Cauchy’s Mean Theorem and the Cauchy-Schwarz Inequality

Benjamin Porter

April 19, 2020
## Contents

1 Cauchy’s Mean Theorem ............................. 3
   1.1 Abstract ......................................... 3
   1.2 Formal proof ....................................... 4
      1.2.1 Collection sum and product ................. 4
      1.2.2 Auxiliary lemma .............................. 5
      1.2.3 Mean and GMean ............................. 5
      1.2.4 list-neq, list-eq .............................. 7
      1.2.5 Element selection ............................ 8
      1.2.6 Abstract properties .......................... 8
      1.2.7 Existence of a new collection .............. 10
      1.2.8 Cauchy’s Mean Theorem ...................... 10

2 The Cauchy-Schwarz Inequality ................. 12
   2.1 Abstract ......................................... 12
   2.2 Formal Proof .................................... 12
      2.2.1 Vector, Dot and Norm definitions. .......... 12
Abstract

This document presents the mechanised proofs of two popular theorems attributed to Augustin Louis Cauchy - Cauchy’s Mean Theorem and the Cauchy-Schwarz Inequality.
Chapter 1

Cauchy’s Mean Theorem

theory CauchysMeanTheorem
imports Complex/Main
begin

1.1 Abstract

The following document presents a proof of Cauchy’s Mean theorem formalised in the Isabelle/Isar theorem proving system.

Theorem: For any collection of positive real numbers the geometric mean is always less than or equal to the arithmetic mean. In mathematical terms:

\[ \sqrt[n]{x_1 x_2 \ldots x_n} \leq \frac{x_1 + \ldots + x_n}{n} \]

We will use the term mean to denote the arithmetic mean and gmean to denote the geometric mean.

Informal Proof:

This proof is based on the proof presented in [1]. First we need an auxiliary lemma (the proof of which is presented formally below) that states:

Given two pairs of numbers of equal sum, the pair with the greater product is the pair with the least difference. Using this lemma we now present the proof -

Given any collection \( C \) of positive numbers with mean \( M \) and product \( P \) and with some element not equal to \( M \) we can choose two elements from the collection, \( a \) and \( b \) where \( a > M \) and \( b < M \). Remove these elements from the collection and replace them with two new elements, \( a' \) and \( b' \) such that \( a' = M \) and \( a' + b' = a + b \). This new collection \( C' \) now has a greater product \( P' \) but equal mean with respect to \( C \). We can continue in this fashion until we have a collection \( C_n \) such that \( P_n > P \) and \( M_n = M \), but \( C_n \) has all its elements equal to \( M \) and thus \( P_n = M^n \). Using the definition of geometric and arithmetic means above we can see that for any collection of positive
CHAPTER 1. CAUCHY’S MEAN THEOREM

elements \( E \) it is always true that \( \text{gmean} E \leq \text{mean} E \). QED.


1.2 Formal proof

1.2.1 Collection sum and product

The finite collections of numbers will be modelled as lists. We then define
sum and product operations over these lists.

Sum and product definitions

\textbf{notation} (input) \( \text{sum-list} \left( \sum: \left[999 \right] 998 \right) \)

\textbf{notation} (input) \( \text{prod-list} \left( \prod: \left[999 \right] 998 \right) \)

Properties of sum and product

We now present some useful properties of sum and product over collections.

These lemmas just state that if all the elements in a collection \( C \) are less
(greater than) than some value \( m \), then the sum will less than (greater than)
\( m \ast \text{length}(C) \).

\textbf{lemma sum-list-mono-lt} [rule-format]:
\textbf{fixes} \( xs::\text{real list} \)
\textbf{shows} \( xs \neq [] \land (\forall x \in \text{set} \; xs. \; x < m) \)
\quad \rightarrow (\left( \sum : xs \right) < (m \ast (\text{real} \; (\text{length} \; xs)))) \)
\textbf{⟨proof⟩}

\textbf{lemma sum-list-mono-gt} [rule-format]:
\textbf{fixes} \( xs::\text{real list} \)
\textbf{shows} \( xs \neq [] \land (\forall x \in \text{set} \; xs. \; x > m) \)
\quad \rightarrow (\left( \sum : xs \right) > (m \ast (\text{real} \; (\text{length} \; xs)))) \)
\textbf{⟨proof⟩}

\textbf{proof omitted}

If \( a \) is in \( C \) then the sum of the collection \( D \) where \( D \) is \( C \) with \( a \) removed
is the sum of \( C \) minus \( a \).

\textbf{lemma sum-list-rmv1}:
\textbf{a} \in \text{set} \; xs \; \Rightarrow \; \sum : (\text{remove} \; 1 \; a \; xs) = \sum : xs - (a :: 'a :: \text{ab-group-add}) \)
\textbf{⟨proof⟩}

A handy addition and division distribution law over collection sums.

\textbf{lemma list-sum-distrib-aux}:
shows \( \sum :xs / (n :: a :: \text{archimedean-field}) + \sum :xs = (1 + (1/n)) * \sum :xs \)

\( \langle \text{proof} \rangle \)

**Lemma** remove1-retains-prod:

fixes \( a \) and \( xs ::'a :: \text{comm-ring-1 list} \)

shows \( a : \text{set xs} \rightarrow \Pi :xs = \Pi : (\text{remove1} \ a \ xs) * a \)

\( \langle \text{proof} \rangle \)

The final lemma of this section states that if all elements are positive and non-zero then the product of these elements is also positive and non-zero.

**Lemma** el-gt0-imp-prod-gt0 [rule-format]:

fixes \( xs ::'a :: \text{archimedean-field list} \)

shows \( \forall y : \text{set xs} \rightarrow y > 0 \Rightarrow \Pi :xs > 0 \)

\( \langle \text{proof} \rangle \)

### 1.2.2 Auxiliary Lemma

This section presents a proof of the auxiliary lemma required for this theorem.

**Lemma** prod-exp:

fixes \( x :: \text{real} \)

shows \( 4 * (x+y) = (x+y)^2 - (x-y)^2 \)

\( \langle \text{proof} \rangle \)

**Lemma** abs-less-imp-sq-less [rule-format]:

fixes \( x :: \text{real} \) and \( y :: \text{real} \) and \( z :: \text{real} \) and \( w :: \text{real} \)

assumes diff: \( \text{abs} (x-y) < \text{abs} (z-w) \)

shows \( (x-y)^2 < (z-w)^2 \)

\( \langle \text{proof} \rangle \)

The required lemma (phrased slightly differently than in the informal proof.)

Here we show that for any two pairs of numbers with equal sums the pair with the least difference has the greater product.

**Lemma** le-diff-imp-gt-prod [rule-format]:

fixes \( x :: \text{real} \) and \( y :: \text{real} \) and \( z :: \text{real} \) and \( w :: \text{real} \)

assumes diff: \( \text{abs} (x-y) < \text{abs} (z-w) \) and sum: \( x+y = z+w \)

shows \( x*y > z*w \)

\( \langle \text{proof} \rangle \)

### 1.2.3 Mean and GMean

Now we introduce definitions and properties of arithmetic and geometric means over collections of real numbers.

**Definitions**

*Arithmetic mean*
CHAPTER 1. CAUCHY’S MEAN THEOREM

**definition**

\[\text{mean} :: (\text{real list}) \Rightarrow \text{real}\]

\[\text{mean } s = (\sum : s / \text{real } (\text{length } s))\]

**Geometric mean**

**definition**

\[\text{gmean} :: (\text{real list}) \Rightarrow \text{real}\]

\[\text{gmean } s = \text{root } (\text{length } s) (\prod : s)\]

**Properties**

Here we present some trivial properties of \textit{mean} and \textit{gmean}.

**lemma** list-sum-mean:

**fixes** \(xs::\text{real list}\)

**shows** \(\sum : xs = ((\text{mean } xs) * (\text{real } (\text{length } xs)))\)

**⟨proof⟩**

**lemma** list-mean-eq-iff:

**fixes** \(\text{one}::\text{real list} \text{ and } \text{two}::\text{real list}\)

**assumes**

\(se: (\sum : \text{one} = \sum : \text{two})\) and

\(le: (\text{length } \text{one} = \text{length } \text{two})\)

**shows** \((\text{mean one} = \text{mean two})\)

**⟨proof⟩**

**lemma** list-gmean-gt-iff:

**fixes** \(\text{one}::\text{real list} \text{ and } \text{two}::\text{real list}\)

**assumes**

\(gz1: \prod : \text{one} > 0\) and \(gz2: \prod : \text{two} > 0\) and

\(nc1: \text{one} \neq []\) and \(nc2: \text{two} \neq []\) and

\(pe: (\prod : \text{one} > \prod : \text{two})\) and

\(le: (\text{length } \text{one} = \text{length } \text{two})\)

**shows** \((\text{gmean one} > \text{gmean two})\)

**⟨proof⟩**

This slightly more complicated lemma shows that for every non-empty collection with mean \(M\), adding another element \(a\) where \(a = M\) results in a new list with the same mean \(M\).

**lemma** list-mean-cons [rule-format]:

**fixes** \(xs::\text{real list}\)

**shows** \(xs \neq [] \rightarrow \text{mean } ((\text{mean } xs) \# xs) = \text{mean } xs\)

**⟨proof⟩**

For a non-empty collection with positive mean, if we add a positive number to the collection then the mean remains positive.

**lemma** mean-gt-0 [rule-format]:

\(xs \neq [] \land 0 < x \land 0 < (\text{mean } xs) \rightarrow 0 < (\text{mean } (x \# xs))\)

**⟨proof⟩**
1.2.4 list-neq, list-eq

This section presents a useful formalisation of the act of removing all the elements from a collection that are equal (not equal) to a particular value. We use this to extract all the non-mean elements from a collection as is required by the proof.

Definitions

list-neq and list-eq just extract elements from a collection that are not equal (or equal) to some value.

abbreviation
list-neq :: ('a list) ⇒ 'a ⇒ ('a list) where
list-neq xs el == filter (λx. x≠el) xs

abbreviation
list-eq :: ('a list) ⇒ 'a ⇒ ('a list) where
list-eq xs el == filter (λx. x=el) xs

Properties

This lemma just proves a required fact about list-neq, remove1 and length.

lemma list-neq-remove1 [rule-format]:
shows a≠m ∧ a : set xs
→ length (list-neq (remove1 a xs) m) < length (list-neq xs m)
⟨proof⟩

We now prove some facts about list-eq, list-neq, length, sum and product.

lemma list-eq-sum [simp]:
fixes xs::real list
shows Σ : (list-eq xs m) = (m * (real (length (list-eq xs m))))
⟨proof⟩

lemma list-eq-prod [simp]:
fixes xs::real list
shows Π: (list-eq xs m) = (m ^ (length (list-eq xs m)))
⟨proof⟩

lemma sum-list-split:
fixes xs::real list
shows Σ : xs = (Σ : (list-neq xs m) + Σ : (list-eq xs m))
⟨proof⟩

lemma prod-list-split:
fixes xs::real list
shows Π : xs = (Π : (list-neq xs m) * Π : (list-eq xs m))
CHAPTER 1. CAUCHY’S MEAN THEOREM

proof

lemma sum-list-length-split:
  fixes xs::real list
  shows length xs = length (list-neq xs m) + length (list-eq xs m)
  ⟨proof⟩

1.2.5 Element selection

We now show that given after extracting all the elements not equal to the mean there exists one that is greater then (or less than) the mean.

lemma pick-one-gt:
  fixes xs::real list and m::real
  defines m: m ≡ (mean xs) and neq: noteq ≡ list-neq xs m
  assumes asum: neq̸= []
  shows ∃ e. e : set neq ∧ e > m
  ⟨proof⟩

lemma pick-one-lt:
  fixes xs::real list and m::real
  defines m: m ≡ (mean xs) and neq: noteq ≡ list-neq xs m
  assumes asum: neq̸= []
  shows ∃ e. e : set neq ∧ e < m
  ⟨proof⟩

1.2.6 Abstract properties

In order to maintain some comprehension of the following proofs we now introduce some properties of collections.

Definitions

het: The heterogeneity of a collection is the number of elements not equal to its mean. A heterogeneity of zero implies the all the elements in the collection are the same (i.e. homogeneous).

definition
  het :: real list ⇒ nat where
  het l = length (list-neq l (mean l))

lemma het-gt-0-imp-noteq-ne: het l > 0 ⇒ list-neq l (mean l) ≠ []
  ⟨proof⟩

lemma het-gt-0I: assumes a: a ∈ set xs and b: b ∈ set xs and neq: a ≠ b
  shows het xs > 0
  ⟨proof⟩

γ−eq: Two lists are γ-equivalent if and only if they both have the same number of elements and the same arithmetic means.
CHAPTER 1. CAUCHY’S MEAN THEOREM

**definition**

\( \gamma\text{-eq} :: ((\text{real list}) \ast (\text{real list})) \Rightarrow \text{bool} \)

where

\( \gamma\text{-eq} a \leftrightarrow \text{mean (fst a)} = \text{mean (snd a)} \land \text{length (fst a)} = \text{length (snd a)} \)

\( \gamma\text{-eq} \) is transitive and symmetric.

**lemma** \( \gamma\text{-eq-sym} \): \( \gamma\text{-eq} (a, b) = \gamma\text{-eq} (b, a) \)

\( \langle \text{proof} \rangle \)

**lemma** \( \gamma\text{-eq-trans} \):

\( \gamma\text{-eq} (x, y) \Longrightarrow \gamma\text{-eq} (y, z) \Longrightarrow \gamma\text{-eq} (x, z) \)

\( \langle \text{proof} \rangle \)

**pos**: A list is positive if all its elements are greater than 0.

**definition**

\( \text{pos} :: \text{real list} \Rightarrow \text{bool} \)

where

\( \text{pos} l \leftrightarrow (\text{if } l = \text{[]} \text{ then False else } \forall e : e \in l \rightarrow e > 0) \)

**lemma** \( \text{pos-empty [simp]} \): \( \text{pos} [] = \text{False} \) \( \langle \text{proof} \rangle \)

**lemma** \( \text{pos-single [simp]} \): \( \text{pos} [x] = (x > 0) \) \( \langle \text{proof} \rangle \)

**lemma** \( \text{pos-imp-ne}: \text{pos} \; x \rightarrow x \neq [] \) \( \langle \text{proof} \rangle \)

**lemma** \( \text{pos-cons [simp]}:

\( x \neq [] \rightarrow \text{pos} (x \# xs) =

\( (\text{if } (x > 0) \text{ then } \text{pos} \; xs \text{ else False}) \)

\( (\text{is } ?F \; x \; xs \; \text{is } ?A \; xs \rightarrow \text{is } ?S \; x \; xs) \) \( \langle \text{proof} \rangle \)

**Properties**

Here we prove some non-trivial properties of the abstract properties.

Two lemmas regarding \( \text{pos} \). The first states the removing an element from a positive collection (of more than 1 element) results in a positive collection. The second asserts that the mean of a positive collection is positive.

**lemma** \( \text{pos-imp-rmv-pos} \):

\( \text{assumes} (\text{remove1 a xs) \neq [] pos xs shows pos (remove1 a xs) \langle \text{proof} \rangle} \)

**lemma** \( \text{pos-mean} \):

\( \text{pos xs} \Rightarrow \text{mean xs} > 0 \) \( \langle \text{proof} \rangle \)

We now show that homogeneity of a non-empty collection \( x \) implies that its product is equal to \((\text{mean } x)^\ast (\text{length } x)\).

**lemma** \( \text{prod-list-het0} \):

\( \text{shows} x \neq [] \land \text{het } x = 0 \Rightarrow \prod : x = (\text{mean } x) ^ \ast (\text{length } x) \) \( \langle \text{proof} \rangle \)

Furthermore we present an important result - that a homogeneous collection has equal geometric and arithmetic means.
CHAPTER 1. CAUCHY’S MEAN THEOREM

lemma het-base:
shows pos x ∧ het x = 0 ⇒ gmean x = mean x
⟨proof⟩

1.2.7 Existence of a new collection

We now present the largest and most important proof in this document. Given any positive and non-homogeneous collection of real numbers there exists a new collection that is γ-equivalent, positive, has a strictly lower heterogeneity and a greater geometric mean.

lemma new-list-gt-gmean:
fixes xs :: real list and m :: real
and neq and eq
defines
   m: m ≡ mean xs
   neq:noteq ≡ list-neq xs m
   eq: eq ≡ list-eq xs m
assumes pos-xs: pos xs and het-gt-0: het xs > 0
shows
   ∃xs', gmean xs' > gmean xs ∧ γ-eq (xs',xs) ∧
   het xs' < het xs ∧ pos xs'
⟨proof⟩

Furthermore we show that for all non-homogeneous positive collections there exists another collection that is γ-equivalent, positive, has a greater geometric mean and is homogeneous.

lemma existence-of-het0 [rule-format]:
shows ∀x. p = het x ∧ p > 0 ∧ pos x →
   (∃y. gmean y > gmean x ∧ γ-eq (x,y) ∧ het y = 0 ∧ pos y)
⟨is-Q p is ∀x. (?A x p → ?S x)⟩
⟨proof⟩

1.2.8 Cauchy’s Mean Theorem

We now present the final proof of the theorem. For any positive collection we show that its geometric mean is less than or equal to its arithmetic mean.

theorem CauchysMeanTheorem:
fixes z::real list
assumes pos z
shows gmean z ≤ mean z
⟨proof⟩

In the equality version we prove that the geometric mean is identical to the arithmetic mean iff the collection is homogeneous.

theorem CauchysMeanTheorem-Eq:
fixes z::real list
assumes pos z
shows $gmean z = mean z \iff het z = 0$
⟨proof⟩
end

**corollary** Cauchy’sMeanTheorem-Less:

fixes $z :: \text{real list}$

assumes pos $z$ and $het z > 0$

shows $gmean z < mean z$
⟨proof⟩
end
Chapter 2

The Cauchy-Schwarz Inequality

theory CauchySchwarz
imports Complex-Main
begin
⟨proof⟩

2.1 Abstract

The following document presents a formalised proof of the Cauchy-Schwarz Inequality for the specific case of $\mathbb{R}^n$. The system used is Isabelle/Isar.

Theorem: Take $V$ to be some vector space possessing a norm and inner product, then for all $a, b \in V$ the following inequality holds: $|a \cdot b| \leq \|a\| \|b\|$. Specifically, in the Real case, the norm is the Euclidean length and the inner product is the standard dot product.

2.2 Formal Proof

2.2.1 Vector, Dot and Norm definitions.

This section presents definitions for a real vector type, a dot product function and a norm function.

Vector

We now define a vector type to be a tuple of (function, length). Where the function is of type $\text{nat} \Rightarrow \text{real}$. We also define some accessor functions and appropriate notation.

type-synonym vector = (nat⇒)nat
definition
$\text{ith} :: \text{vector} \Rightarrow \text{nat} \Rightarrow \text{real} \ ((\text{-} \text{-}) \ [80,100] \ 100) \ where$

$\text{ith} \ v \ i = \text{fst} \ v \ i$

definition
$\text{vlen} :: \text{vector} \Rightarrow \text{nat} \ where$

$vlen \ v = \text{snd} \ v$

Now to access the second element of some vector $v$ the syntax is $v_2$.

Dot and Norm

We now define the dot product and norm operations.

definition
dot :: \text{vector} \Rightarrow \text{vector} \Rightarrow \text{real} \ (\text{infixr} \cdot \ 60) \ where
dot a \ b = \left( \sum_{j \in \{1..(vlen \ a)\}} a_j \cdot b_j \right)$

definition
norm :: \text{vector} \Rightarrow \text{real} \ (\|\| \ 100) \ where

$\text{norm} \ v = \sqrt{\left( \sum_{j \in \{1..(vlen \ v)\}} v_j^2 \right)}$

Another definition of the norm is $\|v\| = \sqrt{v \cdot v}$. We show that our definition leads to this one.

lemma norm-dot:
$\|v\| = \sqrt{v \cdot v}$
⟨proof⟩

A further important property is that the norm is never negative.

lemma norm-pos:
$\|v\| \geq 0$
⟨proof⟩

We now prove an intermediary lemma regarding double summation.

lemma double-sum-aux:
fixes $f :: \text{nat} \Rightarrow \text{real}$

shows
$(\sum_{k \in \{1..n\}} \cdot (\sum_{j \in \{1..n\}} f_k \cdot g_j)) =
(\sum_{k \in \{1..n\}} \cdot (\sum_{j \in \{1..n\}} (f_k \cdot g_j + f_j \cdot g_k) / 2))$
⟨proof⟩

The final theorem can now be proven. It is a simple forward proof that uses properties of double summation and the preceding lemma.

theorem CauchySchwarzReal:
fixes $x :: \text{vector}$

assumes $\text{vlen} \ x = \text{vlen} \ y$

shows $|x \cdot y| \leq \|x\| \cdot \|y\|$
⟨proof⟩

end