Complete Non-Orders and Fixed Points

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Abstract

We develop an Isabelle/HOL library of order-theoretic concepts, such as various completeness conditions and fixed-point theorems. We keep our formalization as general as possible: we reprove several well-known results about complete orders, often with only antisymmetry or attractivity, a mild condition implied by either antisymmetry or transitivity. In particular, we generalize various theorems ensuring the existence of a quasi-fixed point of monotone maps over complete relations, and show that the set of (quasi-)fixed points is itself complete. This result generalizes and strengthens theorems of Knaster–Tarski, Bourbaki–Witt, Kleene, Markowsky, Pataraia, Mashburn, Bhatta–George, and Stouti–Maaden.

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1 Introduction

The main driving force towards mechanizing mathematics using proof assistants has been the reliability they offer, exemplified prominently by [10], [12], [15], etc. In this work, we utilize another aspect of proof assistants: they are also engineering tools for developing mathematical theories.

Fixed-point theorems are important in computer science, such as in denotational semantics [20] and in abstract interpretation [7], as they allow the definition of semantics of loops and recursive functions. The Knaster–Tarski theorem [23] shows that any monotone map $f : A \to A$ over complete lattice $(A, \sqsubseteq)$ has a fixed point, and the set of fixed points forms also a complete lattice. The result was generalized in various ways: Markowsky [16] showed a corresponding result for *chain-complete* posets. The proof uses the Bourbaki–Witt theorem [6], stating that any inflationary map over a chain-complete poset has a fixed point. The original proof of the latter is non-elementary in the sense that it relies on ordinals and Hartogs' theorem. Pataraia [18] gave an elementary proof that monotone maps over *pointed directed-complete* poset has a fixed point. Fixed points are studied also for *pseudo-orders* [21], relaxing transitivity. Stouti and Maaden [22] showed that every monotone map over a complete pseudo-order has a (least) fixed point. Markowsky's result was also generalized to *weak* chain-complete pseudo-orders by Bhatta and George [4, 5].

Another line of order-theoretic fixed points is the *iterative* approach. Kantorovitch showed that for ω*-continuous* map f over a complete lattice,[1] the iteration $\perp, f\perp, f^2\perp, \ldots$ converges to a fixed point [14, Theorem I]. Tarski [23] also claimed a similar result for a *countably distributive* map over a *countably complete Boolean algebra*. Kleene's fixed-point theorem states that, for *Scott-continuous* maps over pointed directed-complete posets, the iteration converges to the least fixed point. Finally, Mashburn [17] proved a version for ω-continuous maps over ω-complete posets, which covers Kantorovitch's, Tarski's and Kleene's claims.

In particular, we provide the following:

- Several *locales* that help organizing the different order-theoretic conditions, such as reflexivity, transitivity, antisymmetry, and their combination, as well as concepts such as connex and well-related sets, analogues of chains and well-ordered sets in a non-ordered context.

- Existence of fixed points: We provide two proof schemes to prove that monotone or inflationary mapping $f : A \to A$ over a complete related set $\langle A, \sqsubseteq\rangle$ has a *quasi-fixed point* $f\,x \sim x$, meaning $x \sqsubseteq f\,x \wedge f\,x \sqsubseteq x$, for various notions of completeness. The first one, similar to the original proof by Tarski [23], does not require any ordering assumptions, but relies on completeness with respect to all subsets. The second one, inspired by a *constructive* approach by Grall [11], is a proof scheme based on the notion of derivations. Here we demand antisymmetry (to avoid the necessity of the axiom of choice), but can be instantiated to *well-complete* sets, a generalization of weak chain-completeness. This also allows us to generalize Bourbaki–Witt theorem [6] to pseudo-orders.

- Completeness of the set of fixed points: if $(A, \sqsubseteq)$ satisfies a mild condition, which we call *attractivity* and which is implied by either transitivity or antisymmetry, then the set of quasi-fixed points inherits the completeness class from $(A, \sqsubseteq)$, if it is at least well-complete. The result instantiates to the full completeness (generalizing Knaster–Tarski and [22]), directed-completeness [18], chain-completeness [16], and weak chain-completeness [5].

[1] More precisely, he assumes a conditionally complete lattice defined over vectors and that $\perp \sqsubseteq f\,\perp$ and $f\,v' \sqsubseteq v'$. Hence f, which is monotone, is a map over the complete lattice $\{v \mid \perp \sqsubseteq v \sqsubseteq v'\}$.

- Iterative construction: For an ω-continuous map over an ω-complete related set, we show that suprema of $\{f^n \perp \mid n \in \mathbb{N}\}$ are quasi-fixed points. Under attractivity, the quasi-fixed points obtained from this scheme are precisely the least quasi-fixed points of f. This generalizes Mashburn's result, and thus ones by Kantorovitch, Tarski and Kleene.

We remark that all these results would have required much more effort than we spent (if possible at all), if we were not with the aforementioned smart assistance by Isabelle. Our workflow was often the following: first we formalize existing proofs, try relaxing assumptions, see where proof breaks, and at some point ask for a counterexample.

Concerning Isabelle formalization, one can easily find several formalizations of complete partial orders or lattices in Isabelle?s standard library. They are, however, defined on partial orders, either in form of classes or locales, and thus not directly reusable for non-orders. Nevertheless we tried to make our formalization compatible with the existing ones, and various correspondences are ensured.

This archive is the third version of this work. The first version has been published in the conference paper [24]. The second version has been published in the journal paper [8]. The third version is a restructuration of the second version for future formalizations, including [25].

2 Binary Relations

We start with basic properties of binary relations.

theory *Binary-Relations*
 imports

 Main
begin

unbundle *lattice-syntax*

lemma *conj-iff-conj-iff-imp-iff*: $\mathit{Trueprop}\ (x \wedge y \longleftrightarrow x \wedge z) \equiv (x \Longrightarrow (y \longleftrightarrow z))$
 ⟨*proof*⟩

lemma *conj-imp-eq-imp-imp*: $(P \wedge Q \Longrightarrow \mathit{PROP}\ R) \equiv (P \Longrightarrow Q \Longrightarrow \mathit{PROP}\ R)$
 ⟨*proof*⟩

lemma *tranclp-trancl*: $r^{++} = (\lambda x\ y.\ (x,y) \in \{(a,b).\ r\ a\ b\}^{+})$
 ⟨*proof*⟩

lemma *tranclp-id*[*simp*]: $\mathit{transp}\ r \Longrightarrow \mathit{tranclp}\ r = r$
 ⟨*proof*⟩

lemma *transp-tranclp*[*simp*]: $\mathit{transp}\ (\mathit{tranclp}\ r)$ ⟨*proof*⟩

lemma *funpow-dom*: f ‘ $A \subseteq A \Longrightarrow (f \widehat{\ } \widehat{\ } n)$ ‘ $A \subseteq A$ ⟨*proof*⟩

lemma *image-subsetD*: f ‘ $A \subseteq B \Longrightarrow a \in A \Longrightarrow f\ a \in B$ ⟨*proof*⟩

Below we introduce an Isabelle-notation for $\{\ldots x \ldots \mid x \in X\}$.

syntax
-range :: $'a \Rightarrow idts \Rightarrow {}'a\ set$ ((*1*{- /|./ -}))
-image :: $'a \Rightarrow pttrn \Rightarrow {}'a\ set \Rightarrow {}'a\ set$ ((*1*{- /|./ (-/ ∈ -)}))
translations
$\{e\ |.\ p\} \rightleftharpoons$ *CONST range* $(\lambda p.\ e)$
$\{e\ |.\ p \in A\} \rightleftharpoons$ *CONST image* $(\lambda p.\ e)\ A$

lemma *image-constant*:
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i = y$
shows f ‘ $I = (\textit{if}\ I = \{\}\ \textit{then}\ \{\}\ \textit{else}\ \{y\})$
⟨*proof*⟩

2.1 Various Definitions

Here we introduce various definitions for binary relations. The first one is our abbreviation for the dual of a relation.

abbreviation(*input*) *dual* ((-$^-$) [*1000*] *1000*) **where** $r^-\ x\ y \equiv r\ y\ x$

lemma *conversep-is-dual*[*simp*]: *conversep* = *dual* ⟨*proof*⟩

lemma *dual-inf*: $(r \sqcap s)^- = r^- \sqcap s^-$ ⟨*proof*⟩

Monotonicity is already defined in the library, but we want one restricted to a domain.

lemmas *monotone-onE* = *monotone-on-def*[*unfolded atomize-eq*, *THEN iffD1*, *elim-format*, *rule-format*]

lemma *monotone-on-dual*: *monotone-on* $X\ r\ s\ f \Longrightarrow$ *monotone-on* $X\ r^-\ s^-\ f$
⟨*proof*⟩

lemma *monotone-on-id*: *monotone-on* $X\ r\ r$ *id*
⟨*proof*⟩

lemma *monotone-on-cmono*: $A \subseteq B \Longrightarrow$ *monotone-on* $B \le$ *monotone-on* A
⟨*proof*⟩

Here we define the following notions in a standard manner

The symmetric part of a relation:

definition *sympartp* **where** *sympartp* $r\ x\ y \equiv r\ x\ y \wedge r\ y\ x$

lemma *sympartpI*[*intro*]:
fixes r (**infix** $\sqsubseteq$ *50*)

assumes $x \sqsubseteq y$ **and** $y \sqsubseteq x$ **shows** *sympartp* $(\sqsubseteq)$ x y
⟨*proof*⟩

lemma *sympartpE*[*elim*]:
fixes r (**infix** $\sqsubseteq$ *50*)
assumes *sympartp* $(\sqsubseteq)$ x y **and** $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow$ *thesis* **shows** *thesis*
⟨*proof*⟩

lemma *sympartp-dual*: *sympartp* r^- = *sympartp* r
⟨*proof*⟩

lemma *sympartp-eq*[*simp*]: *sympartp* $(=) = (=)$ ⟨*proof*⟩

lemma *sympartp-sympartp*[*simp*]: *sympartp* (*sympartp* r) = *sympartp* r ⟨*proof*⟩

lemma *reflclp-sympartp*[*simp*]: (*sympartp* r)$^{==}$ = *sympartp* $r^{==}$ ⟨*proof*⟩

definition *equivpartp* r x y $\equiv$ $x = y \vee r\ x\ y \wedge r\ y\ x$

lemma *sympartp-reflclp-equivp*[*simp*]: *sympartp* $r^{==}$ = *equivpartp* r ⟨*proof*⟩

lemma *equivpartI*[*simp*]: *equivpartp* r x x
and *sympartp-equivpartpI*: *sympartp* r x y $\Longrightarrow$ *equivpartp* r x y
and *equivpartpCI*[*intro*]: $(x \neq y \Longrightarrow$ *sympartp* r x $y) \Longrightarrow$ *equivpartp* r x y
⟨*proof*⟩

lemma *equivpartpE*[*elim*]:
assumes *equivpartp* r x y
and $x = y \Longrightarrow$ *thesis*
and $r\ x\ y \Longrightarrow r\ y\ x \Longrightarrow$ *thesis*
shows *thesis*
⟨*proof*⟩

lemma *equivpartp-eq*[*simp*]: *equivpartp* $(=) = (=)$ ⟨*proof*⟩

lemma *sympartp-equivpartp*[*simp*]: *sympartp* (*equivpartp* r) = (*equivpartp* r)
and *equivpartp-equivpartp*[*simp*]: *equivpartp* (*equivpartp* r) = (*equivpartp* r)
and *equivpartp-sympartp*[*simp*]: *equivpartp* (*sympartp* r) = (*equivpartp* r)
⟨*proof*⟩

lemma *equivpartp-dual*: *equivpartp* r^- = *equivpartp* r
⟨*proof*⟩

The asymmetric part:

definition *asympartp* r x y $\equiv$ $r\ x\ y \wedge \neg\ r\ y\ x$

lemma *asympartpE*[*elim*]:
fixes r (**infix** $\sqsubseteq$ *50*)
shows *asympartp* $(\sqsubseteq)$ x $y \Longrightarrow (x \sqsubseteq y \Longrightarrow \neg y \sqsubseteq x \Longrightarrow$ *thesis*$) \Longrightarrow$ *thesis*

$\langle proof \rangle$

lemmas *asympartpI*[*intro*] = *asympartp-def*[*unfolded atomize-eq*, *THEN iffD2*, *unfolded conj-imp-eq-imp-imp*, *rule-format*]

lemma *asympartp-eq*[*simp*]: $asympartp\ (=) = bot$ $\langle proof \rangle$

lemma *asympartp-sympartp* [*simp*]: $asympartp\ (sympartp\ r) = bot$
and *sympartp-asympartp* [*simp*]: $sympartp\ (asympartp\ r) = bot$
$\langle proof \rangle$

lemma *asympartp-dual*: $asympartp\ r^{-} = (asympartp\ r)^{-}$ $\langle proof \rangle$

Restriction to a set:

definition *Restrp* (**infixl** $\upharpoonright$ *60*) **where** $(r \upharpoonright A)\ a\ b \equiv a \in A \wedge b \in A \wedge r\ a\ b$

lemmas *RestrpI*[*intro*!] = *Restrp-def*[*unfolded atomize-eq*, *THEN iffD2*, *unfolded conj-imp-eq-imp-imp*]
lemmas *RestrpE*[*elim*!] = *Restrp-def*[*unfolded atomize-eq*, *THEN iffD1*, *elim-format*, *unfolded conj-imp-eq-imp-imp*]

lemma *Restrp-simp*[*simp*]: $a \in A \Longrightarrow b \in A \Longrightarrow (r \upharpoonright A)\ a\ b \longleftrightarrow r\ a\ b$ $\langle proof \rangle$

lemma *Restrp-UNIV*[*simp*]: $r \upharpoonright UNIV \equiv r$ $\langle proof \rangle$

lemma *Restrp-Restrp*[*simp*]: $r \upharpoonright A \upharpoonright B \equiv r \upharpoonright A \cap B$ $\langle proof \rangle$

lemma *sympartp-Restrp*[*simp*]: $sympartp\ (r \upharpoonright A) \equiv sympartp\ r \upharpoonright A$
$\langle proof \rangle$

Relational images:

definition *Imagep* (**infixr** *‘‘‘ 59*) **where** $r\ ‘‘‘\ A \equiv \{b.\ \exists\, a \in A.\ r\ a\ b\}$

lemma *Imagep-Image*: $r\ ‘‘‘\ A = \{(a,b).\ r\ a\ b\}\ ‘‘\ A$
$\langle proof \rangle$

lemma *in-Imagep*: $b \in r\ ‘‘‘\ A \longleftrightarrow (\exists\, a \in A.\ r\ a\ b)$ $\langle proof \rangle$

lemma *ImagepI*: $a \in A \Longrightarrow r\ a\ b \Longrightarrow b \in r\ ‘‘‘\ A$ $\langle proof \rangle$

lemma *subset-Imagep*: $B \subseteq r\ ‘‘‘\ A \longleftrightarrow (\forall\, b{\in}B.\ \exists\, a{\in}A.\ r\ a\ b)$
$\langle proof \rangle$

Bounds of a set:

definition $bound\ X\ (\sqsubseteq)\ b \equiv \forall\, x \in X.\ x \sqsubseteq b$ **for** r (**infix** $\sqsubseteq$ *50*)

lemma
fixes r (**infix** $\sqsubseteq$ *50*)
shows *boundI*[*intro*!]: $(\bigwedge x.\ x \in X \Longrightarrow x \sqsubseteq b) \Longrightarrow bound\ X\ (\sqsubseteq)\ b$

and *boundE*[*elim*]: *bound* X ($\sqsubseteq$) $b \Longrightarrow ((\bigwedge x.\ x \in X \Longrightarrow x \sqsubseteq b) \Longrightarrow$ *thesis*) $\Longrightarrow$ *thesis*
and *boundD*: *bound* X ($\sqsubseteq$) $b \Longrightarrow a \in X \Longrightarrow a \sqsubseteq b$
⟨*proof*⟩

lemma *bound-empty*: *bound* {} = ($\lambda r\ x.$ *True*) ⟨*proof*⟩

lemma *bound-cmono*: **assumes** $X \subseteq Y$ **shows** *bound* $Y \leq$ *bound* X
⟨*proof*⟩

lemmas *bound-subset* = *bound-cmono*[*THEN le-funD*, *THEN le-funD*, *THEN le-boolD*, *folded atomize-imp*]

lemma *bound-un*: *bound* $(A \cup B)$ = *bound* $A \sqcap$ *bound* B
⟨*proof*⟩

lemma *bound-insert*[*simp*]:
fixes r (**infix** $\sqsubseteq$ *50*)
shows *bound* (*insert* x X) ($\sqsubseteq$) $b \longleftrightarrow x \sqsubseteq b \wedge$ *bound* X ($\sqsubseteq$) b ⟨*proof*⟩

lemma *bound-cong*:
assumes $A = A'$
and $b = b'$
and $\bigwedge a.\ a \in A' \Longrightarrow$ *le* a b' = *le'* a b'
shows *bound* A *le* b = *bound* A' *le'* b'
⟨*proof*⟩

lemma *bound-subsel*: *le* $\leq$ *le'* $\Longrightarrow$ *bound* A *le* $\leq$ *bound* A *le'*
⟨*proof*⟩

Extreme (greatest) elements in a set:

definition *extreme* X ($\sqsubseteq$) $e \equiv e \in X \wedge (\forall x \in X.\ x \sqsubseteq e)$ **for** r (**infix** $\sqsubseteq$ *50*)

lemma
fixes r (**infix** $\sqsubseteq$ *50*)
shows *extremeI*[*intro*]: $e \in X \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow x \sqsubseteq e) \Longrightarrow$ *extreme* X ($\sqsubseteq$) e
and *extremeD*: *extreme* X ($\sqsubseteq$) $e \Longrightarrow e \in X$ *extreme* X ($\sqsubseteq$) $e \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow x \sqsubseteq e)$
and *extremeE*[*elim*]: *extreme* X ($\sqsubseteq$) $e \Longrightarrow (e \in X \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow x \sqsubseteq e) \Longrightarrow$ *thesis*) $\Longrightarrow$ *thesis*
⟨*proof*⟩

lemma
fixes r (**infix** $\sqsubseteq$ *50*)
shows *extreme-UNIV*[*simp*]: *extreme UNIV* ($\sqsubseteq$) $t \longleftrightarrow (\forall x.\ x \sqsubseteq t)$ ⟨*proof*⟩

lemma *extreme-iff-bound*: *extreme* X r $e \longleftrightarrow$ *bound* X r $e \wedge e \in X$ ⟨*proof*⟩

lemma *extreme-imp-bound*: *extreme* X r $x \Longrightarrow$ *bound* X r x ⟨*proof*⟩

lemma *extreme-inf*: $extreme\ X\ (r \sqcap s)\ x \longleftrightarrow extreme\ X\ r\ x \wedge extreme\ X\ s\ x$
⟨*proof*⟩

lemma *extremes-equiv*: $extreme\ X\ r\ b \Longrightarrow extreme\ X\ r\ c \Longrightarrow sympartp\ r\ b\ c$ ⟨*proof*⟩

lemma *extreme-cong*:
assumes $A = A'$
and $b = b'$
and $\bigwedge a.\ a \in A' \Longrightarrow b' \in A' \Longrightarrow le\ a\ b' = le'\ a\ b'$
shows $extreme\ A\ le\ b = extreme\ A'\ le'\ b'$
⟨*proof*⟩

lemma *extreme-subset*: $X \subseteq Y \Longrightarrow extreme\ X\ r\ x \Longrightarrow extreme\ Y\ r\ y \Longrightarrow r\ x\ y$
⟨*proof*⟩

lemma *extreme-subrel*:
$le \leq le' \Longrightarrow extreme\ A\ le \leq extreme\ A\ le'$ ⟨*proof*⟩

Now suprema and infima are given uniformly as follows. The definition is restricted to a given set.

definition
$extreme\text{-}bound\ A\ (\sqsubseteq)\ X \equiv extreme\ \{b \in A.\ bound\ X\ (\sqsubseteq)\ b\}\ (\sqsubseteq)^{-}$ **for** r (**infix** $\sqsubseteq$ *50*)

lemmas *extreme-boundI-extreme* = *extreme-bound-def*[*unfolded atomize-eq*, *THEN fun-cong*, *THEN iffD2*]

lemmas *extreme-boundD-extreme* = *extreme-bound-def*[*unfolded atomize-eq*, *THEN fun-cong*, *THEN iffD1*]

context
fixes $A :: 'a\ set$ **and** $less\text{-}eq :: 'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)
begin

lemma *extreme-boundI*[*intro*]:
assumes $\bigwedge b.\ bound\ X\ (\sqsubseteq)\ b \Longrightarrow b \in A \Longrightarrow s \sqsubseteq b$ **and** $\bigwedge x.\ x \in X \Longrightarrow x \sqsubseteq s$ **and** $s \in A$
shows $extreme\text{-}bound\ A\ (\sqsubseteq)\ X\ s$
⟨*proof*⟩

lemma *extreme-boundD*:
assumes $extreme\text{-}bound\ A\ (\sqsubseteq)\ X\ s$
shows $x \in X \Longrightarrow x \sqsubseteq s$
and $bound\ X\ (\sqsubseteq)\ b \Longrightarrow b \in A \Longrightarrow s \sqsubseteq b$
and *extreme-bound-in*: $s \in A$
⟨*proof*⟩

lemma *extreme-boundE*[*elim*]:

assumes *extreme-bound* A $(\sqsubseteq)$ X s
and $s \in A \Longrightarrow$ *bound* X $(\sqsubseteq)$ $s \Longrightarrow (\bigwedge b.$ *bound* X $(\sqsubseteq)$ $b \Longrightarrow b \in A \Longrightarrow s \sqsubseteq b)$
$\Longrightarrow$ *thesis*
shows *thesis*
⟨*proof*⟩

lemma *extreme-bound-imp-bound*: *extreme-bound* A $(\sqsubseteq)$ X $s \Longrightarrow$ *bound* X $(\sqsubseteq)$ s
⟨*proof*⟩

lemma *extreme-imp-extreme-bound*:
assumes *Xs*: *extreme* X $(\sqsubseteq)$ s **and** *XA*: $X \subseteq A$ **shows** *extreme-bound* A $(\sqsubseteq)$ X s
⟨*proof*⟩

lemma *extreme-bound-subset-bound*:
assumes *XY*: $X \subseteq Y$
and *sX*: *extreme-bound* A $(\sqsubseteq)$ X s
and *b*: *bound* Y $(\sqsubseteq)$ b **and** *bA*: $b \in A$
shows $s \sqsubseteq b$
⟨*proof*⟩

lemma *extreme-bound-subset*:
assumes *XY*: $X \subseteq Y$
and *sX*: *extreme-bound* A $(\sqsubseteq)$ X sX
and *sY*: *extreme-bound* A $(\sqsubseteq)$ Y sY
shows $sX \sqsubseteq sY$
⟨*proof*⟩

lemma *extreme-bound-iff*:
extreme-bound A $(\sqsubseteq)$ X $s \longleftrightarrow s \in A \wedge (\forall c \in A. (\forall x \in X. x \sqsubseteq c) \longrightarrow s \sqsubseteq c) \wedge$
$(\forall x \in X. x \sqsubseteq s)$
⟨*proof*⟩

lemma *extreme-bound-empty*: *extreme-bound* A $(\sqsubseteq)$ $\{\}$ $x \longleftrightarrow$ *extreme* A $(\sqsubseteq)^-$ x
⟨*proof*⟩

lemma *extreme-bound-singleton-refl*[*simp*]:
extreme-bound A $(\sqsubseteq)$ $\{x\}$ $x \longleftrightarrow x \in A \wedge x \sqsubseteq x$ ⟨*proof*⟩

lemma *extreme-bound-image-const*:
$x \sqsubseteq x \Longrightarrow I \neq \{\} \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow f\ i = x) \Longrightarrow x \in A \Longrightarrow$ *extreme-bound* A
$(\sqsubseteq)$ $(f$ ‘ $I)$ x
⟨*proof*⟩

lemma *extreme-bound-UN-const*:
$x \sqsubseteq x \Longrightarrow I \neq \{\} \Longrightarrow (\bigwedge i\ y.\ i \in I \Longrightarrow P\ i\ y \longleftrightarrow x = y) \Longrightarrow x \in A \Longrightarrow$
extreme-bound A $(\sqsubseteq)$ $(\bigcup i{\in}I.\ \{y.\ P\ i\ y\})$ x
⟨*proof*⟩

lemma *extreme-bounds-equiv*:

assumes s: *extreme-bound* A $(\sqsubseteq)$ X s **and** s': *extreme-bound* A $(\sqsubseteq)$ X s'
shows *sympartp* $(\sqsubseteq)$ s s'
⟨*proof*⟩

lemma *extreme-bound-squeeze*:
assumes XY: $X \subseteq Y$ **and** YZ: $Y \subseteq Z$
and Xs: *extreme-bound* A $(\sqsubseteq)$ X s **and** Zs: *extreme-bound* A $(\sqsubseteq)$ Z s
shows *extreme-bound* A $(\sqsubseteq)$ Y s
⟨*proof*⟩

lemma *bound-closed-imp-extreme-bound-eq-extreme*:
assumes *closed*: $\forall\, b \in A.$ *bound* X $(\sqsubseteq)$ $b \longrightarrow b \in X$ **and** XA: $X \subseteq A$
shows *extreme-bound* A $(\sqsubseteq)$ X = *extreme* X $(\sqsubseteq)$
⟨*proof*⟩

end

lemma *extreme-bound-cong*:
assumes $A = A'$
and $X = X'$
and $\bigwedge a\ b.\ a \in A' \Longrightarrow b \in A' \Longrightarrow le\ a\ b \longleftrightarrow le'\ a\ b$
and $\bigwedge a\ b.\ a \in X' \Longrightarrow b \in A' \Longrightarrow le\ a\ b \longleftrightarrow le'\ a\ b$
shows *extreme-bound* A *le* X s = *extreme-bound* A le' X s
⟨*proof*⟩

Maximal or Minimal

definition *extremal* X $(\sqsubseteq)$ $x \equiv x \in X \wedge (\forall\, y \in X.\ x \sqsubseteq y \longrightarrow y \sqsubseteq x)$ **for** r (**infix** $\sqsubseteq$ *50*)

context
fixes $r :: {'a} \Rightarrow {'a} \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)
begin

lemma *extremalI*:
assumes $x \in X$ $\bigwedge y.\ y \in X \Longrightarrow x \sqsubseteq y \Longrightarrow y \sqsubseteq x$
shows *extremal* X $(\sqsubseteq)$ x
⟨*proof*⟩

lemma *extremalE*:
assumes *extremal* X $(\sqsubseteq)$ x
and $x \in X \Longrightarrow (\bigwedge y.\ y \in X \Longrightarrow x \sqsubseteq y \Longrightarrow y \sqsubseteq x) \Longrightarrow thesis$
shows *thesis*
⟨*proof*⟩

lemma *extremalD*:
assumes *extremal* X $(\sqsubseteq)$ x **shows** $x \in X$ $y \in X \Longrightarrow x \sqsubseteq y \Longrightarrow y \sqsubseteq x$
⟨*proof*⟩

end

context
  fixes *ir* (**infix** $\preceq$ *50*) **and** *r* (**infix** $\sqsubseteq$ *50*) **and** *I f*
  assumes *mono*: *monotone-on I* $(\preceq)$ $(\sqsubseteq)$ *f*
begin

lemma *monotone-image-bound*:
  assumes $X \subseteq I$ **and** $b \in I$ **and** *bound X* $(\preceq)$ *b*
  shows *bound* (*f* ‘ *X*) $(\sqsubseteq)$ (*f b*)
  ⟨*proof*⟩

lemma *monotone-image-extreme*:
  assumes *e*: *extreme I* $(\preceq)$ *e*
  shows *extreme* (*f* ‘ *I*) $(\sqsubseteq)$ (*f e*)
  ⟨*proof*⟩

end

context
  fixes *ir* :: $'i \Rightarrow 'i \Rightarrow$ *bool* (**infix** $\preceq$ *50*)
    and *r* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
    and *f* **and** *A* **and** *e* **and** *I*
  assumes *fIA*: *f* ‘ $I \subseteq A$
    and *mono*: *monotone-on I* $(\preceq)$ $(\sqsubseteq)$ *f*
    and *e*: *extreme I* $(\preceq)$ *e*
begin

lemma *monotone-extreme-imp-extreme-bound*:
  extreme-bound A $(\sqsubseteq)$ (*f* ‘ *I*) (*f e*)
  ⟨*proof*⟩

lemma *monotone-extreme-extreme-boundI*:
  $x = f\ e \Longrightarrow$ *extreme-bound A* $(\sqsubseteq)$ (*f* ‘ *I*) *x*
  ⟨*proof*⟩

end

2.2 Locales for Binary Relations

We now define basic properties of binary relations, in form of *locales* [13, 2].

2.2.1 Syntactic Locales

The following locales do not assume anything, but provide infix notations for relations.

locale *less-eq-syntax* =
  fixes *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)

locale *less-syntax* =
  fixes *less* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubset$ *50*)

locale *equivalence-syntax* =
  fixes *equiv* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sim$ *50*)
begin

abbreviation *equiv-class* ($[\text{-}]_\sim$) **where** $[x]_\sim \equiv \{ y.\ x \sim y \}$

end

Next ones introduce abbreviations for dual etc. To avoid needless constants, one should be careful when declaring them as sublocales.

locale *less-eq-dualize* = *less-eq-syntax*
begin

abbreviation (*input*) *greater-eq* (**infix** $\sqsupseteq$ *50*) **where** $x \sqsupseteq y \equiv y \sqsubseteq x$

end

locale *less-eq-symmetrize* = *less-eq-dualize*
begin

abbreviation *sym* (**infix** $\sim$ *50*) **where** $(\sim) \equiv$ *sympartp* $(\sqsubseteq)$
abbreviation *equiv* (**infix** $(\simeq)$ *50*) **where** $(\simeq) \equiv$ *equivpartp* $(\sqsubseteq)$

end

locale *less-eq-asymmetrize* = *less-eq-symmetrize*
begin

abbreviation *less* (**infix** $\sqsubset$ *50*) **where** $(\sqsubset) \equiv$ *asympartp* $(\sqsubseteq)$
abbreviation *greater* (**infix** $\sqsupset$ *50*) **where** $(\sqsupset) \equiv (\sqsubset)^-$

lemma *asym-cases*[*consumes 1*, *case-names asym sym*]:
  assumes $x \sqsubseteq y$ **and** $x \sqsubset y \Longrightarrow$ *thesis* **and** $x \sim y \Longrightarrow$ *thesis*
  shows *thesis*
  $\langle$*proof*$\rangle$

end

locale *less-dualize* = *less-syntax*
begin

abbreviation (*input*) *greater* (**infix** $\sqsupset$ *50*) **where** $x \sqsupset y \equiv y \sqsubset x$

end

locale *related-set* =

fixes A :: $'a$ *set* **and** *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)

2.2.2 Basic Properties of Relations

In the following we define basic properties in form of locales.

Reflexivity restricted on a set:

locale *reflexive* = *related-set* +
assumes *refl*[*intro*]: $x \in A \Longrightarrow x \sqsubseteq x$
begin

lemma *eq-implies*: $x = y \Longrightarrow x \in A \Longrightarrow x \sqsubseteq y$ ⟨*proof*⟩

lemma *reflexive-subset*: $B \subseteq A \Longrightarrow$ *reflexive* B $(\sqsubseteq)$ ⟨*proof*⟩

lemma *extreme-singleton*[*simp*]: $x \in A \Longrightarrow$ *extreme* $\{x\}$ $(\sqsubseteq)$ $y \longleftrightarrow x = y$ ⟨*proof*⟩

lemma *extreme-bound-singleton*: $x \in A \Longrightarrow$ *extreme-bound* A $(\sqsubseteq)$ $\{x\}$ x ⟨*proof*⟩

lemma *extreme-bound-cone*: $x \in A \Longrightarrow$ *extreme-bound* A $(\sqsubseteq)$ $\{a \in A.\ a \sqsubseteq x\}$ x ⟨*proof*⟩

end

lemmas *reflexiveI*[*intro*!] = *reflexive.intro*

lemma *reflexiveE*[*elim*]:
assumes *reflexive* A r **and** $(\bigwedge x.\ x \in A \Longrightarrow r\ x\ x) \Longrightarrow$ *thesis* **shows** *thesis*
⟨*proof*⟩

lemma *reflexive-cong*:
$(\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b) \Longrightarrow$ *reflexive* A $r \longleftrightarrow$ *reflexive* A r'
⟨*proof*⟩

locale *irreflexive* = *related-set* A $(\sqsubset)$ **for** A **and** *less* (**infix** $\sqsubset$ *50*) +
assumes *irrefl*: $x \in A \Longrightarrow \neg\ x \sqsubset x$
begin

lemma *irreflD*[*simp*]: $x \sqsubset x \Longrightarrow \neg x \in A$ ⟨*proof*⟩

lemma *implies-not-eq*: $x \sqsubset y \Longrightarrow x \in A \Longrightarrow x \neq y$ ⟨*proof*⟩

lemma *Restrp-irreflexive*: *irreflexive* *UNIV* $((\sqsubset)\upharpoonright A)$
⟨*proof*⟩

lemma *irreflexive-subset*: $B \subseteq A \Longrightarrow$ *irreflexive* B $(\sqsubset)$ ⟨*proof*⟩

end

lemmas *irreflexiveI*[*intro*!] = *irreflexive.intro*

lemma *irreflexive-cong*:
$(\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b) \Longrightarrow$ *irreflexive* $A\ r \longleftrightarrow$ *irreflexive* $A\ r'$
⟨*proof*⟩

context *reflexive* **begin**

interpretation *less-eq-asymmetrize*⟨*proof*⟩

lemma *asympartp-irreflexive*: *irreflexive* A $(\sqsubset)$ ⟨*proof*⟩

end

locale *transitive* = *related-set* +
assumes *trans*[*trans*]: $x \sqsubseteq y \Longrightarrow y \sqsubseteq z \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x \sqsubseteq z$
begin

lemma *Restrp-transitive*: *transitive UNIV* $((\sqsubseteq)\restriction A)$
⟨*proof*⟩

lemma *bound-trans*[*trans*]: *bound* X $(\sqsubseteq)$ $b \Longrightarrow b \sqsubseteq c \Longrightarrow X \subseteq A \Longrightarrow b \in A \Longrightarrow c \in A \Longrightarrow$ *bound* X $(\sqsubseteq)$ c
⟨*proof*⟩

lemma *extreme-bound-mono*:
assumes XY: $\forall x{\in}X.\ \exists y{\in}Y.\ x \sqsubseteq y$ **and** XA: $X \subseteq A$ **and** YA: $Y \subseteq A$
and sX: *extreme-bound* A $(\sqsubseteq)$ X sX
and sY: *extreme-bound* A $(\sqsubseteq)$ Y sY
shows $sX \sqsubseteq sY$
⟨*proof*⟩

lemma *transitive-subset*:
assumes BA: $B \subseteq A$ **shows** *transitive* B $(\sqsubseteq)$
⟨*proof*⟩

lemma *asympartp-transitive*: *transitive* A (*asympartp* $(\sqsubseteq)$)
⟨*proof*⟩

lemma *reflclp-transitive*: *transitive* A $(\sqsubseteq)^{==}$
⟨*proof*⟩

The symmetric part is also transitive, but this is done in the later semi-attractive locale

end

lemmas *transitiveI* = *transitive.intro*

lemma *transitive-ball*[*code*]:
transitive A $(\sqsubseteq) \longleftrightarrow (\forall x \in A.\ \forall y \in A.\ \forall z \in A.\ x \sqsubseteq y \longrightarrow y \sqsubseteq z \longrightarrow x \sqsubseteq z)$
for *less-eq* (**infix** $\sqsubseteq$ *50*)
⟨*proof*⟩

lemma *transitive-cong*:
assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$ **shows** *transitive* A r $\longleftrightarrow$ *transitive* A r'
⟨*proof*⟩

lemma *transitive-empty*[*intro*!]: *transitive* {} r ⟨*proof*⟩

lemma *tranclp-transitive*: *transitive* A (*tranclp* r)
⟨*proof*⟩

locale *symmetric* = *related-set* A $(\sim)$ **for** A **and** *equiv* (**infix** $\sim$ *50*) +
assumes *sym*[*sym*]: $x \sim y \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow y \sim x$
begin

lemma *sym-iff*: $x \in A \Longrightarrow y \in A \Longrightarrow x \sim y \longleftrightarrow y \sim x$
⟨*proof*⟩

lemma *Restrp-symmetric*: *symmetric UNIV* $((\sim)\restriction A)$
⟨*proof*⟩

lemma *symmetric-subset*: $B \subseteq A \Longrightarrow$ *symmetric* B $(\sim)$
⟨*proof*⟩

end

lemmas *symmetricI*[*intro*] = *symmetric.intro*

lemma *symmetric-cong*:
$(\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b) \Longrightarrow$ *symmetric* A $r \longleftrightarrow$ *symmetric* A r'
⟨*proof*⟩

lemma *symmetric-empty*[*intro*!]: *symmetric* {} r ⟨*proof*⟩

global-interpretation *sympartp*: *symmetric UNIV sympartp* r
rewrites $\bigwedge r.\ r \restriction UNIV \equiv r$
and $\bigwedge x.\ x \in UNIV \equiv True$
and $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
and $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
⟨*proof*⟩

lemma *sympartp-symmetric*: *symmetric* A (*sympartp* r) ⟨*proof*⟩

locale *antisymmetric* = *related-set* +
  assumes *antisym*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow x = y$
begin

interpretation *less-eq-symmetrize*⟨*proof*⟩

lemma *sym-iff-eq-refl*: $x \in A \Longrightarrow y \in A \Longrightarrow x \sim y \longleftrightarrow x = y \wedge y \sqsubseteq y$ ⟨*proof*⟩

lemma *equiv-iff-eq*[*simp*]: $x \in A \Longrightarrow y \in A \Longrightarrow x \simeq y \longleftrightarrow x = y$ ⟨*proof*⟩

lemma *extreme-unique*: $X \subseteq A \Longrightarrow$ *extreme* X $(\sqsubseteq)$ $x \Longrightarrow$ *extreme* X $(\sqsubseteq)$ $y \longleftrightarrow x = y$
  ⟨*proof*⟩

lemma *ex-extreme-iff-ex1*:
  $X \subseteq A \Longrightarrow$ *Ex* (*extreme* X $(\sqsubseteq)$) $\longleftrightarrow$ *Ex1* (*extreme* X $(\sqsubseteq)$) ⟨*proof*⟩

lemma *ex-extreme-iff-the*:
  $X \subseteq A \Longrightarrow$ *Ex* (*extreme* X $(\sqsubseteq)$) $\longleftrightarrow$ *extreme* X $(\sqsubseteq)$ (*The* (*extreme* X $(\sqsubseteq)$))
  ⟨*proof*⟩

lemma *eq-The-extreme*: $X \subseteq A \Longrightarrow$ *extreme* X $(\sqsubseteq)$ $x \Longrightarrow x =$ *The* (*extreme* X $(\sqsubseteq)$)
  ⟨*proof*⟩

lemma *Restrp-antisymmetric*: *antisymmetric UNIV* $((\sqsubseteq)\restriction A)$
  ⟨*proof*⟩

lemma *antisymmetric-subset*: $B \subseteq A \Longrightarrow$ *antisymmetric* B $(\sqsubseteq)$
  ⟨*proof*⟩

end

lemmas *antisymmetricI*[*intro*] = *antisymmetric.intro*

lemma *antisymmetric-cong*:
  $(\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b) \Longrightarrow$ *antisymmetric* A $r \longleftrightarrow$ *antisymmetric* A r'
  ⟨*proof*⟩

lemma *antisymmetric-empty*[*intro*!]: *antisymmetric* {} r ⟨*proof*⟩

lemma *antisymmetric-union*:
  fixes *less-eq* (**infix** $\sqsubseteq$ *50*)
  assumes A: *antisymmetric* A $(\sqsubseteq)$ **and** B: *antisymmetric* B $(\sqsubseteq)$
    and AB: $\forall a \in A.\ \forall b \in B.\ a \sqsubseteq b \longrightarrow b \sqsubseteq a \longrightarrow a = b$
  shows *antisymmetric* $(A \cup B)$ $(\sqsubseteq)$

⟨*proof*⟩

The following notion is new, generalizing antisymmetry and transitivity.

locale *semiattractive* = *related-set* +
assumes *attract*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow y \sqsubseteq z \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x \sqsubseteq z$
begin

interpretation *less-eq-symmetrize*⟨*proof*⟩

lemma *equiv-order-trans*[*trans*]:
assumes *xy*: $x \simeq y$ **and** *yz*: $y \sqsubseteq z$ **and** *x*: $x \in A$ **and** *y*: $y \in A$ **and** *z*: $z \in A$
shows $x \sqsubseteq z$
⟨*proof*⟩

lemma *equiv-transitive*: *transitive* A $(\simeq)$
⟨*proof*⟩

lemma *sym-order-trans*[*trans*]:
assumes *xy*: $x \sim y$ **and** *yz*: $y \sqsubseteq z$ **and** *x*: $x \in A$ **and** *y*: $y \in A$ **and** *z*: $z \in A$
shows $x \sqsubseteq z$
⟨*proof*⟩

interpretation *sym*: *transitive* A $(\sim)$
⟨*proof*⟩

lemmas *sym-transitive* = *sym.transitive-axioms*

lemma *extreme-bound-quasi-const*:
assumes *C*: $C \subseteq A$ **and** *x*: $x \in A$ **and** *C0*: $C \neq \{\}$ **and** *const*: $\forall y \in C.\ y \sim x$
shows *extreme-bound* A $(\sqsubseteq)$ C x
⟨*proof*⟩

lemma *extreme-bound-quasi-const-iff*:
assumes *C*: $C \subseteq A$ **and** *x*: $x \in A$ **and** *y*: $y \in A$ **and** *C0*: $C \neq \{\}$ **and** *const*: $\forall z \in C.\ z \sim x$
shows *extreme-bound* A $(\sqsubseteq)$ C $y \longleftrightarrow x \sim y$
⟨*proof*⟩

lemma *Restrp-semiattractive*: *semiattractive UNIV* $((\sqsubseteq)\restriction A)$
⟨*proof*⟩

lemma *semiattractive-subset*: $B \subseteq A \Longrightarrow$ *semiattractive* B $(\sqsubseteq)$
⟨*proof*⟩

end

lemmas *semiattractiveI* = *semiattractive.intro*

lemma *semiattractive-cong*:
  assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  shows *semiattractive* $A\ r \longleftrightarrow$ *semiattractive* $A\ r'$ (**is** $?l \longleftrightarrow ?r$)
⟨*proof*⟩

lemma *semiattractive-empty*[*intro*!]: *semiattractive* {} r
  ⟨*proof*⟩

locale *attractive* = *semiattractive* +
  assumes *semiattractive* $A\ (\sqsubseteq)^-$
begin

interpretation *less-eq-symmetrize*⟨*proof*⟩

sublocale *dual*: *semiattractive* $A\ (\sqsubseteq)^-$
  rewrites $\bigwedge r.$ *sympartp* $(r \restriction A) \equiv$ *sympartp* $r \restriction A$
    and $\bigwedge r.$ *sympartp* (*sympartp* r) $\equiv$ *sympartp* r
    and *sympartp* $((\sqsubseteq) \restriction A)^- \equiv (\sim) \restriction A$
    and *sympartp* $(\sqsubseteq)^- \equiv (\sim)$
    and *equivpartp* $(\sqsubseteq)^- \equiv (\simeq)$
  ⟨*proof*⟩

lemma *order-equiv-trans*[*trans*]:
  assumes *xy*: $x \sqsubseteq y$ **and** *yz*: $y \simeq z$ **and** x: $x \in A$ **and** y: $y \in A$ **and** z: $z \in A$
  shows $x \sqsubseteq z$
  ⟨*proof*⟩

lemma *order-sym-trans*[*trans*]:
  assumes *xy*: $x \sqsubseteq y$ **and** *yz*: $y \sim z$ **and** x: $x \in A$ **and** y: $y \in A$ **and** z: $z \in A$
  shows $x \sqsubseteq z$
  ⟨*proof*⟩

lemma *extreme-bound-sym-trans*:
  assumes *XA*: $X \subseteq A$ **and** *Xx*: *extreme-bound* $A\ (\sqsubseteq)\ X\ x$
    and *xy*: $x \sim y$ **and** *yA*: $y \in A$
  shows *extreme-bound* $A\ (\sqsubseteq)\ X\ y$
⟨*proof*⟩

interpretation *Restrp*: *semiattractive* *UNIV* $(\sqsubseteq)\restriction A$ ⟨*proof*⟩
interpretation *dual.Restrp*: *semiattractive* *UNIV* $(\sqsubseteq)^-\restriction A$ ⟨*proof*⟩

lemma *Restrp-attractive*: *attractive* *UNIV* $((\sqsubseteq)\restriction A)$
  ⟨*proof*⟩

lemma *attractive-subset*: $B \subseteq A \Longrightarrow$ *attractive* $B\ (\sqsubseteq)$
  ⟨*proof*⟩

end

lemmas *attractiveI* = *attractive.intro*[*OF* - *attractive-axioms.intro*]

lemma *attractive-cong*:
  assumes *r*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  shows $attractive\ A\ r \longleftrightarrow attractive\ A\ r'$
  ⟨*proof*⟩

lemma *attractive-empty*[*intro*!]: *attractive* {} *r*
  ⟨*proof*⟩

context *antisymmetric* **begin**

sublocale *attractive*
  ⟨*proof*⟩

end

context *transitive* **begin**

sublocale *attractive*
  rewrites $\bigwedge r.\ sympartp\ (r \restriction A) \equiv sympartp\ r \restriction A$
    and $\bigwedge r.\ sympartp\ (sympartp\ r) \equiv sympartp\ r$
    and $sympartp\ (\sqsubseteq)^- \equiv sympartp\ (\sqsubseteq)$
    and $(sympartp\ (\sqsubseteq))^- \equiv sympartp\ (\sqsubseteq)$
    and $(sympartp\ (\sqsubseteq) \restriction A)^- \equiv sympartp\ (\sqsubseteq) \restriction A$
    and $asympartp\ (asympartp\ (\sqsubseteq)) = asympartp\ (\sqsubseteq)$
    and $asympartp\ (sympartp\ (\sqsubseteq)) = bot$
    and $asympartp\ (\sqsubseteq) \restriction A = asympartp\ ((\sqsubseteq) \restriction A)$
  ⟨*proof*⟩

end

2.3 Combined Properties

Some combinations of the above basic properties are given names.

locale *asymmetric* = *related-set* *A* ($\sqsubset$) **for** *A* **and** *less* (**infix** $\sqsubset$ *50*) +
  assumes *asym*: $x \sqsubset y \Longrightarrow y \sqsubset x \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow False$
begin

sublocale *irreflexive*
  ⟨*proof*⟩

lemma *antisymmetric-axioms*: *antisymmetric* *A* ($\sqsubset$)
  ⟨*proof*⟩

lemma *Restrp-asymmetric*: *asymmetric* *UNIV* $((\sqsubset)\restriction A)$
  ⟨*proof*⟩

lemma *asymmetric-subset*: $B \subseteq A \Longrightarrow asymmetric\ B\ (\sqsubset)$

$\langle proof \rangle$

end

lemmas *asymmetricI* = *asymmetric.intro*

lemma *asymmetric-iff-irreflexive-antisymmetric*:
fixes *less* (**infix** $\sqsubset$ *50*)
shows *asymmetric* A $(\sqsubset) \longleftrightarrow$ *irreflexive* A $(\sqsubset) \wedge$ *antisymmetric* A $(\sqsubset)$ (**is** *?l* $\longleftrightarrow$ *?r*)
$\langle proof \rangle$

lemma *asymmetric-cong*:
assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *asymmetric* $A\ r \longleftrightarrow$ *asymmetric* $A\ r'$
$\langle proof \rangle$

lemma *asymmetric-empty*: *asymmetric* $\{\}\ r$
$\langle proof \rangle$

locale *quasi-ordered-set* = *reflexive* + *transitive*
begin

lemma *quasi-ordered-subset*: $B \subseteq A \Longrightarrow$ *quasi-ordered-set* B $(\sqsubseteq)$
$\langle proof \rangle$

end

lemmas *quasi-ordered-setI* = *quasi-ordered-set.intro*

lemma *quasi-ordered-set-cong*:
assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *quasi-ordered-set* $A\ r \longleftrightarrow$ *quasi-ordered-set* $A\ r'$
$\langle proof \rangle$

lemma *quasi-ordered-set-empty*[*intro*!]: *quasi-ordered-set* $\{\}\ r$
$\langle proof \rangle$

lemma *rtranclp-quasi-ordered*: *quasi-ordered-set* A (*rtranclp* r)
$\langle proof \rangle$

locale *near-ordered-set* = *antisymmetric* + *transitive*
begin

interpretation *Restrp*: *antisymmetric UNIV* $(\sqsubseteq)\restriction A$ $\langle proof \rangle$
interpretation *Restrp*: *transitive UNIV* $(\sqsubseteq)\restriction A$ $\langle proof \rangle$

lemma *Restrp-near-order*: *near-ordered-set UNIV* $((\sqsubseteq)\restriction A)\langle proof \rangle$

lemma *near-ordered-subset*: $B \subseteq A \Longrightarrow$ *near-ordered-set* B $(\sqsubseteq)$
$\langle proof \rangle$

end

lemmas *near-ordered-setI* = *near-ordered-set.intro*

lemma *near-ordered-set-cong*:
assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *near-ordered-set* $A\ r \longleftrightarrow$ *near-ordered-set* $A\ r'$
$\langle proof \rangle$

lemma *near-ordered-set-empty*[*intro*!]: *near-ordered-set* $\{\}\ r$
$\langle proof \rangle$

locale *pseudo-ordered-set* = *reflexive* + *antisymmetric*
begin

interpretation *less-eq-symmetrize*$\langle proof \rangle$

lemma *sym-eq*[*simp*]: $x \in A \Longrightarrow y \in A \Longrightarrow x \sim y \longleftrightarrow x = y$
$\langle proof \rangle$

lemma *extreme-bound-singleton-eq*[*simp*]: $x \in A \Longrightarrow$ *extreme-bound* A $(\sqsubseteq)$ $\{x\}$ y $\longleftrightarrow x = y$
$\langle proof \rangle$

lemma *eq-iff*: $x \in A \Longrightarrow y \in A \Longrightarrow x = y \longleftrightarrow x \sqsubseteq y \wedge y \sqsubseteq x$ $\langle proof \rangle$

lemma *extreme-order-iff-eq*: $e \in A \Longrightarrow$ *extreme* $\{x \in A.\ x \sqsubseteq e\}$ $(\sqsubseteq)$ $s \longleftrightarrow e = s$
$\langle proof \rangle$

lemma *pseudo-ordered-subset*: $B \subseteq A \Longrightarrow$ *pseudo-ordered-set* B $(\sqsubseteq)$
$\langle proof \rangle$

end

lemmas *pseudo-ordered-setI* = *pseudo-ordered-set.intro*

lemma *pseudo-ordered-set-cong*:
assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *pseudo-ordered-set* $A\ r \longleftrightarrow$ *pseudo-ordered-set* $A\ r'$
$\langle proof \rangle$

lemma *pseudo-ordered-set-empty*[*intro*!]: *pseudo-ordered-set* $\{\}\ r$
$\langle proof \rangle$

locale *partially-ordered-set* = *reflexive* + *antisymmetric* + *transitive*
begin

sublocale *pseudo-ordered-set* + *quasi-ordered-set* + *near-ordered-set* ⟨*proof*⟩

lemma *partially-ordered-subset*: $B \subseteq A \Longrightarrow$ *partially-ordered-set* B $(\sqsubseteq)$
⟨*proof*⟩

end

lemmas *partially-ordered-setI* = *partially-ordered-set.intro*

lemma *partially-ordered-set-cong*:
assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *partially-ordered-set* $A\ r \longleftrightarrow$ *partially-ordered-set* $A\ r'$
⟨*proof*⟩

lemma *partially-ordered-set-empty*[*intro*!]: *partially-ordered-set* $\{\}$ r
⟨*proof*⟩

locale *strict-ordered-set* = *irreflexive* + *transitive* A $(\sqsubset)$
begin

sublocale *asymmetric*
⟨*proof*⟩

lemma *near-ordered-set-axioms*: *near-ordered-set* A $(\sqsubset)$
⟨*proof*⟩

interpretation *Restrp*: *asymmetric UNIV* $(\sqsubset)\restriction A$ ⟨*proof*⟩
interpretation *Restrp*: *transitive UNIV* $(\sqsubset)\restriction A$ ⟨*proof*⟩

lemma *Restrp-strict-order*: *strict-ordered-set UNIV* $((\sqsubset)\restriction A)$⟨*proof*⟩

lemma *strict-ordered-subset*: $B \subseteq A \Longrightarrow$ *strict-ordered-set* B $(\sqsubset)$
⟨*proof*⟩

end

lemmas *strict-ordered-setI* = *strict-ordered-set.intro*

lemma *strict-ordered-set-cong*:
assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *strict-ordered-set* $A\ r \longleftrightarrow$ *strict-ordered-set* $A\ r'$
⟨*proof*⟩

lemma *strict-ordered-set-empty*[*intro*!]: *strict-ordered-set* $\{\}$ r
⟨*proof*⟩

locale *tolerance* = *symmetric* + *reflexive* A $(\sim)$
begin

lemma *tolerance-subset*: $B \subseteq A \Longrightarrow$ *tolerance* B $(\sim)$
⟨*proof*⟩

end

lemmas *toleranceI* = *tolerance.intro*

lemma *tolerance-cong*:
assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *tolerance* $A\ r \longleftrightarrow$ *tolerance* $A\ r'$
⟨*proof*⟩

lemma *tolerance-empty*[*intro*!]: *tolerance* {} r ⟨*proof*⟩

global-interpretation *equiv*: *tolerance UNIV equivpartp r*
rewrites $\bigwedge r.\ r \upharpoonright UNIV \equiv r$
and $\bigwedge x.\ x \in UNIV \equiv True$
and $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
and $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
⟨*proof*⟩

locale *partial-equivalence* = *symmetric* +
assumes *transitive* A $(\sim)$
begin

sublocale *transitive* A $(\sim)$
rewrites *sympartp* $(\sim)\upharpoonright A \equiv (\sim)\upharpoonright A$
and *sympartp* $((\sim)\upharpoonright A) \equiv (\sim)\upharpoonright A$
⟨*proof*⟩

lemma *partial-equivalence-subset*: $B \subseteq A \Longrightarrow$ *partial-equivalence* B $(\sim)$
⟨*proof*⟩

end

lemmas *partial-equivalenceI* = *partial-equivalence.intro*[*OF - partial-equivalence-axioms.intro*]

lemma *partial-equivalence-cong*:
assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *partial-equivalence* $A\ r \longleftrightarrow$ *partial-equivalence* $A\ r'$
⟨*proof*⟩

lemma *partial-equivalence-empty*[*intro*!]: *partial-equivalence* {} r
⟨*proof*⟩

locale *equivalence* = *symmetric* + *reflexive* A $(\sim)$ + *transitive* A $(\sim)$
begin

sublocale *tolerance* + *partial-equivalence* + *quasi-ordered-set* A $(\sim)\langle proof\rangle$

lemma *equivalence-subset*: $B \subseteq A \Longrightarrow$ *equivalence* B $(\sim)$
$\langle proof\rangle$

end

lemmas *equivalenceI* = *equivalence.intro*

lemma *equivalence-cong*:
assumes *r*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *equivalence* $A\ r \longleftrightarrow$ *equivalence* $A\ r'$
$\langle proof\rangle$

Some combinations lead to uninteresting relations.

context
fixes $r :: {}'a \Rightarrow {}'a \Rightarrow bool$ (**infix** $\bowtie$ *50*)
begin

proposition *reflexive-irreflexive-is-empty*:
assumes *r*: *reflexive* A $(\bowtie)$ **and** *ir*: *irreflexive* A $(\bowtie)$
shows $A = \{\}$
$\langle proof\rangle$

proposition *symmetric-antisymmetric-imp-eq*:
assumes *s*: *symmetric* A $(\bowtie)$ **and** *as*: *antisymmetric* A $(\bowtie)$
shows $(\bowtie)\upharpoonright A \le (=)$
$\langle proof\rangle$

proposition *nontolerance*:
shows *irreflexive* A $(\bowtie)$ $\wedge$ *symmetric* A $(\bowtie)$ $\longleftrightarrow$ *tolerance* A $(\lambda x\ y.\ \neg\ x \bowtie y)$
$\langle proof\rangle$

proposition *irreflexive-transitive-symmetric-is-empty*:
assumes *irr*: *irreflexive* A $(\bowtie)$ **and** *tr*: *transitive* A $(\bowtie)$ **and** *sym*: *symmetric* A $(\bowtie)$
shows $(\bowtie)\upharpoonright A = bot$
$\langle proof\rangle$

end

2.4 Totality

locale *semiconnex* = *related-set* - $(\sqsubset)$ + *less-syntax* +
assumes *semiconnex*: $x \in A \Longrightarrow y \in A \Longrightarrow x \sqsubset y \vee x = y \vee y \sqsubset x$
begin

lemma *cases*[*consumes 2*, *case-names less eq greater*]:

assumes $x \in A$ **and** $y \in A$ **and** $x \sqsubset y \Longrightarrow P$ **and** $x = y \Longrightarrow P$ **and** $y \sqsubset x \Longrightarrow P$
shows P ⟨*proof*⟩

lemma *neqE*:
assumes $x \in A$ **and** $y \in A$
shows $x \neq y \Longrightarrow (x \sqsubset y \Longrightarrow P) \Longrightarrow (y \sqsubset x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *semiconnex-subset*: $B \subseteq A \Longrightarrow$ *semiconnex* B $(\sqsubset)$
⟨*proof*⟩

end

lemmas *semiconnexI*[*intro*] = *semiconnex.intro*

Totality is negated antisymmetry [19, Proposition 2.2.4].

proposition *semiconnex-iff-neg-antisymmetric*:
fixes *less* (**infix** $\sqsubset$ *50*)
shows *semiconnex* A $(\sqsubset) \longleftrightarrow$ *antisymmetric* A $(\lambda x\ y.\ \neg\ x \sqsubset y)$ (**is** $?l \longleftrightarrow ?r$)
⟨*proof*⟩

lemma *semiconnex-cong*:
assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *semiconnex* A $r \longleftrightarrow$ *semiconnex* A r'
⟨*proof*⟩

locale *semiconnex-irreflexive* = *semiconnex* + *irreflexive*
begin

lemma *neq-iff*: $x \in A \Longrightarrow y \in A \Longrightarrow x \neq y \longleftrightarrow x \sqsubset y \vee y \sqsubset x$ ⟨*proof*⟩

lemma *semiconnex-irreflexive-subset*: $B \subseteq A \Longrightarrow$ *semiconnex-irreflexive* B $(\sqsubset)$
⟨*proof*⟩

end

lemmas *semiconnex-irreflexiveI* = *semiconnex-irreflexive.intro*

lemma *semiconnex-irreflexive-cong*:
assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *semiconnex-irreflexive* A $r \longleftrightarrow$ *semiconnex-irreflexive* A r'
⟨*proof*⟩

locale *connex* = *related-set* +
assumes *comparable*: $x \in A \Longrightarrow y \in A \Longrightarrow x \sqsubseteq y \vee y \sqsubseteq x$
begin

interpretation *less-eq-asymmetrize*⟨*proof*⟩

sublocale *reflexive* ⟨*proof*⟩

lemma *comparable-cases*[*consumes 2*, *case-names le ge*]:
  assumes $x \in A$ **and** $y \in A$ **and** $x \sqsubseteq y \Longrightarrow P$ **and** $y \sqsubseteq x \Longrightarrow P$ **shows** P
  ⟨*proof*⟩

lemma *comparable-three-cases*[*consumes 2*, *case-names less eq greater*]:
  assumes $x \in A$ **and** $y \in A$ **and** $x \sqsubset y \Longrightarrow P$ **and** $x \sim y \Longrightarrow P$ **and** $y \sqsubset x \Longrightarrow P$ **shows** P
  ⟨*proof*⟩

lemma
  assumes x: $x \in A$ **and** y: $y \in A$
  shows *not-iff-asym*: $\neg x \sqsubseteq y \longleftrightarrow y \sqsubset x$
    and *not-asym-iff*: $\neg x \sqsubset y \longleftrightarrow y \sqsubseteq x$
  ⟨*proof*⟩

lemma *connex-subset*: $B \subseteq A \Longrightarrow$ *connex* B $(\sqsubseteq)$
  ⟨*proof*⟩

interpretation *less-eq-asymmetrize*⟨*proof*⟩

end

lemmas *connexI*[*intro*] = *connex.intro*

lemmas *connexE* = *connex.comparable-cases*

lemma *connex-empty*: *connex* {} A ⟨*proof*⟩

context
  fixes *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
begin

lemma *connex-iff-semiconnex-reflexive*: *connex* A $(\sqsubseteq)$ $\longleftrightarrow$ *semiconnex* A $(\sqsubseteq)$ $\wedge$ *reflexive* A $(\sqsubseteq)$
  (**is** *?c* $\longleftrightarrow$ *?t* $\wedge$ *?r*)
⟨*proof*⟩

lemma *chain-connect*: *Complete-Partial-Order.chain* r A $\equiv$ *connex* A r
  ⟨*proof*⟩

lemma *connex-union*:
  assumes *connex* X $(\sqsubseteq)$ **and** *connex* Y $(\sqsubseteq)$ **and** $\forall x \in X.\ \forall y \in Y.\ x \sqsubseteq y \vee y \sqsubseteq x$
  shows *connex* $(X \cup Y)$ $(\sqsubseteq)$
  ⟨*proof*⟩

end

lemma *connex-cong*:
  assumes *r*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  shows $connex\ A\ r \longleftrightarrow connex\ A\ r'$
  ⟨*proof*⟩

locale *total-pseudo-ordered-set* = *connex* + *antisymmetric*
begin

sublocale *pseudo-ordered-set* ⟨*proof*⟩

lemma *not-weak-iff*:
  assumes *x*: $x \in A$ **and** *y*: $y \in A$ **shows** $\neg\ y \sqsubseteq x \longleftrightarrow x \sqsubseteq y \wedge x \neq y$
⟨*proof*⟩

lemma *total-pseudo-ordered-subset*: $B \subseteq A \Longrightarrow$ *total-pseudo-ordered-set* B $(\sqsubseteq)$
  ⟨*proof*⟩

interpretation *less-eq-asymmetrize*⟨*proof*⟩

interpretation *asympartp*: *semiconnex-irreflexive* A $(\sqsubset)$
⟨*proof*⟩

lemmas *asympartp-semiconnex* = *asympartp.semiconnex-axioms*
lemmas *asympartp-semiconnex-irreflexive* = *asympartp.semiconnex-irreflexive-axioms*

end

lemmas *total-pseudo-ordered-setI* = *total-pseudo-ordered-set.intro*

lemma *total-pseudo-ordered-set-cong*:
  assumes *r*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
  shows *total-pseudo-ordered-set* $A\ r \longleftrightarrow$ *total-pseudo-ordered-set* $A\ r'$
  ⟨*proof*⟩

locale *total-quasi-ordered-set* = *connex* + *transitive*
begin

sublocale *quasi-ordered-set* ⟨*proof*⟩

lemma *total-quasi-ordered-subset*: $B \subseteq A \Longrightarrow$ *total-quasi-ordered-set* B $(\sqsubseteq)$
  ⟨*proof*⟩

end

lemmas *total-quasi-ordered-setI* = *total-quasi-ordered-set.intro*

lemma *total-quasi-ordered-set-cong*:

assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *total-quasi-ordered-set* $A\ r \longleftrightarrow$ *total-quasi-ordered-set* $A\ r'$
⟨*proof*⟩

locale *total-ordered-set* = *total-quasi-ordered-set* + *antisymmetric*
begin

sublocale *partially-ordered-set* + *total-pseudo-ordered-set* ⟨*proof*⟩

lemma *total-ordered-subset*: $B \subseteq A \Longrightarrow$ *total-ordered-set* B $(\sqsubseteq)$
⟨*proof*⟩

lemma *weak-semiconnex*: *semiconnex* A $(\sqsubseteq)$
⟨*proof*⟩

interpretation *less-eq-asymmetrize*⟨*proof*⟩

end

lemmas *total-ordered-setI* = *total-ordered-set.intro*[*OF total-quasi-ordered-setI*]

lemma *total-ordered-set-cong*:
assumes r: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
shows *total-ordered-set* $A\ r \longleftrightarrow$ *total-ordered-set* $A\ r'$
⟨*proof*⟩

lemma *monotone-connex-image*:
fixes *ir* (**infix** $\preceq$ *50*) **and** r (**infix** $\sqsubseteq$ *50*)
assumes *mono*: *monotone-on* I $(\preceq)$ $(\sqsubseteq)$ f **and** *connex*: *connex* I $(\preceq)$
shows *connex* $(f\ `\ I)$ $(\sqsubseteq)$
⟨*proof*⟩

2.5 Order Pairs

We pair a relation (weak part) with a well-behaving "strict" part. Here no assumption is put on the "weak" part.

locale *compatible-ordering* =
related-set + *irreflexive* +
assumes *strict-implies-weak*: $x \sqsubset y \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow x \sqsubseteq y$
assumes *weak-strict-trans*[*trans*]: $x \sqsubseteq y \Longrightarrow y \sqsubset z \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x \sqsubset z$
assumes *strict-weak-trans*[*trans*]: $x \sqsubset y \Longrightarrow y \sqsubseteq z \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x \sqsubset z$
begin

The following sequence of declarations are in order to obtain fact names in a manner similar to the Isabelle/HOL facts of orders.

The strict part is necessarily transitive.

sublocale *strict*: *transitive* A $(\sqsubset)$
  ⟨*proof*⟩

sublocale *strict-ordered-set* A $(\sqsubset)$ ⟨*proof*⟩

thm *strict.trans asym irrefl*

lemma *Restrp-compatible-ordering*: *compatible-ordering UNIV* $((\sqsubseteq)\restriction A)$ $((\sqsubset)\restriction A)$
  ⟨*proof*⟩

lemma *strict-implies-not-weak*: $x \sqsubset y \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow \neg\ y \sqsubseteq x$
  ⟨*proof*⟩

lemma *weak-implies-not-strict*:
  assumes *xy*: $x \sqsubseteq y$ **and** [*simp*]: $x \in A$ $y \in A$
  shows $\neg y \sqsubset x$
⟨*proof*⟩

lemma *compatible-ordering-subset*: **assumes** $X \subseteq A$ **shows** *compatible-ordering* X $(\sqsubseteq)$ $(\sqsubset)$
  ⟨*proof*⟩

end

context *transitive* **begin**

interpretation *less-eq-asymmetrize*⟨*proof*⟩

lemma *asym-trans*[*trans*]:
  shows $x \sqsubset y \Longrightarrow y \sqsubseteq z \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x \sqsubset z$
    and $x \sqsubseteq y \Longrightarrow y \sqsubset z \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x \sqsubset z$
  ⟨*proof*⟩

lemma *asympartp-compatible-ordering*: *compatible-ordering* A $(\sqsubseteq)$ $(\sqsubset)$
  ⟨*proof*⟩

end

locale *reflexive-ordering* = *reflexive* + *compatible-ordering*

locale *reflexive-attractive-ordering* = *reflexive-ordering* + *attractive*

locale *pseudo-ordering* = *pseudo-ordered-set* + *compatible-ordering*
begin

sublocale *reflexive-attractive-ordering*⟨*proof*⟩

end

locale *quasi-ordering* = *quasi-ordered-set* + *compatible-ordering*
begin

sublocale *reflexive-attractive-ordering*⟨*proof*⟩

lemma *quasi-ordering-subset*: **assumes** $X \subseteq A$ **shows** *quasi-ordering* X ($\sqsubseteq$) ($\sqsubset$)
⟨*proof*⟩

end

context *quasi-ordered-set* **begin**

interpretation *less-eq-asymmetrize*⟨*proof*⟩

lemma *asympartp-quasi-ordering*: *quasi-ordering* A ($\sqsubseteq$) ($\sqsubset$)
⟨*proof*⟩

end

locale *partial-ordering* = *partially-ordered-set* + *compatible-ordering*
begin

sublocale *quasi-ordering* + *pseudo-ordering*⟨*proof*⟩

lemma *partial-ordering-subset*: **assumes** $X \subseteq A$ **shows** *partial-ordering* X ($\sqsubseteq$) ($\sqsubset$)
⟨*proof*⟩

end

context *partially-ordered-set* **begin**

interpretation *less-eq-asymmetrize*⟨*proof*⟩

lemma *asympartp-partial-ordering*: *partial-ordering* A ($\sqsubseteq$) ($\sqsubset$)
⟨*proof*⟩

end

locale *total-quasi-ordering* = *total-quasi-ordered-set* + *compatible-ordering*
begin

sublocale *quasi-ordering*⟨*proof*⟩

lemma *total-quasi-ordering-subset*: **assumes** $X \subseteq A$ **shows** *total-quasi-ordering* X ($\sqsubseteq$) ($\sqsubset$)
⟨*proof*⟩

end

context *total-quasi-ordered-set* **begin**

interpretation *less-eq-asymmetrize*⟨*proof*⟩

lemma *asympartp-total-quasi-ordering*: *total-quasi-ordering* A $(\sqsubseteq)$ $(\sqsubset)$
⟨*proof*⟩

end

Fixing the definition of the strict part is very common, though it looks restrictive to the author.

locale *strict-quasi-ordering* = *quasi-ordered-set* + *less-syntax* +
assumes *strict-iff*: $x \in A \Longrightarrow y \in A \Longrightarrow x \sqsubset y \longleftrightarrow x \sqsubseteq y \wedge \neg y \sqsubseteq x$
begin

sublocale *compatible-ordering*
⟨*proof*⟩

end

locale *strict-partial-ordering* = *strict-quasi-ordering* + *antisymmetric*
begin

sublocale *partial-ordering*⟨*proof*⟩

lemma *strict-iff-neq*: $x \in A \Longrightarrow y \in A \Longrightarrow x \sqsubset y \longleftrightarrow x \sqsubseteq y \wedge x \neq y$
⟨*proof*⟩

end

locale *total-ordering* = *reflexive* + *compatible-ordering* + *semiconnex* A $(\sqsubset)$
begin

sublocale *semiconnex-irreflexive* ⟨*proof*⟩

sublocale *connex*
⟨*proof*⟩

lemma *not-weak*:
assumes $x \in A$ **and** $y \in A$ **shows** $\neg\ x \sqsubseteq y \longleftrightarrow y \sqsubset x$
⟨*proof*⟩

lemma *not-strict*: $x \in A \Longrightarrow y \in A \Longrightarrow \neg\ x \sqsubset y \longleftrightarrow y \sqsubseteq x$
⟨*proof*⟩

sublocale *strict-partial-ordering*
⟨*proof*⟩

sublocale *total-ordered-set*⟨*proof*⟩

context
  fixes s
  assumes s: $\forall\, x \in A.\ x \sqsubset s \longrightarrow (\exists\, z \in A.\ x \sqsubset z \wedge z \sqsubset s)$ **and** sA: $s \in A$
begin

lemma *dense-weakI*:
  assumes *bound*: $\bigwedge x.\ x \sqsubset s \Longrightarrow x \in A \Longrightarrow x \sqsubseteq y$ **and** yA: $y \in A$
  shows $s \sqsubseteq y$
⟨*proof*⟩

lemma *dense-bound-iff*:
  assumes bA: $b \in A$ **shows** *bound* $\{x{\in}A.\ x \sqsubset s\}$ $(\sqsubseteq)$ $b \longleftrightarrow s \sqsubseteq b$
  ⟨*proof*⟩

lemma *dense-extreme-bound*:
  extreme-bound A $(\sqsubseteq)$ $\{x \in A.\ x \sqsubset s\}$ s
  ⟨*proof*⟩

end

lemma *ordinal-cases*[*consumes 1*, *case-names suc lim*]:
  assumes aA: $a \in A$
    and *suc*: $\bigwedge p.$ *extreme* $\{x \in A.\ x \sqsubset a\}$ $(\sqsupseteq)$ $p \Longrightarrow$ *thesis*
    and *lim*: *extreme-bound* A $(\sqsubseteq)$ $\{x \in A.\ x \sqsubset a\}$ $a \Longrightarrow$ *thesis*
  shows *thesis*
⟨*proof*⟩

end

context *total-ordered-set* **begin**

interpretation *less-eq-asymmetrize*⟨*proof*⟩

lemma *asympartp-total-ordering*: *total-ordering* A $(\sqsubseteq)$ $(\sqsubset)$
  ⟨*proof*⟩

end

2.6 Functions

definition *pointwise* $I\ r\ f\ g \equiv \forall\, i \in I.\ r\ (f\ i)\ (g\ i)$

lemmas *pointwiseI* = *pointwise-def*[*unfolded atomize-eq*, *THEN iffD2*, *rule-format*]

lemmas *pointwiseD*[*simp*] = *pointwise-def*[*unfolded atomize-eq*, *THEN iffD1*, *rule-format*]

lemma *pointwise-cong*:

assumes $r = r'$ $\bigwedge i.\ i \in I \Longrightarrow f\ i = f'\ i$ $\bigwedge i.\ i \in I \Longrightarrow g\ i = g'\ i$
shows *pointwise* $I\ r\ f\ g$ = *pointwise* $I\ r'\ f'\ g'$
⟨*proof*⟩

lemma *pointwise-empty*[*simp*]: *pointwise* {} = ⊤ ⟨*proof*⟩

lemma *dual-pointwise*[*simp*]: (*pointwise* $I\ r$)$^-$ = *pointwise* $I\ r^-$
⟨*proof*⟩

lemma *pointwise-dual*: *pointwise* $I\ r^-\ f\ g \Longrightarrow$ *pointwise* $I\ r\ g\ f$ ⟨*proof*⟩

lemma *pointwise-un*: *pointwise* $(I \cup J)\ r$ = *pointwise* $I\ r \sqcap$ *pointwise* $J\ r$
⟨*proof*⟩

lemma *pointwise-unI*[*intro*!]: *pointwise* $I\ r\ f\ g \Longrightarrow$ *pointwise* $J\ r\ f\ g \Longrightarrow$ *pointwise* $(I \cup J)\ r\ f\ g$
⟨*proof*⟩

lemma *pointwise-bound*: *bound* F (*pointwise* $I\ r$) $f \longleftrightarrow (\forall\, i \in I.$ *bound* $\{f\ i \mid .\ f \in F\}\ r\ (f\ i))$
⟨*proof*⟩

lemma *pointwise-extreme*:
shows *extreme* F (*pointwise* $X\ r$) $e \longleftrightarrow e \in F \wedge (\forall\, x \in X.$ *extreme* $\{f\ x \mid .\ f \in F\}\ r\ (e\ x))$
⟨*proof*⟩

lemma *pointwise-extreme-bound*:
fixes r (**infix** $\sqsubseteq$ *50*)
assumes F: $F \subseteq \{f.\ f\ `\ X \subseteq A\}$
shows *extreme-bound* $\{f.\ f\ `\ X \subseteq A\}$ (*pointwise* X $(\sqsubseteq)$) $F\ s \longleftrightarrow$
$(\forall\, x \in X.$ *extreme-bound* A $(\sqsubseteq)$ $\{f\ x \mid .\ f \in F\}\ (s\ x))$ (**is** *?p* $\longleftrightarrow$ *?a*)
⟨*proof*⟩

lemma *dual-pointwise-extreme-bound*:
extreme-bound FA (*pointwise* $X\ r$)$^-$ F = *extreme-bound* FA (*pointwise* $X\ r^-$) F
⟨*proof*⟩

lemma *pointwise-monotone-on*:
fixes *less-eq* (**infix** $\sqsubseteq$ *50*) **and** *prec-eq* (**infix** $\preceq$ *50*)
shows *monotone-on* I $(\preceq)$ (*pointwise* A $(\sqsubseteq)$) $f \longleftrightarrow$
$(\forall\, a \in A.$ *monotone-on* I $(\preceq)$ $(\sqsubseteq)$ $(\lambda i.\ f\ i\ a))$ (**is** *?l* $\longleftrightarrow$ *?r*)
⟨*proof*⟩

lemmas *pointwise-monotone* = *pointwise-monotone-on*[*of UNIV*]

lemma (**in** *reflexive*) *pointwise-reflexive*: *reflexive* $\{f.\ f\ `\ I \subseteq A\}$ (*pointwise* I $(\sqsubseteq)$)
⟨*proof*⟩

lemma (**in** *irreflexive*) *pointwise-irreflexive*:
assumes *I0*: $I \neq \{\}$ **shows** *irreflexive* $\{f.\ f \text{ ` } I \subseteq A\}$ (*pointwise* I $(\sqsubset)$)
⟨*proof*⟩

lemma (**in** *semiattractive*) *pointwise-semiattractive*: *semiattractive* $\{f.\ f \text{ ` } I \subseteq A\}$ (*pointwise* I $(\sqsubseteq)$)
⟨*proof*⟩

lemma (**in** *attractive*) *pointwise-attractive*: *attractive* $\{f.\ f \text{ ` } I \subseteq A\}$ (*pointwise* I $(\sqsubseteq)$)
⟨*proof*⟩

Antisymmetry will not be preserved by pointwise extension over restricted domain.

lemma (**in** *antisymmetric*) *pointwise-antisymmetric*:
antisymmetric $\{f.\ f \text{ ` } I \subseteq A\}$ (*pointwise* I $(\sqsubseteq)$)
⟨*proof*⟩

lemma (**in** *transitive*) *pointwise-transitive*: *transitive* $\{f.\ f \text{ ` } I \subseteq A\}$ (*pointwise* I $(\sqsubseteq)$)
⟨*proof*⟩

lemma (**in** *quasi-ordered-set*) *pointwise-quasi-order*:
quasi-ordered-set $\{f.\ f \text{ ` } I \subseteq A\}$ (*pointwise* I $(\sqsubseteq)$)
⟨*proof*⟩

lemma (**in** *compatible-ordering*) *pointwise-compatible-ordering*:
assumes *I0*: $I \neq \{\}$
shows *compatible-ordering* $\{f.\ f \text{ ` } I \subseteq A\}$ (*pointwise* I $(\sqsubseteq)$) (*pointwise* I $(\sqsubset)$)
⟨*proof*⟩

2.7 Relating to Classes

In Isabelle 2020, we should declare sublocales in class before declaring dual sublocales, since otherwise facts would be prefixed by "dual.dual."

context *ord* **begin**

abbreviation *least* **where** *least* $X \equiv$ *extreme* X $(\lambda x\ y.\ y \leq x)$

abbreviation *greatest* **where** *greatest* $X \equiv$ *extreme* X $(\leq)$

abbreviation *supremum* **where** *supremum* $X \equiv$ *least* (*Collect* (*bound* X $(\leq)$))

abbreviation *infimum* **where** *infimum* $X \equiv$ *greatest* (*Collect* (*bound* X $(\lambda x\ y.\ y \leq x)$))

lemma *supremumI*: *bound* X $(\leq)$ $s \Longrightarrow (\bigwedge b.\ \textit{bound}\ X\ (\leq)\ b \Longrightarrow s \leq b) \Longrightarrow$ *supremum* X s

and *infimumI*: *bound* X $(\geq)$ $i \Longrightarrow (\bigwedge b.$ *bound* X $(\geq)$ $b \Longrightarrow b \leq i) \Longrightarrow$ *infimum* X i
⟨*proof*⟩

lemma *supremumE*: *supremum* X $s \Longrightarrow$
(*bound* X $(\leq)$ $s \Longrightarrow (\bigwedge b.$ *bound* X $(\leq)$ $b \Longrightarrow s \leq b) \Longrightarrow$ *thesis*) $\Longrightarrow$ *thesis*
and *infimumE*: *infimum* X $i \Longrightarrow$
(*bound* X $(\geq)$ $i \Longrightarrow (\bigwedge b.$ *bound* X $(\geq)$ $b \Longrightarrow b \leq i) \Longrightarrow$ *thesis*) $\Longrightarrow$ *thesis*
⟨*proof*⟩

lemma *extreme-bound-supremum*[*simp*]: *extreme-bound UNIV* $(\leq) =$ *supremum*
⟨*proof*⟩
lemma *extreme-bound-infimum*[*simp*]: *extreme-bound UNIV* $(\geq) =$ *infimum* ⟨*proof*⟩

lemma *Least-eq-The-least*: *Least* $P =$ *The* (*least* $\{x.\ P\ x\}$)
⟨*proof*⟩

lemma *Greatest-eq-The-greatest*: *Greatest* $P =$ *The* (*greatest* $\{x.\ P\ x\}$)
⟨*proof*⟩

end

lemma *Ball-UNIV*[*simp*]: *Ball UNIV* = *All* ⟨*proof*⟩
lemma *Bex-UNIV*[*simp*]: *Bex UNIV* = *Ex* ⟨*proof*⟩

lemma *pointwise-UNIV-le*[*simp*]: *pointwise UNIV* $(\leq) = (\leq)$ ⟨*proof*⟩
lemma *pointwise-UNIV-ge*[*simp*]: *pointwise UNIV* $(\geq) = (\geq)$ ⟨*proof*⟩

lemma *fun-supremum-iff*: *supremum* F $e \longleftrightarrow (\forall x.$ *supremum* $\{f\ x\ |.\ f \in F\}$ $(e\ x))$
⟨*proof*⟩

lemma *fun-infimum-iff*: *infimum* F $e \longleftrightarrow (\forall x.$ *infimum* $\{f\ x\ |.\ f \in F\}$ $(e\ x))$
⟨*proof*⟩

class *reflorder* = *ord* + **assumes** *reflexive-ordering UNIV* $(\leq)$ $(<)$
begin

sublocale *order*: *reflexive-ordering UNIV*
rewrites $\bigwedge x.\ x \in$ *UNIV* $\equiv$ *True*
and $\bigwedge X.\ X \subseteq$ *UNIV* $\equiv$ *True*
and $\bigwedge r.\ r \upharpoonright$ *UNIV* $\equiv r$
and $\bigwedge P.$ *True* $\wedge P \equiv P$
and *Ball UNIV* $\equiv$ *All*
and *Bex UNIV* $\equiv$ *Ex*
and *sympartp* $(\leq)^{-} \equiv$ *sympartp* $(\leq)$
and $\bigwedge P1.$ (*True* $\Longrightarrow$ *PROP P1*) $\equiv$ *PROP P1*
and $\bigwedge P1.$ (*True* $\Longrightarrow$ *P1*) $\equiv$ *Trueprop P1*

and $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
$\langle proof \rangle$

end

We should have imported locale-based facts in classes, e.g.:

thm *order.trans order.strict.trans order.refl order.irrefl order.asym order.extreme-bound-singleton*

class *attrorder* = *ord* +
assumes *reflexive-attractive-ordering UNIV* $(\leq)$ $(<)$
begin

We need to declare subclasses before sublocales in order to preserve facts for superclasses.

subclass *reflorder*
$\langle proof \rangle$

sublocale *order*: *reflexive-attractive-ordering UNIV*
rewrites $\bigwedge x.\ x \in UNIV \equiv True$
and $\bigwedge X.\ X \subseteq UNIV \equiv True$
and $\bigwedge r.\ r \restriction UNIV \equiv r$
and $\bigwedge P.\ True \wedge P \equiv P$
and $Ball\ UNIV \equiv All$
and $Bex\ UNIV \equiv Ex$
and $sympartp\ (\leq)^{-} \equiv sympartp\ (\leq)$
and $\bigwedge P1.\ (True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
and $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
and $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
$\langle proof \rangle$

end

thm *order.extreme-bound-quasi-const*

class *psorder* = *ord* + **assumes** *pseudo-ordering UNIV* $(\leq)$ $(<)$
begin

subclass *attrorder*
$\langle proof \rangle$

sublocale *order*: *pseudo-ordering UNIV*
rewrites $\bigwedge x.\ x \in UNIV \equiv True$
and $\bigwedge X.\ X \subseteq UNIV \equiv True$
and $\bigwedge r.\ r \restriction UNIV \equiv r$
and $\bigwedge P.\ True \wedge P \equiv P$
and $Ball\ UNIV \equiv All$
and $Bex\ UNIV \equiv Ex$

and *sympartp* $(\leq)^{-} \equiv$ *sympartp* $(\leq)$
and $\bigwedge P1.$ $(True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
and $\bigwedge P1.$ $(True \Longrightarrow P1) \equiv Trueprop\ P1$
and $\bigwedge P1\ P2.$ $(True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
⟨*proof*⟩

end

class *qorder* = *ord* + **assumes** *quasi-ordering UNIV* $(\leq)$ $(<)$
begin

subclass *attrorder*
⟨*proof*⟩

sublocale *order*: *quasi-ordering UNIV*
rewrites $\bigwedge x.\ x \in UNIV \equiv True$
and $\bigwedge X.\ X \subseteq UNIV \equiv True$
and $\bigwedge r.\ r \upharpoonright UNIV \equiv r$
and $\bigwedge P.\ True \wedge P \equiv P$
and *Ball UNIV* $\equiv$ *All*
and *Bex UNIV* $\equiv$ *Ex*
and *sympartp* $(\leq)^{-} \equiv$ *sympartp* $(\leq)$
and $\bigwedge P1.$ $(True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
and $\bigwedge P1.$ $(True \Longrightarrow P1) \equiv Trueprop\ P1$
and $\bigwedge P1\ P2.$ $(True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
⟨*proof*⟩

lemmas [*intro*!] = *order.quasi-ordered-subset*

end

class *porder* = *ord* + **assumes** *partial-ordering UNIV* $(\leq)$ $(<)$
begin

interpretation *partial-ordering UNIV*
⟨*proof*⟩

subclass *psorder*⟨*proof*⟩

subclass *qorder*⟨*proof*⟩

sublocale *order*: *partial-ordering UNIV*
rewrites $\bigwedge x.\ x \in UNIV \equiv True$
and $\bigwedge X.\ X \subseteq UNIV \equiv True$
and $\bigwedge r.\ r \upharpoonright UNIV \equiv r$
and $\bigwedge P.\ True \wedge P \equiv P$
and *Ball UNIV* $\equiv$ *All*

and *Bex UNIV* ≡ *Ex*
and *sympartp* $(\leq)^{-}$ ≡ *sympartp* $(\leq)$
and $\bigwedge$*P1*. (*True* ⟹ *PROP P1*) ≡ *PROP P1*
and $\bigwedge$*P1*. (*True* ⟹ *P1*) ≡ *Trueprop P1*
and $\bigwedge$*P1 P2*. (*True* ⟹ *PROP P1* ⟹ *PROP P2*) ≡ (*PROP P1* ⟹ *PROP P2*)
⟨*proof*⟩

end

class *linqorder* = *ord* + **assumes** *total-quasi-ordering UNIV* $(\leq)$ $(<)$
begin

interpretation *total-quasi-ordering UNIV*
⟨*proof*⟩

subclass *qorder*⟨*proof*⟩

sublocale *order*: *total-quasi-ordering UNIV*
rewrites $\bigwedge$*x*. $x \in$ *UNIV* ≡ *True*
and $\bigwedge$*X*. $X \subseteq$ *UNIV* ≡ *True*
and $\bigwedge$*r*. $r \restriction$ *UNIV* ≡ *r*
and $\bigwedge$*P*. *True* ∧ *P* ≡ *P*
and *Ball UNIV* ≡ *All*
and *Bex UNIV* ≡ *Ex*
and *sympartp* $(\leq)^{-}$ ≡ *sympartp* $(\leq)$
and $\bigwedge$*P1*. (*True* ⟹ *PROP P1*) ≡ *PROP P1*
and $\bigwedge$*P1*. (*True* ⟹ *P1*) ≡ *Trueprop P1*
and $\bigwedge$*P1 P2*. (*True* ⟹ *PROP P1* ⟹ *PROP P2*) ≡ (*PROP P1* ⟹ *PROP P2*)
⟨*proof*⟩

lemmas *asympartp-le* = *order.not-iff-asym*[*symmetric*, *abs-def*]

end

Isabelle/HOL's *preorder* belongs to *qorder*, but not vice versa.

context *preorder* **begin**

The relation $(<)$ is defined as the antisymmetric part of $(\leq)$.

lemma [*simp*]:
shows *asympartp-le*: *asympartp* $(\leq) = (<)$
and *asympartp-ge*: *asympartp* $(\geq) = (>)$
⟨*proof*⟩

interpretation *strict-quasi-ordering UNIV* $(\leq)$ $(<)$
⟨*proof*⟩

subclass *qorder*⟨*proof*⟩

sublocale *order*: *strict-quasi-ordering UNIV*
rewrites $\bigwedge x.\ x \in \mathit{UNIV} \equiv \mathit{True}$
and $\bigwedge X.\ X \subseteq \mathit{UNIV} \equiv \mathit{True}$
and $\bigwedge r.\ r \upharpoonright \mathit{UNIV} \equiv r$
and $\bigwedge P.\ \mathit{True} \wedge P \equiv P$
and $\mathit{Ball}\ \mathit{UNIV} \equiv \mathit{All}$
and $\mathit{Bex}\ \mathit{UNIV} \equiv \mathit{Ex}$
and $\mathit{sympartp}\ (\leq)^{-} \equiv \mathit{sympartp}\ (\leq)$
and $\bigwedge P1.\ (\mathit{True} \Longrightarrow \mathit{PROP}\ P1) \equiv \mathit{PROP}\ P1$
and $\bigwedge P1.\ (\mathit{True} \Longrightarrow P1) \equiv \mathit{Trueprop}\ P1$
and $\bigwedge P1\ P2.\ (\mathit{True} \Longrightarrow \mathit{PROP}\ P1 \Longrightarrow \mathit{PROP}\ P2) \equiv (\mathit{PROP}\ P1 \Longrightarrow \mathit{PROP}$
$P2)$
⟨*proof*⟩

end

context *order* **begin**

interpretation *strict-partial-ordering UNIV* $(\leq)$ $(<)$
⟨*proof*⟩

subclass *porder*⟨*proof*⟩

sublocale *order*: *strict-partial-ordering UNIV*
rewrites $\bigwedge x.\ x \in \mathit{UNIV} \equiv \mathit{True}$
and $\bigwedge X.\ X \subseteq \mathit{UNIV} \equiv \mathit{True}$
and $\bigwedge r.\ r \upharpoonright \mathit{UNIV} \equiv r$
and $\bigwedge P.\ \mathit{True} \wedge P \equiv P$
and $\mathit{Ball}\ \mathit{UNIV} \equiv \mathit{All}$
and $\mathit{Bex}\ \mathit{UNIV} \equiv \mathit{Ex}$
and $\mathit{sympartp}\ (\leq)^{-} \equiv \mathit{sympartp}\ (\leq)$
and $\bigwedge P1.\ (\mathit{True} \Longrightarrow \mathit{PROP}\ P1) \equiv \mathit{PROP}\ P1$
and $\bigwedge P1.\ (\mathit{True} \Longrightarrow P1) \equiv \mathit{Trueprop}\ P1$
and $\bigwedge P1\ P2.\ (\mathit{True} \Longrightarrow \mathit{PROP}\ P1 \Longrightarrow \mathit{PROP}\ P2) \equiv (\mathit{PROP}\ P1 \Longrightarrow \mathit{PROP}$
$P2)$
⟨*proof*⟩

end

Isabelle/HOL's *linorder* is equivalent to our locale *total-ordering*.

context *linorder* **begin**

subclass *linqorder* ⟨*proof*⟩

sublocale *order*: *total-ordering UNIV*
rewrites $\bigwedge x.\ x \in \mathit{UNIV} \equiv \mathit{True}$
and $\bigwedge X.\ X \subseteq \mathit{UNIV} \equiv \mathit{True}$
and $\bigwedge r.\ r \upharpoonright \mathit{UNIV} \equiv r$

and $\bigwedge P.\ True \wedge P \equiv P$
and $Ball\ UNIV \equiv All$
and $Bex\ UNIV \equiv Ex$
and $sympartp\ (\leq)^{-} \equiv sympartp\ (\leq)$
and $\bigwedge P1.\ (True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
and $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
and $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
$\langle proof \rangle$

end

Tests: facts should be available in the most general classes.

thm *order.strict.trans*[**where** *'a='a::reflorder*]
thm *order.extreme-bound-quasi-const*[**where** *'a='a::attrorder*]
thm *order.extreme-bound-singleton-eq*[**where** *'a='a::psorder*]
thm *order.trans*[**where** *'a='a::qorder*]
thm *order.comparable-cases*[**where** *'a='a::linqorder*]
thm *order.cases*[**where** *'a='a::linorder*]

2.8 Declaring Duals

sublocale *reflexive* $\subseteq$ *sym*: *reflexive A sympartp* $(\sqsubseteq)$
rewrites $sympartp\ (\sqsubseteq)^{-} \equiv sympartp\ (\sqsubseteq)$
and $\bigwedge r.\ sympartp\ (sympartp\ r) \equiv sympartp\ r$
and $\bigwedge r.\ sympartp\ r \upharpoonright A \equiv sympartp\ (r \upharpoonright A)$
$\langle proof \rangle$

sublocale *quasi-ordered-set* $\subseteq$ *sym*: *quasi-ordered-set A sympartp* $(\sqsubseteq)$
rewrites $sympartp\ (\sqsubseteq)^{-} = sympartp\ (\sqsubseteq)$
and $sympartp\ (sympartp\ (\sqsubseteq)) = sympartp\ (\sqsubseteq)$
$\langle proof \rangle$

At this point, we declare dual as sublocales. In the following, “rewrites” eventually cleans up redundant facts.

sublocale *reflexive* $\subseteq$ *dual*: *reflexive A* $(\sqsubseteq)^{-}$
rewrites $sympartp\ (\sqsubseteq)^{-} \equiv sympartp\ (\sqsubseteq)$
and $\bigwedge r.\ sympartp\ (r \upharpoonright A) \equiv sympartp\ r \upharpoonright A$
and $(\sqsubseteq)^{-} \upharpoonright A \equiv ((\sqsubseteq) \upharpoonright A)^{-}$
$\langle proof \rangle$

context *attractive* **begin**

interpretation *less-eq-symmetrize*$\langle proof \rangle$

sublocale *dual*: *attractive A* $(\sqsupseteq)$
rewrites $sympartp\ (\sqsupseteq) = (\sim)$
and $equivpartp\ (\sqsupseteq) \equiv (\simeq)$
and $\bigwedge r.\ sympartp\ (r \upharpoonright A) \equiv sympartp\ r \upharpoonright A$

and $\bigwedge r.\ \mathit{sympartp}\ (\mathit{sympartp}\ r) \equiv \mathit{sympartp}\ r$
and $(\sqsubseteq)^{-} \upharpoonright A \equiv ((\sqsubseteq) \upharpoonright A)^{-}$
$\langle \mathit{proof} \rangle$

end

context *irreflexive* **begin**

sublocale *dual*: *irreflexive* $A\ (\sqsubset)^{-}$
rewrites $(\sqsubset)^{-} \upharpoonright A \equiv ((\sqsubset) \upharpoonright A)^{-}$
$\langle \mathit{proof} \rangle$

end

sublocale *transitive* $\subseteq$ *dual*: *transitive* $A\ (\sqsubseteq)^{-}$
rewrites $(\sqsubseteq)^{-} \upharpoonright A \equiv ((\sqsubseteq) \upharpoonright A)^{-}$
and $\mathit{sympartp}\ (\sqsubseteq)^{-} = \mathit{sympartp}\ (\sqsubseteq)$
and $\mathit{asympartp}\ (\sqsubseteq)^{-} = (\mathit{asympartp}\ (\sqsubseteq))^{-}$
$\langle \mathit{proof} \rangle$

sublocale *antisymmetric* $\subseteq$ *dual*: *antisymmetric* $A\ (\sqsubseteq)^{-}$
rewrites $(\sqsubseteq)^{-} \upharpoonright A \equiv ((\sqsubseteq) \upharpoonright A)^{-}$
and $\mathit{sympartp}\ (\sqsubseteq)^{-} = \mathit{sympartp}\ (\sqsubseteq)$
$\langle \mathit{proof} \rangle$

context *antisymmetric* **begin**

lemma *extreme-bound-unique*:
$\mathit{extreme\text{-}bound}\ A\ (\sqsubseteq)\ X\ x \Longrightarrow \mathit{extreme\text{-}bound}\ A\ (\sqsubseteq)\ X\ y \longleftrightarrow x = y$
$\langle \mathit{proof} \rangle$

lemma *ex-extreme-bound-iff-ex1*:
$\mathit{Ex}\ (\mathit{extreme\text{-}bound}\ A\ (\sqsubseteq)\ X) \longleftrightarrow \mathit{Ex1}\ (\mathit{extreme\text{-}bound}\ A\ (\sqsubseteq)\ X)$
$\langle \mathit{proof} \rangle$

lemma *ex-extreme-bound-iff-the*:
$\mathit{Ex}\ (\mathit{extreme\text{-}bound}\ A\ (\sqsubseteq)\ X) \longleftrightarrow \mathit{extreme\text{-}bound}\ A\ (\sqsubseteq)\ X\ (\mathit{The}\ (\mathit{extreme\text{-}bound}\ A\ (\sqsubseteq)\ X))$
$\langle \mathit{proof} \rangle$

end

sublocale *semiconnex* $\subseteq$ *dual*: *semiconnex* $A\ (\sqsubset)^{-}$
rewrites $\mathit{sympartp}\ (\sqsubset)^{-} = \mathit{sympartp}\ (\sqsubset)$
$\langle \mathit{proof} \rangle$

sublocale *connex* $\subseteq$ *dual*: *connex* $A\ (\sqsubseteq)^{-}$
rewrites $\mathit{sympartp}\ (\sqsubseteq)^{-} = \mathit{sympartp}\ (\sqsubseteq)$
$\langle \mathit{proof} \rangle$

sublocale *semiconnex-irreflexive* $\subseteq$ *dual*: *semiconnex-irreflexive* A $(\sqsubset)^-$
  rewrites *sympartp* $(\sqsubset)^- =$ *sympartp* $(\sqsubset)$
  $\langle$*proof*$\rangle$

sublocale *pseudo-ordered-set* $\subseteq$ *dual*: *pseudo-ordered-set* A $(\sqsubseteq)^-$
  rewrites *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
  $\langle$*proof*$\rangle$

sublocale *quasi-ordered-set* $\subseteq$ *dual*: *quasi-ordered-set* A $(\sqsubseteq)^-$
  rewrites *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
  $\langle$*proof*$\rangle$

sublocale *partially-ordered-set* $\subseteq$ *dual*: *partially-ordered-set* A $(\sqsubseteq)^-$
  rewrites *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
  $\langle$*proof*$\rangle$

sublocale *total-pseudo-ordered-set* $\subseteq$ *dual*: *total-pseudo-ordered-set* A $(\sqsubseteq)^-$
  rewrites *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
  $\langle$*proof*$\rangle$

sublocale *total-quasi-ordered-set* $\subseteq$ *dual*: *total-quasi-ordered-set* A $(\sqsubseteq)^-$
  rewrites *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
  $\langle$*proof*$\rangle$

sublocale *compatible-ordering* $\subseteq$ *dual*: *compatible-ordering* A $(\sqsubseteq)^-$ $(\sqsubset)^-$
  rewrites *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
  $\langle$*proof*$\rangle$

lemmas(**in** *qorder*) [*intro*!] = *order.dual.quasi-ordered-subset*

sublocale *reflexive-ordering* $\subseteq$ *dual*: *reflexive-ordering* A $(\sqsubseteq)^-$ $(\sqsubset)^-$
  rewrites *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
  $\langle$*proof*$\rangle$

sublocale *reflexive-attractive-ordering* $\subseteq$ *dual*: *reflexive-attractive-ordering* A $(\sqsubseteq)^-$
$(\sqsubset)^-$
  rewrites *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
  $\langle$*proof*$\rangle$

sublocale *pseudo-ordering* $\subseteq$ *dual*: *pseudo-ordering* A $(\sqsubseteq)^-$ $(\sqsubset)^-$
  rewrites *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
  $\langle$*proof*$\rangle$

sublocale *quasi-ordering* $\subseteq$ *dual*: *quasi-ordering* A $(\sqsubseteq)^-$ $(\sqsubset)^-$
  rewrites *sympartp* $(\sqsubseteq)^- =$ *sympartp* $(\sqsubseteq)$
  $\langle$*proof*$\rangle$

sublocale *partial-ordering* $\subseteq$ *dual*: *partial-ordering* A $(\sqsubseteq)^-$ $(\sqsubset)^-$

rewrites *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

sublocale *total-quasi-ordering* $\subseteq$ *dual*: *total-quasi-ordering* A $(\sqsubseteq)^{-}$ $(\sqsubset)^{-}$
rewrites *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

sublocale *total-ordering* $\subseteq$ *dual*: *total-ordering* A $(\sqsubseteq)^{-}$ $(\sqsubset)^{-}$
rewrites *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

sublocale *strict-quasi-ordering* $\subseteq$ *dual*: *strict-quasi-ordering* A $(\sqsubseteq)^{-}$ $(\sqsubset)^{-}$
rewrites *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

sublocale *strict-partial-ordering* $\subseteq$ *dual*: *strict-partial-ordering* A $(\sqsubseteq)^{-}$ $(\sqsubset)^{-}$
rewrites *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

sublocale *total-ordering* $\subseteq$ *dual*: *total-ordering* A $(\sqsubseteq)^{-}$ $(\sqsubset)^{-}$
rewrites *sympartp* $(\sqsubseteq)^{-}$ = *sympartp* $(\sqsubseteq)$
⟨*proof*⟩

lemma(**in** *antisymmetric*) *monotone-extreme-imp-extreme-bound-iff*:
fixes *ir* (**infix** $\preceq$ *50*)
assumes f ‘ $C \subseteq A$ **and** *monotone-on* C $(\preceq)$ $(\sqsubseteq)$ f **and** i: *extreme* C $(\preceq)$ i
shows *extreme-bound* A $(\sqsubseteq)$ $(f$ ‘ $C)$ $x \longleftrightarrow f\ i = x$
⟨*proof*⟩

2.9 Instantiations

Finally, we instantiate our classes for sanity check.

instance *nat* :: *linorder* ⟨*proof*⟩

Pointwise ordering of functions are compatible only if the weak part is transitive.

instance *fun* :: (*type*,*qorder*) *reflorder*
⟨*proof*⟩

instance *fun* :: (*type*,*qorder*) *qorder*
⟨*proof*⟩

instance *fun* :: (*type*,*porder*) *porder*
⟨*proof*⟩

end
theory *Well-Relations*
imports *Binary-Relations*
begin

3 Well-Relations

A related set $\langle A, \sqsubseteq\rangle$ is called *topped* if there is a "top" element $\top \in A$, a greatest element in A. Note that there might be multiple tops if $(\sqsubseteq)$ is not antisymmetric.

definition *extremed* $A\ r \equiv \exists\, e.\ extreme\ A\ r\ e$

lemma *extremedI*: $extreme\ A\ r\ e \Longrightarrow extremed\ A\ r$
⟨*proof*⟩

lemma *extremedE*: $extremed\ A\ r \Longrightarrow (\bigwedge e.\ extreme\ A\ r\ e \Longrightarrow thesis) \Longrightarrow thesis$
⟨*proof*⟩

lemma *extremed-imp-ex-bound*: $extremed\ A\ r \Longrightarrow X \subseteq A \Longrightarrow \exists\, b \in A.\ bound\ X\ r\ b$
⟨*proof*⟩

locale *well-founded* = *related-set* - $(\sqsubset)$ + *less-syntax* +
assumes *induct*[*consumes 1*, *case-names less*, *induct set*]:
$a \in A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow (\bigwedge y.\ y \in A \Longrightarrow y \sqsubset x \Longrightarrow P\ y) \Longrightarrow P\ x) \Longrightarrow P\ a$
begin

sublocale *asymmetric*
⟨*proof*⟩

lemma *prefixed-Imagep-imp-empty*:
assumes *a*: $X \subseteq ((\sqsubset)\ ``` X) \cap A$ **shows** $X = \{\}$
⟨*proof*⟩

lemma *nonempty-imp-ex-extremal*:
assumes *QA*: $Q \subseteq A$ **and** *Q*: $Q \neq \{\}$
shows $\exists\, z \in Q.\ \forall\, y \in Q.\ \neg\ y \sqsubset z$
⟨*proof*⟩

interpretation *Restrp*: *well-founded UNIV* $(\sqsubset)\restriction A$
rewrites $\bigwedge x.\ x \in UNIV \equiv True$
and $(\sqsubset)\restriction A\restriction UNIV = (\sqsubset)\restriction A$
and $\bigwedge P1.\ (True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
and $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
and $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
⟨*proof*⟩

lemmas *Restrp-well-founded* = *Restrp.well-founded-axioms*
lemmas *Restrp-induct*[*consumes 0*, *case-names less*] = *Restrp.induct*

interpretation *Restrp.tranclp*: *well-founded UNIV* $((\sqsubset)\restriction A)^{++}$
rewrites $\bigwedge x.\ x \in UNIV \equiv True$
and $((\sqsubset)\restriction A)^{++} \restriction UNIV = ((\sqsubset)\restriction A)^{++}$

and $(((\sqsubset)\upharpoonright A)^{++})^{++} = ((\sqsubset)\upharpoonright A)^{++}$
and $\bigwedge P1.\ (True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
and $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
and $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
⟨*proof*⟩

lemmas *Restrp-tranclp-well-founded* = *Restrp.tranclp.well-founded-axioms*
lemmas *Restrp-tranclp-induct*[*consumes 0*, *case-names less*] = *Restrp.tranclp.induct*

end

context
fixes A :: $'a$ *set* **and** *less* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\sqsubset$ *50*)
begin

lemma *well-foundedI-pf*:
assumes *pre*: $\bigwedge X.\ X \subseteq A \Longrightarrow X \subseteq ((\sqsubset)\ ``\ X) \cap A \Longrightarrow X = \{\}$
shows *well-founded* A $(\sqsubset)$
⟨*proof*⟩

lemma *well-foundedI-extremal*:
assumes *a*: $\bigwedge X.\ X \subseteq A \Longrightarrow X \neq \{\} \Longrightarrow \exists x \in X.\ \forall y \in X.\ \neg\ y \sqsubset x$
shows *well-founded* A $(\sqsubset)$
⟨*proof*⟩

lemma *well-founded-iff-ex-extremal*:
well-founded A $(\sqsubset) \longleftrightarrow (\forall X \subseteq A.\ X \neq \{\} \longrightarrow (\exists x \in X.\ \forall z \in X.\ \neg\ z \sqsubset x))$
⟨*proof*⟩

end

lemma *well-founded-cong*:
assumes *r*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow r\ a\ b \longleftrightarrow r'\ a\ b$
and *A*: $\bigwedge a\ b.\ r'\ a\ b \Longrightarrow a \in A \longleftrightarrow a \in A'$
and *B*: $\bigwedge a\ b.\ r'\ a\ b \Longrightarrow b \in A \longleftrightarrow b \in A'$
shows *well-founded* A $r \longleftrightarrow$ *well-founded* A' r'
⟨*proof*⟩

lemma *wfP-iff-well-founded-UNIV*: *wfP* $r \longleftrightarrow$ *well-founded UNIV* r
⟨*proof*⟩

lemma *well-founded-empty*[*intro*!]: *well-founded* $\{\}$ r
⟨*proof*⟩

lemma *well-founded-singleton*:
assumes $\neg r\ x\ x$ **shows** *well-founded* $\{x\}$ r
⟨*proof*⟩

lemma *well-founded-Restrp*[*simp*]: *well-founded* A ($r\upharpoonright B$) $\longleftrightarrow$ *well-founded* ($A\cap B$) r (**is** *?l* $\longleftrightarrow$ *?r*)
⟨*proof*⟩

lemma *Restrp-tranclp-well-founded-iff*:
  fixes *less* (**infix** $\sqsubset$ *50*)
  shows *well-founded UNIV* $((\sqsubset) \upharpoonright A)^{++} \longleftrightarrow$ *well-founded* A $(\sqsubset)$ (**is** *?l* $\longleftrightarrow$ *?r*)
⟨*proof*⟩

lemma (**in** *well-founded*) *well-founded-subset*:
  assumes $B \subseteq A$ **shows** *well-founded* B $(\sqsubset)$
  ⟨*proof*⟩

lemma *well-founded-extend*:
  fixes *less* (**infix** $\sqsubset$ *50*)
  assumes A: *well-founded* A $(\sqsubset)$
  assumes B: *well-founded* B $(\sqsubset)$
  assumes AB: $\forall a \in A.\ \forall b \in B.\ \neg b \sqsubset a$
  shows *well-founded* $(A \cup B)$ $(\sqsubset)$
⟨*proof*⟩

lemma *closed-UN-well-founded*:
  fixes r (**infix** $\sqsubset$ *50*)
  assumes XX: $\forall X\in XX.$ *well-founded* X $(\sqsubset) \wedge (\forall x\in X.\ \forall y\in\bigcup XX.\ y \sqsubset x \longrightarrow y \in X)$
  shows *well-founded* $(\bigcup XX)$ $(\sqsubset)$
⟨*proof*⟩

lemma *well-founded-cmono*:
  assumes r': $r' \leq r$ **and** *wf*: *well-founded* A r
  shows *well-founded* A r'
⟨*proof*⟩

locale *well-founded-ordered-set* = *well-founded* + *transitive* - $(\sqsubset)$
begin

sublocale *strict-ordered-set*⟨*proof*⟩

interpretation *Restrp*: *strict-ordered-set UNIV* $(\sqsubset)\upharpoonright A$ + *Restrp*: *well-founded UNIV* $(\sqsubset)\upharpoonright A$
  ⟨*proof*⟩

lemma *Restrp-well-founded-order*: *well-founded-ordered-set UNIV* $((\sqsubset)\upharpoonright A)$⟨*proof*⟩

lemma *well-founded-ordered-subset*: $B \subseteq A \Longrightarrow$ *well-founded-ordered-set* B $(\sqsubset)$
  ⟨*proof*⟩

end

lemmas *well-founded-ordered-setI* = *well-founded-ordered-set.intro*

lemma *well-founded-ordered-set-empty*[*intro*!]: *well-founded-ordered-set* {} *r*
⟨*proof*⟩

locale *well-related-set* = *related-set* +
assumes *nonempty-imp-ex-extreme*: $X \subseteq A \Longrightarrow X \neq \{\} \Longrightarrow \exists\, e.\ extreme\ X\ (\sqsubseteq)^{-}\ e$
begin

sublocale *connex*
⟨*proof*⟩

lemmas *connex* = *connex-axioms*

interpretation *less-eq-asymmetrize*⟨*proof*⟩

sublocale *asym*: *well-founded* A $(\sqsubset)$
⟨*proof*⟩

lemma *well-related-subset*: $B \subseteq A \Longrightarrow$ *well-related-set* B $(\sqsubseteq)$
⟨*proof*⟩

lemma *monotone-image-well-related*:
fixes *leB* (**infix** $\trianglelefteq$ *50*)
assumes *mono*: *monotone-on* A $(\sqsubseteq)$ $(\trianglelefteq)$ f **shows** *well-related-set* $(f\ `\ A)$ $(\trianglelefteq)$
⟨*proof*⟩

end

sublocale *well-related-set* ⊆ *reflexive* ⟨*proof*⟩

lemmas *well-related-setI* = *well-related-set.intro*

lemmas *well-related-iff-ex-extreme* = *well-related-set-def*

lemma *well-related-set-empty*[*intro*!]: *well-related-set* {} *r*
⟨*proof*⟩

context
fixes *less-eq* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)
begin

lemma *well-related-iff-neg-well-founded*:
well-related-set A $(\sqsubseteq) \longleftrightarrow$ *well-founded* A $(\lambda x\ y.\ \neg\ y \sqsubseteq x)$
⟨*proof*⟩

lemma *well-related-singleton-refl*:

assumes $x \sqsubseteq x$ **shows** *well-related-set* $\{x\}$ $(\sqsubseteq)$
⟨*proof*⟩

lemma *closed-UN-well-related*:
assumes XX: $\forall X \in XX.$ *well-related-set* X $(\sqsubseteq) \wedge (\forall x \in X. \forall y \in \bigcup XX. \neg x \sqsubseteq y \longrightarrow y \in X)$
shows *well-related-set* $(\bigcup XX)$ $(\sqsubseteq)$
⟨*proof*⟩

end

lemma *well-related-extend*:
fixes r (**infix** $\sqsubseteq$ *50*)
assumes *well-related-set* A $(\sqsubseteq)$ **and** *well-related-set* B $(\sqsubseteq)$ **and** $\forall a \in A. \forall b \in B. a \sqsubseteq b$
shows *well-related-set* $(A \cup B)$ $(\sqsubseteq)$
⟨*proof*⟩

lemma *pair-well-related*:
fixes *less-eq* (**infix** $\sqsubseteq$ *50*)
assumes $i \sqsubseteq i$ **and** $i \sqsubseteq j$ **and** $j \sqsubseteq j$
shows *well-related-set* $\{i, j\}$ $(\sqsubseteq)$
⟨*proof*⟩

locale *pre-well-ordered-set* = *semiattractive* + *well-related-set*
begin

interpretation *less-eq-asymmetrize*⟨*proof*⟩

sublocale *transitive*
⟨*proof*⟩

sublocale *total-quasi-ordered-set*⟨*proof*⟩

end

lemmas *pre-well-ordered-iff-semiattractive-well-related* =
pre-well-ordered-set-def[*unfolded atomize-eq*]

lemma *pre-well-ordered-set-empty*[*intro*!]: *pre-well-ordered-set* {} r
⟨*proof*⟩

lemma *pre-well-ordered-iff*:
pre-well-ordered-set A r $\longleftrightarrow$ *total-quasi-ordered-set* A r $\wedge$ *well-founded* A (*asympartp* r)
(**is** $?p \longleftrightarrow ?t \wedge ?w$)
⟨*proof*⟩

lemma (**in** *semiattractive*) *pre-well-ordered-iff-well-related*:

assumes *XA*: $X \subseteq A$
shows *pre-well-ordered-set* X ($\sqsubseteq$) $\longleftrightarrow$ *well-related-set* X ($\sqsubseteq$) (**is** *?l* $\longleftrightarrow$ *?r*)
⟨*proof*⟩

lemma *semiattractive-extend*:
fixes r (**infix** $\sqsubseteq$ *50*)
assumes *A*: *semiattractive* A ($\sqsubseteq$) **and** *B*: *semiattractive* B ($\sqsubseteq$)
and *AB*: $\forall\, a \in A.\ \forall\, b \in B.\ a \sqsubseteq b \wedge \neg\, b \sqsubseteq a$
shows *semiattractive* ($A \cup B$) ($\sqsubseteq$)
⟨*proof*⟩

lemma *pre-well-order-extend*:
fixes r (**infix** $\sqsubseteq$ *50*)
assumes *A*: *pre-well-ordered-set* A ($\sqsubseteq$) **and** *B*: *pre-well-ordered-set* B ($\sqsubseteq$)
and *AB*: $\forall\, a \in A.\ \forall\, b \in B.\ a \sqsubseteq b \wedge \neg\, b \sqsubseteq a$
shows *pre-well-ordered-set* ($A{\cup}B$) ($\sqsubseteq$)
⟨*proof*⟩

lemma (**in** *well-related-set*) *monotone-image-pre-well-ordered*:
fixes *leB* (**infix** $\sqsubseteq''$ *50*)
assumes *mono*: *monotone-on* A ($\sqsubseteq$) ($\sqsubseteq'$) f
and *image*: *semiattractive* (f ‘ A) ($\sqsubseteq'$)
shows *pre-well-ordered-set* (f ‘ A) ($\sqsubseteq'$)
⟨*proof*⟩

locale *well-ordered-set* = *antisymmetric* + *well-related-set*
begin

sublocale *pre-well-ordered-set*⟨*proof*⟩

sublocale *total-ordered-set*⟨*proof*⟩

lemma *well-ordered-subset*: $B \subseteq A \Longrightarrow$ *well-ordered-set* B ($\sqsubseteq$)
⟨*proof*⟩

sublocale *asym*: *well-founded-ordered-set* A *asympartp* ($\sqsubseteq$)
⟨*proof*⟩

end

lemmas *well-ordered-iff-antisymmetric-well-related* = *well-ordered-set-def*[*unfolded atomize-eq*]

lemma *well-ordered-set-empty*[*intro*!]: *well-ordered-set* {} r
⟨*proof*⟩

lemma (**in** *antisymmetric*) *well-ordered-iff-well-related*:
assumes *XA*: $X \subseteq A$
shows *well-ordered-set* X ($\sqsubseteq$) $\longleftrightarrow$ *well-related-set* X ($\sqsubseteq$) (**is** *?l* $\longleftrightarrow$ *?r*)

⟨*proof*⟩

context
  fixes A :: $'a$ *set* **and** *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
begin

context
  assumes A: $\forall a \in A.\ \forall b \in A.\ a \sqsubseteq b$
begin

interpretation *well-related-set* A $(\sqsubseteq)$
  ⟨*proof*⟩

lemmas *trivial-well-related* = *well-related-set-axioms*

lemma *trivial-pre-well-order*: *pre-well-ordered-set* A $(\sqsubseteq)$
  ⟨*proof*⟩

end

interpretation *less-eq-asymmetrize*⟨*proof*⟩

lemma *well-ordered-iff-well-founded-total-ordered*:
  well-ordered-set A $(\sqsubseteq) \longleftrightarrow$ *total-ordered-set* A $(\sqsubseteq) \wedge$ *well-founded* A $(\sqsubset)$
⟨*proof*⟩

end

context
  fixes *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
begin

lemma *well-order-extend*:
  assumes A: *well-ordered-set* A $(\sqsubseteq)$ **and** B: *well-ordered-set* B $(\sqsubseteq)$
    and ABa: $\forall a \in A.\ \forall b \in B.\ a \sqsubseteq b \longrightarrow b \sqsubseteq a \longrightarrow a = b$
    and AB: $\forall a \in A.\ \forall b \in B.\ a \sqsubseteq b$
  shows *well-ordered-set* $(A \cup B)$ $(\sqsubseteq)$
⟨*proof*⟩

interpretation *singleton*: *antisymmetric* $\{a\}$ $(\sqsubseteq)$ **for** a ⟨*proof*⟩

lemmas *singleton-antisymmetric*[*intro*!] = *singleton.antisymmetric-axioms*

lemma *singleton-well-ordered*[*intro*!]: $a \sqsubseteq a \Longrightarrow$ *well-ordered-set* $\{a\}$ $(\sqsubseteq)$
  ⟨*proof*⟩

lemma *closed-UN-well-ordered*:
  assumes *anti*: *antisymmetric* $(\bigcup XX)$ $(\sqsubseteq)$
    and XX: $\forall X \in XX.$ *well-ordered-set* X $(\sqsubseteq) \wedge (\forall x \in X.\ \forall y \in \bigcup XX.\ \neg\ x \sqsubseteq y \longrightarrow$

$y \in X$)
shows *well-ordered-set* ($\bigcup XX$) ($\sqsubseteq$)
⟨*proof*⟩

end

lemma (**in** *well-related-set*) *monotone-image-well-ordered*:
fixes *leB* (**infix** $\sqsubseteq''$ *50*)
assumes *mono*: *monotone-on* A ($\sqsubseteq$) ($\sqsubseteq'$) f
and *image*: *antisymmetric* (f ‘ A) ($\sqsubseteq'$)
shows *well-ordered-set* (f ‘ A) ($\sqsubseteq'$)
⟨*proof*⟩

3.1 Relating to Classes

locale *well-founded-quasi-ordering* = *quasi-ordering* + *well-founded*
begin

lemma *well-founded-quasi-ordering-subset*:
assumes $X \subseteq A$ **shows** *well-founded-quasi-ordering* X ($\sqsubseteq$) ($\sqsubset$)
⟨*proof*⟩

end

class *wf-qorder* = *ord* +
assumes *well-founded-quasi-ordering UNIV* ($\leq$) ($<$)
begin

interpretation *well-founded-quasi-ordering UNIV*
⟨*proof*⟩

subclass *qorder* ⟨*proof*⟩

sublocale *order*: *well-founded-quasi-ordering UNIV*
rewrites $\bigwedge x.\ x \in UNIV \equiv True$
and $\bigwedge X.\ X \subseteq UNIV \equiv True$
and $\bigwedge r.\ r \restriction UNIV \equiv r$
and $\bigwedge P.\ True \wedge P \equiv P$
and *Ball UNIV* $\equiv$ *All*
and *Bex UNIV* $\equiv$ *Ex*
and *sympartp* $(\leq)^{-} \equiv$ *sympartp* $(\leq)$
and $\bigwedge P1.\ (True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
and $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
and $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
⟨*proof*⟩

end

context *wellorder* **begin**

subclass *wf-qorder*
⟨*proof*⟩

sublocale *order*: *well-ordered-set UNIV*
rewrites $\bigwedge x.\ x \in \mathit{UNIV} \equiv \mathit{True}$
and $\bigwedge X.\ X \subseteq \mathit{UNIV} \equiv \mathit{True}$
and $\bigwedge r.\ r \upharpoonright \mathit{UNIV} \equiv r$
and $\bigwedge P.\ \mathit{True} \wedge P \equiv P$
and $\mathit{Ball\ UNIV} \equiv \mathit{All}$
and $\mathit{Bex\ UNIV} \equiv \mathit{Ex}$
and $\mathit{sympartp}\ (\leq)^{-} \equiv \mathit{sympartp}\ (\leq)$
and $\bigwedge P1.\ (\mathit{True} \Longrightarrow \mathit{PROP\ P1}) \equiv \mathit{PROP\ P1}$
and $\bigwedge P1.\ (\mathit{True} \Longrightarrow P1) \equiv \mathit{Trueprop\ P1}$
and $\bigwedge P1\ P2.\ (\mathit{True} \Longrightarrow \mathit{PROP\ P1} \Longrightarrow \mathit{PROP\ P2}) \equiv (\mathit{PROP\ P1} \Longrightarrow \mathit{PROP}$
$P2)$
⟨*proof*⟩

end

thm *order.nonempty-imp-ex-extreme*

3.2 omega-Chains

definition $\mathit{omega\text{-}chain}\ A\ r \equiv \exists f :: \mathit{nat} \Rightarrow 'a.\ \mathit{monotone}\ (\leq)\ r\ f \wedge \mathit{range}\ f = A$

lemma *omega-chainI*:
fixes $f :: \mathit{nat} \Rightarrow 'a$
assumes $\mathit{monotone}\ (\leq)\ r\ f\ \mathit{range}\ f = A$ **shows** $\mathit{omega\text{-}chain}\ A\ r$
⟨*proof*⟩

lemma *omega-chainE*:
assumes $\mathit{omega\text{-}chain}\ A\ r$
and $\bigwedge f :: \mathit{nat} \Rightarrow 'a.\ \mathit{monotone}\ (\leq)\ r\ f \Longrightarrow \mathit{range}\ f = A \Longrightarrow \mathit{thesis}$
shows *thesis*
⟨*proof*⟩

lemma (**in** *transitive*) *local-chain*:
assumes *CA*: $\mathit{range}\ C \subseteq A$
shows $(\forall\, i{::}\mathit{nat}.\ C\ i \sqsubseteq C\ (\mathit{Suc}\ i)) \longleftrightarrow \mathit{monotone}\ (<)\ (\sqsubseteq)\ C$
⟨*proof*⟩

lemma *pair-omega-chain*:
assumes $r\ a\ a\ r\ b\ b\ r\ a\ b$ **shows** $\mathit{omega\text{-}chain}\ \{a,b\}\ r$
⟨*proof*⟩

Every omega-chain is a well-order.

lemma *omega-chain-imp-well-related*:
fixes *less-eq* (**infix** $\sqsubseteq$ *50*)

assumes A: *omega-chain* A ($\sqsubseteq$) **shows** *well-related-set* A ($\sqsubseteq$)
⟨*proof*⟩

lemma (**in** *semiattractive*) *omega-chain-imp-pre-well-ordered*:
assumes *omega-chain* A ($\sqsubseteq$) **shows** *pre-well-ordered-set* A ($\sqsubseteq$)
⟨*proof*⟩

lemma (**in** *antisymmetric*) *omega-chain-imp-well-ordered*:
assumes *omega-chain* A ($\sqsubseteq$) **shows** *well-ordered-set* A ($\sqsubseteq$)
⟨*proof*⟩

3.2.1 Relation image that preserves well-orderedness.

definition *well-image* f A ($\sqsubseteq$) *fa fb* $\equiv$
$\forall a\ b.$ *extreme* $\{x{\in}A.\ fa = f\ x\}$ $(\sqsubseteq)^-$ $a \longrightarrow$ *extreme* $\{y{\in}A.\ fb = f\ y\}$ $(\sqsubseteq)^-$ $b \longrightarrow$
$a \sqsubseteq b$
for *less-eq* (**infix** $\sqsubseteq$ *50*)

lemmas *well-imageI* = *well-image-def*[*unfolded atomize-eq*, *THEN iffD2*, *rule-format*]
lemmas *well-imageD* = *well-image-def*[*unfolded atomize-eq*, *THEN iffD1*, *rule-format*]

lemma (**in** *pre-well-ordered-set*)
well-image-well-related: *pre-well-ordered-set* (f‘A) (*well-image* f A ($\sqsubseteq$))
⟨*proof*⟩

end
theory *Directedness*
imports *Binary-Relations Well-Relations*
begin

Directed sets:

locale *directed* =
fixes A **and** *less-eq* (**infix** $\sqsubseteq$ *50*)
assumes *pair-bounded*: $x \in A \Longrightarrow y \in A \Longrightarrow \exists z \in A.\ x \sqsubseteq z \wedge y \sqsubseteq z$

lemmas *directedI*[*intro*] = *directed.intro*

lemmas *directedD* = *directed-def*[*unfolded atomize-eq*, *THEN iffD1*, *rule-format*]

context
fixes *less-eq* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)
begin

lemma *directedE*:
assumes *directed* A ($\sqsubseteq$) **and** $x \in A$ **and** $y \in A$
and $\bigwedge z.\ z \in A \Longrightarrow x \sqsubseteq z \Longrightarrow y \sqsubseteq z \Longrightarrow$ *thesis*
shows *thesis*
⟨*proof*⟩

lemma *directed-empty*[*simp*]: *directed* {} ($\sqsubseteq$) ⟨*proof*⟩

lemma *directed-union*:
assumes *dX*: *directed* X ($\sqsubseteq$) **and** *dY*: *directed* Y ($\sqsubseteq$)
and *XY*: $\forall x{\in}X.\ \forall y{\in}Y.\ \exists z \in X \cup Y.\ x \sqsubseteq z \land y \sqsubseteq z$
shows *directed* ($X \cup Y$) ($\sqsubseteq$)
⟨*proof*⟩

lemma *directed-extend*:
assumes *X*: *directed* X ($\sqsubseteq$) **and** *Y*: *directed* Y ($\sqsubseteq$) **and** *XY*: $\forall x{\in}X.\ \forall y{\in}Y.\ x \sqsubseteq y$
shows *directed* ($X \cup Y$) ($\sqsubseteq$)
⟨*proof*⟩

end

sublocale *connex* $\subseteq$ *directed*
⟨*proof*⟩

lemmas(**in** *connex*) *directed* = *directed-axioms*

lemma *monotone-directed-image*:
fixes *ir* (**infix** $\preceq$ *50*) **and** *r* (**infix** $\sqsubseteq$ *50*)
assumes *mono*: *monotone-on* I ($\preceq$) ($\sqsubseteq$) f **and** *dir*: *directed* I ($\preceq$)
shows *directed* (f ‘ I) ($\sqsubseteq$)
⟨*proof*⟩

definition *directed-set* A ($\sqsubseteq$) $\equiv \forall X \subseteq A.$ *finite* $X \longrightarrow (\exists b \in A.$ *bound* X $(\sqsubseteq)$ $b)$
for *less-eq* (**infix** $\sqsubseteq$ *50*)

lemmas *directed-setI* = *directed-set-def*[*unfolded atomize-eq*, *THEN iffD2*, *rule-format*]
lemmas *directed-setD* = *directed-set-def*[*unfolded atomize-eq*, *THEN iffD1*, *rule-format*]

lemma *directed-imp-nonempty*:
fixes *less-eq* (**infix** $\sqsubseteq$ *50*)
shows *directed-set* A ($\sqsubseteq$) $\Longrightarrow A \neq \{\}$
⟨*proof*⟩

lemma *directedD2*:
fixes *less-eq* (**infix** $\sqsubseteq$ *50*)
assumes *dir*: *directed-set* A ($\sqsubseteq$) **and** *xA*: $x \in A$ **and** *yA*: $y \in A$
shows $\exists z \in A.\ x \sqsubseteq z \land y \sqsubseteq z$
⟨*proof*⟩

lemma *monotone-directed-set-image*:
fixes *ir* (**infix** $\preceq$ *50*) **and** *r* (**infix** $\sqsubseteq$ *50*)
assumes *mono*: *monotone-on* I ($\preceq$) ($\sqsubseteq$) f **and** *dir*: *directed-set* I ($\preceq$)
shows *directed-set* (f ‘ I) ($\sqsubseteq$)

⟨*proof*⟩

lemma *directed-set-iff-extremed*:
  fixes *less-eq* (**infix** $\sqsubseteq$ *50*)
  assumes *Dfin*: *finite D*
  shows *directed-set D* $(\sqsubseteq) \longleftrightarrow$ *extremed D* $(\sqsubseteq)$
⟨*proof*⟩

lemma (**in** *transitive*) *directed-iff-nonempty-pair-bounded*:
  directed-set A $(\sqsubseteq) \longleftrightarrow A \neq \{\} \wedge (\forall x{\in}A.\ \forall y{\in}A.\ \exists z{\in}A.\ x \sqsubseteq z \wedge y \sqsubseteq z)$
  (**is** *?l* $\longleftrightarrow$ - $\wedge$ *?r*)
⟨*proof*⟩

lemma (**in** *transitive*) *directed-set-iff-nonempty-directed*:
  directed-set A $(\sqsubseteq) \longleftrightarrow A \neq \{\} \wedge$ *directed A* $(\sqsubseteq)$
  ⟨*proof*⟩

lemma (**in** *well-related-set*) *finite-sets-extremed*:
  assumes *fin*: *finite X* **and** *X0*: $X \neq \{\}$ **and** *XA*: $X \subseteq A$
  shows *extremed X* $(\sqsubseteq)$
⟨*proof*⟩

lemma (**in** *well-related-set*) *directed-set*:
  assumes *A0*: $A \neq \{\}$ **shows** *directed-set A* $(\sqsubseteq)$
⟨*proof*⟩

lemma *prod-directed*:
  fixes *leA* (**infix** $\sqsubseteq_A$ *50*) **and** *leB* (**infix** $\sqsubseteq_B$ *50*)
  assumes *dir*: *directed X* (*rel-prod* $(\sqsubseteq_A)$ $(\sqsubseteq_B)$)
  shows *directed* (*fst* ‘ *X*) $(\sqsubseteq_A)$ **and** *directed* (*snd* ‘ *X*) $(\sqsubseteq_B)$
⟨*proof*⟩

class *dir* = *ord* + **assumes** *directed UNIV* $(\leq)$
begin

sublocale *order*: *directed UNIV* $(\leq)$
  rewrites $\bigwedge x.\ x \in$ *UNIV* $\equiv$ *True*
    and $\bigwedge X.\ X \subseteq$ *UNIV* $\equiv$ *True*
    and $\bigwedge r.\ r \upharpoonright$ *UNIV* $\equiv r$
    and $\bigwedge P.$ *True* $\wedge P \equiv P$
    and *Ball UNIV* $\equiv$ *All*
    and *Bex UNIV* $\equiv$ *Ex*
    and *sympartp* $(\leq)^-$ $\equiv$ *sympartp* $(\leq)$
    and $\bigwedge$*P1*. (*True* $\Longrightarrow$ *PROP P1*) $\equiv$ *PROP P1*
    and $\bigwedge$*P1*. (*True* $\Longrightarrow$ *P1*) $\equiv$ *Trueprop P1*
    and $\bigwedge$*P1 P2*. (*True* $\Longrightarrow$ *PROP P1* $\Longrightarrow$ *PROP P2*) $\equiv$ (*PROP P1* $\Longrightarrow$ *PROP P2*)
  ⟨*proof*⟩

end

class *filt* = *ord* +
  assumes *directed UNIV* $(\geq)$
begin

sublocale *order.dual*: *directed UNIV* $(\geq)$
  rewrites $\bigwedge x.\ x \in UNIV \equiv True$
    and $\bigwedge X.\ X \subseteq UNIV \equiv True$
    and $\bigwedge r.\ r \restriction UNIV \equiv r$
    and $\bigwedge P.\ True \wedge P \equiv P$
    and $Ball\ UNIV \equiv All$
    and $Bex\ UNIV \equiv Ex$
    and $sympartp\ (\leq)^{-} \equiv sympartp\ (\leq)$
    and $\bigwedge P1.\ (True \Longrightarrow PROP\ P1) \equiv PROP\ P1$
    and $\bigwedge P1.\ (True \Longrightarrow P1) \equiv Trueprop\ P1$
    and $\bigwedge P1\ P2.\ (True \Longrightarrow PROP\ P1 \Longrightarrow PROP\ P2) \equiv (PROP\ P1 \Longrightarrow PROP\ P2)$
  ⟨*proof*⟩

end

subclass (**in** *linqorder*) *dir*⟨*proof*⟩

subclass (**in** *linqorder*) *filt*⟨*proof*⟩

thm *order.directed-axioms*[**where** $'a = 'a$ ::*dir*]

thm *order.dual.directed-axioms*[**where** $'a = 'a$ ::*filt*]

end

4 Completeness of Relations

Here we formalize various order-theoretic completeness conditions.

theory *Complete-Relations*
  imports *Well-Relations Directedness*
begin

4.1 Completeness Conditions

Order-theoretic completeness demands certain subsets of elements to admit suprema or infima.

definition *complete* (*-–complete*[*999*]*1000*) **where**
$\mathcal{C}$*–complete* $A\ (\sqsubseteq) \equiv \forall X \subseteq A.\ \mathcal{C}\ X\ (\sqsubseteq) \longrightarrow (\exists s.\ extreme\text{-}bound\ A\ (\sqsubseteq)\ X\ s)$ **for** *less-eq* (**infix** $\sqsubseteq$ *50*)

lemmas *completeI* = *complete-def*[*unfolded atomize-eq*, *THEN iffD2*, *rule-format*]
lemmas *completeD* = *complete-def*[*unfolded atomize-eq*, *THEN iffD1*, *rule-format*]
lemmas *completeE* = *complete-def*[*unfolded atomize-eq*, *THEN iffD1*, *rule-format*, *THEN exE*]

lemma *complete-cmono*: $CC \le DD \Longrightarrow DD{-}complete \le CC{-}complete$
⟨*proof*⟩

lemma *complete-subclass*:
fixes *less-eq* (**infix** $\sqsubseteq$ *50*)
assumes $\mathcal{C}{-}complete\ A\ (\sqsubseteq)$ **and** $\forall X \subseteq A.\ \mathcal{D}\ X\ (\sqsubseteq) \longrightarrow \mathcal{C}\ X\ (\sqsubseteq)$
shows $\mathcal{D}{-}complete\ A\ (\sqsubseteq)$
⟨*proof*⟩

lemma *complete-empty*[*simp*]: $\mathcal{C}{-}complete\ \{\}\ r \longleftrightarrow \neg\ \mathcal{C}\ \{\}\ r$ ⟨*proof*⟩

context
fixes *less-eq* :: $'a \Rightarrow\ 'a \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)
begin

Toppedness can be also seen as a completeness condition, since it is equivalent to saying that the universe has a supremum.

lemma *extremed-iff-UNIV-complete*: $extremed\ A\ (\sqsubseteq) \longleftrightarrow (\lambda X\ r.\ X = A){-}complete\ A\ (\sqsubseteq)$ (**is** $?l \longleftrightarrow ?r$)
⟨*proof*⟩

The dual notion of topped is called "pointed", equivalently ensuring a supremum of the empty set.

lemma *pointed-iff-empty-complete*: $extremed\ A\ (\sqsubseteq) \longleftrightarrow (\lambda X\ r.\ X = \{\}){-}complete\ A\ (\sqsubseteq)^{-}$
⟨*proof*⟩

Downward closure is topped.

lemma *dual-closure-is-extremed*:
assumes *bA*: $b \in A$ **and** $b \sqsubseteq b$
shows $extremed\ \{a \in A.\ a \sqsubseteq b\}\ (\sqsubseteq)$
⟨*proof*⟩

Downward closure preserves completeness.

lemma *dual-closure-is-complete*:
assumes *A*: $\mathcal{C}{-}complete\ A\ (\sqsubseteq)$ **and** *bA*: $b \in A$
shows $\mathcal{C}{-}complete\ \{x{\in}A.\ x \sqsubseteq b\}\ (\sqsubseteq)$
⟨*proof*⟩

interpretation *less-eq-dualize*⟨*proof*⟩

Upward closure preserves completeness, under a condition.

lemma *closure-is-complete*:

assumes A: $\mathcal{C}$−*complete* A $(\sqsubseteq)$ **and** bA: $b \in A$
and Cb: $\forall X \subseteq A.\ \mathcal{C}\ X\ (\sqsubseteq) \longrightarrow$ *bound* $X\ (\sqsupseteq)\ b \longrightarrow \mathcal{C}\ (X \cup \{b\})\ (\sqsubseteq)$
shows $\mathcal{C}$−*complete* $\{x{\in}A.\ b \sqsubseteq x\}\ (\sqsubseteq)$
⟨*proof*⟩

lemma *biclosure-is-complete*:
assumes A: $\mathcal{C}$−*complete* A $(\sqsubseteq)$ **and** aA: $a \in A$ **and** bA: $b \in A$ **and** ab: $a \sqsubseteq b$
and Ca: $\forall X \subseteq A.\ \mathcal{C}\ X\ (\sqsubseteq) \longrightarrow$ *bound* $X\ (\sqsupseteq)\ a \longrightarrow \mathcal{C}\ (X \cup \{a\})\ (\sqsubseteq)$
shows $\mathcal{C}$−*complete* $\{x{\in}A.\ a \sqsubseteq x \wedge x \sqsubseteq b\}\ (\sqsubseteq)$
⟨*proof*⟩

end

One of the most well-studied notion of completeness would be the semilattice condition: every pair of elements x and y has a supremum $x \sqcup y$ (not necessarily unique if the underlying relation is not antisymmetric).

definition *pair-complete* $\equiv (\lambda X\ r.\ \exists\, x\ y.\ X = \{x,y\})$−*complete*

lemma *pair-completeI*:
assumes $\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow \exists\, s.$ *extreme-bound* $A\ r\ \{x,y\}\ s$
shows *pair-complete* $A\ r$
⟨*proof*⟩

lemma *pair-completeD*:
assumes *pair-complete* $A\ r$
shows $x \in A \Longrightarrow y \in A \Longrightarrow \exists\, s.$ *extreme-bound* $A\ r\ \{x,y\}\ s$
⟨*proof*⟩

context
fixes *less-eq* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
begin

lemma *pair-complete-imp-directed*:
assumes *comp*: *pair-complete* $A\ (\sqsubseteq)$ **shows** *directed* $A\ (\sqsubseteq)$
⟨*proof*⟩

end

lemma (**in** *connex*) *pair-complete*: *pair-complete* $A\ (\sqsubseteq)$
⟨*proof*⟩

The next one assumes that every nonempty finite set has a supremum.

abbreviation *finite-complete* $\equiv (\lambda X\ r.$ *finite* $X \wedge X \neq \{\})$−*complete*

lemma *finite-complete-le-pair-complete*: *finite-complete* $\leq$ *pair-complete*
⟨*proof*⟩

The next one assumes that every nonempty bounded set has a supremum. It is also called the Dedekind completeness.

abbreviation *conditionally-complete* $A \equiv (\lambda X\ r.\ \exists\, b \in A.\ bound\ X\ r\ b \wedge X \neq \{\})-complete\ A$

lemma *conditionally-complete-imp-nonempty-imp-ex-extreme-bound-iff-ex-bound*:
assumes *comp*: *conditionally-complete* $A\ r$
assumes $X \subseteq A$ **and** $X \neq \{\}$
shows $(\exists\, s.\ extreme\text{-}bound\ A\ r\ X\ s) \longleftrightarrow (\exists\, b{\in}A.\ bound\ X\ r\ b)$
⟨*proof*⟩

The ω-completeness condition demands a supremum for an ω-chain.

Directed completeness is an important notion in domain theory [1], asserting that every nonempty directed set has a supremum. Here, a set X is *directed* if any pair of two elements in X has a bound in X.

definition *directed-complete* $\equiv (\lambda X\ r.\ directed\ X\ r \wedge X \neq \{\})-complete$

lemma *monotone-directed-complete*:
assumes *comp*: *directed-complete* $A\ r$
assumes *fI*: f ‘ $I \subseteq A$ **and** *dir*: *directed* I *ri* **and** *I0*: $I \neq \{\}$ **and** *mono*: *monotone-on* I *ri* r f
shows $\exists\, s.\ extreme\text{-}bound\ A\ r\ (f$ ‘ $I)\ s$
⟨*proof*⟩

lemma (**in** *reflexive*) *dual-closure-is-directed-complete*:
assumes *comp*: $directed{-}complete\ A\ (\sqsubseteq)$ **and** *bA*: $b \in A$
shows $directed{-}complete\ \{X \in A.\ b \sqsubseteq X\}\ (\sqsubseteq)$
⟨*proof*⟩

The next one is quite complete, only the empty set may fail to have a supremum. The terminology follows [3], although there it is defined more generally depending on a cardinal α such that a nonempty set X of cardinality below α has a supremum.

abbreviation *semicomplete* $\equiv (\lambda X\ r.\ X \neq \{\})-complete$

lemma *semicomplete-nonempty-imp-extremed*:
$semicomplete\ A\ r \Longrightarrow A \neq \{\} \Longrightarrow extremed\ A\ r$
⟨*proof*⟩

lemma *connex-dual-semicomplete*: $semicomplete\ \{C.\ connex\ C\ r\}\ (\supseteq)$
⟨*proof*⟩

4.2 Pointed Ones

The term ‘pointed’ refers to the dual notion of toppedness, i.e., there is a global least element. This serves as the supremum of the empty set.

lemma *complete-sup*: $(CC \sqcup CC')-complete\ A\ r \longleftrightarrow CC-complete\ A\ r \wedge CC'-complete\ A\ r$
⟨*proof*⟩

lemma *pointed-directed-complete*:
$directed{-}complete\ A\ r \longleftrightarrow directed\text{-}complete\ A\ r \wedge extremed\ A\ r^{-}$
⟨*proof*⟩

"Bounded complete" refers to pointed conditional complete, but this notion is just the dual of semicompleteness. We prove this later.

abbreviation $bounded\text{-}complete\ A \equiv (\lambda X\ r.\ \exists\, b{\in}A.\ bound\ X\ r\ b){-}complete\ A$

4.3 Relations between Completeness Conditions

context
fixes $less\text{-}eq :: {'a} \Rightarrow {'a} \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)
begin

interpretation *less-eq-dualize*⟨*proof*⟩

Pair-completeness implies that the universe is directed. Thus, with directed completeness implies toppedness.

proposition *directed-complete-pair-complete-imp-extremed*:
assumes *dc*: $directed\text{-}complete\ A\ (\sqsubseteq)$ **and** *pc*: $pair\text{-}complete\ A\ (\sqsubseteq)$ **and** *A*: $A \neq \{\}$
shows $extremed\ A\ (\sqsubseteq)$
⟨*proof*⟩

Semicomplete is conditional complete and topped.

proposition *semicomplete-iff-conditionally-complete-extremed*:
assumes *A*: $A \neq \{\}$
shows $semicomplete\ A\ (\sqsubseteq) \longleftrightarrow conditionally\text{-}complete\ A\ (\sqsubseteq) \wedge extremed\ A\ (\sqsubseteq)$ (**is** $?l \longleftrightarrow ?r$)
⟨*proof*⟩

proposition *complete-iff-pointed-semicomplete*:
$\top{-}complete\ A\ (\sqsubseteq) \longleftrightarrow semicomplete\ A\ (\sqsubseteq) \wedge extremed\ A\ (\sqsupseteq)$ (**is** $?l \longleftrightarrow ?r$)
⟨*proof*⟩

Conditional completeness only lacks top and bottom to be complete.

proposition *complete-iff-conditionally-complete-extremed-pointed*:
$\top{-}complete\ A\ (\sqsubseteq) \longleftrightarrow conditionally\text{-}complete\ A\ (\sqsubseteq) \wedge extremed\ A\ (\sqsubseteq) \wedge extremed\ A\ (\sqsupseteq)$
⟨*proof*⟩

If the universe is directed, then every pair is bounded, and thus has a supremum. On the other hand, supremum gives an upper bound, witnessing directedness.

proposition *conditionally-complete-imp-pair-complete-iff-directed*:
assumes *comp*: $conditionally\text{-}complete\ A\ (\sqsubseteq)$
shows $pair\text{-}complete\ A\ (\sqsubseteq) \longleftrightarrow directed\ A\ (\sqsubseteq)$ (**is** $?l \longleftrightarrow ?r$)
⟨*proof*⟩

end

4.4 Duality of Completeness Conditions

Conditional completeness is symmetric.

context fixes *less-eq* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)
begin

interpretation *less-eq-dualize*⟨*proof*⟩

lemma *conditionally-complete-dual*:
 assumes *comp*: *conditionally-complete* A $(\sqsubseteq)$ **shows** *conditionally-complete* A $(\sqsupseteq)$
⟨*proof*⟩

Full completeness is symmetric.

lemma *complete-dual*:
 $\top-$*complete* A $(\sqsubseteq) \Longrightarrow \top-$*complete* A $(\sqsupseteq)$
 ⟨*proof*⟩

Now we show that bounded completeness is the dual of semicompleteness.

lemma *bounded-complete-iff-pointed-conditionally-complete*:
 assumes A: $A \neq \{\}$
 shows *bounded-complete* A $(\sqsubseteq) \longleftrightarrow$ *conditionally-complete* A $(\sqsubseteq) \wedge$ *extremed* A $(\sqsupseteq)$
 ⟨*proof*⟩

proposition *bounded-complete-iff-dual-semicomplete*:
 bounded-complete A $(\sqsubseteq) \longleftrightarrow$ *semicomplete* A $(\sqsupseteq)$
⟨*proof*⟩

lemma *bounded-complete-imp-conditionally-complete*:
 assumes *bounded-complete* A $(\sqsubseteq)$ **shows** *conditionally-complete* A $(\sqsubseteq)$
 ⟨*proof*⟩

Completeness in downward-closure:

lemma *conditionally-complete-imp-semicomplete-in-dual-closure*:
 assumes A: *conditionally-complete* A $(\sqsubseteq)$ **and** bA: $b \in A$
 shows *semicomplete* $\{a \in A.\ a \sqsubseteq b\}$ $(\sqsubseteq)$
⟨*proof*⟩

end

Completeness in intervals:

lemma *conditionally-complete-imp-complete-in-interval*:
 fixes *less-eq* (**infix** $\sqsubseteq$ *50*)
 assumes *comp*: *conditionally-complete* A $(\sqsubseteq)$ **and** aA: $a \in A$ **and** bA: $b \in A$

and *aa*: $a \sqsubseteq a$ **and** *ab*: $a \sqsubseteq b$
shows $\top$−*complete* $\{x \in A.\ a \sqsubseteq x \wedge x \sqsubseteq b\}$ $(\sqsubseteq)$
⟨*proof*⟩

lemmas *connex-bounded-complete* = *connex-dual-semicomplete*[*folded bounded-complete-iff-dual-semicomplete*]

4.5 Completeness Results Requiring Order-Like Properties

Above results hold without any assumption on the relation. This part demands some order-like properties.

It is well known that in a semilattice, i.e., a pair-complete partial order, every finite nonempty subset of elements has a supremum. We prove the result assuming transitivity, but only that.

lemma (**in** *transitive*) *pair-complete-iff-finite-complete*:
pair-complete A $(\sqsubseteq)$ $\longleftrightarrow$ *finite-complete A* $(\sqsubseteq)$ (**is** *?l* $\longleftrightarrow$ *?r*)
⟨*proof*⟩

Gierz et al. [9] showed that a directed complete partial order is semicomplete if and only if it is also a semilattice. We generalize the claim so that the underlying relation is only transitive.

proposition(**in** *transitive*) *semicomplete-iff-directed-complete-pair-complete*:
semicomplete A $(\sqsubseteq)$ $\longleftrightarrow$ *directed-complete A* $(\sqsubseteq)$ $\wedge$ *pair-complete A* $(\sqsubseteq)$ (**is** *?l* $\longleftrightarrow$ *?r*)
⟨*proof*⟩

The last argument in the above proof requires transitivity, but if we had reflexivity then x itself is a supremum of $\{x\}$ (see ⟦*reflexive ?A ?less-eq*; *?x* $\in$ *?A*⟧ $\Longrightarrow$ *extreme-bound ?A ?less-eq* $\{$*?x*$\}$ *?x*) and so $x \sqsubseteq s$ would be immediate. Thus we can replace transitivity by reflexivity, but then pair-completeness does not imply finite completeness. We obtain the following result.

proposition (**in** *reflexive*) *semicomplete-iff-directed-complete-finite-complete*:
semicomplete A $(\sqsubseteq)$ $\longleftrightarrow$ *directed-complete A* $(\sqsubseteq)$ $\wedge$ *finite-complete A* $(\sqsubseteq)$ (**is** *?l* $\longleftrightarrow$ *?r*)
⟨*proof*⟩

4.6 Relating to Classes

Isabelle's class *complete-lattice* is $\top$−*complete*.

lemma (**in** *complete-lattice*) $\top$−*complete UNIV* $(\le)$
⟨*proof*⟩

4.7 Set-wise Completeness

lemma *Pow-extreme-bound*: $X \subseteq$ *Pow A* $\Longrightarrow$ *extreme-bound* (*Pow A*) $(\subseteq)$ X $(\bigcup X)$
⟨*proof*⟩

lemma *Pow-complete*: $\mathcal{C}$*−complete* (*Pow A*) (⊆)
⟨*proof*⟩

lemma *directed-directed-Un*:
assumes *ch*: $XX \subseteq \{X.\ directed\ X\ r\}$ **and** *dir*: *directed XX* (⊆)
shows *directed* ($\bigcup XX$) *r*
⟨*proof*⟩

lemmas *directed-connex-Un* = *directed-directed-Un*[*OF - connex.directed*]

lemma *directed-sets-directed-complete*:
assumes *cl*: $\forall DC.\ DC \subseteq AA \longrightarrow (\forall X{\in}DC.\ directed\ X\ r) \longrightarrow (\bigcup DC) \in AA$
shows *directed−complete* $\{X \in AA.\ directed\ X\ r\}$ (⊆)
⟨*proof*⟩

lemma *connex-directed-Un*:
assumes *ch*: $CC \subseteq \{C.\ connex\ C\ r\}$ **and** *dir*: *directed CC* (⊆)
shows *connex* ($\bigcup CC$) *r*
⟨*proof*⟩

lemma *connex-is-directed-complete*: *directed−complete* $\{C.\ C \subseteq A \wedge connex\ C\ r\}$ (⊆)
⟨*proof*⟩

lemma (**in** *well-ordered-set*) *well-ordered-set-insert*:
assumes *aA*: *total-ordered-set* (*insert a A*) (⊑)
shows *well-ordered-set* (*insert a A*) (⊑)
⟨*proof*⟩

The following should be true in general, but here we use antisymmetry to avoid the axiom of choice.

lemma (**in** *antisymmetric*) *pointwise-connex-complete*:
assumes *comp*: *connex−complete A* (⊑)
shows *connex−complete* $\{f.\ f\ \text{‘}\ X \subseteq A\}$ (*pointwise X* (⊑))
⟨*proof*⟩

Our supremum/infimum coincides with those of Isabelle's *complete-lattice*.

lemma *complete-UNIV*: ⊤*−complete* (*UNIV*::′*a*::*complete-lattice set*) (≤)
⟨*proof*⟩

context
fixes *X* :: ′*a* :: *complete-lattice set*
begin

lemma *supremum-Sup*: *supremum X* ($\bigsqcup X$)
⟨*proof*⟩

lemmas *Sup-eq-The-supremum* = *order.dual.eq-The-extreme*[*OF supremum-Sup*]

lemma *supremum-eq-Sup*: *supremum* $X\ x \longleftrightarrow \bigsqcup X = x$
⟨*proof*⟩

lemma *infimum-Inf*:
shows *infimum* $X\ (\bigsqcap X)$
⟨*proof*⟩

lemmas *Inf-eq-The-infimum* = *order.eq-The-extreme*[*OF infimum-Inf*]

lemma *infimum-eq-Inf*: *infimum* $X\ x \longleftrightarrow \bigsqcap X = x$
⟨*proof*⟩

end

end

theory *Fixed-Points*
imports *Complete-Relations Directedness*
begin

5 Existence of Fixed Points in Complete Related Sets

The following proof is simplified and generalized from Stouti–Maaden [22]. We construct some set whose extreme bounds – if they exist, typically when the underlying related set is complete – are fixed points of a monotone or inflationary function on any related set. When the related set is attractive, those are actually the least fixed points. This generalizes [22], relaxing reflexivity and antisymmetry.

locale *fixed-point-proof* = *related-set* +
fixes f
assumes f: f ‘ $A \subseteq A$
begin

sublocale *less-eq-asymmetrize*⟨*proof*⟩

definition AA **where** $AA \equiv$
$\{X.\ X \subseteq A \wedge f$ ‘ $X \subseteq X \wedge (\forall Y\ s.\ Y \subseteq X \longrightarrow$ *extreme-bound* $A\ (\sqsubseteq)\ Y\ s \longrightarrow s \in X)\}$

lemma AA-I:
$X \subseteq A \Longrightarrow f$ ‘ $X \subseteq X \Longrightarrow (\bigwedge Y\ s.\ Y \subseteq X \Longrightarrow$ *extreme-bound* $A\ (\sqsubseteq)\ Y\ s \Longrightarrow s \in X) \Longrightarrow X \in AA$
⟨*proof*⟩

lemma AA-E:

$X \in AA \Longrightarrow$
$(X \subseteq A \Longrightarrow f \text{ ` } X \subseteq X \Longrightarrow (\bigwedge Y\ s.\ Y \subseteq X \Longrightarrow \textit{extreme-bound } A\ (\sqsubseteq)\ Y\ s \Longrightarrow s \in X) \Longrightarrow \textit{thesis}) \Longrightarrow \textit{thesis}$
⟨*proof*⟩

definition C **where** $C \equiv \bigcap\ AA$

lemma *A-AA*: $A \in AA$ ⟨*proof*⟩

lemma *C-AA*: $C \in AA$
⟨*proof*⟩

lemma *CA*: $C \subseteq A$ ⟨*proof*⟩

lemma *fC*: $f \text{ ` } C \subseteq C$ ⟨*proof*⟩

context
fixes c **assumes** *Cc*: *extreme-bound* A $(\sqsubseteq)$ C c
begin

private lemma *cA*: $c \in A$ ⟨*proof*⟩ **lemma** *cC*: $c \in C$ ⟨*proof*⟩ **lemma** *fcC*: $f\ c \in C$ ⟨*proof*⟩ **lemma** *fcA*: $f\ c \in A$ ⟨*proof*⟩

lemma *qfp-as-extreme-bound*:
assumes *infl-mono*: $\forall x \in A.\ x \sqsubseteq f\ x \vee (\forall y \in A.\ y \sqsubseteq x \longrightarrow f\ y \sqsubseteq f\ x)$
shows $f\ c \sim c$
⟨*proof*⟩

lemma *extreme-qfp*:
assumes *attract*: $\forall q \in A.\ \forall x \in A.\ f\ q \sim q \longrightarrow x \sqsubseteq f\ q \longrightarrow x \sqsubseteq q$
and *mono*: *monotone-on* A $(\sqsubseteq)$ $(\sqsubseteq)$ f
shows *extreme* $\{q \in A.\ f\ q \sim q \vee f\ q = q\}$ $(\sqsupseteq)$ c
⟨*proof*⟩

end

lemma *ex-qfp*:
assumes *comp*: $CC-complete$ A $(\sqsubseteq)$ **and** C: CC C $(\sqsubseteq)$
and *infl-mono*: $\forall a \in A.\ a \sqsubseteq f\ a \vee (\forall b \in A.\ b \sqsubseteq a \longrightarrow f\ b \sqsubseteq f\ a)$
shows $\exists s \in A.\ f\ s \sim s$
⟨*proof*⟩

lemma *ex-extreme-qfp-fp*:
assumes *comp*: $CC-complete$ A $(\sqsubseteq)$ **and** C: CC C $(\sqsubseteq)$
and *attract*: $\forall q \in A.\ \forall x \in A.\ f\ q \sim q \longrightarrow x \sqsubseteq f\ q \longrightarrow x \sqsubseteq q$
and *mono*: *monotone-on* A $(\sqsubseteq)$ $(\sqsubseteq)$ f
shows $\exists c.$ *extreme* $\{q \in A.\ f\ q \sim q \vee f\ q = q\}$ $(\sqsupseteq)$ c
⟨*proof*⟩

lemma *ex-extreme-qfp*:
  assumes *comp*: $CC{-}complete\ A\ (\sqsubseteq)$ **and** *C*: $CC\ C\ (\sqsubseteq)$
    and *attract*: $\forall\, q \in A.\ \forall\, x \in A.\ f\ q \sim q \longrightarrow x \sqsubseteq f\ q \longrightarrow x \sqsubseteq q$
    and *mono*: $monotone\text{-}on\ A\ (\sqsubseteq)\ (\sqsubseteq)\ f$
  shows $\exists\, c.\ extreme\ \{q \in A.\ f\ q \sim q\}\ (\sqsupseteq)\ c$
⟨*proof*⟩

end

context
  fixes *less-eq* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*) **and** A :: $'a\ set$ **and** f
  assumes *f*: $f\ `\ A \subseteq A$
begin

interpretation *less-eq-symmetrize*⟨*proof*⟩
interpretation *fixed-point-proof* $A\ (\sqsubseteq)\ f$ ⟨*proof*⟩

theorem *complete-infl-mono-imp-ex-qfp*:
  assumes *comp*: $\top{-}complete\ A\ (\sqsubseteq)$ **and** *infl-mono*: $\forall\, a{\in}A.\ a \sqsubseteq f\ a \vee (\forall\, b{\in}A.\ b \sqsubseteq a \longrightarrow f\ b \sqsubseteq f\ a)$
  shows $\exists\, s{\in}A.\ f\ s \sim s$
  ⟨*proof*⟩

end

corollary (**in** *antisymmetric*) *complete-infl-mono-imp-ex-fp*:
  assumes *comp*: $\top{-}complete\ A\ (\sqsubseteq)$ **and** *f*: $f\ `\ A \subseteq A$
    and *infl-mono*: $\forall\, a{\in}A.\ a \sqsubseteq f\ a \vee (\forall\, b{\in}A.\ b \sqsubseteq a \longrightarrow f\ b \sqsubseteq f\ a)$
  shows $\exists\, s \in A.\ f\ s = s$
⟨*proof*⟩

context *semiattractive* **begin**

interpretation *less-eq-symmetrize*⟨*proof*⟩

theorem *complete-mono-imp-ex-extreme-qfp*:
  assumes *comp*: $\top{-}complete\ A\ (\sqsubseteq)$ **and** *f*: $f\ `\ A \subseteq A$
    and *mono*: $monotone\text{-}on\ A\ (\sqsubseteq)\ (\sqsubseteq)\ f$
  shows $\exists\, s.\ extreme\ \{p \in A.\ f\ p \sim p\}\ (\sqsubseteq)\ s$
⟨*proof*⟩

end

corollary (**in** *antisymmetric*) *complete-mono-imp-ex-extreme-fp*:
  assumes *comp*: $\top{-}complete\ A\ (\sqsubseteq)$ **and** *f*: $f\ `\ A \subseteq A$
    and *mono*: $monotone\text{-}on\ A\ (\sqsubseteq)\ (\sqsubseteq)\ f$
  shows $\exists\, s.\ extreme\ \{s \in A.\ f\ s = s\}\ (\sqsubseteq)^{-}\ s$
⟨*proof*⟩

6 Fixed Points in Well-Complete Antisymmetric Sets

In this section, we prove that an inflationary or monotone map over a well-complete antisymmetric set has a fixed point.

In order to formalize such a theorem in Isabelle, we followed Grall's [11] elementary proof for Bourbaki–Witt and Markowsky's theorems. His idea is to consider well-founded derivation trees over A, where from a set $C \subseteq A$ of premises one can derive $f\ (\bigsqcup C)$ if C is a chain. The main observation is as follows: Let D be the set of all the derivable elements; that is, for each $d \in D$ there exists a well-founded derivation whose root is d. It is shown that D is a chain, and hence one can build a derivation yielding $f\ (\bigsqcup D)$, and $f\ (\bigsqcup D)$ is shown to be a fixed point.

lemma *bound-monotone-on*:
 assumes *mono*: *monotone-on* $A\ r\ s\ f$ **and** *XA*: $X \subseteq A$ **and** *aA*: $a \in A$ **and** *rXa*: *bound* $X\ r\ a$
 shows *bound* $(f`X)\ s\ (f\ a)$
⟨*proof*⟩

context *fixed-point-proof* **begin**

To avoid the usage of the axiom of choice, we carefully define derivations so that any derivable element determines its lower set. This led to the following definition:

definition *derivation* $X \equiv X \subseteq A \land$ *well-ordered-set* $X\ (\sqsubseteq) \land$
 $(\forall x \in X.$ *let* $Y = \{y \in X.\ y \sqsubset x\}$ *in*
  $(\exists y.$ *extreme* $Y\ (\sqsubseteq)\ y \land x = f\ y) \lor$
  $f\ `\ Y \subseteq Y \land$ *extreme-bound* $A\ (\sqsubseteq)\ Y\ x)$

lemma *empty-derivation*: *derivation* $\{\}$ ⟨*proof*⟩

lemma assumes *derivation* P
 shows *derivation-A*: $P \subseteq A$ **and** *derivation-well-ordered*: *well-ordered-set* $P\ (\sqsubseteq)$
 ⟨*proof*⟩

lemma *derivation-cases*[*consumes 2*, *case-names suc lim*]:
 assumes *derivation* X **and** $x \in X$
  and $\bigwedge Y\ y.\ Y = \{y \in X.\ y \sqsubset x\} \Longrightarrow$ *extreme* $Y\ (\sqsubseteq)\ y \Longrightarrow x = f\ y \Longrightarrow$ *thesis*
  and $\bigwedge Y.\ Y = \{y \in X.\ y \sqsubset x\} \Longrightarrow f\ `\ Y \subseteq Y \Longrightarrow$ *extreme-bound* $A\ (\sqsubseteq)\ Y\ x$ $\Longrightarrow$ *thesis*
 shows *thesis*
 ⟨*proof*⟩

definition *derivable* $x \equiv \exists X.$ *derivation* $X \land x \in X$

lemma *derivableI*[*intro?*]: *derivation* $X \Longrightarrow x \in X \Longrightarrow$ *derivable* x ⟨*proof*⟩

lemma *derivableE*: *derivable* $x \Longrightarrow (\bigwedge P.$ *derivation* $P \Longrightarrow x \in P \Longrightarrow$ *thesis*$) \Longrightarrow$ *thesis*
⟨*proof*⟩

lemma *derivable-A*: *derivable* $x \Longrightarrow x \in A$ ⟨*proof*⟩

lemma *UN-derivations-eq-derivable*: $(\bigcup \{P.$ *derivation* $P\}) = \{x.$ *derivable* $x\}$
⟨*proof*⟩

end

locale *fixed-point-proof2* = *fixed-point-proof* + *antisymmetric* +
assumes *derivation-infl*: $\forall X\ x\ y.$ *derivation* $X \longrightarrow x \in X \longrightarrow y \in X \longrightarrow x \sqsubseteq y \longrightarrow x \sqsubseteq f\ y$
and *derivation-f-refl*: $\forall X\ x.$ *derivation* $X \longrightarrow x \in X \longrightarrow f\ x \sqsubseteq f\ x$
begin

lemma *derivation-lim*:
assumes *P*: *derivation* P **and** *fP*: f ' $P \subseteq P$ **and** *Pp*: *extreme-bound* A $(\sqsubseteq)$ P p
shows *derivation* $(P \cup \{p\})$
⟨*proof*⟩

lemma *derivation-suc*:
assumes *P*: *derivation* P **and** *Pp*: *extreme* P $(\sqsubseteq)$ p **shows** *derivation* $(P \cup \{f\ p\})$
⟨*proof*⟩

lemma *derivable-closed*:
assumes *x*: *derivable* x **shows** *derivable* $(f\ x)$
⟨*proof*⟩

The following lemma is derived from Grall's proof. We simplify the claim so that we consider two elements from one derivation, instead of two derivations.

lemma *derivation-useful*:
assumes *X*: *derivation* X **and** *xX*: $x \in X$ **and** *yX*: $y \in X$ **and** *xy*: $x \sqsubset y$
shows $f\ x \sqsubseteq y$
⟨*proof*⟩

Next one is the main lemma of this section, stating that elements from two possibly different derivations are comparable, and moreover the lower one is in the derivation of the upper one. The latter claim, not found in Grall's proof, is crucial in proving that the union of all derivations is well-related.

lemma *derivations-cross-compare*:
assumes *X*: *derivation* X **and** *Y*: *derivation* Y **and** *xX*: $x \in X$ **and** *yY*: $y \in Y$
shows $(x \sqsubset y \wedge x \in Y) \vee x = y \vee (y \sqsubset x \wedge y \in X)$
⟨*proof*⟩

sublocale *derivable*: *well-ordered-set* $\{x.\ derivable\ x\}$ $(\sqsubseteq)$
$\langle proof \rangle$

lemma *pred-unique*:
assumes X: *derivation* X **and** xX: $x \in X$
shows $\{z \in X.\ z \sqsubset x\} = \{z.\ derivable\ z \wedge z \sqsubset x\}$
$\langle proof \rangle$

The set of all derivable elements is itself a derivation.

lemma *derivation-derivable*: *derivation* $\{x.\ derivable\ x\}$
$\langle proof \rangle$

Finally, if the set of all derivable elements admits a supremum, then it is a fixed point.

context
fixes p
assumes p: *extreme-bound* A $(\sqsubseteq)$ $\{x.\ derivable\ x\}$ p
begin

lemma *sup-derivable-derivable*: *derivable* p
$\langle proof \rangle$ **lemmas** *sucp* = *sup-derivable-derivable*[*THEN derivable-closed*]

lemma *sup-derivable-prefixed*: $f\ p \sqsubseteq p$ $\langle proof \rangle$

lemma *sup-derivable-postfixed*: $p \sqsubseteq f\ p$
$\langle proof \rangle$

lemma *sup-derivable-qfp*: $f\ p \sim p$
$\langle proof \rangle$

lemma *sup-derivable-fp*: $f\ p = p$
$\langle proof \rangle$

end

end

The assumptions are satisfied by monotone functions.

context *fixed-point-proof* **begin**

context
assumes *ord*: *antisymmetric* A $(\sqsubseteq)$
begin

interpretation *antisymmetric* $\langle proof \rangle$

context
assumes *mono*: *monotone-on* A $(\sqsubseteq)$ $(\sqsubseteq)$ f
begin

interpretation *fixed-point-proof2*
⟨*proof*⟩

lemmas *mono-imp-fixed-point-proof2* = *fixed-point-proof2-axioms*

corollary *mono-imp-sup-derivable-fp*:
  assumes *p*: *extreme-bound* A ($\sqsubseteq$) $\{x.$ *derivable* $x\}$ p
  shows $f\ p = p$
  ⟨*proof*⟩

lemma *mono-imp-sup-derivable-lfp*:
  assumes *p*: *extreme-bound* A ($\sqsubseteq$) $\{x.$ *derivable* $x\}$ p
  shows *extreme* $\{q \in A.\ f\ q = q\}$ ($\sqsupseteq$) p
⟨*proof*⟩

lemma *mono-imp-ex-least-fp*:
  assumes *comp*: *well-related-set−complete* A ($\sqsubseteq$)
  shows $\exists\, p.$ *extreme* $\{q \in A.\ f\ q = q\}$ ($\sqsupseteq$) p
⟨*proof*⟩

end

end

end

Bourbaki-Witt Theorem on well-complete pseudo-ordered set:

theorem (**in** *pseudo-ordered-set*) *well-complete-infl′-imp-ex-fp*:
  assumes *comp*: *well-related-set−complete* A ($\sqsubseteq$)
    and *f*: f ‘ $A \subseteq A$ **and** *infl*: $\forall\, x \in A.\ \forall\, y \in A.\ x \sqsubseteq y \longrightarrow x \sqsubseteq f\ y$
  shows $\exists\, p \in A.\ f\ p = p$
⟨*proof*⟩

Bourbaki-Witt Theorem on posets:

corollary (**in** *partially-ordered-set*) *well-complete-infl-imp-ex-fp*:
  assumes *comp*: *well-related-set−complete* A ($\sqsubseteq$)
    and *f*: f ‘ $A \subseteq A$ **and** *infl*: $\forall\, x \in A.\ x \sqsubseteq f\ x$
  shows $\exists\, p \in A.\ f\ p = p$
⟨*proof*⟩

7 Completeness of (Quasi-)Fixed Points

We now prove that, under attractivity, the set of quasi-fixed points is complete.

definition *setwise* **where** *setwise* $r\ X\ Y \equiv \forall\, x{\in}X.\ \forall\, y{\in}Y.\ r\ x\ y$

lemmas *setwiseI*[*intro*] = *setwise-def*[*unfolded atomize-eq*, *THEN iffD2*, *rule-format*]

lemmas *setwiseE*[*elim*] = *setwise-def*[*unfolded atomize-eq*, *THEN iffD1*, *elim-format*, *rule-format*]

context *fixed-point-proof* **begin**

abbreviation *setwise-less-eq* (**infix** $\sqsubseteq^s$ *50*) **where** $(\sqsubseteq^s) \equiv$ *setwise* $(\sqsubseteq)$

7.1 Least Quasi-Fixed Points for Attractive Relations.

lemma *attract-mono-imp-least-qfp*:
  assumes *attract*: *attractive* A $(\sqsubseteq)$
    and *comp*: *well-related-set−complete* A $(\sqsubseteq)$
    and *mono*: *monotone-on* A $(\sqsubseteq)$ $(\sqsubseteq)$ f
  shows $\exists\, c.$ *extreme* $\{p \in A.\ f\ p \sim p \vee f\ p = p\}$ $(\sqsupseteq)$ $c \wedge f\ c \sim c$
⟨*proof*⟩

7.2 General Completeness

lemma *attract-mono-imp-fp-qfp-complete*:
  assumes *attract*: *attractive* A $(\sqsubseteq)$
    and *comp*: *CC−complete* A $(\sqsubseteq)$
    and *wr-CC*: $\forall\, C \subseteq A.$ *well-related-set* C $(\sqsubseteq) \longrightarrow CC\ C\ (\sqsubseteq)$
    and *extend*: $\forall\, X\ Y.\ CC\ X\ (\sqsubseteq) \longrightarrow CC\ Y\ (\sqsubseteq) \longrightarrow X \sqsubseteq^s Y \longrightarrow CC\ (X \cup Y)$
$(\sqsubseteq)$
    and *mono*: *monotone-on* A $(\sqsubseteq)$ $(\sqsubseteq)$ f
    and *P*: $P \subseteq \{x \in A.\ f\ x = x\}$
  shows *CC−complete* $(\{q \in A.\ f\ q \sim q\} \cup P)$ $(\sqsubseteq)$
⟨*proof*⟩

lemma *attract-mono-imp-qfp-complete*:
  assumes *attractive* A $(\sqsubseteq)$
    and *CC−complete* A $(\sqsubseteq)$
    and $\forall\, C \subseteq A.$ *well-related-set* C $(\sqsubseteq) \longrightarrow CC\ C\ (\sqsubseteq)$
    and $\forall\, X\ Y.\ CC\ X\ (\sqsubseteq) \longrightarrow CC\ Y\ (\sqsubseteq) \longrightarrow X \sqsubseteq^s Y \longrightarrow CC\ (X \cup Y)\ (\sqsubseteq)$
    and *monotone-on* A $(\sqsubseteq)$ $(\sqsubseteq)$ f
  shows *CC−complete* $\{p \in A.\ f\ p \sim p\}$ $(\sqsubseteq)$
  ⟨*proof*⟩

lemma *antisym-mono-imp-fp-complete*:
  assumes *anti*: *antisymmetric* A $(\sqsubseteq)$
    and *comp*: *CC−complete* A $(\sqsubseteq)$
    and *wr-CC*: $\forall\, C \subseteq A.$ *well-related-set* C $(\sqsubseteq) \longrightarrow CC\ C\ (\sqsubseteq)$
    and *extend*: $\forall\, X\ Y.\ CC\ X\ (\sqsubseteq) \longrightarrow CC\ Y\ (\sqsubseteq) \longrightarrow X \sqsubseteq^s Y \longrightarrow CC\ (X \cup Y)$
$(\sqsubseteq)$
    and *mono*: *monotone-on* A $(\sqsubseteq)$ $(\sqsubseteq)$ f
  shows *CC−complete* $\{p \in A.\ f\ p = p\}$ $(\sqsubseteq)$
⟨*proof*⟩

end

7.3 Instances

7.3.1 Instances under attractivity

context *attractive* **begin**

interpretation *less-eq-symmetrize*⟨*proof*⟩

Full completeness

theorem *mono-imp-qfp-complete*:
assumes *comp*: $\top$−*complete* A ($\sqsubseteq$) **and** f: f ‘ $A \subseteq A$ **and** *mono*: *monotone-on* A ($\sqsubseteq$) ($\sqsubseteq$) f
shows $\top$−*complete* $\{p \in A.\ f\ p \sim p\}$ ($\sqsubseteq$)
⟨*proof*⟩

Connex completeness

theorem *mono-imp-qfp-connex-complete*:
assumes *comp*: *connex*−*complete* A ($\sqsubseteq$)
and f: f ‘ $A \subseteq A$ **and** *mono*: *monotone-on* A ($\sqsubseteq$) ($\sqsubseteq$) f
shows *connex*−*complete* $\{p \in A.\ f\ p \sim p\}$ ($\sqsubseteq$)
⟨*proof*⟩

Directed completeness

theorem *mono-imp-qfp-directed-complete*:
assumes *comp*: *directed*−*complete* A ($\sqsubseteq$)
and f: f ‘ $A \subseteq A$ **and** *mono*: *monotone-on* A ($\sqsubseteq$) ($\sqsubseteq$) f
shows *directed*−*complete* $\{p \in A.\ f\ p \sim p\}$ ($\sqsubseteq$)
⟨*proof*⟩

Well Completeness

theorem *mono-imp-qfp-well-complete*:
assumes *comp*: *well-related-set*−*complete* A ($\sqsubseteq$)
and f: f ‘ $A \subseteq A$ **and** *mono*: *monotone-on* A ($\sqsubseteq$) ($\sqsubseteq$) f
shows *well-related-set*−*complete* $\{p \in A.\ f\ p \sim p\}$ ($\sqsubseteq$)
⟨*proof*⟩

end

7.3.2 Usual instances under antisymmetry

context *antisymmetric* **begin**

Knaster–Tarski

theorem *mono-imp-fp-complete*:
assumes *comp*: $\top$−*complete* A ($\sqsubseteq$)
and f: f ‘ $A \subseteq A$ **and** *mono*: *monotone-on* A ($\sqsubseteq$) ($\sqsubseteq$) f
shows $\top$−*complete* $\{p \in A.\ f\ p = p\}$ ($\sqsubseteq$)
⟨*proof*⟩

Markowsky 1976

theorem *mono-imp-fp-connex-complete*:
  assumes *comp*: *connex−complete* A ($\sqsubseteq$)
    and f: f ‘ $A \subseteq A$ **and** *mono*: *monotone-on* A ($\sqsubseteq$) ($\sqsubseteq$) f
  shows *connex−complete* $\{p \in A.\ f\ p = p\}$ ($\sqsubseteq$)
⟨*proof*⟩

Pataraia

theorem *mono-imp-fp-directed-complete*:
  assumes *comp*: *directed−complete* A ($\sqsubseteq$)
    and f: f ‘ $A \subseteq A$ **and** *mono*: *monotone-on* A ($\sqsubseteq$) ($\sqsubseteq$) f
  shows *directed−complete* $\{p \in A.\ f\ p = p\}$ ($\sqsubseteq$)
⟨*proof*⟩

Bhatta & George 2011

theorem *mono-imp-fp-well-complete*:
  assumes *comp*: *well-related-set−complete* A ($\sqsubseteq$)
    and f: f ‘ $A \subseteq A$ **and** *mono*: *monotone-on* A ($\sqsubseteq$) ($\sqsubseteq$) f
  shows *well-related-set−complete* $\{p \in A.\ f\ p = p\}$ ($\sqsubseteq$)
⟨*proof*⟩

end

end
theory *Continuity*
  imports *Complete-Relations*
begin

7.4 Scott Continuity, ω-Continuity

In this Section, we formalize Scott continuity and ω-continuity. We then prove that a Scott continuous map is ω-continuous and that an ω-continuous map is “nearly” monotone.

definition *continuous* (-*−continuous* [*1000*]*1000*) **where**
  $\mathcal{C}$*−continuous* A ($\sqsubseteq$) B ($\trianglelefteq$) $f \equiv$
   f ‘ $A \subseteq B \wedge$
    ($\forall X\ s.\ \mathcal{C}\ X$ ($\sqsubseteq$) $\longrightarrow X \neq \{\} \longrightarrow X \subseteq A \longrightarrow$ *extreme-bound* A ($\sqsubseteq$) $X\ s \longrightarrow$ *extreme-bound* B ($\trianglelefteq$) (f‘X) ($f\ s$))
  for *leA* (**infix** $\sqsubseteq$ *50*) **and** *leB* (**infix** $\trianglelefteq$ *50*)

lemmas *continuousI*[*intro?*] =
  continuous-def[*unfolded atomize-eq*, *THEN iffD2*, *unfolded conj-imp-eq-imp-imp*, *rule-format*]

lemmas *continuousE* =
  continuous-def[*unfolded atomize-eq*, *THEN iffD1*, *elim-format*, *unfolded conj-imp-eq-imp-imp*, *rule-format*]

lemma
  fixes *prec-eq* (**infix** $\preceq$ *50*) **and** *less-eq* (**infix** $\sqsubseteq$ *50*)

assumes $\mathcal{C}-continuous\ I\ (\preceq)\ A\ (\sqsubseteq)\ f$
shows *continuous-carrierD*[*dest*]: $f\ `\ I \subseteq A$
and *continuousD*: $\mathcal{C}\ X\ (\preceq) \Longrightarrow X \neq \{\} \Longrightarrow X \subseteq I \Longrightarrow extreme\text{-}bound\ I\ (\preceq)\ X\ b \Longrightarrow extreme\text{-}bound\ A\ (\sqsubseteq)\ (f\ `\ X)\ (f\ b)$
⟨*proof*⟩

lemma *continuous-comp*:
fixes *leA* (**infix** $\sqsubseteq_A$ *50*) **and** *leB* (**infix** $\sqsubseteq_B$ *50*) **and** *leC* (**infix** $\sqsubseteq_C$ *50*)
assumes *KfL*: $\forall X \subseteq A.\ \mathcal{K}\ X\ (\sqsubseteq_A) \longrightarrow \mathcal{L}\ (f\ `\ X)\ (\sqsubseteq_B)$
assumes *f*: $\mathcal{K}-continuous\ A\ (\sqsubseteq_A)\ B\ (\sqsubseteq_B)\ f$ **and** *g*: $\mathcal{L}-continuous\ B\ (\sqsubseteq_B)\ C\ (\sqsubseteq_C)\ g$
shows $\mathcal{K}-continuous\ A\ (\sqsubseteq_A)\ C\ (\sqsubseteq_C)\ (g \circ f)$
⟨*proof*⟩

lemma *continuous-comp-top*:
fixes *leA* (**infix** $\sqsubseteq_A$ *50*) **and** *leB* (**infix** $\sqsubseteq_B$ *50*) **and** *leC* (**infix** $\sqsubseteq_C$ *50*)
assumes *f*: $\mathcal{K}-continuous\ A\ (\sqsubseteq_A)\ B\ (\sqsubseteq_B)\ f$ **and** *g*: $\top-continuous\ B\ (\sqsubseteq_B)\ C\ (\sqsubseteq_C)\ g$
shows $\mathcal{K}-continuous\ A\ (\sqsubseteq_A)\ C\ (\sqsubseteq_C)\ (g \circ f)$
⟨*proof*⟩

lemma *id-continuous*:
fixes *leA* (**infix** $\sqsubseteq_A$ *50*)
shows $\mathcal{K}-continuous\ A\ (\sqsubseteq_A)\ A\ (\sqsubseteq_A)\ (\lambda x.\ x)$
⟨*proof*⟩

lemma *cst-continuous*:
fixes *leA* (**infix** $\sqsubseteq_A$ *50*) **and** *leB* (**infix** $\sqsubseteq_B$ *50*)
assumes $b \in B$ **and** *bb*: $b \sqsubseteq_B b$
shows $\mathcal{K}-continuous\ A\ (\sqsubseteq_A)\ B\ (\sqsubseteq_B)\ (\lambda x.\ b)$
⟨*proof*⟩

lemma *continuous-cmono*:
assumes *CD*: $\mathcal{C} \leq \mathcal{D}$ **shows** $\mathcal{D}-continuous \leq \mathcal{C}-continuous$
⟨*proof*⟩

context
fixes *prec-eq* :: $'i \Rightarrow 'i \Rightarrow bool$ (**infix** $\preceq$ *50*) **and** *less-eq* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\sqsubseteq$ *50*)
begin

lemma *continuous-subclass*:
assumes *CD*: $\forall X \subseteq I.\ X \neq \{\} \longrightarrow \mathcal{C}\ X\ (\preceq) \longrightarrow \mathcal{D}\ X\ (\preceq)$ **and** *cont*: $\mathcal{D}-continuous\ I\ (\preceq)\ A\ (\sqsubseteq)\ f$
shows $\mathcal{C}-continuous\ I\ (\preceq)\ A\ (\sqsubseteq)\ f$
⟨*proof*⟩

lemma *chain-continuous-imp-well-continuous*:

assumes *cont*: *connex*−*continuous* I ($\preceq$) A ($\sqsubseteq$) f
shows *well-related-set*−*continuous* I ($\preceq$) A ($\sqsubseteq$) f
⟨*proof*⟩

lemma *well-continuous-imp-omega-continous*:
assumes *cont*: *well-related-set*−*continuous* I ($\preceq$) A ($\sqsubseteq$) f
shows *omega-chain*−*continuous* I ($\preceq$) A ($\sqsubseteq$) f
⟨*proof*⟩

end

abbreviation *scott-continuous* I ($\preceq$) $\equiv$ *directed-set*−*continuous* I ($\preceq$)
for *prec-eq* (**infix** $\preceq$ *50*)

lemma *scott-continuous-imp-well-continuous*:
fixes *prec-eq* :: $'i \Rightarrow\ 'i \Rightarrow$ *bool* (**infix** $\preceq$ *50*) **and** *less-eq* :: $'a \Rightarrow\ 'a \Rightarrow$ *bool* (**infix** $\sqsubseteq$ *50*)
assumes *cont*: *scott-continuous* I ($\preceq$) A ($\sqsubseteq$) f
shows *well-related-set*−*continuous* I ($\preceq$) A ($\sqsubseteq$) f
⟨*proof*⟩

lemmas *scott-continuous-imp-omega-continuous* =
scott-continuous-imp-well-continuous[*THEN well-continuous-imp-omega-continous*]

7.4.1 Continuity implies monotonicity

lemma *continuous-imp-mono-refl*:
fixes *prec-eq* (**infix** $\preceq$ *50*) **and** *less-eq* (**infix** $\sqsubseteq$ *50*)
assumes *cont*: $\mathcal{C}$−*continuous* I ($\preceq$) A ($\sqsubseteq$) f **and** *xyC*: $\mathcal{C}$ $\{x,y\}$ ($\preceq$)
and *xy*: $x \preceq y$ **and** *yy*: $y \preceq y$
and *x*: $x \in I$ **and** *y*: $y \in I$
shows $f\,x \sqsubseteq f\,y$
⟨*proof*⟩

lemma *omega-continuous-imp-mono-refl*:
fixes *prec-eq* (**infix** $\preceq$ *50*) **and** *less-eq* (**infix** $\sqsubseteq$ *50*)
assumes *cont*: *omega-chain*−*continuous* I ($\preceq$) A ($\sqsubseteq$) f
and *xx*: $x \preceq x$ **and** *xy*: $x \preceq y$ **and** *yy*: $y \preceq y$
and *x*: $x \in I$ **and** *y*: $y \in I$
shows $f\,x \sqsubseteq f\,y$
⟨*proof*⟩

context *reflexive* **begin**

lemma *continuous-imp-monotone-on*:
fixes *leB* (**infix** $\trianglelefteq$ *50*)
assumes *cont*: $\mathcal{C}$−*continuous* A ($\sqsubseteq$) B ($\trianglelefteq$) f
and *II*: $\forall\, i \in A.\ \forall\ j \in A.\ i \sqsubseteq j \longrightarrow \mathcal{C}\ \{i,j\}$ ($\sqsubseteq$)
shows *monotone-on* A ($\sqsubseteq$) ($\trianglelefteq$) f

⟨*proof*⟩

lemma *well-complete-imp-monotone-on*:
fixes *leB* (**infix** $\trianglelefteq$ *50*)
assumes *cont*: *well-related-set−continuous* A ($\sqsubseteq$) B ($\trianglelefteq$) f
shows *monotone-on* A ($\sqsubseteq$) ($\trianglelefteq$) f
⟨*proof*⟩

end

end
theory *Kleene-Fixed-Point*
imports *Complete-Relations Continuity*
begin

8 Iterative Fixed Point Theorem

Kleene's fixed-point theorem states that, for a pointed directed complete partial order $\langle A, \sqsubseteq\rangle$ and a Scott-continuous map $f : A \to A$, the supremum of $\{f^n(\bot) \mid n \in \mathbb{N}\}$ exists in A and is a least fixed point. Mashburn [17] generalized the result so that $\langle A, \sqsubseteq\rangle$ is a ω-complete partial order and f is ω-continuous.

In this section we further generalize the result and show that for ω-complete relation $\langle A, \sqsubseteq\rangle$ and for every bottom element $\bot \in A$, the set $\{f^n(\bot) \mid n \in \mathbb{N}\}$ has suprema (not necessarily unique, of course) and, they are quasi-fixed points. Moreover, if ($\sqsubseteq$) is attractive, then the suprema are precisely the least quasi-fixed points.

8.1 Existence of Iterative Fixed Points

The first part of Kleene's theorem demands to prove that the set $\{f^n(\bot) \mid n \in \mathbb{N}\}$ has a supremum and that all such are quasi-fixed points. We prove this claim without assuming anything on the relation $\sqsubseteq$ besides ω-completeness and one bottom element.

notation *compower* (-^-[*1000*,*1000*]*1000*)

lemma *monotone-on-funpow*: **assumes** f: f ‘ $A \subseteq A$ **and** *mono*: *monotone-on* A r r f
shows *monotone-on* A r r (f^n)
⟨*proof*⟩

no-notation *bot* ($\bot$)

context
fixes A **and** *less-eq* (**infix** $\sqsubseteq$ *50*) **and** *bot* ($\bot$) **and** f
assumes *bot*: $\bot \in A$ $\forall q \in A.\ \bot \sqsubseteq q$

assumes *cont*: *omega-chain−continuous A* (⊑) *A* (⊑) *f*
begin

interpretation *less-eq-symmetrize*⟨*proof*⟩ **lemma** *f*: *f* ' *A* ⊆ *A* ⟨*proof*⟩ **abbreviation**(*input*) *Fn* ≡ {*f*^*n* ⊥ |. *n* :: *nat*}

private lemma *fn-ref*: *f*^*n* ⊥ ⊑ *f*^*n* ⊥ **and** *fnA*: *f*^*n* ⊥ ∈ *A*
⟨*proof*⟩ **lemma** *FnA*: *Fn* ⊆ *A* ⟨*proof*⟩ **lemma** *Fn-chain*: *omega-chain Fn* (⊑)
⟨*proof*⟩ **lemma** *Fn*: *Fn* = *range* (λ*n*. *f*^*n* ⊥) ⟨*proof*⟩

theorem *kleene-qfp*:
assumes *q*: *extreme-bound A* (⊑) *Fn q*
shows *f q* ∼ *q*
⟨*proof*⟩

lemma *ex-kleene-qfp*:
assumes *comp*: *omega-chain−complete A* (⊑)
shows ∃ *p*. *extreme-bound A* (⊑) *Fn p*
⟨*proof*⟩

8.2 Iterative Fixed Points are Least.

Kleene's theorem also states that the quasi-fixed point found this way is a least one. Again, attractivity is needed to prove this statement.

lemma *kleene-qfp-is-least*:
assumes *attract*: ∀ *q* ∈ *A*. ∀ *x* ∈ *A*. *f q* ∼ *q* ⟶ *x* ⊑ *f q* ⟶ *x* ⊑ *q*
assumes *q*: *extreme-bound A* (⊑) *Fn q*
shows *extreme* {*s* ∈ *A*. *f s* ∼ *s*} (⊒) *q*
⟨*proof*⟩

lemma *kleene-qfp-iff-least*:
assumes *comp*: *omega-chain−complete A* (⊑)
assumes *attract*: ∀ *q* ∈ *A*. ∀ *x* ∈ *A*. *f q* ∼ *q* ⟶ *x* ⊑ *f q* ⟶ *x* ⊑ *q*
assumes *dual-attract*: ∀ *p* ∈ *A*. ∀ *q* ∈ *A*. ∀ *x* ∈ *A*. *p* ∼ *q* ⟶ *q* ⊑ *x* ⟶ *p* ⊑ *x*
shows *extreme-bound A* (⊑) *Fn* = *extreme* {*s* ∈ *A*. *f s* ∼ *s*} (⊒)
⟨*proof*⟩

end

context *attractive* **begin**

interpretation *less-eq-dualize* + *less-eq-symmetrize*⟨*proof*⟩

theorem *kleene-qfp-is-dual-extreme*:
assumes *comp*: *omega-chain−complete A* (⊑)
and *cont*: *omega-chain−continuous A* (⊑) *A* (⊑) *f* **and** *bA*: *b* ∈ *A* **and** *bot*: ∀ *x* ∈ *A*. *b* ⊑ *x*
shows *extreme-bound A* (⊑) {*f*^*n b* |. *n* :: *nat*} = *extreme* {*s* ∈ *A*. *f s* ∼ *s*} (⊒)
⟨*proof*⟩

end

corollary(**in** *antisymmetric*) *kleene-fp*:
  assumes *cont*: *omega-chain−continuous* A (⊑) A (⊑) f
    and b: $b \in A$ $\forall x \in A.\ b \sqsubseteq x$
    and p: *extreme-bound* A (⊑) {f^n b |. n :: *nat*} p
  shows $f\ p = p$
  ⟨*proof*⟩

no-notation *compower* (-^-[*1000*,*1000*]*1000*)

end

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