# Complete Non-Orders and Fixed Points

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**Abstract**

We develop an Isabelle/HOL library of order-theoretic concepts, such as various completeness conditions and fixed-point theorems. We keep our formalization as general as possible: we reprove several well-known results about complete orders, often without any properties of ordering, thus complete non-orders. In particular, we generalize the Knaster–Tarski theorem so that we ensure the existence of a quasi-fixed point of monotone maps over complete non-orders, and show that the set of quasi-fixed points is complete under a mild condition—attractivity—which is implied by either antisymmetry or transitivity. This result generalizes and strengthens a result by Stauti and Maaden. Finally, we recover Kleene's fixed-point theorem for omega-complete non-orders, again using attractivity to prove that Kleene's fixed points are least quasi-fixed points.

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# 1 Introduction

The main driving force towards mechanizing mathematics using proof assistants has been the reliability they offer, exemplified prominently by [7], [8], [10], etc. In this work, we utilize another aspect of Isabelle/JEdit [17] as engineering tools for developing mathematical theories. We formalize order-theoretic concepts and results, adhering to an *as-general-as-possible* approach: most results concerning order-theoretic completeness and fixed-point theorems are proved without assuming the underlying relations to be orders (non-orders). In particular, we provide the following:

- A locale-based library for binary relations, as partly depicted in Figure 1.

- Various completeness results that generalize known theorems in order theory: Actually most relationships and duality of completeness conditions are proved without *any* properties of the underlying relations.

- Existence of fixed points: We show that a relation-preserving mapping $f : A \to A$ over a complete non-order $\langle A, \sqsubseteq \rangle$ admits a *quasi-fixed point*

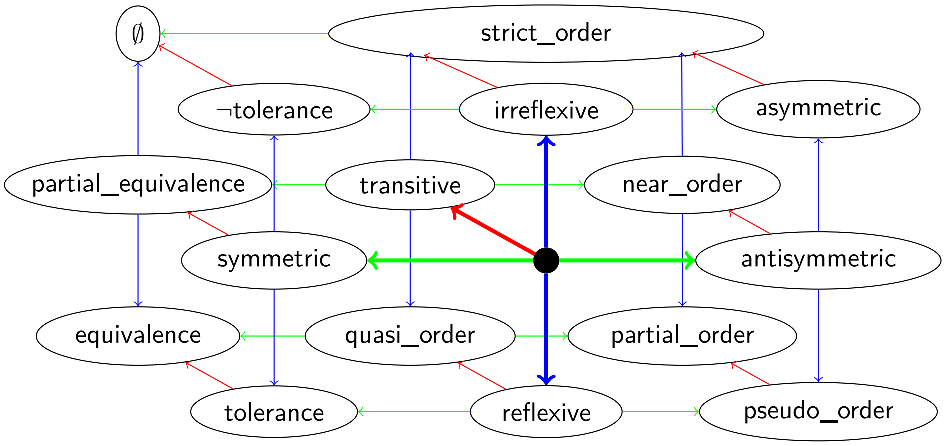


Figure 1: Combinations of basic properties. The black dot around the center represents arbitrary binary relations, and the five outgoing arrows indicate atomic assumptions. We do not present the combination of reflexive and irreflexive, which is empty, and one of symmetric and antisymmetric, which is a subset of equality. Node "¬tolerance" indicates the negated relation is tolerance, and "∅" is the empty relation.

$f(x) \sim x$, meaning $x \sqsubseteq f(x) \wedge f(x) \sqsubseteq x$. Clearly if $\sqsubseteq$ is antisymmetric then this implies the existence of fixed points $f(x) = x$.

- Completeness of the set of fixed points: We further show that if $\sqsubseteq$ satisfies a mild condition, which we call *attractivity* and which is implied by either transitivity or antisymmetry, then the set of quasi-fixed points is complete. Furthermore, we also show that if $\sqsubseteq$ is antisymmetric, then the set of *strict* fixed points $f(x) = x$ is complete.

- Kleene-style fixed-point theorems: For an $\omega$-complete non-order $\langle A, \sqsubseteq \rangle$ with a bottom element $\bot \in A$ (not necessarily unique) and for every $\omega$-continuous map $f : A \to A$, a supremum exists for the set $\{f^n(\bot) \mid n \in \mathbb{N}\}$, and it is a quasi-fixed point. If $\sqsubseteq$ is attractive, then the quasi-fixed points obtained this way are precisely the least quasi-fixed points.

We remark that all these results would have required much more effort than we spent (if possible at all), if we were not with the smart assistance by Isabelle. Our workflow was often the following: first we formalize existing proofs, try relaxing assumptions, see where proof breaks, and at some point ask for a counterexample.

**Related Work** Many attempts have been made to generalize the notion of completeness for lattices, conducted in different directions: by relaxing the notion of order itself, removing transitivity (pseudo-orders [15]); by relaxing the notion of lattice, considering minimal upper bounds instead of least upper bounds ($\chi$-posets [11]); by relaxing the notion of completeness, requiring the existence of least upper bounds for restricted classes of subsets (e.g., directed complete and $\omega$-complete, see [5] for a textbook). Considering those generalizations, it was natural to prove new versions of classical fixed-point theorems for maps preserving those structures, e.g., existence of least fixed points for monotone maps on (weak chain) complete pseudo-orders [4, 16], construction of least fixed points for $\omega$-continuous functions for $\omega$-complete lattices [12], (weak chain) completeness of the set of fixed points for monotone functions on (weak chain) complete pseudo-orders [13].

Concerning Isabelle formalization, one can easily find several formalizations of complete partial orders or lattices in Isabelle's standard library. They are, however, defined on partial orders, either in form of classes or locales, and thus not directly reusable for non-orders. Nevertheless we tried to make our formalization compatible with the existing ones, and various correspondences are ensured.

This work has been published in the conference paper [18].

## 2 Binary Relations

We start with basic properties of binary relations.

**theory** *Binary-Relations*
**imports**
*Main*
**begin**

**lemma** *conj-imp-eq-imp-imp*: $(P \wedge Q \Longrightarrow PROP\ R) \equiv (P \Longrightarrow Q \Longrightarrow PROP\ R)$
$\langle proof \rangle$

**lemma** *tranclp-trancl*: $r^{++} = (\lambda x\ y.\ (x,y) \in \{(a,b).\ r\ a\ b\}^{+})$
$\langle proof \rangle$

**lemma** *tranclp-id*[*simp*]: $transp\ r \Longrightarrow tranclp\ r = r$
$\langle proof \rangle$

**lemma** *transp-tranclp*[*simp*]: $transp\ (tranclp\ r)$ $\langle proof \rangle$

**lemma** *funpow-dom*: $f\ `\ A \subseteq A \Longrightarrow (f\widehat{\ }\widehat{\ }n)\ `\ A \subseteq A$ $\langle proof \rangle$

Below we introduce an Isabelle-notation for $\{\ldots x \ldots \mid x \in X\}$.

**syntax**
*-range* :: $'a \Rightarrow idts \Rightarrow 'a\ set$ $((1\{\text{- }/|./\text{ -}\}))$
*-image* :: $'a \Rightarrow pttrn \Rightarrow 'a\ set \Rightarrow 'a\ set$ $((1\{\text{- }/|./\ (\text{-}/ \in \text{-})\}))$
**translations**
$\{e\ |.\ p\} \rightleftharpoons \{e \mid p.\ CONST\ True\}$

$\{e \mid. \ p \in A\} \rightleftharpoons CONST\ image\ (\lambda p.\ e)\ A$

**lemma** *image-constant*:
  **assumes** $\bigwedge i.\ i \in I \Longrightarrow f\ i = y$
  **shows** $f\ `\ I = (if\ I = \{\}\ then\ \{\}\ else\ \{y\})$
  ⟨*proof*⟩

## 2.1 Various Definitions

Here we introduce various definitions for binary relations. The first one is our abbreviation for the dual of a relation.

**abbreviation**(*input*) *dual* $((\text{-}^{-})\ [1000]\ 1000)$ **where** $r^{-}\ x\ y \equiv r\ y\ x$

**lemma** *conversep-is-dual*[*simp*]: *conversep* = *dual* ⟨*proof*⟩

Monotonicity is already defined in the library, but we want one restricted to a domain.

**definition** *monotone-on* $X\ r\ s\ f \equiv \forall x\ y.\ x \in X \longrightarrow y \in X \longrightarrow r\ x\ y \longrightarrow s\ (f\ x)\ (f\ y)$

**lemmas** *monotone-onI* = *monotone-on-def*[*unfolded atomize-eq*, *THEN iffD2*, *rule-format*]
**lemma** *monotone-onD*: $monotone\text{-}on\ X\ r\ s\ f \Longrightarrow r\ x\ y \Longrightarrow x \in X \Longrightarrow y \in X \Longrightarrow s\ (f\ x)\ (f\ y)$
  ⟨*proof*⟩

**lemmas** *monotone-onE* = *monotone-on-def*[*unfolded atomize-eq*, *THEN iffD1*, *elim-format*, *rule-format*]

**lemma** *monotone-on-UNIV*[*simp*]: *monotone-on UNIV* = *monotone*
  ⟨*proof*⟩

**lemma** *monotone-on-dual*: $monotone\text{-}on\ X\ r\ s\ f \Longrightarrow monotone\text{-}on\ X\ r^{-}\ s^{-}\ f$
  ⟨*proof*⟩

**lemma** *monotone-on-id*: *monotone-on X r r id*
  ⟨*proof*⟩

**lemma** *monotone-on-cmono*: $A \subseteq B \Longrightarrow monotone\text{-}on\ B \le monotone\text{-}on\ A$
  ⟨*proof*⟩

Here we define the following notions in a standard manner

The symmetric part of a relation:

**definition** *sympartp* **where** $sympartp\ r\ x\ y \equiv r\ x\ y \land r\ y\ x$

**lemma** *sympartpI*[*intro*]:
  **fixes** $r$ (**infix** $\sqsubseteq$ *50*)
  **assumes** $x \sqsubseteq y$ **and** $y \sqsubseteq x$ **shows** $sympartp\ (\sqsubseteq)\ x\ y$
  ⟨*proof*⟩

**lemma** *sympartpE*[*elim*]:
  **fixes** $r$ (**infix** $\sqsubseteq$ *50*)
  **assumes** *sympartp* $(\sqsubseteq)$ $x$ $y$ **and** $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow$ *thesis* **shows** *thesis*
  ⟨*proof*⟩

**lemma** *sympartp-dual*: *sympartp* $r^- =$ *sympartp* $r$
  ⟨*proof*⟩

**lemma** *sympartp-eq*[*simp*]: *sympartp* $(=) = (=)$ ⟨*proof*⟩

**lemma** *reflclp-sympartp*[*simp*]: $($*sympartp* $r)^{==} =$ *sympartp* $r^{==}$ ⟨*proof*⟩

**definition** *equivpartp* $r\ x\ y \equiv x = y \lor r\ x\ y \land r\ y\ x$

**lemma** *sympartp-reflclp-equivp*[*simp*]: *sympartp* $r^{==} =$ *equivpartp* $r$ ⟨*proof*⟩

**lemma** *equivpartI*[*simp*]: *equivpartp* $r\ x\ x$
  **and** *sympartp-equivpartpI*: *sympartp* $r\ x\ y \Longrightarrow$ *equivpartp* $r\ x\ y$
  **and** *equivpartpCI*[*intro*]: $(x \neq y \Longrightarrow$ *sympartp* $r\ x\ y) \Longrightarrow$ *equivpartp* $r\ x\ y$
  ⟨*proof*⟩

**lemma** *equivpartpE*[*elim*]:
  **assumes** *equivpartp* $r\ x\ y$
    **and** $x = y \Longrightarrow$ *thesis*
    **and** $r\ x\ y \Longrightarrow r\ y\ x \Longrightarrow$ *thesis*
  **shows** *thesis*
  ⟨*proof*⟩

**lemma** *equivpartp-eq*[*simp*]: *equivpartp* $(=) = (=)$ ⟨*proof*⟩

**lemma** *sympartp-equivpartp*[*simp*]: *sympartp* (*equivpartp* $r$) = (*equivpartp* $r$)
  **and** *equivpartp-equivpartp*[*simp*]: *equivpartp* (*equivpartp* $r$) = (*equivpartp* $r$)
  **and** *equivpartp-sympartp*[*simp*]: *equivpartp* (*sympartp* $r$) = (*equivpartp* $r$)
  ⟨*proof*⟩

**lemma** *equivpartp-dual*: *equivpartp* $r^- =$ *equivpartp* $r$
  ⟨*proof*⟩

The asymmetric part:

**definition** *asympartp* $r\ x\ y \equiv r\ x\ y \land \neg\ r\ y\ x$

**lemma** *asympartpE*[*elim*]:
  **fixes** $r$ (**infix** $\sqsubseteq$ *50*)
  **shows** *asympartp* $(\sqsubseteq)$ $x$ $y \Longrightarrow (x \sqsubseteq y \Longrightarrow \neg y \sqsubseteq x \Longrightarrow$ *thesis*$) \Longrightarrow$ *thesis*
  ⟨*proof*⟩

**lemmas** *asympartpI*[*intro*] = *asympartp-def*[*unfolded atomize-eq*, *THEN iffD2*, *unfolded conj-imp-eq-imp-imp*, *rule-format*]

**lemma** *asympartp-eq*[*simp*]: *asympartp* (=) = *bot* ⟨*proof*⟩

**lemma** *asympartp-sympartp* [*simp*]: *asympartp* (*sympartp* *r*) = *bot*
**and** *sympartp-asympartp* [*simp*]: *sympartp* (*asympartp* *r*) = *bot*
⟨*proof*⟩

Restriction to a set:

**definition** *Restrp* (**infixl** ↾ *60*) **where** $(r \upharpoonright A)\ a\ b \equiv a \in A \wedge b \in A \wedge r\ a\ b$

**lemmas** *RestrpI*[*intro*!] = *Restrp-def*[*unfolded atomize-eq*, *THEN iffD2*, *unfolded conj-imp-eq-imp-imp*]
**lemmas** *RestrpE*[*elim*!] = *Restrp-def*[*unfolded atomize-eq*, *THEN iffD1*, *elim-format*, *unfolded conj-imp-eq-imp-imp*]

**lemma** *Restrp-UNIV*[*simp*]: $r \upharpoonright UNIV \equiv r$ ⟨*proof*⟩

**lemma** *Restrp-Restrp*[*simp*]: $r \upharpoonright A \upharpoonright B \equiv r \upharpoonright A \cap B$ ⟨*proof*⟩

**lemma** *sympartp-Restrp*[*simp*]: *sympartp* $(r \upharpoonright A) \equiv$ *sympartp* $r \upharpoonright A$
⟨*proof*⟩

Relational images:

**definition** *Imagep* (**infixr** ''' *59*) **where** $r$ ''' $A \equiv \{b.\ \exists\, a \in A.\ r\ a\ b\}$

**lemma** *Imagep-Image*: $r$ ''' $A = \{(a,b).\ r\ a\ b\}$ '' $A$
⟨*proof*⟩

**lemma** *in-Imagep*: $b \in r$ ''' $A \longleftrightarrow (\exists\, a \in A.\ r\ a\ b)$ ⟨*proof*⟩

**lemma** *ImagepI*: $a \in A \Longrightarrow r\ a\ b \Longrightarrow b \in r$ ''' $A$ ⟨*proof*⟩

**lemma** *subset-Imagep*: $B \subseteq r$ ''' $A \longleftrightarrow (\forall\, b{\in}B.\ \exists\, a{\in}A.\ r\ a\ b)$
⟨*proof*⟩

Bounds of a set:

**definition** *bound* $X\ r\ b \equiv \forall\, x \in X.\ r\ x\ b$

**lemma**
**fixes** $r$ (**infix** ⊑ *50*)
**shows** *boundI*[*intro*!]: $(\bigwedge x.\ x \in X \Longrightarrow x \sqsubseteq b) \Longrightarrow$ *bound* $X$ (⊑) $b$
**and** *boundE*[*elim*]: *bound* $X$ (⊑) $b \Longrightarrow ((\bigwedge x.\ x \in X \Longrightarrow x \sqsubseteq b) \Longrightarrow$ *thesis*$) \Longrightarrow$ *thesis*
⟨*proof*⟩

**lemma** *bound-empty*: *bound* {} = ($\lambda r\ x.$ *True*) ⟨*proof*⟩

**lemma** *bound-insert*[*simp*]:
**fixes** $r$ (**infix** ⊑ *50*)

**shows** *bound* (*insert* $x$ $X$) ($\sqsubseteq$) $b \longleftrightarrow x \sqsubseteq b \wedge$ *bound* $X$ ($\sqsubseteq$) $b$ ⟨*proof*⟩

Extreme (greatest) elements in a set:

**definition** *extreme* $X\ r\ e \equiv e \in X \wedge (\forall x \in X.\ r\ x\ e)$

**lemma**
**fixes** $r$ (**infix** $\sqsubseteq$ *50*)
**shows** *extremeI*[*intro*]: $e \in X \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow x \sqsubseteq e) \Longrightarrow$ *extreme* $X$ ($\sqsubseteq$) $e$
**and** *extremeD*: *extreme* $X$ ($\sqsubseteq$) $e \Longrightarrow e \in X$ *extreme* $X$ ($\sqsubseteq$) $e \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow x \sqsubseteq e)$
**and** *extremeE*[*elim*]: *extreme* $X$ ($\sqsubseteq$) $e \Longrightarrow (e \in X \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow x \sqsubseteq e) \Longrightarrow$ *thesis*) $\Longrightarrow$ *thesis*
⟨*proof*⟩

**lemma**
**fixes** $r$ (**infix** $\sqsubseteq$ *50*)
**shows** *extreme-UNIV*[*simp*]: *extreme UNIV* ($\sqsubseteq$) $t \longleftrightarrow (\forall x.\ x \sqsubseteq t)$ ⟨*proof*⟩

**lemma** *extremes-equiv*: *extreme* $X\ r\ b \Longrightarrow$ *extreme* $X\ r\ c \Longrightarrow$ *sympartp* $r\ b\ c$ ⟨*proof*⟩

**lemma** *bound-cmono*: **assumes** $X \subseteq Y$ **shows** *bound* $Y \le$ *bound* $X$
⟨*proof*⟩

**lemma** *sympartp-sympartp*[*simp*]: *sympartp* (*sympartp* $r$) = *sympartp* $r$ ⟨*proof*⟩

Now suprema and infima are given uniformly as follows. The definition is restricted to a given set.

**context**
**fixes** $A$ :: *'a set* **and** *less-eq* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
**begin**

**abbreviation** *extreme-bound* $X \equiv$ *extreme* $\{b \in A.$ *bound* $X$ ($\sqsubseteq$) $b\}$ $(\lambda x\ y.\ y \sqsubseteq x)$

**lemma** *extreme-boundI*[*intro*]:
**assumes** $\bigwedge b.$ *bound* $X$ ($\sqsubseteq$) $b \Longrightarrow b \in A \Longrightarrow s \sqsubseteq b$ **and** $\bigwedge x.\ x \in X \Longrightarrow x \sqsubseteq s$ **and** $s \in A$
**shows** *extreme-bound* $X\ s$
⟨*proof*⟩

**lemma** *extreme-bound-bound*: *extreme-bound* $X\ y \Longrightarrow x \in X \Longrightarrow x \sqsubseteq y$ ⟨*proof*⟩

**lemma** *extreme-bound-mono*:
**assumes** $XY$: $X \subseteq Y$
**and** $sX$: *extreme-bound* $X\ sX$
**and** $sY$: *extreme-bound* $Y\ sY$
**shows** $sX \sqsubseteq sY$
⟨*proof*⟩

**lemma** *extreme-bound-iff*:

**shows** *extreme-bound* $X\ s \longleftrightarrow s \in A \wedge (\forall c \in A.\ (\forall x \in X.\ x \sqsubseteq c) \longrightarrow s \sqsubseteq c) \wedge (\forall x \in X.\ x \sqsubseteq s)$
⟨*proof*⟩

**lemma** *extreme-bound-singleton-refl*[*simp*]:
*extreme-bound* $\{x\}\ x \longleftrightarrow x \in A \wedge x \sqsubseteq x$ ⟨*proof*⟩

**lemma** *extreme-bound-image-const*:
$x \sqsubseteq x \Longrightarrow I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f\ i = x) \Longrightarrow x \in A \Longrightarrow$ *extreme-bound* $(f\ \text{`}\ I)\ x$
⟨*proof*⟩

**lemma** *extreme-bound-UN-const*:
$x \sqsubseteq x \Longrightarrow I \neq \{\} \Longrightarrow (\bigwedge i\ y.\ i \in I \Longrightarrow P\ i\ y \longleftrightarrow x = y) \Longrightarrow x \in A \Longrightarrow$
*extreme-bound* $(\bigcup i{\in}I.\ \{y.\ P\ i\ y\})\ x$
⟨*proof*⟩

**end**

**context**
**fixes** $ir :: {'i} \Rightarrow {'i} \Rightarrow bool$ (**infix** $\preceq$ *50*)
**and** $r :: {'a} \Rightarrow {'a} \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)
**and** $f$ **and** $A$ **and** $e$ **and** $I$
**assumes** *fIA*: $f\ \text{`}\ I \subseteq A$
**and** *mono*: *monotone-on* $I\ (\preceq)\ (\sqsubseteq)\ f$
**and** *e*: *extreme* $I\ (\preceq)\ e$
**begin**

**lemma** *monotone-extreme-imp-extreme-bound*:
*extreme-bound* $A\ (\sqsubseteq)\ (f\ \text{`}\ I)\ (f\ e)$
⟨*proof*⟩

**lemma** *monotone-extreme-extreme-boundI*:
$x = f\ e \Longrightarrow$ *extreme-bound* $A\ (\sqsubseteq)\ (f\ \text{`}\ I)\ x$
⟨*proof*⟩

**end**

## 2.2 Locales for Binary Relations

We now define basic properties of binary relations, in form of *locales* [9, 2].

### 2.2.1 Syntactic Locales

The following locales do not assume anything, but provide infix notations for relations.

**locale** *less-eq-syntax* =
**fixes** *less-eq* $:: {'a} \Rightarrow {'a} \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)

**locale** *less-syntax* =
  **fixes** *less* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubset$ *50*)

**locale** *equivalence-syntax* =
  **fixes** *equiv* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sim$ *50*)
**begin**

**abbreviation** *equiv-class* ($[\text{-}]_\sim$) **where** $[x]_\sim \equiv \{\ y.\ x \sim y\ \}$

**end**

Next ones introduce abbreviations for dual etc. To avoid needless constants, one should be careful when declaring them as sublocales.

**locale** *less-eq-notations* = *less-eq-syntax*
**begin**

**abbreviation** (*input*) *greater-eq* (**infix** $\sqsupseteq$ *50*) **where** $x \sqsupseteq y \equiv y \sqsubseteq x$
**abbreviation** *sym* (**infix** $\sim$ *50*) **where** $(\sim) \equiv$ *sympartp* $(\sqsubseteq)$
**abbreviation** *less* (**infix** $\sqsubset$ *50*) **where** $(\sqsubset) \equiv$ *asympartp* $(\sqsubseteq)$
**abbreviation** *greater* (**infix** $\sqsupset$ *50*) **where** $(\sqsupset) \equiv (\sqsubset)^-$
**abbreviation** *equiv* (**infix** $(\simeq)$ *50*) **where** $(\simeq) \equiv$ *equivpartp* $(\sqsubseteq)$

**lemma** *asym-cases*[*consumes 1*, *case-names asym sym*]:
  **assumes** $x \sqsubseteq y$ **and** $x \sqsubset y \Longrightarrow$ *thesis* **and** $x \sim y \Longrightarrow$ *thesis*
  **shows** *thesis*
  ⟨*proof*⟩

**end**

**locale** *less-notations* = *less-syntax*
**begin**

**abbreviation** (*input*) *greater* (**infix** $\sqsupset$ *50*) **where** $x \sqsupset y \equiv y \sqsubset x$

**end**

**locale** *related-set* =
  **fixes** *A* :: $'a$ *set* **and** *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)

### 2.2.2 Basic Properties of Relations

In the following we define basic properties in form of locales.

Reflexivity restricted on a set:

**locale** *reflexive* = *related-set* +
  **assumes** *refl*[*intro*]: $x \in A \Longrightarrow x \sqsubseteq x$
**begin**

**lemma** *eq-implies*: $x = y \Longrightarrow x \in A \Longrightarrow x \sqsubseteq y$ ⟨*proof*⟩

**lemma** *extreme-singleton*[*simp*]: $x \in A \Longrightarrow$ *extreme* $\{x\}$ $(\sqsubseteq)$ $y \longleftrightarrow x = y$ ⟨*proof*⟩

**lemma** *extreme-bound-singleton*: $x \in A \Longrightarrow$ *extreme-bound* $A$ $(\sqsubseteq)$ $\{x\}$ $x$ ⟨*proof*⟩

**lemma** *reflexive-subset*: $B \subseteq A \Longrightarrow$ *reflexive* $B$ $(\sqsubseteq)$ ⟨*proof*⟩

**end**

**declare** *reflexive.intro*[*intro*!]

**lemma** *reflexiveE*[*elim*]:
  **assumes** *reflexive* $A$ $r$ **and** $(\bigwedge x.\ x \in A \Longrightarrow r\ x\ x) \Longrightarrow$ *thesis* **shows** *thesis*
  ⟨*proof*⟩

**lemma** *reflexive-cong*:
  $(\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b) \Longrightarrow$ *reflexive* $A$ $r \longleftrightarrow$ *reflexive* $A$
$r'$
  ⟨*proof*⟩

**locale** *irreflexive* = *related-set* $A$ $(\sqsubset)$ **for** $A$ **and** *less* (**infix** $\sqsubset$ *50*) +
  **assumes** *irrefl*: $x \in A \Longrightarrow \neg\ x \sqsubset x$
**begin**

**lemma** *irreflD*[*simp*]: $x \sqsubset x \Longrightarrow \neg x \in A$ ⟨*proof*⟩

**lemma** *implies-not-eq*: $x \sqsubset y \Longrightarrow x \in A \Longrightarrow x \neq y$ ⟨*proof*⟩

**lemma** *Restrp-irreflexive*: *irreflexive* *UNIV* $((\sqsubset)\restriction A)$
  ⟨*proof*⟩

**lemma** *irreflexive-subset*: $B \subseteq A \Longrightarrow$ *irreflexive* $B$ $(\sqsubset)$ ⟨*proof*⟩

**end**

**declare** *irreflexive.intro*[*intro*!]

**lemma** *irreflexive-cong*:
  $(\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b) \Longrightarrow$ *irreflexive* $A$ $r \longleftrightarrow$ *irreflexive*
$A$ $r'$
  ⟨*proof*⟩

**locale** *transitive* = *related-set* +
  **assumes** *trans*[*trans*]: $x \sqsubseteq y \Longrightarrow y \sqsubseteq z \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x$
$\sqsubseteq z$
**begin**

**interpretation** *less-eq-notations*⟨*proof*⟩

**lemma** *Restrp-transitive*: *transitive UNIV* $((\sqsubseteq)\restriction A)$
⟨*proof*⟩

**lemma** *bound-trans*[*trans*]: *bound* $X$ $(\sqsubseteq)$ $b \Longrightarrow b \sqsubseteq c \Longrightarrow X \subseteq A \Longrightarrow b \in A \Longrightarrow c \in A \Longrightarrow$ *bound* $X$ $(\sqsubseteq)$ $c$
⟨*proof*⟩

**lemma** *transitive-subset*:
**assumes** *BA*: $B \subseteq A$ **shows** *transitive* $B$ $(\sqsubseteq)$
⟨*proof*⟩

**lemma** *asympartp-transitive*: *transitive* $A$ $(\sqsubset)$
⟨*proof*⟩

**lemma** *reflclp-transitive*: *transitive* $A$ $(\sqsubseteq)^{==}$
⟨*proof*⟩

The symmetric part is also transitive, but this is done in the later semi-attractive locale

**end**

**declare** *transitive.intro*[*intro?*]

**lemma** *transitive-cong*:
**assumes** *r*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$ **shows** *transitive* $A$ $r$ $\longleftrightarrow$ *transitive* $A$ $r'$
⟨*proof*⟩

**lemma** *tranclp-transitive*: *transitive* $A$ (*tranclp* $r$)
⟨*proof*⟩

**locale** *symmetric* = *related-set* $A$ $(\sim)$ **for** $A$ **and** *equiv* (**infix** $\sim$ *50*) +
**assumes** *sym*[*sym*]: $x \sim y \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow y \sim x$
**begin**

**lemma** *sym-iff*: $x \in A \Longrightarrow y \in A \Longrightarrow x \sim y \longleftrightarrow y \sim x$
⟨*proof*⟩

**lemma** *Restrp-symmetric*: *symmetric UNIV* $((\sim)\restriction A)$
⟨*proof*⟩

**lemma** *symmetric-subset*: $B \subseteq A \Longrightarrow$ *symmetric* $B$ $(\sim)$
⟨*proof*⟩

**end**

**declare** *symmetric.intro*[*intro*]

**lemma** *symmetric-cong*:
$(\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b) \Longrightarrow$ *symmetric* $A\ r \longleftrightarrow$ *symmetric* $A\ r'$
⟨*proof*⟩

**global-interpretation** *sympartp*: *symmetric UNIV sympartp r*
**rewrites** $\bigwedge r.\ r \upharpoonright UNIV \equiv r$
**and** $\bigwedge x.\ x \in UNIV \equiv True$
**and** $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
**and** $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
⟨*proof*⟩

**lemma** *sympartp-symmetric*: *symmetric* $A$ (*sympartp* $r$) ⟨*proof*⟩

**locale** *antisymmetric* = *related-set* +
**assumes** *antisym*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow x = y$
**begin**

**interpretation** *less-eq-notations*⟨*proof*⟩

**lemma** *sym-iff-eq-refl*: $x \in A \Longrightarrow y \in A \Longrightarrow x \sim y \longleftrightarrow x = y \land y \sqsubseteq y$ ⟨*proof*⟩

**lemma** *equiv-iff-eq*[*simp*]: $x \in A \Longrightarrow y \in A \Longrightarrow x \simeq y \longleftrightarrow x = y$ ⟨*proof*⟩

**lemma** *extreme-unique*: $X \subseteq A \Longrightarrow$ *extreme* $X\ (\sqsubseteq)\ x \Longrightarrow$ *extreme* $X\ (\sqsubseteq)\ y \longleftrightarrow x = y$
⟨*proof*⟩

**lemma** *ex-extreme-iff-ex1*:
$X \subseteq A \Longrightarrow Ex$ (*extreme* $X\ (\sqsubseteq)) \longleftrightarrow Ex1$ (*extreme* $X\ (\sqsubseteq)$) ⟨*proof*⟩

**lemma** *ex-extreme-iff-the*:
$X \subseteq A \Longrightarrow Ex$ (*extreme* $X\ (\sqsubseteq)) \longleftrightarrow$ *extreme* $X\ (\sqsubseteq)$ (*The* (*extreme* $X\ (\sqsubseteq)$))
⟨*proof*⟩

**lemma** *Restrp-antisymmetric*: *antisymmetric UNIV* $((\sqsubseteq)\upharpoonright A)$
⟨*proof*⟩

**lemma** *antisymmetric-subset*: $B \subseteq A \Longrightarrow$ *antisymmetric* $B\ (\sqsubseteq)$
⟨*proof*⟩

**end**

**declare** *antisymmetric.intro*[*intro*]

**lemma** *antisymmetric-cong*:
$(\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b) \Longrightarrow$ *antisymmetric* $A\ r \longleftrightarrow$

*antisymmetric A r′*
⟨*proof*⟩

**lemma** *antisymmetric-union*:
**fixes** *less-eq* (**infix** $\sqsubseteq$ *50*)
**assumes** *A*: *antisymmetric A* ($\sqsubseteq$) **and** *B*: *antisymmetric B* ($\sqsubseteq$)
**and** *AB*: $\forall a \in A.\ \forall b \in B.\ a \sqsubseteq b \longrightarrow b \sqsubseteq a \longrightarrow a = b$
**shows** *antisymmetric* $(A \cup B)$ ($\sqsubseteq$)
⟨*proof*⟩

The following notion is new, generalizing antisymmetry and transitivity.

**locale** *semiattractive* = *related-set* +
**assumes** *attract*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow y \sqsubseteq z \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x \sqsubseteq z$
**begin**

**interpretation** *less-eq-notations*⟨*proof*⟩

**lemma** *equiv-order-trans*[*trans*]:
**assumes** *xy*: $x \simeq y$ **and** *yz*: $y \sqsubseteq z$ **and** *x*: $x \in A$ **and** *y*: $y \in A$ **and** *z*: $z \in A$
**shows** $x \sqsubseteq z$
⟨*proof*⟩

**lemma** *equiv-transitive*: *transitive A* ($\simeq$)
⟨*proof*⟩

**lemma** *sym-order-trans*[*trans*]:
**assumes** *xy*: $x \sim y$ **and** *yz*: $y \sqsubseteq z$ **and** *x*: $x \in A$ **and** *y*: $y \in A$ **and** *z*: $z \in A$
**shows** $x \sqsubseteq z$
⟨*proof*⟩

**interpretation** *sym*: *transitive A* ($\sim$)
⟨*proof*⟩

**lemmas** *sym-transitive* = *sym.transitive-axioms*

**lemma** *extreme-bound-quasi-const*:
**assumes** *C*: $C \subseteq A$ **and** *x*: $x \in A$ **and** *C0*: $C \neq \{\}$ **and** *const*: $\forall y \in C.\ y \sim x$
**shows** *extreme-bound A* ($\sqsubseteq$) *C x*
⟨*proof*⟩

**lemma** *extreme-bound-quasi-const-iff*:
**assumes** *C*: $C \subseteq A$ **and** *x*: $x \in A$ **and** *y*: $y \in A$ **and** *C0*: $C \neq \{\}$ **and** *const*: $\forall z \in C.\ z \sim x$
**shows** *extreme-bound A* ($\sqsubseteq$) *C y* $\longleftrightarrow x \sim y$
⟨*proof*⟩

**lemma** *Restrp-semiattractive*: *semiattractive UNIV* $((\sqsubseteq)\restriction A)$
⟨*proof*⟩

**lemma** *semiattractive-subset*: $B \subseteq A \Longrightarrow$ *semiattractive* $B$ $(\sqsubseteq)$
  ⟨*proof*⟩

**end**

**lemma** *semiattractive-cong*:
  **assumes** $r$: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  **shows** *semiattractive* $A\ r \longleftrightarrow$ *semiattractive* $A\ r'$ (**is** $?l \longleftrightarrow ?r$)
⟨*proof*⟩

**locale** *attractive* = *semiattractive* +
  **assumes** *semiattractive* $A$ $(\sqsubseteq)^-$
**begin**

**interpretation** *less-eq-notations*⟨*proof*⟩

**sublocale** *dual*: *semiattractive* $A$ $(\sqsubseteq)^-$
  **rewrites** $\bigwedge r.$ *sympartp* $(r \restriction A) \equiv$ *sympartp* $r \restriction A$
    **and** $\bigwedge r.$ *sympartp* (*sympartp* $r$) $\equiv$ *sympartp* $r$
    **and** *sympartp* $((\sqsubseteq) \restriction A)^- \equiv (\sim) \restriction A$
    **and** *sympartp* $(\sqsubseteq)^- \equiv (\sim)$
    **and** *equivpartp* $(\sqsubseteq)^- \equiv (\simeq)$
  ⟨*proof*⟩

**lemma** *order-equiv-trans*[*trans*]:
  **assumes** *xy*: $x \sqsubseteq y$ **and** *yz*: $y \simeq z$ **and** $x$: $x \in A$ **and** $y$: $y \in A$ **and** $z$: $z \in A$
  **shows** $x \sqsubseteq z$
  ⟨*proof*⟩

**lemma** *order-sym-trans*[*trans*]:
  **assumes** *xy*: $x \sqsubseteq y$ **and** *yz*: $y \sim z$ **and** $x$: $x \in A$ **and** $y$: $y \in A$ **and** $z$: $z \in A$
  **shows** $x \sqsubseteq z$
  ⟨*proof*⟩

**interpretation** *Restrp*: *semiattractive* *UNIV* $(\sqsubseteq)\restriction A$ ⟨*proof*⟩
**interpretation** *dual.Restrp*: *semiattractive* *UNIV* $(\sqsubseteq)^-\restriction A$ ⟨*proof*⟩

**lemma** *Restrp-attractive*: *attractive* *UNIV* $((\sqsubseteq)\restriction A)$
  ⟨*proof*⟩

**lemma** *attractive-subset*: $B \subseteq A \Longrightarrow$ *attractive* $B$ $(\sqsubseteq)$
  ⟨*proof*⟩

**end**

**lemma** *attractive-cong*:
  **assumes** $r$: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  **shows** *attractive* $A\ r \longleftrightarrow$ *attractive* $A\ r'$

$\langle proof \rangle$

**context** *antisymmetric* **begin**

**sublocale** *attractive*
$\langle proof \rangle$

**end**

**context** *transitive* **begin**

**sublocale** *attractive*
**rewrites** $\bigwedge r.$ *sympartp* $(r \upharpoonright A) \equiv$ *sympartp* $r \upharpoonright A$
**and** $\bigwedge r.$ *sympartp* (*sympartp* $r$) $\equiv$ *sympartp* $r$
**and** *sympartp* $(\sqsubseteq)^- \equiv$ *sympartp* $(\sqsubseteq)$
**and** (*sympartp* $(\sqsubseteq))^- \equiv$ *sympartp* $(\sqsubseteq)$
**and** (*sympartp* $(\sqsubseteq) \upharpoonright A)^- \equiv$ *sympartp* $(\sqsubseteq) \upharpoonright A$
**and** *asympartp* (*asympartp* $(\sqsubseteq)$) $=$ *asympartp* $(\sqsubseteq)$
**and** *asympartp* (*sympartp* $(\sqsubseteq)$) $=$ *bot*
**and** *asympartp* $(\sqsubseteq) \upharpoonright A =$ *asympartp* $((\sqsubseteq) \upharpoonright A)$
$\langle proof \rangle$

**end**

### 2.3 Combined Properties

Some combinations of the above basic properties are given names.

**locale** *asymmetric* = *related-set* $A$ $(\sqsubset)$ **for** $A$ **and** *less* (**infix** $\sqsubset$ *50*) +
**assumes** *asym*: $x \sqsubset y \Longrightarrow y \sqsubset x \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow$ *False*
**begin**

**sublocale** *irreflexive*
$\langle proof \rangle$

**lemma** *antisymmetric-axioms*: *antisymmetric* $A$ $(\sqsubset)$
$\langle proof \rangle$

**lemma** *Restrp-asymmetric*: *asymmetric UNIV* $((\sqsubset)\upharpoonright A)$
$\langle proof \rangle$

**lemma** *asymmetric-subset*: $B \subseteq A \Longrightarrow$ *asymmetric* $B$ $(\sqsubset)$
$\langle proof \rangle$

**end**

**lemma** *asymmetric-iff-irreflexive-antisymmetric*:
**fixes** *less* (**infix** $\sqsubset$ *50*)
**shows** *asymmetric* $A$ $(\sqsubset) \longleftrightarrow$ *irreflexive* $A$ $(\sqsubset) \wedge$ *antisymmetric* $A$ $(\sqsubset)$ (**is** *?l* $\longleftrightarrow$ *?r*)

⟨*proof*⟩

**lemma** *asymmetric-cong*:
  **assumes** *r*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  **shows** *asymmetric* $A\ r \longleftrightarrow$ *asymmetric* $A\ r'$
  ⟨*proof*⟩

**locale** *quasi-ordered-set* = *reflexive* + *transitive*
**begin**

**lemma** *quasi-ordered-subset*: $B \subseteq A \Longrightarrow$ *quasi-ordered-set* $B\ (\sqsubseteq)$
  ⟨*proof*⟩

**end**

**lemma** *quasi-ordered-set-cong*:
  **assumes** *r*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  **shows** *quasi-ordered-set* $A\ r \longleftrightarrow$ *quasi-ordered-set* $A\ r'$
  ⟨*proof*⟩

**locale** *near-ordered-set* = *antisymmetric* + *transitive*
**begin**

**interpretation** *Restrp*: *antisymmetric UNIV* $(\sqsubseteq)\restriction A$ ⟨*proof*⟩
**interpretation** *Restrp*: *transitive UNIV* $(\sqsubseteq)\restriction A$ ⟨*proof*⟩

**lemma** *Restrp-near-order*: *near-ordered-set UNIV* $((\sqsubseteq)\restriction A)$⟨*proof*⟩

**lemma** *near-ordered-subset*: $B \subseteq A \Longrightarrow$ *near-ordered-set* $B\ (\sqsubseteq)$
  ⟨*proof*⟩

**end**

**lemma** *near-ordered-set-cong*:
  **assumes** *r*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  **shows** *near-ordered-set* $A\ r \longleftrightarrow$ *near-ordered-set* $A\ r'$
  ⟨*proof*⟩

**locale** *pseudo-ordered-set* = *reflexive* + *antisymmetric*
**begin**

**interpretation** *less-eq-notations*⟨*proof*⟩

**lemma** *sym-eq*[*simp*]: $x \in A \Longrightarrow y \in A \Longrightarrow x \sim y \longleftrightarrow x = y$
  ⟨*proof*⟩

**lemma** *extreme-bound-singleton-eq*[*simp*]: $x \in A \Longrightarrow$ *extreme-bound* $A\ (\sqsubseteq)\ \{x\}\ y$
$\longleftrightarrow x = y$
  ⟨*proof*⟩

**lemma** *eq-iff*: $x \in A \Longrightarrow y \in A \Longrightarrow x = y \longleftrightarrow x \sqsubseteq y \wedge y \sqsubseteq x$ ⟨*proof*⟩

**lemma** *extreme-order-iff-eq*: $e \in A \Longrightarrow$ *extreme* $\{x \in A.\ x \sqsubseteq e\}$ $(\sqsubseteq)$ $s \longleftrightarrow e = s$
⟨*proof*⟩

**lemma** *pseudo-ordered-subset*: $B \subseteq A \Longrightarrow$ *pseudo-ordered-set* $B$ $(\sqsubseteq)$
⟨*proof*⟩

**end**

**lemma** *pseudo-ordered-set-cong*:
**assumes** $r$: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
**shows** *pseudo-ordered-set* $A$ $r \longleftrightarrow$ *pseudo-ordered-set* $A$ $r'$
⟨*proof*⟩

**locale** *partially-ordered-set* = *reflexive* + *antisymmetric* + *transitive*
**begin**

**sublocale** *pseudo-ordered-set* + *quasi-ordered-set* + *near-ordered-set* ⟨*proof*⟩

**lemma** *partially-ordered-subset*: $B \subseteq A \Longrightarrow$ *partially-ordered-set* $B$ $(\sqsubseteq)$
⟨*proof*⟩

**end**

**lemma** *partially-ordered-set-cong*:
**assumes** $r$: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
**shows** *partially-ordered-set* $A$ $r \longleftrightarrow$ *partially-ordered-set* $A$ $r'$
⟨*proof*⟩

**locale** *strict-ordered-set* = *irreflexive* + *transitive* $A$ $(\sqsubset)$
**begin**

**sublocale** *asymmetric*
⟨*proof*⟩

**lemma** *near-ordered-set-axioms*: *near-ordered-set* $A$ $(\sqsubset)$
⟨*proof*⟩

**interpretation** *Restrp*: *asymmetric UNIV* $(\sqsubset)\restriction A$ ⟨*proof*⟩
**interpretation** *Restrp*: *transitive UNIV* $(\sqsubset)\restriction A$ ⟨*proof*⟩

**lemma** *Restrp-strict-order*: *strict-ordered-set UNIV* $((\sqsubset)\restriction A)$⟨*proof*⟩

**lemma** *strict-ordered-subset*: $B \subseteq A \Longrightarrow$ *strict-ordered-set* $B$ $(\sqsubset)$
⟨*proof*⟩

**end**

**lemma** *strict-ordered-set-cong*:
  **assumes** *r*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  **shows** *strict-ordered-set* $A\ r \longleftrightarrow$ *strict-ordered-set* $A\ r'$
  ⟨*proof*⟩

**locale** *tolerance* = *symmetric* + *reflexive* $A$ $(\sim)$
**begin**

**lemma** *tolerance-subset*: $B \subseteq A \Longrightarrow$ *tolerance* $B$ $(\sim)$
  ⟨*proof*⟩

**end**

**lemma** *tolerance-cong*:
  **assumes** *r*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  **shows** *tolerance* $A\ r \longleftrightarrow$ *tolerance* $A\ r'$
  ⟨*proof*⟩

**global-interpretation** *equiv*: *tolerance UNIV equivpartp r*
  **rewrites** $\bigwedge r.\ r \upharpoonright UNIV \equiv r$
    **and** $\bigwedge x.\ x \in UNIV \equiv True$
    **and** $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
    **and** $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP$
$P2)$
  ⟨*proof*⟩

**locale** *partial-equivalence* = *symmetric* +
  **assumes** *transitive* $A$ $(\sim)$
**begin**

**sublocale** *transitive* $A$ $(\sim)$
  **rewrites** *sympartp* $(\sim)\upharpoonright A \equiv (\sim)\upharpoonright A$
    **and** *sympartp* $((\sim)\upharpoonright A) \equiv (\sim)\upharpoonright A$
  ⟨*proof*⟩

**lemma** *partial-equivalence-subset*: $B \subseteq A \Longrightarrow$ *partial-equivalence* $B$ $(\sim)$
  ⟨*proof*⟩

**end**

**lemma** *partial-equivalence-cong*:
  **assumes** *r*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  **shows** *partial-equivalence* $A\ r \longleftrightarrow$ *partial-equivalence* $A\ r'$
  ⟨*proof*⟩

**locale** *equivalence* = *symmetric* + *reflexive* $A$ $(\sim)$ + *transitive* $A$ $(\sim)$
**begin**

**sublocale** *tolerance* + *partial-equivalence* + *quasi-ordered-set* $A$ $(\sim)\langle proof\rangle$

**lemma** *equivalence-subset*: $B \subseteq A \Longrightarrow$ *equivalence* $B$ $(\sim)$
$\langle proof\rangle$

**end**

**lemma** *equivalence-cong*:
**assumes** $r$: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
**shows** *equivalence* $A\ r \longleftrightarrow$ *equivalence* $A\ r'$
$\langle proof\rangle$

Some combinations lead to uninteresting relations.

**context**
**fixes** $r :: 'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\bowtie$ *50*)
**begin**

**proposition** *reflexive-irreflexive-is-empty*:
**assumes** $r$: *reflexive* $A$ $(\bowtie)$ **and** $ir$: *irreflexive* $A$ $(\bowtie)$
**shows** $A = \{\}$
$\langle proof\rangle$

**proposition** *symmetric-antisymmetric-imp-eq*:
**assumes** $s$: *symmetric* $A$ $(\bowtie)$ **and** $as$: *antisymmetric* $A$ $(\bowtie)$
**shows** $(\bowtie)\restriction A \le (=)$
$\langle proof\rangle$

**proposition** *nontolerance*:
**shows** *irreflexive* $A$ $(\bowtie) \wedge$ *symmetric* $A$ $(\bowtie) \longleftrightarrow$ *tolerance* $A$ $(\lambda x\ y.\ \neg\ x \bowtie y)$
$\langle proof\rangle$

**proposition** *irreflexive-transitive-symmetric-is-empty*:
**assumes** $irr$: *irreflexive* $A$ $(\bowtie)$ **and** $tr$: *transitive* $A$ $(\bowtie)$ **and** $sym$: *symmetric* $A$ $(\bowtie)$
**shows** $(\bowtie)\restriction A = bot$
$\langle proof\rangle$

**end**

## 2.4 Totality

**locale** *semiconnex* = *related-set* $A$ $(\sqsubset)$ **for** $A$ **and** *less* (**infix** $\sqsubset$ *50*) +
**assumes** *semiconnex*: $x \in A \Longrightarrow y \in A \Longrightarrow x \sqsubset y \vee x = y \vee y \sqsubset x$
**begin**

**lemma** *cases*[*consumes 2*, *case-names less eq greater*]:
**assumes** $x \in A$ **and** $y \in A$ **and** $x \sqsubset y \Longrightarrow P$ **and** $x = y \Longrightarrow P$ **and** $y \sqsubset x \Longrightarrow P$
**shows** $P$ $\langle proof\rangle$

**lemma** *neqE*:
  **assumes** $x \in A$ **and** $y \in A$
  **shows** $x \neq y \Longrightarrow (x \sqsubset y \Longrightarrow P) \Longrightarrow (y \sqsubset x \Longrightarrow P) \Longrightarrow P$
  ⟨*proof*⟩

**lemma** *semiconnex-subset*: $B \subseteq A \Longrightarrow$ *semiconnex* $B$ $(\sqsubset)$
  ⟨*proof*⟩

**end**

**declare** *semiconnex.intro*[*intro*]

Totality is negated antisymmetry [14, Proposition 2.2.4].

**proposition** *semiconnex-iff-neg-antisymmetric*:
  **fixes** *less* (**infix** $\sqsubset$ *50*)
  **shows** *semiconnex* $A$ $(\sqsubset) \longleftrightarrow$ *antisymmetric* $A$ $(\lambda x\ y.\ \neg\ x \sqsubset y)$ (**is** *?l* $\longleftrightarrow$ *?r*)
⟨*proof*⟩

**lemma** *semiconnex-cong*:
  **assumes** $r$: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  **shows** *semiconnex* $A\ r \longleftrightarrow$ *semiconnex* $A\ r'$
  ⟨*proof*⟩

**locale** *semiconnex-irreflexive* = *semiconnex* + *irreflexive*
**begin**

**lemma** *neq-iff*: $x \in A \Longrightarrow y \in A \Longrightarrow x \neq y \longleftrightarrow x \sqsubset y \vee y \sqsubset x$ ⟨*proof*⟩

**lemma** *semiconnex-irreflexive-subset*: $B \subseteq A \Longrightarrow$ *semiconnex-irreflexive* $B$ $(\sqsubset)$
  ⟨*proof*⟩

**end**

**lemma** *semiconnex-irreflexive-cong*:
  **assumes** $r$: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  **shows** *semiconnex-irreflexive* $A\ r \longleftrightarrow$ *semiconnex-irreflexive* $A\ r'$
  ⟨*proof*⟩

**locale** *connex* = *related-set* +
  **assumes** *comparable*: $x \in A \Longrightarrow y \in A \Longrightarrow x \sqsubseteq y \vee y \sqsubseteq x$
**begin**

**interpretation** *less-eq-notations*⟨*proof*⟩

**sublocale** *reflexive* ⟨*proof*⟩

**lemma** *comparable-cases*[*consumes 2*, *case-names le ge*]:
  **assumes** $x \in A$ **and** $y \in A$ **and** $x \sqsubseteq y \Longrightarrow P$ **and** $y \sqsubseteq x \Longrightarrow P$ **shows** $P$
  ⟨*proof*⟩

**lemma** *comparable-three-cases*[*consumes 2*, *case-names less eq greater*]:
  **assumes** $x \in A$ **and** $y \in A$ **and** $x \sqsubset y \Longrightarrow P$ **and** $x \sim y \Longrightarrow P$ **and** $y \sqsubset x \Longrightarrow P$ **shows** $P$
  ⟨*proof*⟩

**lemma**
  **assumes** $x$: $x \in A$ **and** $y$: $y \in A$
  **shows** *not-iff-asym*: $\neg x \sqsubseteq y \longleftrightarrow y \sqsubset x$
    **and** *not-asym-iff*[*simp*]: $\neg x \sqsubset y \longleftrightarrow y \sqsubseteq x$
  ⟨*proof*⟩

**lemma** *connex-subset*: $B \subseteq A \Longrightarrow$ *connex* $B$ $(\sqsubseteq)$
  ⟨*proof*⟩

**end**

**lemmas** *connexE* = *connex.comparable-cases*

**lemmas** *connexI*[*intro*] = *connex.intro*

**context**
  **fixes** *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
**begin**

**lemma** *connex-iff-semiconnex-reflexive*: *connex* $A$ $(\sqsubseteq) \longleftrightarrow$ *semiconnex* $A$ $(\sqsubseteq) \wedge$ *reflexive* $A$ $(\sqsubseteq)$
  (**is** $?c \longleftrightarrow ?t \wedge ?r$)
⟨*proof*⟩

**lemma** *chain-connect*: *Complete-Partial-Order.chain* $r$ $A \equiv$ *connex* $A$ $r$
  ⟨*proof*⟩

**lemma** *connex-union*:
  **assumes** *connex* $X$ $(\sqsubseteq)$ **and** *connex* $Y$ $(\sqsubseteq)$ **and** $\forall x \in X.\ \forall y \in Y.\ x \sqsubseteq y \vee y \sqsubseteq x$
  **shows** *connex* $(X \cup Y)$ $(\sqsubseteq)$
  ⟨*proof*⟩

**end**

**lemma** *connex-cong*:
  **assumes** $r$: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  **shows** *connex* $A$ $r \longleftrightarrow$ *connex* $A$ $r'$
  ⟨*proof*⟩

**locale** *total-pseudo-ordered-set* = *connex* + *antisymmetric*
**begin**

**sublocale** *pseudo-ordered-set* ⟨*proof*⟩

**lemma** *not-weak-iff*:
**assumes** $x$: $x \in A$ **and** $y$: $y \in A$ **shows** $\neg\ y \sqsubseteq x \longleftrightarrow x \sqsubseteq y \wedge x \neq y$
⟨*proof*⟩

**lemma** *total-pseudo-ordered-subset*: $B \subseteq A \Longrightarrow$ *total-pseudo-ordered-set* $B$ $(\sqsubseteq)$
⟨*proof*⟩

**end**

**lemma** *total-pseudo-ordered-set-cong*:
**assumes** $r$: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
**shows** *total-pseudo-ordered-set* $A$ $r$ $\longleftrightarrow$ *total-pseudo-ordered-set* $A$ $r'$
⟨*proof*⟩

**locale** *total-quasi-ordered-set* = *connex* + *transitive*
**begin**

**sublocale** *quasi-ordered-set* ⟨*proof*⟩

**lemma** *total-quasi-ordered-subset*: $B \subseteq A \Longrightarrow$ *total-quasi-ordered-set* $B$ $(\sqsubseteq)$
⟨*proof*⟩

**end**

**lemma** *total-quasi-ordered-set-cong*:
**assumes** $r$: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
**shows** *total-quasi-ordered-set* $A$ $r$ $\longleftrightarrow$ *total-quasi-ordered-set* $A$ $r'$
⟨*proof*⟩

**locale** *total-ordered-set* = *total-quasi-ordered-set* + *antisymmetric*
**begin**

**sublocale** *partially-ordered-set* + *total-pseudo-ordered-set* ⟨*proof*⟩

**lemma** *total-ordered-subset*: $B \subseteq A \Longrightarrow$ *total-ordered-set* $B$ $(\sqsubseteq)$
⟨*proof*⟩

**end**

**lemma** *total-ordered-set-cong*:
**assumes** $r$: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
**shows** *total-ordered-set* $A$ $r$ $\longleftrightarrow$ *total-ordered-set* $A$ $r'$
⟨*proof*⟩

## 2.5 Well-Foundedness

**locale** *well-founded* = *related-set* $A$ $(\sqsubset)$ **for** $A$ **and** *less* (**infix** $\sqsubset$ *50*) +

**assumes** *induct*[*consumes 1*, *case-names less*, *induct set*]:
$a \in A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow (\bigwedge y.\ y \in A \Longrightarrow y \sqsubset x \Longrightarrow P\ y) \Longrightarrow P\ x) \Longrightarrow P\ a$
**begin**

**sublocale** *asymmetric*
⟨*proof*⟩

**lemma** *prefixed-Imagep-imp-empty*:
**assumes** *a*: $X \subseteq ((\sqsubset)$ ‘‘‘ $X) \cap A$ **shows** $X = \{\}$
⟨*proof*⟩

**lemma** *nonempty-imp-ex-extremal*:
**assumes** *QA*: $Q \subseteq A$ **and** *Q*: $Q \neq \{\}$
**shows** $\exists z \in Q.\ \forall y \in Q.\ \neg\ y \sqsubset z$
⟨*proof*⟩

**interpretation** *Restrp*: *well-founded UNIV* $(\sqsubset)\restriction A$
**rewrites** $\bigwedge x.\ x \in \mathit{UNIV} \equiv \mathit{True}$
**and** $(\sqsubset)\restriction A\restriction \mathit{UNIV} = (\sqsubset)\restriction A$
**and** $\bigwedge P1.\ (\mathit{True} \Longrightarrow \mathit{PROP}\ P1) \equiv \mathit{PROP}\ P1$
**and** $\bigwedge P1.\ (\mathit{True} \Longrightarrow P1) \equiv \mathit{Trueprop}\ P1$
**and** $\bigwedge P1\ P2.\ (\mathit{True} \Longrightarrow \mathit{PROP}\ P1 \Longrightarrow \mathit{PROP}\ P2) \equiv (\mathit{PROP}\ P1 \Longrightarrow \mathit{PROP}\ P2)$
⟨*proof*⟩

**lemmas** *Restrp-well-founded* = *Restrp.well-founded-axioms*
**lemmas** *Restrp-induct*[*consumes 0*, *case-names less*] = *Restrp.induct*

**interpretation** *Restrp.tranclp*: *well-founded UNIV* $((\sqsubset)\restriction A)^{++}$
**rewrites** $\bigwedge x.\ x \in \mathit{UNIV} \equiv \mathit{True}$
**and** $((\sqsubset)\restriction A)^{++} \restriction \mathit{UNIV} = ((\sqsubset)\restriction A)^{++}$
**and** $(((\sqsubset)\restriction A)^{++})^{++} = ((\sqsubset)\restriction A)^{++}$
**and** $\bigwedge P1.\ (\mathit{True} \Longrightarrow \mathit{PROP}\ P1) \equiv \mathit{PROP}\ P1$
**and** $\bigwedge P1.\ (\mathit{True} \Longrightarrow P1) \equiv \mathit{Trueprop}\ P1$
**and** $\bigwedge P1\ P2.\ (\mathit{True} \Longrightarrow \mathit{PROP}\ P1 \Longrightarrow \mathit{PROP}\ P2) \equiv (\mathit{PROP}\ P1 \Longrightarrow \mathit{PROP}\ P2)$
⟨*proof*⟩

**lemmas** *Restrp-tranclp-well-founded* = *Restrp.tranclp.well-founded-axioms*
**lemmas** *Restrp-tranclp-induct*[*consumes 0*, *case-names less*] = *Restrp.tranclp.induct*

**end**

**context**
**fixes** $A$ :: *'a set* **and** *less* :: *'a* ⇒ *'a* ⇒ *bool* (**infix** ⊏ *50*)
**begin**

**lemma** *well-foundedI-pf*:
**assumes** *pre*: $\bigwedge X.\ X \subseteq A \Longrightarrow X \subseteq ((\sqsubset)$ ‘‘‘ $X) \cap A \Longrightarrow X = \{\}$

**shows** *well-founded* $A$ $(\sqsubset)$
⟨*proof*⟩

**lemma** *well-foundedI-extremal*:
**assumes** $a$: $\bigwedge X.\ X \subseteq A \Longrightarrow X \neq \{\} \Longrightarrow \exists\, x \in X.\ \forall\, y \in X.\ \neg\ y \sqsubset x$
**shows** *well-founded* $A$ $(\sqsubset)$
⟨*proof*⟩

**lemma** *well-founded-iff-ex-extremal*:
*well-founded* $A$ $(\sqsubset) \longleftrightarrow (\forall\, X \subseteq A.\ X \neq \{\} \longrightarrow (\exists\, x \in X.\ \forall\, z \in X.\ \neg\ z \sqsubset x))$
⟨*proof*⟩

**end**

**lemma** *well-founded-cong*:
**assumes** $r$: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
**and** $A$: $\bigwedge a\ b.\ r'\ a\ b \Longrightarrow a \in A \longleftrightarrow a \in A'$
**and** $B$: $\bigwedge a\ b.\ r'\ a\ b \Longrightarrow b \in A \longleftrightarrow b \in A'$
**shows** *well-founded* $A\ r \longleftrightarrow$ *well-founded* $A'\ r'$
⟨*proof*⟩

**lemma** *wfP-iff-well-founded-UNIV*: *wfP* $r \longleftrightarrow$ *well-founded UNIV* $r$
⟨*proof*⟩

**lemma** *well-founded-singleton*:
**assumes** $\neg r\ x\ x$ **shows** *well-founded* $\{x\}$ $r$
⟨*proof*⟩

**lemma** *well-founded-Restrp*[*simp*]: *well-founded* $A$ $(r{\restriction}B) \longleftrightarrow$ *well-founded* $(A {\cap} B)$ $r$ (**is** *?l* $\longleftrightarrow$ *?r*)
⟨*proof*⟩

**lemma** (**in** *well-founded*) *well-founded-subset*:
**assumes** $B \subseteq A$ **shows** *well-founded* $B$ $(\sqsubset)$
⟨*proof*⟩

**lemma** *well-founded-extend*:
**fixes** *less* (**infix** $\sqsubset$ *50*)
**assumes** $A$: *well-founded* $A$ $(\sqsubset)$
**assumes** $B$: *well-founded* $B$ $(\sqsubset)$
**assumes** $AB$: $\forall\, a \in A.\ \forall\, b \in B.\ \neg b \sqsubset a$
**shows** *well-founded* $(A \cup B)$ $(\sqsubset)$
⟨*proof*⟩

**lemma** *closed-UN-well-founded*:
**fixes** $r$ (**infix** $\sqsubset$ *50*)
**assumes** $XX$: $\forall\, X{\in}XX.$ *well-founded* $X$ $(\sqsubset) \wedge (\forall\, x{\in}X.\ \forall\, y{\in}\bigcup XX.\ y \sqsubset x \longrightarrow y \in X)$
**shows** *well-founded* $(\bigcup XX)$ $(\sqsubset)$

⟨*proof*⟩

**lemma** *well-founded-cmono*:
  **assumes** $r'$: $r' \leq r$ **and** *wf*: *well-founded A r*
  **shows** *well-founded A* $r'$
⟨*proof*⟩

**locale** *well-founded-ordered-set* = *well-founded* + *transitive* - $(\sqsubset)$
**begin**

**sublocale** *strict-ordered-set*⟨*proof*⟩

**interpretation** *Restrp*: *strict-ordered-set UNIV* $(\sqsubset)\restriction A$ + *Restrp*: *well-founded UNIV* $(\sqsubset)\restriction A$
  ⟨*proof*⟩

**lemma** *Restrp-well-founded-order*: *well-founded-ordered-set UNIV* $((\sqsubset)\restriction A)$⟨*proof*⟩

**lemma** *well-founded-ordered-subset*: $B \subseteq A \Longrightarrow$ *well-founded-ordered-set B* $(\sqsubset)$
  ⟨*proof*⟩

**end**

**lemma** (**in** *well-founded*) *Restrp-tranclp-well-founded-ordered*: *well-founded-ordered-set UNIV* $((\sqsubset)\restriction A)^{++}$
  ⟨*proof*⟩

**locale** *well-related-set* = *related-set* +
  **assumes** *nonempty-imp-ex-extreme*: $X \subseteq A \Longrightarrow X \neq \{\} \Longrightarrow \exists\, e.$ *extreme X* $(\sqsubseteq)^{-}$ $e$
**begin**

**sublocale** *connex*
⟨*proof*⟩

**lemmas** *connex-axioms* = *connex-axioms*

**interpretation** *less-eq-notations*⟨*proof*⟩

**sublocale** *asym*: *well-founded A* $(\sqsubset)$
⟨*proof*⟩

**lemma** *well-related-subset*: $B \subseteq A \Longrightarrow$ *well-related-set B* $(\sqsubseteq)$
  ⟨*proof*⟩

**end**

**context**
  **fixes** *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)

**begin**

**lemma** *well-related-iff-neg-well-founded*:
$\quad$ *well-related-set* $A$ $(\sqsubseteq)$ $\longleftrightarrow$ *well-founded* $A$ $(\lambda x\ y.\ \neg\ y \sqsubseteq x)$
$\quad\langle proof\rangle$

**lemma** *well-related-singleton-refl*:
$\quad$ **assumes** $x \sqsubseteq x$ **shows** *well-related-set* $\{x\}$ $(\sqsubseteq)$
$\quad\langle proof\rangle$

**lemma** *closed-UN-well-related*:
$\quad$ **assumes** $XX$: $\forall X \in XX.$ *well-related-set* $X$ $(\sqsubseteq)$ $\wedge$ $(\forall x \in X.\ \forall y \in \bigcup XX.\ \neg x \sqsubseteq y \longrightarrow y \in X)$
$\quad$ **shows** *well-related-set* $(\bigcup XX)$ $(\sqsubseteq)$
$\quad\langle proof\rangle$

**end**

**lemma** *well-related-extend*:
$\quad$ **fixes** $r$ (**infix** $\sqsubseteq$ *50*)
$\quad$ **assumes** *well-related-set* $A$ $(\sqsubseteq)$ **and** *well-related-set* $B$ $(\sqsubseteq)$ **and** $\forall a \in A.\ \forall b \in B.\ a \sqsubseteq b$
$\quad$ **shows** *well-related-set* $(A \cup B)$ $(\sqsubseteq)$
$\quad\langle proof\rangle$

**locale** *pre-well-ordered-set* = *semiattractive* + *well-related-set*
**begin**

**interpretation** *less-eq-notations*$\langle proof\rangle$

**sublocale** *transitive*
$\langle proof\rangle$

**sublocale** *total-quasi-ordered-set*$\langle proof\rangle$

**lemmas** *connex-axioms* = *connex-axioms*

**lemma** *strict-weak-trans*[*trans*]:
$\quad$ **assumes** *xy*: $x \sqsubset y$ **and** *yz*: $y \sqsubseteq z$ **and** *x*: $x \in A$ **and** *y*: $y \in A$ **and** *z*: $z \in A$
$\quad$ **shows** $x \sqsubset z$
$\langle proof\rangle$

**lemma** *weak-strict-trans*[*trans*]:
$\quad$ **assumes** *xy*: $x \sqsubseteq y$ **and** *yz*: $y \sqsubset z$ **and** *x*: $x \in A$ **and** *y*: $y \in A$ **and** *z*: $z \in A$
$\quad$ **shows** $x \sqsubset z$
$\langle proof\rangle$

**end**

**lemma** *pre-well-ordered-iff*:
*pre-well-ordered-set* $A$ $r \longleftrightarrow$ *total-quasi-ordered-set* $A$ $r \wedge$ *well-founded* $A$ (*asympartp* $r$)
(**is** $?p \longleftrightarrow ?t \wedge ?w$)
$\langle proof \rangle$

**lemma** (**in** *semiattractive*) *pre-well-ordered-iff-well-related*:
**assumes** $XA$: $X \subseteq A$
**shows** *pre-well-ordered-set* $X$ $(\sqsubseteq) \longleftrightarrow$ *well-related-set* $X$ $(\sqsubseteq)$ (**is** $?l \longleftrightarrow ?r$)
$\langle proof \rangle$

**lemma** *semiattractive-extend*:
**fixes** $r$ (**infix** $\sqsubseteq$ *50*)
**assumes** $A$: *semiattractive* $A$ $(\sqsubseteq)$ **and** $B$: *semiattractive* $B$ $(\sqsubseteq)$
**and** $AB$: $\forall a \in A.\ \forall b \in B.\ a \sqsubseteq b \wedge \neg\ b \sqsubseteq a$
**shows** *semiattractive* $(A \cup B)$ $(\sqsubseteq)$
$\langle proof \rangle$

**lemma** *pre-well-order-extend*:
**fixes** $r$ (**infix** $\sqsubseteq$ *50*)
**assumes** $A$: *pre-well-ordered-set* $A$ $(\sqsubseteq)$ **and** $B$: *pre-well-ordered-set* $B$ $(\sqsubseteq)$
**and** $AB$: $\forall a \in A.\ \forall b \in B.\ a \sqsubseteq b \wedge \neg\ b \sqsubseteq a$
**shows** *pre-well-ordered-set* $(A \cup B)$ $(\sqsubseteq)$
$\langle proof \rangle$

**locale** *well-ordered-set* = *antisymmetric* + *well-related-set*
**begin**

**sublocale** *pre-well-ordered-set*$\langle proof \rangle$

**sublocale** *total-ordered-set*$\langle proof \rangle$

**lemma** *well-ordered-subset*: $B \subseteq A \Longrightarrow$ *well-ordered-set* $B$ $(\sqsubseteq)$
$\langle proof \rangle$

**lemmas** *connex-axioms* = *connex-axioms*

**end**

**lemma** (**in** *antisymmetric*) *well-ordered-iff-well-related*:
**assumes** $XA$: $X \subseteq A$
**shows** *well-ordered-set* $X$ $(\sqsubseteq) \longleftrightarrow$ *well-related-set* $X$ $(\sqsubseteq)$ (**is** $?l \longleftrightarrow ?r$)
$\langle proof \rangle$

**context**
**fixes** $A$ **and** *less-eq* (**infix** $\sqsubseteq$ *50*)
**assumes** $A$: $\forall a \in A.\ \forall b \in A.\ a \sqsubseteq b$
**begin**

**interpretation** *well-related-set* $A$ $(\sqsubseteq)$
  ⟨*proof*⟩

**lemmas** *trivial-well-related* = *well-related-set-axioms*

**lemma** *trivial-pre-well-order*: *pre-well-ordered-set* $A$ $(\sqsubseteq)$
  ⟨*proof*⟩

**end**

**context**
  **fixes** *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
**begin**

**interpretation** *less-eq-notations*⟨*proof*⟩

**lemma** *well-order-extend*:
  **assumes** $A$: *well-ordered-set* $A$ $(\sqsubseteq)$ **and** $B$: *well-ordered-set* $B$ $(\sqsubseteq)$
    **and** $ABa$: $\forall a \in A.\ \forall b \in B.\ a \sqsubseteq b \longrightarrow b \sqsubseteq a \longrightarrow a = b$
    **and** $AB$: $\forall a \in A.\ \forall b \in B.\ a \sqsubseteq b$
  **shows** *well-ordered-set* $(A \cup B)$ $(\sqsubseteq)$
⟨*proof*⟩

**interpretation** *singleton*: *antisymmetric* $\{a\}$ $(\sqsubseteq)$ **for** $a$ ⟨*proof*⟩

**lemmas** *singleton-antisymmetric*[*intro*!] = *singleton.antisymmetric-axioms*

**lemma** *singleton-well-ordered*[*intro*!]: $a \sqsubseteq a \Longrightarrow$ *well-ordered-set* $\{a\}$ $(\sqsubseteq)$
  ⟨*proof*⟩

**lemma** *closed-UN-well-ordered*:
  **assumes** *anti*: *antisymmetric* $(\bigcup\ XX)$ $(\sqsubseteq)$
    **and** $XX$: $\forall X \in XX.$ *well-ordered-set* $X$ $(\sqsubseteq) \wedge (\forall x \in X.\ \forall y \in \bigcup XX.\ \neg\ x \sqsubseteq y \longrightarrow$
$y \in X)$
  **shows** *well-ordered-set* $(\bigcup XX)$ $(\sqsubseteq)$
  ⟨*proof*⟩

**end**

Directed sets:

**definition** *directed* $A$ $r \equiv \forall x \in A.\ \forall y \in A.\ \exists z \in A.\ r\ x\ z \wedge r\ y\ z$

**lemmas** *directedI*[*intro*] = *directed-def*[*unfolded atomize-eq*, *THEN iffD2*, *rule-format*]

**lemmas** *directedD* = *directed-def*[*unfolded atomize-eq*, *THEN iffD1*, *rule-format*]

**context**
  **fixes** *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
**begin**

**lemma** *directedE*:
  **assumes** *directed* $A$ ($\sqsubseteq$) **and** $x \in A$ **and** $y \in A$
    **and** $\bigwedge z.\ z \in A \Longrightarrow x \sqsubseteq z \Longrightarrow y \sqsubseteq z \Longrightarrow$ *thesis*
  **shows** *thesis*
  $\langle proof \rangle$

**lemma** *directed-empty*[*simp*]: *directed* {} ($\sqsubseteq$) $\langle proof \rangle$

**lemma** *directed-union*:
  **assumes** *dX*: *directed* $X$ ($\sqsubseteq$) **and** *dY*: *directed* $Y$ ($\sqsubseteq$)
    **and** *XY*: $\forall x{\in}X.\ \forall y{\in}Y.\ \exists z \in X \cup Y.\ x \sqsubseteq z \wedge y \sqsubseteq z$
  **shows** *directed* ($X \cup Y$) ($\sqsubseteq$)
  $\langle proof \rangle$

**lemma** *directed-extend*:
  **assumes** *X*: *directed* $X$ ($\sqsubseteq$) **and** *Y*: *directed* $Y$ ($\sqsubseteq$) **and** *XY*: $\forall x{\in}X.\ \forall y{\in}Y.\ x$
$\sqsubseteq y$
  **shows** *directed* ($X \cup Y$) ($\sqsubseteq$)
$\langle proof \rangle$

**end**

**context** *connex* **begin**

**lemma** *directed*: *directed* $A$ ($\sqsubseteq$)
$\langle proof \rangle$

**end**

**context**
  **fixes** *ir* :: $'i \Rightarrow 'i \Rightarrow$ *bool* (**infix** $\preceq$ *50*)
  **fixes** *r* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
**begin**

**lemma** *monotone-connex-image*:
  **assumes** *mono*: *monotone-on* $I$ ($\preceq$) ($\sqsubseteq$) $f$
  **assumes** *connex*: *connex* $I$ ($\preceq$)
  **shows** *connex* ($f$ ‘ $I$) ($\sqsubseteq$)
$\langle proof \rangle$

**lemma** *monotone-directed-image*:
  **assumes** *mono*: *monotone-on* $I$ ($\preceq$) ($\sqsubseteq$) $f$
  **assumes** *dir*: *directed* $I$ ($\preceq$) **shows** *directed* ($f$ ‘ $I$) ($\sqsubseteq$)
$\langle proof \rangle$

**end**

## 2.6 Order Pairs

**locale** *compatible* = *related-set* + *related-set* $A$ ($\sqsubset$) **for** *less* (**infix** $\sqsubset$ *50*) +
  **assumes** *compat-right*[*trans*]: $x \sqsubseteq y \Longrightarrow y \sqsubset z \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x \sqsubset z$
  **assumes** *compat-left*[*trans*]: $x \sqsubset y \Longrightarrow y \sqsubseteq z \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x \sqsubset z$

**locale** *compatible-ordering* = *reflexive* + *irreflexive* + *compatible* +
  **assumes** *strict-implies-weak*: $x \sqsubset y \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow x \sqsubseteq y$
**begin**

The strict part is necessarily transitive.

The following sequence of declarations are in order to obtain fact names in a manner similar to the Isabelle/HOL facts of orders.

**sublocale** *strict*: *transitive* $A$ ($\sqsubset$)
  ⟨*proof*⟩

**sublocale** *strict-ordered-set* $A$ ($\sqsubset$) ⟨*proof*⟩

**thm** *strict.trans asym irrefl*

**lemma** *strict-implies-not-weak*: $x \sqsubset y \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow \neg\, y \sqsubseteq x$
  ⟨*proof*⟩

**end**

**context** *transitive* **begin**

**interpretation** *less-eq-notations*⟨*proof*⟩

**lemma** *asym-trans*[*trans*]:
  **shows** $x \sqsubset y \Longrightarrow y \sqsubseteq z \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x \sqsubset z$
    **and** $x \sqsubseteq y \Longrightarrow y \sqsubset z \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x \sqsubset z$
  ⟨*proof*⟩

**end**

**locale** *attractive-ordering* = *compatible-ordering* + *attractive*

**locale** *pseudo-ordering* = *compatible-ordering* + *pseudo-ordered-set*
**begin**

**sublocale** *attractive-ordering* ⟨*proof*⟩

**end**

**locale** *quasi-ordering* = *compatible-ordering* + *quasi-ordered-set*
**begin**

**sublocale** *attractive-ordering* ⟨*proof*⟩

**end**

**locale** *partial-ordering* = *compatible-ordering* + *partially-ordered-set*
**begin**

**sublocale** *pseudo-ordering* + *quasi-ordering* ⟨*proof*⟩

**end**

**locale** *well-founded-ordering* = *quasi-ordering* + *well-founded*

**locale** *total-ordering* = *compatible-ordering* + *total-ordered-set*
**begin**

**sublocale** *partial-ordering* ⟨*proof*⟩

**end**

**locale** *strict-total-ordering* = *partial-ordering* + *semiconnex* $A$ $(\sqsubset)$
**begin**

**sublocale** *semiconnex-irreflexive* ⟨*proof*⟩

**sublocale** *connex*
⟨*proof*⟩

**sublocale** *total-ordering* ⟨*proof*⟩

**lemma** *not-weak*[*simp*]:
**assumes** $x \in A$ **and** $y \in A$ **shows** $\neg\, x \sqsubseteq y \longleftrightarrow y \sqsubset x$
⟨*proof*⟩

**lemma** *not-strict*[*simp*]: $x \in A \Longrightarrow y \in A \Longrightarrow \neg\, x \sqsubset y \longleftrightarrow y \sqsubseteq x$
⟨*proof*⟩

**end**

## 2.7 Relating to Classes

In Isabelle 2020, we should declare sublocales in class before declaring dual sublocales, since otherwise facts would be prefixed by "dual.dual."

**context** *ord* **begin**

**abbreviation** *least* **where** *least* $X \equiv$ *extreme* $X$ $(\lambda x\ y.\ y \le x)$

**abbreviation** *greatest* **where** *greatest* $X \equiv$ *extreme* $X$ $(\leq)$

**abbreviation** *supremum* **where** *supremum* $X \equiv$ *least* (*Collect* (*bound* $X$ $(\leq)$))

**abbreviation** *infimum* **where** *infimum* $X \equiv$ *greatest* (*Collect* (*bound* $X$ ($\lambda x\ y.\ y \leq x$)))

**lemma** *Least-eq-The-least*: *Least* $P$ = *The* (*least* $\{x.\ P\ x\}$)
⟨*proof*⟩

**lemma** *Greatest-eq-The-greatest*: *Greatest* $P$ = *The* (*greatest* $\{x.\ P\ x\}$)
⟨*proof*⟩

**end**

**lemma** *Ball-UNIV*[*simp*]: *Ball UNIV* = *All* ⟨*proof*⟩
**lemma** *Bex-UNIV*[*simp*]: *Bex UNIV* = *Ex* ⟨*proof*⟩

**class** *compat* = *ord* + **assumes** *compatible-ordering UNIV* $(\leq)$ $(<)$
**begin**

**sublocale** *order*: *compatible-ordering UNIV*
**rewrites** $\bigwedge x.\ x \in$ *UNIV* $\equiv$ *True*
**and** $\bigwedge X.\ X \subseteq$ *UNIV* $\equiv$ *True*
**and** $\bigwedge r.\ r \restriction$ *UNIV* $\equiv r$
**and** $\bigwedge P.$ *True* $\wedge P \equiv P$
**and** *Ball UNIV* $\equiv$ *All*
**and** *Bex UNIV* $\equiv$ *Ex*
**and** *sympartp* $(\leq)^{-} \equiv$ *sympartp* $(\leq)$
**and** $\bigwedge P1.$ (*True* $\Longrightarrow$ *PROP P1*) $\equiv$ *PROP P1*
**and** $\bigwedge P1.$ (*True* $\Longrightarrow$ *P1*) $\equiv$ *Trueprop P1*
**and** $\bigwedge P1\ P2.$ (*True* $\Longrightarrow$ *PROP P1* $\Longrightarrow$ *PROP P2*) $\equiv$ (*PROP P1* $\Longrightarrow$ *PROP P2*)
⟨*proof*⟩

**end**

We should have imported locale-based facts in classes, e.g.:

**thm** *order.trans order.strict.trans order.refl order.irrefl order.asym order.extreme-bound-singleton*

**class** *attractive-order* = *ord* + **assumes** *attractive-ordering UNIV* $(\leq)$ $(<)$
**begin**

We need to declare subclasses before sublocales in order to preserve facts for superclasses.

**interpretation** *attractive-ordering UNIV*
⟨*proof*⟩

**subclass** *compat* ⟨*proof*⟩

**sublocale** *order*: *attractive-ordering UNIV*
  **rewrites** $\bigwedge x.\ x \in UNIV \equiv True$
    **and** $\bigwedge X.\ X \subseteq UNIV \equiv True$
    **and** $\bigwedge r.\ r \upharpoonright UNIV \equiv r$
    **and** $\bigwedge P.\ True \wedge P \equiv P$
    **and** $Ball\ UNIV \equiv All$
    **and** $Bex\ UNIV \equiv Ex$
    **and** $sympartp\ (\leq)^{-} \equiv sympartp\ (\leq)$
    **and** $\bigwedge P1.\ (True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
    **and** $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
    **and** $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
  ⟨*proof*⟩

**end**

**thm** *order.extreme-bound-quasi-const*

**class** *psorder* = *ord* + **assumes** *pseudo-ordering UNIV* $(\leq)$ $(<)$
**begin**

**interpretation** *pseudo-ordering UNIV* ⟨*proof*⟩

**subclass** *attractive-order* ⟨*proof*⟩

**sublocale** *order*: *pseudo-ordering UNIV*
  **rewrites** $\bigwedge x.\ x \in UNIV \equiv True$
    **and** $\bigwedge X.\ X \subseteq UNIV \equiv True$
    **and** $\bigwedge r.\ r \upharpoonright UNIV \equiv r$
    **and** $\bigwedge P.\ True \wedge P \equiv P$
    **and** $Ball\ UNIV \equiv All$
    **and** $Bex\ UNIV \equiv Ex$
    **and** $sympartp\ (\leq)^{-} \equiv sympartp\ (\leq)$
    **and** $\bigwedge P1.\ (True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
    **and** $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
    **and** $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
  ⟨*proof*⟩

**end**

**class** *qorder* = *ord* + **assumes** *quasi-ordering UNIV* $(\leq)$ $(<)$
**begin**

**interpretation** *quasi-ordering UNIV* ⟨*proof*⟩

**subclass** *attractive-order* ⟨*proof*⟩

**sublocale** *order*: *quasi-ordering UNIV*
**rewrites** $\bigwedge x.\ x \in \mathit{UNIV} \equiv \mathit{True}$
**and** $\bigwedge X.\ X \subseteq \mathit{UNIV} \equiv \mathit{True}$
**and** $\bigwedge r.\ r \upharpoonright \mathit{UNIV} \equiv r$
**and** $\bigwedge P.\ \mathit{True} \wedge P \equiv P$
**and** $\mathit{Ball\ UNIV} \equiv \mathit{All}$
**and** $\mathit{Bex\ UNIV} \equiv \mathit{Ex}$
**and** $\mathit{sympartp}\ (\leq)^{-} \equiv \mathit{sympartp}\ (\leq)$
**and** $\bigwedge P1.\ (\mathit{True} \Longrightarrow \mathit{PROP\ P1}) \equiv \mathit{PROP\ P1}$
**and** $\bigwedge P1.\ (\mathit{True} \Longrightarrow P1) \equiv \mathit{Trueprop\ P1}$
**and** $\bigwedge P1\ P2.\ (\mathit{True} \Longrightarrow \mathit{PROP\ P1} \Longrightarrow \mathit{PROP\ P2}) \equiv (\mathit{PROP\ P1} \Longrightarrow \mathit{PROP\ P2})$
⟨*proof*⟩

**end**

**class** *porder* = *ord* + **assumes** *partial-ordering UNIV* $(\leq)$ $(<)$
**begin**

**interpretation** *partial-ordering UNIV*
⟨*proof*⟩

**subclass** *psorder* ⟨*proof*⟩
**subclass** *qorder* ⟨*proof*⟩

**sublocale** *order*: *partial-ordering UNIV*
**rewrites** $\bigwedge x.\ x \in \mathit{UNIV} \equiv \mathit{True}$
**and** $\bigwedge X.\ X \subseteq \mathit{UNIV} \equiv \mathit{True}$
**and** $\bigwedge r.\ r \upharpoonright \mathit{UNIV} \equiv r$
**and** $\bigwedge P.\ \mathit{True} \wedge P \equiv P$
**and** $\mathit{Ball\ UNIV} \equiv \mathit{All}$
**and** $\mathit{Bex\ UNIV} \equiv \mathit{Ex}$
**and** $\mathit{sympartp}\ (\leq)^{-} \equiv \mathit{sympartp}\ (\leq)$
**and** $\bigwedge P1.\ (\mathit{True} \Longrightarrow \mathit{PROP\ P1}) \equiv \mathit{PROP\ P1}$
**and** $\bigwedge P1.\ (\mathit{True} \Longrightarrow P1) \equiv \mathit{Trueprop\ P1}$
**and** $\bigwedge P1\ P2.\ (\mathit{True} \Longrightarrow \mathit{PROP\ P1} \Longrightarrow \mathit{PROP\ P2}) \equiv (\mathit{PROP\ P1} \Longrightarrow \mathit{PROP\ P2})$
⟨*proof*⟩

**end**

**class** *wf-qorder* = *ord* + **assumes** *well-founded-ordering UNIV* $(\leq)$ $(<)$
**begin**

**interpretation** *well-founded-ordering UNIV*
⟨*proof*⟩

**subclass** *qorder* ⟨*proof*⟩

**sublocale** *order*: *well-founded-ordering UNIV*
**rewrites** $\bigwedge x.\ x \in UNIV \equiv True$
**and** $\bigwedge X.\ X \subseteq UNIV \equiv True$
**and** $\bigwedge r.\ r \restriction UNIV \equiv r$
**and** $\bigwedge P.\ True \wedge P \equiv P$
**and** $Ball\ UNIV \equiv All$
**and** $Bex\ UNIV \equiv Ex$
**and** $sympartp\ (\leq)^{-} \equiv sympartp\ (\leq)$
**and** $\bigwedge P1.\ (True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
**and** $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
**and** $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
⟨*proof*⟩

**end**

**class** *totalorder* = *ord* + **assumes** *total-ordering UNIV* $(\leq)$ $(<)$
**begin**

**interpretation** *total-ordering UNIV*
⟨*proof*⟩

**subclass** *porder* ⟨*proof*⟩

**sublocale** *order*: *total-ordering UNIV*
**rewrites** $\bigwedge x.\ x \in UNIV \equiv True$
**and** $\bigwedge X.\ X \subseteq UNIV \equiv True$
**and** $\bigwedge r.\ r \restriction UNIV \equiv r$
**and** $\bigwedge P.\ True \wedge P \equiv P$
**and** $Ball\ UNIV \equiv All$
**and** $Bex\ UNIV \equiv Ex$
**and** $sympartp\ (\leq)^{-} \equiv sympartp\ (\leq)$
**and** $\bigwedge P1.\ (True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
**and** $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
**and** $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
⟨*proof*⟩

**end**

Isabelle/HOL's *preorder* belongs to *qorder*, but not vice versa.

**subclass** (**in** *preorder*) *qorder*
⟨*proof*⟩

**subclass** (**in** *order*) *porder* ⟨*proof*⟩

**subclass** (**in** *wellorder*) *wf-qorder*
⟨*proof*⟩

Isabelle/HOL's *linorder* is equivalent to our locale *strict-total-ordering*.

**context** *linorder* **begin**

**interpretation** *strict-total-ordering UNIV*
⟨*proof*⟩

**subclass** *totalorder* ⟨*proof*⟩

**sublocale** *order*: *strict-total-ordering UNIV*
**rewrites** $\bigwedge x.\ x \in UNIV \equiv True$
**and** $\bigwedge X.\ X \subseteq UNIV \equiv True$
**and** $\bigwedge r.\ r \upharpoonright UNIV \equiv r$
**and** $\bigwedge P.\ True \wedge P \equiv P$
**and** $Ball\ UNIV \equiv All$
**and** $Bex\ UNIV \equiv Ex$
**and** $sympartp\ (\le)^{-} \equiv sympartp\ (\le)$
**and** $\bigwedge P1.\ (True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
**and** $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
**and** $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
⟨*proof*⟩

**end**

Tests: facts should be available in the most general classes.

**thm** *order.strict.trans*[**where** *'a='a::compat*]
**thm** *order.extreme-bound-quasi-const*[**where** *'a='a::attractive-order*]
**thm** *order.extreme-bound-singleton-eq*[**where** *'a='a::psorder*]
**thm** *order.trans*[**where** *'a='a::qorder*]
**thm** *order.comparable-cases*[**where** *'a='a::totalorder*]
**thm** *order.cases*[**where** *'a='a::linorder*]

## 2.8 Declaring Duals

**sublocale** *reflexive* ⊆ *sym*: *reflexive A sympartp* (⊑)
**rewrites** $sympartp\ (\sqsubseteq)^{-} \equiv sympartp\ (\sqsubseteq)$
**and** $\bigwedge r.\ sympartp\ (sympartp\ r) \equiv sympartp\ r$
**and** $\bigwedge r.\ sympartp\ r \upharpoonright A \equiv sympartp\ (r \upharpoonright A)$
⟨*proof*⟩

**sublocale** *quasi-ordered-set* ⊆ *sym*: *quasi-ordered-set A sympartp* (⊑)
**rewrites** $sympartp\ (\sqsubseteq)^{-} = sympartp\ (\sqsubseteq)$
**and** $sympartp\ (sympartp\ (\sqsubseteq)) = sympartp\ (\sqsubseteq)$
⟨*proof*⟩

At this point, we declare dual as sublocales. In the following, "rewrites" eventually cleans up redundant facts.

**sublocale** *reflexive* ⊆ *dual*: *reflexive A* $(\sqsubseteq)^{-}$
**rewrites** $sympartp\ (\sqsubseteq)^{-} \equiv sympartp\ (\sqsubseteq)$
**and** $\bigwedge r.\ sympartp\ (r \upharpoonright A) \equiv sympartp\ r \upharpoonright A$

**and** $(\sqsubseteq)^- \restriction A \equiv ((\sqsubseteq) \restriction A)^-$
⟨*proof*⟩

**context** *attractive* **begin**

**interpretation** *less-eq-notations*⟨*proof*⟩

**sublocale** *dual*: *attractive* $A$ $(\sqsupseteq)$
**rewrites** *sympartp* $(\sqsupseteq) = (\sim)$
**and** *equivpartp* $(\sqsupseteq) \equiv (\simeq)$
**and** $\bigwedge r.$ *sympartp* $(r \restriction A) \equiv$ *sympartp* $r \restriction A$
**and** $\bigwedge r.$ *sympartp* (*sympartp* $r$) $\equiv$ *sympartp* $r$
**and** $(\sqsubseteq)^- \restriction A \equiv ((\sqsubseteq) \restriction A)^-$
⟨*proof*⟩

**end**

**context** *irreflexive* **begin**

**sublocale** *dual*: *irreflexive* $A$ $(\sqsubset)^-$
**rewrites** $(\sqsubset)^- \restriction A \equiv ((\sqsubset) \restriction A)^-$
⟨*proof*⟩

**end**

**sublocale** *transitive* $\subseteq$ *dual*: *transitive* $A$ $(\sqsubseteq)^-$
**rewrites** $(\sqsubseteq)^- \restriction A \equiv ((\sqsubseteq) \restriction A)^-$
**and** *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
**and** *asympartp* $(\sqsubseteq)^- =$ (*asympartp* $(\sqsubseteq))^-$
⟨*proof*⟩

**sublocale** *antisymmetric* $\subseteq$ *dual*: *antisymmetric* $A$ $(\sqsubseteq)^-$
**rewrites** $(\sqsubseteq)^- \restriction A \equiv ((\sqsubseteq) \restriction A)^-$
**and** *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**sublocale** *semiconnex* $\subseteq$ *dual*: *semiconnex* $A$ $(\sqsubset)^-$
**rewrites** *sympartp* $(\sqsubset)^- =$ *sympartp* $(\sqsubset)$
⟨*proof*⟩

**sublocale** *connex* $\subseteq$ *dual*: *connex* $A$ $(\sqsubseteq)^-$
**rewrites** *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**sublocale** *semiconnex-irreflexive* $\subseteq$ *dual*: *semiconnex-irreflexive* $A$ $(\sqsubset)^-$
**rewrites** *sympartp* $(\sqsubset)^- =$ *sympartp* $(\sqsubset)$
⟨*proof*⟩

**sublocale** *pseudo-ordered-set* $\subseteq$ *dual*: *pseudo-ordered-set* $A$ $(\sqsubseteq)^-$

**rewrites** *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**sublocale** *quasi-ordered-set* $\subseteq$ *dual*: *quasi-ordered-set* $A$ $(\sqsubseteq)^{-}$
**rewrites** *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**sublocale** *partially-ordered-set* $\subseteq$ *dual*: *partially-ordered-set* $A$ $(\sqsubseteq)^{-}$
**rewrites** *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**sublocale** *total-pseudo-ordered-set* $\subseteq$ *dual*: *total-pseudo-ordered-set* $A$ $(\sqsubseteq)^{-}$
**rewrites** *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**sublocale** *total-quasi-ordered-set* $\subseteq$ *dual*: *total-quasi-ordered-set* $A$ $(\sqsubseteq)^{-}$
**rewrites** *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**sublocale** *compatible-ordering* $\subseteq$ *dual*: *compatible-ordering* $A$ $(\sqsubseteq)^{-}$ $(\sqsubset)^{-}$
**rewrites** *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**sublocale** *attractive-ordering* $\subseteq$ *dual*: *attractive-ordering* $A$ $(\sqsubseteq)^{-}$ $(\sqsubset)^{-}$
**rewrites** *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**sublocale** *pseudo-ordering* $\subseteq$ *dual*: *pseudo-ordering* $A$ $(\sqsubseteq)^{-}$ $(\sqsubset)^{-}$
**rewrites** *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**sublocale** *quasi-ordering* $\subseteq$ *dual*: *quasi-ordering* $A$ $(\sqsubseteq)^{-}$ $(\sqsubset)^{-}$
**rewrites** *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**sublocale** *partial-ordering* $\subseteq$ *dual*: *partial-ordering* $A$ $(\sqsubseteq)^{-}$ $(\sqsubset)^{-}$
**rewrites** *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**sublocale** *total-ordering* $\subseteq$ *dual*: *total-ordering* $A$ $(\sqsubseteq)^{-}$ $(\sqsubset)^{-}$
**rewrites** *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

**lemma**(**in** *antisymmetric*) *monotone-extreme-imp-extreme-bound-iff*:
**fixes** *ir* (**infix** $\preceq$ *50*)
**assumes** $f$ ‘ $C \subseteq A$ **and** *monotone-on* $C$ $(\preceq)$ $(\sqsubseteq)$ $f$ **and** $i$: *extreme* $C$ $(\preceq)$ $i$
**shows** *extreme-bound* $A$ $(\sqsubseteq)$ ($f$ ‘ $C$) $x \longleftrightarrow f\ i = x$
⟨*proof*⟩

## 2.9 Instantiations

Finally, we instantiate our classes for sanity check.

**instance** *nat* :: *linorder* ⟨*proof*⟩

Pointwise ordering of functions are compatible only if the weak part is transitive.

**instance** *fun* :: (*type*,*qorder*) *compat*
⟨*proof*⟩

**instance** *fun* :: (*type*,*qorder*) *qorder*
⟨*proof*⟩

**instance** *fun* :: (*type*,*porder*) *porder*
⟨*proof*⟩

**end**

# 3 Completeness of Relations

Here we formalize various order-theoretic completeness conditions.

**theory** *Complete-Relations*
**imports** *Binary-Relations*
**begin**

## 3.1 Completeness Conditions

Order-theoretic completeness demands certain subsets of elements to admit suprema or infima.

**definition** *complete* (*-−complete*[*999*]*1000*) **where**
$CC{-}complete\ A\ r \equiv \forall X \subseteq A.\ X \in CC \longrightarrow (\exists s.\ extreme\text{-}bound\ A\ r\ X\ s)$

**lemmas** *completeI* = *complete-def*[*unfolded atomize-eq*, *THEN iffD2*, *rule-format*]
**lemmas** *completeD* = *complete-def*[*unfolded atomize-eq*, *THEN iffD1*, *rule-format*]

**lemma** *complete-cmono*: $CC \subseteq DD \Longrightarrow DD{-}complete \leq CC{-}complete$
⟨*proof*⟩

**lemma** *complete-empty*[*simp*]: $CC{-}complete\ \{\}\ r \longleftrightarrow \{\} \notin CC$ ⟨*proof*⟩

A related set $\langle A, \sqsubseteq \rangle$ is called *topped* if there is a “top” element $\top \in A$, a greatest element in $A$. Note that there might be multiple tops if $(\sqsubseteq)$ is not antisymmetric.

**definition** *extremed* $A\ r \equiv \exists e.\ extreme\ A\ r\ e$

**lemma** *extremed-imp-ex-bound*: $extremed\ A\ r \Longrightarrow X \subseteq A \Longrightarrow \exists b \in A.\ bound\ X\ r\ b$

$\langle proof \rangle$

**context**
**fixes** *less-eq* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)
**begin**

Toppedness can be also seen as a completeness condition, since it is equivalent to saying that the universe has a supremum.

**lemma** *extremed-iff-UNIV-complete*: *extremed* $A$ $(\sqsubseteq) \longleftrightarrow \{A\}-complete\ A\ (\sqsubseteq)$ (**is** $?l \longleftrightarrow ?r$)
$\langle proof \rangle$

The dual notion of topped is called "pointed", equivalently ensuring a supremum of the empty set.

**lemma** *pointed-iff-empty-complete*: *extremed* $A$ $(\sqsubseteq) \longleftrightarrow \{\{\}\}-complete\ A\ (\sqsubseteq)^{-}$
$\langle proof \rangle$

**end**

One of the most well-studied notion of completeness would be the semilattice condition: every pair of elements $x$ and $y$ has a supremum $x \sqcup y$ (not necessarily unique if the underlying relation is not antisymmetric).

**definition** *pair-complete* $\equiv \{\{x,y\} \mid .\ x\ y :: 'a\}-complete$

**lemma** *pair-completeI*:
**assumes** $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow \exists s.\ extreme\text{-}bound\ A\ r\ \{x,y\}\ s$
**shows** *pair-complete* $A$ $r$
$\langle proof \rangle$

**lemma** *pair-completeD*:
**assumes** *pair-complete* $A$ $r$
**shows** $x \in A \Longrightarrow y \in A \Longrightarrow \exists s.\ extreme\text{-}bound\ A\ r\ \{x,y\}\ s$
$\langle proof \rangle$

**context**
**fixes** *less-eq* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)
**begin**

**lemma** *pair-complete-imp-directed*:
**assumes** *comp*: *pair-complete* $A$ $(\sqsubseteq)$ **shows** *directed* $A$ $(\sqsubseteq)$
$\langle proof \rangle$

**end**

**lemma** (**in** *connex*) *pair-complete*: *pair-complete* $A$ $(\sqsubseteq)$
$\langle proof \rangle$

The next one assumes that every nonempty finite set has a supremum.

**abbreviation** *finite-complete* $\equiv$ $\{X.$ *finite* $X \wedge X \neq \{\}\}$*−complete*

**lemma** *finite-complete-le-pair-complete*: *finite-complete* $\leq$ *pair-complete*
⟨*proof*⟩

The next one assumes that every nonempty bounded set has a supremum. It is also called the Dedekind completeness.

**abbreviation** *conditionally-complete* $A$ $r$ $\equiv$ $\{X.\ \exists\, b \in A.$ *bound* $X$ $r$ $b \wedge X \neq \{\}\}$*−complete* $A$ $r$

**lemma** *conditionally-complete-imp-nonempty-imp-ex-extreme-bound-iff-ex-bound*:
**assumes** *comp*: *conditionally-complete* $A$ $r$
**assumes** $X \subseteq A$ **and** $X \neq \{\}$
**shows** $(\exists\, s.$ *extreme-bound* $A$ $r$ $X$ $s) \longleftrightarrow (\exists\, b{\in}A.$ *bound* $X$ $r$ $b)$
⟨*proof*⟩

The $\omega$-completeness condition demands a supremum for an $\omega$-chain, $a_1 \sqsubseteq a_2 \sqsubseteq \ldots$. We model $\omega$-chain as the range of a monotone map $f : i \mapsto a_i$.

**definition** *omega-complete* $A$ $r$ $\equiv$ $\{$*range* $f$ $|$ $f$ :: *nat* $\Rightarrow$ $'a.$ *monotone* $(\leq)$ $r$ $f\}$*−complete* $A$ $r$

**lemma** (**in** *transitive*) *local-chain*:
**assumes** *CA*: *range* $C \subseteq A$
**shows** $(\forall\, i{::}nat.\ C\ i \sqsubseteq C\ (Suc\ i)) \longleftrightarrow$ *monotone* $(<)$ $(\sqsubseteq)$ $C$
⟨*proof*⟩

*Directed completeness* is an important notion in domain theory [1], asserting that every nonempty directed set has a supremum. Here, a set $X$ is *directed* if any pair of two elements in $X$ has a bound in $X$.

**definition** *directed-complete* $A$ $r$ $\equiv$ $\{X.$ *directed* $X$ $r \wedge X \neq \{\}\}$*−complete* $A$ $r$

**lemma** *monotone-directed-complete*:
**assumes** *comp*: *directed-complete* $A$ $r$
**assumes** *fI*: $f$ ‘ $I \subseteq A$ **and** *dir*: *directed* $I$ *ri* **and** *I0*: $I \neq \{\}$ **and** *mono*: *monotone-on* $I$ *ri* $r$ $f$
**shows** $\exists\, s.$ *extreme-bound* $A$ $r$ $(f$ ‘ $I)$ $s$
⟨*proof*⟩

The next one is quite complete, only the empty set may fail to have a supremum. The terminology follows [3], although there it is defined more generally depending on a cardinal $\alpha$ such that a nonempty set $X$ of cardinality below $\alpha$ has a supremum.

**abbreviation** *semicomplete* $\equiv$ $\{X.\ X \neq \{\}\}$*−complete*

**lemma** *semicomplete-nonempty-imp-extremed*:
*semicomplete* $A$ $r \Longrightarrow A \neq \{\} \Longrightarrow$ *extremed* $A$ $r$
⟨*proof*⟩

**lemma** *connex-dual-semicomplete*: *semicomplete* $\{C.\ connex\ C\ r\}$ $(\supseteq)$
$\langle proof \rangle$

### 3.2 Pointed Ones

The term 'pointed' refers to the dual notion of toppedness, i.e., there is a global least element. This serves as the supremum of the empty set.

**lemma** *complete-union*: $(CC \cup CC')-complete\ A\ r \longleftrightarrow CC-complete\ A\ r \wedge CC'-complete\ A\ r$
$\langle proof \rangle$

**lemma** *pointed-directed-complete*:
$\{X.\ directed\ X\ r\}-complete\ A\ r \longleftrightarrow directed\text{-}complete\ A\ r \wedge extremed\ A\ r^{-}$
$\langle proof \rangle$

"Bounded complete" refers to pointed conditional complete, but this notion is just the dual of semicompleteness. We prove this later.

**abbreviation** *bounded-complete* $A\ r \equiv \{X.\ \exists\, b{\in}A.\ bound\ X\ r\ b\}-complete\ A\ r$

### 3.3 Relations between Completeness Conditions

**context**
**fixes** *less-eq* :: $'a \Rightarrow\ 'a \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)
**begin**

**interpretation** *less-eq-notations*$\langle proof \rangle$

Pair-completeness implies that the universe is directed. Thus, with directed completeness implies toppedness.

**proposition** *directed-complete-pair-complete-imp-extremed*:
**assumes** *dc*: *directed-complete* $A$ $(\sqsubseteq)$ **and** *pc*: *pair-complete* $A$ $(\sqsubseteq)$ **and** $A$: $A \neq \{\}$
**shows** *extremed* $A$ $(\sqsubseteq)$
$\langle proof \rangle$

Semicomplete is conditional complete and topped.

**proposition** *semicomplete-iff-conditionally-complete-extremed*:
**assumes** $A$: $A \neq \{\}$
**shows** *semicomplete* $A$ $(\sqsubseteq) \longleftrightarrow$ *conditionally-complete* $A$ $(\sqsubseteq) \wedge$ *extremed* $A$ $(\sqsubseteq)$ (**is** $?l \longleftrightarrow ?r$)
$\langle proof \rangle$

**proposition** *complete-iff-pointed-semicomplete*:
$UNIV-complete\ A\ (\sqsubseteq) \longleftrightarrow semicomplete\ A\ (\sqsubseteq) \wedge extremed\ A\ (\sqsupseteq)$ (**is** $?l \longleftrightarrow ?r$)
$\langle proof \rangle$

Conditional completeness only lacks top and bottom to be complete.

**proposition** *complete-iff-conditionally-complete-extremed-pointed*:

*UNIV*−*complete A* ($\sqsubseteq$) $\longleftrightarrow$ *conditionally-complete A* ($\sqsubseteq$) $\wedge$ *extremed A* ($\sqsubseteq$) $\wedge$ *extremed A* ($\sqsupseteq$)
⟨*proof*⟩

If the universe is directed, then every pair is bounded, and thus has a supremum. On the other hand, supremum gives an upper bound, witnessing directedness.

**proposition** *conditionally-complete-imp-pair-complete-iff-directed*:
**assumes** *comp*: *conditionally-complete A* ($\sqsubseteq$)
**shows** *pair-complete A* ($\sqsubseteq$) $\longleftrightarrow$ *directed A* ($\sqsubseteq$) (**is** *?l* $\longleftrightarrow$ *?r*)
⟨*proof*⟩

**end**

Following is a new generalization of (weak) chain-completeness:

**abbreviation** *well-complete A r* $\equiv$ {*X*. *well-related-set X r*}−*complete A r*

## 3.4 Duality of Completeness Conditions

Conditional completeness is symmetric.

**context fixes** *less-eq* :: *′a* $\Rightarrow$ *′a* $\Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
**begin**

**interpretation** *less-eq-notations*⟨*proof*⟩

**lemma** *conditionally-complete-dual*:
**assumes** *comp*: *conditionally-complete A* ($\sqsubseteq$) **shows** *conditionally-complete A* ($\sqsupseteq$)
⟨*proof*⟩

Full completeness is symmetric.

**lemma** *complete-dual*:
*UNIV*−*complete A* ($\sqsubseteq$) $\Longrightarrow$ *UNIV*−*complete A* ($\sqsupseteq$)
⟨*proof*⟩

Now we show that bounded completeness is the dual of semicompleteness.

**lemma** *pointed-conditionally-complete-iff-bounded-complete*:
**assumes** *A*: *A* $\neq$ {}
**shows** *conditionally-complete A* ($\sqsubseteq$) $\wedge$ *extremed A* ($\sqsupseteq$) $\longleftrightarrow$ *bounded-complete A* ($\sqsubseteq$)
⟨*proof*⟩

**proposition** *bounded-complete-iff-dual-semicomplete*:
*bounded-complete A* ($\sqsubseteq$) $\longleftrightarrow$ *semicomplete A* ($\sqsupseteq$)
⟨*proof*⟩

**end**

**lemmas** *connex-bounded-complete* = *connex-dual-semicomplete*[*folded bounded-complete-iff-dual-semicomplete*]

### 3.5 Completeness Results Requiring Order-Like Properties

Above results hold without any assumption on the relation. This part demands some order-like properties.

It is well known that in a semilattice, i.e., a pair-complete partial order, every finite nonempty subset of elements has a supremum. We prove the result assuming transitivity, but only that.

**lemma** (**in** *transitive*) *pair-complete-iff-finite-complete*:
*pair-complete* $A$ ($\sqsubseteq$) $\longleftrightarrow$ *finite-complete* $A$ ($\sqsubseteq$) (**is** $?l \longleftrightarrow ?r$)
⟨*proof*⟩

Gierz et al. [6] showed that a directed complete partial order is semicomplete if and only if it is also a semilattice. We generalize the claim so that the underlying relation is only transitive.

**proposition**(**in** *transitive*) *semicomplete-iff-directed-complete-pair-complete*:
*semicomplete* $A$ ($\sqsubseteq$) $\longleftrightarrow$ *directed-complete* $A$ ($\sqsubseteq$) $\wedge$ *pair-complete* $A$ ($\sqsubseteq$) (**is** $?l$ $\longleftrightarrow$ $?r$)
⟨*proof*⟩

The last argument in the above proof requires transitivity, but if we had reflexivity then $x$ itself is a supremum of $\{x\}$ (see $[\![$*reflexive* $?A$ *?less-eq*; $?x \in ?A]\!] \Longrightarrow$ *extreme-bound* $?A$ *?less-eq* $\{?x\}$ $?x$) and so $x \sqsubseteq s$ would be immediate. Thus we can replace transitivity by reflexivity, but then pair-completeness does not imply finite completeness. We obtain the following result.

**proposition** (**in** *reflexive*) *semicomplete-iff-directed-complete-finite-complete*:
*semicomplete* $A$ ($\sqsubseteq$) $\longleftrightarrow$ *directed-complete* $A$ ($\sqsubseteq$) $\wedge$ *finite-complete* $A$ ($\sqsubseteq$) (**is** $?l$ $\longleftrightarrow$ $?r$)
⟨*proof*⟩

### 3.6 Relating to Classes

Isabelle's class *complete-lattice* is $UNIV-complete$.

**lemma** (**in** *complete-lattice*) $UNIV-complete$ $UNIV$ ($\leq$)
⟨*proof*⟩

### 3.7 Set-wise Completeness

**lemma** *directed-sets-directed-complete*:
**assumes** *cl*: $\forall DC.\ DC \subseteq AA \longrightarrow (\forall X{\in}DC.\ directed\ X\ r) \longrightarrow (\bigcup DC) \in AA$
**shows** $\{DC.\ directed\ DC\ (\subseteq)\}-complete\ \{X \in AA.\ directed\ X\ r\}\ (\subseteq)$
⟨*proof*⟩

**lemma** *connex-directed-Un*:

**assumes** *ch*: $CC \subseteq \{C.\ connex\ C\ r\}$ **and** *dir*: *directed* $CC$ $(\subseteq)$
**shows** *connex* $(\bigcup CC)\ r$
$\langle proof \rangle$

**lemma** *connex-directed-complete*: $\{DC.\ directed\ DC\ (\subseteq)\}-complete\ \{C.\ connex\ C\ r\}\ (\subseteq)$
$\langle proof \rangle$

**end**

**theory** *Fixed-Points*
**imports** *Complete-Relations*
**begin**

# 4 Existence of Fixed Points in Complete Related Sets

The following proof is simplified and generalized from Stouti–Maaden [16]. We construct some set whose extreme bounds – if they exist, typically when the underlying related set is complete – are fixed points of a monotone or inflationary function on any related set. When the related set is attractive, those are actually the least fixed points. This generalizes [16], relaxing reflexivity and antisymmetry.

**locale** *fixed-point-proof* = *related-set* +
**fixes** $f$
**assumes** $f$: $f\ `\ A \subseteq A$
**begin**

**sublocale** *less-eq-notations*$\langle proof \rangle$

**definition** $AA$ **where** $AA \equiv$
$\{X.\ X \subseteq A \wedge f\ `\ X \subseteq X \wedge (\forall\ Y\ s.\ Y \subseteq X \longrightarrow extreme\text{-}bound\ A\ (\sqsubseteq)\ Y\ s \longrightarrow s \in X)\}$

**lemma** $AA$-$I$:
$X \subseteq A \Longrightarrow f\ `\ X \subseteq X \Longrightarrow (\bigwedge Y\ s.\ Y \subseteq X \Longrightarrow extreme\text{-}bound\ A\ (\sqsubseteq)\ Y\ s \Longrightarrow s \in X) \Longrightarrow X \in AA$
$\langle proof \rangle$

**lemma** $AA$-$E$:
$X \in AA \Longrightarrow$
$(X \subseteq A \Longrightarrow f\ `\ X \subseteq X \Longrightarrow (\bigwedge Y\ s.\ Y \subseteq X \Longrightarrow extreme\text{-}bound\ A\ (\sqsubseteq)\ Y\ s \Longrightarrow s \in X) \Longrightarrow thesis) \Longrightarrow thesis$
$\langle proof \rangle$

**definition** $C$ **where** $C \equiv \bigcap\ AA$

**lemma** *A-AA*: $A \in AA$ ⟨*proof*⟩

**lemma** *C-AA*: $C \in AA$
⟨*proof*⟩

**lemma** *CA*: $C \subseteq A$ ⟨*proof*⟩

**lemma** *fC*: $f$ ‘ $C \subseteq C$ ⟨*proof*⟩

**context**
  **fixes** $c$ **assumes** *Cc*: *extreme-bound* $A$ $(\sqsubseteq)$ $C$ $c$
**begin**

**private lemma** *cA*: $c \in A$ ⟨*proof*⟩ **lemma** *cC*: $c \in C$ ⟨*proof*⟩ **lemma** *fcC*: $f\ c \in C$ ⟨*proof*⟩ **lemma** *fcA*: $f\ c \in A$ ⟨*proof*⟩

**lemma** *qfp-as-extreme-bound*:
  **assumes** *infl-mono*: $\forall x \in A.\ x \sqsubseteq f\ x \vee (\forall y \in A.\ y \sqsubseteq x \longrightarrow f\ y \sqsubseteq f\ x)$
  **shows** $f\ c \sim c$
⟨*proof*⟩

**lemma** *extreme-qfp*:
  **assumes** *attract*: $\forall q \in A.\ \forall x \in A.\ f\ q \sim q \longrightarrow x \sqsubseteq f\ q \longrightarrow x \sqsubseteq q$
    **and** *mono*: *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
  **shows** *extreme* $\{q \in A.\ f\ q \sim q \vee f\ q = q\}$ $(\sqsupseteq)$ $c$
⟨*proof*⟩

**end**

**lemma** *ex-qfp*:
  **assumes** *comp*: $CC{-}complete$ $A$ $(\sqsubseteq)$ **and** $C$: $C \in CC$
    **and** *infl-mono*: $\forall a \in A.\ a \sqsubseteq f\ a \vee (\forall b \in A.\ b \sqsubseteq a \longrightarrow f\ b \sqsubseteq f\ a)$
  **shows** $\exists s \in A.\ f\ s \sim s$
  ⟨*proof*⟩

**lemma** *ex-extreme-qfp-fp*:
  **assumes** *comp*: $CC{-}complete$ $A$ $(\sqsubseteq)$ **and** $C$: $C \in CC$
    **and** *attract*: $\forall q \in A.\ \forall x \in A.\ f\ q \sim q \longrightarrow x \sqsubseteq f\ q \longrightarrow x \sqsubseteq q$
    **and** *mono*: *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
  **shows** $\exists c.$ *extreme* $\{q \in A.\ f\ q \sim q \vee f\ q = q\}$ $(\sqsupseteq)$ $c$
  ⟨*proof*⟩

**lemma** *ex-extreme-qfp*:
  **assumes** *comp*: $CC{-}complete$ $A$ $(\sqsubseteq)$ **and** $C$: $C \in CC$
    **and** *attract*: $\forall q \in A.\ \forall x \in A.\ f\ q \sim q \longrightarrow x \sqsubseteq f\ q \longrightarrow x \sqsubseteq q$
    **and** *mono*: *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
  **shows** $\exists c.$ *extreme* $\{q \in A.\ f\ q \sim q\}$ $(\sqsupseteq)$ $c$
⟨*proof*⟩

**end**

**context**
  **fixes** *less-eq* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*) **and** $A$ :: $'a$ *set* **and** $f$
  **assumes** $f$: $f$ ‘ $A \subseteq A$
**begin**

**interpretation** *less-eq-notations*⟨*proof*⟩
**interpretation** *fixed-point-proof* $A$ ($\sqsubseteq$) $f$ ⟨*proof*⟩

**theorem** *complete-infl-mono-imp-ex-qfp*:
  **assumes** *comp*: *UNIV−complete* $A$ ($\sqsubseteq$) **and** *infl-mono*: $\forall a \in A.\ a \sqsubseteq f\ a \vee (\forall b \in A.\ b \sqsubseteq a \longrightarrow f\ b \sqsubseteq f\ a)$
  **shows** $\exists s \in A.\ f\ s \sim s$
  ⟨*proof*⟩

**end**

**corollary** (**in** *antisymmetric*) *complete-infl-mono-imp-ex-fp*:
  **assumes** *comp*: *UNIV−complete* $A$ ($\sqsubseteq$) **and** $f$: $f$ ‘ $A \subseteq A$
    **and** *infl-mono*: $\forall a \in A.\ a \sqsubseteq f\ a \vee (\forall b \in A.\ b \sqsubseteq a \longrightarrow f\ b \sqsubseteq f\ a)$
  **shows** $\exists s \in A.\ f\ s = s$
⟨*proof*⟩

**context** *semiattractive* **begin**

**interpretation** *less-eq-notations*⟨*proof*⟩

**theorem** *complete-mono-imp-ex-extreme-qfp*:
  **assumes** *comp*: *UNIV−complete* $A$ ($\sqsubseteq$) **and** $f$: $f$ ‘ $A \subseteq A$
    **and** *mono*: *monotone-on* $A$ ($\sqsubseteq$) ($\sqsubseteq$) $f$
  **shows** $\exists s.$ *extreme* $\{p \in A.\ f\ p \sim p\}$ ($\sqsubseteq$) $s$
⟨*proof*⟩

**end**

**corollary** (**in** *antisymmetric*) *complete-mono-imp-ex-extreme-fp*:
  **assumes** *comp*: *UNIV−complete* $A$ ($\sqsubseteq$) **and** $f$: $f$ ‘ $A \subseteq A$
    **and** *mono*: *monotone-on* $A$ ($\sqsubseteq$) ($\sqsubseteq$) $f$
  **shows** $\exists s.$ *extreme* $\{s \in A.\ f\ s = s\}$ $(\sqsubseteq)^-$ $s$
⟨*proof*⟩

# 5 Fixed Points in Well-Complete Antisymmetric Sets

In this section, we prove that an inflationary or monotone map over a well-complete antisymmetric set has a fixed point.

In order to formalize such a theorem in Isabelle, we followed Grall's [**?**] elementary proof for Bourbaki–Witt and Markowsky's theorems. His idea is to consider well-founded derivation trees over $A$, where from a set $C \subseteq A$ of premises one can derive $f\ (\bigsqcup C)$ if $C$ is a chain. The main observation is as follows: Let $D$ be the set of all the derivable elements; that is, for each $d \in D$ there exists a well-founded derivation whose root is $d$. It is shown that $D$ is a chain, and hence one can build a derivation yielding $f\ (\bigsqcup D)$, and $f\ (\bigsqcup D)$ is shown to be a fixed point.

**lemma** *bound-monotone-on*:
**assumes** *mono*: *monotone-on A r s f* **and** *XA*: $X \subseteq A$ **and** *aA*: $a \in A$ **and** *rXa*: *bound X r a*
**shows** *bound* $(f`X)$ *s* $(f\ a)$
⟨*proof*⟩

**context** *fixed-point-proof* **begin**

To avoid the usage of the axiom of choice, we carefully define derivations so that any derivable element determines its lower set. This led to the following definition:

**definition** *derivation* $X \equiv X \subseteq A \wedge$ *well-ordered-set* $X\ (\sqsubseteq) \wedge$
$(\forall x \in X.$ *let* $Y = \{y \in X.\ y \sqsubset x\}$ *in*
$(\exists y.$ *extreme* $Y\ (\sqsubseteq)\ y \wedge x = f\ y) \vee$
$f\ `\ Y \subseteq Y \wedge$ *extreme-bound* $A\ (\sqsubseteq)\ Y\ x)$

**lemma assumes** *derivation P*
**shows** *derivation-A*: $P \subseteq A$ **and** *derivation-well-ordered*: *well-ordered-set* $P\ (\sqsubseteq)$
⟨*proof*⟩

**lemma** *derivation-cases*[*consumes 2*, *case-names suc lim*]:
**assumes** *derivation X* **and** $x \in X$
**and** $\bigwedge Y\ y.\ Y = \{y \in X.\ y \sqsubset x\} \Longrightarrow$ *extreme* $Y\ (\sqsubseteq)\ y \Longrightarrow x = f\ y \Longrightarrow$ *thesis*
**and** $\bigwedge Y.\ Y = \{y \in X.\ y \sqsubset x\} \Longrightarrow f\ `\ Y \subseteq Y \Longrightarrow$ *extreme-bound* $A\ (\sqsubseteq)\ Y\ x$ $\Longrightarrow$ *thesis*
**shows** *thesis*
⟨*proof*⟩

**definition** *derivable* $x \equiv \exists X.$ *derivation* $X \wedge x \in X$

**lemma** *derivableI*[*intro?*]: *derivation* $X \Longrightarrow x \in X \Longrightarrow$ *derivable* $x$ ⟨*proof*⟩
**lemma** *derivableE*: *derivable* $x \Longrightarrow (\bigwedge P.$ *derivation* $P \Longrightarrow x \in P \Longrightarrow$ *thesis*$) \Longrightarrow$ *thesis*
⟨*proof*⟩

**lemma** *derivable-A*: *derivable* $x \Longrightarrow x \in A$ ⟨*proof*⟩

**lemma** *UN-derivations-eq-derivable*: $(\bigcup \{P.$ *derivation* $P\}) = \{x.$ *derivable* $x\}$
⟨*proof*⟩

**context**
**assumes** *ord*: *antisymmetric* $A$ ($\sqsubseteq$)
**begin**

**interpretation** *antisymmetric* ⟨*proof*⟩

**lemma** *derivation-lim*:
**assumes** $P$: *derivation* $P$ **and** *fP*: $f$ ‘ $P \subseteq P$ **and** *Pp*: *extreme-bound* $A$ ($\sqsubseteq$) $P$ $p$
**shows** *derivation* $(P \cup \{p\})$
⟨*proof*⟩

**context**
**assumes** *derivation-infl*: $\forall X\ x\ y.$ *derivation* $X \longrightarrow x \in X \longrightarrow y \in X \longrightarrow x \sqsubseteq y \longrightarrow x \sqsubseteq f\ y$
**and** *derivation-f-refl*: $\forall X\ x.$ *derivation* $X \longrightarrow x \in X \longrightarrow f\ x \sqsubseteq f\ x$
**begin**

**lemma** *derivation-suc*:
**assumes** $P$: *derivation* $P$ **and** *Pp*: *extreme* $P$ ($\sqsubseteq$) $p$ **shows** *derivation* $(P \cup \{f\ p\})$
⟨*proof*⟩

**lemma** *derivable-closed*:
**assumes** $x$: *derivable* $x$ **shows** *derivable* $(f\ x)$
⟨*proof*⟩

The following lemma is derived from Grall's proof. We simplify the claim so that we consider two elements from one derivation, instead of two derivations.

**lemma** *derivation-useful*:
**assumes** $X$: *derivation* $X$ **and** *xX*: $x \in X$ **and** *yX*: $y \in X$ **and** *xy*: $x \sqsubset y$
**shows** $f\ x \sqsubseteq y$
⟨*proof*⟩

Next one is the main lemma of this section, stating that elements from two possibly different derivations are comparable, and moreover the lower one is in the derivation of the upper one. The latter claim, not found in Grall's proof, is crucial in proving that the union of all derivations is well-related.

**lemma** *derivations-cross-compare*:
**assumes** $X$: *derivation* $X$ **and** $Y$: *derivation* $Y$ **and** *xX*: $x \in X$ **and** *yY*: $y \in Y$
**shows** $(x \sqsubset y \wedge x \in Y) \vee x = y \vee (y \sqsubset x \wedge y \in X)$
⟨*proof*⟩

**interpretation** *derivable*: *well-ordered-set* $\{x.$ *derivable* $x\}$ ($\sqsubseteq$)
⟨*proof*⟩

**lemmas** *derivable-well-ordered* = *derivable.well-ordered-set-axioms*
**lemmas** *derivable-total-ordered* = *derivable.total-ordered-set-axioms*

**lemmas** *derivable-well-related* = *derivable.well-related-set-axioms*

**lemma** *pred-unique*:
**assumes** *X*: *derivation X* **and** *xX*: $x \in X$
**shows** $\{z \in X.\ z \sqsubset x\} = \{z.\ derivable\ z \wedge z \sqsubset x\}$
⟨*proof*⟩

The set of all derivable elements is itself a derivation.

**lemma** *derivation-derivable*: *derivation* $\{x.\ derivable\ x\}$
⟨*proof*⟩

Finally, if the set of all derivable elements admits a supremum, then it is a fixed point.

**lemma**
**assumes** *p*: *extreme-bound A* $(\sqsubseteq)$ $\{x.\ derivable\ x\}$ *p*
**shows** *sup-derivable-qfp*: $f\ p \sim p$ **and** *sup-derivable-fp*: $f\ p = p$
⟨*proof*⟩

**end**

The assumptions are satisfied by monotone functions.

**context**
**assumes** *mono*: *monotone-on A* $(\sqsubseteq)$ $(\sqsubseteq)$ *f*
**begin**

**lemma** *mono-imp-derivation-infl*:
$\forall X\ x\ y.\ derivation\ X \longrightarrow x \in X \longrightarrow y \in X \longrightarrow x \sqsubseteq y \longrightarrow x \sqsubseteq f\ y$
⟨*proof*⟩

**lemma** *mono-imp-derivation-f-refl*:
$\forall X\ x.\ derivation\ X \longrightarrow x \in X \longrightarrow f\ x \sqsubseteq f\ x$
⟨*proof*⟩

**corollary** *mono-imp-sup-derivable-fp*:
**assumes** *p*: *extreme-bound A* $(\sqsubseteq)$ $\{x.\ derivable\ x\}$ *p*
**shows** $f\ p = p$
⟨*proof*⟩

**lemma** *mono-imp-sup-derivable-lfp*:
**assumes** *p*: *extreme-bound A* $(\sqsubseteq)$ $\{x.\ derivable\ x\}$ *p*
**shows** *extreme* $\{q \in A.\ f\ q = q\}$ $(\sqsupseteq)$ *p*
⟨*proof*⟩

**lemma** *mono-imp-ex-least-fp*:
**assumes** *comp*: *well-complete A* $(\sqsubseteq)$
**shows** $\exists p.\ extreme\ \{q \in A.\ f\ q = q\}\ (\sqsupseteq)\ p$
⟨*proof*⟩

**end**

**end**

**end**

Bourbaki-Witt Theorem on well-complete pseudo-ordered set:

**theorem** (**in** *pseudo-ordered-set*) *well-complete-infl-imp-ex-fp*:
**assumes** *comp*: *well-complete* $A$ ($\sqsubseteq$)
**and** $f$: $f$ ‘ $A \subseteq A$ **and** *infl*: $\forall x \in A.\ \forall y \in A.\ x \sqsubseteq y \longrightarrow x \sqsubseteq f\ y$
**shows** $\exists p \in A.\ f\ p = p$
$\langle$*proof*$\rangle$

# 6 Completeness of (Quasi-)Fixed Points

We now prove that, under attractivity, the set of quasi-fixed points is complete.

**definition** *setwise* **where** *setwise* $r\ X\ Y \equiv \forall x{\in}X.\ \forall y{\in}Y.\ r\ x\ y$

**lemmas** *setwiseI*[*intro*] = *setwise-def*[*unfolded atomize-eq*, *THEN iffD2*, *rule-format*]
**lemmas** *setwiseE*[*elim*] = *setwise-def*[*unfolded atomize-eq*, *THEN iffD1*, *elim-format*, *rule-format*]

**context** *fixed-point-proof* **begin**

**abbreviation** *setwise-less-eq* (**infix** $\sqsubseteq^s$ *50*) **where** ($\sqsubseteq^s$) $\equiv$ *setwise* ($\sqsubseteq$)

## 6.1 Least Quasi-Fixed Points for Attractive Relations.

**lemma** *attract-mono-imp-least-qfp*:
**assumes** *attract*: *attractive* $A$ ($\sqsubseteq$)
**and** *comp*: *well-complete* $A$ ($\sqsubseteq$)
**and** *mono*: *monotone-on* $A$ ($\sqsubseteq$) ($\sqsubseteq$) $f$
**shows** $\exists c.$ *extreme* $\{p \in A.\ f\ p \sim p \vee f\ p = p\}$ ($\sqsupseteq$) $c \wedge f\ c \sim c$
$\langle$*proof*$\rangle$

## 6.2 General Completeness

**lemma** *attract-mono-imp-fp-qfp-complete*:
**assumes** *attract*: *attractive* $A$ ($\sqsubseteq$)
**and** *comp*: $CC-$*complete* $A$ ($\sqsubseteq$)
**and** *wr-CC*: $\forall C \subseteq A.$ *well-related-set* $C$ ($\sqsubseteq$) $\longrightarrow C \in CC$
**and** *extend*: $\forall X \in CC.\ \forall Y \in CC.\ X \sqsubseteq^s Y \longrightarrow X \cup Y \in CC$
**and** *mono*: *monotone-on* $A$ ($\sqsubseteq$) ($\sqsubseteq$) $f$
**and** $P$: $P \subseteq \{x \in A.\ f\ x = x\}$
**shows** $CC-$*complete* ($\{q \in A.\ f\ q \sim q\} \cup P$) ($\sqsubseteq$)
$\langle$*proof*$\rangle$

**lemma** *attract-mono-imp-qfp-complete*:

**assumes** *attractive* $A$ $(\sqsubseteq)$
**and** $CC-complete$ $A$ $(\sqsubseteq)$
**and** $\forall\, C \subseteq A.$ *well-related-set* $C$ $(\sqsubseteq) \longrightarrow C \in CC$
**and** $\forall\, X \in CC.\ \forall\, Y \in CC.\ X \sqsubseteq^s Y \longrightarrow X \cup Y \in CC$
**and** *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
**shows** $CC-complete$ $\{p \in A.\ f\, p \sim p\}$ $(\sqsubseteq)$
$\langle proof \rangle$

**lemma** *antisym-mono-imp-fp-complete*:
**assumes** *anti*: *antisymmetric* $A$ $(\sqsubseteq)$
**and** *comp*: $CC-complete$ $A$ $(\sqsubseteq)$
**and** *wr-CC*: $\forall\, C \subseteq A.$ *well-related-set* $C$ $(\sqsubseteq) \longrightarrow C \in CC$
**and** *extend*: $\forall\, X \in CC.\ \forall\, Y \in CC.\ X \sqsubseteq^s Y \longrightarrow X \cup Y \in CC$
**and** *mono*: *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
**shows** $CC-complete$ $\{p \in A.\ f\, p = p\}$ $(\sqsubseteq)$
$\langle proof \rangle$

**end**

## 6.3 Instances

### 6.3.1 Instances under attractivity

**context** *attractive* **begin**

**interpretation** *less-eq-notations*$\langle proof \rangle$

Full completeness

**theorem** *mono-imp-qfp-UNIV-complete*:
**assumes** *comp*: $UNIV-complete$ $A$ $(\sqsubseteq)$ **and** $f$: $f$ ‘ $A \subseteq A$ **and** *mono*: *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
**shows** $UNIV-complete$ $\{p \in A.\ f\, p \sim p\}$ $(\sqsubseteq)$
$\langle proof \rangle$

Connex completeness

**theorem** *mono-imp-qfp-connex-complete*:
**assumes** *comp*: $\{X.$ *connex* $X$ $(\sqsubseteq)\}-complete$ $A$ $(\sqsubseteq)$
**and** $f$: $f$ ‘ $A \subseteq A$ **and** *mono*: *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
**shows** $\{X.$ *connex* $X$ $(\sqsubseteq)\}-complete$ $\{p \in A.\ f\, p \sim p\}$ $(\sqsubseteq)$
$\langle proof \rangle$

Directed completeness

**theorem** *mono-imp-qfp-directed-complete*:
**assumes** *comp*: $\{X.$ *directed* $X$ $(\sqsubseteq)\}-complete$ $A$ $(\sqsubseteq)$
**and** $f$: $f$ ‘ $A \subseteq A$ **and** *mono*: *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
**shows** $\{X.$ *directed* $X$ $(\sqsubseteq)\}-complete$ $\{p \in A.\ f\, p \sim p\}$ $(\sqsubseteq)$
$\langle proof \rangle$

Well Completeness

**theorem** *mono-imp-qfp-well-complete*:

**assumes** *comp*: *well-complete* $A$ $(\sqsubseteq)$
**and** $f$: $f$ ` $A \subseteq A$ **and** *mono*: *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
**shows** *well-complete* $\{p \in A.\ f\ p \sim p\}$ $(\sqsubseteq)$
$\langle proof \rangle$

**end**

### 6.3.2 Usual instances under antisymmetry

**context** *antisymmetric* **begin**

Knaster–Tarski

**theorem** *mono-imp-fp-complete*:
**assumes** *comp*: $UNIV-complete$ $A$ $(\sqsubseteq)$
**and** $f$: $f$ ` $A \subseteq A$ **and** *mono*: *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
**shows** $UNIV-complete$ $\{p \in A.\ f\ p = p\}$ $(\sqsubseteq)$
$\langle proof \rangle$

Markowsky 1976

**theorem** *mono-imp-fp-connex-complete*:
**assumes** *comp*: $\{X.\ connex\ X\ (\sqsubseteq)\}-complete$ $A$ $(\sqsubseteq)$
**and** $f$: $f$ ` $A \subseteq A$ **and** *mono*: *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
**shows** $\{X.\ connex\ X\ (\sqsubseteq)\}-complete$ $\{p \in A.\ f\ p = p\}$ $(\sqsubseteq)$
$\langle proof \rangle$

Pataraia

**theorem** *mono-imp-fp-directed-complete*:
**assumes** *comp*: $\{X.\ directed\ X\ (\sqsubseteq)\}-complete$ $A$ $(\sqsubseteq)$
**and** $f$: $f$ ` $A \subseteq A$ **and** *mono*: *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
**shows** $\{X.\ directed\ X\ (\sqsubseteq)\}-complete$ $\{p \in A.\ f\ p = p\}$ $(\sqsubseteq)$
$\langle proof \rangle$

Bhatta & George 2011

**theorem** *mono-imp-fp-well-complete*:
**assumes** *comp*: *well-complete* $A$ $(\sqsubseteq)$
**and** $f$: $f$ ` $A \subseteq A$ **and** *mono*: *monotone-on* $A$ $(\sqsubseteq)$ $(\sqsubseteq)$ $f$
**shows** *well-complete* $\{p \in A.\ f\ p = p\}$ $(\sqsubseteq)$
$\langle proof \rangle$

**end**

**end**
**theory** *Kleene-Fixed-Point*
**imports** *Complete-Relations*
**begin**

## 7 Iterative Fixed Point Theorem

Kleene's fixed-point theorem states that, for a pointed directed complete partial order $\langle A, \sqsubseteq \rangle$ and a Scott-continuous map $f : A \to A$, the supremum of $\{f^n(\bot) \mid n \in \mathbb{N}\}$ exists in $A$ and is a least fixed point. Mashburn [12] generalized the result so that $\langle A, \sqsubseteq \rangle$ is a $\omega$-complete partial order and $f$ is $\omega$-continuous.

In this section we further generalize the result and show that for $\omega$-complete relation $\langle A, \sqsubseteq \rangle$ and for every bottom element $\bot \in A$, the set $\{f^n(\bot) \mid n \in \mathbb{N}\}$ has suprema (not necessarily unique, of course) and, they are quasi-fixed points. Moreover, if $(\sqsubseteq)$ is attractive, then the suprema are precisely the least quasi-fixed points.

### 7.1 Scott Continuity, $\omega$-Completeness, $\omega$-Continuity

In this Section, we formalize $\omega$-completeness, Scott continuity and $\omega$-continuity. We then prove that a Scott continuous map is $\omega$-continuous and that an $\omega$-continuous map is "nearly" monotone.

**context**
**fixes** $A$ :: $'a$ *set* **and** *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
**begin**

**definition** *omega-continuous f* $\equiv$
$f$ ‘ $A \subseteq A \wedge$
$(\forall\, c :: nat \Rightarrow 'a.\ \forall\ b \in A.$
*range* $c \subseteq A \longrightarrow$
*monotone* $(\leq)$ $(\sqsubseteq)$ $c \longrightarrow$
*extreme-bound* $A$ $(\sqsubseteq)$ (*range* $c$) $b \longrightarrow$ *extreme-bound* $A$ $(\sqsubseteq)$ ($f$ ‘ *range* $c$) ($f$ $b$))

**lemmas** *omega-continuousI*[*intro?*] =
*omega-continuous-def*[*unfolded atomize-eq*, *THEN iffD2*, *unfolded conj-imp-eq-imp-imp*, *rule-format*]

**lemmas** *omega-continuousDdom* =
*omega-continuous-def*[*unfolded atomize-eq*, *THEN iffD1*, *unfolded conj-imp-eq-imp-imp*, *THEN conjunct1*]

**lemmas** *omega-continuousD* =
*omega-continuous-def*[*unfolded atomize-eq*, *THEN iffD1*, *unfolded conj-imp-eq-imp-imp*, *THEN conjunct2*, *rule-format*]

**lemmas** *omega-continuousE*[*elim*] =
*omega-continuous-def*[*unfolded atomize-eq*, *THEN iffD1*, *elim-format*, *unfolded conj-imp-eq-imp-imp*, *rule-format*]

**lemma** *omega-continuous-imp-mono-refl*:
**assumes** *cont*: *omega-continuous f*

**and** $x$: $x \in A$ **and** $y$: $y \in A$
**and** $xy$: $x \sqsubseteq y$ **and** $xx$: $x \sqsubseteq x$ **and** $yy$: $y \sqsubseteq y$
**shows** $f\ x \sqsubseteq f\ y$
⟨*proof*⟩

**definition** *scott-continuous* $f \equiv$
$f$ ' $A \subseteq A \wedge$
$(\forall X\ s.\ X \subseteq A \longrightarrow$ *directed* $X$ $(\sqsubseteq) \longrightarrow X \neq \{\} \longrightarrow$ *extreme-bound* $A$ $(\sqsubseteq)$ $X$ $s$
$\longrightarrow$ *extreme-bound* $A$ $(\sqsubseteq)$ ($f$ ' $X$) $(f\ s))$

**lemmas** *scott-continuousI*[*intro?*] =
*scott-continuous-def*[*unfolded atomize-eq*, *THEN iffD2*, *unfolded conj-imp-eq-imp-imp*, *rule-format*]

**lemmas** *scott-continuousE* =
*scott-continuous-def*[*unfolded atomize-eq*, *THEN iffD1*, *elim-format*, *unfolded conj-imp-eq-imp-imp*, *rule-format*]

**lemma** *scott-continuous-imp-mono-refl*:
**assumes** *scott*: *scott-continuous* $f$
**and** $x$: $x \in A$ **and** $y$: $y \in A$ **and** $xy$: $x \sqsubseteq y$ **and** $yy$: $y \sqsubseteq y$
**shows** $f\ x \sqsubseteq f\ y$
⟨*proof*⟩

**lemma** *scott-continuous-imp-omega-continuous*:
**assumes** *scott*: *scott-continuous* $f$ **shows** *omega-continuous* $f$
⟨*proof*⟩

**end**

## 7.2 Existence of Iterative Fixed Points

The first part of Kleene's theorem demands to prove that the set $\{f^n(\bot) \mid n \in \mathbb{N}\}$ has a supremum and that all such are quasi-fixed points. We prove this claim without assuming anything on the relation $\sqsubseteq$ besides $\omega$-completeness and one bottom element.

**notation** *compower* (-^-[*1000*,*1000*]*1000*)

**lemma** *mono-funpow*: **assumes** $f$: $f$ ' $A \subseteq A$ **and** *mono*: *monotone-on* $A$ $r$ $r$ $f$
**shows** *monotone-on* $A$ $r$ $r$ $(f$^$n)$
⟨*proof*⟩

**no-notation** *bot* $(\bot)$

**context**
**fixes** $A$ **and** *less-eq* (**infix** $\sqsubseteq$ *50*) **and** *bot* $(\bot)$ **and** $f$
**assumes** *bot*: $\bot \in A\ \forall q \in A.\ \bot \sqsubseteq q$
**assumes** *cont*: *omega-continuous* $A$ $(\sqsubseteq)$ $f$
**begin**

**interpretation** *less-eq-notations*⟨*proof*⟩ **lemma** *f*: $f$ ‘ $A \subseteq A$ ⟨*proof*⟩ **abbreviation**(*input*) $Fn \equiv \{f\hat{}n \perp \mid. n :: nat\}$

**private lemma** *fn-ref*: $f\hat{}n \perp \sqsubseteq f\hat{}n \perp$ **and** *fnA*: $f\hat{}n \perp \in A$
⟨*proof*⟩ **lemma** *FnA*: $Fn \subseteq A$ ⟨*proof*⟩ **lemma** *fn-monotone*: *monotone* $(\leq)$ $(\sqsubseteq)$ $(\lambda n.\ f\hat{}n \perp)$
⟨*proof*⟩ **lemma** *Fn*: $Fn = range\ (\lambda n.\ f\hat{}n \perp)$ ⟨*proof*⟩

**theorem** *kleene-qfp*:
  **assumes** *q*: *extreme-bound* $A$ $(\sqsubseteq)$ *Fn* $q$
  **shows** $f\ q \sim q$
⟨*proof*⟩

**lemma** *ex-kleene-qfp*:
  **assumes** *comp*: *omega-complete* $A$ $(\sqsubseteq)$
  **shows** $\exists p.$ *extreme-bound* $A$ $(\sqsubseteq)$ *Fn* $p$
  ⟨*proof*⟩

## 7.3 Iterative Fixed Points are Least.

Kleene's theorem also states that the quasi-fixed point found this way is a least one. Again, attractivity is needed to prove this statement.

**lemma** *kleene-qfp-is-least*:
  **assumes** *attract*: $\forall q \in A.\ \forall x \in A.\ f\ q \sim q \longrightarrow x \sqsubseteq f\ q \longrightarrow x \sqsubseteq q$
  **assumes** *q*: *extreme-bound* $A$ $(\sqsubseteq)$ *Fn* $q$
  **shows** *extreme* $\{s \in A.\ f\ s \sim s\}$ $(\sqsupseteq)$ $q$
⟨*proof*⟩

**lemma** *kleene-qfp-iff-least*:
  **assumes** *comp*: *omega-complete* $A$ $(\sqsubseteq)$
  **assumes** *attract*: $\forall q \in A.\ \forall x \in A.\ f\ q \sim q \longrightarrow x \sqsubseteq f\ q \longrightarrow x \sqsubseteq q$
  **assumes** *dual-attract*: $\forall p \in A.\ \forall q \in A.\ \forall x \in A.\ p \sim q \longrightarrow q \sqsubseteq x \longrightarrow p \sqsubseteq x$
  **shows** *extreme-bound* $A$ $(\sqsubseteq)$ *Fn* = *extreme* $\{s \in A.\ f\ s \sim s\}$ $(\sqsupseteq)$
⟨*proof*⟩

**end**

**context** *attractive* **begin**

**interpretation** *less-eq-notations*⟨*proof*⟩

**theorem** *kleene-qfp-is-dual-extreme*:
  **assumes** *comp*: *omega-complete* $A$ $(\sqsubseteq)$
    **and** *cont*: *omega-continuous* $A$ $(\sqsubseteq)$ $f$ **and** *bA*: $b \in A$ **and** *bot*: $\forall x \in A.\ b \sqsubseteq x$
  **shows** *extreme-bound* $A$ $(\sqsubseteq)$ $\{f\hat{}n\ b \mid. n :: nat\}$ = *extreme* $\{s \in A.\ f\ s \sim s\}$ $(\sqsupseteq)$
  ⟨*proof*⟩

**end**

**corollary**(**in** *antisymmetric*) *kleene-fp*:
  **assumes** *cont*: *omega-continuous* $A$ ($\sqsubseteq$) $f$
    **and** $b$: $b \in A$ $\forall x \in A.\ b \sqsubseteq x$
    **and** $p$: *extreme-bound* $A$ ($\sqsubseteq$) $\{f\hat{\ }n\ b\ |.\ n :: nat\}$ $p$
  **shows** $f\ p = p$
  $\langle proof \rangle$

**no-notation** *compower* (-$\hat{\ }$-[*1000*,*1000*]*1000*)

**end**

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