Combinatorics on Words formalized
Lyndon Words

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Contents

theory *Lyndon*
imports *Combinatorics-Words.CoWBasic*
begin

Chapter 1

Lyndon words

A Lyndon word is a non-empty word that is lexicographically strictly smaller than any other word in its conjugacy class, i.e., than any its rotations. They are named after R. Lyndon who introduced them in [4] as "standard" sequences.

We present elementary results on Lyndon words, mostly covered by results in [3, Chapter 5] and [1, 2].

This definition assumes a linear order on letters given by the context.

1.1 Definition and elementary properties

1.1.1 Underlying order

lemma (**in** *linorder*) *lexordp-mid-pref*: *ord-class.lexordp u v* $\Longrightarrow$ *ord-class.lexordp v* (*u*·*s*) $\Longrightarrow$
  u $\leq_p$ *v*
  ⟨*proof*⟩

lemma (**in** *linorder*) *lexordp-ext*: *ord-class.lexordp u v* $\Longrightarrow$ $\neg$ *u* $\leq_p$ *v* $\Longrightarrow$
  ord-class.lexordp (*u*·*w*) (*v*·*z*)
  ⟨*proof*⟩

lemmas [*code*] = *lexordp-simps*

context *linorder*
begin

abbreviation *Lyndon-less* :: *'a list* ⇒ *'a list* ⇒ *bool* (**infixl** <*lex 50*)
  where *Lyndon-less xs ys* ≡ *ord-class.lexordp xs ys*

abbreviation *Lyndon-le* :: *'a list* ⇒ *'a list* ⇒ *bool* (**infixl** ≤*lex 50*)
  where *Lyndon-le xs ys* ≡ *ord-class.lexordp-eq xs ys*

interpretation *rlex*: *linorder* ($\leq_{lex}$) ($<_{lex}$)
⟨*proof*⟩

interpretation *dual-rlex*: *linorder* $\lambda\ x\ y.\ y \leq_{lex} x\ \lambda\ x\ y.\ y <_{lex} x$
⟨*proof*⟩

lemma *sorted-dual-rev-iff*: *dual-rlex.sorted ws* $\longleftrightarrow$ *rlex.sorted* (*rev ws*)
⟨*proof*⟩

Several useful lemmas that are formulated for relations, interpreted for the default linear order.

lemmas *lexord-suf-linorder* = *lexord-sufE*[*of* - - - - {$(x,\ y).\ x < y$}, *folded lexordp-conv-lexord*]
and *lexord-append-leftI-linorder* = *lexord-append-leftI*[*of* - - {$(x,\ y).\ x < y$} -, *folded lexordp-conv-lexord*]
and *lexord-app-right-linorder* = *lexord-sufI*[*of* - - {$(x,\ y).\ x < y$} -, *folded lexordp-conv-lexord*]
and *lexord-take-index-conv-linorder* = *lexord-take-index-conv*[*of* - - {$(x,\ y).\ x < y$}, *folded lexordp-conv-lexord*]
and *mismatch-lexord-linorder* = *mismatch-lexord*[*of* - - {$(x,\ y).\ x < y$}, *folded lexordp-conv-lexord*]
and *lexord-cancel-right-linorder* = *lexord-cancel-right*[*of* - - - - {$(a,b).\ a < b$}, *folded lexordp-conv-lexord*]

1.1.2 Lyndon word definition

fun *Lyndon* :: *'a list* ⇒ *bool*
where *Lyndon w* = ($w \neq \varepsilon \wedge (\forall\, n.\ 0 < n \wedge n < |w| \longrightarrow w <_{lex} rotate\ n\ w)$)

lemma *LyndonD*: *Lyndon w* $\Longrightarrow$ $0 < n \Longrightarrow n < |w| \Longrightarrow w <_{lex} rotate\ n\ w$
⟨*proof*⟩

lemma *LyndonD-nemp*: *Lyndon w* $\Longrightarrow$ $w \neq \varepsilon$
⟨*proof*⟩

lemma *LyndonI*: $w \neq \varepsilon \Longrightarrow \forall\ n.\ 0 < n \wedge n < |w| \longrightarrow w <_{lex} rotate\ n\ w \Longrightarrow$ *Lyndon w*
⟨*proof*⟩

lemma *Lyndon-sing*: *Lyndon* [*a*]
⟨*proof*⟩

lemma *Lyndon-prim*: **assumes** *Lyndon w*
shows *primitive w*
⟨*proof*⟩

lemma *Lyndon-conj-greater*: *Lyndon* ($u{\cdot}v$) $\Longrightarrow u \neq \varepsilon \Longrightarrow v \neq \varepsilon \Longrightarrow u{\cdot}v <_{lex} v{\cdot}u$
⟨*proof*⟩

1.1.3 Code equations for Lyndon words

primrec *Lyndon-rec* :: *'a list* ⇒ *nat* ⇒ *bool* **where**
Lyndon-rec w 0 = True |
Lyndon-rec w (Suc n) = (if w <lex rotate (Suc n) w then Lyndon-rec w n else False)

lemma *Lyndon-rec-all*: **assumes** *Lyndon-rec (a # w) (|w|)*
shows *n < |a#w| ⟹ 0 < n ⟹ Lyndon-rec (a#w) n*
⟨*proof*⟩

lemma *Lyndon-Lyndon-rec*: **assumes** *Lyndon w*
shows *0 < n ⟹ n < |w| ⟹ Lyndon-rec w n*
⟨*proof*⟩

lemma *Lyndon-code* [*code*]:
Lyndon Nil = False
Lyndon (a # w) = Lyndon-rec (a # w) (|w|)
⟨*proof*⟩

1.1.4 Properties of Lyndon words

Lyndon words are unbordered

theorem *Lyndon-unbordered*: **assumes** *Lyndon w* **shows** ¬ *bordered w*
⟨*proof*⟩

Each conjugacy class contains a Lyndon word

lemma *conjug-Lyndon-ex*: **assumes** *primitive w*
obtains *n* **where** *Lyndon (rotate n w)*
⟨*proof*⟩

lemma *conjug-Lyndon-ex'*: **assumes** *primitive w*
obtains *v* **where** *w ∼ v* **and** *Lyndon v*
⟨*proof*⟩

1.2 Characterization by suffixes

lemma *Lyndon-suf-less*: **assumes** *Lyndon w s ≤ns w s ≠ w*
shows *w <lex s*
⟨*proof*⟩

lemma *Lyndon-pref-suf-less*: **assumes** *Lyndon w p ≤p w s ≤ns w s ≠ w*
shows *p <lex s*
⟨*proof*⟩

lemma *suf-less-Lyndon*: **assumes** *w ≠ ε* **and** ∀ *s*. (*s ≤ns w ⟶ s ≠ w ⟶ w <lex s*)

shows *Lyndon w*
⟨*proof*⟩

corollary *Lyndon-suf-char*: $w \neq \varepsilon \Longrightarrow$ *Lyndon* $w \longleftrightarrow (\forall s.\ s \leq_{ns} w \longrightarrow s \neq w \longrightarrow w <_{lex} s)$
⟨*proof*⟩

lemma *Lyndon-suf-le*: *Lyndon* $w \Longrightarrow s \leq_{ns} w \Longrightarrow w \leq_{lex} s$
⟨*proof*⟩

1.3 Unbordered prefix of a Lyndon word is Lyndon

lemma *unbordered-pref-Lyndon*: *Lyndon* $(u{\cdot}v) \Longrightarrow u \neq \varepsilon \Longrightarrow \neg$ *bordered* $u \Longrightarrow$ *Lyndon* u
⟨*proof*⟩

1.4 Concatenation of Lyndon words

theorem *Lyndon-concat*: **assumes** *Lyndon* u **and** *Lyndon* v **and** $u <_{lex} v$
shows *Lyndon* $(u{\cdot}v)$
⟨*proof*⟩

1.5 Longest Lyndon suffix

fun *longest-Lyndon-suffix*:: *'a list* ⇒ *'a list* (*LynSuf*) **where**
longest-Lyndon-suffix $\varepsilon = \varepsilon$ |
longest-Lyndon-suffix $(a\#w) =$ (*if Lyndon* $(a\#w)$ *then* $a\#w$ *else longest-Lyndon-suffix* w)

lemma *longest-Lyndon-suf-ext*: $\neg$ *Lyndon* $(a\ \#\ w) \Longrightarrow$ *LynSuf* $w =$ *LynSuf* $(a\ \#\ w)$
⟨*proof*⟩

lemma *longest-Lyndon-suf-suf*: $w \neq \varepsilon \Longrightarrow$ *LynSuf* $w \leq_s w$
⟨*proof*⟩

lemma *longest-Lyndon-suf-max*:
$v \leq_s w \Longrightarrow$ *Lyndon* $v \Longrightarrow v \leq_s$ (*LynSuf* w)
⟨*proof*⟩

lemma *longest-Lyndon-suf-Lyndon-id*: **assumes** *Lyndon* w
shows *LynSuf* $w = w$
⟨*proof*⟩

lemma *longest-Lyndon-suf-longest*: $w \neq \varepsilon \Longrightarrow v' \leq_s w \Longrightarrow$ *Lyndon* $v' \Longrightarrow |v'| \leq |(\mathit{LynSuf}\ w)|$
⟨*proof*⟩

lemma *longest-Lyndon-suf-sing*: $LynSuf\ [a] = [a]$
⟨*proof*⟩

lemma *longest-Lyndon-suf-Lyndon*: $w \neq \varepsilon \Longrightarrow Lyndon\ (LynSuf\ w)$
⟨*proof*⟩

lemma *longest-Lyndon-suf-nemp*: $w \neq \varepsilon \Longrightarrow LynSuf\ w \neq \varepsilon$
⟨*proof*⟩

lemma *longest-Lyndon-sufI*:
assumes $q \leq_s w$ **and** *Lyndon* q **and** *all-s*: $(\forall\ s.\ (s \leq_s w \wedge Lyndon\ s) \longrightarrow s \leq_s q)$
shows $LynSuf\ w = q$
⟨*proof*⟩

corollary *longest-Lyndon-sufI'*:
assumes $q \leq_s w$ **and** *Lyndon* q **and** *all-s*: $\forall\ s.\ (s \leq_s w \wedge Lyndon\ s) \longrightarrow |s| \leq |q|$
shows $LynSuf\ w = q$
⟨*proof*⟩

The next lemma is fabricated to suit the upcoming definition of longest Lyndon factorization.

lemma *longest-Lyndon-suf-shorter*: **assumes** $w \neq \varepsilon$
shows $|w^{<-1}(LynSuf\ w)| < |w|$
⟨*proof*⟩

1.6 Lyndon factorizations

function *Lyndon-fac*::*'a list* ⇒ *'a list list* (*LynFac*)
where $Lyndon\text{-}fac\ w = (if\ w \neq \varepsilon\ then\ ((Lyndon\text{-}fac\ (w\ ^{<-1}(LynSuf\ w))) \cdot [LynSuf\ w])\ else\ \varepsilon)$
⟨*proof*⟩
termination
⟨*proof*⟩

The factorization $LynFac\ w$ obtained by taking always the longest Lyndon suffix is well defined, and called "Lyndon factorization (of w)".

lemma *Lyndon-fac-simp*: $w \neq \varepsilon \Longrightarrow Lyndon\text{-}fac\ w = Lyndon\text{-}fac\ (w^{<-1} LynSuf\ w) \cdot (LynSuf\ w\ \#\ \varepsilon)$
⟨*proof*⟩

lemma *Lyndon-fac-emp*: $Lyndon\text{-}fac\ \varepsilon = \varepsilon$
⟨*proof*⟩

Note that the Lyndon factorization of a Lyndon word is trivial.

lemma *Lyndon-fac-longest-Lyndon-id*: $Lyndon\ w \Longrightarrow Lyndon\text{-}fac\ w = [w]$
⟨*proof*⟩

Lyndon factorization is composed of Lyndon words ...

lemma *Lyndon-fac-set*: $z \in set\ (Lyndon\text{-}fac\ w) \Longrightarrow Lyndon\ z$
⟨*proof*⟩

...and it indeed is a factorization of the argument.

lemma *Lyndon-fac-longest-dec*: $concat\ (Lyndon\text{-}fac\ w) = w$
⟨*proof*⟩

The following lemma makes explicit the inductive character of the definition of *LynFac*.

lemma *Lyndon-fac-longest-pref*: $us \leq p\ Lyndon\text{-}fac\ w \Longrightarrow Lyndon\text{-}fac\ (concat\ us) = us$
⟨*proof*⟩

We give name to an important predicate: monotone (nonincreasing) list of Lyndon words.

definition *Lyndon-mono* :: *'a list list* ⇒ *bool* **where**
Lyndon-mono ws ⟷ (∀ *u* ∈ *set ws*. *Lyndon u*) ∧ (*rlex.sorted* (*rev ws*))

lemma *Lyndon-mono-set*: *Lyndon-mono ws* ⟹ *u* ∈ *set ws* ⟹ *Lyndon u*
⟨*proof*⟩

lemma *Lyndon-mono-sorted*: *Lyndon-mono ws* ⟹ *rlex.sorted* (*rev ws*)
⟨*proof*⟩

lemma *Lyndon-mono-nth*: *Lyndon-mono ws* ⟹ $i \leq j \Longrightarrow j < |ws| \Longrightarrow ws!j \leq_{lex} ws!i$
⟨*proof*⟩

lemma *Lyndon-mono-empty*[*simp*]: *Lyndon-mono* ε
⟨*proof*⟩

lemma *Lyndon-mono-sing*: *Lyndon u* ⟹ *Lyndon-mono* [*u*]
⟨*proof*⟩

lemma *Lyndon-mono-fac-Lyndon-mono*:
assumes $ps \leq_f ws$ **and** *Lyndon-mono ws* **shows** *Lyndon-mono ps*
⟨*proof*⟩

Lyndon factorization is monotone! Altogether, we have shown that the Lyndon factorization is a monotone factorization into Lyndon words.

theorem *fac-Lyndon-mono*: *Lyndon-mono* (*Lyndon-fac w*)
⟨*proof*⟩

Now we want to show the converse: any monotone factorization into Lyndon words is the Lyndon factorization

The last element in the Lyndon factorization is the smallest suffix.

lemma *Lyndon-mono-last-smallest*: *Lyndon-mono ws* $\Longrightarrow$ *s* $\leq_{ns}$ (*concat ws*) $\Longrightarrow$ *last ws* $\leq_{lex}$ *s*
⟨*proof*⟩

A monotone list, if seen as a factorization, must end with the longest suffix

lemma *Lyndon-mono-last-longest*: **assumes** *ws* $\neq \varepsilon$ **and** *Lyndon-mono ws*
shows *LynSuf* (*concat ws*) = *last ws*
⟨*proof*⟩

Therefore, by construction, any monotone list is the Lyndon factorization of its concatenation

lemma *Lyndon-mono-fac*:
Lyndon-mono ws $\Longrightarrow$ *ws* = *Lyndon-fac* (*concat ws*)
⟨*proof*⟩

This implies that the Lyndon factorization can be characterized in two equivalent ways: as the (unique) monotone factorization (into Lyndon words) or as the "suffix greedy" factorization (into Lyndon words).

corollary *Lyndon-mono-fac-iff*: *Lyndon-mono ws* $\longleftrightarrow$ *ws* = *LynFac* (*concat ws*)
⟨*proof*⟩

corollary *Lyndon-mono-unique*: **assumes** *Lyndon-mono ws* **and** *Lyndon-mono zs* **and** *concat ws* = *concat zs*
shows *ws* = *zs*
⟨*proof*⟩

1.6.1 Standard factorization

lemma *Lyndon-std*: **assumes** *Lyndon w 1* < |*w*|
obtains *l m* **where** *w* = *l*·*m* **and** *Lyndon l* **and** *Lyndon m* **and** *l* $<_{lex}$ *m*
⟨*proof*⟩

corollary *Lyndon-std-iff*:
Lyndon w $\longleftrightarrow$ (|*w*| = *1* $\vee$ ($\exists$ *l m*. *w* = *l*·*m* $\wedge$ *Lyndon l* $\wedge$ *Lyndon m* $\wedge$ *l* $<_{lex}$ *m*))
(**is** *?L* $\longleftrightarrow$ *?R*)
⟨*proof*⟩

end — end context linorder

end

theory *Lyndon-Addition*
imports *Lyndon Szpilrajn.Szpilrajn*

begin

1.6.2 The minimal relation

We define the minimal relation which guarantees the lexicographic minimality of w compared to its nontrivial conjugates.

inductive-set *rotate-rel* :: *'a list* ⇒ *'a rel* **for** *w*
where $0 < n \Longrightarrow n < |w| \Longrightarrow (\textit{mismatch-pair}\ w\ (\textit{rotate}\ n\ w)) \in \textit{rotate-rel}\ w$

A word is Lyndon iff the corresponding order of letters is compatible with *rotate-rel w*.

lemma (**in** *linorder*) *rotate-rel-iff*: **assumes** $w \neq \varepsilon$
shows *Lyndon w* ⟷ *rotate-rel w* ⊆ {(*x*,*y*). *x* < *y*} (**is** *?L* ⟷ *?R*)
⟨*proof*⟩

It is well known that an acyclic order can be extended to a total strict linear order. This means that a word is Lyndon with respect to some order iff its *rotate-rel w* is acyclic.

lemma *Lyndon-rotate-rel-iff*:
acyclic (*rotate-rel w*) ⟷ (∃ *r*. *strict-linear-order r* ∧ *rotate-rel w* ⊆ *r*) (**is** *?L* ⟷ *?R*)
⟨*proof*⟩

lemma *slo-linorder*: *strict-linear-order r* ⟹ *class.linorder* (λ *a b*. (*a*,*b*) ∈ $r^{=}$) (λ *a b*. (*a*,*b*) ∈ *r*)
⟨*proof*⟩

Application examples

lemma assumes $w \neq \varepsilon$ **and** *acyclic* (*rotate-rel w*) **shows** *primitive w*
⟨*proof*⟩

lemma assumes $w \neq \varepsilon$ **and** *acyclic* (*rotate-rel w*) **shows** ¬ *bordered w*
⟨*proof*⟩

end

References

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