Ceva's Theorem

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Abstract

This entry contains a definition of the area the triangle constructed by three points. Building on this, some basic geometric properties about the area of a triangle are derived. These properties are used to prove Ceva's theorem.

Contents

theory *Ceva*
imports
  Triangle.Triangle
begin

definition *Triangle-area* :: $'a$::*real-inner* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ $\Rightarrow$ *real*
  where *Triangle-area x y z* = *abs*(*sin* (*angle x y z*)) $*$ *dist x y* $*$ *dist y z*

lemma *Triangle-area-per1* : *Triangle-area a b c* = *Triangle-area b c a*
⟨*proof*⟩

lemma *Triangle-area-per2* : *Triangle-area a b c* = *Triangle-area b a c*
⟨*proof*⟩

lemma *collinear-angle*:
  fixes *a b c* :: $'a$::*euclidean-space*
  shows *collinear* $\{a, b, c\} \Longrightarrow a \neq b \Longrightarrow b \neq c \Longrightarrow$ *angle a b c* $\in \{0, pi\}$
⟨*proof*⟩

lemma *Triangle-area-0* :
  fixes *c* :: $'a$::*euclidean-space*
  shows *Triangle-area a b c* = *0* $\longleftrightarrow$ *collinear* $\{a,b,c\}$
⟨*proof*⟩

lemma *Angle-longer-side* :
  fixes *a* :: $'a$ :: *euclidean-space*

assumes *Col* : *between* (b,d) *c*
assumes *NeqBC* : $b \neq c$
shows *angle a b c* = *angle a b d*
⟨*proof*⟩

lemma *Triangle-area-comb* :
fixes *c* :: *'a::euclidean-space*
assumes *Col* : *between* (b,c) *m*
shows *Triangle-area a b m* + *Triangle-area a c m* = *Triangle-area a b c*
⟨*proof*⟩

lemma *Triangle-area-cal* :
fixes *a* :: *'a::euclidean-space*
assumes *Col* : *collinear* $\{a,m,b\}$
shows ∃ *k*. *dist a m* ∗ *k* = *Triangle-area a c m* ∧ *dist b m* ∗ *k* = *Triangle-area b c m*
⟨*proof*⟩

lemma *Triangle-area-comb-alt* :
fixes *a* :: *'a::euclidean-space*
assumes *Col1* : *collinear* $\{a,m,b\}$
assumes *Col2* : *collinear* $\{c,k,m\}$
shows *Goal* : ∃ *h*. *dist a m* ∗ *h* = *Triangle-area a c k* ∧ *dist b m* ∗ *h* = *Triangle-area b c k*
⟨*proof*⟩

lemma *Cevas* :
fixes *a* :: *'a::euclidean-space*
assumes *MidCol* : *collinear* $\{a,k,d\}$ ∧ *collinear* $\{b,k,e\}$ ∧ *collinear* $\{c,k,f\}$
assumes *TriCol* : *collinear* $\{a,f,b\}$ ∧ *collinear* $\{a,e,c\}$ ∧ *collinear* $\{b,d,c\}$
assumes *Triangle* : ¬ *collinear* $\{a,b,c\}$
shows *dist a f* ∗ *dist b d* ∗ *dist c e* = *dist f b* ∗ *dist d c* ∗ *dist e a*
⟨*proof*⟩

end