Ceva's Theorem

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Abstract

This entry contains a definition of the area the triangle constructed by three points. Building on this, some basic geometric properties about the area of a triangle are derived. These properties are used to prove Ceva's theorem.

Contents

theory *Ceva*
imports
Triangle.Triangle
begin

definition *Triangle-area* :: *'a*::*real-inner* ⇒ *'a* ⇒ *'a* ⇒ *real*
where *Triangle-area x y z* = *abs*(*sin* (*angle x y z*)) ∗ *dist x y* ∗ *dist y z*

lemma *Triangle-area-per1* : *Triangle-area a b c* = *Triangle-area b c a*
⟨*proof*⟩

lemma *Triangle-area-per2* : *Triangle-area a b c* = *Triangle-area b a c*
⟨*proof*⟩

lemma *collinear-angle*:
fixes *a b c* :: *'a*::*euclidean-space*
shows *collinear* {*a*, *b*, *c*} ⟹ *a* ≠ *b* ⟹ *b* ≠ *c* ⟹ *angle a b c* ∈ {*0*, *pi*}
⟨*proof*⟩

lemma *Triangle-area-0* :
fixes *c* :: *'a*::*euclidean-space*
shows *Triangle-area a b c* = *0* ⟷ *collinear* {*a*,*b*,*c*}
⟨*proof*⟩

lemma *Angle-longer-side* :
fixes *a* :: *'a* :: *euclidean-space*

assumes *Col* : *between* (*b*,*d*) *c*
assumes *NeqBC* : $b \neq c$
shows *angle a b c* = *angle a b d*
⟨*proof*⟩

lemma *Triangle-area-comb* :
fixes *c* :: *'a*::*euclidean-space*
assumes *Col* : *between* (*b*,*c*) *m*
shows *Triangle-area a b m* + *Triangle-area a c m* = *Triangle-area a b c*
⟨*proof*⟩

lemma *Triangle-area-cal* :
fixes *a* :: *'a*::*euclidean-space*
assumes *Col* : *collinear* {*a*,*m*,*b*}
shows ∃ *k*. *dist a m* ∗ *k* = *Triangle-area a c m* ∧ *dist b m* ∗ *k* = *Triangle-area b c m*
⟨*proof*⟩

lemma *Triangle-area-comb-alt* :
fixes *a* :: *'a*::*euclidean-space*
assumes *Col1* : *collinear* {*a*,*m*,*b*}
assumes *Col2* : *collinear* {*c*,*k*,*m*}
shows *Goal* : ∃ *h*. *dist a m* ∗ *h* = *Triangle-area a c k* ∧ *dist b m* ∗ *h* = *Triangle-area b c k*
⟨*proof*⟩

lemma *Cevas* :
fixes *a* :: *'a*::*euclidean-space*
assumes *MidCol* : *collinear* {*a*,*k*,*d*} ∧ *collinear* {*b*,*k*,*e*} ∧ *collinear* {*c*,*k*,*f*}
assumes *TriCol* : *collinear* {*a*,*f*,*b*} ∧ *collinear* {*a*,*e*,*c*} ∧ *collinear* {*b*,*d*,*c*}
assumes *Triangle* : ¬ *collinear* {*a*,*b*,*c*}
shows *dist a f* ∗ *dist b d* ∗ *dist c e* = *dist f b* ∗ *dist d c* ∗ *dist e a*
⟨*proof*⟩

end