult-SubdisL*:
assumes *p*: *p* ∈ *Subdis A*
shows (λ*a*. *p a* ∗ *ind* (*f a*) *a*′) ∈ *Subdis A*
⟨*proof*⟩

lemma *ind-mult-SubdisR*:
assumes *p*: *p* ∈ *Subdis A*
shows (λ*a*. *ind* (*f a*) *a*′ ∗ *p a*) ∈ *Subdis A*
⟨*proof*⟩

lemma *isum-ind-multL*: *a*′ ∈ *A* ⟹ *f a*′ ≥ *0* ⟹ *isum* (λ*a*. *f a* ∗ *ind a*′ *a*) *A* = *f a*′
⟨*proof*⟩

lemma *isum-ind-multR*: *a*′ ∈ *A* ⟹ *f a*′ ≥ *0* ⟹ *isum* (λ*a*. *ind a*′ *a* ∗ *f a*) *A* = *f a*′
⟨*proof*⟩

definition *ext* :: ′*a set* ⇒ (′*a* ⇒ (′*b* ⇒ *real*)) ⇒ ((′*a* ⇒ *real*) ⇒ (′*b* ⇒ *real*))
where
ext A f ≡ λ*p b*. *isum* (λ*a*. *p a* ∗ *f a b*) *A*

lemma *ext-ge-0*:
assumes *f*: ∀ *a*∈*A*. *f a* ∈ *Subdis B* **and** *p*: *p* ∈ *Subdis A* **and** *b*: *b* ∈ *B*
shows *ext A f p b* ≥ *0*
⟨*proof*⟩

lemma *Subdis-sum-isum-le-1*:
assumes *B*: *finite B* **and** *f*: ∀ *a*∈*A*. *f a* ∈ *Subdis B* **and** *p*: *p* ∈ *Subdis A*
shows (∑ *b*∈*B*. *isum* (λ*a*. *p a* ∗ *f a b*) *A*) ≤ *1*
⟨*proof*⟩

lemma *sbounded-prod-Subdis*:
assumes *f*: ∀ *a*∈*A*. *f a* ∈ *Subdis B* **and** *p*: *p* ∈ *Subdis A*
shows *sbounded* (λ(*a*, *b*). *p b* ∗ *f b a*) (*B* × *A*)
⟨*proof*⟩

lemma *ext-eq*: $\forall a \in A.\ p1\ a = p2\ a \Longrightarrow \forall a \in A.\ \forall b \in B.\ f1\ a\ b = f2\ a\ b \Longrightarrow b \in B \Longrightarrow ext\ A\ f1\ p1\ b = ext\ A\ f2\ p2\ b$
⟨*proof*⟩

lemma *ext-Subdis*:
assumes *f*: $\forall a \in A.\ f\ a \in Subdis\ B$ **and** *p*: $p \in Subdis\ A$
shows $ext\ A\ f\ p \in Subdis\ B$
⟨*proof*⟩

lemma *ext-Dis*:
assumes *f*: $\forall a \in A.\ f\ a \in Dis\ B$ **and** *p*: $p \in Dis\ A$
shows $ext\ A\ f\ p \in Dis\ B$
⟨*proof*⟩

lemma *ext-ind*: $p \in Subdis\ A \Longrightarrow a \in A \Longrightarrow ext\ A\ ind\ p\ a = p\ a$
⟨*proof*⟩

lemma *ext-o*:
assumes *f*: $\forall a \in A.\ f\ a \in B$ **and** *gg*: $\forall b \in B.\ gg\ b \in Subdis\ C$ **and** *p*: $p \in Subdis\ A$ **and** *c*: $c \in C$
shows $ext\ A\ (gg\ o\ f)\ p\ c = ext\ B\ gg\ (ext\ A\ (ind\ o\ f)\ p)\ c$
⟨*proof*⟩

definition *lift* :: $'a\ set \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow real) \Rightarrow ('b \Rightarrow real)$ **where**
$lift\ A\ f\ p \equiv \lambda b.\ isum\ (\lambda a.\ p\ a)\ \{a.\ a \in A \wedge f\ a = b\}$

lemma *lift-ext*:
assumes *p*: $p \in Subdis\ A$
shows $lift\ A\ f\ p = ext\ A\ (ind\ o\ f)\ p$
⟨*proof*⟩

lemma *lift-eq*:
assumes *f*: $\forall a \in A.\ f1\ a = f2\ a$ **and** *p*: $\forall a \in A.\ p1\ a = p2\ a$ **and** *b*: $b \in B$
shows $lift\ A\ f1\ p1\ b = lift\ A\ f2\ p2\ b$
⟨*proof*⟩

lemma *lift-Subdis*:
assumes *p*: $p \in Subdis\ A$

shows *lift A f p* $\in$ *Subdis B*
⟨*proof*⟩

lemma *lift-Dis*:
assumes *f*: $\forall a \in A.$ *f a* $\in$ *B* **and** *p*: *p* $\in$ *Dis A*
shows *lift A f p* $\in$ *Dis B*
⟨*proof*⟩

lemma *lift-id*[*simp*]:
assumes *p*: *p* $\in$ *Subdis A* **and** $a \in A$
shows *lift A id p a* = *p a*
⟨*proof*⟩

lemma *lift-o*[*simp*]:
assumes *f*: $\forall a \in A.$ *f a* $\in$ *B* **and** *g*: $\forall b \in B.$ *g b* $\in$ *C* **and** *p*: *p* $\in$ *Subdis A* **and** *c*: $c \in C$
shows *lift A* (*g o f*) *p c* = *lift B g* (*lift A f p*) *c*
⟨*proof*⟩

lemma *lift-ind*:
assumes *a*: *a* $\in$ *A*
shows *lift A f* (*ind a*) = *ind* (*f a*)
⟨*proof*⟩

lemma *isum-lift*:
assumes *f*: $\forall a \in A.$ *f a* $\in$ *B* **and** *p*: *p* $\in$ *Subdis A*
shows *isum* (*lift A f p*) *B* = *isum p A*
⟨*proof*⟩

lemma *lift-reflects-Dis*:
assumes *f*: $\forall a \in A.$ *f a* $\in$ *B* **and** *p*: *p* $\in$ *Subdis A*
shows *lift A f p* $\in$ *Dis B* $\longleftrightarrow$ *p* $\in$ *Dis A*
⟨*proof*⟩

definition *flatP* :: ('*a* ⇒ *real*) *set* ⇒
(('*a* ⇒ *real*) ⇒ *real*) ⇒ ('*a* ⇒ *real*) **where**

$flatP\ Da\ pp \equiv \lambda a.\ isum\ (\lambda p.\ pp\ p * p\ a)\ Da$

lemma *flatP-ext*: $flatP\ Da = ext\ Da\ id$
⟨*proof*⟩

lemma *flatP-eq*: $\forall p \in Da.\ pp1\ p = pp2\ p \Longrightarrow a \in A \Longrightarrow flatP\ Da\ pp1\ a = flatP\ Da\ pp2\ a$
⟨*proof*⟩

lemma *flatP-Subdis*: $Da \subseteq Subdis\ A \Longrightarrow$
$pp \in Subdis\ Da \Longrightarrow flatP\ Da\ pp \in Subdis\ A$
⟨*proof*⟩

lemma *flatP-Da*: $\forall pp \in Dis\ Da.\ ext\ Da\ id\ pp \in Da \Longrightarrow$
$pp \in Dis\ Da \Longrightarrow flatP\ Da\ pp \in Da$
⟨*proof*⟩

lemma *flatP-lift-ind*:
assumes *Da*: $Da \subseteq Subdis\ A\ ind\ `\ A \subseteq Da$
and *p*: $p \in Subdis\ A$ **and** *a*: $a \in A$
shows $flatP\ Da\ (lift\ A\ ind\ p)\ a = p\ a$
⟨*proof*⟩

lemma *flatP-ind*:
assumes *Da*: $Da \subseteq Subdis\ A$
and $p \in Da$ **and** $a \in A$
shows $flatP\ Da\ (ind\ p)\ a = p\ a$
⟨*proof*⟩

lemma *flatP-lift*:
assumes *Da*: $Da \subseteq Subdis\ A$
and *Db*: $Db \subseteq Subdis\ B$
and *Dab*: $\forall p \in Da.\ lift\ A\ f\ p \in Db$
assumes *f*: $\forall a \in A.\ f\ a \in B$ **and** *pp*: $pp \in Subdis\ Da$ **and** *b*: $b \in B$
shows $flatP\ Db\ (lift\ Da\ (lift\ A\ f)\ pp)\ b = lift\ A\ f\ (flatP\ Da\ pp)\ b$
⟨*proof*⟩

lemma *flatP-flatP-lift*:

assumes *Da*: $Da \subseteq Subdis\ A$
and *fDa*: $\forall\, pp \in Daa.\ flatP\ Da\ pp \in Da$
and *Daa*: $Daa \subseteq Subdis\ Da$
assumes *ppp*: $ppp \in Subdis\ Daa$ **and** *a*: $a \in A$
shows $flatP\ Da\ (flatP\ Daa\ ppp)\ a = flatP\ Da\ (lift\ Daa\ (flatP\ Da)\ ppp)\ a$
⟨*proof*⟩

definition *flat* :: $'a\ set \Rightarrow (('a \Rightarrow real) \Rightarrow real) \Rightarrow ('a \Rightarrow real)$ **where**
$flat\ A\ pp \equiv \lambda a.\ isum\ (\lambda p.\ pp\ p * p\ a)\ (Subdis\ A)$

lemma *flat-flatP*: $flat\ A = flatP\ (Subdis\ A)$
⟨*proof*⟩

lemma *flat-ext*: $flat\ A = ext\ (Subdis\ A)\ id$
⟨*proof*⟩

lemma *flat-eq*: $\forall\, p{\in}Subdis\ A.\ pp1\ p = pp2\ p \Longrightarrow a \in A \Longrightarrow flat\ A\ pp1\ a = flat\ A\ pp2\ a$
⟨*proof*⟩

lemma *flat-Subdis*: $pp \in Subdis\ (Subdis\ A) \Longrightarrow flat\ A\ pp \in Subdis\ A$
⟨*proof*⟩

lemma *flat-lift-ind*:
assumes *p*: $p \in Subdis\ A$ **and** *a*: $a \in A$
shows $flat\ A\ (lift\ A\ ind\ p)\ a = p\ a$
⟨*proof*⟩

lemma *flat-ind*:
assumes $p \in Subdis\ A$ **and** $a \in A$
shows $flat\ A\ (ind\ p)\ a = p\ a$
⟨*proof*⟩

lemma *flat-lift*:
assumes *f*: $\forall\, a{\in}A.\ f\ a \in B$ **and** *pp*: $pp \in Subdis\ (Subdis\ A)$ **and** *b*: $b \in B$
shows $flat\ B\ (lift\ (Subdis\ A)\ (lift\ A\ f)\ pp)\ b = lift\ A\ f\ (flat\ A\ pp)\ b$
⟨*proof*⟩

lemma *flat-flat-lift*:
assumes *ppp*: *ppp* ∈ *Subdis* (*Subdis* (*Subdis A*)) **and** *a*: *a* ∈ *A*
shows *flat A* (*flat* (*Subdis A*) *ppp*) *a* = *flat A* (*lift* (*Subdis* (*Subdis A*)) (*flat A*) *ppp*) *a*
⟨*proof*⟩

definition *dflat* :: ′*a set* ⇒ ((′*a* ⇒ *real*) ⇒ *real*) ⇒ (′*a* ⇒ *real*) **where**
dflat A pp ≡ λ*a*. *isum* (λ*p*. *pp p* ∗ *p a*) (*Dis A*)

lemma *dflat-flatP*: *dflat A* = *flatP* (*Dis A*)
⟨*proof*⟩

lemma *dflat-ext*: *dflat A* = *ext* (*Dis A*) *id*
⟨*proof*⟩

lemma *dflat-eq*: ∀ *p*∈*Dis A*. *pp1 p* = *pp2 p* ⟹ *a* ∈ *A* ⟹ *dflat A pp1 a* = *dflat A pp2 a*
⟨*proof*⟩

lemma *dflat-Subdis*: *pp* ∈ *Subdis* (*Dis A*) ⟹ *dflat A pp* ∈ *Subdis A*
⟨*proof*⟩

lemma *dflat-Dis*: *pp* ∈ *Dis* (*Dis A*) ⟹ *dflat A pp* ∈ *Dis A*
⟨*proof*⟩

lemma *dflat-lift-ind*:
assumes *p*: *p* ∈ *Dis A* **and** *a*: *a* ∈ *A*
shows *dflat A* (*lift A ind p*) *a* = *p a*
⟨*proof*⟩

lemma *dflat-ind*:
assumes *p* ∈ *Dis A* **and** *a* ∈ *A*
shows *dflat A* (*ind p*) *a* = *p a*
⟨*proof*⟩

lemma *dflat-lift-Subdis*:
assumes *f*: ∀ *a*∈*A*. *f a* ∈ *B* **and** *pp*: *pp* ∈ *Subdis* (*Dis A*) **and** *b*: *b* ∈ *B*
shows *dflat B* (*lift* (*Dis A*) (*lift A f*) *pp*) *b* = *lift A f* (*dflat A pp*) *b*
⟨*proof*⟩

corollary *dflat-lift*:
assumes *f*: $\forall a{\in}A.\ f\ a \in B$ **and** *pp*: $pp \in Dis\ (Dis\ A)$ **and** *b*: $b \in B$
shows $dflat\ B\ (lift\ (Dis\ A)\ (lift\ A\ f)\ pp)\ b = lift\ A\ f\ (dflat\ A\ pp)\ b$
⟨*proof*⟩

lemma *dflat-dflat-lift-Subdis*:
assumes *ppp*: $ppp \in Subdis\ (Dis\ (Dis\ A))$ **and** *a*: $a \in A$
shows $dflat\ A\ (dflat\ (Dis\ A)\ ppp)\ a = dflat\ A\ (lift\ (Dis\ (Dis\ A))\ (dflat\ A)\ ppp)\ a$
⟨*proof*⟩

corollary *dflat-dflat-lift*:
assumes *ppp*: $ppp \in Dis\ (Dis\ (Dis\ A))$ **and** *a*: $a \in A$
shows $dflat\ A\ (dflat\ (Dis\ A)\ ppp)\ a = dflat\ A\ (lift\ (Dis\ (Dis\ A))\ (dflat\ A)\ ppp)\ a$
⟨*proof*⟩

lemma *dflat-from-flat*:
assumes *pp*: $pp \in Subdis\ (Dis\ A)$ **and** *a*: $a \in A$
shows $dflat\ A\ pp\ a = flat\ A\ (\lambda p.\ if\ p \in Dis\ A\ then\ pp\ p\ else\ 0)\ a$
⟨*proof*⟩

lemma *dflat-flat*:
assumes *pp*: $pp \in Subdis\ (Dis\ A)$ **and** *a*: $a \in A$ **and** $\forall p{\in}Subdis\ A - Dis\ A.\ pp\ p = 0$
shows $dflat\ A\ pp\ a = flat\ A\ pp\ a$
⟨*proof*⟩

lemma *dflat-flat′*:
assumes *pp*: $pp \in Dis\ (Dis\ A)$ **and** *a*: $a \in A$ **and** $\forall p{\in}Subdis\ A - Dis\ A.\ pp\ p = 0$
shows $dflat\ A\ pp\ a = flat\ A\ pp\ a$
⟨*proof*⟩

2.3 Expectation

definition *expd* :: $'a\ set \Rightarrow ('a \Rightarrow real) \Rightarrow ('a \Rightarrow real) \Rightarrow real$ **where**
$expd\ A\ p\ X \equiv isum\ (\lambda a.\ p\ a * X\ a)\ A$

lemma *ext-expd*: $ext\ A\ f\ p\ b = expd\ A\ p\ (\lambda a.\ f\ a\ b)$
⟨*proof*⟩

lemma *expd-ge-0'*:
assumes $p \in Subdis\ A$ **and** *positive f A* **and** *sbounded* $(\lambda a.\ p\ a * f\ a)\ A$
shows $expd\ A\ p\ f \geq 0$
⟨*proof*⟩

lemma *expd-ge-0*:
assumes p: $p \in Subdis\ A$ **and** f: *positive f A* $\forall a{\in}A.\ f\ a \leq x$
shows $expd\ A\ p\ f \geq 0$
⟨*proof*⟩

lemma *expd-le-upper*:
assumes p: $p \in Subdis\ A$ **and** f: *positive f A* $\forall a{\in}A.\ f\ a \leq x$ **and** x: $x \geq 0$
shows $expd\ A\ p\ f \leq x$
⟨*proof*⟩

lemma *expd-ge-lower-Subdis*:
assumes p: $p \in Subdis\ A$ **and** f: $\forall a{\in}A.\ f\ a \geq x$ **and** x: $x \geq 0$
and *pf*: *sbounded* $(\lambda a.\ p\ a * f\ a)\ A$
shows $expd\ A\ p\ f \geq x * isum\ p\ A$
⟨*proof*⟩

lemma *expd-ge-lower-Dis'*:
assumes p: $p \in Dis\ A$ **and** f: $\forall a{\in}A.\ f\ a \geq x$ **and** x: $x \geq 0$
and *pf*: *sbounded* $(\lambda a.\ p\ a * f\ a)\ A$
shows $expd\ A\ p\ f \geq x$
⟨*proof*⟩

lemma *expd-ge-lower-Dis*:
assumes p: $p \in Dis\ A$ **and** f: $\forall a{\in}A.\ f\ a \geq x$ $\forall a{\in}A.\ f\ a \leq y$
and *xy*: $x \geq 0$ $y \geq 0$
shows $expd\ A\ p\ f \geq x$
⟨*proof*⟩

lemma *expd-ge01*:
assumes p: $p \in Subdis\ A$ **and** f: $\forall a{\in}A.\ f\ a \geq 0$ $\forall a{\in}A.\ f\ a \leq 1$
shows $expd\ A\ p\ f \geq 0$
⟨*proof*⟩

lemma *expd-le01*:
assumes p: $p \in Subdis\ A$ **and** f: $\forall a{\in}A.\ f\ a \geq 0$ $\forall a{\in}A.\ f\ a \leq 1$
shows $expd\ A\ p\ f \leq 1$
⟨*proof*⟩

lemma *expd-const-Subdis*[*simp*]:
assumes p: $p \in Subdis\ A$ **and** $c \geq 0$
shows $expd\ A\ p\ (\lambda\text{-}.\ c) = c * isum\ p\ A$

⟨*proof*⟩

lemma *expd-const-le*:
assumes *p*: *p* ∈ *Subdis A* **and** *c* ≥ *0*
shows *expd A p* (λ-. *c*) ≤ *c*
⟨*proof*⟩

lemma *expd-const-Dis*[*simp*]:
assumes *p*: *p* ∈ *Dis A* **and** *c* ≥ *0*
shows *expd A p* (λ-. *c*) = *c*
⟨*proof*⟩

lemma *expd-eq-ct-iff* [*simp*]:
assumes *p* ∈ *Subdis A c* > *0*
shows *expd A p* (λ-. *c*) = *c* ⟷ *p* ∈ *Dis A*
⟨*proof*⟩

lemma *expd-0* [*simp*]: *expd A p* (λ-. *0*) = *0*
⟨*proof*⟩

lemma *expd-1-le-1*: *p* ∈ *Subdis A* ⟹ *expd A p* (λ-. *1*) ≤ *1*
⟨*proof*⟩

lemma *expd-1-eq-1* [*simp*]: *p* ∈ *Dis A* ⟹ *expd A p* (λ-. *1*) = *1*
⟨*proof*⟩

lemma *expd-plus'*:
assumes *p*: *p* ∈ *Subdis A*
and *f1*: *positive f1 A sbounded* (λ*a*. *p a* ∗ *f1 a*) *A*
and *f2*: *positive f2 A sbounded* (λ*a*. *p a* ∗ *f2 a*) *A*
shows *expd A p* (λ*a*. *f1 a* + *f2 a*) = *expd A p f1* + *expd A p f2*
⟨*proof*⟩

lemma *expd-plus*:
assumes *p*: *p* ∈ *Subdis A*
and *f1*: *positive f1 A bdd-above* (*f1*‘*A*)
and *f2*: *positive f2 A bdd-above* (*f2*‘*A*)
shows *expd A p* (λ*a*. *f1 a* + *f2 a*) = *expd A p f1* + *expd A p f2*
⟨*proof*⟩

lemma *expd-mult'*:
assumes *p*: *p* ∈ *Subdis A*
and *f*: *positive f A sbounded* (λ*a*. *p a* ∗ *f a*) *A* **and** *c*: *c* ≥ *0*
shows *expd A p* (λ*a*. *c* ∗ *f a*) = *c* ∗ *expd A p f*
⟨*proof*⟩

lemma *expd-mult*:
assumes *p*: *p* ∈ *Subdis A*
and *f*: *positive f A bdd-above* (*f*‘*A*) **and** *c*: *c* ≥ *0*

shows $expd\ A\ p\ (\lambda a.\ c * f\ a) = c * expd\ A\ p\ f$
$\langle proof \rangle$

end