

# Intersecting Chords Theorem

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## Abstract

This entry provides a geometric proof of the intersecting chords theorem. The theorem states that when two chords intersect each other inside a circle, the products of their segments are equal.

After a short review of existing proofs in the literature [1, 3–5], I decided to use a proof approach that employs reasoning about lengths of line segments, the orthogonality of two lines and Pythagoras Law. Hence, one can understand the formalized proof easily with the knowledge of a few general geometric facts that are commonly taught in high-school.

This theorem is the 55th theorem of the Top 100 Theorems list.

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## 1 Introduction

The intersecting chords theorem states:

When two chords intersect each other inside a circle, the products of their segments are equal.

To prove this theorem in Isabelle, I reviewed existing formalizations in theorem provers and proofs in the literature [1, 3–5]. At the time of this AFP submission, the formalization of geometry in Isabelle is limited to only a few concepts and theorems. Hence, I selected to formalize the proof approach that fitted best to the already existing geometry formalizations.

The proof in HOL Light [3] simply unfolds the involved geometric predicates and then proves the theorem using only algebraic computations on real numbers. By a quick and shallow inspection of the proof script without executing the proof script step by step in HOL Light, I could not understand the proof script well enough to re-write the proof in Isabelle. As running the script in HOL Light seemed too involved to me, I ignored HOL Light’s proof approach and considered the other approaches in the literature.

The first proof approach [5] that I found in the literature employs similarity of triangles, the inscribed angle theorem, and basic reasoning with angles. The intersecting chords theorem only consists of two reasoning steps after stating the geometric observations about angles. However, the proof requires to formalize the concept of similarity of triangles, extend the existing formalization of angles, and prove the inscribed angle theorem. So, I abandoned this proof approach and considered the second proof approach.

The second proof approach [5] needs only basic geometric reasoning about lengths of line segments, the orthogonality of two lines and Pythagoras Law. More specifically, one must prove that the line that goes through the apex and the midpoint of the base in an isosceles triangle is orthogonal to the base. This is easily derived from the property of an isosceles triangle using the congruence properties of triangles, which is already formalized in AFP’s Triangle entry [2]. Furthermore, Pythagoras Law is a special case of the Law of Cosines, which is already formalized in AFP’s Triangle entry.

Ultimately, I decided to use this second proof approach, which I sketch in more detail in the next subsection.

## 1.1 Informal Proof Sketch

The proof of the intersecting chords theorem relies on the following observation which is depicted in Figure 1.

Instead of considering *two* arbitrary chords intersecting, consider *one* arbitrary chord with endpoints  $S$  and  $T$  on a circle with center  $C$  and one arbitrary point  $X$  on the chord  $ST$ . This point  $X$  on the chord creates two line segments on this chord, the left part  $SX$ , and the right part  $XT$ . Without loss of generality, we can assume that  $SX$  is longer than  $XT$ , as shown in Figure 1.

The key lemma for the intersecting chords theorem provides a closed expression for the length of these two line segments using the distances of the chord endpoints and the point to the center  $C$ , i.e., the lemma states:

$$\overline{SX} \cdot \overline{XT} = \overline{SC}^2 - \overline{XC}^2.$$

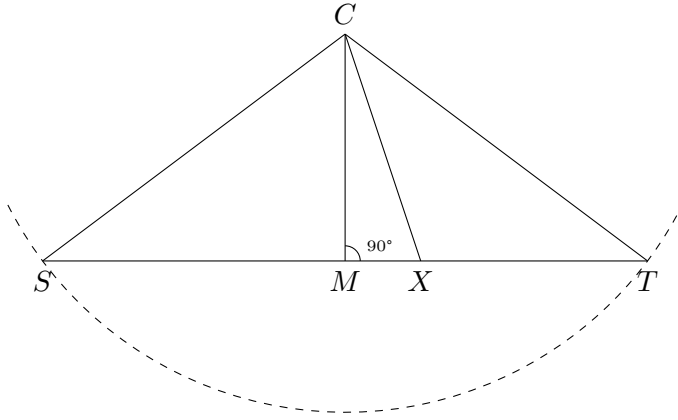


Figure 1: Key Lemma states  $\overline{SX} \cdot \overline{XT} = \overline{SC}^2 - \overline{XC}^2$

To prove this fact, we consider the midpoint  $M$  of the chord  $ST$ . First, as  $M$  is the midpoint,  $\overline{SM}$  and  $\overline{TM}$  are equal. Second, we observe that the lengths of the line segments  $SX$  and  $XT$  are:

$$\overline{SX} = \overline{SM} + \overline{MX} \text{ and } \overline{XT} = \overline{TM} - \overline{MX} = \overline{SM} - \overline{MX}.$$

Third, the Pythagoras law for the triangles  $SMC$  and  $XMC$  states:

$$\overline{SM}^2 + \overline{MC}^2 = \overline{SC}^2 \text{ and } \overline{XM}^2 + \overline{MC}^2 = \overline{XC}^2.$$

Finally, the product can be expressed as:

$$\begin{aligned} \overline{SX} \cdot \overline{XT} &= (\overline{SM} + \overline{MX}) \cdot (\overline{TM} - \overline{MX}) = \overline{SM}^2 - \overline{MX}^2 = \\ &= (\overline{SC}^2 - \overline{MC}^2) - (\overline{XC}^2 - \overline{MC}^2) = \overline{SC}^2 - \overline{XC}^2. \end{aligned}$$

The intersecting chord theorem now follows directly from this lemma: as the distances  $SC$  and  $XC$  for two arbitrary chords intersecting at  $X$  are equal, also the products of the chord segments are equal.

## 2 Intersecting Chord Theorem

```

theory Chord-Segments
imports
  ~/src/HOL/Analysis/Analysis
  ../Triangle/Triangle
begin

```

### 2.1 Preliminaries

#### 2.1.1 Addition to Fields

```

lemma divide-eq-minus-1-iff:

```

**fixes**  $a\ b :: 'a :: field$   
**shows**  $(a / b = - 1) \longleftrightarrow b \neq 0 \wedge a = - b$   
 $\langle proof \rangle$

### 2.1.2 Addition to Power

**lemma** *power2-eq-iff-nonneg*:  
**fixes**  $x\ y :: 'a :: linordered-semidom$   
**assumes**  $0 \leq x\ 0 \leq y$   
**shows**  $(x ^ 2 = y ^ 2) \longleftrightarrow x = y$   
 $\langle proof \rangle$

### 2.1.3 Addition to Transcendental

**lemma** *arccos-minus-abs*:  
**assumes**  $|x| \leq 1$   
**shows**  $arccos (- x) = pi - arccos x$   
 $\langle proof \rangle$

### 2.1.4 Additions to Convex Euclidean Space

**lemma** *between-same*:  
**assumes**  $between (a, a) x$   
**shows**  $x = a$   
 $\langle proof \rangle$

**lemma** *betweenI*:  
**assumes**  $0 \leq u\ u \leq 1\ x = (1 - u) *_R a + u *_R b$   
**shows**  $between (a, b) x$   
 $\langle proof \rangle$

**lemma** *betweenE*:  
**assumes**  $between (a, b) x$   
**obtains**  $u$  **where**  $0 \leq u\ u \leq 1\ x = (1 - u) *_R a + u *_R b$   
 $\langle proof \rangle$

**lemma** *betweenI'*:  
**assumes**  $0 \leq u\ u \leq 1\ x = u *_R a + (1 - u) *_R b$   
**shows**  $between (a, b) x$   
 $\langle proof \rangle$

**lemma** *betweenE'*:  
**assumes**  $between (a, b) x$   
**obtains**  $u$  **where**  $0 \leq u\ u \leq 1\ x = u *_R a + (1 - u) *_R b$   
 $\langle proof \rangle$

**lemma** *between-implies-scaled-diff*:  
**assumes**  $between (S, T) X\ between (S, T) Y\ S \neq Y$   
**obtains**  $c$  **where**  $(X - Y) = c *_R (S - Y)$   
 $\langle proof \rangle$

**lemma** *between-swap*:  
**fixes**  $A B X Y :: 'a::\text{euclidean-space}$   
**assumes** *between*  $(A, B) X$   
**assumes** *between*  $(A, B) Y$   
**shows** *between*  $(X, B) Y \longleftrightarrow \text{between } (A, Y) X$   
 $\langle \text{proof} \rangle$

**lemma** *betweenE-if-dist-leq*:  
**fixes**  $A B X :: 'a::\text{euclidean-space}$   
**assumes** *between*  $(A, B) X$   
**assumes**  $\text{dist } A X \leq \text{dist } B X$   
**obtains**  $u$  **where**  $1 / 2 \leq u \leq 1$  **and**  $X = u *_R A + (1 - u) *_R B$   
 $\langle \text{proof} \rangle$

**lemma** *dist-geq-iff-midpoint-in-between*:  
**fixes**  $A B X :: 'a::\text{euclidean-space}$   
**assumes** *between*  $(A, B) X$   
**shows**  $\text{dist } A X \leq \text{dist } B X \longleftrightarrow \text{between } (X, B) (\text{midpoint } A B)$   
 $\langle \text{proof} \rangle$

### 2.1.5 Additions to Angles

**lemma** *vangle-scales*:  
**assumes**  $0 < c$   
**shows**  $\text{vangle } (c *_R v_1) v_2 = \text{vangle } v_1 v_2$   
 $\langle \text{proof} \rangle$

**lemma** *vangle-inverse*:  
 $\text{vangle } (- v_1) v_2 = \text{pi} - \text{vangle } v_1 v_2$   
 $\langle \text{proof} \rangle$

**lemma** *orthogonal-iff-angle*:  
**shows** *orthogonal*  $(A - B) (C - B) \longleftrightarrow \text{angle } A B C = \text{pi} / 2$   
 $\langle \text{proof} \rangle$

**lemma** *angle-inverse*:  
**assumes** *between*  $(A, C) B$   
**assumes**  $A \neq B B \neq C$   
**shows**  $\text{angle } A B D = \text{pi} - \text{angle } C B D$   
 $\langle \text{proof} \rangle$

**lemma** *strictly-between-implies-angle-eq-pi*:  
**assumes** *between*  $(A, C) B$   
**assumes**  $A \neq B B \neq C$   
**shows**  $\text{angle } A B C = \text{pi}$   
 $\langle \text{proof} \rangle$

### 2.1.6 Additions to Triangles

**lemma** *pythagoras*:

**fixes**  $A B C :: 'a :: euclidean-space$

**assumes** *orthogonal*  $(A - C) (B - C)$

**shows**  $(\text{dist } B C)^2 + (\text{dist } C A)^2 = (\text{dist } A B)^2$

*<proof>*

**lemma** *isosceles-triangle-orthogonal-on-midpoint*:

**fixes**  $A B C :: 'a :: euclidean-space$

**assumes**  $\text{dist } C A = \text{dist } C B$

**shows** *orthogonal*  $(C - \text{midpoint } A B) (A - \text{midpoint } A B)$

*<proof>*

## 2.2 Properties of Chord Segments

**lemma** *chord-property*:

**fixes**  $S C :: 'a :: euclidean-space$

**assumes**  $\text{dist } C S = \text{dist } C T$

**assumes** *between*  $(S, T) X$

**shows**  $\text{dist } S X * \text{dist } X T = (\text{dist } C S)^2 - (\text{dist } C X)^2$

*<proof>*

**theorem** *product-of-chord-segments*:

**fixes**  $S_1 T_1 S_2 T_2 X C :: 'a :: euclidean-space$

**assumes** *between*  $(S_1, T_1) X$  *between*  $(S_2, T_2) X$

**assumes**  $\text{dist } C S_1 = r$   $\text{dist } C T_1 = r$

**assumes**  $\text{dist } C S_2 = r$   $\text{dist } C T_2 = r$

**shows**  $\text{dist } S_1 X * \text{dist } X T_1 = \text{dist } S_2 X * \text{dist } X T_2$

*<proof>*

**end**

## References

- [1] G. D. Birkhoff and R. Beatley. *Basic Geometry*, page 150. AMS Chelsea Publishing, Providence, Rhode Island, third edition, 1959.
- [2] M. Eberl. Basic geometric properties of triangles. *Archive of Formal Proofs*, Dec. 2015. <http://isa-afp.org/entries/Triangle.shtml>, Formal proof development.
- [3] J. Harrison. Theorem on product of segments of chords (in HOL light), 2010. <https://github.com/jrh13/hol-light/blob/master/100/chords.ml>.
- [4] W. R. Knorr. *Textual Studies in Ancient and Medieval Geometry*, pages 34–36. Birkhäuser, Boston, first edition, 1989.

- [5] Proofwiki. Intersecting chord theorem, 2014. [https://proofwiki.org/wiki/Intersecting\\_Chord\\_Theorem](https://proofwiki.org/wiki/Intersecting_Chord_Theorem) [Online; accessed = 10-October-2016].