# Intersecting Chords Theorem 

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#### Abstract

This entry provides a geometric proof of the intersecting chords theorem. The theorem states that when two chords intersect each other inside a circle, the products of their segments are equal.

After a short review of existing proofs in the literature [1, 3-5], I decided to use a proof approach that employs reasoning about lengths of line segments, the orthogonality of two lines and Pythagoras Law. Hence, one can understand the formalized proof easily with the knowledge of a few general geometric facts that are commonly taught in high-school.

This theorem is the 55 th theorem of the Top 100 Theorems list.


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## 1 Introduction

The intersecting chords theorem states:
When two chords intersect each other inside a circle, the products of their segments are equal.

To prove this theorem in Isabelle, I reviewed existing formalizations in theorem provers and proofs in the literature [1,3-5]. At the time of this AFP submission, the formalization of geometry in Isabelle is limited to only a few concepts and theorems. Hence, I selected to formalize the proof approach that fitted best to the already existing geometry formalizations.

The proof in HOL Light [3] simply unfolds the involved geometric predicates and then proves the theorem using only algebraic computations on


Figure 1: Key Lemma states $\overline{S X} \cdot \overline{X T}=\overline{S C}^{2}-\overline{X C}^{2}$
real numbers. By a quick and shallow inspection of the proof script without executing the proof script step by step in HOL Light, I could not understand the proof script well enough to re-write the proof in Isabelle. As running the script in HOL Light seemed too involved to me, I ignored HOL Light's proof approach and considered the other approaches in the literature.

The first proof approach [5] that I found in the literature employs similarity of triangles, the inscribed angle theorem, and basic reasoning with angles. The intersecting chords theorem only consists of two reasoning steps after stating the geometric observations about angles. However, the proof requires to formalize the concept of similarity of triangles, extend the existing formalization of angles, and prove the inscribed angle theorem. So, I abandoned this proof approach and considered the second proof approach.

The second proof approach [5] needs only basic geometric reasoning about lengths of line segments, the orthogonality of two lines and Pythagoras Law. More specifically, one must prove that the line that goes through the apex and the midpoint of the base in an isosceles triangle is orthogonal to the base. This is easily derived from the property of an isosceles triangle using the congruence properties of triangles, which is already formalized in AFP's Triangle entry [2]. Furthermore, Pythagoras Law is a special case of the Law of Cosines, which is already formalized in AFP's Triangle entry.

Ultimately, I decided to use this second proof approach, which I sketch in more detail in the next subsection.

### 1.1 Informal Proof Sketch

The proof of the intersecting chords theorem relies on the following observation which is depicted in Figure 1.

Instead of considering two arbitrary chords intersecting, consider one arbitrary chord with endpoints $S$ and $T$ on a circle with center $C$ and one arbitrary point $X$ on the chord $S T$. This point $X$ on the chord creates two line segments on this chord, the left part $S X$, and the right part $X T$. Without loss of generality, we can assume that $S X$ is longer that $X T$, as shown in Figure 1.

The key lemma for the intersecting chords theorem provides a closed expression for the length of these two line segment using the distances of the chord endpoints and the point to the center $C$, i.e., the lemma states:

$$
\overline{S X} \cdot \overline{X T}=\overline{S C}^{2}-\overline{X C}^{2} .
$$

To prove this fact, we consider the midpoint $M$ of the chord $S T$. First, as $M$ is the midpoint, $\overline{S M}$ and $\overline{T M}$ are equal. Second, we observe that the lengths of the line segments $S X$ and $X T$ are:

$$
\overline{S X}=\overline{S M}+\overline{M X} \text { and } \overline{X T}=\overline{T M}-\overline{M X}=\overline{S M}-\overline{M X} .
$$

Third, the Pythagoras law for the triangles SMC and XMC states:

$$
\overline{S M}^{2}+\overline{M C}^{2}=\overline{S C}^{2} \text { and } \overline{X M}^{2}+\overline{M C}^{2}=\overline{X C}^{2} .
$$

Finally, the product can be expressed as:

$$
\begin{aligned}
& \overline{S X} \cdot \overline{X T}=(\overline{S M}+\overline{M X}) \cdot(\overline{T M}-\overline{M X})=\overline{S M}^{2}-\overline{M X}^{2}= \\
& \left(\overline{S C}^{2}-\overline{M C}^{2}\right)-\left(\overline{X C}^{2}-\overline{M C}^{2}\right)=\overline{S C}^{2}-\overline{X C}^{2} .
\end{aligned}
$$

The intersecting chord theorem now follows directly from this lemma: as the distances $S C$ and $X C$ for two arbitrary chords intersecting at $X$ are equal, also the products of the chord segments are equal.

## 2 Intersecting Chord Theorem

theory Chord-Segments
imports Triangle.Triangle
begin

### 2.1 Preliminaries

```
lemma betweenE-if-dist-leq:
    fixes \(A B X\) :: 'a::euclidean-space
    assumes between \((A, B) X\)
    assumes dist \(A X \leq\) dist \(B X\)
    obtains \(u\) where \(1 / 2 \leq u u \leq 1\) and \(X=u *_{R} A+(1-u) *_{R} B\)
proof (cases \(A=B\) )
    assume \(A \neq B\)
    from 〈between \((A, B) X\rangle\) obtain \(u\) where \(u: u \geq 0 u \leq 1\) and \(X: X=u *_{R}\)
\(A+(1-u) *_{R} B\)
```

```
    by (metis add.commute betweenE between-commute)
    from X have }X=B+u\mp@subsup{*}{R}{}(A-B)\mathrm{ and }X=A+(u-1)\mp@subsup{*}{R}{}(A-B
    by (simp add: scaleR-diff-left real-vector.scale-right-diff-distrib)+
    from }\langleX=B+u\mp@subsup{*}{R}{}(A-B)\rangle\mathrm{ have dist-B: dist B X = norm (u**R}(A-B)
    by (auto simp add: dist-norm)
    from }\langleX=A+(u-1)\mp@subsup{*}{R}{}(A-B)\rangle\mathrm{ have dist-A: dist A X = norm ((u-1)
*R
    by (auto simp add: dist-norm)
    from }\langleA\not=B\rangle\mathrm{ have norm (A-B)>0 by auto
    from this <dist A X\leq dist B X> have u\geq1/ 2
        using dist- }A\mathrm{ dist- }B\mathrm{ by simp
    from this <u\leq1\rangle}X\mathrm{ that show thesis by blast
next
    assume }A=
    define u :: real where }u=
```



```
u) }\mp@subsup{*}{R}{}
    unfolding u-def by auto
    with that show thesis by blast
qed
lemma dist-geq-iff-midpoint-in-between:
    fixes A B X :: 'a::euclidean-space
    assumes between (A,B) X
    shows dist A X dist B X \longleftrightarrow between ( }X,B\mathrm{ ) (midpoint A B)
proof
    assume dist A X dist B X
    from <between (A,B) X〉 this obtain u
        where u:1/2\lequu\leq1 and X:X=u*R}A+(1-u)*\mp@subsup{*}{R}{}
        using betweenE-if-dist-leq by blast
    have M: midpoint }AB=(1/2)**RA+(1/2) * *R B
        unfolding midpoint-def by (simp add: scaleR-add-right)
    from <1/2 <u> have 1: midpoint A B = (1/(2*u))** X + (1-(1/(2
* u))) *R B
    proof -
        have (2-u*2) / (2*u)=1/u-u/u
            using u(1) by (simp add: diff-divide-distrib)
        also have ... = 1/u-1 using u(1) by auto
        finally have (2 - u* 2) / (2*u) = 1/u-1.
        from <1 / 2 \lequ\rangle this show ?thesis
            using X M by (simp add: scaleR-add-right scaleR-add-left[symmetric])
    qed
    moreover from u have 2: (1/(2*u))\geq0(1/(2*u))\leq1 by auto
    ultimately show between ( }X,B\mathrm{ ) (midpoint A B)
        using betweenI [of concl: B X] by (metis add.commute between-commute)
next
    assume between ( }X,B\mathrm{ ) (midpoint A B)
    then have between ( }A\mathrm{ , midpoint A B) X
        using <between (A, B) X> between-midpoint(1) between-swap by blast
```

```
    then have dist A X \leq dist A (midpoint A B)
    using between zero-le-dist by force
    also have dist A (midpoint A B)\leqdist B(midpoint A B)
    by (simp add: dist-midpoint)
    also from <between (X,B) (midpoint A B)> have dist B (midpoint A B)\leqdist
B X
    using between zero-le-dist by (metis add.commute dist-commute le-add-same-cancel1)
    finally show dist A X dist B X .
qed
```


## 2．2 Properties of Chord Segments

lemma chord－property：
fixes $S C$ ：：＇$a$ ：：euclidean－space
assumes dist $C S=\operatorname{dist} C T$
assumes between $(S, T) X$
shows dist $S X *$ dist $X T=($ dist $C S)$ へ2 $-($ dist $C X)$ へ2
proof－
define $M$ where $M=$ midpoint $S T$
have between $(S, T) M$
unfolding $M$－def by（simp add：between－midpoint（1））
have dist $T M=\operatorname{dist} S M$
unfolding $M$－def by（simp add：dist－midpoint）
have distances： $\max ($ dist $S X)($ dist $X T)=($ dist $S M)+($ dist $X M) \wedge$
$\min (\operatorname{dist} S X)($ dist $X T)=(\operatorname{dist} S M)-(\operatorname{dist} X M)$
proof cases
assume dist $S X \leq \operatorname{dist} X T$
then have between $(X, T) M$ using 〈between $(S, T) X>M$－def by（simp add：dist－geq－iff－midpoint－in－between dist－commute）
then have between $(S, M) X$
using 〈between $(S, T) X\rangle\langle$ between $(S, T) M\rangle$ between－swap by blast
from 〈between $(X, T) M\rangle$ have dist $X T=\operatorname{dist} X M+\operatorname{dist} M T$
using between by auto
moreover from 〈between $(S, M) X$ have dist $S X=$ dist $S M-$ dist $M X$ using between dist－commute by force
ultimately show ？thesis
using 〈dist $S X \leq$ dist $X T\rangle\langle$ dist $T M=$ dist $S M\rangle$
by（simp add：add．commute dist－commute max－def min－def）
next
assume $\neg($ dist $S X \leq \operatorname{dist} X T)$
then have dist $T X \leq$ dist $S X$ by（simp add：dist－commute）
then have between $(S, X) M$
using＜between $(S, T) X>M$－def
by（simp add：dist－geq－iff－midpoint－in－between midpoint－sym between－commute）
then have between $(T, M) X$
using 〈between $(S, T) X\rangle\langle$ between $(S, T) M\rangle$ between－swap between－commute
by metis
from 〈between $(S, X) M\rangle$ have dist $S X=\operatorname{dist} S M+\operatorname{dist} M X$ using between by auto
moreover from＜between $(T, M) X\rangle$ have dist $T X=$ dist $T M-d i s t M X$
using between dist－commute by force
ultimately show ？thesis
using $\langle\neg$ dist $S X \leq \operatorname{dist} X T\rangle\langle$ dist $T M=\operatorname{dist} S M\rangle$
by（metis dist－commute max－def min－def）
qed
have orthogonal $(C-M)(S-M)$
using 〈dist $C S=\operatorname{dist} C T\rangle M$－def
by（auto simp add：isosceles－triangle－orthogonal－on－midpoint）
have orthogonal $(C-M)(X-M)$
proof－
have between $(S, T) M$
using $M$－def between－midpoint（1）by blast
obtain $c$ where $(X-M)=c *_{R}(S-M)$
proof（cases $S=M$ ）
assume $S \neq M$
then obtain $c$ where $(X-M)=c *_{R}(S-M)$
using between－implies－scaled－diff $[O F\langle$ between $(S, T) X\rangle\langle$ between $(S, T)$
$M>$ ］by metis
from this that show thesis by blast
next
assume $S=M$
from this $\langle$ between $(S, T) X\rangle$ have $X=M$
by（simp add：midpoint－between $M$－def）
from $\langle X=M\rangle\langle S=M\rangle$ have $(X-M)=0 *_{R}(S-M)$ by simp
from this that show thesis by blast
qed
from this＜orthogonal $(C-M)(S-M)$ 〉show ？thesis
by（auto intro：orthogonal－clauses（2））
qed
from 〈orthogonal $(C-M)(S-M)\rangle\langle o r t h o g o n a l(C-M)(X-M)\rangle$ have
$(\text { dist } S M)^{\wedge} 2+(\text { dist } M C)^{\wedge} 2=(\text { dist } C S)^{\wedge} 2$
$(\text { dist } X M)^{\wedge} 2+(\operatorname{dist} M C) \wedge^{\wedge} 2=($ dist $C X) \wedge^{\wedge}$
by（auto simp only：Pythagoras）
then have geometric－observation：
$(\text { dist } S M)^{\wedge} 2=(\text { dist } C S)^{\wedge 2}-($ dist $M C) \wedge^{\wedge} 2$
$(\text { dist } X M)^{\wedge} 2=(\text { dist } C X)^{\wedge} 2-(\operatorname{dist} M C)^{\wedge} 2$
by auto
have dist $S X * \operatorname{dist} X T=\max (\operatorname{dist} S X)(\operatorname{dist} X T) * \min (\operatorname{dist} S X)(d i s t X$
T）
by（auto split：split－max）
also have $\ldots=(($ dist $S M)+(\operatorname{dist} X M)) *(($ dist $S M)-($ dist $X M))$
using distances by simp
also have $\ldots=($ dist $\left.S M){ }^{\text {－} 2-(d i s t ~} X M\right)$ へ2
by（simp add：field－simps power2－eq－square）

```
        also have ... = ((dist CS)^2 - (distMC)^2) - ((dist CX)^2 - (dist
MC)^2)
            using geometric-observation by simp
    also have ... = (dist CS)^2 - (dist CX)^2 by simp
    finally show ?thesis.
qed
theorem product-of-chord-segments:
    fixes S S T T }\mp@subsup{\mp@code{S}}{2}{}\mp@subsup{T}{2}{}XC\mathrm{ :: 'a :: euclidean-space
    assumes between (S , T1 ) X between ( }\mp@subsup{S}{2}{},\mp@subsup{T}{2}{})
    assumes dist C S S =r dist C T T = r
    assumes dist C S S =r dist C T T 
    shows dist S}\mp@subsup{S}{1}{}X*\mathrm{ dist X T}\mp@subsup{T}{1}{}=\mathrm{ dist }\mp@subsup{S}{2}{}X*\mathrm{ dist X T
proof -
    from \dist C S S = r\rangle\langledist C T T = r\rangle\langlebetween ( }\mp@subsup{S}{1}{},\mp@subsup{T}{1}{})X
    have dist S}\mp@subsup{S}{1}{}X*\mathrm{ dist X T T = r^2 - (dist CX) ^2
            by (subst chord-property) auto
    also from <dist C S S = r\rangle\langledist C T T 
    have ... = dist S}\mp@subsup{S}{2}{}X*\mathrm{ dist X T T 
        by (subst chord-property) auto
    finally show ?thesis.
qed
end
```


## References

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