A Proof from THE BOOK: The Partial Fraction Expansion of the Cotangent

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Abstract

In this article, I formalise a proof from THE BOOK [1, Chapter 23]; namely a formula that was called 'one of the most beautiful formulas involving elementary functions':

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right)$$

The proof uses Herglotz's trick to show the real case and analytic continuation for the complex case.

Contents

1 The Partial-Fraction Formula for the Cotangent Function

theory *Cotangent-PFD-Formula*
imports *HOL−Complex-Analysis.Complex-Analysis HOL−Real-Asymp.Real-Asymp*

begin

1.1 Auxiliary lemmas

lemma *uniformly-on-image*:
uniformly-on $(f \text{ ` } A)$ g = *filtercomap* $(\lambda h.\ h \circ f)$ (*uniformly-on* A $(g \circ f)$)
⟨*proof*⟩

lemma *uniform-limit-image*:
uniform-limit $(f \text{ ` } A)$ g h $F \longleftrightarrow$ *uniform-limit* A $(\lambda x\ y.\ g\ x\ (f\ y))$ $(\lambda x.\ h\ (f\ x))$ F
⟨*proof*⟩

lemma *Ints-add-iff1* [*simp*]: $x \in \mathbb{Z} \Longrightarrow x + y \in \mathbb{Z} \longleftrightarrow y \in \mathbb{Z}$
⟨*proof*⟩

lemma *Ints-add-iff2* [*simp*]: $y \in \mathbb{Z} \Longrightarrow x + y \in \mathbb{Z} \longleftrightarrow x \in \mathbb{Z}$
⟨*proof*⟩

If a set is discrete (i.e. the difference between any two points is bounded from below), it has no limit points:

lemma *discrete-imp-not-islimpt*:
assumes *e*: $0 < e$
and *d*: $\forall x \in S.\ \forall y \in S.\ \mathit{dist}\ y\ x < e \longrightarrow y = x$
shows $\neg x$ *islimpt* S
⟨*proof*⟩

In particular, the integers have no limit point:

lemma *Ints-not-limpt*: $\neg((x :: 'a :: \textit{real-normed-algebra-1})$ *islimpt* $\mathbb{Z})$
⟨*proof*⟩

The following lemma allows evaluating telescoping sums of the form

$$\sum_{n=0}^{\infty} (f(n) - f(n+k))$$

where $f(n) \longrightarrow 0$, i.e. where all terms except for the first k are cancelled by later summands.

lemma *sums-long-telescope*:
fixes $f :: \mathit{nat} \Rightarrow 'a :: \{$*topological-group-add*, *topological-comm-monoid-add*, *ab-group-add*$\}$
assumes *lim*: $f \longrightarrow 0$
shows $(\lambda n.\ f\ n - f\ (n + c))$ *sums* $(\sum k<c.\ f\ k)$ (**is** - *sums* ?*S*)
⟨*proof*⟩

1.2 Definition of auxiliary function

The following function is the infinite sum appearing on the right-hand side of the cotangent formula. It can be written either as

$$\sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right)$$

or as

$$2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2} \ .$$

definition *cot-pfd* :: *'a* :: {*real-normed-field*, *banach*} ⇒ *'a* **where**
cot-pfd x = (∑ *n*. *2* ∗ *x* / (*x* ^ *2* − *of-nat* (*Suc n*) ^ *2*))

The sum in the definition of *cot-pfd* converges uniformly on compact sets. This implies, in particular, that *cot-pfd* is holomorphic (and thus also continuous).

lemma *uniform-limit-cot-pfd-complex*:
assumes *R* ≥ *0*
shows *uniform-limit* (*cball 0 R* :: *complex set*)
(λ*N x*. ∑ *n*<*N*. *2* ∗ *x* / (*x* ^ *2* − *of-nat* (*Suc n*) ^ *2*)) *cot-pfd sequentially*
⟨*proof*⟩

lemma *sums-cot-pfd-complex*:
fixes *x* :: *complex*
shows (λ*n*. *2* ∗ *x* / (*x* ^ *2* − *of-nat* (*Suc n*) ^ *2*)) *sums cot-pfd x*
⟨*proof*⟩

lemma *sums-cot-pfd-complex'-aux*:
fixes *x* :: *'a* :: {*banach*, *real-normed-field*, *field-char-0*}
assumes *x* ∉ ℤ − {*0*}
shows *2* ∗ *x* / (*x* ^ *2* − *of-nat* (*Suc n*) ^ *2*) =
1 / (*x* + *of-nat* (*Suc n*)) + *1* / (*x* − *of-nat* (*Suc n*))
⟨*proof*⟩

lemma *sums-cot-pfd-complex'*:
fixes *x* :: *complex*
assumes *x* ∉ ℤ − {*0*}
shows (λ*n*. *1* / (*x* + *of-nat* (*Suc n*)) + *1* / (*x* − *of-nat* (*Suc n*))) *sums cot-pfd x*
⟨*proof*⟩

lemma *summable-cot-pfd-complex*:
fixes *x* :: *complex*
shows *summable* (λ*n*. *2* ∗ *x* / (*x* ^ *2* − *of-nat* (*Suc n*) ^ *2*))
⟨*proof*⟩

lemma *summable-cot-pfd-real*:
fixes x :: *real*
shows *summable* ($\lambda n.$ *2* $* x$ / (x ^ *2* $-$ *of-nat* (*Suc* n) ^ *2*))
⟨*proof*⟩

lemma *sums-cot-pfd-real*:
fixes x :: *real*
shows ($\lambda n.$ *2* $* x$ / (x ^ *2* $-$ *of-nat* (*Suc* n) ^ *2*)) *sums cot-pfd* x
⟨*proof*⟩

lemma *cot-pfd-complex-of-real* [*simp*]: *cot-pfd* (*complex-of-real* x) = *of-real* (*cot-pfd* x)
⟨*proof*⟩

lemma *uniform-limit-cot-pfd-real*:
assumes $R \geq 0$
shows *uniform-limit* (*cball 0 R* :: *real set*)
($\lambda N\, x.\ \sum n<N.$ *2* $* x$ / (x ^ *2* $-$ *of-nat* (*Suc* n) ^ *2*)) *cot-pfd sequentially*
⟨*proof*⟩

1.3 Holomorphicity and continuity

lemma *has-field-derivative-cot-pfd-complex*:
fixes z :: *complex*
assumes z: $z \in -(\mathbb{Z} - \{0\})$
shows (*cot-pfd has-field-derivative* ($-$*Polygamma 1* (*1* + z) $-$ *Polygamma 1* (*1* $-$ z))) (*at* z)
⟨*proof*⟩

lemma *has-field-derivative-cot-pfd-complex′* [*derivative-intros*]:
assumes (g *has-field-derivative* g') (*at* x *within* A) **and** $g\ x \notin \mathbb{Z} - \{0\}$
shows (($\lambda x.$ *cot-pfd* ($g\ x$ :: *complex*)) *has-field-derivative*
($-$*Polygamma 1* (*1* + $g\ x$) $-$ *Polygamma 1* (*1* $-$ $g\ x$)) $* g'$) (*at* x *within* A)
⟨*proof*⟩

lemma *Polygamma-real-conv-complex*: $x \neq 0 \Longrightarrow$ *Polygamma* $n\ x$ = *Re* (*Polygamma* n (*of-real* x))
⟨*proof*⟩

lemma *has-field-derivative-cot-pfd-real* [*derivative-intros*]:
assumes (g *has-field-derivative* g') (*at* x *within* A) **and** $g\ x \notin \mathbb{Z} - \{0\}$
shows (($\lambda x.$ *cot-pfd* ($g\ x$ :: *real*)) *has-field-derivative*
($-$*Polygamma 1* (*1* + $g\ x$) $-$ *Polygamma 1* (*1* $-$ $g\ x$)) $* g'$) (*at* x *within* A)
⟨*proof*⟩

lemma *holomorphic-on-cot-pfd* [*holomorphic-intros*]:
assumes $A \subseteq -(\mathbb{Z} - \{0\})$

shows *cot-pfd holomorphic-on A*
⟨*proof*⟩

lemma *holomorphic-on-cot-pfd'* [*holomorphic-intros*]:
assumes *f holomorphic-on A* $\bigwedge x.\ x \in A \Longrightarrow f\ x \notin \mathbb{Z} - \{0\}$
shows $(\lambda x.\ \textit{cot-pfd}\ (f\ x))$ *holomorphic-on A*
⟨*proof*⟩

lemma *continuous-on-cot-pfd-complex* [*continuous-intros*]:
assumes *continuous-on A f* $\bigwedge z.\ z \in A \Longrightarrow f\ z \notin \mathbb{Z} - \{0\}$
shows *continuous-on A* $(\lambda x.\ \textit{cot-pfd}\ (f\ x :: \textit{complex}))$
⟨*proof*⟩

lemma *continuous-on-cot-pfd-real* [*continuous-intros*]:
assumes *continuous-on A f* $\bigwedge z.\ z \in A \Longrightarrow f\ z \notin \mathbb{Z} - \{0\}$
shows *continuous-on A* $(\lambda x.\ \textit{cot-pfd}\ (f\ x :: \textit{real}))$
⟨*proof*⟩

1.4 Functional equations

In this section, we will show three few functional equations for the function *cot-pfd*. The first one is trivial; the other two are a bit tedious and not very insightful, so I will not comment on them.

cot-pfd is an odd function:

lemma *cot-pfd-complex-minus* [*simp*]: $\textit{cot-pfd}\ (-x :: \textit{complex}) = -\textit{cot-pfd}\ x$
⟨*proof*⟩

lemma *cot-pfd-real-minus* [*simp*]: $\textit{cot-pfd}\ (-x :: \textit{real}) = -\textit{cot-pfd}\ x$
⟨*proof*⟩

$1\ /\ x + \textit{cot-pfd}\ x$ is periodic with period 1:

lemma *cot-pfd-plus-1-complex*:
assumes $x \notin \mathbb{Z}$
shows $\textit{cot-pfd}\ (x + 1 :: \textit{complex}) = \textit{cot-pfd}\ x - 1\ /\ (x + 1) + 1\ /\ x$
⟨*proof*⟩

lemma *cot-pfd-plus-1-real*:
assumes $x \notin \mathbb{Z}$
shows $\textit{cot-pfd}\ (x + 1 :: \textit{real}) = \textit{cot-pfd}\ x - 1\ /\ (x + 1) + 1\ /\ x$
⟨*proof*⟩

cot-pfd satisfies the following functional equation:

$$2f(x) = f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) + \frac{2}{x+1}$$

lemma *cot-pfd-funeq-complex*:

fixes x :: *complex*
assumes $x \notin \mathbb{Z}$
shows $2 * cot\text{-}pfd\ x = cot\text{-}pfd\ (x\ /\ 2) + cot\text{-}pfd\ ((x + 1)\ /\ 2) + 2\ /\ (x + 1)$
⟨*proof*⟩

lemma *cot-pfd-funeq-real*:
fixes x :: *real*
assumes $x \notin \mathbb{Z}$
shows $2 * cot\text{-}pfd\ x = cot\text{-}pfd\ (x\ /\ 2) + cot\text{-}pfd\ ((x + 1)\ /\ 2) + 2\ /\ (x + 1)$
⟨*proof*⟩

1.5 The limit at 0

lemma *cot-pfd-real-tendsto-0*: $cot\text{-}pfd\ {-0}{\rightarrow}\ (0 :: real)$
⟨*proof*⟩

1.6 Final result

To show the final result, we first prove the real case using Herglotz's trick, following the presentation in 'Proofs from THE BOOK'. [1, Chapter 23].

lemma *cot-pfd-formula-real*:
assumes $x \notin \mathbb{Z}$
shows $pi * cot\ (pi * x) = 1\ /\ x + cot\text{-}pfd\ x$
⟨*proof*⟩

We now lift the result from the domain $\mathbb{R}\backslash\mathbb{Z}$ to $\mathbb{C}\backslash\mathbb{Z}$. We do this by noting that $\mathbb{C}\backslash\mathbb{Z}$ is connected and the point $\frac{1}{2}$ is both in $\mathbb{C}\backslash\mathbb{Z}$ and a limit point of $\mathbb{R}\backslash\mathbb{Z}$.

lemma *one-half-limit-point-Reals-minus-Ints*: $(1\ /\ 2 :: complex)$ *islimpt* $\mathbb{R} - \mathbb{Z}$
⟨*proof*⟩

theorem *cot-pfd-formula-complex*:
fixes z :: *complex*
assumes $z \notin \mathbb{Z}$
shows $pi * cot\ (pi * z) = 1\ /\ z + cot\text{-}pfd\ z$
⟨*proof*⟩

end

References

[1] M. Aigner and G. M. Ziegler. *Proofs from THE BOOK*. Springer, 4th edition, 2009.