Category Theory to Yoneda's Lemma

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This development proves Yoneda's lemma and aims to be readable by humans. It only defines what is needed for the lemma: categories, functors and natural transformations. Limits, adjunctions and other important concepts are not included.

There is no explanation or discussion in this document. See [O'K04] for this and a survey of category theory formalisations.

Contents

1 Categories

theory *Cat*
imports *HOL−Library.FuncSet*
begin

1.1 Definitions

record $('o, 'a)$ *category* =
  ob :: $'o$ *set* (*Ob*$_1$ *70*)
  ar :: $'a$ *set* (*Ar*$_1$ *70*)
  dom :: $'a \Rightarrow 'o$ (*Dom*$_1$ - [*81*] *70*)
  cod :: $'a \Rightarrow 'o$ (*Cod*$_1$ - [*81*] *70*)
  id :: $'o \Rightarrow 'a$ (*Id*$_1$ - [*81*] *80*)
  comp :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\cdot_1$ *60*)

definition
  hom :: [$('o,'a,'m)$ *category-scheme*, $'o$, $'o$] $\Rightarrow$ $'a$ *set*
    (*Hom*$_1$ - - [*81*,*81*] *80*) **where**
  hom CC A B = { *f*. *f*∈*ar CC* & *dom CC f* = *A* & *cod CC f* = *B* }

locale *category* =
  fixes *CC* (**structure**)
  assumes *dom-object* [*intro*]:
  $f \in Ar \Longrightarrow Dom\ f \in Ob$
  and *cod-object* [*intro*]:
  $f \in Ar \Longrightarrow Cod\ f \in Ob$
  and *id-left* [*simp*]:
  $f \in Ar \Longrightarrow Id\ (Cod\ f) \cdot f = f$
  and *id-right* [*simp*]:
  $f \in Ar \Longrightarrow f \cdot Id\ (Dom\ f) = f$
  and *id-hom* [*intro*]:
  $A \in Ob \Longrightarrow Id\ A \in Hom\ A\ A$
  and *comp-types* [*intro*]:
  $\bigwedge A\ B\ C.\ (comp\ CC) : (Hom\ B\ C) \to (Hom\ A\ B) \to (Hom\ A\ C)$
  and *comp-associative* [*simp*]:
  $f \in Ar \Longrightarrow g \in Ar \Longrightarrow h \in Ar$
  $\Longrightarrow Cod\ h = Dom\ g \Longrightarrow Cod\ g = Dom\ f$
  $\Longrightarrow f \cdot (g \cdot h) = (f \cdot g) \cdot h$

1.2 Lemmas

lemma (**in** *category*) *homI*:
  assumes $f \in Ar$ **and** $Dom\ f = A$ **and** $Cod\ f = B$
  shows $f \in Hom\ A\ B$
  ⟨*proof*⟩

lemma (**in** *category*) *homE*:
  assumes $A \in Ob$ **and** $B \in Ob$ **and** $f \in Hom\ A\ B$
  shows $Dom\ f = A$ **and** $Cod\ f = B$

⟨*proof*⟩

lemma (**in** *category*) *id-arrow* [*intro*]:
assumes $A \in Ob$
shows $Id\ A \in Ar$
⟨*proof*⟩

lemma (**in** *category*) *id-dom-cod*:
assumes $A \in Ob$
shows $Dom\ (Id\ A) = A$ **and** $Cod\ (Id\ A) = A$
⟨*proof*⟩

lemma (**in** *category*) *compI* [*intro*]:
assumes *f*: $f \in Ar$ **and** *g*: $g \in Ar$ **and** $Cod\ f = Dom\ g$
shows $g \cdot f \in Ar$
and $Dom\ (g \cdot f) = Dom\ f$
and $Cod\ (g \cdot f) = Cod\ g$
⟨*proof*⟩

end

2 Set is a Category

theory *SetCat*
imports *Cat*
begin

2.1 Definitions

record *'c set-arrow* =
set-dom :: *'c set*
set-func :: *'c* ⇒ *'c*
set-cod :: *'c set*

definition
set-arrow :: [*'c set*, *'c set-arrow*] ⇒ *bool* **where**
set-arrow U f ⟷ *set-dom f* ⊆ *U* & *set-cod f* ⊆ *U*
& (*set-func f*): (*set-dom f*) → (*set-cod f*)
& *set-func f* ∈ *extensional* (*set-dom f*)

definition
set-id :: [*'c set*, *'c set*] ⇒ *'c set-arrow* **where**
set-id U = (λ*s*∈*Pow U*. (|*set-dom*=*s*, *set-func*=λ*x*∈*s*. *x*, *set-cod*=*s*|))

definition
set-comp :: [*'c set-arrow*, *'c set-arrow*] ⇒ *'c set-arrow* (**infix** ⊙ *70*) **where**
set-comp g f =
(|

set-dom = *set-dom f*,
set-func = *compose* (*set-dom f*) (*set-func g*) (*set-func f*),
set-cod = *set-cod g*
⦇

definition
set-cat :: *'c set* ⇒ (*'c set*, *'c set-arrow*) *category* **where**
set-cat U =
⦇
ob = *Pow U*,
ar = {*f*. *set-arrow U f*},
dom = *set-dom*,
cod = *set-cod*,
id = *set-id U*,
comp = *set-comp*
⦈

2.2 Simple Rules and Lemmas

lemma *set-objectI* [*intro*]: *A* ⊆ *U* ⟹ *A* ∈ *ob* (*set-cat U*)
⟨*proof*⟩

lemma *set-objectE* [*intro*]: *A* ∈ *ob* (*set-cat U*) ⟹ *A* ⊆ *U*
⟨*proof*⟩

lemma *set-homI* [*intro*]:
assumes *A* ⊆ *U*
and *B* ⊆ *U*
and *f* : *A*→*B*
and *f* ∈ *extensional A*
shows ⦇*set-dom*=*A*, *set-func*=*f*, *set-cod*=*B*⦈ ∈ *hom* (*set-cat U*) *A B*
⟨*proof*⟩

lemma *set-dom* [*simp*]: *dom* (*set-cat U*) *f* = *set-dom f*
⟨*proof*⟩

lemma *set-cod* [*simp*]: *cod* (*set-cat U*) *f* = *set-cod f*
⟨*proof*⟩

lemma *set-id* [*simp*]: *id* (*set-cat U*) *A* = *set-id U A*
⟨*proof*⟩

lemma *set-comp* [*simp*]: *comp* (*set-cat U*) *g f* = *g* ⊙ *f*
⟨*proof*⟩

lemma *set-dom-cod-object-subset* [*intro*]:
assumes *f*: *f* ∈ *ar* (*set-cat U*)
shows *dom* (*set-cat U*) *f* ∈ *ob* (*set-cat U*)

and *cod* (*set-cat U*) *f* ∈ *ob* (*set-cat U*)
and *set-cod f* ⊆ *U*
and *set-dom f* ⊆ *U*
⟨*proof*⟩

In this context, $f \in hom\ A\ B$ is quite a strong claim.

lemma *set-homE* [*intro*]:
assumes *f*: $f \in hom\ (set\text{-}cat\ U)\ A\ B$
shows $A \subseteq U$
and $B \subseteq U$
and *set-dom f* = *A*
and *set-func f* : $A \to B$
and *set-cod f* = *B*
⟨*proof*⟩

2.3 Set is a Category

lemma *set-id-left*:
assumes *f*: $f \in ar\ (set\text{-}cat\ U)$
shows *set-id U* (*set-cod f*) ⊙ *f* = *f*
⟨*proof*⟩

lemma *set-id-right*:
assumes *f*: $f \in ar\ (set\text{-}cat\ U)$
shows *f* ⊙ (*set-id U* (*set-dom f*)) = *f*
⟨*proof*⟩

lemma *set-id-hom*:
assumes $A \in ob\ (set\text{-}cat\ U)$
shows $id\ (set\text{-}cat\ U)\ A \in hom\ (set\text{-}cat\ U)\ A\ A$
⟨*proof*⟩

lemma *set-comp-types*:
comp (*set-cat U*) ∈ *hom* (*set-cat U*) *B C* → *hom* (*set-cat U*) *A B* → *hom* (*set-cat U*) *A C*
⟨*proof*⟩

We reason explicitly about the function component of the composite arrow, leaving the rest to the simplifier.

lemma *set-comp-associative*:
fixes *f* **and** *g* **and** *h*
assumes *f*: $f \in ar\ (set\text{-}cat\ U)$
and *g*: $g \in ar\ (set\text{-}cat\ U)$
and *h*: $h \in ar\ (set\text{-}cat\ U)$
and *hg*: $cod\ (set\text{-}cat\ U)\ h = dom\ (set\text{-}cat\ U)\ g$
and *gf*: $cod\ (set\text{-}cat\ U)\ g = dom\ (set\text{-}cat\ U)\ f$
shows $comp\ (set\text{-}cat\ U)\ f\ (comp\ (set\text{-}cat\ U)\ g\ h) =$
$comp\ (set\text{-}cat\ U)\ (comp\ (set\text{-}cat\ U)\ f\ g)\ h$

⟨*proof*⟩

theorem *set-cat-cat*: *category* (*set-cat U*)
⟨*proof*⟩

end

3 Functors

theory *Functors*
imports *Cat*
begin

3.1 Definitions

record ('*o1*,'*a1*,'*o2*,'*a2*) *functor* =
  om :: '*o1* ⇒ '*o2*
  am :: '*a1* ⇒ '*a2*

abbreviation
  om-syn (- o [*81*]) **where**
  $F_{\mathrm{o}} \equiv om\ F$

abbreviation
  am-syn (- a [*81*]) **where**
  $F_{\mathrm{a}} \equiv am\ F$

locale *two-cats* = *AA?*: *category AA* + *BB?*: *category BB*
    for *AA* :: ('*o1*,'*a1*,'*m1*)*category-scheme* (**structure**)
    and *BB* :: ('*o2*,'*a2*,'*m2*)*category-scheme* (**structure**) +
  fixes *preserves-dom* :: ('*o1*,'*a1*,'*o2*,'*a2*)*functor* ⇒ *bool*
    and *preserves-cod* :: ('*o1*,'*a1*,'*o2*,'*a2*)*functor* ⇒ *bool*
    and *preserves-id* :: ('*o1*,'*a1*,'*o2*,'*a2*)*functor* ⇒ *bool*
    and *preserves-comp* :: ('*o1*,'*a1*,'*o2*,'*a2*)*functor* ⇒ *bool*
  defines *preserves-dom* $G \equiv \forall f{\in}Ar_{AA}.\ G_{\mathrm{o}}\ (Dom_{AA}\ f) = Dom_{BB}\ (G_{\mathrm{a}}\ f)$
    and *preserves-cod* $G \equiv \forall f{\in}Ar_{AA}.\ G_{\mathrm{o}}\ (Cod_{AA}\ f) = Cod_{BB}\ (G_{\mathrm{a}}\ f)$
    and *preserves-id* $G \equiv \forall A{\in}Ob_{AA}.\ G_{\mathrm{a}}\ (Id_{AA}\ A) = Id_{BB}\ (G_{\mathrm{o}}\ A)$
    and *preserves-comp* $G \equiv$
      $\forall f{\in}Ar_{AA}.\ \forall g{\in}Ar_{AA}.\ Cod_{AA}\ f = Dom_{AA}\ g \longrightarrow G_{\mathrm{a}}\ (g \cdot_{AA} f) = (G_{\mathrm{a}}\ g) \cdot_{BB} (G_{\mathrm{a}}\ f)$

locale *functor* = *two-cats* +
  fixes *F* (**structure**)
  assumes *F-preserves-arrows*: $F_{\mathrm{a}} : Ar_{AA} \to Ar_{BB}$
    and *F-preserves-objects*: $F_{\mathrm{o}} : Ob_{AA} \to Ob_{BB}$
    and *F-preserves-dom*: *preserves-dom F*
    and *F-preserves-cod*: *preserves-cod F*
    and *F-preserves-id*: *preserves-id F*

and *F-preserves-comp*: *preserves-comp F*
begin

lemmas *F-axioms* = *F-preserves-arrows F-preserves-objects F-preserves-dom F-preserves-cod F-preserves-id F-preserves-comp*

lemmas *func-pred-defs* = *preserves-dom-def preserves-cod-def preserves-id-def preserves-comp-def*

end

This gives us nicer notation for asserting that things are functors.

abbreviation
Functor (*Functor* - : - ⟶ - [*81*]) **where**
Functor F : *AA* ⟶ *BB* ≡ *functor AA BB F*

3.2 Simple Lemmas

For example:

lemma (**in** *functor*) *Functor F* : *AA* ⟶ *BB* ⟨*proof*⟩

lemma *functors-preserve-arrows* [*intro*]:
assumes *Functor F* : *AA* ⟶ *BB*
and *f* ∈ *ar AA*
shows F_{a} *f* ∈ *ar BB*
⟨*proof*⟩

lemma (**in** *functor*) *functors-preserve-homsets*:
assumes *1*: *A* ∈ Ob_{AA}
and *2*: *B* ∈ Ob_{AA}
and *3*: *f* ∈ Hom_{AA} *A B*
shows F_{a} *f* ∈ Hom_{BB} (F_{o} *A*) (F_{o} *B*)
⟨*proof*⟩

lemma *functors-preserve-objects* [*intro*]:
assumes *Functor F* : *AA* ⟶ *BB*
and *A* ∈ *ob AA*
shows F_{o} *A* ∈ *ob BB*
⟨*proof*⟩

3.3 Identity Functor

definition
id-func :: ('*o*,'*a*,'*m*) *category-scheme* ⇒ ('*o*,'*a*,'*o*,'*a*) *functor* **where**
id-func CC = ⦇*om*=(λ*A*∈*ob CC*. *A*), *am*=(λ*f*∈*ar CC*. *f*)⦈

locale *one-cat* = *two-cats* +
  assumes *endo*: $BB = AA$

lemma (**in** *one-cat*) *id-func-preserves-arrows*:
  shows $(id\text{-}func\ AA)_{\mathrm{a}} : Ar \to Ar$
  ⟨*proof*⟩

lemma (**in** *one-cat*) *id-func-preserves-objects*:
  shows $(id\text{-}func\ AA)_{\mathrm{o}} : Ob \to Ob$
  ⟨*proof*⟩

lemma (**in** *one-cat*) *id-func-preserves-dom*:
  shows *preserves-dom* (*id-func AA*)
⟨*proof*⟩

lemma (**in** *one-cat*) *id-func-preserves-cod*:
  preserves-cod (*id-func AA*)
⟨*proof*⟩

lemma (**in** *one-cat*) *id-func-preserves-id*:
  preserves-id (*id-func AA*)
⟨*proof*⟩

lemma (**in** *one-cat*) *id-func-preserves-comp*:
  preserves-comp (*id-func AA*)
⟨*proof*⟩

theorem (**in** *one-cat*) *id-func-functor*:
  Functor (*id-func AA*) : $AA \longrightarrow AA$
⟨*proof*⟩

end

4 HomFunctors

theory *HomFunctors*
imports *SetCat Functors*
begin

locale *into-set* = *two-cats AA BB*
    for *AA* :: $('o,'a,'m)$*category-scheme* (**structure**)
    and *BB* (**structure**) +
  fixes *U* **and** *Set*
  defines $U \equiv (UNIV::'a\ set)$
  defines $Set \equiv set\text{-}cat\ U$

assumes *BB-Set*: $BB = Set$
fixes *homf* $(Hom'(\text{-},'\text{-}'))$
defines $homf\ A \equiv (\!|$
$om = (\lambda B{\in}Ob.\ Hom\ A\ B)$,
$am = (\lambda f{\in}Ar.\ (\!|set\text{-}dom{=}Hom\ A\ (Dom\ f), set\text{-}func{=}(\lambda g{\in}Hom\ A\ (Dom\ f).\ f \cdot g), set\text{-}cod{=}Hom\ A\ (Cod\ f)|\!))$
$|\!)$

lemma (**in** *into-set*) *homf-preserves-arrows*:
$Hom(A,\text{-})_{\mathrm{a}} : Ar \to ar\ Set$
⟨*proof*⟩

lemma (**in** *into-set*) *homf-preserves-objects*:
$Hom(A,\text{-})_{\mathrm{o}} : Ob \to ob\ Set$
⟨*proof*⟩

lemma (**in** *into-set*) *homf-preserves-dom*:
assumes *f*: $f \in Ar$
shows $Hom(A,\text{-})_{\mathrm{o}}\ (Dom\ f) = dom\ Set\ (Hom(A,\text{-})_{\mathrm{a}}\ f)$
⟨*proof*⟩

lemma (**in** *into-set*) *homf-preserves-cod*:
assumes *f*: $f \in Ar$
shows $Hom(A,\text{-})_{\mathrm{o}}\ (Cod\ f) = cod\ Set\ (Hom(A,\text{-})_{\mathrm{a}}\ f)$
⟨*proof*⟩

lemma (**in** *into-set*) *homf-preserves-id*:
assumes *B*: $B \in Ob$
shows $Hom(A,\text{-})_{\mathrm{a}}\ (Id\ B) = id\ Set\ (Hom(A,\text{-})_{\mathrm{o}}\ B)$
⟨*proof*⟩

lemma (**in** *into-set*) *homf-preserves-comp*:
assumes *f*: $f \in Ar$
and *g*: $g \in Ar$
and *fg*: $Cod\ f = Dom\ g$
shows $Hom(A,\text{-})_{\mathrm{a}}\ (g \cdot f) = (Hom(A,\text{-})_{\mathrm{a}}\ g) \odot (Hom(A,\text{-})_{\mathrm{a}}\ f)$
⟨*proof*⟩

theorem (**in** *into-set*) *homf-into-set*:
$Functor\ Hom(A,\text{-}) : AA \longrightarrow Set$
⟨*proof*⟩

end

5 Natural Transformations

theory *NatTrans*
imports *Functors*
begin

locale *natural-transformation* = *two-cats* +
  fixes F **and** G **and** u
  assumes *Functor* $F : AA \longrightarrow BB$
  and *Functor* $G : AA \longrightarrow BB$
  and $u : ob\ AA \to ar\ BB$
  and $u \in extensional\ (ob\ AA)$
  and $\forall A{\in}Ob.\ u\ A \in Hom_{BB}\ (F_{\mathrm{o}}\ A)\ (G_{\mathrm{o}}\ A)$
  and $\forall A{\in}Ob.\ \forall B{\in}Ob.\ \forall f{\in}Hom\ A\ B.\ (G_{\mathrm{a}}\ f) \cdot_{BB} (u\ A) = (u\ B) \cdot_{BB} (F_{\mathrm{a}}\ f)$

abbreviation
  nt-syn (- : - ⇒ - *in Func* '(- , - ') [*81*]) **where**
  $u : F \Rightarrow G$ *in Func*$(AA, BB) \equiv$ *natural-transformation* $AA\ BB\ F\ G\ u$

locale *endoNT* = *natural-transformation* + *one-cat*

theorem (**in** *endoNT*) *id-restrict-natural*:
  $(\lambda A{\in}Ob.\ Id\ A) : (id\text{-}func\ AA) \Rightarrow (id\text{-}func\ AA)$ *in Func*(AA,AA)
⟨*proof*⟩

end

6 Yonedas Lemma

theory *Yoneda*
imports *HomFunctors NatTrans*
begin

6.1 The Sandwich Natural Transformation

locale *Yoneda* = *functor* + *into-set* +
  assumes *TERM* $(AA :: ('o,'a,'m)category\text{-}scheme)$
  fixes *sandwich* :: $['o,'a,'o] \Rightarrow 'a\ set\text{-}arrow$ $(\sigma'(\text{-},\text{-}'))$
  defines *sandwich* $A\ a \equiv (\lambda B{\in}Ob.$ ⦇
  set-dom=$Hom\ A\ B$,
  set-func=$(\lambda f{\in}Hom\ A\ B.\ set\text{-}func\ (F_{\mathrm{a}}\ f)\ a)$,
  set-cod=$F_{\mathrm{o}}\ B$
  ⦈)
  fixes *unsandwich* :: $['o,'o \Rightarrow 'a\ set\text{-}arrow] \Rightarrow 'a$ $(\sigma^{\leftarrow}'(\text{-},\text{-}'))$
  defines *unsandwich* $A\ u \equiv set\text{-}func\ (u\ A)\ (Id\ A)$

lemma (**in** *Yoneda*) *F-into-set*:

Functor $F : AA \longrightarrow Set$
⟨*proof*⟩

lemma (**in** *Yoneda*) *F-comp-func*:
assumes *1*: $A \in Ob$ **and** *2*: $B \in Ob$ **and** *3*: $C \in Ob$
and *4*: $g \in Hom\ A\ B$ **and** *5*: $f \in Hom\ B\ C$
shows *set-func* $(F_{\mathrm{a}}\ (f \cdot g)) = compose\ (F_{\mathrm{o}}\ A)\ (set\text{-}func\ (F_{\mathrm{a}}\ f))\ (set\text{-}func\ (F_{\mathrm{a}}\ g))$
⟨*proof*⟩

lemma (**in** *Yoneda*) *sandwich-funcset*:
assumes *A*: $A \in Ob$
and $a \in F_{\mathrm{o}}\ A$
shows $\sigma(A,a) : Ob \to ar\ Set$
⟨*proof*⟩

lemma (**in** *Yoneda*) *sandwich-type*:
assumes *A*: $A \in Ob$ **and** *B*: $B \in Ob$
and $a \in F_{\mathrm{o}}\ A$
shows $\sigma(A,a)\ B \in hom\ Set\ (Hom\ A\ B)\ (F_{\mathrm{o}}\ B)$
⟨*proof*⟩

lemma (**in** *Yoneda*) *sandwich-commutes*:
assumes *AOb*: $A \in Ob$ **and** *BOb*: $B \in Ob$ **and** *COb*: $C \in Ob$
and *aFa*: $a \in F_{\mathrm{o}}\ A$
and *fBC*: $f \in Hom\ B\ C$
shows $(F_{\mathrm{a}}\ f) \odot (\sigma(A,a)\ B) = (\sigma(A,a)\ C) \odot (Hom(A,\text{-})_{\mathrm{a}}\ f)$
⟨*proof*⟩

lemma (**in** *Yoneda*) *sandwich-natural*:
assumes $A \in Ob$
and $a \in F_{\mathrm{o}}\ A$
shows $\sigma(A,a) : Hom(A,\text{-}) \Rightarrow F\ in\ Func(AA,Set)$
⟨*proof*⟩

6.2 Sandwich Components are Bijective

lemma (**in** *Yoneda*) *unsandwich-left-inverse*:
assumes *1*: $A \in Ob$
and *2*: $a \in F_{\mathrm{o}}\ A$
shows $\sigma^{\leftarrow}(A,\sigma(A,a)) = a$
⟨*proof*⟩

lemma (**in** *Yoneda*) *unsandwich-right-inverse*:

assumes *1*: $A \in Ob$
and *2*: $u : Hom(A,\text{-}) \Rightarrow F$ *in* $Func(AA,Set)$
shows $\sigma(A,\sigma^{\leftarrow}(A,u)) = u$
⟨*proof*⟩

In order to state the lemma, we must rectify a curious omission from the Isabelle/HOL library. They define the idea of injectivity on a given set, but surjectivity is only defined relative to the entire universe of the target type.

definition
surj-on :: $['a \Rightarrow 'b, 'a\ set, 'b\ set] \Rightarrow bool$ **where**
surj-on f A B $\longleftrightarrow (\forall\, y{\in}B.\ \exists\, x{\in}A.\ f(x){=}y)$

definition
bij-on :: $['a \Rightarrow 'b, 'a\ set, 'b\ set] \Rightarrow bool$ **where**
bij-on f A B $\longleftrightarrow$ *inj-on f A* & *surj-on f A B*

definition
equinumerous :: $['a\ set, 'b\ set] \Rightarrow bool$ (**infix** $\cong$ *40*) **where**
equinumerous A B $\longleftrightarrow (\exists f.$ *bij-betw f A B*$)$

lemma *bij-betw-eq*:
bij-betw f A B $\longleftrightarrow$
inj-on f A $\wedge\ (\forall\, y{\in}B.\ \exists\, x{\in}A.\ f(x){=}y) \wedge (\forall\, x{\in}A.\ f\ x \in B)$
⟨*proof*⟩

theorem (**in** *Yoneda*) *Yoneda*:
assumes *1*: $A \in Ob$
shows $F_{\mathrm{O}}\ A \cong \{u.\ u : Hom(A,\text{-}) \Rightarrow F$ *in* $Func(AA,Set)\}$
⟨*proof*⟩

end

References

[O'K04] Greg O'Keefe. Towards a readable formalisation of category theory. In Mike Atkinson, editor, *Computing: The Australasian Theory Symposium*, volume 91 of *Electronic Notes in Theoretical Computer Science*, pages 212–228. Elsevier, 2004.