Isabelle/Circus

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Abstract

The Circus specification language combines elements for complex data and behavior specifications, using an integration of Z and CSP with a refinement calculus. Its semantics is based on Hoare and He's unifying theories of programming (UTP).

Isabelle/Circus is a formalization of the UTP and the Circus language in Isabelle/HOL. It contains proof rules and tactic support that allows for proofs of refinement for Circus processes (involving both data and behavioral aspects).

This environment supports a syntax for the semantic definitions which is close to textbook presentations of Circus.

These theories are presented with details in [9]. This document is a technical appendix of this report.

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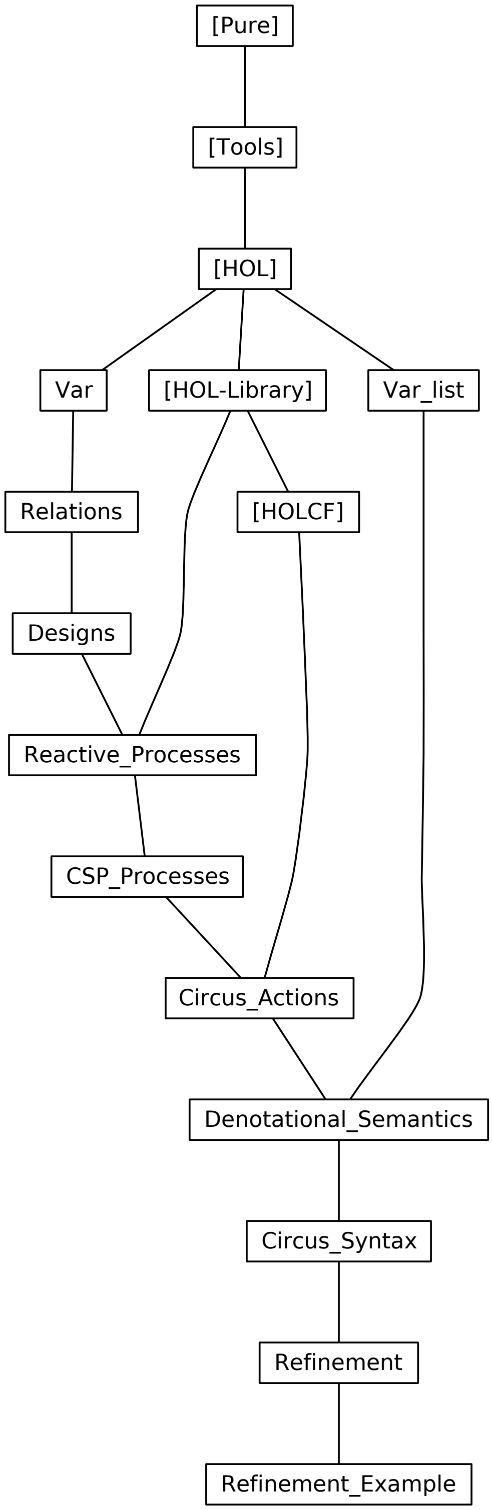
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1 Introduction

Many systems involve both complex (sometimes infinite) data structures and interactions between concurrent processes. Refinement of abstract specifications of such systems into more concrete ones, requires an appropriate formalisation of refinement and appropriate proof support.

There are several combinations of process-oriented modeling languages with data-oriented specification formalisms such as Z or B or CASL; examples are discussed in [3, 10, 17, 14]. In this paper, we consider *Circus* [18], a language for refinement, that supports modeling of high-level specifications, designs, and concrete programs. It is representative of a class of languages that provide facilities to model data types, using a predicate-based notation, and patterns of interactions, without imposing architectural restrictions. It is this feature that makes it suitable for reasoning about both abstract and low-level designs.

We present a "shallow embedding" of the *Circus* semantics enabling state variables and channels in *Circus* to have arbitrary HOL types. Therefore, the entire handling of typing can be completely shifted to the (efficiently implemented) Isabelle type-checker and is therefore implicit in proofs. This drastically simplifies definitions and proofs, and makes the reuse of standardized proof procedures possible. Compared to implementations based on a "deep embedding" such as [19] this significantly improves the usability of the resulting proof environment.

Our representation brings particular technical challenges and contributions concerning some important notions about variables. The main challenge was to represent alphabets and bindings in a typed way that preserves the semantics and improves deduction. We provide a representation of bindings without an explicit management of alphabets. However, the representation of some core concepts in the unifying theories of programming (UTP) and *Circus* constructs (variable scopes and renaming) became challenging. Thus, we propose a (stack-based) solution that allows the coding of state variables scoping with no need for renaming. This solution is even a contribution to the UTP theory that does not allow nested variable scoping. Some challenging and tricky definitions (e.g. channels and name sets) are explained in this paper.

This paper is organized as follows. The next section gives an introduction to the basics of our work: Isabelle/HOL, UTP and *Circus* with a short example of a *Circus* process. In Section 3, we present our embedding of the basic concepts of *Circus* (alphabet, variables ...). We introduce the representation of some *Circus* actions and process, with an overview of the Isabelle/*Circus* syntax. In Section 4, we show on an example, how Isabelle/*Circus* can be used to write specifications. We give some details on what is happening "behind the scenes" when the system parses each part of the specification. In the last part of this section, we show how to write proofs based on specifications, and give a refinement proof example. A more developed version of this paper can be found in [9].

2 Background

2.1 Isabelle, HOL and Isabelle/HOL

2.1.1 isar

[12] is a generic theorem prover implemented in SML. It is based on the so-called "LCF-style architecture", which makes it possible to extend a small trusted logical kernel by user-programmed procedures in a logically safe way. New object logics can be introduced to Isabelle by specifying their syntax and semantics, by deriving its inference rules from there and program specific tactic support for the object logic. Isabelle is based on a typed λ-calculus including a Haskell-style type-system with type-classes (e.g. in $\alpha :: \text{order}$, the type-variable ranges over all types that posses a partial ordering.)

2.1.2 Higher-order logic (HOL)

[7, 1] is a classical logic based on a simple type system. It provides the usual logical connectives like $\_ \wedge \_$, $\_ \rightarrow \_$, $\neg \_$ as well as the object-logical quantifiers $\forall\, x \bullet P\ x$ and $\exists\, x \bullet P\ x$; in contrast to first-order logic, quantifiers may range over arbitrary types, including total functions $f :: \alpha \Rightarrow \beta$. HOL is centered around extensional equality $\_ = \_ :: \alpha \Rightarrow \alpha \Rightarrow \text{bool}$. HOL is more expressive than first-order logic, since, *e.g.* , induction schemes can be expressed inside the logic. Being based on some polymorphically typed λ-calculus, HOL can be viewed as a combination of a programming language like SML or Haskell and a specification language providing powerful logical quantifiers ranging over elementary and function types.

2.1.3 Isabelle/HOL

is an instance of Isabelle with higher-order logic. It provides a rich collection of library theories like sets, pairs, relations, partial functions lists, multi-sets, orderings, and various arithmetic theories which only contain rules derived from conservative, *i.e.* logically safe definitions. Setups for the automated proof procedures like `simp`, `auto`, and arithmetic types such as `int` are provided.

2.2 Advanced Specification Constructs in Isabelle/HOL

2.2.1 Constant definitions.

In its easiest form, constant definitions are definitional logical axioms of the form $c \equiv E$ where c is a fresh constant symbol not occurring in E which is closed (both wrt. variables and type variables). For example:

```
definition upd::(α⇒β)⇒α⇒β⇒(α⇒β)     ("_(|_ := _|)")
where       "upd f x v ≡ λ z. if x=z then v else f z"
```

The pragma `("_(| _ := _|)")` for the Isabelle syntax engine introduces the notation `f(|x:=y|)` for `upd f x y`. Moreover, some elaborate preprocessing allows for recursive definitions, provided that a termination ordering can be established. Such recursive definitions are thus internally reduced to definitional axioms.

2.2.2 Type definitions.

Types can be introduced in Isabelle/HOL in different ways. The most general way to safely introduce new types is using the **typedef** construct. This allows introducing a type as a non-empty subset of an existing type. More precisely, the new type is specified to be isomorphic to this non-empty subset. For instance:

```
typedef mytype = "{x::nat. x < 10}"
```

This definition requires that the set is non-empty: `∃x. x∈{x::nat. x<10}`, which is easy to prove in this case:

```
by (rule_tac x = 1 in exI, simp)
```

where *rule_tac* is a tactic that applies an introduction rule, and `exI` corresponds to the introduction of the existential quantification.

Similarly, the **datatype** command allows the definition of inductive datatypes. It introduces a datatype using a list of *constructors*. For instance, a logical compiler is invoked for the following introduction of the type `option`:

```
datatype α option  = None | Some α
```

which generates the underlying type definition and derives distinctness rules and induction principles. Besides the *constructors* `None` and `Some`, the following match-operator and his rules are also generated:

case x of None ⇒... | Some a ⇒...

2.2.3 Extensible records.

Isabelle/HOL's support for *extensible records* is of particular importance for our work. Record types are denoted, for example, by:

```
record T = a::T1
           b::T2
```

which implicitly introduces the record constructor (|a:=e_1,b:=e_2|) and the update of record r in field a, written as `r(|a:= x|)`. Extensible records are represented internally by cartesian products with an implicit free component

δ, i.e. in this case by a triple of the type $\mathtt{T}_1 \times \mathtt{T}_2 \times \delta$. The third component can be referenced by a *special selector* `more` available on extensible records. Thus, the record `T` can be extended later on using the syntax:

```
record ET = T + c::T3
```

The key point is that theorems can be established, once and for all, on `T` types, even if future parts of the record are not yet known, and reused in the later definition and proofs over `ET`-values. Using this feature, we can model the effect of defining the alphabet of UTP processes incrementally while maintaining the full expressivity of HOL wrt. the types of $\mathtt{T}_1$, $\mathtt{T}_2$ and $\mathtt{T}_3$.

2.3 *Circus* and its UTP Foundation

Circus is a formal specification language [18] which integrates the notions of states and complex data types (in a Z-like style) and communicating parallel processes inspired from CSP. From Z, the language inherits the notion of a schema used to model sets of (ground) states as well as syntactic machinery to describe pre-states and post-states; from CSP, the language inherits the concept of *communication events* and typed communication channels, the concepts of deterministic and non-deterministic choice (reflected by the process combinators $P \Box P'$ and $P \sqcap P'$), the concept of concealment (hiding) $P \backslash A$ of events in A occurring in in the evolution of process P. Due to the presence of state variables, the *Circus* synchronous communication operator syntax is slightly different frome CSP: $P \llbracket\ n \mid c \mid n' \rrbracket P'$ means that P and P' communicate via the channels mentioned in c; moreover, P may modify the variables mentioned in n only, and P' in n' only, n and n' are disjoint name sets.

Moreover, the language comes with a formal notion of refinement based on a denotational semantics. It follows the failure/divergence semantics [15], (but coined in terms of the UTP [13]) providing a notion of execution trace `tr`, refusals `ref`, and divergences. It is expressed in terms of the UTP [11] which makes it amenable to other refinement-notions in UTP. Figure 1 presents a simple *Circus* specification, `FIG`, the fresh identifiers generator.

2.3.1 Predicates and Relations.

The UTP is a semantic framework based on an alphabetized relational calculus. An *alphabetized predicate* is a pair (*alphabet*, *predicate*) where the free variables appearing in the predicate are all in the alphabet, e.g. $(\{x, y\}, x > y)$. As such, it is very similar to the concept of a *schema* in Z. In the base theory Isabelle/UTP of this work, we represent alphabetized predicates by sets of (extensible) records, e.g. `{A. x A > y A}`.

An *alphabetized relation* is an alphabetized predicate where the alphabet is composed of input (undecorated) and output (dashed) variables. In this

$[ID]$

channel *req*
channel *ret*, *out* : *ID*

process *FIG* $\widehat{=}$ **begin**

state S == $[\, idS : \mathbb{P}\ ID \,]$

$Init \ \widehat{=}\ idS := \emptyset$

Out
ΔS $v! : ID$
$v! \notin idS$ $idS' = idS \cup \{v!\}$

Remove
ΔS $x? : ID$
$idS' = idS \setminus \{x?\}$

$\bullet\ Init\,;\ \textbf{var}\ v : ID\ \bullet$
$(\mu\ X \bullet\ (req \rightarrow Out\,;\ out!v \rightarrow Skip\ \ \square\ \ ret?x \rightarrow Remove)\,;\ X)$

end

Figure 1: The Fresh Identifiers Generator in (Textbook) *Circus*

case the predicate describes a relation between input and output variables, for example $(\{x, x', y, y'\}, x' = x + y)$ which is a notation for: `{(A,A').x A' = x A + y A}`, which is a set of pairs, thus a relation.

Standard predicate calculus operators are used to combine alphabetized predicates. The definition of these operators is very similar to the standard one, with some additional constraints on the alphabets.

2.3.2 Designs and processes.

In UTP, in order to explicitly record the termination of a program, a subset of alphabetized relations is introduced. These relations are called *designs* and their alphabet should contain the special boolean observational variable `ok`. It is used to record the start and termination of a program. A UTP design is defined as follows in Isabelle:

```
(P ⊢ Q) ≡ λ (A,A'). (ok A ∧ P (A,A')) ⟶ (ok A' ∧ Q (A,A'))
```

Following the way of UTP to describe reactive processes, more observational variables are needed to record the interaction with the environment. Three observational variables are defined for this subset of relations: `wait`, `tr` and `ref`. The boolean variable `wait` records if the process is waiting for an interaction or has terminated. `tr` records the list (trace) of interactions the process has performed so far. The variable `ref` contains the set of interactions (events) the process may refuse to perform. These observational variables defines the basic alphabet of all reactive processes called "`alpha_rp`".

Some healthiness conditions are defined over `wait`, `tr` and `ref` to ensure that a recative process satisfies some properties [6] (see Table 2 in [9]).

A CSP process is a UTP reactive process that satisfies two additional healthiness conditions(all well-formedness conditions can be found in [9]). A process that satisfies these conditions is said to be CSP healthy.

3 Isabelle/*Circus*

Process	::=	**circusprocess** Tpar* name = PParagraph* **where** Action
PParagraph	::=	AlphabetP | StateP | ChannelP | NamesetP | ChansetP | SchemaP
	|	ActionP
AlphabetP	::=	**alphabet** [vardecl+]
vardecl	::=	name :: type
StateP	::=	**state** [vardecl+]
ChannelP	::=	**channel** [chandecl+]
chandecl	::=	name | name type
NamesetP	::=	**nameset** name = [name+]
ChansetP	::=	**chanset** name = [name+]
SchemaP	::=	**schema** name = SchemaExpression
ActionP	::=	**action** name = Action
Action	::=	**Skip** | **Stop** | Action ; Action | Action □ Action | Action ⊓ Action
	|	Action \ chansetN | var := expr | guard & Action | comm → Action
	|	**Schema** name | ActionName | μ var @ Action | **var** var @ Action
	|	Action ⟦ namesetN | chansetN | namesetN ⟧ Action

Figure 2: Isabelle/*Circus* syntax

The Isabelle/*Circus* environment allows a syntax of processes which is close to the textbook presentations of *Circus* (see Fig. 2). Similar to other specification constructs in Isabelle/HOL, this syntax is "parsed away", *i.e.* compiled into an internal representation of the denotational semantics of *Circus*, which is a formalization in form of a shallow embedding of the (essentially untyped) paper-and-pencil definitions by Oliveira et al. [13], based on UTP. *Circus* actions are defined as CSP healthy reactive processes.

In the UTP representation of reactive processes we have given in a previous paper [8], the process type is generic. It contains two type parameters that represent the channel type and the alphabet of the process. These parameters are very general, and they are instantiated for each specific process. This could be problematic when representing the *Circus* semantics, since some definitions rely directly on variables and channels (e.g assignment and communication). In this section we present our solution to deal with this kind of problems, and our representation of the *Circus* actions and processes.

We now describe the foundation as well as the semantic definition of some process operators of *Circus*. A distinguishing feature of *Circus* processes are explicit state variables which do not exist in other process algebras like, e.g., CSP. These can be:

- *global* state variables, *i.e.* they are declared via alphabetized predicates in the **state** section, or Z-like Δ operations on global states that generate alphabetized relations, or
- *local* state variables, *i.e.* they are result of the variable declaration statement **var** var @ Action. The scope of local variables is restricted to Action.

On both kind of state variables, logical constraints may be expressed.

3.1 Alphabets and Variables

In order to define the set of variables of a specification, the *Circus* semantics considers the alphabet of its components, be it on the level of alphabetized predicates, alphabetized relations or actions. We recall that these items are represented by sets of records or sets of pairs of records. The *alphabet of a process* is defined by extending the basic reactive process alphabet (cf. Section 2.3.2) by its variable names and types. For the example *FIG*, where the global state variable

idS is defined, this is reflected in Isabelle/Circus by the extension of the process alphabet by this variable, i.e. by the extension of the Isabelle/HOL record:

```
record α alpha = α alpha_rp +  idS :: ID set
```

This introduces the record type `alpha` that contains the observational variables of a reactive process, plus the variable `idS`. Note that our *Circus* semantic representation allows "built-in" bindings of alphabets in a typed way. Moreover, there is no restriction on the associated HOL type. However, the inconvenience of this representation is that variables cannot be introduced "on the fly"; they must be known statically i.e. at type inference time. Another consequence is that a "syntactic" operation such as variable renaming has to be expressed as a "semantic" operation that maps one record type into another.

3.1.1 Updating and accessing global variables.

Since the alphabets are represented by HOL records, i.e. a kind binding "*name* $\mapsto$ *value*", we need a certain infrastructure to access data in them and to update them. The Isabelle representation as records gives us already two functions (for each record)"select" and "update". The "select" function returns the value of a given variable name, and the "update" functions updates the value of this variable. Since we may have different HOL types for different variables, a unique definition for select and update cannot be provided. There is an instance of these functions for each variable in the record. The name of the variable is used to distinguish the different instances: for the select function the name is used directly and for the update function the name is used as a prefix e.g. for a variable named "x" the names of the *select* and *update* functions are respectively `x` of type α and `x_update`. Since a variable is characterized essentially by these functions, we define a general type (synonym) called `var` which represents a variable as a pair of its select and update function (in the underlying state σ).

```
types (β, σ) var = "(σ ⇒ β) * ((β ⇒β) ⇒ σ ⇒ σ)"
```

For a given alphabet (record) of type σ, `(β, the type σ)var` represents the type of the variables whose value type is β. One can then extract the select and update functions from a given variable with the following functions:

```
definition select :: "(β, σ) var ⇒σ ⇒ β"
  where select f ≡ (fst f)

definition update :: "(β, σ) var ⇒ β ⇒ σ ⇒ σ"
  where update f v ≡ (snd f) (λ _ . v)
```

Finally, we introduce a function called `VAR` to implement a syntactic translation of a variable name to an entity of type `var`.

```
syntax "_VAR" :: "id ⇒(β, σ) var" ("VAR _")
translations VAR x => (x, _update_ name x)
```

Note that in this syntactic translation rule, `_update_ name x` stands for the concatenation of the string `_update_` with the content of the variable `x`; the resulting `_update_x` in this example is mapped to the field-update function of the extensible record `x_update` by a default mechanism. On this basis, the assignment notation can be written as usual:

```
syntax
  "_assign" :: "id ⇒(σ ⇒β) ⇒(α, σ) action" ("_ ‘:=‘ _")
translations
  "x ‘:=‘ E"  => "CONST ASSIGN (VAR x) E"
```

and mapped to the *semantics* of the program variable `(x,x_update)` together with the universal `ASSIGN` operator defined later on, in Section 3.3.2.

3.1.2 Updating and accessing local variables.

In *Circus*, local program variables can be introduced on the fly, and their scopes are explicitly defined, as can be seen in the *FIG* example. In textbook

Circus, nested scopes are handled by variable renaming which is not possible in our representation due to the implicit representation of variable names. We represent local program variables by global variables, using the `var` type defined above, where selection and update involve an explicit stack discipline. Each variable is mapped to a list of values, and not to one value only (as for state variables). Entering the scope of a variable is just adding a new value as the head of the corresponding values list. Leaving a variable scope is just removing the head of the values list. The select and update functions correspond to selecting and updating the head of the list. This ensures dynamic scoping, as it is stated by the *Circus* semantics.

Note that this encoding scheme requires to make local variables lexically distinct from global variables; local variable instances are just distinguished from the global ones by the stack discipline.

3.2 Synchronization infrastructure: Name sets and channels.

3.2.1 Name sets.

An important notion, used in the definition of parallel *Circus* actions, is name sets as seen in Section 2.3. A name set is a set of variable names, which is a subset of the alphabet. This notion cannot be directly expressed in our representation since variable names are not explicitly represented. Thus its definition relies on the characterization of the variables in our representation. As for variables, name sets are defined by their functional characterization. They are used in the definition of the binding merge function *MSt* below:
$\forall v @ (v \in ns1 \Rightarrow v' = (1.v)) \wedge (v \in ns2 \Rightarrow v' = (2.v)) \wedge (v \notin ns1 \cup ns2 \Rightarrow v' = v)$.

The disjoint name sets $ns1$ and $ns2$ are used to determine which variable values (extracted from local bindings of the parallel components) are used to update the global binding of the process. A name set can be functionally defined as a binding update function, that copies values from a local binding to the global one. For example, a name set *NS* that only contains the variable x can be defined as follows in Isabelle/Circus:

```
definition NS lb gb ≡ x_update (x lb) gb
```

where `lb` and `gb` stands for local and global bindings, `x` and `x_update` are the select and update functions of variable `x`. Then the merge function can be defined by composing the application of the name sets to the global binding.

3.2.2 Channels.

Reactive processes interact with the environment via synchronizations and communications. A synchronization is an interaction via a channel without any exchange of data. A communication is a synchronization with data exchange. In order to reason about communications in the same way, a datatype *channels* is defined using the channels names as constructors. For instance, in:

```
datatype channels = chan1 | chan2 nat | chan3 bool
```

we declare three channels: `chan1` that synchronizes without data , `chan2` that communicates natural values and `chan3` that exchanges boolean values.

This definition makes it possible to reason globally about communications since they have the same type. However, the channels may not have the same type: in the example above, the types of `chan1`, `chan2` and `chan3` are respectively `channels`, `nat` $\Rightarrow$ `channels` and `bool` $\Rightarrow$ `channels`. In the definition of some *Circus* operators, we need to compare two channels, and one can't compare for example `chan1` with `chan2` since they don't have the same type. A solution would be to compare `chan1` with (`chan2 v`). The types are equivalent in this case, but the problem remains because comparing (`chan2 0`) to (`chan2 1`) will state inequality just because the communicated values are not equal. We could define an inductive function over the datatype `channels` to compare channels, but this is only possible when all the channels are known *a priori*.

Thus, we add some constraint to the generic channels type: we require the `channels` type to implement a function `chan_eq` that tests the equality of two channels. Fortunately, Isabelle/HOL provides a construct for this kind of restriction: the type classes (sorts) mentioned in Section 2.1. We define a type class (interface) `chan_eq` that contains a signature of the `chan_eq` function.

```
class chan_eq =
  fixes chan_eq :: "α ⇒α ⇒ bool"
begin end
```

Concrete channels type must implement the interface (class) " `chan_eq`" that can be easily defined for this concrete type. Moreover, one can use this class to add some definition that depends on the channel equivalence function. For example, a trace equivalence function can be defined as follows:

```
fun tr_eq where
  tr_eq [] [] = True | tr_eq xs [] = False | tr_eq [] ys = False
| tr_eq (x#xs) (y#ys) = if chan_eq x y then tr_eq xs ys else False
```

It is applicable to traces of elements whose type belongs to the sort `chan_eq`.

3.3 Actions and Processes

The *Circus* actions type is defined as the set of all the CSP healthy reactive processes. The type `(`α`,`σ`)relation_rp` is the reactive process type where α is of `channels` type and σ is a record extensions of `action_rp`, *i.e.* the global state variables. On this basis, we can encode the concept of a process for a family of possible state instances. We introduce below the vital type **action**:

```
typedef(Action)
 (α::chan_eq,σ) action = {p::(α,σ)relation_rp. is_CSP_process p}
proof - {...}
qed
```

As mentioned before, a type-definition introduces a new type by stating a set. In our case it is the set of reactive processes that satisfy the healthiness-conditions for CSP-processes, isomorphic to the new type.

Technically, this construct introduces two constants definitions `Abs_Action` and `Rep_Action` respectively of type `(α,σ) relation_rp ⇒(α,σ) action` and `(α,σ)action ⇒(α,σ)relation_rp` as well as the usual two axioms expressing the bijection `Abs_Action(Rep_Action(X))=X` and `is_CSP_process p ⟹Rep_Action(Abs_Action(p))=p` where `is_CSP_process` captures the healthiness conditions.

Every *Circus* action is an abstraction of an alphabetized predicate. In [9], we introduce the definitions of all the actions and operators using their denotational semantics. The environment contains, for each action, the proof that this predicate is CSP healthy.

In this section, we present some of the important definitions, namely: basic actions, assignments, communications, hiding, and recursion.

3.3.1 Basic actions.

`Stop` is defined as a reactive design, with a precondition `true` and a postcondition stating that the system deadlocks and the traces are not evolving.

```
definition
Stop ≡  Abs_Action (R (true ⊢λ(A, A'). tr A' = tr A ∧wait A'))
```

`Skip` is defined as a reactive design, with a precondition *true* and a postcondition stating that the system terminates and all the state variables are not changed. We represent this fact by stating that the `more` field (seen in Section 2.2) is not changed, since this field is mapped to all the state variables. Note that using the `more`-field is a tribute to our encoding of alphabets by extensible records and stands for all future extensions of the alphabet (e.g. state variables).

```
definition Skip ≡  Abs_Action (R (true ⊢ λ (A, A'). tr A' = tr A
                                    ∧ ¬ wait A' ∧ more A = more A'))
```

3.3.2 The universal assignment action.

In Section 3.1.1, we described how global and local variables are represented by access- and updates functions introduced by fields in extensible records.

In these terms, the "lifting" to the assignment action in *Circus* processes is straightforward:

```
definition
  ASSIGN::"(β, σ) var ⇒(σ ⇒ β) ⇒(α::ev_eq, σ) action"
where
  ASSIGN x e ≡ Abs_Action (R (true ⊢ Y))
where
 Y = λ(A, A'). tr A' = tr A ∧ ¬ wait A' ∧
                 more A' = (assign x (e (more A))) (more A)
```

where `assign` is the projection into the update operation of a semantic variable described in section 3.1.1.

3.3.3 Communications.

The definition of prefixed actions is based on the definition of a special relation `do_I`. In the *Circus* denotational semantics [13], various forms of prefixing were defined. In our theory, we define one general form, and the other forms are defined as special cases.

```
definition do_I c x P ≡  X  ◁ wait o fst ▷  Y
where
X = (λ (A, A'). tr A = tr A' ∧ ((c ‘ P) ∩ ref A') = {})
and
Y = (λ (A, A'). hd ((tr A') - (tr A)) ∈ (c ‘ P) ∧
     (c (select x (more A))) = (last (tr A')))
```

where `c` is a channel constructor, `x` is a variable (of `var` type) and `P` is a predicate. The `do_I` relation gives the semantics of an interaction: if the system is ready to interact, the trace is unchanged and the waiting channel is not refused. After performing the interaction, the new event in the trace corresponds to this interaction.

The semantics of the whole action is given by the following definition:

```
definition Prefix c x P S ≡Abs_Action(R (true ⊢ Y)) ; S
where
Y =  do_I c x P ∧  (λ (A, A'). more A' = more A)
```

where `c` is a channel constructor, `x` is a variable (of type var), `P` is a predicate and `S` is an action. This definition states that the prefixed action semantics is given by the interaction semantics (`do_I`) sequentially composed with the semantics of the continuation (action `S`).

Different types of communication are considered:

- Inputs: the communication is done over a variable.
- Constrained Inputs: the input variable value is constrained with a predicate.

- Outputs: the communications exchanges only one value.
- Synchronizations: only the channel name is considered (no data).

The semantics of these different forms of communications is based on the general definition above.

```
definition read c x P ≡ Prefix c x true P
definition write1 c a P ≡ Prefix c (λs. a s, (λ x. λy. y)) true P
definition write0 c P ≡ Prefix (λ_.c) (λ_._, (λ x. λy. y)) true P
```

where `read`, `write1` and `write0` respectively correspond to inputs, outputs and synchronization. Constrained inputs correspond to the general definition.

We configure the Isabelle syntax-engine such that it parses the usual communication primitives and gives the corresponding semantics:

```
translations
  c ? p →P       == CONST read c (VAR p) P
  c ? p : b →P   == CONST Prefix c (VAR p) b P
  c ! p →P       == CONST write1 c p P
  a → P          == CONST write0 (TYPE(_)) a P
```

3.3.4 Hiding.

The hiding operator is interesting because it depends on a channel set. This operator `P \ cs` is used to encapsulate the events that are in the channel set `cs`. These events become no longer visible from the environment. The semantics of the hiding operator is given by the following reactive process:

```
definition
Hide ::"[(α, σ) action , α set] ⇒(α, σ) action" (infixl "\")
where
P \ cs ≡ Abs_Action( R(λ (A, A').
          ∃ s. (Rep_Action P)(A, A'⦇tr :=s, ref := (ref A') ∪ cs⦈)
              ∧ (tr A' - tr A) = (tr_filter (s - tr A) cs))); Skip
```

The definition uses a filtering function `tr_filter` that removes from a trace the events whose channels belong to a given set. The definition of this function is based on the function `chan_eq` we defined in the class `chan_eq`. This explains the presence of the constraint on the type of the action channels in the hiding definition, and in the definition of the filtering function below:

```
fun tr_filter::"a::chan_eq list ⇒a set ⇒a list" where
  tr_filter [] cs = []
| tr_filter (x#xs) cs = (if (¬ chan-in_set x cs)
                            then (x#(tr_filter xs cs))
                              else (tr_filter xs cs))
```

where the `chan-in_set` function checks if a given channel belongs to a channel set using `chan_eq` as equality function.

3.3.5 Recursion.

To represent the recursion operator "μ" over actions, we use the universal least fix-point operator "*lfp*" defined in the HOL library for lattices and we follow again [13]. The use of least fix-points in [13] is the most substantial deviation from the standard CSP denotational semantics, which requires Scott-domains and complete partial orderings. The operator *lfp* is inherited from the "*Complete Lattice class*" under some conditions, and all theorems defined over this operator can be reused. In order to reuse this operator, we have to show that the least-fixpoint over functionals that enrich pairs of failure - and divergence trace sets monotonely, produces an **`action`** that satisfies the CSP healthiness conditions. This consistency proof for the recursion operator is the largest contained in the Isabelle/*Circus* library.

Therefore, we must prove that the *Circus* actions type defines a complete lattice. This leads to prove that the actions type belongs to the HOL "*Complete Lattice class*". Since type classes in HOL are hierarchic, the proof is in three steps: first, a proof that the *Circus* actions type forms a lattice by instantiating the HOL "*Lattice class*"; second, a proof that actions type instantiates a subclass of lattices called "*Bounded Lattice class*"; third, proof of the instantiation from the "*Complete Lattice class*". More on these proofs can be found in [9].

3.3.6 *Circus* Processes.

A *Circus* process is defined in our environment as a local theory by introducing qualified names for all its components. This is very similar to the notion of *namespaces* popular in programming languages. Defining a *Circus* process locally makes it possible to encapsulate definitions of alphabet, channels, schema expressions and actions in the same namespace. It is important for the foundation of Isabelle/*Circus* to avoid the ambiguity between local process entities definitions (e.g. `FIG.Out` and `DFIG.Out` in the example of Section 4).

4 Using Isabelle/*Circus*

We describe the front-end interface of Isabelle/*Circus*. In order to support a maximum of common *Circus* syntactic look-and-feel, we have programmed at the SML level of Isabelle a compiler that parses and (partially) pretty prints *Circus* process given in the syntax presented in Figure 2.

4.1 Writing specifications

A specification is a sequence of paragraphs. Each paragraph may be a declaration of alphabet, state, channels, name sets, channel sets, schema expressions or actions. The main action is introduced by the keyword **where**. Below, we illustrate how to use the environment to write a *Circus* specification using the `FIG` process example presented in Figure 1.

```
circusprocess FIG =
  alphabet = [v::nat, x::nat]
  state = [idS::nat set]
  channel = [req, ret nat, out nat]
  schema Init = idS := {}
  schema Out = ∃a. v' = a ∧ v' ∉ idS ∧ idS' = idS ∪{v'}
  schema Remove = x ∉ idS ∧ idS' = idS - {x}
  where var v · Schema Init; (μ X ·(req →Schema Out; out!v →Skip)
                                    □ (ret?x →Schema Remove); X)
```

Each line of the specification is translated into the corresponding semantic operator given in Section 3.3. We describe below the result of executing each command of `FIG`:

- the compiler introduces a scope of local components whose names are qualified by the process name (`FIG` in the example).
- **alphabet** generates a list of record fields to represent the binding. These fields map names to value lists.
- **state** generates a list of record fields that corresponds to the state variables. The names are mapped to single values. This command, together with **alphabet** command, generates a record that represents all the variables (for the `FIG` example the command generates the record `FIG_alphabet`, that contains the fields `v` and `x` of type `nat list` and the field `idS` of type `nat set`).
- **channel** introduces a datatype of typed communication channels (for the `FIG` example the command generates the datatype `FIG_channels` that contains the constructors `req` without communicated value and `ret` and `out` that communicate natural values).
- **schema** allows the definition of schema expressions represented as an alphabetized relation over the process variables (in the example the schema expressions `FIG.Init`, `FIG.Out` and `FIG.Remove` are generated).
- **action** introduces definitions for *Circus* actions in the process. These definitions are based on the denotational semantics of *Circus* actions.

The type parameters of the action type are instantiated with the locally defined channels and alphabet types.

- **where** introduces the main action as in **action** command (in the example the main action is `FIG.FIG` of type `(FIG_channels, FIG_alphabet)action`).

4.2 Relational and Functional Refinement in Circus

The main goal of Isabelle/*Circus* is to provide a proof environment for *Circus* processes. The "shallow-embedding" of *Circus* and UTP in Isabelle/HOL offers the possibility to reuse proof procedures, infrastructure and theorem libraries already existing in Isabelle/HOL. Moreover, once a process specification is encoded and parsed in Isabelle/*Circus*, proofs of, e. g., refinement properties can be developped using the ISAR language for structured proofs.

To show in more details how to use Isabelle/*Circus*, we provide a small example of action refinement proof. The refinement relation is defined as the universal reverse implication in the UTP. In *Circus*, it is defined as follows:

```
definition A1 ⊑c A2 ≡(Rep_Action A1) ⊑utp (Rep_Action A2)
```

where A1 and A2 are *Circus* actions, `⊑c` and `⊑utp` stands respectively for refinement relation on *Circus* actions and on UTP predicate.

This definition assumes that the actions A1 and A2 share the same alphabet (binding) and the same channels. In general, refinement involves an important data evolution and growth. The data refinement is defined in [16, 5] by backwards and forwards simulations. In this paper, we restrict ourselves to a special case, the so-called *functional* backwards simulation. This refers to the fact that the abstraction relation `R` that relates concrete and abstract actions is just a function:

```
definition Simulation ("_ ⪯_ _") where
 A1 ⪯R A2 = ∀a b.(Rep_Action A2)(a,b) ⟶(Rep_Action A1)(R a,R b)
```

where `A1` and `A2` are *Circus* actions and `R` is a function mapping the corresponding `A1` alphabet to the `A2` alphabet.

4.3 Refinement Proofs

We can use the definition of simulation to transform the proof of refinement to a simple proof of implication by unfolding the operators in terms of their underlying relational semantics. The problem with this approach is that the size of proofs will grow exponentially with the size of the processes. To avoid this problem, some general refinement laws were defined in [5] to deal with the refinement of *Circus* actions at operators level and not at UTP level. We introduced and proved a subset of theses laws in our environment (see Table 1).

$$\frac{P \preceq_S Q \qquad P' \preceq_S Q'}{P;\ P' \preceq_S Q;\ Q'}\ \text{SeqI} \qquad \frac{P \preceq_S Q \qquad g_1 \simeq_S g_2}{g_1 \& P \preceq_S g_2 \& Q}\ \text{GrdI}$$

$$\frac{P \preceq_S Q \qquad x \sim_S y}{var\ x \bullet P \preceq_S var\ y \bullet Q}\ \text{VarI} \qquad \frac{P \preceq_S Q \qquad x \sim_S y}{c?x \rightarrow P \preceq_S c?y \rightarrow Q}\ \text{InpI}$$

$$\frac{P \preceq_S Q \qquad P' \preceq_S Q'}{P \sqcap P' \preceq_S Q \sqcap Q'}\ \text{NdetI} \qquad \frac{P \preceq_S Q \qquad x \sim_S y}{c!x \rightarrow P \preceq_S c!y \rightarrow Q}\ \text{OutI}$$

$$\frac{\begin{matrix}[X \preceq_S Y] \\ \vdots \\ P\ X \preceq_S Q\ Y\end{matrix} \qquad mono\ P \qquad mono\ Q}{\mu\ X \bullet P\ X \preceq_S \mu\ Y \bullet Q\ Y}\ \text{MuI} \qquad \frac{P \preceq_S Q \qquad P' \preceq_S Q'}{P \Box P' \preceq_S Q \Box Q'}\ \text{DetI}$$

$$\frac{\begin{matrix}[Pre\ sc_1\ (S\ A)] \\ \vdots \\ Pre\ sc_2\ A\end{matrix} \qquad \begin{matrix}[Pre\ sc_1\ (S\ A) \quad sc_2\ (A, A')] \\ \vdots \\ sc_1\ (S\ A, S\ A')\end{matrix}}{schema\ sc_1 \preceq_S schema\ sc_2}\ \text{SchI} \qquad \frac{P \preceq_S Q}{a \rightarrow P \preceq_S a \rightarrow Q}\ \text{SyncI}$$

$$\frac{P \preceq_S Q \quad P' \preceq_S Q' \quad ns_1 \sim_S ns_1' \quad ns_2 \sim_S ns_2'}{P[\![ns_1 \mid cs \mid ns_2]\!]P' \preceq_S Q[\![ns_1' \mid cs \mid ns_2']\!]Q'}\ \text{ParI} \qquad \frac{}{Skip \preceq_S Skip}\ \text{SkipI}$$

Table 1: Proved refinement laws

In Table 1, the relations "$x \sim_S y$" and "$g_1 \simeq_S g_2$" record the fact that the variable x (repectively the guard g_1) is refined by the variable y (repectively by the guard g_2) w.r.t the simulation function S.

These laws can be used in complex refinement proofs to simplify them at the *Circus* level. More rules can be defined and proved to deal with more complicated statements like combination of operators for example. Using these laws, and exploiting the advantages of a shallow embedding, the automated proof of refinement becomes surprisingly simple.

Coming back to our example, let us consider the `DFIG` specification below, where the management of the identifiers via the set `idS` is refined into a set of removed identifiers `retidS` and a number `max`, which is the rank of the last issued identifier.

```
circusprocess DFIG =
  alphabet = [w::nat, y::nat]
  state = [retidS::nat set, max::nat]
  schema Init = retidS' = {} ∧max' = 0
  schema Out = w' = max ∧ max' = max+1 ∧ retidS' = retidS - {max}
  schema Remove = y < max ∧ y ∉ retidS ∧  retidS' = retidS ∪ {y}
                    ∧ max' = max
  where var w · Schema Init; (μ X ·(req →Schema Out; out!w →Skip)
                                 □ (ret?y →Schema Remove); X)
```

We provide the proof of refinement of `FIG` by `DFIG` just instantiating the simulation function `R` by the following abstraction function, that maps the underlying concrete states to abstract states:

```
definition Sim A = FIG_alphabet.make (w A) (y A)
                                      ({a. a < (max A) ∧ a ∉ (retidS A)})
```

where A is the alphabet of `DFIG`, and `FIG_alphabet.make` yields an alphabet of type `FIG_Alphabet` initializing the values of `v`, `x` and `idS` by their corresponding values from `DFIG_alphabet`: `w`, `y` and `{a. a < max ∧ a ∉ retidS}`).

To prove that `DFIG` is a refinement of `FIG` one must prove that the main action `DFIG.DFIG` refines the main action `FIG.FIG`. The definition is then simplified, and the refinement laws are applied to simplify the proof goal. Thus, the full proof consists of a few lines in ISAR:

```
theorem "FIG.FIG ≼Sim DFIG.DFIG"
  apply (auto simp: DFIG.DFIG_def FIG.FIG_def mono_Seq
              intro!: VarI SeqI MuI DetI SyncI InpI OutI SkipI)
  apply (simp_all add: SimRemove SimOut SimInit Sim_def)
done
```

First, the definitions of `FIG.FIG` and `DFIG.DFIG` are simplified and the defined refinement laws are used by the `auto` tactic as introduction rules. The second step replaces the definition of the simulation function and uses some proved lemmas to finish the proof. The three lemmas used in this proof: `SimInit`, `SimOut` and `SimRemove` give proofs of simulation for the schema `Init`, `Out` and `Remove`.

5 Conclusions

We have shown for the language *Circus*, which combines data-oriented modeling in the style of Z and behavioral modeling in the style of CSP, a semantics in form of a shallow embedding in Isabelle/HOL. In particular, by representing the somewhat non-standard concept of the *alphabet* in UTP in form of extensible records in HOL, we achieved a fairly compact, typed presentation of the language. In contrast to previous work based on some deep embedding [19], this shallow embedding allows arbitrary (higher-order) HOL-types for channels, events, and state-variables, such as, e.g., sets of relations etc. Besides, systematic renaming of local variables is avoided by compiling them essentially to global variables using a stack of variable instances. The necessary proofs for showing that the definitions are consistent — *i.e.* satisfy altogether `is_CSP_healthy` — have been done, together with a number of algebraic simplification laws on *Circus* processes.

Since the encoding effort can be hidden behind the scene by flexible extension mechanisms of the Isabelle, it is possible to have a compact notation for both specifications and proofs. Moreover, existing standard tactics of Isabelle such as `auto`, `simp` and `metis` can be reused since our *Circus* semantics is representationally close to HOL. Thus, we provide an environment that can cope with combined refinements concerning data and behavior. Finally, we demonstrate its power — w.r.t. both expressivity and proof automation — with a small, but prototypic example of a process-refinement.

In the future, we intend to use Isabelle/*Circus* for the generation of test-cases, on the basis of [4], using the HOL-TestGen-environment [2].

6 Acknowledgement

We warmly thank Markarius Wenzel for his valuable help with the Isabelle framework. Furthermore, we are greatly indebted to Ana Cavalcanti for her comments on the semantic foundation of this work.

7 UTP variables

theory *Var*
imports *Main*
begin

UTP variables are characterized by two functions, *select* and *update*. The variable type is then defined as a tuple (*select* * *update*).

type-synonym $('a, 'r)\ var = ('r \Rightarrow 'a) * (('a \Rightarrow 'a) \Rightarrow 'r \Rightarrow 'r)$

The *lookup* function returns the corrsponding *select* function of a variable.

definition $lookup :: ('a, 'r)\ var \Rightarrow 'r \Rightarrow 'a$
  where $lookup\ f \equiv (fst\ f)$

The *assign* function uses the *update* function of a variable to update its value.

definition $assign :: ('a, 'r)\ var \Rightarrow 'a \Rightarrow 'r \Rightarrow 'r$
  where $assign\ f\ v \equiv (snd\ f)\ (\lambda\ \text{-}\ .\ v)$

The *VAR* function allows to retrieve a variable given its name.

syntax -*VAR* :: *id* ⇒ ('*a*, '*r*) *var* (‹*VAR* -›)
translations *VAR x* => (*x*, -*update-name x*)

end

8 Predicates and relations

theory *Relations*
imports *Var*
begin
default-sort *type*

Unifying Theories of Programming (UTP) is a semantic framework based on an alphabetized relational calculus. An alphabetized predicate is a pair (alphabet, predicate) where the free variables appearing in the predicate are all in the alphabet.

An alphabetized relation is an alphabetized predicate where the alphabet is composed of input (undecorated) and output (dashed) variables. In this case the predicate describes a relation between input and output variables.

8.1 Definitions

In this section, the definitions of predicates, relations and standard operators are given.

type-synonym $'\alpha\ alphabet = '\alpha$

type-synonym $'\alpha$ *predicate* = $'\alpha$ *alphabet* ⇒ *bool*

definition *true*::$'\alpha$ *predicate*
where *true* ≡ λ*A*. *True*

definition *false*::$'\alpha$ *predicate*
where *false* ≡ λ*A*. *False*

definition *not*::$'\alpha$ *predicate* ⇒ $'\alpha$ *predicate* (‹¬ -› [*40*] *40*)
where ¬ *P* ≡ λ*A*. ¬ (*P A*)

definition *conj*::$'\alpha$ *predicate* ⇒ $'\alpha$ *predicate* ⇒ $'\alpha$ *predicate* (**infixr** ‹∧› *35*)
where *P* ∧ *Q* ≡ λ*A*. *P A* ∧ *Q A*

definition *disj*::$'\alpha$ *predicate* ⇒ $'\alpha$ *predicate* ⇒ $'\alpha$ *predicate* (**infixr** ‹∨› *30*)
where *P* ∨ *Q* ≡ λ*A*. *P A* ∨ *Q A*

definition *impl*::$'\alpha$ *predicate* ⇒ $'\alpha$ *predicate* ⇒ $'\alpha$ *predicate* (**infixr** ‹⟶› *25*)
where *P* ⟶ *Q* ≡ λ*A*. *P A* ⟶ *Q A*

definition *iff*::$'\alpha$ *predicate* ⇒ $'\alpha$ *predicate* ⇒ $'\alpha$ *predicate* (**infixr** ‹⟷› *25*)
where *P* ⟷ *Q* ≡ λ*A*. *P A* ⟷ *Q A*

definition *ex*::[$'\beta$ ⇒$'\alpha$ *predicate*] ⇒ $'\alpha$ *predicate* (**binder** ‹∃ › *10*)
where ∃ *x*. *P x* ≡ λ*A*. ∃ *x*. (*P x*) *A*

definition *all*::[$'\beta$ ⇒$'\alpha$ *predicate*] ⇒ $'\alpha$ *predicate* (**binder** ‹∀ › *10*)
where ∀ *x*. *P x* ≡ λ *A*. ∀ *x*. (*P x*) *A*

type-synonym $'\alpha$ *condition* = ($'\alpha$ × $'\alpha$) ⇒ *bool*
type-synonym $'\alpha$ *relation* = ($'\alpha$ × $'\alpha$) ⇒ *bool*

definition *cond*::$'\alpha$ *relation* ⇒ $'\alpha$ *condition* ⇒ $'\alpha$ *relation* ⇒ $'\alpha$ *relation*
(‹(*3*- ◃ - ▹ / -)› [*14*,*0*,*15*] *14*)
where (*P* ◃ *b* ▹ *Q*) ≡ (*b* ∧ *P*) ∨ ((¬ *b*) ∧ *Q*)

definition *comp*::(($'\alpha$ × $'\beta$) ⇒ *bool*) ⇒ (($'\beta$ × $'\gamma$) ⇒ *bool*) ⇒ ($'\alpha$ × $'\gamma$) ⇒ *bool*
(**infixr** ‹; ; › *25*)
where *P* ; ; *Q* ≡ λ*r*. *r* : ({*p*. *P p*} *O* {*q*. *Q q*})

definition *Assign*::($'a$, $'b$) *var* ⇒ $'a$ ⇒ $'b$ *relation*
where *Assign x a* ≡ λ(*A*, *A'*). *A'* = (*assign x a*) *A*

syntax
-assignment :: *id* ⇒ $'a$ ⇒ $'b$ *relation* (‹- :== -›)
translations
y :== *vv* => *CONST Assign* (*VAR y*) *vv*

abbreviation (*input*) *closure*::$'\alpha$ *predicate* ⇒ *bool* (‹[-]›)

where $[\ P\ ] \equiv \forall\ A.\ P\ A$

abbreviation (*input*) *ndet*::'α *relation* ⇒ 'α *relation* ⇒ 'α *relation* (‹(- ⊓ -)›)
where $P \sqcap Q \equiv P \lor Q$

abbreviation (*input*) *join*::'α *relation* ⇒ 'α *relation* ⇒ 'α *relation* (‹(- ⊔ -)›)
where $P \sqcup Q \equiv P \land Q$

abbreviation (*input*) *ndetS*::'α *relation set* ⇒ 'α *relation* (‹(⨅ -)›)
where $\bigsqcap\ S \equiv \lambda A.\ A \in \bigcup\{\{p.\ P\ p\} \mid P.\ P \in S\}$

abbreviation (*input*) *conjS*::'α *relation set* ⇒ 'α *relation* (‹(⨆ -)›)
where $\bigsqcup\ S \equiv \lambda A.\ A \in \bigcap\{\{p.\ P\ p\} \mid P.\ P \in S\}$

abbreviation (*input*) *skip-r*::'α *relation* (‹Π*r*›)
where $\Pi r \equiv \lambda\ (A,\ A')\ .\ A = A'$

abbreviation (*input*) *Bot*::'α *relation*
where *Bot* ≡ *true*

abbreviation (*input*) *Top*::'α *relation*
where *Top* ≡ *false*

lemmas *utp-defs* = *true-def false-def conj-def disj-def not-def impl-def iff-def ex-def all-def cond-def comp-def Assign-def*

8.2 Proofs

All useful proved lemmas over predicates and relations are presented here. First, we introduce the most important lemmas that will be used by automatic tools to simplify proofs. In the second part, other lemmas are proved using these basic ones.

8.2.1 Setup of automated tools

lemma *true-intro*: *true x* **by** (*simp add*: *utp-defs*)
lemma *false-elim*: *false x* ⟹ *C* **by** (*simp add*: *utp-defs*)
lemma *true-elim*: *true x* ⟹ *C* ⟹ *C* **by** (*simp add*: *utp-defs*)

lemma *not-intro*: (*P x* ⟹ *false x*) ⟹ (¬ *P*) *x* **by** (*auto simp add*: *utp-defs*)
lemma *not-elim*: (¬ *P*) *x* ⟹ *P x* ⟹ *C* **by** (*auto simp add*: *utp-defs*)
lemma *not-dest*: (¬ *P*) *x* ⟹ ¬ *P x* **by** (*auto simp add*: *utp-defs*)

lemma *conj-intro*: *P x* ⟹ *Q x* ⟹ (*P* ∧ *Q*) *x* **by** (*auto simp add*: *utp-defs*)
lemma *conj-elim*: (*P* ∧ *Q*) *x* ⟹ (*P x* ⟹ *Q x* ⟹ *C*) ⟹ *C* **by** (*auto simp add*: *utp-defs*)

lemma *disj-introC*: (¬ *Q x* ⟹ *P x*) ⟹ (*P* ∨ *Q*) *x* **by** (*auto simp add*: *utp-defs*)

lemma *disj-elim*: $(P \lor Q)\ x \Longrightarrow (P\ x \Longrightarrow C) \Longrightarrow (Q\ x \Longrightarrow C) \Longrightarrow C$ **by** (*auto simp add*: *utp-defs*)

lemma *impl-intro*: $(P\ x \Longrightarrow Q\ x) \Longrightarrow (P \longrightarrow Q)\ x$ **by** (*auto simp add*: *utp-defs*)
lemma *impl-elimC*: $(P \longrightarrow Q)\ x \Longrightarrow (\neg\ P\ x \Longrightarrow R) \Longrightarrow (Q\ x \Longrightarrow R) \Longrightarrow R$ **by** (*auto simp add*: *utp-defs*)

lemma *iff-intro*: $(P\ x \Longrightarrow Q\ x) \Longrightarrow (Q\ x \Longrightarrow P\ x) \Longrightarrow (P \longleftrightarrow Q)\ x$ **by** (*auto simp add*: *utp-defs*)
lemma *iff-elimC*:
$(P \longleftrightarrow Q)\ x \Longrightarrow (P\ x \Longrightarrow Q\ x \Longrightarrow R) \Longrightarrow (\neg\ P\ x \Longrightarrow \neg\ Q\ x \Longrightarrow R) \Longrightarrow R$ **by** (*auto simp add*: *utp-defs*)

lemma *all-intro*: $(\bigwedge a.\ P\ a\ x) \Longrightarrow (\forall\ a.\ P\ a)\ x$ **by** (*auto simp add*: *utp-defs*)
lemma *all-elim*: $(\forall\ a.\ P\ a)\ x \Longrightarrow (P\ a\ x \Longrightarrow R) \Longrightarrow R$ **by** (*auto simp add*: *utp-defs*)

lemma *ex-intro*: $P\ a\ x \Longrightarrow (\exists\ a.\ P\ a)\ x$ **by** (*auto simp add*: *utp-defs*)
lemma *ex-elim*: $(\exists\ a.\ P\ a)\ x \Longrightarrow (\bigwedge a.\ P\ a\ x \Longrightarrow Q) \Longrightarrow Q$ **by** (*auto simp add*: *utp-defs*)

lemma *comp-intro*: $P\ (a,\ b) \Longrightarrow Q\ (b,\ c) \Longrightarrow (P\ ;;\ Q)\ (a,\ c)$
by (*auto simp add*: *comp-def*)

lemma *comp-elim*:
$(P\ ;;\ Q)\ ac \Longrightarrow (\bigwedge a\ b\ c.\ ac = (a,\ c) \Longrightarrow P\ (a,\ b) \Longrightarrow Q\ (b,\ c) \Longrightarrow C) \Longrightarrow C$
by (*auto simp add*: *comp-def*)

declare *not-def* [*simp*]

declare *iff-intro* [*intro*!]
and *not-intro* [*intro*!]
and *impl-intro* [*intro*!]
and *disj-introC* [*intro*!]
and *conj-intro* [*intro*!]
and *true-intro* [*intro*!]
and *comp-intro* [*intro*]

declare *not-dest* [*dest*!]
and *iff-elimC* [*elim*!]
and *false-elim* [*elim*!]
and *impl-elimC* [*elim*!]
and *disj-elim* [*elim*!]
and *conj-elim* [*elim*!]
and *comp-elim* [*elim*!]
and *true-elim* [*elim*!]

declare *all-intro* [*intro*!] **and** *ex-intro* [*intro*]
declare *ex-elim* [*elim*!] **and** *all-elim* [*elim*]

lemmas *relation-rules* = *iff-intro not-intro impl-intro disj-introC conj-intro true-intro comp-intro not-dest iff-elimC false-elim impl-elimC all-elim disj-elim conj-elim comp-elim all-intro ex-intro ex-elim*

lemma *split-cond*:
$A\ ((P \triangleleft b \triangleright Q)\ x) = ((b\ x \longrightarrow A\ (P\ x)) \wedge (\neg\ b\ x \longrightarrow A\ (Q\ x)))$
by (*cases b x*) (*auto simp add*: *utp-defs*)

lemma *split-cond-asm*:
$A\ ((P \triangleleft b \triangleright Q)\ x) = (\neg\ ((b\ x \wedge \neg\ A\ (P\ x)) \vee (\neg\ b\ x \wedge \neg\ A\ (Q\ x))))$
by (*cases b x*) (*auto simp add*: *utp-defs*)

lemmas *cond-splits* = *split-cond split-cond-asm*

8.2.2 Misc lemmas

lemma *cond-idem*:$(P \triangleleft b \triangleright P) = P$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-symm*:$(P \triangleleft b \triangleright Q) = (Q \triangleleft \neg\ b \triangleright P)$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-assoc*: $((P \triangleleft b \triangleright Q) \triangleleft c \triangleright R) = (P \triangleleft b \wedge c \triangleright (Q \triangleleft c \triangleright R))$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-distr*: $(P \triangleleft b \triangleright (Q \triangleleft c \triangleright R)) = ((P \triangleleft b \triangleright Q) \triangleleft c \triangleright (P \triangleleft b \triangleright R))$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-unit-T*:$(P \triangleleft \mathit{true} \triangleright Q) = P$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-unit-F*:$(P \triangleleft \mathit{false} \triangleright Q) = Q$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-L6*: $(P \triangleleft b \triangleright (Q \triangleleft b \triangleright R)) = (P \triangleleft b \triangleright R)$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-L7*: $(P \triangleleft b \triangleright (P \triangleleft c \triangleright Q)) = (P \triangleleft b \vee c \triangleright Q)$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-and-distr*: $((P \wedge Q) \triangleleft b \triangleright (R \wedge S)) = ((P \triangleleft b \triangleright R) \wedge (Q \triangleleft b \triangleright S))$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-or-distr*: $((P \vee Q) \triangleleft b \triangleright (R \vee S)) = ((P \triangleleft b \triangleright R) \vee (Q \triangleleft b \triangleright S))$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-imp-distr*:

$((P \longrightarrow Q) \triangleleft b \triangleright (R \longrightarrow S)) = ((P \triangleleft b \triangleright R) \longrightarrow (Q \triangleleft b \triangleright S))$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-eq-distr*:
$((P \longleftrightarrow Q) \triangleleft b \triangleright (R \longleftrightarrow S)) = ((P \triangleleft b \triangleright R) \longleftrightarrow (Q \triangleleft b \triangleright S))$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *comp-assoc*: $(P ;; (Q ;; R)) = ((P ;; Q) ;; R)$
by (*rule ext*) *blast*

lemma *conj-comp*:
$(\bigwedge a\ b\ c.\ P\ (a, b) = P\ (a, c)) \Longrightarrow (P \wedge (Q ;; R)) = ((P \wedge Q) ;; R)$
by (*rule ext*) *blast*

lemma *comp-cond-left-distr*:
assumes $\bigwedge x\ y\ z.\ b\ (x, y) = b\ (x, z)$
shows $((P \triangleleft b \triangleright Q) ;; R) = ((P ;; R) \triangleleft b \triangleright (Q ;; R))$
using *assms* **by** (*auto simp*: *fun-eq-iff utp-defs*)

lemma *ndet-symm*: $(P::'a\ relation) \sqcap Q = Q \sqcap P$
by (*rule ext*) *blast*

lemma *ndet-assoc*: $P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap R$
by (*rule ext*) *blast*

lemma *ndet-idemp*: $P \sqcap P = P$
by (*rule ext*) *blast*

lemma *ndet-distr*: $P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap (P \sqcap R)$
by (*rule ext*) *blast*

lemma *cond-ndet-distr*: $(P \triangleleft b \triangleright (Q \sqcap R)) = ((P \triangleleft b \triangleright Q) \sqcap (P \triangleleft b \triangleright R))$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *ndet-cond-distr*: $(P \sqcap (Q \triangleleft b \triangleright R)) = ((P \sqcap Q) \triangleleft b \triangleright (P \sqcap R))$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *comp-ndet-l-distr*: $((P \sqcap Q) ;; R) = ((P ;; R) \sqcap (Q ;; R))$
by (*auto simp*: *fun-eq-iff utp-defs*)

lemma *comp-ndet-r-distr*: $(P ;; (Q \sqcap R)) = ((P ;; Q) \sqcap (P ;; R))$
by (*auto simp*: *fun-eq-iff utp-defs*)

lemma *l2-5-1-A*: $\forall X \in S.\ [X \longrightarrow (\bigsqcap S)]$
by *blast*

lemma *l2-5-1-B*: $(\forall X \in S.\ [X \longrightarrow P]) \longrightarrow [(\bigsqcap S) \longrightarrow P]$
by *blast*

lemma *l2-5-1*: $[(\bigsqcap\ S) \longrightarrow P] \longleftrightarrow (\forall\ X \in S.\ [X \longrightarrow P])$
  by *blast*

lemma *empty-disj*: $\bigsqcap\ \{\} = Top$
  by (*rule ext*) *blast*

lemma *l2-5-1-2*: $[P \longrightarrow (\bigsqcup\ S)] \longleftrightarrow (\forall\ X \in S.\ [P \longrightarrow X])$
  by *blast*

lemma *empty-conj*: $\bigsqcup\ \{\} = Bot$
  by (*rule ext*) *blast*

lemma *l2-5-2*: $((\bigsqcup\ S) \sqcap Q) = (\bigsqcup\{P \sqcap Q \mid P.\ P{\in}S\})$
  by (*rule ext*) *blast*

lemma *l2-5-3*: $((\bigsqcap\ S) \sqcup Q) = (\bigsqcap\{P \sqcup Q \mid P.\ P \in S\})$
  by (*rule ext*) *blast*

lemma *l2-5-4*: $((\bigsqcap\ S)\ ;;\ Q) = (\bigsqcap\{P\ ;;\ Q \mid P.\ P \in S\})$
  by (*rule ext*) *blast*

lemma *l2-5-5*: $(Q\ ;;\ (\bigsqcap\ S)) = (\bigsqcap\{Q\ ;;\ P \mid P.\ P \in S\})$
  by (*rule ext*) *blast*

lemma *all-idem* :$(\forall\, b.\ \forall\, a.\ P\ a) = (\forall\, a.\ P\ a)$
  by (*simp add*: *all-def*)

lemma *comp-unit-R* [*simp*]: $(P\ ;;\ \Pi r) = P$
  by (*auto simp*: *fun-eq-iff utp-defs*)

lemma *comp-unit-L* [*simp*]: $(\Pi r\ ;;\ P) = P$
  by (*auto simp*: *fun-eq-iff utp-defs*)

lemmas *comp-unit-simps* = *comp-unit-R comp-unit-L*

lemma *not-cond*: $(\neg(P \triangleleft b \triangleright Q)) = ((\neg\ P) \triangleleft b \triangleright (\neg\ Q))$
  by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-conj-not-distr*:
$((P \triangleleft b \triangleright Q) \wedge \neg(R \triangleleft b \triangleright S)) = ((P \wedge \neg R) \triangleleft b \triangleright (Q \wedge \neg S))$
  by (*rule ext*) (*auto split*: *cond-splits*)

lemma *imp-cond-distr*: $(R \longrightarrow (P \triangleleft b \triangleright Q)) = ((R \longrightarrow P) \triangleleft b \triangleright (R \longrightarrow Q))$
  by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-imp-dist*: $((P \triangleleft b \triangleright Q) \longrightarrow R) = ((P \longrightarrow R) \triangleleft b \triangleright (Q \longrightarrow R))$
  by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-conj-distr*: $((P \triangleleft b \triangleright Q) \wedge R) = ((P \wedge R) \triangleleft b \triangleright (Q \wedge R))$

by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-disj-distr*: $((P \triangleleft b \triangleright Q) \vee R) = ((P \vee R) \triangleleft b \triangleright (Q \vee R))$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-know-b*: $(b \wedge (P \triangleleft b \triangleright Q)) = (b \wedge P)$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-know-nb*: $((\neg (b)) \wedge (P \triangleleft b \triangleright Q)) = ((\neg (b)) \wedge Q)$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-ass-if*: $(P \triangleleft b \triangleright Q) = (((b) \wedge P \triangleleft b \triangleright Q))$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *cond-ass-else*: $(P \triangleleft b \triangleright Q) = (P \triangleleft b \triangleright ((\neg b) \wedge Q))$
by (*rule ext*) (*auto split*: *cond-splits*)

lemma *not-true-eq-false*: $(\neg\ \mathit{true}) = \mathit{false}$
by (*rule ext*) *blast*

lemma *not-false-eq-true*: $(\neg\ \mathit{false}) = \mathit{true}$
by (*rule ext*) *blast*

lemma *conj-idem*: $((P{::}'\alpha\ \mathit{predicate}) \wedge P) = P$
by (*rule ext*) *blast*

lemma *disj-idem*: $((P{::}'\alpha\ \mathit{predicate}) \vee P) = P$
by (*rule ext*) *blast*

lemma *conj-comm*: $((P{::}'\alpha\ \mathit{predicate}) \wedge Q) = (Q \wedge P)$
by (*rule ext*) *blast*

lemma *disj-comm*: $((P{::}'\alpha\ \mathit{predicate}) \vee Q) = (Q \vee P)$
by (*rule ext*) *blast*

lemma *conj-subst*: $P = R \Longrightarrow ((P{::}'\alpha\ \mathit{predicate}) \wedge Q) = (R \wedge Q)$
by (*rule ext*) *blast*

lemma *disj-subst*: $P = R \Longrightarrow ((P{::}'\alpha\ \mathit{predicate}) \vee Q) = (R \vee Q)$
by (*rule ext*) *blast*

lemma *conj-assoc*:$(((P{::}'\alpha\ \mathit{predicate}) \wedge Q) \wedge S) = (P \wedge (Q \wedge S))$
by (*rule ext*) *blast*

lemma *disj-assoc*:$(((P{::}'\alpha\ \mathit{predicate}) \vee Q) \vee S) = (P \vee (Q \vee S))$
by (*rule ext*) *blast*

lemma *conj-disj-abs*:$((P{::}'\alpha\ \mathit{predicate}) \wedge (P \vee Q)) = P$
by (*rule ext*) *blast*

lemma *disj-conj-abs*:$((P::'\alpha\ predicate) \lor (P \land Q)) = P$
by (*rule ext*) *blast*

lemma *conj-disj-distr*:$((P::'\alpha\ predicate) \land (Q \lor R)) = ((P \land Q) \lor (P \land R))$
by (*rule ext*) *blast*

lemma *disj-conj-dsitr*:$((P::'\alpha\ predicate) \lor (Q \land R)) = ((P \lor Q) \land (P \lor R))$
by (*rule ext*) *blast*

lemma *true-conj-id*:$(P \land true) = P$
by (*rule ext*) *blast*

lemma *true-dsij-zero*:$(P \lor true) = true$
by (*rule ext*) *blast*

lemma *true-conj-zero*:$(P \land false) = false$
by (*rule ext*) *blast*

lemma *true-dsij-id*:$(P \lor false) = P$
by (*rule ext*) *blast*

lemma *imp-vacuous*: $(false \longrightarrow u) = true$
by (*rule ext*) *blast*

lemma *p-and-not-p*: $(P \land \neg\ P) = false$
by (*rule ext*) *blast*

lemma *conj-disj-not-abs*: $((P::'\alpha\ predicate) \land ((\neg P) \lor Q)) = (P \land Q)$
by (*rule ext*) *blast*

lemma *p-or-not-p*: $(P \lor \neg\ P) = true$
by (*rule ext*) *blast*

lemma *double-negation*: $(\neg\ \neg\ (P::'\alpha\ predicate)) = P$
by (*rule ext*) *blast*

lemma *not-conj-deMorgans*: $(\neg\ ((P::'\alpha\ predicate) \land Q)) = ((\neg\ P) \lor (\neg\ Q))$
by (*rule ext*) *blast*

lemma *not-disj-deMorgans*: $(\neg\ ((P::'\alpha\ predicate) \lor Q)) = ((\neg\ P) \land (\neg\ Q))$
by (*rule ext*) *blast*

lemma *p-imp-p*: $(P \longrightarrow P) = true$
by (*rule ext*) *blast*

lemma *imp-imp*: $((P::'\alpha\ predicate) \longrightarrow (Q \longrightarrow R)) = ((P \land Q) \longrightarrow R)$
by (*rule ext*) *blast*

lemma *imp-trans*: $((P \longrightarrow Q) \wedge (Q \longrightarrow R) \longrightarrow P \longrightarrow R) = \mathit{true}$
by (*rule ext*) *blast*

lemma *p-equiv-p*: $(P \longleftrightarrow P) = \mathit{true}$
by (*rule ext*) *blast*

lemma *equiv-eq*: $((((P::'\alpha\ \mathit{predicate}) \wedge Q) \vee (\neg P \wedge \neg Q)) = \mathit{true}) \longleftrightarrow (P = Q)$
by (*auto simp add*: *fun-eq-iff utp-defs*)

lemma *equiv-eq1*: $(((P::'\alpha\ \mathit{predicate}) \longleftrightarrow Q) = \mathit{true}) \longleftrightarrow (P = Q)$
by (*auto simp add*: *fun-eq-iff utp-defs*)

lemma *cond-subst*: $b = c \Longrightarrow (P \triangleleft b \triangleright Q) = (P \triangleleft c \triangleright Q)$
by *simp*

lemma *ex-disj-distr*: $((\exists\, x.\ P\ x) \vee (\exists\, x.\ Q\ x)) = (\exists\, x.\ (P\ x \vee Q\ x))$
by (*rule ext*) *blast*

lemma *all-disj-distr*: $((\forall\, x.\ P\ x) \vee (\forall\, x.\ Q)) = (\forall\, x.\ (P\ x \vee Q))$
by (*rule ext*) *blast*

lemma *all-conj-distr*: $((\forall\, x.\ P\ x) \wedge (\forall\, x.\ Q\ x)) = (\forall\, x.\ (P\ x \wedge Q\ x))$
by (*rule ext*) *blast*

lemma *all-triv*: $(\forall\, x.\ P) = P$
by (*rule ext*) *blast*

lemma *closure-true*: $[\mathit{true}]$
by *blast*

lemma *closure-p-eq-true*: $[P] \longleftrightarrow (P = \mathit{true})$
by (*simp add*: *fun-eq-iff utp-defs*)

lemma *closure-equiv-eq*: $[P \longleftrightarrow Q] \longleftrightarrow (P = Q)$
by (*simp add*: *fun-eq-iff utp-defs*)

lemma *closure-conj-distr*: $([P] \wedge [Q]) = [P \wedge Q]$
by *blast*

lemma *closure-imp-distr*: $[P \longrightarrow Q] \longrightarrow [P] \longrightarrow [Q]$
by *blast*

lemma *true-iff*[*simp*]: $(P \longleftrightarrow \mathit{true}) = P$
by *blast*

lemma *true-imp*[*simp*]: $(\mathit{true} \longrightarrow P) = P$
by *blast*

end

9 Designs

theory *Designs*
imports *Relations*
begin

In UTP, in order to explicitly record the termination of a program, a subset of alphabetized relations is introduced. These relations are called designs and their alphabet should contain the special boolean observational variable ok. It is used to record the start and termination of a program.

9.1 Definitions

In the following, the definitions of designs alphabets, designs and healthiness (well-formedness) conditions are given. The healthiness conditions of designs are defined by $H1$, $H2$, $H3$ and $H4$.

record *alpha-d* = *ok*::*bool*

type-synonym *'α alphabet-d* = *'α alpha-d-scheme alphabet*
type-synonym *'α relation-d* = *'α alphabet-d relation*

definition *design*::*'α relation-d* ⇒ *'α relation-d* ⇒ *'α relation-d* (‹'(- ⊢ -')›)
where (*P* ⊢ *Q*) ≡ λ (*A*, *A'*) . (*ok A* ∧ *P* (*A*,*A'*)) ⟶ (*ok A'* ∧ *Q* (*A*,*A'*))

definition *skip-d* :: *'α relation-d* (‹Π*d*›)
where Π*d* ≡ (*true* ⊢ Π*r*)

definition *J*
where *J* ≡ λ (*A*, *A'*) . (*ok A* ⟶ *ok A'*) ∧ *more A* = *more A'*

type-synonym *'α Healthiness-condition* = *'α relation* ⇒ *'α relation*

definition
Healthy::*'α relation* ⇒ *'α Healthiness-condition* ⇒ *bool* (‹- *is* - *healthy*›)
where *P is H healthy* ≡ (*P* = *H P*)

lemma *Healthy-def'*: *P is H healthy* = (*H P* = *P*)
  unfolding *Healthy-def* **by** *auto*

definition *H1*::(*'α alphabet-d*) *Healthiness-condition*
where *H1* (*P*) ≡ (*ok o fst* ⟶ *P*)

definition *H2*::(*'α alphabet-d*) *Healthiness-condition*
where *H2* (*P*) ≡ *P* ; ; *J*

definition *H3*::(*'α alphabet-d*) *Healthiness-condition*
where *H3* (*P*) ≡ *P* ; ; Π*d*

definition $H4$::('α *alphabet-d*) *Healthiness-condition*
where $H4$ (P) $\equiv$ $((P; ; true) \longleftrightarrow true)$

definition σf::'α *relation-d* $\Rightarrow$ 'α *relation-d*
where $\sigma f\ D \equiv \lambda\ (A,\ A')\ .\ D\ (A,\ A'(|ok{:=}False|))$

definition σt::'α *relation-d* $\Rightarrow$ 'α *relation-d*
where $\sigma t\ D \equiv \lambda\ (A,\ A')\ .\ D\ (A,\ A'(|ok{:=}True|))$

definition $OKAY$::'α *relation-d*
where $OKAY \equiv \lambda\ (A,\ A')\ .\ ok\ A$

definition $OKAY'$::'α *relation-d*
where $OKAY' \equiv \lambda\ (A,\ A')\ .\ ok\ A'$

lemmas *design-defs* = *design-def skip-d-def J-def Healthy-def H1-def H2-def H3-def H4-def* σ*f-def* σ*t-def OKAY-def OKAY'-def*

9.2 Proofs

Proof of theorems and properties of designs and their healthiness conditions are given in the following.

lemma *t-comp-lz-d*: $(true; ; (P \vdash Q)) = true$
  apply (*auto simp*: *fun-eq-iff design-defs*)
  apply (*rule-tac* $b{=}b(|ok{:=}False|)$ **in** *comp-intro*, *auto*)
done

lemma *pi-comp-left-unit*: $(\Pi d; ; (P \vdash Q)) = (P \vdash Q)$
by (*auto simp*: *fun-eq-iff design-defs*)

theorem *t3-1-4-2*:
$((P1 \vdash Q1) \triangleleft b \triangleright (P2 \vdash Q2)) = ((P1 \triangleleft b \triangleright P2) \vdash (Q1 \triangleleft b \triangleright Q2))$
by (*auto simp*: *fun-eq-iff design-defs split*: *cond-splits*)

lemma *conv-conj-distr*: $\sigma t\ (P \wedge Q) = (\sigma t\ P \wedge \sigma t\ Q)$
by (*auto simp*: *design-defs fun-eq-iff*)

lemma *conv-disj-distr*: $\sigma t\ (P \vee Q) = (\sigma t\ P \vee \sigma t\ Q)$
by (*auto simp*: *design-defs fun-eq-iff*)

lemma *conv-imp-distr*: $\sigma t\ (P \longrightarrow Q) = ((\sigma t\ P) \longrightarrow \sigma t\ Q)$
by (*auto simp*: *design-defs fun-eq-iff*)

lemma *conv-not-distr*: $\sigma t\ (\neg\ P) = (\neg(\sigma t\ P))$
by (*auto simp*: *design-defs fun-eq-iff*)

lemma *div-conj-distr*: $\sigma f\ (P \wedge Q) = (\sigma f\ P \wedge \sigma f\ Q)$
by (*auto simp*: *design-defs fun-eq-iff*)

lemma *div-disj-distr*: $\sigma f\ (P \vee Q) = (\sigma f\ P \vee \sigma f\ Q)$
by (*auto simp*: *design-defs fun-eq-iff*)

lemma *div-imp-distr*: $\sigma f\ (P \longrightarrow Q) = ((\sigma f\ P) \longrightarrow \sigma f\ Q)$
by (*auto simp*: *design-defs fun-eq-iff*)

lemma *div-not-distr*: $\sigma f\ (\neg\ P) = (\neg(\sigma f\ P))$
by (*auto simp*: *design-defs fun-eq-iff*)

lemma *ok-conv*: $\sigma t\ OKAY = OKAY$
by (*auto simp*: *design-defs fun-eq-iff*)

lemma *ok-div*: $\sigma f\ OKAY = OKAY$
by (*auto simp*: *design-defs fun-eq-iff*)

lemma *ok′-conv*: $\sigma t\ OKAY' = true$
by (*auto simp*: *design-defs fun-eq-iff*)

lemma *ok′-div*: $\sigma f\ OKAY' = false$
by (*auto simp*: *design-defs fun-eq-iff*)

lemma *H2-J-1*:
 assumes *A*: *P is H2 healthy*
 shows $[(\lambda\ (A,\ A').\ (P(A,\ A'(\!|ok := False|\!)) \longrightarrow P(A,\ A'(\!|ok := True|\!))))]$
using *A* **by** (*auto simp*: *design-defs fun-eq-iff*)

lemma *H2-J-2-a* : $P\ (a,b) \longrightarrow (P\ ;;\ J)\ (a,b)$
  unfolding *J-def* **by** *auto*

lemma *ok-or-not-ok* : $[\![P(a,\ b(\!|ok := True|\!));\ P(a,\ b(\!|ok := False|\!))]\!] \Longrightarrow P(a,\ b)$
  apply (*case-tac ok b*)
  apply (*subgoal-tac* $b(\!|ok{:=}True|\!) = b$)
  apply (*simp-all*)
  apply (*subgoal-tac* $b(\!|ok{:=}False|\!) = b$)
  apply (*simp-all*)
done

lemma *H2-J-2-b* :
  assumes *A*: $[(\lambda\ (A,\ A').\ (P(A,\ A'(\!|ok := False|\!)) \longrightarrow P(A,\ A'(\!|ok := True|\!))))]$
  and *B* : $(P\ ;;\ J)\ (a,b)$
  shows $P\ (a,b)$
  using *B*
  apply (*auto simp*: *design-defs fun-eq-iff*)
  apply (*case-tac ok b*)
  apply (*subgoal-tac* $b = ba(\!|ok{:=}True|\!)$, *auto intro*!: *A*[*simplified*, *rule-format*])
  apply (*rule-tac* $s{=}ba$ **and** $t{=}ba(\!|ok{:=}False|\!)$ **in** *subst*, *simp-all*)
  apply (*subgoal-tac* $b = ba$, *simp-all*)
  apply (*case-tac ok ba*)
  apply (*subgoal-tac* $b = ba$, *simp-all*)

apply (*subgoal-tac b = ba(|ok:=True|), auto intro!: A[simplified, rule-format]*)
apply (*rule-tac s=ba* **and** *t=ba(|ok:=False|)* **in** *subst, simp-all*)
done

lemma *H2-J-2* :
assumes *A*: $[(\lambda\ (A, A'). (P(A, A'(|ok := False|)) \longrightarrow P(A, A'(|ok := True|))))]$
shows *P is H2 healthy*
apply (*auto simp add: H2-def Healthy-def fun-eq-iff*)
apply (*simp add: H2-J-2-a*)
apply (*rule H2-J-2-b [OF A]*)
apply *auto*
done

lemma *H2-J*:
$[\lambda\ (A, A'). P(A, A'(|ok := False|)) \longrightarrow P(A, A'(|ok := True|))] = P$ *is H2 healthy*
using *H2-J-1 H2-J-2* **by** *blast*

lemma *design-eq1*: $(P \vdash Q) = (P \vdash P \wedge Q)$
by (*rule ext*) (*auto simp: design-defs*)

lemma *H1-idem*: *H1 o H1 = H1*
by (*auto simp: design-defs fun-eq-iff*)

lemma *H1-idem2*: (*H1* (*H1 P*)) = (*H1 P*)
by (*simp add: H1-idem[simplified fun-eq-iff Fun.comp-def, rule-format] fun-eq-iff*)

lemma *H2-idem*: *H2 o H2 = H2*
by (*auto simp: design-defs fun-eq-iff*)

lemma *H2-idem2*: (*H2* (*H2 P*)) = (*H2 P*)
by (*simp add: H2-idem[simplified fun-eq-iff Fun.comp-def, rule-format] fun-eq-iff*)

lemma *H1-H2-commute*: *H1 o H2 = H2 o H1*
by (*auto simp: design-defs fun-eq-iff split: cond-splits*)

lemma *H1-H2-commute2*: *H1* (*H2 P*) = *H2* (*H1 P*)
by (*simp add: H1-H2-commute[simplified fun-eq-iff Fun.comp-def, rule-format] fun-eq-iff*)

lemma *alpha-d-eqD*: $r = r' \Longrightarrow ok\ r = ok\ r' \wedge alpha\text{-}d.more\ r = alpha\text{-}d.more\ r'$
by (*auto simp: alpha-d.equality*)

lemma *design-H1*: $(P \vdash Q)$ *is H1 healthy*
by (*auto simp: design-defs fun-eq-iff*)

lemma *design-H2*:
$(\forall\ a\ b.\ P\ (a, b(|ok := True|)) \longrightarrow P\ (a, b(|ok := False|))) \Longrightarrow (P \vdash Q)$ *is H2 healthy*
by (*rule H2-J-2*) (*auto simp: design-defs fun-eq-iff*)

end

10 Reactive processes

theory *Reactive-Processes*
imports *Designs HOL−Library.Sublist*

begin

Following the way of UTP to describe reactive processes, more observational variables are needed to record the interaction with the environment. Three observational variables are defined for this subset of relations: *wait*, *tr* and *ref*. The boolean variable *wait* records if the process is waiting for an interaction or has terminated. *tr* records the list (trace) of interactions the process has performed so far. The variable *ref* contains the set of interactions (events) the process may refuse to perform.

In this section, we introduce first some preliminary notions, useful for trace manipulations. The definitions of reactive process alphabets and healthiness conditions are also given. Finally, proved lemmas and theorems are listed.

10.1 Preliminaries

type-synonym *'α trace* = *'α list*

fun *list-diff*::*'α list* ⇒ *'α list* ⇒ *'α list option* **where**
  list-diff l [] = *Some l*
| *list-diff* [] *l* = *None*
| *list-diff* (*x#xs*) (*y#ys*) = (*if* (*x* = *y*) *then* (*list-diff xs ys*) *else None*)

instantiation *list* :: (*type*) *minus*
begin
definition *list-minus* : *l1* − *l2* ≡ *the* (*list-diff l1 l2*)
instance ..
end

lemma *list-diff-empty* [*simp*]: *the* (*list-diff l* []) = *l*
by (*cases l*) *auto*

lemma *prefix-diff-empty* [*simp*]: *l* − [] = *l*
by (*induct l*) (*auto simp*: *list-minus*)

lemma *prefix-diff-eq* [*simp*]: *l* − *l* = []
by (*induct l*) (*auto simp*: *list-minus*)

lemma *prefix-diff* [*simp*]: (*l* @ *t*) − *l* = *t*
by (*induct l*) (*auto simp*: *list-minus*)

lemma *prefix-subst* [*simp*]: *l* @ *t* = *m* ⟹ *m* − *l* = *t*
by (*auto*)

lemma *prefix-subst1* [*simp*]: $m = l @ t \Longrightarrow m - l = t$
by (*auto*)

lemma *prefix-diff1* [*simp*]: $((l @ m) @ t) - (l @ m) = t$
by (*rule prefix-diff*)

lemma *prefix-diff2* [*simp*]: $(l @ (m @ t)) - (l @ m) = t$
apply (*simp only*: *append-assoc* [*symmetric*])
apply (*rule prefix-diff1*)
done

lemma *prefix-diff3* [*simp*]: $(l @ m) - (l @ t) = (m - t)$
by (*induct l*, *auto simp*: *list-minus*)

lemma *prefix-diff4* [*simp*]: $(a \# m) - (a \# t) = (m - t)$
by (*auto simp*: *list-minus*)

class *ev-eq* =
  fixes *ev-eq* :: $'a \Rightarrow 'a \Rightarrow bool$
  assumes *refl*: *ev-eq a a*
  assumes *comm*: *ev-eq a b* = *ev-eq b a*

definition *filter-chan-set a cs* = (¬ (∃ *e*∈*cs*. *ev-eq a e*))

lemma *in-imp-not-fcs*:
x∈*S* ⟹ ¬ *filter-chan-set x S*
apply (*auto simp*: *filter-chan-set-def*)
apply (*rule-tac bexI*, *auto simp*: *refl*)
done

fun *tr-filter*::'*a*::*ev-eq list* ⇒ '*a set* ⇒ '*a list* **where**
    tr-filter [] *cs* = []
  | *tr-filter* (*x*#*xs*) *cs* = (*if* (*filter-chan-set x cs*) *then* (*x*#(*tr-filter xs cs*))
                                                            else (*tr-filter xs cs*))

lemma *tr-filter-conc*: (*tr-filter* (*a*@*b*) *cs*) = ((*tr-filter a cs*) @ (*tr-filter b cs*))
by (*induct a*, *auto*)

lemma *filter-chan-set-hd-tr-filter*:
tr-filter l cs ≠ [] −−> *filter-chan-set* (*hd* (*tr-filter l cs*)) *cs*
by (*induct l*, *auto*)

lemma *tr-filter-conc-eq1*:
(*a*@*b* = (*tr-filter* (*a*@*c*) *cs*)) ⟶ (*b* = (*tr-filter c cs*))
apply (*induct a*, *auto*)
apply (*case-tac tr-filter* (*a2* @ *c*) *cs* = [], *simp-all*)
apply (*drule filter-chan-set-hd-tr-filter*[*rule-format*])

apply (*case-tac tr-filter* (*a2* @ *c*) *cs*, *simp-all*)
done

lemma *tr-filter-conc-eq2*:
(*a*@*b* = (*tr-filter* (*a*@*c*) *cs*)) ⟶ (*a* = (*tr-filter a cs*))
apply (*induct a*, *auto*)
apply (*case-tac tr-filter* (*a2* @ *c*) *cs* = [], *simp-all*)
apply (*drule filter-chan-set-hd-tr-filter*[*rule-format*])
apply (*case-tac tr-filter* (*a2* @ *c*) *cs*, *simp-all*)
apply (*case-tac tr-filter* (*a2* @ *c*) *cs* = [], *simp-all*)
apply (*drule filter-chan-set-hd-tr-filter*[*rule-format*])
apply (*case-tac tr-filter* (*a2* @ *c*) *cs*, *simp-all*)
done

lemma *tr-filter-conc-eq*:
(*a*@*b* = (*tr-filter* (*a*@*c*) *cs*)) = (*b* = (*tr-filter c cs*) & *a* = (*tr-filter a cs*))
apply (*rule*, *rule*)
apply (*rule tr-filter-conc-eq1*[*rule-format*, *of a*], *clarsimp*)
apply (*rule tr-filter-conc-eq2*[*rule-format*, *of a b c*], *clarsimp*)
apply (*clarsimp simp*: *tr-filter-conc*)
done

lemma *tr-filter-conc-eq3*:
(*b* = (*tr-filter* (*a*@*c*) *cs*)) = (∃ *b1 b2*. *b*=*b1*@*b2* & *b2* = (*tr-filter c cs*) & *b1* = (*tr-filter a cs*))
by (*rule*, *auto simp*: *tr-filter-conc*)

lemma *tr-filter-un*:
tr-filter l (*s1* ∪ *s2*) = *tr-filter* (*tr-filter l s1*) *s2*
by (*induct l*, *auto simp*: *filter-chan-set-def*)

instantiation *list* :: (*ev-eq*) *ev-eq*
begin
fun *ev-eq-list* **where**
ev-eq-list [] [] = *True*
| *ev-eq-list l* [] = *False*
| *ev-eq-list* [] *l* = *False*
| *ev-eq-list* (*x*#*xs*) (*y*#*ys*) = (*if* (*ev-eq x y*) *then* (*ev-eq-list xs ys*) *else False*)
instance
proof
fix *a*::′*a*::*ev-eq list* **show** *ev-eq a a*
by (*induct a*, *auto simp*: *ev-eq-class.refl*)
next
fix *a b*::′*a*::*ev-eq list* **show** *ev-eq a b* = *ev-eq b a*
apply (*cases a*)
apply (*cases b*, *simp-all add*: *ev-eq-class.comm*)
apply (*hypsubst-thin*)
apply (*induct b*, *simp-all add*: *ev-eq-class.comm*)

apply (*case-tac ev-eq aa a*, *simp-all add*: *ev-eq-class.comm*)
apply (*case-tac list* = [], *simp-all*)
apply (*case-tac b*, *simp-all*)
apply (*atomize*)
apply (*erule-tac x*=*hd list* **in** *allE*)
apply (*erule-tac x*=*tl list* **in** *allE*)
apply (*subst* (*asm*) *hd-Cons-tl*, *simp-all*)
done
qed
end

10.2 Definitions

abbreviation *subl*::′*a list* ⇒ ′*a list* ⇒ *bool* (‹- ≤ -›)
where *l1* ≤ *l2* == *Sublist.prefix l1 l2*

lemma *list-diff-empty-eq*: *l1* − *l2* = [] ⟹ *l2* ≤ *l1* ⟹ *l1* = *l2*
by (*auto simp*: *prefix-def*)

The definitions of reactive process alphabets and healthiness conditions are given in the following. The healthiness conditions of reactive processes are defined by $R1$, $R2$, $R3$ and their composition R.

type-synonym ′ϑ *refusal* = ′ϑ *set*

record ′ϑ *alpha-rp* = *alpha-d* +
wait:: *bool*
tr :: ′ϑ *trace*
ref :: ′ϑ *refusal*

Note that we define here the class of UTP alphabets that contain *wait*, *tr* and *ref*, or, in other words, we define here the class of reactive process alphabets.

type-synonym (′ϑ,′σ) *alphabet-rp* = (′ϑ,′σ) *alpha-rp-scheme alphabet*
type-synonym (′ϑ,′σ) *relation-rp* = (′ϑ,′σ) *alphabet-rp relation*

definition *diff-tr s1 s2* = ((*tr s1*) − (*tr s2*))

definition *spec* :: [*bool*, *bool*, (′ϑ,′σ) *relation-rp*] ⇒ (′ϑ,′σ) *relation-rp*
where *spec b b′ P* ≡ λ (*A*, *A′*). *P* (*A*⦇*wait* := *b′*⦈, *A′*⦇*ok* := *b*⦈)

abbreviation *Speciftt* (‹-$^t{}_t$›) **where** $(P)^t{}_t$ ≡ *spec True True P*

abbreviation *Speciff* (‹-$^f{}_f$›) **where** $(P)^f{}_f$ ≡ *spec False False P*

abbreviation *Speciftf* (‹-$^t{}_f$›) **where** $(P)^t{}_f$ ≡ *spec True False P*

abbreviation *Specifft* (‹-$^f{}_t$›) **where** $(P)^f{}_t$ ≡ *spec False True P*

definition *R1*::((′ϑ,′σ) *alphabet-rp*) *Healthiness-condition*

where $R1\ (P)\ \equiv\ \lambda(A,\ A').\ (P\ (A,\ A')) \wedge (tr\ A \leq tr\ A')$

definition $R2$::$(('\vartheta,'\sigma)\ alphabet\text{-}rp)\ Healthiness\text{-}condition$
where $R2\ (P)\ \equiv \lambda(A,\ A').\ (P\ (A(\!|tr{:=}[]|\!),A'(\!|tr{:=}\ tr\ A' - tr\ A|\!)) \wedge tr\ A \leq tr\ A')$

definition Πrea
where $\Pi rea\ \equiv \lambda(A,\ A').\ (\neg ok\ A \wedge tr\ A \leq tr\ A') \vee (ok\ A' \wedge tr\ A = tr\ A'$
$\wedge\ (wait\ A = wait\ A') \wedge ref\ A = ref\ A' \wedge more\ A = more\ A')$

definition $R3$::$(('\vartheta,'\sigma)\ alphabet\text{-}rp)\ Healthiness\text{-}condition$
where $R3\ (P)\ \equiv (\Pi rea \triangleleft wait\ o\ fst \triangleright P)$

definition R::$(('\vartheta,'\sigma)\ alphabet\text{-}rp)\ Healthiness\text{-}condition$
where $R\ \equiv R3\ o\ R2\ o\ R1$

lemmas *rp-defs* = *R1-def R2-def* Π*rea-def R3-def R-def spec-def*

10.3 Proofs

lemma *tr-filter-empty* [*simp*]: *tr-filter l* {} = *l*
by (*induct l*) (*auto simp*: *filter-chan-set-def*)

lemma *trf-imp-filtercs*: ⟦*xs* = *tr-filter ys cs*; *xs* ≠ []⟧ ⟹ *filter-chan-set* (*hd xs*) *cs*
apply (*induct xs*, *auto*)
apply (*induct ys*, *auto*)
apply (*case-tac filter-chan-set a cs*, *auto*)
done

lemma *filtercs-imp-trf*:
⟦*filter-chan-set x cs*; *xs* = *tr-filter ys cs*⟧ ⟹ *x#xs* = *tr-filter* (*x#ys*) *cs*
by (*induct xs*) *auto*

lemma *alpha-d-more-eqI*:
  assumes *tr r* = *tr r′ wait r* = *wait r′ ref r* = *ref r′ more r* = *more r′*
  shows *alpha-d.more r* = *alpha-d.more r′*
  using *assms* **by** (*cases r*, *cases r′*) *auto*

lemma *alpha-d-more-eqE*:
  assumes *alpha-d.more r* = *alpha-d.more r′*
  obtains *tr r* = *tr r′ wait r* = *wait r′ ref r* = *ref r′ more r* = *more r′*
  using *assms* **by** (*cases r*, *cases r′*) *auto*

lemma *alpha-rp-eqE*:
  assumes *r* = *r′*
  obtains *ok r* = *ok r′ tr r* = *tr r′ wait r* = *wait r′ ref r* = *ref r′ more r* = *more r′*
  using *assms* **by** (*cases r*, *cases r′*) *auto*

lemma *R-idem*: *R o R* = *R*

by (*auto simp*: *rp-defs design-defs fun-eq-iff split*: *cond-splits*)

lemma *R-idem2*: *R* (*R P*) = *R P*
by (*auto simp*: *rp-defs design-defs fun-eq-iff split*: *cond-splits*)

lemma *R1-idem*: *R1 o R1* = *R1*
by (*auto simp*: *rp-defs design-defs*)

lemma *R1-idem2*: *R1* (*R1 x*) = *R1 x*
by (*auto simp*: *rp-defs design-defs*)

lemma *R2-idem*: *R2 o R2* = *R2*
by (*auto simp*: *rp-defs design-defs fun-eq-iff prefix-def*)

lemma *R2-idem2*: *R2* (*R2 x*) = *R2 x*
by (*auto simp*: *rp-defs design-defs fun-eq-iff prefix-def*)

lemma *R3-idem*: *R3 o R3* = *R3*
by (*auto simp*: *rp-defs design-defs fun-eq-iff split*: *cond-splits*)

lemma *R3-idem2*: *R3* (*R3 x*) = *R3 x*
by (*auto simp*: *R3-idem*[*simplified Fun.comp-def fun-eq-iff*] *fun-eq-iff*)

lemma *R1-R2-commute*: (*R1 o R2*) = (*R2 o R1*)
by (*auto simp*: *rp-defs design-defs fun-eq-iff prefix-def*)

lemma *R1-R3-commute*: (*R1 o R3*) = (*R3 o R1*)
by (*auto simp*: *rp-defs design-defs fun-eq-iff split*: *cond-splits*)

lemma *R2-R3-commute*: *R2 o R3* = *R3 o R2*
by (*auto simp*: *rp-defs design-defs fun-eq-iff prefix-def split*: *cond-splits*)

lemma *R-abs-R1*: *R o R1* = *R*
apply (*auto simp*: *R-def*)
apply (*subst* (*3*) *R1-idem*[*symmetric*])
apply (*auto*)
done

lemma *R-abs-R2*: *R o R2* = *R*
by (*auto simp*: *rp-defs design-defs fun-eq-iff*)

lemma *R-abs-R3*: *R o R3* = *R*
by (*auto simp*: *rp-defs design-defs fun-eq-iff split*: *cond-splits*)

lemma *R-is-R1*:
  assumes *A*: *P is R healthy*
  shows *P is R1 healthy*
proof −
  have *R P* = *P*

using *assms* **by** (*simp-all only*: *Healthy-def*)
moreover
have (*R P*) *is R1 healthy*
by (*auto simp add*: *design-defs rp-defs fun-eq-iff split*: *cond-splits*)
ultimately show *?thesis* **by** *simp*
qed

lemma *R-is-R2*:
assumes *A*: *P is R healthy*
shows *P is R2 healthy*
proof –
have *R P* = *P*
using *assms* **by** (*simp-all only*: *Healthy-def*)
moreover
have (*R P*) *is R2 healthy*
by (*auto simp add*: *design-defs rp-defs fun-eq-iff prefix-def split*: *cond-splits*)
ultimately show *?thesis* **by** *simp*
qed

lemma *R-is-R3*:
assumes *A*: *P is R healthy*
shows *P is R3 healthy*
proof –
have *R P* = *P*
using *assms* **by** (*simp-all only*: *Healthy-def*)
moreover
have (*R P*) *is R3 healthy*
by (*auto simp add*: *design-defs rp-defs fun-eq-iff split*: *cond-splits*)
ultimately show *?thesis* **by** *simp*
qed

lemma *R-disj*:
assumes *A*: *P is R healthy*
assumes *B*: *Q is R healthy*
shows (*P* ∨ *Q*) *is R healthy*
proof –
have *R P* = *P* **and** *R Q* = *Q*
using *assms* **by** (*simp-all only*: *Healthy-def*)
moreover
have ((*R P*) ∨ (*R Q*)) *is R healthy*
by (*auto simp add*: *design-defs rp-defs fun-eq-iff split*: *cond-splits*)
ultimately show *?thesis* **by** *simp*
qed

lemma *R-disj2*: *R* (*P* ∨ *Q*) = (*R P* ∨ *R Q*)
apply (*subst R-disj*[*simplified Healthy-def*, **where** *P*=*R P*])
apply (*simp-all add*: *R-idem2*)
apply (*auto simp*: *fun-eq-iff rp-defs split*: *cond-splits*)
done

```
lemma R1-comp:
  assumes P is R1 healthy
    and Q is R1 healthy
  shows (P; ; Q) is R1 healthy
proof -
  have R1 P = P and R1 Q = Q
    using assms by (simp-all only: Healthy-def)
  moreover
  have ((R1 P) ; ; (R1 Q)) is R1 healthy
    by (auto simp add: design-defs rp-defs fun-eq-iff split: cond-splits)
  ultimately show ?thesis by simp
qed

lemma R1-comp2:
  assumes A: P is R1 healthy
  assumes B: Q is R1 healthy
  shows R1 (P; ; Q) = ((R1 P); ; Q)
using A B
apply (subst R1-comp[simplified Healthy-def, symmetric])
apply (auto simp: fun-eq-iff rp-defs design-defs)
done

lemma J-is-R1: J is R1 healthy
  by (auto simp: rp-defs design-defs fun-eq-iff elim: alpha-d-more-eqE)

lemma J-is-R2: J is R2 healthy
  by (auto simp: rp-defs design-defs fun-eq-iff prefix-def
    elim!: alpha-d-more-eqE intro!: alpha-d-more-eqI)

lemma R1-H2-commute2: R1 (H2 P) = H2 (R1 P)
  by (auto simp add: H2-def R1-def J-def fun-eq-iff
    elim!: alpha-d-more-eqE intro!: alpha-d-more-eqI)

lemma R1-H2-commute: R1 o H2 = H2 o R1
by (auto simp: R1-H2-commute2)

lemma R2-H2-commute2: R2 (H2 P) = H2 (R2 P)
apply (auto simp add: fun-eq-iff rp-defs design-defs strict-prefix-def)
apply (rule-tac b=ba(|tr := tr a @ tr ba|) in comp-intro)
apply (auto simp: fun-eq-iff prefix-def
  elim!: alpha-d-more-eqE alpha-rp-eqE intro!: alpha-d-more-eqI alpha-rp.equality)
apply (rule-tac b=ba(|tr := tr a @ tr ba|) in comp-intro,
 auto simp: elim: alpha-d-more-eqE alpha-rp-eqE intro: alpha-d-more-eqI alpha-rp.equality)
apply (rule-tac b=ba(|tr := tr a @ tr ba|) in comp-intro,
 auto simp: elim: alpha-d-more-eqE alpha-rp-eqE intro: alpha-d-more-eqI alpha-rp.equality)
apply (rule-tac x=zs in exI, auto)+

done
```

lemma *R2-H2-commute*: *R2 o H2* = *H2 o R2*
by (*auto simp*: *R2-H2-commute2*)

lemma *R3-H2-commute2*: *R3* (*H2 P*) = *H2* (*R3 P*)
apply (*auto simp*: *fun-eq-iff rp-defs design-defs strict-prefix-def*
elim: *alpha-d-more-eqE split*: *cond-splits*)
done

lemma *R3-H2-commute*: *R3 o H2* = *H2 o R3*
by (*auto simp*: *R3-H2-commute2*)

lemma *R-join*:
assumes *x is R healthy*
and *y is R healthy*
shows (*x* ⊓ *y*) *is R healthy*
proof −
have *R x* = *x* **and** *R y* = *y*
using *assms* **by** (*simp-all only*: *Healthy-def*)
moreover
have ((*R x*) ⊓ (*R y*)) *is R healthy*
by (*auto simp add*: *design-defs rp-defs fun-eq-iff split*: *cond-splits*)
ultimately show *?thesis* **by** *simp*
qed

lemma *R-meet*:
assumes *A*: *x is R healthy*
and *B*:*y is R healthy*
shows (*x* ⊔ *y*) *is R healthy*
proof −
have *R x* = *x* **and** *R y* = *y*
using *assms* **by** (*simp-all only*: *Healthy-def*)
moreover
have ((*R x*) ⊔ (*R y*)) *is R healthy*
by (*auto simp add*: *design-defs rp-defs fun-eq-iff split*: *cond-splits*)
ultimately show *?thesis* **by** *simp*
qed

lemma *R-H2-commute*: *R o H2* = *H2 o R*
apply (*auto simp add*: *rp-defs design-defs fun-eq-iff split*: *cond-splits*
elim: *alpha-d-more-eqE*)
apply (*rule-tac b*=*ba*(|*tr* := *tr b*|) **in** *comp-intro*, *auto split*: *cond-splits*
elim!: *alpha-d-more-eqE alpha-rp-eqE intro*!: *alpha-d-more-eqI alpha-rp.equality*)
apply (*rule-tac s*=*ba* **in** *subst*, *auto intro*!: *alpha-d-more-eqI alpha-rp.equality*)
apply (*rule-tac s*=*ba* **in** *subst*, *auto intro*!: *alpha-d-more-eqI alpha-rp.equality*)
apply (*rule-tac b*=*ba*(|*tr* := *tr b*|) **in** *comp-intro*, *auto split*: *cond-splits*)
apply (*rule-tac s*=*ba* **in** *subst*,
auto elim: *alpha-d-more-eqE alpha-rp-eqE intro*: *alpha-d-more-eqI alpha-rp.equality*)

apply (*rule-tac b=ba*⦇*tr := tr b*⦈ **in** *comp-intro*,
auto elim: *alpha-d-more-eqE alpha-rp-eqE intro*: *alpha-d-more-eqI alpha-rp.equality*
split: *cond-splits*)
apply (*rule-tac s=ba* **in** *subst*,
auto elim: *alpha-d-more-eqE alpha-rp-eqE intro*: *alpha-d-more-eqI alpha-rp.equality*)
done

lemma *R-H2-commute2*: *R* (*H2 P*) = *H2* (*R P*)
by (*auto simp*: *fun-eq-iff R-H2-commute*[*simplified fun-eq-iff Fun.comp-def*])

end

11 CSP processes

theory *CSP-Processes*
imports *Reactive-Processes*
begin

A CSP process is a UTP reactive process that satisfies two additional healthiness conditions called $CSP1$ and $CSP2$. A reactive process that satisfies $CSP1$ and $CSP2$ is said to be CSP healthy.

11.1 Definitions

We introduce here the definitions of the CSP healthiness conditions.

definition *CSP1*::(('ϑ,'σ) *alphabet-rp*) *Healthiness-condition*
where *CSP1* (*P*) $\equiv$ *P* $\vee$ (λ(*A*, *A'*). $\neg$*ok A* $\wedge$ *tr A* $\leq$ *tr A'*)

definition *J-csp*
where *J-csp* $\equiv$ λ(*A*, *A'*). (*ok A* $\longrightarrow$ *ok A'*) $\wedge$ *tr A* = *tr A'* $\wedge$ *wait A* = *wait A'*
$\wedge$ *ref A* = *ref A'* $\wedge$ *more A* = *more A'*

definition *CSP2*::(('ϑ,'σ) *alphabet-rp*) *Healthiness-condition*
where *CSP2* (*P*) $\equiv$ *P* ;; *J-csp*

definition *is-CSP-process*::('ϑ,'σ) *relation-rp* $\Rightarrow$ *bool* **where**
is-CSP-process P $\equiv$ *P is CSP1 healthy* $\wedge$ *P is CSP2 healthy* $\wedge$ *P is R healthy*

lemmas *csp-defs* = *CSP1-def J-csp-def CSP2-def is-CSP-process-def*

lemma *is-CSP-processE1* [*elim?*]:
assumes *is-CSP-process P*
obtains *P is CSP1 healthy P is CSP2 healthy P is R healthy*
using *assms* **unfolding** *is-CSP-process-def* **by** *simp*

lemma *is-CSP-processE2* [*elim?*]:
assumes *is-CSP-process P*
obtains *CSP1 P* = *P CSP2 P* = *P R P* = *P*

using *assms* **unfolding** *is-CSP-process-def* **by** (*simp add*: *Healthy-def′*)

11.2 Proofs

Theorems and lemmas relative to CSP processes are introduced here.

lemma *CSP1-CSP2-commute*: *CSP1 o CSP2* = *CSP2 o CSP1*
by (*auto simp*: *csp-defs fun-eq-iff*)

lemma *CSP2-is-H2*: *H2* = *CSP2*
apply (*clarsimp simp add*: *csp-defs design-defs rp-defs fun-eq-iff*)
apply (*rule iffI*)
apply (*erule-tac* [!] *comp-elim*)
apply (*rule-tac* [!] *b*=*ba* **in** *comp-intro*)
apply (*auto elim*!: *alpha-d-more-eqE intro*!: *alpha-d-more-eqI*)
done

lemma *H2-CSP1-commute*: *H2 o CSP1* = *CSP1 o H2*
apply (*subst CSP2-is-H2*[*simplified Healthy-def*])+
apply (*rule CSP1-CSP2-commute*[*symmetric*])
done

lemma *H2-CSP1-commute2*: *H2* (*CSP1 P*) = *CSP1* (*H2 P*)
by (*simp add*: *H2-CSP1-commute*[*simplified Fun.comp-def fun-eq-iff*, *rule-format*] *fun-eq-iff*)

lemma *CSP1-R-commute*:
  CSP1 (*R P*) = *R* (*CSP1 P*)
by (*auto simp*: *csp-defs rp-defs fun-eq-iff prefix-def split*: *cond-splits*)

lemma *CSP2-R-commute*:
  CSP2 (*R P*) = *R* (*CSP2 P*)
apply (*subst CSP2-is-H2*[*symmetric*])+
apply (*rule R-H2-commute2*[*symmetric*])
done

lemma *CSP1-idem*: *CSP1* = *CSP1 o CSP1*
by (*auto simp*: *csp-defs fun-eq-iff*)

lemma *CSP2-idem*: *CSP2* = *CSP2 o CSP2*
by (*auto simp*: *csp-defs fun-eq-iff*)

lemma *CSP-is-CSP1*:
  assumes *A*: *is-CSP-process P*
  shows *P is CSP1 healthy*
using *A* **by** (*auto simp*: *is-CSP-process-def design-defs*)

lemma *CSP-is-CSP2*:
  assumes *A*: *is-CSP-process P*
  shows *P is CSP2 healthy*

using *A* **by** (*simp add*: *design-defs prefix-def is-CSP-process-def*)

lemma *CSP-is-R*:
  assumes *A*: *is-CSP-process P*
  shows *P is R healthy*
using *A* **by** (*simp add*: *design-defs prefix-def is-CSP-process-def*)

lemma *t-or-f-a*: $P(a, b) \Longrightarrow ((P(a, b(|ok := True|))) \vee (P(a, b(|ok := False|))))$
apply (*case-tac ok b*, *auto*)
apply (*rule-tac t=b(|ok := True|)* **and** *s=b* **in** *ssubst*, *simp-all*)
by (*subgoal-tac b = b(|ok := False|)*, *simp-all*)

lemma *CSP2-ok-a*:
$(CSP2\ P)(a, b(|ok{:=}True|)) \Longrightarrow (P(a, b(|ok{:=}True|)) \vee P(a, b(|ok{:=}False|)))$
apply (*clarsimp simp*: *csp-defs design-defs rp-defs split*: *cond-splits elim*: *prefixE*)
apply (*case-tac ok ba*)
apply (*rule-tac t=b(|ok := True|)* **and** *s=ba* **in** *ssubst*, *simp-all*)
apply (*drule-tac b=b(|ok := False|)* **and** *a=ba* **in** *back-subst*)
apply (*auto intro*: *alpha-rp.equality*)
done

lemma *CSP2-ok-b*:
$(P(a, b(|ok{:=}True|)) \vee P(a, b(|ok{:=}False|))) \Longrightarrow (CSP2\ P)(a, b(|ok{:=}True|))$
by (*auto simp*: *csp-defs design-defs rp-defs*)

lemma *CSP2-ok*:
$(CSP2\ P)(a, b(|ok{:=}True|)) = (P(a, b(|ok{:=}True|)) \vee P(a, b(|ok{:=}False|)))$
apply (*rule iffI*)
apply (*simp add*: *CSP2-ok-a*)
by (*simp add*: *CSP2-ok-b*)

lemma *CSP2-notok-a*: $(CSP2\ P)(a, b(|ok{:=}False|)) \Longrightarrow P(a, b(|ok{:=}False|))$
apply (*clarsimp simp*: *csp-defs design-defs rp-defs*)
apply (*case-tac ok ba*)
apply (*rule-tac t=b(|ok := True|)* **and** *s=ba* **in** *ssubst*, *simp-all*)
apply (*drule-tac b=b(|ok := False|)* **and** *a=ba* **in** *back-subst*)
apply (*auto intro*: *alpha-rp.equality*)
done

lemma *CSP2-notok-b*: $P(a, b(|ok{:=}False|)) \Longrightarrow (CSP2\ P)(a, b(|ok{:=}False|))$
by (*auto simp*: *csp-defs design-defs rp-defs*)

lemma *CSP2-notok*: $(CSP2\ P)(a, b(|ok{:=}False|)) = P(a, b(|ok{:=}False|))$
apply (*rule iffI*)
apply (*simp add*: *CSP2-notok-a*)
by (*simp add*: *CSP2-notok-b*)

lemma *CSP2-t-f*:
  assumes *A*:$(CSP2\ (R\ (r \vdash p)))(a, b)$

and *B*: ((*CSP2* (*R* (*r* ⊢ *p*)))(*a*, *b*(|*ok*:=*False*|))) ∨
((*CSP2* (*R* (*r* ⊢ *p*)))(*a*, *b*(|*ok*:=*True*|))) ⟹ *Q*
shows *Q*
apply (*rule B*)
apply (*rule disjI2*)
apply (*insert A*)
apply (*auto simp add*: *csp-defs design-defs rp-defs*)
done

lemma *disj-CSP1*:
assumes *P is CSP1 healthy*
and *Q is CSP1 healthy*
shows (*P* ∨ *Q*) *is CSP1 healthy*
using *assms* **by** (*auto simp*: *csp-defs design-defs rp-defs fun-eq-iff*)

lemma *disj-CSP2*:
P is CSP2 healthy ==> Q is CSP2 healthy ==> (*P* ∨ *Q*) *is CSP2 healthy*
by (*simp add*: *CSP2-is-H2*[*symmetric*] *Healthy-def′ design-defs comp-ndet-l-distr*)

lemma *disj-CSP*:
assumes *A*: *is-CSP-process P*
assumes *B*: *is-CSP-process Q*
shows *is-CSP-process* (*P* ∨ *Q*)
apply (*simp add*: *is-CSP-process-def Healthy-def*)
apply (*subst disj-CSP2*[*simplified Healthy-def*, *symmetric*])
apply (*rule A*[*THEN CSP-is-CSP2*, *simplified Healthy-def*])
apply (*rule B*[*THEN CSP-is-CSP2*, *simplified Healthy-def*], *simp*)
apply (*subst disj-CSP1*[*simplified Healthy-def*, *symmetric*])
apply (*rule A*[*THEN CSP-is-CSP1*, *simplified Healthy-def*])
apply (*rule B*[*THEN CSP-is-CSP1*, *simplified Healthy-def*], *simp*)
apply (*subst R-disj*[*simplified Healthy-def*])
apply (*rule A*[*THEN CSP-is-R*, *simplified Healthy-def*])
apply (*rule B*[*THEN CSP-is-R*, *simplified Healthy-def*], *simp*)
done

lemma *seq-CSP1*:
assumes *A*: *P is CSP1 healthy*
assumes *B*: *Q is CSP1 healthy*
shows (*P* ;; *Q*) *is CSP1 healthy*
using *A B* **by** (*auto simp*: *csp-defs design-defs rp-defs fun-eq-iff*)

lemma *seq-CSP2*:
assumes *A*: *Q is CSP2 healthy*
shows (*P* ;; *Q*) *is CSP2 healthy*
using *A*
by (*auto simp*: *CSP2-is-H2*[*symmetric*] *H2-J*[*symmetric*])

lemma *seq-R*:
assumes *P is R healthy*

```
  and Q is R healthy
  shows (P ;;  Q) is R healthy
proof −
  have R P = P and R Q = Q
    using assms by (simp-all only: Healthy-def)
  moreover
  have (R P ;;  R Q) is R healthy
    apply (auto simp add: design-defs rp-defs prefix-def fun-eq-iff split: cond-splits)
           apply (rule-tac b=a in comp-intro, auto split: cond-splits)
       apply (rule-tac x=zs in exI, auto split: cond-splits)
     apply (rule-tac b=ba(|tr := tr a @ tr ba|) in comp-intro, auto split: cond-splits)+
    done
  ultimately show ?thesis by simp
qed


lemma seq-CSP:
  assumes A: P is CSP1 healthy
  and B: P is R healthy
  and C: is-CSP-process Q
  shows is-CSP-process (P ;;  Q)
apply (auto simp add: is-CSP-process-def)
apply (subst seq-CSP1[simplified Healthy-def])
apply (rule A[simplified Healthy-def])
apply (rule CSP-is-CSP1[OF C, simplified Healthy-def])
apply (simp add: Healthy-def, subst CSP1-idem, auto)
apply (subst seq-CSP2[simplified Healthy-def])
apply (rule CSP-is-CSP2[OF C, simplified Healthy-def])
apply (simp add: Healthy-def, subst CSP2-idem, auto)
apply (subst seq-R[simplified Healthy-def])
apply (rule B[simplified Healthy-def])
apply (rule CSP-is-R[OF C, simplified Healthy-def])
apply (simp add: Healthy-def, subst R-idem2, auto)
done

lemma rd-ind-wait: (R(¬(P f_f) ⊢ (P t_f)))
                    = (R((¬(λ (A, A'). P (A, A'(|ok := False|))))
                           ⊢ (λ (A, A'). P (A, A'(|ok := True|)))))
apply (auto simp: design-defs rp-defs fun-eq-iff split: cond-splits)
apply (subgoal-tac a(|tr := [], wait := False|) = a(|tr := []|), auto)
apply (subgoal-tac a(|tr := [], wait := False|) = a(|tr := []|), auto)
apply (subgoal-tac a(|tr := [], wait := False|) = a(|tr := []|), auto)
apply (subgoal-tac a(|tr := [], wait := False|) = a(|tr := []|), auto)
apply (subgoal-tac a(|tr := [], wait := False|) = a(|tr := []|), auto)
apply (rule-tac t=a(|tr := [], wait := False|) and s=a(|tr := []|) in subst, simp-all)
done

lemma rd-H1: (R((¬(λ (A, A'). P (A, A'(|ok := False|))))
                 ⊢ (λ (A, A'). P (A, A'(|ok := True|))))) =
```

(*R* ((¬ *H1* (λ (*A*, *A*′). *P* (*A*, *A*′(|*ok* := *False*|))))
⊢ *H1* (λ (*A*, *A*′). *P* (*A*, *A*′(|*ok* := *True*|)))))
by (*auto simp*: *design-defs rp-defs fun-eq-iff split*: *cond-splits*)

lemma *rd-H1-H2*: (*R*((¬ *H1* (λ (*A*, *A*′). *P* (*A*, *A*′(|*ok* := *False*|))))
⊢ *H1* (λ (*A*, *A*′). *P* (*A*, *A*′(|*ok* := *True*|))))) =
(*R*((¬(*H1 o H2*) (λ (*A*, *A*′). *P* (*A*, *A*′(|*ok* := *False*|))))
⊢ (*H1 o H2*) (λ (*A*, *A*′). *P* (*A*, *A*′(|*ok* := *True*|)))))
apply (*auto simp*: *design-defs rp-defs prefix-def fun-eq-iff split*: *cond-splits elim*: *alpha-d-more-eqE*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *False*|) = *ba*(|*ok* := *False*|), *auto intro*: *alpha-d.equality*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *False*|) = *ba*(|*ok* := *False*|), *auto intro*: *alpha-d.equality*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *False*|) = *ba*(|*ok* := *False*|), *auto intro*: *alpha-d.equality*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *True*|) = *ba*(|*ok* := *True*|), *auto intro*: *alpha-d.equality*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *True*|) = *ba*(|*ok* := *True*|), *auto intro*: *alpha-d.equality*)
done

lemma *rd-H1-H2-R-H1-H2*:
(*R* ((¬ (*H1 o H2*) (λ (*A*, *A*′). *P* (*A*, *A*′(|*ok* := *False*|))))
⊢ (*H1 o H2*) (λ (*A*, *A*′). *P* (*A*, *A*′(|*ok* := *True*|))))) =
(*R o H1 o H2*) *P*
apply (*auto simp*: *design-defs rp-defs fun-eq-iff split*: *cond-splits*)
apply (*erule notE*) **back back**
apply (*rule-tac b*=*ba* **in** *comp-intro*, *auto*)
apply (*rule-tac t*=*ba*(|*ok* := *False*|) **and** *s*=*ba* **in** *subst*, *auto intro*: *alpha-d.equality*)
apply (*erule notE*) **back back**
apply (*rule-tac b*=*ba* **in** *comp-intro*, *auto*)
apply (*rule-tac t*=*ba*(|*ok* := *False*|) **and** *s*=*ba* **in** *subst*, *auto intro*: *alpha-d.equality*)
apply (*case-tac ok ba*)
apply (*rule-tac b*=*ba* **in** *comp-intro*, *auto*)
apply (*rule-tac t*=*ba*(|*ok* := *True*|) **and** *s*=*ba* **in** *subst*, *auto*)
apply (*erule notE*) **back**
apply (*rule-tac b*=*ba* **in** *comp-intro*, *auto*)
apply (*rule-tac t*=*ba*(|*ok* := *False*|) **and** *s*=*ba* **in** *subst*, *auto intro*: *alpha-d.equality*)
done

lemma *CSP1-is-R1-H1*:
assumes *P is R1 healthy*
shows *CSP1 P* = *R1* (*H1 P*)
using *assms*
by (*auto simp*: *csp-defs design-defs rp-defs fun-eq-iff split*: *cond-splits*)

lemma *CSP1-is-R1-H1-2*: *CSP1* (*R1 P*) = *R1* (*H1 P*)
by (*auto simp*: *csp-defs design-defs rp-defs fun-eq-iff split*: *cond-splits*)

lemma *CSP1-R1-commute*: *CSP1 o R1* = *R1 o CSP1*
by (*auto simp*: *csp-defs design-defs rp-defs fun-eq-iff split*: *cond-splits*)

lemma *CSP1-R1-commute2*: *CSP1* (*R1 P*) = *R1* (*CSP1 P*)
by (*auto simp*: *csp-defs design-defs rp-defs fun-eq-iff split*: *cond-splits*)

lemma *CSP1-is-R1-H1-b*:
$(P = (R \circ R1 \circ H1 \circ H2)\ P) = (P = (R \circ CSP1 \circ H2)\ P)$
apply (*simp add*: *fun-eq-iff*)
apply (*subst H1-H2-commute2*)
apply (*subst R1-H2-commute2*)
apply (*subst CSP1-is-R1-H1-2*[*symmetric*])
apply (*subst H2-CSP1-commute2*)
apply (*subst R1-H2-commute2*[*symmetric*])
apply (*subst CSP1-R1-commute2*)
apply (*subst R-abs-R1*[*simplified Fun.comp-def fun-eq-iff*])
apply (*auto*)
done

lemma *CSP1-join*:
  assumes *A*: *x is CSP1 healthy*
  and *B*: *y is CSP1 healthy*
  shows ($x \sqcap y$) *is CSP1 healthy*
  using *A B*
  by (*simp add*: *Healthy-def CSP1-def fun-eq-iff utp-defs*)

lemma *CSP2-join*:
  assumes *A*: *x is CSP2 healthy*
  and *B*: *y is CSP2 healthy*
  shows ($x \sqcap y$) *is CSP2 healthy*
  using *A B*
  apply (*simp add*: *design-defs csp-defs fun-eq-iff*)
  apply (*rule allI*)
  apply (*rule allI*)
  apply (*erule-tac x=a* **in** *allE*)
  apply (*erule-tac x=a* **in** *allE*)
  apply (*erule-tac x=b* **in** *allE*)+
  by (*auto*)

lemma *CSP1-meet*:
  assumes *A*: *x is CSP1 healthy*
  and *B*: *y is CSP1 healthy*
  shows ($x \sqcup y$) *is CSP1 healthy*
  using *A B*
  apply (*simp add*: *Healthy-def CSP1-def fun-eq-iff utp-defs*)
  apply (*rule allI*)
  apply (*rule allI*)
  apply (*erule-tac x=a* **in** *allE*)

apply (*erule-tac x=a* **in** *allE*)
apply (*erule-tac x=b* **in** *allE*)+
by (*auto*)

lemma *CSP2-meet*:
assumes *A*: *x is CSP2 healthy*
and *B*: *y is CSP2 healthy*
shows ($x \sqcup y$) *is CSP2 healthy*
using *A B*
apply (*simp add*: *Healthy-def CSP2-def fun-eq-iff*)
apply (*rule allI*)+
apply (*erule-tac x=a* **in** *allE*)
apply (*erule-tac x=a* **in** *allE*)
apply (*erule-tac x=b* **in** *allE*)+
apply (*auto*)
apply (*rule-tac b=ca* **in** *comp-intro*)
apply (*auto simp*: *J-csp-def*)
done

lemma *CSP-join*:
assumes *A*: *is-CSP-process x*
and *B*: *is-CSP-process y*
shows *is-CSP-process* ($x \sqcap y$)
using *A B*
by (*simp add*: *is-CSP-process-def CSP1-join CSP2-join R-join*)

lemma *CSP-meet*:
assumes *A*: *is-CSP-process x*
and *B*: *is-CSP-process y*
shows *is-CSP-process* ($x \sqcup y$)
using *A B*
by (*simp add*: *is-CSP-process-def CSP1-meet CSP2-meet R-meet*)

11.3 CSP processes and reactive designs

In this section, we prove the relation between CSP processes and reactive designs.

lemma *rd-is-CSP1*: (R ($r \vdash p$)) *is CSP1 healthy*
by (*auto simp*: *csp-defs design-defs rp-defs fun-eq-iff split*: *cond-splits elim*: *prefixE*)

lemma *rd-is-CSP2*:
assumes *A*: $\forall$ *a b*. *r* (*a*, *b*⦇*ok* := *True*⦈) $\longrightarrow$ *r* (*a*, *b*⦇*ok* := *False*⦈)
shows (R ($r \vdash p$)) *is CSP2 healthy*
apply (*subst CSP2-is-H2*[*symmetric*])
apply (*simp add*: *Healthy-def*)
apply (*subst R-H2-commute2*[*symmetric*])
apply (*subst design-H2*[*simplified Healthy-def*], *auto simp*: *A*)
done

lemma *rd-is-CSP*:
  assumes *A*: ∀ *a b*. *r* (*a*, *b*(|*ok* := *True*|)) ⟶ *r* (*a*, *b*(|*ok* := *False*|))
  shows *is-CSP-process* (*R* (*r* ⊢ *p*))
apply (*simp add*: *is-CSP-process-def Healthy-def fun-eq-iff*)
apply (*subst R-idem2*)
apply (*subst rd-is-CSP2*[*simplified Healthy-def*, *symmetric*], *rule A*)
apply (*subst rd-is-CSP1*[*simplified Healthy-def*, *symmetric*], *simp*)
done

lemma *CSP-is-rd*:
  assumes *A*: *is-CSP-process P*
  shows $P = (R\ (\neg(P\ {}^{f}{}_{f}) \vdash (P\ {}^{t}{}_{f})))$
  apply (*subst rd-ind-wait*)
  apply (*subst rd-H1*)
  apply (*subst rd-H1-H2*)
  apply (*subst rd-H1-H2-R-H1-H2*)
  apply (*subst R-abs-R1*[*symmetric*])
  apply (*subst CSP1-is-R1-H1-b*)
  apply (*subst CSP2-is-H2*)
  apply (*simp*)
  apply (*subst CSP-is-CSP2*[*OF A*, *simplified Healthy-def*, *symmetric*])
  apply (*subst CSP-is-CSP1*[*OF A*, *simplified Healthy-def*, *symmetric*])
  apply (*subst CSP-is-R*[*OF A*, *simplified Healthy-def*, *symmetric*], *simp*)
done

end

12 Circus actions

theory *Circus-Actions*
imports *HOLCF CSP-Processes*
begin

In this section, we introduce definitions for Circus actions with some useful theorems and lemmas.

default-sort *type*

12.1 Definitions

The Circus actions type is defined as the set of all the CSP healthy reactive processes.

typedef ('ϑ::*ev-eq*,'σ) *action* = {*p*::('ϑ,'σ) *relation-rp*. *is-CSP-process p*}
   morphisms *relation-of action-of*
proof –
   have *R* (*false* ⊢ *true*) ∈ {*p* :: ('ϑ,'σ) *relation-rp*. *is-CSP-process p*}
      by (*auto intro*: *rd-is-CSP*)
   thus *?thesis* **by** *auto*

qed

print-theorems

The type-definition introduces a new type by stating a set. In our case, it is the set of reactive processes that satisfy the healthiness-conditions for CSP-processes, isomorphic to the new type. Technically, this construct introduces two constants (morphisms) definitions *relation_of* and *action_of* as well as the usual axioms expressing the bijection *action-of* (*action.relation-of ?x*) = *?x* and *?y* ∈ {*p. is-CSP-process p*} ⟹ *action.relation-of* (*action-of ?y*) = *?y*.

lemma *relation-of-CSP*: *is-CSP-process* (*relation-of x*)
proof −
have (*relation-of x*) :{*p. is-CSP-process p*} **by** (*rule relation-of*)
then show *is-CSP-process* (*relation-of x*) **..**
qed

lemma *relation-of-CSP1*: (*relation-of x*) *is CSP1 healthy*
by (*rule CSP-is-CSP1*[*OF relation-of-CSP*])

lemma *relation-of-CSP2*: (*relation-of x*) *is CSP2 healthy*
by (*rule CSP-is-CSP2*[*OF relation-of-CSP*])

lemma *relation-of-R*: (*relation-of x*) *is R healthy*
by (*rule CSP-is-R*[*OF relation-of-CSP*])

12.2 Proofs

In the following, Circus actions are proved to be an instance of the *Complete_Lattice* class.

lemma *relation-of-spec-f-f*:
∀ *a b*. (*relation-of y* ⟶ *relation-of x*) (*a*, *b*) ⟹
(*relation-of y*)$^f{}_f$ (*a*(|*tr* := []|), *b*) ⟹
(*relation-of x*)$^f{}_f$ (*a*(|*tr* := []|), *b*)
by (*auto simp*: *spec-def*)

lemma *relation-of-spec-t-f*:
∀ *a b*. (*relation-of y* ⟶ *relation-of x*) (*a*, *b*) ⟹
(*relation-of y*)$^t{}_f$ (*a*(|*tr* := []|), *b*) ⟹
(*relation-of x*)$^t{}_f$ (*a*(|*tr* := []|), *b*)
by (*auto simp*: *spec-def*)

instantiation *action*::(*ev-eq*, *type*) *below*
begin
definition *ref-def* : *P* ⊑ *Q* ≡ [(*relation-of Q*) ⟶ (*relation-of P*)]
instance ..
end

```
instance action :: (ev-eq, type) po
proof
  fix x y z::('a, 'b) action
  {
    show x ⊑ x by (simp add: ref-def utp-defs)
  next
    assume x ⊑ y and y ⊑ z then show  x ⊑ z
      by (simp add: ref-def utp-defs)
  next
    assume A:x ⊑ y and B:y ⊑ x then show  x = y
      by (auto simp add: ref-def relation-of-inject[symmetric] fun-eq-iff)
  }
qed

instantiation action  ::  (ev-eq, type) lattice
begin

definition inf-action : (inf P Q ≡ action-of ((relation-of P) ⊓ (relation-of Q)))
definition sup-action : (sup P Q ≡ action-of ((relation-of P) ⊔ (relation-of Q)))
definition less-eq-action : (less-eq (P::('a, 'b) action) Q ≡ P ⊑ Q)
definition less-action : (less (P::('a, 'b) action) Q ≡ P ⊑ Q ∧ ¬ Q ⊑ P)

instance
proof
  fix x y z::('a, 'b) action
  {
    show (x < y) = (x ≤ y ∧ ¬ y ≤ x)
      by (simp add: less-action less-eq-action)
  next
    show (x ≤ x) by (simp add: less-eq-action)
  next
    assume x ≤ y and y ≤ z
    then show  x ≤ z
      by (simp add: less-eq-action ref-def utp-defs)
  next
    assume x ≤ y and y ≤ x
    then show  x = y
     by (auto simp add: less-eq-action ref-def relation-of-inject[symmetric] utp-defs)
  next
    show inf x y ≤ x
      apply (auto simp add: less-eq-action inf-action ref-def
        csp-defs design-defs rp-defs)
      apply (subst action-of-inverse, simp add: Healthy-def)
      apply (insert relation-of-CSP[where x=x])
      apply (insert relation-of-CSP[where x=y])
      apply (simp-all add: CSP-join)
      apply (simp add: utp-defs)
      done
  next
```

```
  show inf x y ≤ y
    apply (auto simp add: less-eq-action inf-action ref-def csp-defs)
    apply (subst action-of-inverse, simp add: Healthy-def)
    apply (insert relation-of-CSP[where x=x])
    apply (insert relation-of-CSP[where x=y])
    apply (simp-all add: CSP-join)
    apply (simp add: utp-defs)
    done
next
  assume x ≤ y and x ≤ z
  then show x ≤ inf y z
    apply (auto simp add: less-eq-action inf-action ref-def impl-def csp-defs)
    apply (erule-tac x=a in allE, erule-tac x=a in allE)
    apply (erule-tac x=b in allE)+
    apply (subst (asm) action-of-inverse)
    apply (simp add: Healthy-def)
    apply (insert relation-of-CSP[where x=z])
    apply (insert relation-of-CSP[where x=y])
    apply (auto simp add: CSP-join)
    done
next
  show x ≤ sup x y
    apply (auto simp add: less-eq-action sup-action ref-def
       impl-def csp-defs)
    apply (subst (asm) action-of-inverse)
    apply (simp add: Healthy-def)
    apply (insert relation-of-CSP[where x=x])
    apply (insert relation-of-CSP[where x=y])
    apply (auto simp add: CSP-meet)
    done
next
  show y ≤ sup x y
    apply (auto simp add: less-eq-action sup-action ref-def
       impl-def csp-defs)
    apply (subst (asm) action-of-inverse)
    apply (simp add: Healthy-def)
    apply (insert relation-of-CSP[where x=x])
    apply (insert relation-of-CSP[where x=y])
    apply (auto simp add: CSP-meet)
    done
next
  assume y ≤ x and z ≤ x
  then show sup y z ≤ x
    apply (auto simp add: less-eq-action sup-action ref-def  impl-def csp-defs)
    apply (erule-tac x=a in allE)
    apply (erule-tac x=a in allE)
    apply (erule-tac x=b in allE)+
    apply (subst action-of-inverse)
    apply (simp add: Healthy-def)
```

```
      apply (insert relation-of-CSP[where x=z])
      apply (insert relation-of-CSP[where x=y])
      apply (auto simp add: CSP-meet)
      done
  }
qed

end

lemma bot-is-action: R (false ⊢ true) ∈ {p. is-CSP-process p}
  by (auto intro: rd-is-CSP)

lemma bot-eq-true: R (false ⊢ true) = R true
  by (auto simp: fun-eq-iff design-defs rp-defs split: cond-splits)

instantiation action :: (ev-eq, type) bounded-lattice
begin

definition bot-action : (bot::('a, 'b) action) ≡ action-of (R(false ⊢ true))
definition top-action : (top::('a, 'b) action) ≡ action-of (R(true ⊢ false))

instance
proof
  fix x::('a, 'b) action
  {
    show bot ≤ x
      unfolding bot-action
      apply (auto simp add: less-action less-eq-action ref-def bot-action)
      apply (subst action-of-inverse) apply (rule bot-is-action)
      apply (subst bot-eq-true)
      apply (subst (asm) CSP-is-rd)
      apply (rule relation-of-CSP)
      apply (auto simp add: csp-defs rp-defs fun-eq-iff split: cond-splits)
      done
  next
    show x ≤ top
      apply (auto simp add: less-action less-eq-action ref-def top-action)
      apply (subst (asm) action-of-inverse)
      apply (simp)
      apply (rule rd-is-CSP)
      apply auto
      apply (subst action-of-cases[where x=x], simp-all)
      apply (subst action-of-inverse, simp-all)
      apply (subst CSP-is-rd[where P=y], simp-all)
      apply (auto simp: rp-defs design-defs fun-eq-iff split: cond-splits)
      done
  }
qed
```

end

lemma *relation-of-top*: *relation-of top* = *R*(*true* ⊢ *false*)
  apply (*simp add*: *top-action*)
  apply (*subst action-of-inverse*)
  apply (*simp*)
  apply (*rule rd-is-CSP*)
  apply (*auto simp add*: *utp-defs design-defs rp-defs*)
  done

lemma *relation-of-bot*: *relation-of bot* = *R true*
  apply (*simp add*: *bot-action*)
  apply (*subst action-of-inverse*)
  apply (*simp add*: *bot-is-action*[*simplified*], *rule bot-eq-true*)
  done

lemma *non-emptyE*: **assumes** $A \neq \{\}$ **obtains** x **where** $x : A$
  using *assms* **by** (*auto simp add*: *ex-in-conv* [*symmetric*])

lemma *CSP1-Inf*:
assumes $*{:}A \neq \{\}$
shows ($\bigsqcap$ *relation-of* ‘ *A*) *is CSP1 healthy*
proof −
  have ($\bigsqcap$ *relation-of* ‘ *A*) = *CSP1* ($\bigsqcap$ *relation-of* ‘ *A*)
  proof
    fix *P*
    note ∗ **then**
    show ($P \in \bigcup\{\{p.\ P\ p\}\ |P.\ P \in$ *relation-of* ‘ $A\}$) = *CSP1* ($\lambda Aa.\ Aa \in \bigcup\{\{p.$
$P\ p\}\ |P.\ P \in$ *relation-of* ‘ $A\}$) *P*
    apply (*intro iffI*)
    apply (*simp-all add*: *csp-defs*)
    apply (*rule disj-introC*, *simp*)
    apply (*erule disj-elim*, *simp-all*)
    apply (*cases P*, *simp-all*)
    apply (*erule non-emptyE*)
    apply (*rule-tac x*=*Collect* (*relation-of x*) **in** *exI*, *simp*)
    apply (*rule conjI*)
    apply (*rule-tac x*=(*relation-of x*) **in** *exI*, *simp*)
    apply (*subst CSP-is-rd*, *simp add*: *relation-of-CSP*)
    apply (*auto simp add*: *csp-defs design-defs rp-defs fun-eq-iff split*: *cond-splits*)
    done
  qed
  then show ($\bigsqcap$ *relation-of* ‘ *A*) *is CSP1 healthy* **by** (*simp add*: *design-defs*)
qed

lemma *CSP2-Inf*:
assumes $*{:}A \neq \{\}$
shows ($\bigsqcap$ *relation-of* ‘ *A*) *is CSP2 healthy*
proof −

have (⨅ *relation-of* ‘ *A*) = *CSP2* (⨅ *relation-of* ‘ *A*)
proof
fix *P*
note ∗ **then**
show (*P* ∈ ⋃{{*p*. *P p*} |*P*. *P* ∈ *relation-of* ‘ *A*}) = *CSP2* (λ*Aa*. *Aa* ∈ ⋃{{*p*. *P p*} |*P*. *P* ∈ *relation-of* ‘ *A*}) *P*
apply (*intro iffI*)
apply (*simp-all add*: *csp-defs*)
apply (*cases P*, *simp-all*)
apply (*erule exE*)
apply (*rule-tac b*=*b* **in** *comp-intro*, *simp-all*)
apply (*rule-tac x*=*x* **in** *exI*, *simp*)
apply (*erule comp-elim*, *simp-all*)
apply (*erule exE* | *erule conjE*)+
apply (*simp-all*)
apply (*rule-tac x*=*Collect Pa* **in** *exI*, *simp*)
apply (*rule conjI*)
apply (*rule-tac x*=*Pa* **in** *exI*, *simp*)
apply (*erule Set.imageE*, *simp add*: *relation-of*)
apply (*subst CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*subst* (*asm*) *CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*auto simp add*: *csp-defs rp-defs prefix-def design-defs fun-eq-iff split*: *cond-splits*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *False*|) = *c*(|*tr* := *zs*, *ok* := *False*|), *auto*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *False*|) = *c*(|*tr* := *zs*, *ok* := *False*|), *auto*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *False*|) = *c*(|*tr* := *zs*, *ok* := *False*|), *auto*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *False*|) = *c*(|*tr* := *zs*, *ok* := *False*|), *auto*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *False*|) = *c*(|*tr* := *zs*, *ok* := *False*|), *auto*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *False*|) = *c*(|*tr* := *zs*, *ok* := *False*|), *auto*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *True*|) = *c*(|*tr* := *zs*, *ok* := *True*|), *auto*)
apply (*subgoal-tac b*(|*tr* := *zs*, *ok* := *True*|) = *c*(|*tr* := *zs*, *ok* := *True*|), *auto*)
done
qed
then show (⨅ *relation-of* ‘ *A*) *is CSP2 healthy* **by** (*simp add*: *design-defs*)
qed

lemma *R-Inf*:
assumes ∗:*A* ≠ {}
shows (⨅ *relation-of* ‘ *A*) *is R healthy*
proof −
have (⨅ *relation-of* ‘ *A*) = *R* (⨅ *relation-of* ‘ *A*)
proof
fix *P*
show (*P* ∈ ⋃{{*p*. *P p*} |*P*. *P* ∈ *relation-of* ‘ *A*}) = *R* (λ*Aa*. *Aa* ∈ ⋃{{*p*. *P p*} |*P*. *P* ∈ *relation-of* ‘ *A*}) *P*
apply (*cases P*, *simp-all*)
apply (*rule*)
apply (*simp-all add*: *csp-defs rp-defs split*: *cond-splits*)
apply (*erule exE*)

apply (*erule exE* | *erule conjE*)+
apply (*simp-all*)
apply (*erule Set.imageE*, *simp add*: *relation-of*)
apply (*subst* (*asm*) *CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*simp add*: *csp-defs prefix-def design-defs rp-defs fun-eq-iff split*: *cond-splits*)
apply (*rule-tac x*=*x* **in** *exI*, *simp*)
apply (*rule conjI*)
apply (*rule-tac x*=*relation-of xa* **in** *exI*, *simp*)
apply (*subst CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*simp add*: *csp-defs prefix-def design-defs rp-defs fun-eq-iff split*: *cond-splits*)
apply (*insert* ∗)
apply (*erule non-emptyE*)
apply (*rule-tac x*=*Collect* (*relation-of x*) **in** *exI*, *simp*)
apply (*rule conjI*)
apply (*rule-tac x*=(*relation-of x*) **in** *exI*, *simp*)
apply (*subst CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*simp add*: *csp-defs prefix-def design-defs rp-defs fun-eq-iff split*: *cond-splits*)
apply (*erule exE* | *erule conjE*)+
apply (*simp-all*)
apply (*erule Set.imageE*, *simp add*: *relation-of*)
apply (*rule-tac x*=*x* **in** *exI*, *simp*)
apply (*rule conjI*)
apply (*rule-tac x*=(*relation-of xa*) **in** *exI*, *simp*)
apply (*subst CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*simp add*: *csp-defs prefix-def design-defs rp-defs fun-eq-iff split*: *cond-splits*)
apply (*subst* (*asm*) *CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*simp add*: *csp-defs prefix-def design-defs rp-defs fun-eq-iff split*: *cond-splits*)
done
qed
then show (⨅ *relation-of* ‘ *A*) *is R healthy* **by** (*simp add*: *design-defs*)
qed

lemma *CSP-Inf*:
assumes $A \neq \{\}$
shows *is-CSP-process* (⨅ *relation-of* ‘ *A*)
unfolding *is-CSP-process-def*
using *assms CSP1-Inf CSP2-Inf R-Inf*
by *auto*

lemma *Inf-is-action*: $A \neq \{\} \Longrightarrow$ ⨅ *relation-of* ‘ *A* ∈ {*p*. *is-CSP-process p*}
by (*auto dest*!: *CSP-Inf*)

lemma *CSP1-Sup*: $A \neq \{\} \Longrightarrow$ (⨆ *relation-of* ‘ *A*) *is CSP1 healthy*
apply (*auto simp add*: *design-defs csp-defs fun-eq-iff*)
apply (*subst CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*simp add*: *csp-defs prefix-def design-defs rp-defs split*: *cond-splits*)
done

lemma *CSP2-Sup*: $A \neq \{\} \Longrightarrow$ (⨆ *relation-of* ‘ *A*) *is CSP2 healthy*

supply [[*simproc del*: *defined-all*]]
apply (*simp add*: *design-defs csp-defs fun-eq-iff*)
apply (*rule allI*)+
apply (*rule*)
apply (*rule-tac b*=*b* **in** *comp-intro*, *simp-all*)
apply (*erule comp-elim*, *simp-all*)
apply (*rule allI*)
apply (*erule-tac x*=*x* **in** *allE*)
apply (*rule impI*)
apply (*case-tac* (∃ *P*. *x* = *Collect P* & *P* ∈ *relation-of* ' *A*), *simp-all*)
apply (*erule exE* | *erule conjE*)+
apply (*simp-all*)
apply (*erule Set.imageE*, *simp add*: *relation-of*)
apply (*subst* (*asm*) *CSP-is-rd*, *simp add*: *relation-of-CSP*, *subst CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*auto simp add*: *csp-defs design-defs rp-defs split*: *cond-splits*)
apply (*subgoal-tac ba*(|*tr* := *tr c* − *tr aa*, *ok* := *False*|) = *c*(|*tr* := *tr c* − *tr aa*, *ok* := *False*|), *simp-all*)
apply (*subgoal-tac ba*(|*tr* := *tr c* − *tr aa*, *ok* := *False*|) = *c*(|*tr* := *tr c* − *tr aa*, *ok* := *False*|), *simp-all*)
apply (*subgoal-tac ba*(|*tr* := *tr c* − *tr aa*, *ok* := *False*|) = *c*(|*tr* := *tr c* − *tr aa*, *ok* := *False*|), *simp-all*)
apply (*subgoal-tac ba*(|*tr* := *tr c* − *tr aa*, *ok* := *False*|) = *c*(|*tr* := *tr c* − *tr aa*, *ok* := *False*|), *simp-all*)
apply (*subgoal-tac ba*(|*tr* := *tr c* − *tr aa*, *ok* := *False*|) = *c*(|*tr* := *tr c* − *tr aa*, *ok* := *False*|), *simp-all*)
apply (*subgoal-tac ba*(|*tr* := *tr c* − *tr aa*, *ok* := *False*|) = *c*(|*tr* := *tr c* − *tr aa*, *ok* := *False*|), *simp-all*)
apply (*subgoal-tac ba*(|*tr* := *tr c* − *tr aa*, *ok* := *True*|) = *c*(|*tr* := *tr c* − *tr aa*, *ok* := *True*|), *simp-all*)
apply (*subgoal-tac ba*(|*tr* := *tr c* − *tr aa*, *ok* := *True*|) = *c*(|*tr* := *tr c* − *tr aa*, *ok* := *True*|), *simp-all*)
done

lemma *R-Sup*: *A* ≠ {} ⟹ (⨆ *relation-of* ' *A*) *is R healthy*
apply (*simp add*: *rp-defs design-defs csp-defs fun-eq-iff*)
apply (*rule allI*)+
apply (*rule*)
apply (*simp split*: *cond-splits*)
apply (*case-tac wait a*, *simp-all*)
apply (*erule non-emptyE*)
apply (*erule-tac x*=*Collect* (*relation-of x*) **in** *allE*, *simp-all*)
apply (*case-tac relation-of x* (*a*, *b*), *simp-all*)
apply (*subst* (*asm*) *CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*simp add*: *csp-defs design-defs rp-defs split*: *cond-splits*)
apply (*erule-tac x*=(*relation-of x*) **in** *allE*, *simp-all*)
apply (*rule conjI*)
apply (*rule allI*)
apply (*erule-tac x*=*x* **in** *allE*)

apply (*rule impI*)
apply (*case-tac* (∃ *P*. *x* = *Collect P* & *P* ∈ *relation-of* ‘ *A*), *simp-all*)
apply (*erule exE* | *erule conjE*)+
apply (*simp-all*)
apply (*erule Set.imageE*, *simp add*: *relation-of*)
apply (*subst* (*asm*) *CSP-is-rd*, *simp add*: *relation-of-CSP*, *subst CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*simp add*: *csp-defs design-defs rp-defs split*: *cond-splits*)
apply (*erule non-emptyE*)
apply (*erule-tac x*=*Collect* (*relation-of x*) **in** *allE*, *simp-all*)
apply (*case-tac relation-of x* (*a*, *b*), *simp-all*)
apply (*subst* (*asm*) *CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*simp add*: *csp-defs design-defs rp-defs split*: *cond-splits*)
apply (*erule-tac x*=(*relation-of x*) **in** *allE*, *simp-all*)
apply (*simp split*: *cond-splits*)
apply (*rule allI*)
apply (*rule impI*)
apply (*erule exE* | *erule conjE*)+
apply (*simp-all*)
apply (*erule Set.imageE*, *simp add*: *relation-of*)
apply (*subst* (*asm*) *CSP-is-rd*, *simp add*: *relation-of-CSP*, *subst CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*simp add*: *csp-defs design-defs rp-defs split*: *cond-splits*)
apply (*rule allI*)
apply (*rule impI*)
apply (*erule exE* | *erule conjE*)+
apply (*simp-all*)
apply (*erule Set.imageE*, *simp add*: *relation-of*)
apply (*erule-tac x*=*x* **in** *allE*, *simp-all*)
apply (*case-tac relation-of xa* (*a*(|*tr* := []|), *b*(|*tr* := *tr b* − *tr a*|)), *simp-all*)
apply (*subst* (*asm*) *CSP-is-rd*) **back back back**
apply (*simp add*: *relation-of-CSP*, *subst CSP-is-rd*, *simp add*: *relation-of-CSP*)
apply (*simp add*: *csp-defs design-defs rp-defs split*: *cond-splits*)
apply (*erule-tac x*=*P* **in** *allE*, *simp-all*)
done

lemma *CSP-Sup*: *A* ≠ {} ⟹ *is-CSP-process* (⨆ *relation-of* ‘ *A*)
unfolding *is-CSP-process-def* **using** *CSP1-Sup CSP2-Sup R-Sup* **by** *auto*

lemma *Sup-is-action*: *A* ≠ {} ⟹ ⨆ *relation-of* ‘ *A* ∈ {*p*. *is-CSP-process p*}
by (*auto dest*!: *CSP-Sup*)

lemma *relation-of-Sup*:
A ≠ {} ⟹ *relation-of* (*action-of* ⨆ *relation-of* ‘ *A*) = ⨆ *relation-of* ‘ *A*
by (*auto simp*: *action-of-inverse dest*!: *Sup-is-action*)

instantiation *action* :: (*ev-eq*, *type*) *complete-lattice*
begin

definition *Sup-action* :
(*Sup* (*S*:: ('*a*, '*b*) *action set*) ≡ *if S*={} *then bot else action-of* $\bigsqcup$ (*relation-of* ‘ *S*))
definition *Inf-action* :
(*Inf* (*S*:: ('*a*, '*b*) *action set*) ≡ *if S*={} *then top else action-of* $\bigsqcap$ (*relation-of* ‘ *S*))

instance
proof
  fix *A*::('*a*, '*b*) *action set* **and** *z*::('*a*, '*b*) *action*
  {
    fix *x*::('*a*, '*b*) *action*
    assume $x \in A$
    then show *Inf A* ≤ *x*
      apply (*auto simp add*: *less-action less-eq-action Inf-action ref-def*)
      apply (*subst* (*asm*) *action-of-inverse*)
      apply (*auto intro*: *Inf-is-action*[*simplified*])
      done
  } **note** *rule1* = *this*
  {
    assume ∗: $\bigwedge x.\ x \in A \Longrightarrow z \le x$
    show *z* ≤ *Inf A*
    proof (*cases A* = {})
      case *True*
      then show *?thesis* **by** (*simp add*: *Inf-action*)
    next
      case *False*
      show *?thesis*
        using ∗
        apply (*auto simp add*: *Inf-action*)
        using ‹*A* ≠ {}›
        apply (*simp add*: *less-eq-action Inf-action ref-def*)
        apply (*subst* (*asm*) *action-of-inverse*)
        apply (*subst* (*asm*) *ex-in-conv*[*symmetric*])
        apply (*erule exE*)
        apply (*auto intro*: *Inf-is-action*[*simplified*])
        done
    qed
  }{
    fix *x*::('*a*, '*b*) *action*
    assume $x \in A$
    then show *x* ≤ (*Sup A*)
      apply (*auto simp add*: *less-action less-eq-action Sup-action ref-def*)
      apply (*subst* (*asm*) *action-of-inverse*)
      apply (*auto intro*: *Sup-is-action*[*simplified*])
      done
  } **note** *rule2* = *this*
  {
    assume $\bigwedge x.\ x \in A \Longrightarrow x \le z$
    then show *Sup A* ≤ *z*
      apply (*auto simp add*: *Sup-action*)

apply *atomize*
apply (*case-tac A* = {}, *simp-all*)
apply (*insert rule2*)
apply (*auto simp add*: *less-action less-eq-action Sup-action ref-def*)
apply (*subst* (*asm*) *action-of-inverse*)
apply (*auto intro*: *Sup-is-action*[*simplified*])
apply (*subst* (*asm*) *action-of-inverse*)
apply (*auto intro*: *Sup-is-action*[*simplified*])
done
}
{ show *Inf* ({}::(′*a*, ′*b*) *action set*) = *top* **by**(*simp add*:*Inf-action*) **}**
{ show *Sup* ({}::(′*a*, ′*b*) *action set*) = *bot* **by**(*simp add*:*Sup-action*) **}**
qed

end

end

13 Circus variables

theory *Var-list*
imports *Main*
begin

Circus variables are represented by a stack (list) of values. they are characterized by two functions, *select* and *update*. The Circus variable type is defined as a tuple (*select* * *update*) with a list of values instead of a single value.

type-synonym (′*a*, ′*σ*) *var-list* = (′*σ* ⇒ ′*a list*) ∗ ((′*a list* ⇒ ′*a list*) ⇒ ′*σ* ⇒ ′*σ*)

The *select* function returns the top value of the stack.

definition *select* :: (′*a*, ′*r*) *var-list* ⇒ ′*r* ⇒ ′*a*
where *select f* ≡ λ *A*. *hd* ((*fst f*) *A*)

The *increase* function pushes a new value to the top of the stack.

definition *increase* :: (′*a*, ′*r*) *var-list* ⇒ ′*a* ⇒ ′*r* ⇒ ′*r*
where *increase f val* ≡ (*snd f*) (λ *l*. *val*#*l*)

The *increase*0 function pushes an arbitrary value to the top of the stack.

definition *increase0* :: (′*a*, ′*r*) *var-list* ⇒ ′*r* ⇒ ′*r*
where *increase0 f* ≡ (*snd f*) (λ *l*. ((*SOME val*. *True*)#*l*))

The *decrease* function pops the top value of the stack.

definition *decrease* :: (′*a*, ′*r*) *var-list* ⇒ ′*r* ⇒ ′*r*
where *decrease f* ≡ (*snd f*) (λ *l*. (*tl l*))

The *update* function updates the top value of the stack.

definition *update* :: ('*a*, '*r*) *var-list* ⇒ ('*a* ⇒ '*a*) ⇒ '*r* ⇒ '*r*
where *update f upd* ≡ (*snd f*) (λ *l*. (*upd* (*hd l*))#(*tl l*))

The *update*0 function initializes the top of the stack with an arbitrary value.

definition *update0* :: ('*a*, '*r*) *var-list* ⇒ '*r* ⇒ '*r*
where *update0 f* ≡ (*snd f*) (λ *l*. ((*SOME upd*. *True*) (*hd l*))#(*tl l*))

axiomatization **where** *select-increase*: (*select v* (*increase v a s*)) = *a*

The $VAR-LIST$ function allows to retrieve a Circus variable from its name.

syntax *-VAR-LIST* :: *id* ⇒ ('*a*, '*r*) *var-list* (‹*VAR'-LIST* -›)
translations *VAR-LIST x* => (*x*, *-update-name x*)

end

14 Denotational semantics of Circus actions

theory *Denotational-Semantics*
imports *Circus-Actions Var-list*
begin

In this section, we introduce the definitions of Circus actions denotational semantics. We provide the proof of well-formedness of every action. We also provide proofs concerning the monotonicity of operators over actions.

14.1 Skip

definition *Skip* :: ('ϑ::*ev-eq*,'σ) *action* **where**
Skip ≡ *action-of*
(*R* (*true* ⊢ λ(*A*, *A*'). *tr A*' = *tr A* ∧ ¬*wait A*' ∧ *more A* = *more A*'))

lemma *Skip-is-action*:
(*R* (*true* ⊢ λ(*A*, *A*'). *tr A*' = *tr A* ∧ ¬*wait A*' ∧ *more A* = *more A*')) ∈ {*p*. *is-CSP-process p*}
apply (*simp*)
apply (*rule rd-is-CSP*)
by *auto*

lemmas *Skip-is-CSP* = *Skip-is-action*[*simplified*]

lemma *relation-of-Skip*:
relation-of Skip =
(*R* (*true* ⊢ λ(*A*, *A*'). *tr A*' = *tr A* ∧ ¬*wait A*' ∧ *more A* = *more A*'))
by (*simp add*: *Skip-def action-of-inverse Skip-is-CSP*)

definition *CSP3*::(('ϑ::*ev-eq*,'σ) *alphabet-rp*) *Healthiness-condition*
where *CSP3* (*P*) ≡ *relation-of Skip* ; ; *P*

definition *CSP4*::(('ϑ::*ev-eq*,'σ) *alphabet-rp*) *Healthiness-condition*
where *CSP4* (*P*) ≡ *P* ; ; *relation-of Skip*

lemma *Skip-is-CSP3*: (*relation-of Skip*) *is CSP3 healthy*
apply (*auto simp*: *relation-of-Skip rp-defs design-defs fun-eq-iff CSP3-def*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b*=*b* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b*=*b* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
prefer *3*
apply (*split cond-splits*, *simp-all*)+
apply (*auto simp add*: *prefix-def*)
done

lemma *Skip-is-CSP4*: (*relation-of Skip*) *is CSP4 healthy*
apply (*auto simp*: *relation-of-Skip rp-defs design-defs fun-eq-iff CSP4-def*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b*=*b* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b*=*b* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
prefer *3*
apply (*split cond-splits*, *simp-all*)+
apply (*auto simp add*: *prefix-def*)
done

lemma *Skip-comp-absorb*: (*relation-of Skip* ; ; *relation-of Skip*) = *relation-of Skip*
apply (*auto simp*: *relation-of-Skip fun-eq-iff rp-defs true-def design-defs*)
apply (*clarsimp split*: *cond-splits*)+
apply (*case-tac ok aa*, *simp-all*)
apply (*erule disjE*)+
apply (*clarsimp simp*: *prefix-def*)
apply (*clarsimp simp*: *prefix-def*)
apply (*erule disjE*)+
apply (*clarsimp simp*: *prefix-def*)
apply (*clarsimp simp*: *prefix-def*)
apply (*erule disjE*)+
apply (*clarsimp simp*: *prefix-def*)
apply (*clarsimp simp*: *prefix-def*)
apply (*case-tac ok aa*, *simp-all*)
apply (*clarsimp simp*: *prefix-def*)
apply (*clarsimp split*: *cond-splits*)+
apply (*rule-tac b*=*a* **in** *comp-intro*)
apply (*clarsimp split*: *cond-splits*)+
apply (*rule-tac b*=*a* **in** *comp-intro*)
apply (*clarsimp split*: *cond-splits*)+
done

14.2 Stop

definition *Stop* :: ('ϑ::*ev-eq*,'σ) *action*
where *Stop* ≡ *action-of* (*R* (*true* ⊢ λ(*A*, *A'*). *tr A'* = *tr A* ∧ *wait A'*))

lemma *Stop-is-action*:
(*R* (*true* ⊢ λ(*A*, *A'*). *tr A'* = *tr A* ∧ *wait A'*)) ∈ {*p*. *is-CSP-process p*}
apply (*simp*)
apply (*rule rd-is-CSP*)
by *auto*

lemmas *Stop-is-CSP* = *Stop-is-action*[*simplified*]

lemma *relation-of-Stop*:
relation-of Stop = (*R* (*true* ⊢ λ(*A*, *A'*). *tr A'* = *tr A* ∧ *wait A'*))
by (*simp add*: *Stop-def action-of-inverse Stop-is-CSP*)

lemma *Stop-is-CSP3*: (*relation-of Stop*) *is CSP3 healthy*
apply (*auto simp*: *relation-of-Stop relation-of-Skip rp-defs design-defs fun-eq-iff CSP3-def*)
apply (*rule-tac b*=*a* **in** *comp-intro*)
apply (*split cond-splits*, *auto*)
apply (*split cond-splits*)+
apply (*simp-all*)
apply (*case-tac ok aa*, *simp-all*)
apply (*case-tac tr aa* ≤ *tr ba*, *simp-all*)
apply (*case-tac ok ba*, *simp-all*)
apply (*case-tac tr ba* ≤ *tr c*, *simp-all*)
apply (*rule disjI1*)
apply (*simp add*: *prefix-def*)
apply (*erule exE*)+
apply (*rule-tac x*=*zs*@*zsa* **in** *exI*, *simp*)
apply (*rule disjI1*)
apply (*simp add*: *prefix-def*)
apply (*erule exE* | *erule conjE*)+
apply (*rule-tac x*=*zs*@*zsa* **in** *exI*, *simp*)
apply (*split cond-splits*)+
apply (*simp-all add*: *true-def*)
apply (*erule disjE*)
apply (*simp add*: *prefix-def*)
apply (*erule exE* | *erule conjE*)+
apply (*rule-tac x*=*zs*@*zsa* **in** *exI*, *simp*)
apply (*auto simp add*: *prefix-def*)
done

lemma *Stop-is-CSP4*: (*relation-of Stop*) *is CSP4 healthy*
apply (*auto simp*: *relation-of-Stop relation-of-Skip rp-defs design-defs fun-eq-iff CSP4-def*)

apply (*rule-tac b=b* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*case-tac ok aa*, *simp-all*)
apply (*case-tac tr aa* ≤ *tr ba*, *simp-all*)
apply (*case-tac ok ba*, *simp-all*)
apply (*case-tac tr ba* ≤ *tr c*, *simp-all*)
apply (*rule disjI1*)
apply (*simp add*: *prefix-def*)
apply (*erule exE*)+
apply (*rule-tac x=zs@zsa* **in** *exI*, *simp*)
apply (*rule disjI1*)
apply (*simp add*: *prefix-def*)
apply (*erule exE* | *erule conjE*)+
apply (*rule-tac x=zs@zsa* **in** *exI*, *simp*)
apply (*split cond-splits*)+
apply (*simp-all add*: *true-def*)
apply (*erule disjE*)
apply (*simp add*: *prefix-def*)
apply (*erule exE* | *erule conjE*)+
apply (*rule-tac x=zs@zsa* **in** *exI*, *simp*)
apply (*auto simp add*: *prefix-def*)
done

14.3 Chaos

definition *Chaos* :: ('ϑ::*ev-eq*,'σ) *action*
where *Chaos* ≡ *action-of* (*R*(*false* ⊢ *true*))

lemma *Chaos-is-action*: (*R*(*false* ⊢ *true*)) ∈ {*p*. *is-CSP-process p*}
apply (*simp*)
apply (*rule rd-is-CSP*)
by *auto*

lemmas *Chaos-is-CSP* = *Chaos-is-action*[*simplified*]

lemma *relation-of-Chaos*: *relation-of Chaos* = (*R*(*false* ⊢ *true*))
by (*simp add*: *Chaos-def action-of-inverse Chaos-is-CSP*)

14.4 State update actions

definition *Pre* ::'σ *relation* ⇒ 'σ *predicate*
where *Pre sc* ≡ λ*A*. ∃ *A*′. *sc* (*A*, *A*′)

definition *state-update-before* :: 'σ *relation* ⇒ ('ϑ::*ev-eq*,'σ) *action* ⇒ ('ϑ,'σ) *action*
where *state-update-before sc Ac* = *action-of*(*R* ((λ(*A*, *A*′). (*Pre sc*) (*more A*)) ⊢
(λ(*A*, *A*′). *sc* (*more A*, *more A*′) & ¬*wait A*′ & *tr A* = *tr A*′))
; ; *relation-of Ac*)

lemma *state-update-before-is-action*:
(*R* ((λ(*A*, *A′*). (*Pre sc*) (*more A*)) ⊢
(λ(*A*, *A′*).*sc* (*more A*, *more A′*) & ¬*wait A′* & *tr A* = *tr A′*)) ; ; *relation-of Ac*) ∈ {*p*. *is-CSP-process p*}
apply (*simp*)
apply (*rule seq-CSP*)
apply (*rule rd-is-CSP1*)
apply (*auto simp*: *R-idem2 Healthy-def relation-of-CSP*)
done

lemmas *state-update-before-is-CSP* = *state-update-before-is-action*[*simplified*]

lemma *relation-of-state-update-before*:
relation-of (*state-update-before sc Ac*) = (*R* ((λ(*A*, *A′*). (*Pre sc*) (*more A*)) ⊢
(λ(*A*, *A′*). *sc* (*more A*, *more A′*) & ¬*wait A′* & *tr A* = *tr A′*)) ; ; *relation-of Ac*)
by (*simp add*: *state-update-before-def action-of-inverse state-update-before-is-CSP*)

lemma *mono-state-update-before*: *mono* (*state-update-before sc*)
by (*auto simp*: *mono-def less-eq-action ref-def relation-of-state-update-before design-defs rp-defs fun-eq-iff*
split: *cond-splits dest*: *relation-of-spec-f-f*[*simplified*]
relation-of-spec-t-f[*simplified*])

lemma *state-update-before-is-CSP3*: *relation-of* (*state-update-before sc Ac*) *is CSP3 healthy*
apply (*auto simp*: *relation-of-state-update-before relation-of-Skip rp-defs design-defs fun-eq-iff CSP3-def*)
apply (*rule-tac b*=*aa* **in** *comp-intro*)
apply (*split cond-splits*, *auto*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b*=*bb* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*case-tac ok aa*, *simp-all*)
apply (*case-tac tr aa* ≤ *tr ab*, *simp-all*)
apply (*case-tac ok ab*, *simp-all*)
apply (*case-tac tr ab* ≤ *tr bb*, *simp-all*)
apply (*rule disjI1*)
apply (*simp add*: *prefix-def*)
apply (*erule exE*)+
apply (*rule-tac x*=*zs*@*zsa* **in** *exI*, *simp*)
apply (*rule-tac b*=*bb* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule disjI1*)
apply (*simp add*: *prefix-def*)
apply (*erule exE* | *erule conjE*)+
apply (*rule-tac x*=*zs*@*zsa* **in** *exI*, *simp*)
apply (*rule-tac b*=*bb* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+

apply (*simp-all add*: *true-def*)
apply (*erule disjE*)
apply (*simp add*: *prefix-def*)
apply (*erule exE* | *erule conjE*)+
apply (*rule-tac x=zs@zsa* **in** *exI*, *simp*)
apply (*auto simp add*: *prefix-def*)
done

lemma *state-update-before-is-CSP4*:
assumes *A* : *relation-of Ac is CSP4 healthy*
shows *relation-of* (*state-update-before sc Ac*) *is CSP4 healthy*
apply (*auto simp*: *relation-of-state-update-before relation-of-Skip rp-defs design-defs fun-eq-iff CSP4-def*)
apply (*rule-tac b=c* **in** *comp-intro*)
apply (*rule-tac b=ba* **in** *comp-intro*, *simp-all*)
apply (*split cond-splits*, *simp-all*)
apply (*rule-tac b=bb* **in** *comp-intro*, *simp-all*)
apply (*subst A*[*simplified design-defs rp-defs CSP4-def relation-of-Skip*])
apply (*auto simp*: *rp-defs*)
done

definition *state-update-after* :: $'\sigma$ *relation* $\Rightarrow$ $('\vartheta{::}ev\text{-}eq,'\sigma)$ *action* $\Rightarrow$ $('\vartheta,'\sigma)$ *action*
where *state-update-after sc Ac* = *action-of*(*relation-of Ac* ; ; *R* (*true* ⊢ (λ(*A*, *A'*). *sc* (*more A*, *more A'*) & ¬*wait A'* & *tr A* = *tr A'*)))

lemma *state-update-after-is-action*:
(*relation-of Ac* ; ; *R* (*true* ⊢ (λ(*A*, *A'*). *sc* (*more A*, *more A'*) & ¬*wait A'* & *tr A* = *tr A'*))) ∈ {*p*. *is-CSP-process p*}
apply (*simp*)
apply (*rule seq-CSP*)
apply (*auto simp*: *relation-of-CSP*[*simplified is-CSP-process-def*])
apply (*rule rd-is-CSP*, *auto*)
done

lemmas *state-update-after-is-CSP* = *state-update-after-is-action*[*simplified*]

lemma *relation-of-state-update-after*:
relation-of (*state-update-after sc Ac*) = (*relation-of Ac* ; ; *R* (*true* ⊢ (λ(*A*, *A'*). *sc* (*more A*, *more A'*) & ¬*wait A'* & *tr A* = *tr A'*)))
by (*simp add*: *state-update-after-def action-of-inverse state-update-after-is-CSP*)

lemma *mono-state-update-after*: *mono* (*state-update-after sc*)
by (*auto simp*: *mono-def less-eq-action ref-def relation-of-state-update-after design-defs rp-defs fun-eq-iff*
split: *cond-splits dest*: *relation-of-spec-f-f*[*simplified*]
relation-of-spec-t-f[*simplified*])

lemma *state-update-after-is-CSP3*:
  assumes *A* : *relation-of Ac is CSP3 healthy*
  shows *relation-of* (*state-update-after sc Ac*) *is CSP3 healthy*
apply (*auto simp*: *relation-of-state-update-after relation-of-Skip rp-defs design-defs fun-eq-iff CSP3-def*)
apply (*rule-tac b=aa* **in** *comp-intro*)
apply (*split cond-splits*, *auto*)
apply (*rule-tac b=bb* **in** *comp-intro*, *simp-all*)
apply (*subst A*[*simplified design-defs rp-defs CSP3-def relation-of-Skip*])
apply (*auto simp*: *rp-defs*)
done

lemma *state-update-after-is-CSP4*: *relation-of* (*state-update-after sc Ac*) *is CSP4 healthy*
apply (*auto simp*: *relation-of-state-update-after relation-of-Skip rp-defs design-defs fun-eq-iff CSP4-def*)
apply (*rule-tac b=c* **in** *comp-intro*)
apply (*rule-tac b=ba* **in** *comp-intro*, *simp-all*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=bb* **in** *comp-intro*, *simp-all*)
apply (*split cond-splits*, *simp-all*)+
apply (*case-tac ok bb*, *simp-all*)
apply (*case-tac tr bb* ≤ *tr c*, *simp-all*)
apply (*case-tac ok ca*, *simp-all*)
apply (*case-tac tr ca* ≤ *tr c*, *simp-all*)
apply (*auto simp add*: *prefix-def comp-def true-def split*: *cond-splits*)
done

14.5 Sequential composition

definition
Seq::(′ϑ::*ev-eq*,′σ) *action* ⇒ (′ϑ,′σ) *action* ⇒ (′ϑ,′σ) *action* (**infixl** ‹‘; ‘› *24*)
where *P* ‘; ‘ *Q* ≡ *action-of* (*relation-of P* ; ; *relation-of Q*)

lemma *Seq-is-action*: (*relation-of P* ; ; *relation-of Q*) ∈ {*p*. *is-CSP-process p*}
apply (*simp*)
apply (*rule seq-CSP*[*OF relation-of-CSP*[*THEN CSP-is-CSP1*] *relation-of-CSP*[*THEN CSP-is-R*] *relation-of-CSP*])
done

lemmas *Seq-is-CSP* = *Seq-is-action*[*simplified*]

lemma *relation-of-Seq*: *relation-of* (*P* ‘; ‘ *Q*) = (*relation-of P* ; ; *relation-of Q*)
by (*simp add*: *Seq-def action-of-inverse Seq-is-CSP*)

lemma *mono-Seq*: *mono* ((‘; ‘) *P*)
  by (*auto simp*: *mono-def less-eq-action ref-def relation-of-Seq*)

lemma *CSP3-imp-left-Skip*:
  assumes *A*: *relation-of P is CSP3 healthy*
  shows (*Skip* ‘; ‘ *P*) = *P*
apply (*subst relation-of-inject*[*symmetric*])
apply (*simp add*: *relation-of-Seq A*[*simplified design-defs CSP3-def*, *symmetric*])
done

lemma *CSP4-imp-right-Skip*:
  assumes *A*: *relation-of P is CSP4 healthy*
  shows (*P* ‘; ‘ *Skip*) = *P*
apply (*subst relation-of-inject*[*symmetric*])
apply (*simp add*: *relation-of-Seq A*[*simplified design-defs CSP4-def*, *symmetric*])
done

lemma *Seq-assoc*: (*A* ‘; ‘ (*B* ‘; ‘ *C*)) = ((*A* ‘; ‘ *B*) ‘; ‘ *C*)
by (*auto simp*: *relation-of-inject*[*symmetric*] *fun-eq-iff relation-of-Seq rp-defs design-defs*)

lemma *Skip-absorb*: (*Skip* ‘; ‘ *Skip*) = *Skip*
by (*auto simp*: *Skip-comp-absorb relation-of-inject*[*symmetric*] *relation-of-Seq*)

14.6 Internal choice

definition
Ndet::($'\vartheta$::*ev-eq*,$'\sigma$) *action* ⇒ ($'\vartheta$,$'\sigma$) *action* ⇒ ($'\vartheta$,$'\sigma$) *action* (**infixl** ‹⊓› *18*)
where *P* ⊓ *Q* ≡ *action-of* ((*relation-of P*) ∨ (*relation-of Q*))

lemma *Ndet-is-action*: ((*relation-of P*) ∨ (*relation-of Q*)) ∈ {*p*. *is-CSP-process p*}
apply (*simp*)
apply (*rule disj-CSP*)
apply (*simp-all add*: *relation-of-CSP*)
done

lemmas *Ndet-is-CSP* = *Ndet-is-action*[*simplified*]

lemma *relation-of-Ndet*: *relation-of* (*P* ⊓ *Q*) = ((*relation-of P*) ∨ (*relation-of Q*))
by (*simp add*: *Ndet-def action-of-inverse Ndet-is-CSP*)

lemma *mono-Ndet*: *mono* ((⊓) *P*)
by (*auto simp*: *mono-def less-eq-action ref-def relation-of-Ndet*)

14.7 External choice

definition
Det::($'\vartheta$::*ev-eq*,$'\sigma$) *action* ⇒ ($'\vartheta$,$'\sigma$) *action* ⇒ ($'\vartheta$,$'\sigma$) *action* (**infixl** ‹□› *18*)
where *P* □ *Q* ≡ *action-of*(*R*((¬((*relation-of P*)$^f{}_f$) ∧ ¬((*relation-of Q*)$^f{}_f$)) ⊢
(((*relation-of P*)$^t{}_f$ ∧ ((*relation-of Q*)$^t{}_f$))
◃ λ(*A*, *A'*). *tr A* = *tr A'* ∧ *wait A'* ▹
((*relation-of P*)$^t{}_f$ ∨ ((*relation-of Q*)$^t{}_f$)))))

lemma *Det-is-action*:
$(R((\neg((relation\text{-}of\ P)^f{}_f) \wedge \neg((relation\text{-}of\ Q)^f{}_f)) \vdash$
$(((relation\text{-}of\ P)^t{}_f \wedge ((relation\text{-}of\ Q)^t{}_f))$
$\triangleleft\ \lambda(A,\ A').\ tr\ A = tr\ A' \wedge wait\ A' \triangleright$
$((relation\text{-}of\ P)^t{}_f \vee ((relation\text{-}of\ Q)^t{}_f))))) \in \{p.\ is\text{-}CSP\text{-}process\ p\}$
apply (*simp add*: *spec-def*)
apply (*rule rd-is-CSP*)
apply (*auto*)
done

lemmas *Det-is-CSP* = *Det-is-action*[*simplified*]

lemma *relation-of-Det*:
$relation\text{-}of\ (P \sqcap Q) = (R((\neg((relation\text{-}of\ P)^f{}_f) \wedge \neg((relation\text{-}of\ Q)^f{}_f)) \vdash$
$(((relation\text{-}of\ P)^t{}_f \wedge ((relation\text{-}of\ Q)^t{}_f))$
$\triangleleft\ \lambda(A,\ A').\ tr\ A = tr\ A' \wedge wait\ A' \triangleright$
$((relation\text{-}of\ P)^t{}_f \vee ((relation\text{-}of\ Q)^t{}_f)))))$
apply (*unfold Det-def*)
apply (*rule action-of-inverse*)
apply (*rule Det-is-action*)
done

lemma *mono-Det*: *mono* $((\square)\ P)$
by (*auto simp*: *mono-def less-eq-action ref-def relation-of-Det design-defs rp-defs fun-eq-iff*
split: *cond-splits dest*: *relation-of-spec-f-f*[*simplified*]
relation-of-spec-t-f[*simplified*])

14.8 Reactive design assignment

definition
$rd\text{-}assign\ s = action\text{-}of\ (R\ (true \vdash \lambda(A,\ A').\ ref\ A' = ref\ A \wedge tr\ A' = tr\ A \wedge \neg wait\ A' \wedge more\ A' = s))$

lemma *rd-assign-is-action*:
$(R\ (true \vdash \lambda(A,\ A').\ ref\ A' = ref\ A \wedge tr\ A' = tr\ A \wedge \neg wait\ A' \wedge more\ A' = s)) \in \{p.\ is\text{-}CSP\text{-}process\ p\}$
apply (*auto simp*:)
apply (*rule rd-is-CSP*)
by *auto*

lemmas *rd-assign-is-CSP* = *rd-assign-is-action*[*simplified*]

lemma *relation-of-rd-assign*:
$relation\text{-}of\ (rd\text{-}assign\ s) =$
$(R\ (true \vdash \lambda(A,\ A').\ ref\ A' = ref\ A \wedge tr\ A' = tr\ A \wedge \neg wait\ A' \wedge more\ A' = s))$

by (*simp add*: *rd-assign-def action-of-inverse rd-assign-is-CSP*)

14.9 Local state external choice

definition
Loc::$'\sigma \Rightarrow ('\vartheta$::*ev-eq*,$'\sigma$) *action* $\Rightarrow$ $'\sigma \Rightarrow ('\vartheta,'\sigma)$ *action* $\Rightarrow ('\vartheta,'\sigma)$ *action*
(‹'(()*loc* - • - ') ⊞ '(()*loc* - • - ')›)
where (*loc s1* • *P*) ⊞ (*loc s2* • *Q*) ≡
((*rd-assign s1*)‘; ‘*P*) □ ((*rd-assign s2*)‘; ‘ *Q*)

14.10 Schema expression

definition *Schema* :: $'\sigma$ *relation* $\Rightarrow ('\vartheta$::*ev-eq*,$'\sigma$) *action* **where**
Schema sc ≡ *action-of*(*R* ((λ(*A*, *A'*). (*Pre sc*) (*more A*)) ⊢
(λ(*A*, *A'*). *sc* (*more A*, *more A'*) ∧ ¬*wait A'* ∧ *tr A* = *tr A'*)))

lemma *Schema-is-action*:
(*R* ((λ(*A*, *A'*). (*Pre sc*) (*more A*)) ⊢
(λ(*A*, *A'*). *sc* (*more A*, *more A'*) & ¬*wait A'* & *tr A* = *tr A'*))) ∈ {*p*.
is-CSP-process p}
apply (*simp*)
apply (*rule rd-is-CSP*)
apply (*auto*)
done

lemmas *Schema-is-CSP* = *Schema-is-action*[*simplified*]

lemma *relation-of-Schema*:
relation-of (*Schema sc*) = (*R* ((λ(*A*, *A'*). (*Pre sc*) (*more A*)) ⊢
(λ(*A*, *A'*). *sc* (*more A*, *more A'*) ∧ ¬*wait A'* ∧ *tr A* = *tr A'*)))
by (*simp add*: *Schema-def action-of-inverse Schema-is-CSP*)

lemma *Schema-is-state-update-before*: *Schema u* = *state-update-before u Skip*
apply (*subst relation-of-inject*[*symmetric*])
apply (*auto simp*: *relation-of-Schema relation-of-state-update-before relation-of-Skip rp-defs fun-eq-iff*
design-defs)
apply (*split cond-splits*, *simp-all*)
apply (*rule comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule comp-intro*)
apply (*split cond-splits*, *simp-all*)+
prefer *3*
apply (*split cond-splits*, *simp-all*)+
apply (*auto simp*: *prefix-def*)
done

14.11 Parallel composition

type-synonym $'\sigma$ *local-state* = $('\sigma \times ('\sigma \Rightarrow '\sigma \Rightarrow '\sigma))$

fun *MergeSt* :: $'\sigma$ *local-state* $\Rightarrow$ $'\sigma$ *local-state* $\Rightarrow$ $('\vartheta,'\sigma)$ *relation-rp* **where**
MergeSt $(s1,s1')$ $(s2,s2')$ = $((\lambda(S, S').\ (s1'\ s1)\ (more\ S) = more\ S');\ ;$
$(\lambda(S::('\vartheta,'\sigma)\ alphabet\text{-}rp,\ S').\ (s2'\ s2)\ (more\ S) = more\ S'))$

definition *listCons* :: $'\vartheta \Rightarrow\ '\vartheta$ *list list* $\Rightarrow\ '\vartheta$ *list list* (‹- ## -›) **where**
a ## l = ((*map* (*Cons a*)) *l*)

fun *ParMergel* :: $'\vartheta$::*ev-eq list* $\Rightarrow\ '\vartheta$ *list* $\Rightarrow\ '\vartheta$ *set* $\Rightarrow\ '\vartheta$ *list list* **where**
ParMergel [] [] *cs* = [[]]
| *ParMergel* [] (*b#tr2*) *cs* = (*if* (*filter-chan-set b cs*) *then* [[]]
else (*b* ## (*ParMergel* [] *tr2 cs*)))
| *ParMergel* (*a#tr1*) [] *cs* = (*if* (*filter-chan-set a cs*) *then* [[]]
else (*a* ## (*ParMergel tr1* [] *cs*)))
| *ParMergel* (*a#tr1*) (*b#tr2*) *cs* =
(*if* (*filter-chan-set a cs*)
then (*if* (*ev-eq a b*)
then (*a* ## (*ParMergel tr1 tr2 cs*))
else (*if* (*filter-chan-set b cs*)
then [[]]
else (*b* ## (*ParMergel* (*a#tr1*) *tr2 cs*))))
else (*if* (*filter-chan-set b cs*)
then (*a* ## (*ParMergel tr1* (*b#tr2*) *cs*))
else (*a* ## (*ParMergel tr1* (*b#tr2*) *cs*))
@ (*b* ## (*ParMergel* (*a#tr1*) *tr2 cs*))))

definition *ParMerge*:: $'\vartheta$::*ev-eq list* $\Rightarrow\ '\vartheta$ *list* $\Rightarrow\ '\vartheta$ *set* $\Rightarrow\ '\vartheta$ *list set* **where**
ParMerge tr1 tr2 cs = *set* (*ParMergel tr1 tr2 cs*)

lemma *set-Cons1*: *tr1* $\in$ *set l* $\Longrightarrow$ *a # tr1* $\in$ (#) *a* ‘ *set l*
by (*auto*)

lemma *tr-in-set-eq*: (*tr1* $\in$ (#) *b* ‘ *set l*) = (*tr1* $\neq$ [] $\wedge$ *hd tr1* = *b* $\wedge$ *tl tr1* $\in$ *set l*)
by (*induct l*) *auto*

definition *M-par*::$(('\vartheta$::*ev-eq*$),\ '\sigma)$ *alpha-rp-scheme* $\Rightarrow ('\sigma \Rightarrow\ '\sigma \Rightarrow\ '\sigma)$
$\Rightarrow ('\vartheta,\ '\sigma)$ *alpha-rp-scheme* $\Rightarrow ('\sigma \Rightarrow\ '\sigma \Rightarrow\ '\sigma)$
$\Rightarrow ('\vartheta\ set) \Rightarrow ('\vartheta,\ '\sigma)$ *relation-rp* **where**
M-par s1 x1 s2 x2 cs =
$((\lambda(S, S').\ ((diff\text{-}tr\ S'\ S) \in ParMerge\ (diff\text{-}tr\ s1\ S)\ (diff\text{-}tr\ s2\ S)\ cs\ \&$
$ev\text{-}eq\ (tr\text{-}filter\ (tr\ s1)\ cs)\ (tr\text{-}filter\ (tr\ s2)\ cs))) \wedge$
$((\lambda(S, S').\ (wait\ s1 \vee wait\ s2) \wedge$
$ref\ S' \subseteq ((((ref\ s1)\cup(ref\ s2))\cap cs)\cup(((ref\ s1)\cap(ref\ s2))-cs)))$
$\triangleleft$ *wait o snd* $\triangleright$
$((\lambda(S, S').\ (\neg wait\ s1 \vee \neg wait\ s2)) \wedge MergeSt\ ((more\ s1),\ x1)\ ((more\ s2),\ x2))))$

definition *Par*::(′ϑ::*ev-eq*,′σ) *action* ⇒
(′σ ⇒ ′σ ⇒ ′σ) ⇒ ′ϑ *set* ⇒ (′σ ⇒ ′σ ⇒ ′σ) ⇒
(′ϑ,′σ) *action* ⇒ (′ϑ,′σ) *action* (‹- ⟦ - | - | - ⟧ -›) **where**
A1 ⟦ *ns1* | *cs* | *ns2* ⟧ *A2* ≡ (*action-of* (*R* ((λ (*S*, *S′*).
¬ (∃ *tr1 tr2*. ((*relation-of A1*)$^f{}_f$; ; (λ (*S*, *S′*). *tr1* = (*tr S*))) (*S*, *S′*)
∧ (*spec False* (*wait S*) (*relation-of A2*) ; ; (λ (*S*, -). *tr2* = (*tr S*))) (*S*, *S′*)
∧ ((*tr-filter tr1 cs*) = (*tr-filter tr2 cs*))) ∧
¬ (∃ *tr1 tr2*. (*spec False* (*wait S*) (*relation-of A1*); ; (λ(*S*, -). *tr1* = *tr S*)) (*S*, *S′*)
∧ ((*relation-of A2*)$^f{}_f$; ; (λ(*S*, *S′*). *tr2* = (*tr S*))) (*S*, *S′*)
∧ ((*tr-filter tr1 cs*) = (*tr-filter tr2 cs*)))) ⊢
(λ (*S*, *S′*). (∃ *s1 s2*. ((λ (*A*, *A′*). (*relation-of A1*)$^t{}_f$ (*A*, *s1*)
∧ ((*relation-of A2*)$^t{}_f$ (*A*, *s2*))); ; *M-par s1 ns1 s2 ns2 cs*) (*S*, *S′*))))))

lemma *Par-is-action*: (*R* ((λ (*S*, *S′*).
¬ (∃ *tr1 tr2*. ((*relation-of A1*)$^f{}_f$; ; (λ (*S*, *S′*). *tr1* = (*tr S*))) (*S*, *S′*)
∧ (*spec False* (*wait S*) (*relation-of A2*) ; ; (λ (*S*, *S′*). *tr2* = (*tr S*))) (*S*, *S′*)
∧ ((*tr-filter tr1 cs*) = (*tr-filter tr2 cs*))) ∧
¬ (∃ *tr1 tr2*. (*spec False* (*wait S*) (*relation-of A1*); ; (λ(*S*, -). *tr1* = *tr S*)) (*S*, *S′*)
∧ ((*relation-of A2*)$^f{}_f$; ; (λ (*S*, *S′*). *tr2* = (*tr S*))) (*S*, *S′*)
∧ ((*tr-filter tr1 cs*) = (*tr-filter tr2 cs*)))) ⊢
(λ (*S*, *S′*). (∃ *s1 s2*. ((λ (*A*, *A′*). (*relation-of A1*)$^t{}_f$ (*A*, *s1*)
∧ ((*relation-of A2*)$^t{}_f$ (*A*, *s2*))); ; *M-par s1 ns1 s2 ns2 cs*) (*S*, *S′*))))) ∈ {*p*.
is-CSP-process p}
apply (*simp*)
apply (*rule rd-is-CSP*)
apply (*blast*)
done

lemmas *Par-is-CSP* = *Par-is-action*[*simplified*]

lemma *relation-of-Par*:
relation-of (*A1* ⟦ *ns1* | *cs* | *ns2* ⟧ *A2*) = (*R* ((λ (*S*, *S′*).
¬ (∃ *tr1 tr2*. ((*relation-of A1*)$^f{}_f$; ; (λ (*S*, *S′*). *tr1* = (*tr S*))) (*S*, *S′*)
∧ (*spec False* (*wait S*) (*relation-of A2*) ; ; (λ (*S*, *S′*). *tr2* = (*tr S*))) (*S*, *S′*)
∧ ((*tr-filter tr1 cs*) = (*tr-filter tr2 cs*))) ∧
¬ (∃ *tr1 tr2*. (*spec False* (*wait S*) (*relation-of A1*); ; (λ(*S*, -). *tr1* = *tr S*)) (*S*, *S′*)

∧ ((*relation-of A2*)$^f{}_f$; ; (λ (*S*, *S′*). *tr2* = (*tr S*))) (*S*, *S′*)
∧ ((*tr-filter tr1 cs*) = (*tr-filter tr2 cs*)))) ⊢
(λ (*S*, *S′*). (∃ *s1 s2*. ((λ (*A*, *A′*). (*relation-of A1*)$^t{}_f$ (*A*, *s1*)
∧ ((*relation-of A2*)$^t{}_f$ (*A*, *s2*))); ; *M-par s1 ns1 s2 ns2 cs*) (*S*, *S′*)))))
apply (*unfold Par-def*)
apply (*rule action-of-inverse*)
apply (*rule Par-is-action*)
done

lemma *mono-Par*: *mono* (λ*Q*. *P* ⟦ *ns1* | *cs* | *ns2* ⟧ *Q*)
apply (*auto simp*: *mono-def less-eq-action ref-def relation-of-Par design-defs fun-eq-iff rp-defs*

split: *cond-splits*)
apply (*auto simp*: *rp-defs dest*: *relation-of-spec-f-f*[*simplified*] *relation-of-spec-t-f*[*simplified*])
apply (*erule-tac x*=*tr ba* **in** *allE*, *auto*)
apply (*erule notE*)
apply (*auto dest*: *relation-of-spec-f-f relation-of-spec-t-f*)
done

14.12 Local parallel block

definition
ParLoc::′σ ⇒ (′σ ⇒ ′σ ⇒ ′σ) ⇒ (′ϑ::*ev-eq*, ′σ) *action* ⇒ ′ϑ *set* ⇒ ′σ ⇒ (′σ ⇒ ′σ ⇒ ′σ) ⇒ (′ϑ,′σ) *action* ⇒ (′ϑ,′σ) *action*
(‹′(()*par* - | - • - ′) ⟦ - ⟧ ′(()*par* - | - • - ′)›)
where
(*par s1* | *ns1* • *P*) ⟦ *cs* ⟧ (*par s2* | *ns2* • *Q*) ≡ ((*rd-assign s1*)‘; ‘*P*) ⟦ *ns1* | *cs* | *ns2* ⟧ ((*rd-assign s2*)‘; ‘ *Q*)

14.13 Assignment

definition *ASSIGN*::(′*v*, ′σ) *var-list* ⇒ (′σ ⇒ ′*v*) ⇒ (′ϑ::*ev-eq*, ′σ) *action* **where**
ASSIGN x e ≡ *action-of* (*R* (*true* ⊢ (λ (*S*, *S*′). *tr S*′ = *tr S* ∧ ¬*wait S*′ ∧
(*more S*′ = (*update x* (λ-. (*e* (*more S*)))) (*more S*)))))

syntax *-assign*::*id* ⇒ (′σ ⇒ ′*v*) ⇒ (′ϑ, ′σ) *action* (‹- ‘:=‘ -›)
translations *y* ‘:=‘ *vv* => *CONST ASSIGN* (*VAR y*) *vv*

lemma *Assign-is-action*:
(*R* (*true* ⊢ (λ (*S*, *S*′). *tr S*′ = *tr S* ∧ ¬*wait S*′ ∧
(*more S*′ = (*update x* (λ-. (*e* (*more S*)))) (*more S*))))) ∈ {*p*. *is-CSP-process p*}
apply (*simp*)
apply (*rule rd-is-CSP*)
apply (*blast*)
done

lemmas *Assign-is-CSP* = *Assign-is-action*[*simplified*]

lemma *relation-of-Assign*:
relation-of (*ASSIGN x e*) = (*R* (*true* ⊢ (λ (*S*, *S*′). *tr S*′ = *tr S* ∧ ¬*wait S*′ ∧
(*more S*′ = (*update x* (λ-. (*e* (*more S*)))) (*more S*)))))
by (*simp add*: *ASSIGN-def action-of-inverse Assign-is-CSP*)

lemma *Assign-is-state-update-before*: *ASSIGN x e* = *state-update-before* (λ (*s*, *s*′) . *s*′ = (*update x* (λ-. (*e s*))) *s*) *Skip*
apply (*subst relation-of-inject*[*symmetric*])
apply (*auto simp*: *relation-of-Assign relation-of-state-update-before relation-of-Skip rp-defs fun-eq-iff*
Pre-def update-def design-defs)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b*=*b* **in** *comp-intro*)

apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b*=*b* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
defer
apply (*split cond-splits*, *simp-all*)+
prefer *3*
apply (*split cond-splits*, *simp-all*)+
apply (*auto simp add*: *prefix-def*)
done

14.14 Variable scope

definition *Var*::(′*v*, ′*σ*) *var-list* ⇒(′*ϑ*, ′*σ*) *action* ⇒ (′*ϑ*::*ev-eq*,′*σ*) *action* **where**
Var v A ≡ *action-of*(
(*R*(*true* ⊢ (λ (*A*, *A*′). ∃ *a*. *tr A*′ = *tr A* ∧ ¬*wait A*′ ∧ *more A*′ = (*increase v a* (*more A*))))); ;
(*relation-of A*; ;
(*R*(*true* ⊢ (λ (*A*, *A*′). *tr A*′ = *tr A* ∧ ¬*wait A*′ ∧ *more A*′ = (*decrease v* (*more A*)))))))

syntax *-var*::*idt* ⇒ (′*ϑ*, ′*σ*) *action* ⇒ (′*ϑ*, ′*σ*) *action* (‹*var* - • -› [*1000*] *999*)
translations *var y* • *Act* => *CONST Var* (*VAR-LIST y*) *Act*

lemma *Var-is-action*:
((*R*(*true* ⊢ (λ (*A*, *A*′). ∃ *a*. *tr A*′ = *tr A* ∧ ¬*wait A*′ ∧ *more A*′ = (*increase v a* (*more A*))))); ;
(*relation-of A*; ;
(*R*(*true* ⊢ (λ (*A*, *A*′). *tr A*′ = *tr A* ∧ ¬*wait A*′ ∧ *more A*′ = (*decrease v* (*more A*))))))) ∈ {*p*. *is-CSP-process p*}
apply (*simp*)
apply (*rule seq-CSP*)
prefer *3*
apply (*rule seq-CSP*)
apply (*auto simp*: *relation-of-CSP1 relation-of-R*)
apply (*rule rd-is-CSP*)
apply (*auto simp*: *csp-defs rp-defs design-defs fun-eq-iff prefix-def increase-def decrease-def*
split: *cond-splits*)
done

lemmas *Var-is-CSP* = *Var-is-action*[*simplified*]

lemma *relation-of-Var*:
relation-of (*Var v A*) =
((*R*(*true* ⊢ (λ (*A*, *A*′). ∃ *a*. *tr A*′ = *tr A* ∧ ¬*wait A*′ ∧ *more A*′ = (*increase v a* (*more A*))))); ;
(*relation-of A*; ;
(*R*(*true* ⊢ (λ (*A*, *A*′). *tr A*′ = *tr A* ∧ ¬*wait A*′ ∧ *more A*′ = (*decrease v* (*more A*)))))))

apply (*simp only*: *Var-def*)
apply (*rule action-of-inverse*)
apply (*rule Var-is-action*)
done

lemma *mono-Var* : *mono* (*Var x*)
  by (*auto simp*: *mono-def less-eq-action ref-def relation-of-Var*)

definition *Let*::($'v$, $'\sigma$) *var-list* ⇒($'\vartheta$, $'\sigma$) *action* ⇒ ($'\vartheta$::*ev-eq*,$'\sigma$) *action* **where**
Let v A ≡ *action-of*((*relation-of A*; ;
  (*R*(*true* ⊢ (λ (*A*, *A'*). *tr A'* = *tr A* ∧ ¬*wait A'* ∧ *more A'* = (*decrease v* (*more A*)))))))

syntax *-let*::*idt* ⇒ ($'\vartheta$, $'\sigma$) *action* ⇒ ($'\vartheta$, $'\sigma$) *action* (‹*let* - • -› [*1000*] *999*)
translations *let y* • *Act* => *CONST Let* (*VAR-LIST y*) *Act*

lemma *Let-is-action*:
(*relation-of A*; ;
  (*R*(*true* ⊢ (λ (*A*, *A'*). *tr A'* = *tr A* ∧ ¬*wait A'* ∧ *more A'* = (*decrease v* (*more A*)))))) ∈ {*p*. *is-CSP-process p*}
  apply (*simp*)
  apply (*rule seq-CSP*)
  apply (*auto simp*: *relation-of-CSP1 relation-of-R*)
  apply (*rule rd-is-CSP*)
  apply (*auto*)
done

lemmas *Let-is-CSP* = *Let-is-action*[*simplified*]

lemma *relation-of-Let*:
relation-of (*Let v A*) =
  (*relation-of A*; ;
  (*R*(*true* ⊢ (λ (*A*, *A'*). *tr A'* = *tr A* ∧ ¬*wait A'* ∧ *more A'* = (*decrease v* (*more A*))))))
by (*simp add*: *Let-def action-of-inverse Let-is-CSP*)

lemma *mono-Let* : *mono* (*Let x*)
  by (*auto simp*: *mono-def less-eq-action ref-def relation-of-Let*)

lemma *Var-is-state-update-before*: *Var v A* = *state-update-before* (λ (*s*, *s'*). ∃ *a*. *s'* = *increase v a s*) (*Let v A*)
apply (*subst relation-of-inject*[*symmetric*])
apply (*auto simp*: *relation-of-Var relation-of-Let relation-of-state-update-before relation-of-Skip fun-eq-iff*)
apply (*auto simp*: *rp-defs fun-eq-iff Pre-def design-defs*)

apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=ab* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=bb* **in** *comp-intro*, *simp-all*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=ab* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=bb* **in** *comp-intro*, *simp-all*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=ab* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+ **defer**
apply (*rule-tac b=bb* **in** *comp-intro*, *simp-all*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=ab* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+ **defer**
apply (*rule-tac b=bb* **in** *comp-intro*, *simp-all*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=ab* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=bb* **in** *comp-intro*, *simp-all*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=ab* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=bb* **in** *comp-intro*, *simp-all*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=ab* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*case-tac* $\exists A'$ *a*. A' = *increase v a* (*alpha-rp.more aa*), *simp-all add*: *true-def*)
apply (*erule-tac x=increase v a* (*alpha-rp.more aa*) **in** *allE*)
apply (*erule-tac x=a* **in** *allE*, *simp*)
apply (*rule-tac b=bb* **in** *comp-intro*, *simp-all*)
apply (*split cond-splits*, *simp-all*)+
apply (*rule-tac b=ab* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*case-tac* $\exists A'$ *a*. A' = *increase v a* (*alpha-rp.more aa*), *simp-all add*: *true-def*)
apply (*erule-tac x=increase v a* (*alpha-rp.more aa*) **in** *allE*)
apply (*erule-tac x=a* **in** *allE*, *simp*)
apply (*rule-tac b=bb* **in** *comp-intro*, *simp-all*)
apply (*split cond-splits*, *simp-all*)+
done

lemma *Let-is-state-update-after*: *Let v A* = *state-update-after* (λ (*s*, s'). s' = *decrease v s*) *A*
apply (*subst relation-of-inject*[*symmetric*])
apply (*auto simp*: *relation-of-Var relation-of-Let relation-of-state-update-after relation-of-Skip fun-eq-iff*)
apply (*auto simp*: *rp-defs fun-eq-iff Pre-def design-defs*)
apply (*auto split*: *cond-splits*)

done

14.15 Guarded action

definition *Guard*::$'\sigma$ *predicate* $\Rightarrow$ ($'\vartheta$::*ev-eq*, $'\sigma$) *action* $\Rightarrow$ ($'\vartheta$, $'\sigma$) *action* (‹- '&' -›)
where g '&' $P \equiv$ *action-of*(R (((g *o more o fst*) $\longrightarrow \neg$ ((*relation-of* P)$^f{}_f$)) $\vdash$
(((g *o more o fst*) $\wedge$ ((*relation-of* P)$^t{}_f$)) $\vee$
(($\neg$(g *o more o fst*)) $\wedge$ (λ (A, A'). *tr* A' = *tr* A $\wedge$ *wait* A')))))

lemma *Guard-is-action*:
(R (((g *o more o fst*) $\longrightarrow \neg$ ((*relation-of* P)$^f{}_f$)) $\vdash$
(((g *o more o fst*) $\wedge$ ((*relation-of* P)$^t{}_f$)) $\vee$
(($\neg$(g *o more o fst*)) $\wedge$ (λ (A, A'). *tr* A' = *tr* A $\wedge$ *wait* A'))))) $\in$ {p. *is-CSP-process* p}
by (*auto simp add*: *spec-def intro*: *rd-is-CSP*)

lemmas *Guard-is-CSP* = *Guard-is-action*[*simplified*]

lemma *relation-of-Guard*:
relation-of (g '&' P) = (R (((g *o more o fst*) $\longrightarrow$ $\neg$ ((*relation-of* P)$^f{}_f$)) $\vdash$
(((g *o more o fst*) $\wedge$ ((*relation-of* P)$^t{}_f$)) $\vee$
(($\neg$(g *o more o fst*)) $\wedge$ (λ (A, A'). *tr* A' = *tr* A $\wedge$ *wait* A')))))
apply (*unfold Guard-def*)
apply (*subst action-of-inverse*)
apply (*simp-all only*: *Guard-is-action*)
done

lemma *mono-Guard* : *mono* (*Guard* g)
apply (*auto simp*: *mono-def less-eq-action ref-def rp-defs design-defs relation-of-Guard*
split: *cond-splits*)
apply (*auto dest*: *relation-of-spec-f-f relation-of-spec-t-f*)
done

lemma *false-Guard*: *false* '&' P = *Stop*
apply (*subst relation-of-inject*[*symmetric*])
apply (*subst relation-of-Stop*)
apply (*subst relation-of-Guard*)
apply (*simp add*: *fun-eq-iff utp-defs csp-defs design-defs rp-defs*)
done

lemma *false-Guard1*: $\bigwedge$ a b. g (*alpha-rp.more* a) = *False* $\Longrightarrow$
(*relation-of* (g '&' P)) (a, b) = (*relation-of Stop*) (a, b)
apply (*subst relation-of-Guard*)
apply (*subst relation-of-Stop*)
apply (*auto simp*: *fun-eq-iff csp-defs design-defs rp-defs split*: *cond-splits*)
done

lemma *true-Guard*: *true* '&' *P* = *P*
apply (*subst relation-of-inject*[*symmetric*])
apply (*subst relation-of-Guard*)
apply (*subst CSP-is-rd*[*OF relation-of-CSP*]) **back back**
apply (*simp add*: *fun-eq-iff utp-defs csp-defs design-defs rp-defs*)
done

lemma *true-Guard1*: $\bigwedge$ *a b*. *g* (*alpha-rp.more a*) = *True* $\Longrightarrow$
(*relation-of* (*g* '&' *P*)) (*a*, *b*) = (*relation-of P*) (*a*, *b*)
apply (*subst relation-of-Guard*)
apply (*subst CSP-is-rd*[*OF relation-of-CSP*]) **back back**
apply (*auto simp*: *fun-eq-iff csp-defs design-defs rp-defs split*: *cond-splits*)
done

lemma *Guard-is-state-update-before*: *g* '&' *P* = *state-update-before* (λ (*s*, *s'*) . *g s*) *P*
apply (*subst relation-of-inject*[*symmetric*])
apply (*auto simp*: *relation-of-Guard relation-of-state-update-before relation-of-Skip rp-defs fun-eq-iff*
Pre-def update-def design-defs)
apply (*rule-tac b*=*a* **in** *comp-intro*)
apply (*split cond-splits*, *simp-all*)+
apply (*subst CSP-is-rd*)
apply (*simp-all add*: *relation-of-CSP rp-defs design-defs fun-eq-iff*)
apply (*split cond-splits*, *simp-all*)+
apply (*auto*)
apply (*subst* (*asm*) *CSP-is-rd*)
apply (*simp-all add*: *relation-of-CSP rp-defs design-defs fun-eq-iff*)
apply (*split cond-splits*, *simp-all*)+
apply (*subst* (*asm*) *CSP-is-rd*)
apply (*simp-all add*: *relation-of-CSP rp-defs design-defs fun-eq-iff*)
apply (*split cond-splits*, *simp-all*)+
apply (*subst CSP-is-rd*)
apply (*simp-all add*: *relation-of-CSP rp-defs design-defs fun-eq-iff*)
apply (*split cond-splits*, *simp-all*)+
apply (*subst CSP-is-rd*)
apply (*simp-all add*: *relation-of-CSP rp-defs design-defs fun-eq-iff*)
apply (*split cond-splits*, *simp-all*)+
apply (*auto*) **defer**
apply (*split cond-splits*, *simp-all*)+
apply (*subst* (*asm*) *CSP-is-rd*)
apply (*simp-all add*: *relation-of-CSP rp-defs design-defs fun-eq-iff*)
apply (*split cond-splits*, *simp-all*)+ **defer**
apply (*rule disjI1*) **defer**
apply (*case-tac g* (*alpha-rp.more aa*), *simp-all*)
apply (*rule*)+
apply (*simp add*: *impl-def*) **defer**
oops

14.16 Prefixed action

definition *do* **where**
$do\ e \equiv (\lambda(A, A').\ tr\ A = tr\ A' \wedge (e\ (more\ A)) \notin (ref\ A')) \lhd wait\ o\ snd \rhd$
$(\lambda(A, A').\ tr\ A' = (tr\ A\ @[(e\ (more\ A))]))$

definition $do\text{-}I{::}('\sigma \Rightarrow '\vartheta) \Rightarrow '\vartheta\ set \Rightarrow ('\vartheta,\ '\sigma)\ relation\text{-}rp$
where $do\text{-}I\ c\ S \equiv\ ((\lambda(A, A').\ tr\ A = tr\ A'\ \&\ S \cap (ref\ A') = \{\})$
$\lhd wait\ o\ snd \rhd$
$(\lambda(A, A').\ hd\ (tr\ A' - tr\ A) \in S\ \&\ (c\ (more\ A) = (last\ (tr\ A')))))$

definition
$iPrefix{::}('\sigma \Rightarrow '\vartheta{::}ev\text{-}eq) \Rightarrow ('\sigma\ relation) \Rightarrow (('\vartheta,\ '\sigma)\ action \Rightarrow ('\vartheta,\ '\sigma)\ action) \Rightarrow ('\sigma \Rightarrow\ '\vartheta\ set) \Rightarrow ('\vartheta,\ '\sigma)\ action \Rightarrow ('\vartheta,\ '\sigma)\ action$ **where**
$iPrefix\ c\ i\ j\ S\ P \equiv action\text{-}of(R(true \vdash (\lambda\ (A, A').\ (do\text{-}I\ c\ (S\ (more\ A)))\ (A, A')\ \&\ more\ A' = more\ A)))\ `;`\ P$

definition
$oPrefix{::}('\sigma \Rightarrow '\vartheta) \Rightarrow ('\vartheta{::}ev\text{-}eq,\ '\sigma)\ action \Rightarrow ('\vartheta,\ '\sigma)\ action$ **where**
$oPrefix\ c\ P \equiv action\text{-}of(R(true \vdash (do\ c) \wedge (\lambda\ (A, A').\ more\ A' = more\ A)))\ `;`\ P$

definition $Prefix0{::}'\vartheta \Rightarrow ('\vartheta{::}ev\text{-}eq,\ '\sigma)\ action \Rightarrow ('\vartheta,\ '\sigma)\ action$ **where**
$Prefix0\ c\ P \equiv action\text{-}of(R(true \vdash (do\ (\lambda\ \text{-}.\ c)) \wedge (\lambda\ (A, A').\ more\ A' = more\ A)))\ `;`\ P$

definition
$read{::}('v \Rightarrow\ '\vartheta) \Rightarrow ('v,\ '\sigma)\ var\text{-}list \Rightarrow ('\vartheta{::}ev\text{-}eq,\ '\sigma)\ action \Rightarrow ('\vartheta,\ '\sigma)\ action$
where $read\ c\ x\ P\ \equiv iPrefix\ (\lambda\ A.\ c\ (select\ x\ A))\ (\lambda\ (s, s').\ \exists\ a.\ s' = increase\ x\ a\ s)\ (Let\ x)\ (\lambda\text{-}.\ range\ c)\ P$

definition
$read1{::}('v \Rightarrow\ '\vartheta) \Rightarrow ('v,\ '\sigma)\ var\text{-}list \Rightarrow ('\sigma \Rightarrow\ 'v\ set) \Rightarrow ('\vartheta{::}ev\text{-}eq,\ '\sigma)\ action \Rightarrow ('\vartheta,\ '\sigma)\ action$
where $read1\ c\ x\ S\ P\ \equiv iPrefix\ (\lambda\ A.\ c\ (select\ x\ A))\ (\lambda\ (s, s').\ \exists\ a.\ a{\in}(S\ s)\ \&\ s' = increase\ x\ a\ s)\ (Let\ x)\ (\lambda s.\ c`(S\ s))\ P$

definition
$write1{::}('v \Rightarrow\ '\vartheta) \Rightarrow ('\sigma \Rightarrow\ 'v) \Rightarrow ('\vartheta{::}ev\text{-}eq,\ '\sigma)\ action \Rightarrow ('\vartheta,\ '\sigma)\ action$
where $write1\ c\ a\ P \equiv oPrefix\ (\lambda\ A.\ c\ (a\ A))\ P$

definition
$write0{::}'\vartheta \Rightarrow ('\vartheta{::}ev\text{-}eq,\ '\sigma)\ action \Rightarrow ('\vartheta,\ '\sigma)\ action$
where $write0\ c\ P \equiv Prefix0\ c\ P$

syntax
-read ::[*id*, *pttrn*, ('ϑ, 'σ) *action*] => ('ϑ, 'σ) *action* (‹(-‘?‘- /→ -)›)
-readS ::[*id*, *pttrn*, 'ϑ *set*,('ϑ, 'σ) *action*] => ('ϑ, 'σ) *action* (‹(-‘?‘-‘:‘- /→ -)›)

-readSS ::[*id*, *pttrn*, $'\sigma$ => $'\vartheta$ *set*,($'\vartheta$, $'\sigma$) *action*] => ($'\vartheta$, $'\sigma$) *action* (‹(-‘?‘-‘∈‘-/→ -)›)
-write ::[*id*, $'\sigma$, ($'\vartheta$, $'\sigma$) *action*] => ($'\vartheta$, $'\sigma$) *action* (‹(-‘!‘- /→ -)›)
-writeS::[$'\vartheta$, ($'\vartheta$, $'\sigma$) *action*] => ($'\vartheta$, $'\sigma$) *action* (‹(- /→ -)›)

translations
-read c p P == *CONST read c* (*VAR-LIST p*) *P*
-readS c p b P == *CONST read1 c* (*VAR-LIST p*) (λ-. *b*) *P*
-readSS c p b P == *CONST read1 c* (*VAR-LIST p*) *b P*
-write c p P == *CONST write1 c p P*
-writeS a P == *CONST write0 a P*

lemma *Prefix-is-action*:
$(R(true \vdash (do\ c) \wedge (\lambda\ (A,\ A').\ more\ A' = more\ A))) \in \{p.\ is\text{-}CSP\text{-}process\ p\}$
by (*auto intro*: *rd-is-CSP*)

lemma *Prefix1-is-action*:
$(R(true \vdash \lambda(A,\ A').\ do\text{-}I\ c\ (S\ (alpha\text{-}rp.more\ A))\ (A,\ A') \wedge alpha\text{-}rp.more\ A' = alpha\text{-}rp.more\ A)) \in \{p.\ is\text{-}CSP\text{-}process\ p\}$
by (*auto intro*: *rd-is-CSP*)

lemma *Prefix0-is-action*:
$(R(true \vdash (do\ c) \wedge (\lambda\ (A,\ A').\ more\ A' = more\ A))) \in \{p.\ is\text{-}CSP\text{-}process\ p\}$
by (*auto intro*: *rd-is-CSP*)

lemmas *Prefix-is-CSP* = *Prefix-is-action*[*simplified*]

lemmas *Prefix1-is-CSP* = *Prefix1-is-action*[*simplified*]

lemmas *Prefix0-is-CSP* = *Prefix0-is-action*[*simplified*]

lemma *relation-of-iPrefix*:
relation-of (*iPrefix c i j S P*) =
$((R(true \vdash (\lambda\ (A,\ A').\ (do\text{-}I\ c\ (S\ (more\ A)))\ (A,\ A')\ \&\ more\ A' = more\ A)));;$
relation-of P)
by (*simp add*: *iPrefix-def relation-of-Seq action-of-inverse Prefix1-is-CSP*)

lemma *relation-of-oPrefix*:
relation-of (*oPrefix c P*) =
$((R(true \vdash (do\ c) \wedge (\lambda\ (A,\ A').\ more\ A' = more\ A)));;$ *relation-of P*)
by (*simp add*: *oPrefix-def relation-of-Seq action-of-inverse Prefix-is-CSP*)

lemma *relation-of-Prefix0*:
relation-of (*Prefix0 c P*) =
$((R(true \vdash (do\ (\lambda\ \text{-}.\ c)) \wedge (\lambda\ (A,\ A').\ more\ A' = more\ A)));;$ *relation-of P*)
by (*simp add*: *Prefix0-def relation-of-Seq action-of-inverse Prefix0-is-CSP*)

lemma *mono-iPrefix* : *mono* (*iPrefix c i j s*)

by (*auto simp*: *mono-def less-eq-action ref-def relation-of-iPrefix*)

lemma *mono-oPrefix* : *mono* (*oPrefix c*)
by (*auto simp*: *mono-def less-eq-action ref-def relation-of-oPrefix*)

lemma *mono-Prefix0* : *mono*(*Prefix0 c*)
by (*auto simp*: *mono-def less-eq-action ref-def relation-of-Prefix0*)

14.17 Hiding

definition *Hide*::($'\vartheta$::*ev-eq*, $'\sigma$) *action* ⇒ $'\vartheta$ *set* ⇒ ($'\vartheta$, $'\sigma$) *action* (**infixl** ‹\› *18*)
where
$P \setminus cs \equiv$ *action-of*($R(\lambda(S, S').\ \exists\ s.$ (*diff-tr* S' S) = (*tr-filter* (s − (*tr* S)) *cs*) &
(*relation-of* P)(S, S'⦇*tr* := s, *ref* := (*ref* S') ∪ *cs* ⦈)); ; (*relation-of* *Skip*))

definition
hid P cs == ($R(\lambda(S, S').\ \exists\ s.$ (*diff-tr* S' S) = (*tr-filter* (s − (*tr* S)) *cs*) &
(*relation-of* P)(S, S'⦇*tr* := s, *ref* := (*ref* S') ∪ *cs* ⦈)) ; ; (*relation-of Skip*))

lemma *hid-is-R*: *hid P cs is R healthy*
apply (*simp add*: *hid-def*)
apply (*rule seq-R*)
apply (*simp add*: *Healthy-def R-idem2*)
apply (*rule CSP-is-R*)
apply (*rule relation-of-CSP*)
done

lemma *hid-Skip*: *hid P cs* = (*hid P cs* ; ; *relation-of Skip*)
by (*simp add*: *hid-def comp-assoc*[*symmetric*] *Skip-comp-absorb*)

lemma *hid-is-CSP1*: *hid P cs is CSP1 healthy*
apply (*auto simp*: *design-defs CSP1-def hid-def rp-defs fun-eq-iff*)
apply (*rule-tac b*=*a* **in** *comp-intro*)
apply (*clarsimp split*: *cond-splits*)
apply (*subst CSP-is-rd*, *auto simp*: *rp-defs relation-of-CSP design-defs fun-eq-iff split*: *cond-splits*)
apply (*auto simp*: *diff-tr-def relation-of-Skip rp-defs design-defs true-def split*: *cond-splits*)
apply (*rule-tac x*=[] **in** *exI*, *auto*)
done

lemma *hid-is-CSP2*: *hid P cs is CSP2 healthy*
apply (*simp add*: *hid-def*)
apply (*rule seq-CSP2*)
apply (*rule CSP-is-CSP2*)
apply (*rule relation-of-CSP*)
done

lemma *hid-is-CSP*: *is-CSP-process* (*hid P cs*)
by (*auto simp*: *csp-defs hid-is-CSP1 hid-is-R hid-is-CSP2*)

lemma *Hide-is-action*:
$(R(\lambda(S, S'). \exists\ s.\ (\textit{diff-tr}\ S'\ S) = (\textit{tr-filter}\ (s - (tr\ S))\ cs)\ \&$
$(\textit{relation-of}\ P)(S, S'(\!|tr := s, ref := (ref\ S') \cup cs\ |\!))); ;\ (\textit{relation-of Skip})) \in \{p.$
$\textit{is-CSP-process}\ p\}$
by (*simp add*: *hid-is-CSP*[*simplified hid-def*])

lemmas *Hide-is-CSP* = *Hide-is-action*[*simplified*]

lemma *relation-of-Hide*:
$\textit{relation-of}\ (P \setminus cs) = (R(\lambda(S, S'). \exists\ s.\ (\textit{diff-tr}\ S'\ S) = (\textit{tr-filter}\ (s - (tr\ S))\ cs)$
$\&\ (\textit{relation-of}\ P)(S, S'(\!|tr := s, ref := (ref\ S') \cup cs\ |\!))); ;\ (\textit{relation-of Skip}))$
by (*simp add*: *Hide-def action-of-inverse Hide-is-CSP*)

lemma *mono-Hide* : $mono(\lambda\ P.\ P \setminus cs)$
by (*auto simp*: *mono-def less-eq-action ref-def prefix-def utp-defs relation-of-Hide rp-defs*)

14.18 Recursion

To represent the recursion operator "μ" over actions, we use the universal least fix-point operator "*lfp*" defined in the HOL library for lattices. The operator "*lfp*" is inherited from the "Complete Lattice class" under some conditions. All theorems defined over this operator can be reused.

In the *Circus.Circus-Actions* theory, we presented the proof that Circus actions form a complete lattice. The Knaster-Tarski Theorem (in its simplest formulation) states that any monotone function on a complete lattice has a least fixed-point. This is a consequence of the basic boundary properties of the complete lattice operations. Instantiating the complete lattice class allows one to inherit these properties with the definition of the least fixed-point for monotonic functions over Circus actions.

syntax *-MU*::[*idt*, *idt* ⇒ ('ϑ, 'σ) *action*] ⇒ ('ϑ, 'σ) *action* (‹μ - • -›)
translations *-MU X P* == *CONST lfp* (λ *X*. *P*)

end

15 Circus syntax

theory *Circus-Syntax*
imports *Denotational-Semantics*
keywords *alphabet state channel nameset chanset schema action* **and**
circus-process :: *thy-defn*
begin

abbreviation *list-select*::[′*r* ⇒ ′*a list*] ⇒ (′*r* ⇒ ′*a*) **where**
list-select Sel ≡ *hd o Sel*

abbreviation *list-update*::[(′*a list* ⇒ ′*a list*) ⇒ ′*r* ⇒ ′*r*]
⇒ (′*a* ⇒ ′*a*) ⇒ ′*r* ⇒ ′*r* **where**
list-update Upd ≡ λ *e*. *Upd* (λ *l*. (*e* (*hd l*))#(*tl l*))

abbreviation *list-update-const*::[(′*a list* ⇒ ′*a list*) ⇒ ′*r* ⇒ ′*r*]
⇒ ′*a* ⇒ ′*r relation* **where**
list-update-const Upd ≡ λ *e*. λ (*A*, *A*′). *A*′ = *Upd* (λ *l*. *e*#(*tl l*)) *A*

abbreviation *update-const*::[(′*a* ⇒ ′*a*) ⇒ ′*r* ⇒ ′*r*]
⇒ ′*a* ⇒ ′*r relation* **where**
update-const Upd ≡ λ *e*. λ (*A*, *A*′). *A*′ = *Upd* (λ -. *e*) *A*

syntax
  -synt-assign :: *id* ⇒ ′*a* ⇒ ′*b relation* (‹- := -›)

ML ‹
structure VARs-Data = *Proof-Data*
(
  type T = {*State-vars*: *string list*, *Alpha-vars*: *string list*}
  fun init - : *T* = {*State-vars* = [], *Alpha-vars* = []}
)
›

nonterminal *circus-action* **and** *circus-schema*

syntax
  -circus-action :: ′*a* => *circus-action* (‹-›)
  -circus-schema :: ′*a* => *circus-schema* (‹-›)

parse-translation ‹
  let
    fun antiquote-tr ctxt =
      let
        val {*State-vars*=*sv*, *Alpha-vars*=*av*} = *VARs-Data.get ctxt*

        fun get-selector x =
          let val c = *Consts.intern* (*Proof-Context.consts-of ctxt*) *x*
          in
                  if member (=) *av x then SOME* (*Const* (*Circus-Syntax.list-select*,
dummyT) $ (*Syntax.const c*)) *else*
                  if member (=) *sv x then SOME* (*Syntax.const c*) *else NONE end*;

        fun get-update x =
          let val c = *Consts.intern* (*Proof-Context.consts-of ctxt*) *x*
          in

```
            if member (=) av x then SOME (Const (Circus-Syntax.list-update-const,
dummyT) $ (Syntax.const (c^Record.updateN))) else
              if member (=) sv x then SOME (Const (Circus-Syntax.update-const,
dummyT) $ (Syntax.const (c^Record.updateN))) else NONE end;

    fun print text = (fn x => let val - = writeln text; in x end);

    val rel-op-type = @{typ ('a × 'b ⇒ bool) ⇒ ('b × 'c ⇒ bool) ⇒ 'a × 'c ⇒
bool};

    fun tr i (t as Free (x, -)) =
          (case get-selector x of
            SOME c => c $ Bound (i + 1)
          | NONE =>
              (case try (unsuffix ') x of
                SOME y =>
                  (case get-selector y of SOME c => c $ Bound i | NONE => t)
              | NONE => t))
      | tr i (t as (Const (-synt-assign, -) $ Free (x, -) $ r)) =
          (case get-update x of
            SOME c => c $ (tr i r) $ (Const (Product-Type.Pair, dummyT) $
Bound (i + 1) $ Bound i)
          | NONE => t)
(*        | tr i (t as (Const (c, rel-op-type) $ l $ r)) = print c
          ((Syntax.const @{const-name case-prod} $
            Abs (B, dummyT, Abs (B', dummyT, Const (c, rel-op-type)))) $ tr i
l $ tr i r)
              $ (Const (Product-Type.Pair, dummyT) $ Bound (i + 1) $ Bound
i)*)
      | tr i (t $ u) = tr i t $ tr i u
      | tr i (Abs (x, T, t)) = Abs (x, T, tr (i + 1) t)
      | tr - a = a;
   in tr 0 end;

  fun quote-tr ctxt [t] =
        Syntax.const @{const-name case-prod} $
       Abs (A, dummyT, Abs (A', dummyT, antiquote-tr ctxt (Term.incr-boundvars
2 t)))
    | quote-tr - ts = raise TERM (quote-tr, ts);
 in [(@{syntax-const -circus-schema}, quote-tr)] end
›
```

ML ‹

```
fun get-fields (SOME ({fields, parent, ...}: Record.info)) thy =
  (case parent of
      SOME ( -,y) => fields @ get-fields (Record.get-info thy y) thy
    | NONE => fields)
  | get-fields NONE - = []
```

```
val dummy = Term.dummy-pattern dummyT;
fun mk-eq (l, r) = HOLogic.Trueprop $ ((HOLogic.eq-const dummyT) $ l $ r)

fun add-datatype (params, binding) constr-specs thy =
  let
    val ([dt-name], thy') = thy
      |> BNF-LFP-Compat.add-datatype [BNF-LFP-Compat.Keep-Nesting]
        [((binding, params, NoSyn), constr-specs)];
    val constr-names =
      map fst (the-single (map (#3 o snd)
        (#descr (BNF-LFP-Compat.the-info thy' [BNF-LFP-Compat.Keep-Nesting]
dt-name))));
    fun constr (c, Ts) = (Const (c, dummyT), length Ts);
    val constrs = map #1 constr-specs ~~ map constr (constr-names ~~ map #2
constr-specs);
   in ((dt-name, constrs), thy') end;

fun define-channels (params, binding) typesyn channels thy =
  case typesyn of
  NONE =>
  let
    val dt-binding = Binding.suffix-name -channels binding;

    val constr-specs = map (fn (b, opt-T) => (b, the-list opt-T, NoSyn)) channels;
    val ((dt-name, constrs), thy1) =
      add-datatype (params, dt-binding) constr-specs thy;

    val T = Type (dt-name, []);

    val fun-name = ev-eq ^ - ^ Long-Name.base-name dt-name;

    val ev-equ = Free (fun-name, T --> T --> HOLogic.boolT);

    val eqs = map-product (fn (-, (c, n)) => (fn (-, (c1,n1)) =>
      let
        val t = Term.list-comb (c, replicate n dummy);
        val t1 = Term.list-comb (c1, replicate n1 dummy);
      in (if c = c1 then mk-eq ((ev-equ $ t $ t1), @{term True}) else mk-eq ((ev-equ
$ t $ t1), @{term False})) end)) constrs constrs;

  fun case-tac x ctxt =
    resolve-tac ctxt [Thm.instantiate' [] [SOME x]
    (#exhaust (BNF-LFP-Compat.the-info (Proof-Context.theory-of ctxt) [BNF-LFP-Compat.Keep-Nesting]
dt-name))];

  fun proof ctxt = (Class.intro-classes-tac ctxt [] THEN
                    Subgoal.FOCUS (fn {context = ctxt', params = [(-, x)], ...} =>
```

```
(case-tac x ctxt') 1
        THEN auto-tac ctxt') ctxt 1 THEN
    Subgoal.FOCUS (fn {context = ctxt', params = [(-, x), (-, y)],
...} =>
        ((case-tac x ctxt') THEN-ALL-NEW (case-tac y
ctxt')) 1
        THEN auto-tac ctxt') ctxt 1);

 val thy2 =
    thy1
    |> Class.instantiation ([dt-name], params, @{sort ev-eq})
    |> Local-Theory.begin-nested
    |> snd
    |> Function-Fun.add-fun [(Binding.name fun-name, NONE, NoSyn)]
      (map (fn t => ((Binding.empty-atts, t), [], [])) eqs) Function-Fun.fun-config
    |> Local-Theory.end-nested
    |> Class.prove-instantiation-exit (fn ctxt => proof ctxt);
  in
   ((dt-name, constrs), thy2)
  end
  | (SOME typn) =>
  let
   val dt-binding = Binding.suffix-name -channels binding;

   val (dt-name, thy1) =
       thy
       |> Named-Target.theory-init
         |> (fn ctxt => Typedecl.abbrev (dt-binding, map fst params, NoSyn)
(Proof-Context.read-typ ctxt typn) ctxt);

   val thy2 = thy1 |> Local-Theory.exit-global;
  in
   ((dt-name, []), thy2)
  end;

fun define-chanset binding channel-constrs (name, chans) thy  =
  let
   val constrs =
     filter (fn (b, -) => exists (fn a => a = Binding.name-of b) chans) chan-
nel-constrs;
   val bad-chans =
   filter-out (fn a => exists (fn (b, -) => a = Binding.name-of b) channel-constrs)
chans;
   val - = null bad-chans orelse
     error (Bad elements  ^ commas-quote bad-chans ^  in chanset:  ^ quote
(Binding.print name));
   val base-name = Binding.name-of name;
  val cs = map (fn (-, (c, n)) => Term.list-comb (c, replicate n (Const (@{const-name
undefined}, dummyT)))) constrs;
```

val chanset-eq = mk-eq ((Free (base-name, dummyT)), (HOLogic.mk-set dummyT cs));
in
thy
|> Named-Target.theory-init
|> Specification.definition (SOME (Binding.qualify-name true binding base-name, NONE, NoSyn))
[] [] (Binding.empty-atts, chanset-eq)
|> snd |> Local-Theory.exit-global
end;

fun define-nameset binding (rec-binding, alphabet) (ns-binding, names) thy =
let
val all-selectors = get-fields (Record.get-info thy (Sign.full-name thy rec-binding)) thy
val bad-names =
filter-out (fn a => exists (fn (b, -) => String.isSuffix a b) all-selectors) names;
val - = null bad-names orelse
error (Bad elements ^ commas-quote bad-names ^ in nameset: ^ quote (Binding.print ns-binding));
val selectors =
filter (fn (b, -) => exists (fn a => String.isSuffix a b) names) all-selectors;
val updates = map (fn x => (fst x, ((suffix Record.updateN) o fst) x)) selectors;
val selectors′ = map (fn x => (fst x, Const(fst x, dummyT))) selectors;
val updates′ = map (fn (x, y) => (x, Const(y, dummyT))) updates;
val l =
map (fn (b, -) => Binding.name-of b) alphabet;
val formulas = map2 (fn (nx, x) =>
fn (ny, y) =>
if (exists (fn b => String.isSuffix b nx) l)
then Abs (A, dummyT, (Const(Circus-Syntax.list-update, dummyT) $ x)
$ (Abs (-, dummyT, (Const(Circus-Syntax.list-select, dummyT) $ y) $ (Bound 1))))
else Abs (A, dummyT, x $ (Abs (-, dummyT, y $ (Bound 1))))) updates′ selectors′;
val base-name = Binding.name-of ns-binding;
fun comp [a] = a $ (Bound 1) $ (Bound 0)
| comp (a::l) = a $ (Bound 1) $ (comp l);
val nameset-eq = mk-eq ((Free (base-name, dummyT)), (Abs (-, dummyT, (Abs (-, dummyT, comp formulas)))));
in
thy
|> Named-Target.theory-init
|> Specification.definition (SOME (Binding.qualify-name true binding base-name, NONE, NoSyn))
[] [] (Binding.empty-atts, nameset-eq)
|> snd |> Local-Theory.exit-global

```
  end;


fun define-schema binding (ex-binding, expr) (alph-bind, alpha, state) thy =
  let
        val fields-names = (map (fn (x, T) => (Binding.name-of x, T)) (alpha @
state));
       val alpha' = (map (fn (x, T) => (Binding.name-of x, T)) alpha);
       val state' = (map (fn (x, T) => (Binding.name-of x, T)) state);
     val all-selectors = get-fields (Record.get-info thy (Sign.full-name thy alph-bind))
thy
       val base-name = Binding.name-of ex-binding;
       val ctxt = Proof-Context.init-global thy;
       val term =
         Syntax.read-term
           (ctxt
              |> VARs-Data.put ({State-vars=(map fst state'), Alpha-vars=(map fst
alpha')})
             |> Config.put Syntax.root @{nonterminal circus-schema}) expr;
       val sc-eq = mk-eq ((Free (base-name, dummyT)), term);
  in
    thy
    |> Named-Target.theory-init
   |> Specification.definition (SOME (Binding.qualify-name true binding base-name,
NONE, NoSyn))
        [] [] (Binding.empty-atts, sc-eq)
    |> snd
    |> Local-Theory.exit-global
  end;


fun define-action binding (ex-binding, expr) alph-bind chan-bind thy =
  let
    val base-name = Binding.name-of ex-binding;
    val ctxt = Proof-Context.init-global thy;
    val actT = Circus-Actions.action;
    val action-eq =
      mk-eq
        ((Free (base-name,
            Type (actT, [(Proof-Context.read-type-name {proper=true, strict=false}
ctxt (Sign.full-name thy chan-bind)),
         (Proof-Context.read-type-name {proper=true, strict=false} ctxt (Sign.full-name
thy alph-bind))]))),
        (Syntax.parse-term ctxt expr));
  in
    thy
    |> Named-Target.theory-init
   |> Specification.definition (SOME (Binding.qualify-name true binding base-name,
NONE, NoSyn))
```

```
      [] [] (Binding.empty-atts, action-eq)
    |> snd
    |> Local-Theory.exit-global
  end;


fun define-expr binding (alph-bind, alpha, state) chan-bind (ex-binding, (is-schema,
expr)) =
   if is-schema then define-schema binding (ex-binding, expr) (alph-bind, alpha,
state)
  else define-action binding (ex-binding, expr) alph-bind chan-bind;


fun prep-field prep-typ (b: binding, raw-T) ctxt =
  let
    val T = prep-typ ctxt raw-T;
    val ctxt' = Variable.declare-typ T ctxt;
  in ((b, T), ctxt') end;

fun prep-constr prep-typ (b: binding, raw-T) ctxt =
  let
    val T = Option.map (prep-typ ctxt) raw-T;
    val ctxt' = fold Variable.declare-typ (the-list T) ctxt;
  in ((b, T), ctxt') end;


fun gen-circus-process prep-constraint prep-typ
    (raw-params, binding) raw-alphabet raw-state (typesyn, raw-channels) namesets
chansets
    exprs act thy =
  let
    val ctxt = Proof-Context.init-global thy;


    (* internalize arguments *)

    val params = map (prep-constraint ctxt) raw-params;
    val ctxt0 = fold (Variable.declare-typ o TFree) params ctxt;

    val (alphabet, ctxt1) = fold-map (prep-field prep-typ) raw-alphabet ctxt0;
    val (state, ctxt2) = fold-map (prep-field prep-typ) raw-state ctxt1;
    val (channels, ctxt3) = fold-map (prep-constr prep-typ) raw-channels ctxt2;

    val params' = map (Proof-Context.check-tfree ctxt3) params;


    (* type definitions *)
```

```
    val fields =
      map (fn (b, T) => (b, T, NoSyn)) (map (apsnd HOLogic.listT) alphabet @
state);

    val thy1 = thy
      |> not (null fields) ?
        Record.add-record {overloaded = false}
          (params', Binding.suffix-name -alphabet binding) NONE fields;
    val (channel-constrs, thy2) =
      if not (null channels) orelse is-some typesyn
      then apfst snd (define-channels (params', binding) typesyn channels thy1)
      else ([], thy1);
    val thy3 = thy2
     |> not (null chansets) ? fold (define-chanset binding channel-constrs) chansets
      |> not (null namesets) ?
         fold (define-nameset binding ((Binding.suffix-name -alphabet binding), al-
phabet)) namesets
      |> not (null exprs) ?
        fold (define-expr binding ((Binding.suffix-name -alphabet binding), alphabet,
state)
          (Binding.suffix-name -channels binding)) exprs
      |> define-action binding (binding, act)
       (Binding.suffix-name -alphabet binding) (Binding.suffix-name -channels bind-
ing);
  in
    thy3
  end;

fun circus-process x = gen-circus-process (K I) Syntax.check-typ x;
fun circus-process-cmd x = gen-circus-process (apsnd o Typedecl.read-constraint)
Syntax.read-typ x;


local

val fields =
  @{keyword [} |-- Parse.enum1 , (Parse.binding -- (@{keyword ::} |-- Parse.!!!
Parse.typ))
    --| @{keyword ]};

val constrs =
  (@{keyword [} |-- Parse.enum1 , (Parse.binding -- Scan.option Parse.typ)
--| @{keyword ]}) >> pair NONE
  || Parse.typ >> (fn b => (SOME b, []));

val names =
  @{keyword [} |-- Parse.enum1 , Parse.name --| @{keyword ]};
```

in

val - =
Outer-Syntax.command @{*command-keyword circus-process*} *Circus process specification*
((*Parse.type-args-constrained* −− *Parse.binding* −−| @{*keyword* =}) −−
Scan.optional (@{*keyword alphabet*} |−− *Parse.*!!! (@{*keyword* =} |−− *fields*)) [] −−
Scan.optional (@{*keyword state*} |−− *Parse.*!!! (@{*keyword* =} |−− *fields*)) [] −−
Scan.optional (@{*keyword channel*} |−− *Parse.*!!! (@{*keyword* =} |−− *constrs*)) (*NONE*, []) −−
Scan.repeat (@{*keyword nameset*} |−− *Parse.*!!! ((*Parse.binding* −−| @{*keyword* =}) −− *names*)) −−
Scan.repeat (@{*keyword chanset*} |−− *Parse.*!!! ((*Parse.binding* −−| @{*keyword* =}) −− *names*)) −−
Scan.repeat ((@{*keyword schema*} |−− *Parse.*!!! ((*Parse.binding* −−| @{*keyword* =}) −− (*Parse.term* >> *pair true*))) ||
(@{*keyword action*} |−− *Parse.*!!! ((*Parse.binding* −−| @{*keyword* =}) −− (*Parse.term* >> *pair false*)))) −−
(*Parse.where-* |−− *Parse.*!!! *Parse.term*)
>> (*fn* (((((((*a*, *b*), *c*), *d*), *e*), *f*), *g*), *h*) =>
Toplevel.theory (*circus-process-cmd a b c d e f g h*)));

end;
›

end

16 Refinement and Simulation

theory *Refinement*
imports *Denotational-Semantics Circus-Syntax*
begin

16.1 Definitions

In the following, data (state) simulation and functional backwards simulation are defined. The simulation is defined as a function S, that corresponds to a state abstraction function.

definition *Simul S b* = *extend* (*make* (*ok b*) (*wait b*) (*tr b*) (*ref b*)) (*S* (*more b*))

definition
Simulation::($'\vartheta$::*ev-eq*,$'\sigma$) *action* ⇒ ($'\sigma 1$ ⇒ $'\sigma$) ⇒ ($'\vartheta$, $'\sigma 1$) *action* ⇒ *bool* (‹- ⪯- -›)
where
$A \preceq S\ B \equiv \forall\ a\ b.$ (*relation-of B*) (*a*, *b*) ⟶ (*relation-of A*) (*Simul S a*, *Simul S b*)

16.2 Proofs

In order to simplify refinement proofs, some general refinement laws are defined to deal with the refinement of Circus actions at operators level and not at UTP level. Using these laws, and exploiting the advantages of a shallow embedding, the automated proof of refinement becomes surprisingly simple.

lemma *Stop-Sim*: *Stop* $\preceq$*S Stop*
by (*auto simp*: *Simulation-def relation-of-Stop rp-defs design-defs Simul-def alpha-rp.defs*
    split: *cond-splits*)

lemma *Skip-Sim*: *Skip* $\preceq$*S Skip*
by (*auto simp*: *Simulation-def relation-of-Skip design-def rp-defs Simul-def alpha-rp.defs*

    split: *cond-splits*)

lemma *Chaos-Sim*: *Chaos* $\preceq$*S Chaos*
by (*auto simp*: *Simulation-def relation-of-Chaos rp-defs design-defs Simul-def alpha-rp.defs*
    split: *cond-splits*)

lemma *Ndet-Sim*:
  assumes *A*: $P \preceq S\ Q$ **and** *B*: $P' \preceq S\ Q'$
  shows $(P \sqcap P') \preceq S\ (Q \sqcap Q')$
by (*insert A B*, *auto simp*: *Simulation-def relation-of-Ndet*)

lemma *Det-Sim*:
  assumes *A*: $P \preceq S\ Q$ **and** *B*: $P' \preceq S\ Q'$
  shows $(P \Box P') \preceq S\ (Q \Box Q')$
by (*auto simp*: *Simulation-def relation-of-Det design-def rp-defs Simul-def alpha-rp.defs spec-def*
    split: *cond-splits*
    dest: *A*[*simplified Simulation-def Simul-def*, *rule-format*]
        B[*simplified Simulation-def Simul-def*, *rule-format*])

lemma *Schema-Sim*:
  assumes *A*: $\bigwedge a.\ (Pre\ sc1)\ (S\ a) \Longrightarrow (Pre\ sc2)\ a$
  and *B*: $\bigwedge a\ b.\ [\![Pre\ sc1\ (S\ a)\ ;\ sc2\ (a, b)]\!] \Longrightarrow sc1\ (S\ a, S\ b)$
  shows $(Schema\ sc1) \preceq S\ (Schema\ sc2)$
by (*auto simp*: *Simulation-def Simul-def relation-of-Schema rp-defs design-defs alpha-rp.defs A B*
    split: *cond-splits*)

lemma *SUb-Sim*:
  assumes *A*: $\bigwedge a.\ (Pre\ sc1)\ (S\ a) \Longrightarrow (Pre\ sc2)\ a$
  and *B*: $\bigwedge a\ b.\ [\![Pre\ sc1\ (S\ a)\ ;\ sc2\ (a, b)]\!] \Longrightarrow sc1\ (S\ a, S\ b)$
  and *C*: $P \preceq S\ Q$

shows (*state-update-before sc1 P*) $\preceq$*S* (*state-update-before sc2 Q*)
apply (*auto simp*: *Simulation-def Simul-def relation-of-state-update-before rp-defs design-defs alpha-rp.defs A B*
split: *cond-splits*)
apply (*drule C*[*simplified Simulation-def*, *rule-format*])
apply (*rule-tac b*=*Simul S ba* **in** *comp-intro*)
apply (*auto simp*: *A B Simul-def alpha-rp.defs*)
apply (*clarsimp split*: *cond-splits*)+
apply (*drule C*[*simplified Simulation-def*, *rule-format*])
apply (*rule-tac b*=*Simul S ba* **in** *comp-intro*)
apply (*auto simp*: *A B Simul-def alpha-rp.defs*)
apply (*clarsimp split*: *cond-splits*)+
apply (*case-tac ok aa*, *simp-all*)
apply (*erule notE*) **back**
apply (*drule C*[*simplified Simulation-def*, *rule-format*])
apply (*rule-tac b*=*Simul S ba* **in** *comp-intro*)
apply (*auto simp*: *A B Simul-def alpha-rp.defs*)
apply (*clarsimp split*: *cond-splits*)+
apply (*rule A*)
apply (*case-tac Pre sc1* (*S* (*alpha-rp.more aa*)), *simp-all*)
apply (*erule notE*) **back**
apply (*drule C*[*simplified Simulation-def*, *rule-format*])
apply (*rule-tac b*=*Simul S ba* **in** *comp-intro*)
apply (*auto simp*: *A B Simul-def alpha-rp.defs*)
apply (*clarsimp split*: *cond-splits*)+
apply (*drule C*[*simplified Simulation-def*, *rule-format*])
apply (*rule-tac b*=*Simul S ba* **in** *comp-intro*)
apply (*auto simp*: *A B Simul-def alpha-rp.defs*)
apply (*clarsimp split*: *cond-splits*)+
apply (*rule B*, *auto*)
done

lemma *Seq-Sim*:
assumes *A*: *P* $\preceq$*S Q* **and** *B*: *P'* $\preceq$*S Q'*
shows (*P* ‘; ‘ *P'*) $\preceq$*S* (*Q* ‘; ‘ *Q'*)
by (*auto simp*: *Simulation-def relation-of-Seq dest*: *A*[*simplified Simulation-def*, *rule-format*]
B[*simplified Simulation-def*, *rule-format*])

lemma *Par-Sim*:
assumes *A*: *P* $\preceq$*S Q* **and** *B*: *P'* $\preceq$*S Q'*
and *C*: $\bigwedge$ *a b*. *S* (*ns'2 a b*) = *ns2* (*S a*) (*S b*)
and *D*: $\bigwedge$ *a b*. *S* (*ns'1 a b*) = *ns1* (*S a*) (*S b*)
shows (*P* ⟦ *ns1* | *cs* | *ns2* ⟧ *P'*) $\preceq$*S* (*Q* ⟦ *ns'1* | *cs* | *ns'2* ⟧ *Q'*)
apply (*auto simp*: *Simulation-def relation-of-Par fun-eq-iff rp-defs Simul-def design-defs spec-def*
alpha-rp.defs
dest: *A*[*simplified Simulation-def Simul-def*, *rule-format*]

B[*simplified Simulation-def Simul-def*, *rule-format*])
apply (*split cond-splits*)+
apply (*simp*, *erule disjE*, *rule disjI1*, *simp*, *rule disjI2*, *simp-all*, *rule impI*)
apply (*auto*)
apply (*erule-tac x=tr ba* **in** *allE*, *auto*)
apply (*erule notE*) **back**
apply (*rule-tac b=Simul S ba*(|*ok* := *False*|) **in** *comp-intro*)
apply (*auto simp*: *Simul-def alpha-rp.defs dest*: *A*[*simplified Simulation-def Simul-def*, *rule-format*])
apply (*erule-tac x=tr bb* **in** *allE*, *auto*)
apply (*erule notE*) **back**
apply (*rule-tac b=Simul S bb*(|*ok* := *False*|) **in** *comp-intro*)
apply (*auto simp*: *Simul-def alpha-rp.defs dest*: *B*[*simplified Simulation-def Simul-def*, *rule-format*])
apply (*erule-tac x=tr ba* **in** *allE*, *auto*)
apply (*erule notE*) **back**
apply (*rule-tac b=Simul S ba*(|*ok* := *False*|) **in** *comp-intro*)
apply (*auto simp*: *Simul-def alpha-rp.defs dest*: *A*[*simplified Simulation-def Simul-def*, *rule-format*])
apply (*erule-tac x=tr bb* **in** *allE*, *auto*)
apply (*erule notE*) **back**
apply (*rule-tac b=Simul S bb*(|*ok* := *False*|) **in** *comp-intro*)
apply (*auto simp*: *Simul-def alpha-rp.defs dest*: *B*[*simplified Simulation-def Simul-def*, *rule-format*])
apply (*rule-tac x=Simul S s1* **in** *exI*)
apply (*rule-tac x=Simul S s2* **in** *exI*)
apply (*auto simp*: *Simul-def alpha-rp.defs*
dest!: *B*[*simplified Simulation-def Simul-def*, *rule-format*]
A[*simplified Simulation-def Simul-def*, *rule-format*]
split: *cond-splits*)
apply (*rule-tac b=Simul S ba* **in** *comp-intro*)
apply (*auto simp add*: *M-par-def alpha-rp.defs diff-tr-def fun-eq-iff ParMerge-def Simul-def*
split : *cond-splits*)
apply (*rule-tac b=*(|*ok* = *ok bb*, *wait* = *wait bb*, *tr* = *tr bb*, *ref* = *ref bb*,
. . . = *S* (*alpha-rp.more bb*)|) **in** *comp-intro*, *auto*)
apply (*subst D*[**where** *a*=(*alpha-rp.more s1*) **and** *b*=(*alpha-rp.more aa*), *symmetric*], *simp*)
apply (*subst C*[**where** *a*=(*alpha-rp.more s2*) **and** *b*=(*alpha-rp.more bb*), *symmetric*], *simp*)
apply (*rule-tac b=*(|*ok* = *ok bb*, *wait* = *wait bb*, *tr* = *tr bb*, *ref* = *ref bb*,
. . . = *S* (*alpha-rp.more bb*)|) **in** *comp-intro*, *auto*)
apply (*subst D*[**where** *a*=(*alpha-rp.more s1*) **and** *b*=(*alpha-rp.more aa*), *symmetric*], *simp*)
apply (*subst C*[**where** *a*=(*alpha-rp.more s2*) **and** *b*=(*alpha-rp.more bb*), *symmetric*], *simp*)
done

lemma *Assign-Sim*:

assumes *A*: ⋀ *A*. *vy A* = *vx* (*S A*)
and *B*: ⋀ *ff A*. (*S* (*y-update ff A*)) = *x-update ff* (*S A*)
shows (*x* ':=' *vx*) ⪯*S* (*y* ':=' *vy*)
by (*auto simp*: *Simulation-def relation-of-Assign update-def rp-defs design-defs Simul-def A B*
alpha-rp.defs split: *cond-splits*)

lemma *Var-Sim*:
assumes *A*: *P* ⪯*S Q* **and** *B*: ⋀ *ff A*. (*S* ((*snd b*) *ff A*)) = (*snd a*) *ff* (*S A*)
shows (*Var a P*) ⪯*S* (*Var b Q*)
apply (*auto simp*: *Simulation-def relation-of-Var rp-defs design-defs fun-eq-iff Simul-def B*
alpha-rp.defs increase-def decrease-def)
apply (*rule-tac b=Simul S ab* **in** *comp-intro*)
apply (*split cond-splits*)+
apply (*auto simp*: *B alpha-rp.defs Simul-def elim*!: *alpha-rp-eqE*)
apply (*rule-tac b=Simul S bb* **in** *comp-intro*)
apply (*split cond-splits*)+
apply (*auto simp*: *B alpha-rp.defs Simul-def*
elim!: *alpha-rp-eqE dest*!: *A*[*simplified Simulation-def Simul-def*, *rule-format*])
apply (*split cond-splits*)+
apply (*simp add*: *alpha-rp.defs*)
apply (*erule disjE*, *rule disjI1*, *simp*, *rule disjI2*, *simp*)
apply (*simp-all add*: *alpha-rp.defs true-def*)
apply (*rule impI*, (*erule conjE* | *simp*)+)
apply (*simp add*: *B*)
apply (*split cond-splits*)+
apply (*simp add*: *alpha-rp.defs*)
apply (*erule disjE*, *rule disjI1*, *simp*, *rule disjI2*, *simp-all*)
apply (*rule impI*, (*erule conjE* | *simp*)+)
apply (*simp add*: *B*)
done

lemma *Guard-Sim*:
assumes *A*: *P* ⪯*S Q* **and** *B*: ⋀ *A*. *h A* = *g* (*S A*)
shows (*g* '&' *P*) ⪯*S* (*h* '&' *Q*)
apply (*auto simp*: *Simulation-def*)
apply (*case-tac h* (*alpha-rp.more a*))
defer
apply (*case-tac g* (*S* (*alpha-rp.more a*)))
apply (*auto simp*: *true-Guard1 false-Guard1 Simul-def alpha-rp.defs Simulation-def B*
dest!: *A*[*simplified*, *rule-format*] *Stop-Sim*[*simplified*, *rule-format*])
done

lemma *Write0-Sim*:
assumes *A*: *P* ⪯*S Q*
shows *a* → *P* ⪯*S a* → *Q*

using *A*
apply (*auto simp*: *Simulation-def write0-def relation-of-Prefix0 design-defs rp-defs*)
apply (*erule-tac x=ba* **in** *allE*)
apply (*erule-tac x=c* **in** *allE*, *auto*)
apply (*rule-tac b=Simul S ba* **in** *comp-intro*)
apply (*auto split*: *cond-splits simp*: *Simul-def alpha-rp.defs do-def*)
done

lemma *Read-Sim*:
assumes *A*: $P \preceq S\ Q$ **and** *B*: $\bigwedge A.\ (d\ A) = c\ (S\ A)$
shows $a`?`c \to P \preceq S\ a`?`d \to Q$
using *A*
apply (*auto simp*: *Simulation-def read-def relation-of-iPrefix design-defs rp-defs*)
apply (*erule-tac x=ba* **in** *allE*, *erule-tac x=ca* **in** *allE*, *simp*)
apply (*rule-tac b=Simul S ba* **in** *comp-intro*)
apply (*auto split*: *cond-splits simp*: *Simul-def alpha-rp.defs do-I-def select-def B*)
done

lemma *Read1-Sim*:
assumes *A*: $P \preceq S\ Q$ **and** *B*: $\bigwedge A.\ (d\ A) = c\ (S\ A)$
shows $a`?`c`:`s \to P \preceq S\ a`?`d`:`s \to Q$
using *A*
apply (*auto simp*: *Simulation-def read1-def relation-of-iPrefix design-defs rp-defs*)
apply (*erule-tac x=ba* **in** *allE*, *erule-tac x=ca* **in** *allE*, *simp*)
apply (*rule-tac b=Simul S ba* **in** *comp-intro*)
apply (*auto split*: *cond-splits simp*: *Simul-def alpha-rp.defs do-I-def select-def B*)
done

lemma *Read1S-Sim*:
assumes *A*: $P \preceq S\ Q$ **and** *B*: $\bigwedge A.\ (d\ A) = c\ (S\ A)$ **and** *C*: $\bigwedge A.\ (s'\ A) = s\ (S\ A)$
shows $a`?`c`\in`s \to P \preceq S\ a`?`d`\in`s' \to Q$
using *A*
apply (*auto simp*: *Simulation-def read1-def relation-of-iPrefix design-defs rp-defs*)
apply (*erule-tac x=ba* **in** *allE*, *erule-tac x=ca* **in** *allE*, *simp*)
apply (*rule-tac b=Simul S ba* **in** *comp-intro*)
apply (*auto split*: *cond-splits simp*: *Simul-def alpha-rp.defs do-I-def select-def B C*)
done

lemma *Write-Sim*:
assumes *A*: $P \preceq S\ Q$ **and** *B*: $\bigwedge A.\ (d\ A) = c\ (S\ A)$
shows $a`!`c \to P \preceq S\ a`!`d \to Q$
using *A*
apply (*auto simp*: *Simulation-def write1-def relation-of-oPrefix design-defs rp-defs*)
apply (*erule-tac x=ba* **in** *allE*, *erule-tac x=ca* **in** *allE*, *simp*)
apply (*rule-tac b=Simul S ba* **in** *comp-intro*)
apply (*auto split*: *cond-splits simp*: *Simul-def alpha-rp.defs do-def select-def B*)
done

lemma *Hide-Sim*:
assumes *A*: $P \preceq S\ Q$
shows $(P \setminus cs) \preceq S\ (Q \setminus cs)$
apply (*auto simp*: *Simulation-def relation-of-Hide design-defs rp-defs Simul-def alpha-rp.defs*)
apply (*rule-tac b=Simul S ba* **in** *comp-intro*)
apply (*split cond-splits*)+
apply (*auto simp*: *Simul-def alpha-rp.defs Simulation-def*
dest!: *A*[*simplified*, *rule-format*] *Skip-Sim*[*simplified*, *rule-format*])
apply (*rule-tac x=s* **in** *exI*, *auto simp*: *diff-tr-def*)
done

lemma *lfp-Siml*:
assumes *A*: $\bigwedge X.\ (X \preceq S\ Q) \Longrightarrow ((P\ X) \preceq S\ Q)$ **and** *B*: *mono P*
shows $(lfp\ P) \preceq S\ Q$
apply (*rule lfp-ordinal-induct*, *auto simp*: *B A*)
apply (*auto simp add*: *Simulation-def Sup-action relation-of-bot relation-of-Sup*[*simplified*])
apply (*subst* (*asm*) *CSP-is-rd*[*OF relation-of-CSP*])
apply (*auto simp*: *rp-defs fun-eq-iff Simul-def alpha-rp.defs decrease-def split*: *cond-splits*)
done

lemma *Mu-Sim*:
assumes *A*: $\bigwedge X\ Y.\ X \preceq S\ Y \Longrightarrow (P\ X) \preceq S\ (Q\ Y)$
and *B*: *mono P* **and** *C*: *mono Q*
shows $(lfp\ P) \preceq S\ (lfp\ Q)$
apply (*rule lfp-Siml*, *drule A*)
apply (*subst lfp-unfold*, *simp-all add*: *B C*)
done

lemma *bot-Sim*: $bot \preceq S\ bot$
by (*auto simp*: *Simulation-def rp-defs Simul-def relation-of-bot alpha-rp.defs split*: *cond-splits*)

lemma *sim-is-ref*: $P \sqsubseteq Q = P \preceq (id)\ Q$
apply (*auto simp*: *ref-def Simulation-def Simul-def alpha-rp.defs*)
apply (*erule-tac x=a* **in** *allE*)
apply (*erule-tac x=b* **in** *allE*, *auto*)
apply (*rule-tac t=*(|*ok = ok a*, *wait = wait a*, *tr = tr a*, *ref = ref a*, ... = *alpha-rp.more a*|) **and** *s=a* **in** *subst*, *simp*)
apply (*rule-tac t=*(|*ok = ok b*, *wait = wait b*, *tr = tr b*, *ref = ref b*, ... = *alpha-rp.more b*|) **and** *s=b* **in** *subst*, *simp-all*)
apply (*erule-tac x=a* **in** *allE*)
apply (*erule-tac x=b* **in** *allE*, *auto*)
apply (*rule-tac s=*(|*ok = ok a*, *wait = wait a*, *tr = tr a*, *ref = ref a*, ... = *alpha-rp.more a*|) **and** *t=a* **in** *subst*, *simp*)
apply (*rule-tac s=*(|*ok = ok b*, *wait = wait b*, *tr = tr b*, *ref = ref b*, ... = *alpha-rp.more b*|) **and** *t=b* **in** *subst*, *simp-all*)

done

lemma *ref-eq*: $((P::('a::ev\text{-}eq,'b)\ action) = Q) = (P \sqsubseteq Q\ \&\ Q \sqsubseteq P)$
apply (*rule*)
apply (*simp add*: *ref-def*)
apply (*auto simp add*: *ref-def fun-eq-iff relation-of-inject*[*symmetric*])
done

lemma *rd-ref*:
assumes $A{:}R\ (P \vdash Q) \in \{p.\ is\text{-}CSP\text{-}process\ p\}$
and $B{:}R\ (P' \vdash Q') \in \{p.\ is\text{-}CSP\text{-}process\ p\}$
and $C{:}\bigwedge\ a\ b.\ P\ (a,\ b) \Longrightarrow P'\ (a,\ b)$
and $D{:}\bigwedge\ a\ b.\ Q'\ (a,\ b) \Longrightarrow Q\ (a,\ b)$
shows $(action\text{-}of\ (R\ (P \vdash Q))) \sqsubseteq (action\text{-}of\ (R\ (P' \vdash Q')))$
apply (*auto simp*: *ref-def*)
apply (*subst* (*asm*) *action-of-inverse*, *simp add*: *B*[*simplified*])
apply (*subst action-of-inverse*, *simp add*: *A*[*simplified*])
apply (*auto simp*: *rp-defs design-defs C D split*: *cond-splits*)
done

lemma *rd-impl*:
assumes $A{:}R\ (P \vdash Q) \in \{p.\ is\text{-}CSP\text{-}process\ p\}$
and $B{:}R\ (P' \vdash Q') \in \{p.\ is\text{-}CSP\text{-}process\ p\}$
and $C{:}\bigwedge\ a\ b.\ P\ (a,\ b) \Longrightarrow P'\ (a,\ b)$
and $D{:}\bigwedge\ a\ b.\ Q'\ (a,\ b) \Longrightarrow Q\ (a,\ b)$
shows $R\ (P' \vdash Q')\ (a,\ b) \longrightarrow R\ (P \vdash Q)\ (a{::}('a{::}ev\text{-}eq,\ 'b)\ alpha\text{-}rp\text{-}scheme,\ b)$
apply (*insert rd-ref*[*of P Q P′ Q′*, *OF A B C D*])
apply (*auto simp*: *ref-def*)
apply (*subst* (*asm*) *action-of-inverse*, *simp add*: *B*[*simplified*])
apply (*subst* (*asm*) *action-of-inverse*, *simp add*: *A*[*simplified*])
apply (*erule-tac x=a* **in** *allE*)
apply (*erule-tac x=b* **in** *allE*)
apply (*auto*)
done

end

17 Concrete example

theory *Refinement-Example*
imports *Refinement*
begin

In this section, we present a concrete example ofthe use of our environment. We define two Circus processes FIG and DFIG, using our syntax. we give the proof of refinement (simulation) of the first processby the second one using the simulation function *Sim*.

17.1 Process definitions

circus-process *FIG* =
  alphabet = [*v*::*nat*, *x*::*nat*]
  state = [*idS*::*nat set*]
  channel = [*out nat* , *req* , *ret nat*]
  schema *Init* = *idS′* = {}
  schema *Out* = ∃ *a*. *v′* = *a* ∧ *a* ∉ *idS* ∧ *idS′* = *idS* ∪ {*v′*}
  schema *Remove* = *x* ∈ *idS* ∧ *idS′* = *idS* − {*x*}
  where *var v* • (*Schema FIG.Init*‘; ‘
        μ *X* • (((((*req* → (*Schema FIG.Out*))‘; ‘ *out*‘!‘(*hd o v*) → *Skip*))
            □ (*ret*‘?‘*x* → (*Schema FIG.Remove*)))‘; ‘ *X*))

circus-process *DFIG* =
  alphabet = [*v*::*nat*, *x*::*nat*]
  state = [*retidS*::*nat set*, *max*::*nat*]
  channel = *FIG-channels*
  schema *Init* = *retidS′* = {} ∧ *max′* = *0*
  schema *Out* = *v′* = *max* ∧ *max′* = (*max* + *1*) ∧ *retidS′* = *retidS* − {*v′*}
  schema *Remove* = *x* < *max* ∧ *retidS′* = *retidS* ∪ {*x*} ∧ *max′* = *max*
  where *var v* • (*Schema DFIG.Init*‘; ‘
        μ *X* • ((((*req* → (*Schema DFIG.Out*))‘; ‘ (*out*‘!‘(*hd o v*) → *Skip*))
            □ (*ret*‘?‘*x* → (*Schema DFIG.Remove*)))‘; ‘ *X*))

definition *Sim* **where**
  Sim A = *FIG-alphabet.make* (*DFIG-alphabet.v A*) (*DFIG-alphabet.x A*)
  ({*a*. *a* < (*DFIG-alphabet.max A*) ∧ *a* ∉ (*DFIG-alphabet.retidS A*)})

17.2 Simulation proofs

For the simulation proof, we give first proofs for simulation over the schema expressions. The proof is then given over the main actions of the processes.

lemma *SimInit*: (*Schema FIG.Init*) ⪯*Sim* (*Schema DFIG.Init*)
  apply (*auto simp*: *Sim-def Pre-def design-defs DFIG.Init-def FIG.Init-def rp-defs alpha-rp.defs*
          DFIG-alphabet.defs FIG-alphabet.defs intro!: *Schema-Sim*)
  apply (*rule-tac x*=*A*⦇*max* := *0*, *retidS* := {}⦈ **in** *exI*, *simp*)
done

lemma *SimOut*: (*Schema FIG.Out*) ⪯*Sim* (*Schema DFIG.Out*)
  apply (*rule Schema-Sim*)
  apply (*auto simp*: *Pre-def DFIG-alphabet.defs FIG-alphabet.defs*
          alpha-rp.defs Sim-def FIG.Out-def DFIG.Out-def)
  apply (*rule-tac x*=*a*⦇*v* := [*DFIG-alphabet.max a*], *max* := (*Suc* (*DFIG-alphabet.max a*)),
          retidS := *retidS a* − {*DFIG-alphabet.max a*}⦈ **in** *exI*, *simp*)
  apply (*rule-tac x*=*a*⦇*v* := [*DFIG-alphabet.max a*], *max* := (*Suc* (*DFIG-alphabet.max*

a)),
retidS := *retidS a* − {*DFIG-alphabet.max a*}⦈ **in** *exI*, *simp*)
done

lemma *SimRemove*: (*Schema FIG.Remove*) ⪯*Sim* (*Schema DFIG.Remove*)
apply (*rule Schema-Sim*)
apply (*auto simp*: *Pre-def DFIG-alphabet.defs FIG-alphabet.defs alpha-rp.defs Sim-def*)
apply (*clarsimp simp add*: *DFIG.Remove-def FIG.Remove-def*)
apply (*rule-tac x*=*a*⦇*retidS* := *insert* (*hd* (*DFIG-alphabet.x a*)) (*retidS a*)⦈ **in** *exI*, *simp*)
apply (*auto simp add*: *DFIG.Remove-def FIG.Remove-def*)
done

lemma *FIG.FIG* ⪯*Sim DFIG.DFIG*
by (*auto simp*: *DFIG.DFIG-def FIG.FIG-def mono-Seq SimRemove SimOut SimInit Sim-def FIG-alphabet.defs*
intro!: *Var-Sim Seq-Sim Mu-Sim Det-Sim Write0-Sim Write-Sim Read-Sim Skip-Sim*)

end

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