

Generating Cases from Labeled Subgoals

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Abstract

Isabelle/Isar provides *named cases* to structure proofs. This article contains an implementation of a proof method `casify`, which can be used to easily extend proof tools with support for named cases. Such a proof tool must produce labeled subgoals, which are then interpreted by `casify`.

As examples, this work contains verification condition generators producing named cases for three languages: The Hoare language from `HOL/Library`, a monadic language for computations with failure (inspired by the `AutoCorres` tool), and a language of conditional expressions. These VCGs are demonstrated by a number of example programs.

```

theory Case-Labeling
imports Main
keywords print-nested-cases :: diag
begin

```

1 Labeling Subgoals

context begin

qualified type-synonym *prg-ctxt-var* = *unit*

qualified type-synonym *prg-ctxt* = *string* × *nat* × *prg-ctxt-var list*

Embed variables in terms

qualified definition *VAR* :: *'v* ⇒ *prg-ctxt-var* **where**

VAR - = ()

Labeling of a subgoal

qualified definition *VC* :: *prg-ctxt list* ⇒ *'a* ⇒ *'a* **where**

VC *ct* *P* ≡ *P*

Computing the statement numbers and context

qualified definition *CTXT* :: *nat* ⇒ *prg-ctxt list* ⇒ *nat* ⇒ *'a* ⇒ *'a* **where**

CTXT *inp* *ct* *outp* *P* ≡ *P*

Labeling of a term binding or assumption

qualified definition *BIND* :: *string* ⇒ *nat* ⇒ *'a* ⇒ *'a* **where**

BIND *name* *inp* *P* ≡ *P*

Hierarchy labeling

qualified definition *HIER* :: *prg-ctxt list* ⇒ *'a* ⇒ *'a* **where**

HIER *ct* *P* ≡ *P*

Split Labeling. This is used as an assumption

qualified definition *SPLIT* :: *'a* ⇒ *'a* ⇒ *bool* **where**

SPLIT *v* *w* ≡ *v* = *w*

Disambiguation Labeling. This is used as an assumption

qualified definition *DISAMBIG* :: *nat* ⇒ *bool* **where**

DISAMBIG *n* ≡ *True*

lemmas *LABEL-simps* = *BIND-def* *CTXT-def* *HIER-def* *SPLIT-def* *VC-def*

lemma *Initial-Label*: *CTXT* 0 [] *outp* *P* ⇒ *P*

by (*simp* *add*: *Case-Labeling.CTXT-def*)

lemma

BIND-I: *P* ⇒ *BIND* *name* *inp* *P* **and**

BIND-D: *BIND* *name* *inp* *P* ⇒ *P* **and**

VC-I: $P \implies VC \text{ ct } P$
unfolding *Case-Labeling.BIND-def Case-Labeling.VC-def* .

lemma *DISAMBIG-I*: $(DISAMBIG \ n \implies P) \implies P$
by (*auto simp: DISAMBIG-def Case-Labeling.VC-def*)

lemma *DISAMBIG-E*: $(DISAMBIG \ n \implies P) \implies P$
by (*auto simp: DISAMBIG-def*)

Lemmas for the tuple postprocessing

lemma *SPLIT-reflection*: $SPLIT \ x \ y \implies (x \equiv y)$
unfolding *SPLIT-def* **by** (*rule eq-reflection*)

lemma *rev-SPLIT-reflection*: $(x \equiv y) \implies SPLIT \ x \ y$
unfolding *SPLIT-def* ..

lemma *SPLIT-sym*: $SPLIT \ x \ y \implies SPLIT \ y \ x$
unfolding *SPLIT-def* **by** (*rule sym*)

lemma *SPLIT-thin-refl*: $\llbracket SPLIT \ x \ x; PROP \ W \rrbracket \implies PROP \ W$.

lemma *SPLIT-subst*: $\llbracket SPLIT \ x \ y; P \ x \rrbracket \implies P \ y$
unfolding *SPLIT-def* **by** *hypsubst*

lemma *SPLIT-prodE*:
assumes *SPLIT* $(x1, y1) (x2, y2)$
obtains *SPLIT* $x1 \ x2 \ SPLIT \ y1 \ y2$
using *assms* **unfolding** *SPLIT-def* **by** *auto*

end

The labeling constants were qualified to not interfere with any other theory.
The following locale allows using a nice syntax in other theories

locale *Labeling-Syntax* **begin**

abbreviation *VAR* **where** $VAR \equiv Case-Labeling.VAR$

abbreviation *VC* $(V(\langle 2-, - \rangle))$ **where** $VC \text{ bl } ct \equiv Case-Labeling.VC \text{ (bl } \# ct)$

abbreviation *CTXT* $(C(\langle 2-, - \rangle))$ **where** $CTXT \equiv Case-Labeling.CTXT$

abbreviation *BIND* $(B(\langle 2-, - \rangle))$ **where** $BIND \equiv Case-Labeling.BIND$

abbreviation *HIER* $(H(\langle 2-, - \rangle))$ **where** $HIER \equiv Case-Labeling.HIER$

abbreviation *SPLIT* **where** $SPLIT \equiv Case-Labeling.SPLIT$

end

Lemmas for converting terms from *Suc/0* notation to numerals

lemma *Suc-numerals-conv*:

Suc $0 = Numeral1$

Suc $(numeral \ n) = numeral \ (n + num.One)$

by *auto*

lemmas *Suc-numeral-simps = Suc-numerals-conv add-num-simps*

2 Casify

Introduces a command **print-nested-cases**. This is similar to **print-cases**, but shows also the nested cases.

ML-file *<print-nested-cases.ML>*

ML-file *<util.ML>*

Introduces the proof method.

ML-file *<casify.ML>*

```
ML <
  val casify-defs = Casify.Options { simp-all-cases=true, split-right-only=true, protect-subgoals=false }
>
```

```
method-setup prepare-labels = <
  Scan.succeed (fn ctxt => SIMPLE-METHOD (ALLGOALS (Casify.prepare-labels-tac ctxt)))
> VCG labelling: prepare labels
```

```
method-setup casify = <Casify.casify-method-setup casify-defs>
  VCG labelling: Turn the labels into cases
```

end

3 Examples

3.1 A labeling VCG for a monadic language

theory *Monadic-Language*

imports

```
  Complex-Main
  ../Case-Labeling
  HOL-Eisbach.Eisbach
```

begin

ML-file *<../util.ML>*

```
ML <
  fun vcg-tac nt-rules nt-comb ctxt =
    let
      val rules = Named-Theorems.get ctxt nt-rules
```

```

    val comb = Named-Theorems.get ctxt nt-comb
    in REPEAT-ALL-NEW-FWD ( resolve-tac ctxt rules ORELSE' (resolve-tac
    ctxt comb THEN' resolve-tac ctxt rules)) end
  >

```

This language is inspired by the languages used in AutoCorres [1]

```

consts bind :: 'a option  $\Rightarrow$  ('a  $\Rightarrow$  'b option)  $\Rightarrow$  'b option (infixr |>> 4)
consts return :: 'a  $\Rightarrow$  'a option
consts while :: ('a  $\Rightarrow$  bool)  $\Rightarrow$  ('a  $\Rightarrow$  bool)  $\Rightarrow$  ('a  $\Rightarrow$  'a option)  $\Rightarrow$  ('a  $\Rightarrow$  'a option)
consts valid :: bool  $\Rightarrow$  'a option  $\Rightarrow$  ('a  $\Rightarrow$  bool)  $\Rightarrow$  bool

```

```

named-theorems vcg
named-theorems vcg-comb

```

```

method-setup vcg = <
  Scan.succeed (fn ctxt => SIMPLE-METHOD (FIRSTGOAL (vcg-tac @ {named-theorems
  vcg} @ {named-theorems vcg-comb} ctxt)))
  >

```

axiomatization where

```

return[vcg]: valid (Q x) (return x) Q and
bind[vcg]:  $\llbracket \bigwedge x. \text{valid } (R x) (c2 x) Q; \text{valid } P c1 R \rrbracket \Longrightarrow \text{valid } P (bind c1 c2) Q$ 
and
while[vcg]:  $\bigwedge c. \llbracket \bigwedge x. \text{valid } (I x \wedge b x) (c x) I; \bigwedge x. I x \wedge \neg b x \Longrightarrow Q x \rrbracket \Longrightarrow$ 
valid (I x) (while b I c x) Q and
cond[vcg]:  $\bigwedge b c1 c2. \text{valid } P1 c1 Q \Longrightarrow \text{valid } P2 c2 Q \Longrightarrow \text{valid } (\text{if } b \text{ then } P1$ 
else P2) (if b then c1 else c2) Q and
case-prod[vcg]:  $\bigwedge P. \llbracket \bigwedge x y. v = (x,y) \Longrightarrow \text{valid } (P x y) (B x y) Q \rrbracket$ 
 $\Longrightarrow \text{valid } (\text{case } v \text{ of } (x,y) \Rightarrow P x y) (\text{case } v \text{ of } (x,y) \Rightarrow B x y) Q$  and
conseq[vcg-comb]:  $\llbracket \text{valid } P' c Q; P \Longrightarrow P' \rrbracket \Longrightarrow \text{valid } P c Q$ 

```

Labeled rules

```

named-theorems vcg-l
named-theorems vcg-l-comb
named-theorems vcg-elim

```

```

method-setup vcg-l = <
  Scan.succeed (fn ctxt => SIMPLE-METHOD (FIRSTGOAL (vcg-tac @ {named-theorems
  vcg-l} @ {named-theorems vcg-l-comb} ctxt)))
  >

```

```

method vcg-l' = (vcg-l; (elim vcg-elim)?)

```

context begin

```

interpretation Labeling-Syntax .

```

```

lemma L-return[vcg-l]: CTXT inp ct (Suc inp) (valid (P x) (return x) P)
unfolding LABEL-simps by (rule return)

```

lemma *L-bind*[vcg-l]:
assumes $\bigwedge x. \text{CTXT } (\text{Suc } \text{outp}') \text{ (}"bind", \text{outp}', [\text{VAR } x]) \# ct) \text{ outp } (\text{valid } (R \ x) \ (c2 \ x) \ Q)$
assumes $\text{CTXT } \text{inp } ct \ \text{outp}' \ (\text{valid } P \ c1 \ R)$
shows $\text{CTXT } \text{inp } ct \ \text{outp } (\text{valid } P \ (\text{bind } c1 \ c2) \ Q)$
using *assms unfolding LABEL-simps by (rule bind)*

lemma *L-while*[vcg-l]:
fixes *inp ct* **defines** $ct' \equiv \lambda x. \text{"while", inp, [VAR } x]) \# ct$
assumes $\bigwedge x. \text{CTXT } (\text{Suc } \text{inp}) \ (ct' \ x) \ \text{outp}'$
 $(\text{valid } (\text{BIND } \text{"inv-pre"} \ \text{inp } (I \ x) \wedge \text{BIND } \text{"lcond"} \ \text{inp } (b \ x)) \ (c \ x) \ (\lambda x. \text{BIND } \text{"inv-post"} \ \text{inp } (I \ x)))$
assumes $\bigwedge x. B\langle \text{"inv-pre"}, \text{inp}: I \ x \rangle \wedge B\langle \text{"lcond"}, \text{inp}: \neg b \ x \rangle \implies \text{VC } (\text{"post"}, \text{outp}'$
 $, []) \ (ct' \ x) \ (P \ x)$
shows $\text{CTXT } \text{inp } ct \ (\text{Suc } \text{outp}') \ (\text{valid } (I \ x) \ (\text{while } b \ I \ c \ x) \ P)$
using *assms(2-) unfolding LABEL-simps by (rule while)*

lemma *L-cond*[vcg-l]:
fixes *inp ct* **defines** $ct' \equiv (\text{"if", inp, []}) \# ct$
assumes $C\langle \text{Suc } \text{inp}, (\text{"then"}, \text{inp}, []) \# ct', \text{outp}: \text{valid } P1 \ c1 \ Q \rangle$
assumes $C\langle \text{Suc } \text{outp}, (\text{"else"}, \text{outp}, []) \# ct', \text{outp}': \text{valid } P2 \ c2 \ Q \rangle$
shows $C\langle \text{inp}, ct, \text{outp}': \text{valid } (\text{if } B\langle \text{"cond"}, \text{inp}: b \rangle \text{ then } B\langle \text{"then"}, \text{inp}: P1 \rangle \ \text{else } B\langle \text{"else"}, \text{inp}: P2 \rangle) \ (\text{if } b \ \text{then } c1 \ \text{else } c2) \ Q \rangle$
using *assms(2-) unfolding LABEL-simps by (rule cond)*

lemma *L-case-prod*[vcg-l]:
assumes $\bigwedge x \ y. v = (x, y) \implies \text{CTXT } \text{inp } ct \ \text{outp } (\text{valid } (P \ x \ y) \ (B \ x \ y) \ Q)$
shows $\text{CTXT } \text{inp } ct \ \text{outp } (\text{valid } (\text{case } v \ \text{of } (x, y) \Rightarrow P \ x \ y) \ (\text{case } v \ \text{of } (x, y) \Rightarrow B \ x \ y) \ Q)$
using *assms unfolding LABEL-simps by (rule case-prod)*

lemma *L-conseq*[vcg-l-comb]:
assumes $\text{CTXT } (\text{Suc } \text{inp}) \ ct \ \text{outp } (\text{valid } P' \ c \ Q)$
assumes $P \implies \text{VC } (\text{"conseq"}, \text{inp}, []) \ ct \ P'$
shows $\text{CTXT } \text{inp } ct \ \text{outp } (\text{valid } P \ c \ Q)$
using *assms unfolding LABEL-simps by (rule conseq)*

lemma *L-assm-conjE*[vcg-elim]:
assumes $\text{BIND } \text{name } \text{inp } (P \wedge Q)$ **obtains** $\text{BIND } \text{name } \text{inp } P \ \text{BIND } \text{name } \text{inp } Q$
using *assms unfolding LABEL-simps by auto*

declare *conjE*[vcg-elim]

end

```

lemma dvd-div:
  fixes a b c :: int
  assumes a dvd b c dvd b coprime a c
  shows a dvd (b div c)
  using assms coprime-dvd-mult-left-iff by fastforce

lemma divides:
  valid
  (0 < (a :: int))
  (
    return a
    |>> (λn.
      while
        (λn. even n)
        (λn. 0 < n ∧ n dvd a ∧ (∀ m. odd m ∧ m dvd a → m dvd n))
        (λn. return (n div 2))
        n
      )
    )
  (λr. odd r ∧ r dvd a ∧ (∀ m. odd m ∧ m dvd a → m ≤ r))

  apply vcg
  apply (auto simp add: zdvd-imp-le dvd-div div-positive-int elim!
    evenE intro: dvd-mult-right)
  done

lemma L-divides:
  valid
  (0 < (a :: int))
  (
    return a
    |>> (λn.
      while
        (λn. even n)
        (λn. 0 < n ∧ n dvd a ∧ (∀ m. odd m ∧ m dvd a → m dvd n))
        (λn. return (n div 2))
        n
      )
    )
  (λr. odd r ∧ r dvd a ∧ (∀ m. odd m ∧ m dvd a → m ≤ r))

  apply (rule Initial-Label)
  apply vcg-l'
proof casify
print-nested-cases
  case bind
  { case (while n)
    { case post then show ?case by (auto simp: zdvd-imp-le)
    }
  }
  next

```

```

case conseq
from  $\langle 0 < n \rangle$   $\langle \text{even } n \rangle$  have  $0 < n \text{ div } 2$ 
  by (simp add: pos-imp-zdiv-pos-iff z dvd-imp-le)
moreover
from  $\langle n \text{ dvd } a \rangle$   $\langle \text{even } n \rangle$  have  $n \text{ div } 2 \text{ dvd } a$ 
  by (metis dvd-div-mult-self dvd-mult-left)
moreover
{ fix m assume  $\text{odd } m$   $m \text{ dvd } a$ 
  then have  $m \text{ dvd } n$  using conseq.inv-pre(3) by simp
  moreover note  $\langle \text{even } n \rangle$ 
  moreover from  $\langle \text{odd } m \rangle$  have coprime m 2
by (metis dvd-eq-mod-eq-0 invertible-coprime mult-cancel-left2 not-mod-2-eq-1-eq-0)
  ultimately
  have  $m \text{ dvd } n \text{ div } 2$  by (rule dvd-div)
}
ultimately show ?case by auto
}
next
case conseq then show ?case by auto
}
qed

```

lemma *add:*

```

valid
  True
  (
    while
      — COND:  $(\lambda(r,j). j < (b :: \text{nat}))$ 
      — INV:  $(\lambda(r,j). j \leq b \wedge r = a + j)$ 
      — BODY:  $(\lambda(r,j). \text{return } (r + 1, j + 1))$ 
      — START:  $(a, 0)$ 
    |>>  $(\lambda(r,-). \text{return } r)$ 
  )
   $(\lambda r. r = a + b)$ 

```

by *vcg auto*

lemma *mult:*

```

valid
  True
  (
    while
      — COND:  $(\lambda(r,i). i < (a :: \text{nat}))$ 
      — INV:  $(\lambda(r,i). i \leq a \wedge r = i * b)$ 
      — BODY:  $(\lambda(r,i). \text{return } (r + b, i + 1))$ 
    while

```



```

      — COND:  $(\lambda(r,j). j < b)$ 
      — INV:  $(\lambda(r,j). i < a \wedge j \leq b \wedge r = i * b + j)$ 
      — BODY:  $(\lambda(r,j). \text{return } (r + 1, j + 1))$ 
      — START:  $(r, 0)$ 
    |>>  $(\lambda(r,-). \text{return } (r, i + 1))$ 
  )
  — START:  $(0, 0)$ 
|>>  $(\lambda(r,-). \text{return } r)$ 
)
 $(\lambda r. r = a * b)$ 

```

by *vcg auto*

4 Labeled

lemma *L-mult*:

valid

True

```

(
  while
  — COND:  $(\lambda(r,i). i < (a :: \text{nat}))$ 
  — INV:  $(\lambda(r,i). i \leq a \wedge r = i * b)$ 
  — BODY:  $(\lambda(r,i).$ 
    while
    — COND:  $(\lambda(r,j). j < b)$ 
    — INV:  $(\lambda(r,j). i < a \wedge j \leq b \wedge r = i * b + j)$ 
    — BODY:  $(\lambda(r,j). \text{return } (r + 1, j + 1))$ 
    — START:  $(r, 0)$ 
  |>>  $(\lambda(r,-). \text{return } (r, i + 1))$ 
  )
  — START:  $(0, 0)$ 
|>>  $(\lambda(r,-). \text{return } r)$ 
)
 $(\lambda r. r = a * b)$ 

```

apply (*rule Initial-Label*)

apply *vcg-l'*

proof *casify*

case *while*

{ **case** *while*

{ **case** *post* then show ?*case* by *auto*

next

case *conseq* then show ?*case* by *auto*

}

next

case *post* then show ?*case* by *auto*

next

case *conseq* then show ?*case* by *auto*

}

```

next
  case conseq then show ?case by auto
qed

lemma L-paths:
valid
  (path ≠ [])
  ( while
    — COND:  $(\lambda(p,r). p \neq [])$ 
    — INV:  $(\lambda(p,r). \text{distinct } r \wedge \text{hd } (r @ p) = \text{hd } \textit{path} \wedge \text{last } (r @ p) = \text{last } \textit{path})$ 
    — BODY:  $(\lambda(p,r).$ 
      return (hd p)
      |>>  $(\lambda x.$ 
        if  $(r \neq [] \wedge x = \text{hd } r)$ 
          then return []
          else (if  $x \in \text{set } r$ 
            then return (takeWhile  $(\lambda y. y \neq x) r$ )
            else return (r))
          )
        |>>  $(\lambda r'. \text{return } (\text{tl } p, r' @ [x])$ 
        )
      )
    — START: (path, [])
    |>>  $(\lambda(-,r). \text{return } r)$ 
  )
  ( $\lambda r. \text{distinct } r \wedge \text{hd } r = \text{hd } \textit{path} \wedge \text{last } r = \text{last } \textit{path}$ )

apply (rule Initial-Label)
apply vcg-l'
proof casify
  case conseq then show ?case by auto
next
  case (while p r)
  { case conseq
    from conseq have  $r = [] \implies ?\textit{case}$  by (cases p) auto
    moreover
    from conseq have  $r \neq [] \implies \text{hd } p = \text{hd } r \implies ?\textit{case}$  by (cases p) auto
    moreover
    { assume A:  $r \neq [] \wedge \text{hd } p \neq \text{hd } r$ 
      have  $\text{hd } (\text{takeWhile } (\lambda y. y \neq \text{hd } p) r @ \text{hd } p \# \text{tl } p) = \text{hd } r$ 
      using A by (cases r) auto
      moreover
      have  $\text{last } (\text{takeWhile } (\lambda y. y \neq \text{hd } p) r @ \text{hd } p \# \text{tl } p) = \text{last } (r @ p)$ 
      using A  $\langle p \neq [] \rangle$  by auto
      moreover
      have  $\text{distinct } (\text{takeWhile } (\lambda y. y \neq \text{hd } p) r @ [\text{hd } p])$ 
      using conseq by (auto dest: set-takeWhileD)
    }
    ultimately
    have ?case using A conseq by auto
  }

```

```

    }
    ultimately show ?case by blast
next
  case post then show ?case by auto
}
qed

end

```

4.1 Decomposing Conditionals

```
theory Conditionals
```

```
imports
```

```
  Complex-Main
```

```
  ../Case-Labeling
```

```
  HOL-Eisbach.Eisbach
```

```
begin
```

```
context begin
```

```
  interpretation Labeling-Syntax .
```

```
lemma DC-conj:
```

```
  assumes  $C\langle inp, ct, outp': a \rangle C\langle outp', ct, outp: b \rangle$ 
```

```
  shows  $C\langle inp, ct, outp: a \wedge b \rangle$ 
```

```
  using assms unfolding LABEL-simps by auto
```

```
lemma DC-if:
```

```
  fixes  $ct$  defines  $ct' \equiv \lambda pos \text{ name. } (name, pos, []) \# ct$ 
```

```
  assumes  $H\langle ct' \text{ inp } "then": a \rangle \implies C\langle Suc \text{ inp}, ct' \text{ inp } "then", outp': b \rangle$ 
```

```
  assumes  $H\langle ct' \text{ outp}' "else": \neg a \rangle \implies C\langle Suc \text{ outp}', ct' \text{ outp}' "else", outp: c \rangle$ 
```

```
  shows  $C\langle inp, ct, outp: \text{if } a \text{ then } b \text{ else } c \rangle$ 
```

```
  using assms(2-) unfolding LABEL-simps by auto
```

```
lemma DC-final:
```

```
  assumes  $V\langle ("g", inp, []), ct: a \rangle$ 
```

```
  shows  $C\langle inp, ct, Suc \text{ inp}: a \rangle$ 
```

```
  using assms unfolding LABEL-simps by auto
```

```
end
```

```
method vcg-dc = (intro DC-conj DC-if; rule DC-final)
```

```
lemma
```

```
  assumes  $a: a$ 
```

```
  and  $d: b \implies c \implies d$ 
```

```
  and  $d': b \implies c \implies d'$ 
```

```
  and  $e: b \implies \neg c \implies e$ 
```

```
  and  $f: \neg b \implies f$ 
```

```
  shows  $a \wedge (\text{if } b \text{ then } (\text{if } c \text{ then } d \wedge d' \text{ else } e) \text{ else } f)$ 
```

```

apply (rule Initial-Label)
apply vcg-dc
proof casify
  case g show ?case by (fact a)
next
  case then note  $b = \langle b \rangle$ 
  { case then note  $c = \langle c \rangle$ 
    { case g show ?case using  $b\ c$  by (fact d)
      next
      case ga show ?case using  $b\ c$  by (fact d')
    }
  }
  next
  case else
  { case g show ?case using then else by (fact e) }
}
next
case else
{ case g show ?case using else by (fact f) }
qed

```

4.2 Protecting similar subgoals

The proof below fails if the `disambig_subgoals` option is omitted: all three subgoals have the same conclusion and can be discharged without using their assumptions. If the case `g` is solved first, it discharges instead the subgoal $a \implies b$, making the case `then` fail afterwards.

The `disambig_subgoals` options prevents this by inserting vacuous assumptions.

```

lemma
  assumes b
  shows (if a then b else b)  $\wedge b$ 
  apply (rule Initial-Label)
  apply vcg-dc
proof (casify (disambig-subgoals))
  case g show ?case by (fact  $\langle b \rangle$ )
next
  case then case g show ?case by (fact  $\langle b \rangle$ )
next
  case else case g show ?case by (fact  $\langle b \rangle$ )
qed

```

4.3 Unnamed Cases

```

lemma
  assumes  $a \implies b \neg a \implies c\ d$ 
  shows (if a then b else c)  $\wedge d$ 
  apply (rule Initial-Label)
  apply vcg-dc

```

```

  apply (simp-all only: LABEL-simps)[2]
proof (casify (disambig-subgoals))
  case unnamed from ⟨a⟩ show ?case by fact
next
  case unnamed_a from ⟨¬a⟩ show ?case by fact
next
  case g show ?case by fact
qed

end
theory Labeled-Hoare
imports
  .././Case-Labeling
  HOL-Hoare.Hoare-Logic
begin

```

4.4 A labeling VCG for HOL/Hoare

```

context begin
  interpretation Labeling-Syntax .

```

```

lemma LSeqRule:
  assumes C⟨IC,CT,OC1: Valid P c1 a1 Q⟩
  and C⟨Suc OC1,CT,OC: Valid Q c2 a2 R⟩
  shows C⟨IC,CT,OC: Valid P (Seq c1 c2) (Aseq a1 a2) R⟩
  using assms unfolding LABEL-simps by (rule SeqRule)

```

```

lemma LSkipRule:
  assumes V⟨("weaken", IC, []),CT: p ⊆ q⟩
  shows C⟨IC,CT,IC: Valid p SKIP a q⟩
  using assms unfolding LABEL-simps by (rule SkipRule)

```

```

lemmas LAbortRule = LSkipRule — dummy version

```

```

lemma LBasicRule:
  assumes V⟨("basic", IC, []),CT: p ⊆ {s. f s ∈ q}⟩
  shows C⟨IC,CT,IC: Valid p (Basic f) a q⟩
  using assms unfolding LABEL-simps by (rule BasicRule)

```

```

lemma LCondRule:
  fixes IC CT defines CT' ≡ ("cond", IC, []) # CT
  assumes V⟨("vc", IC, []),("cond", IC, []) # CT: p ⊆ {s. (s ∈ b → s ∈ w)
  ∧ (s ∉ b → s ∈ w)}⟩
  and C⟨Suc IC,("then", IC, []) # ("cond", IC, []) # CT,OC1: Valid w c1 a1
  q⟩
  and C⟨Suc OC1,("else", Suc OC1, []) # ("cond", IC, []) # CT,OC: Valid
  w' c2 a2 q⟩
  shows C⟨IC,CT,OC: Valid p (Cond b c1 c2) (Acond a1 a2) q⟩
  using assms(2-) unfolding LABEL-simps by (rule CondRule)

```

lemma *LWhileRule*:
fixes *IC CT* **defines** $CT' \equiv ("while", IC, []) \# CT$
assumes $V\langle("precondition", IC, []), ("while", IC, []) \# CT: p \subseteq i\rangle$
and $C\langle Suc\ IC, ("invariant", Suc\ IC, []) \# ("while", IC, []) \# CT, OC: Valid$
 $(i \cap b) c (A\ 0) i\rangle$
and $V\langle("postcondition", IC, []), ("while", IC, []) \# CT: i \cap - b \subseteq q\rangle$
shows $C\langle IC, CT, OC: Valid\ p (While\ b\ c) (Awhile\ i\ v\ A) q\rangle$
using *assms(2-)* **unfolding** *LABEL-simps* **by** (rule *WhileRule*)

lemma *LABELs-to-prems*:
 $C\langle IC, CT, OC: True\rangle \Longrightarrow P \Longrightarrow C\langle IC, CT, OC: P\rangle$
 $V\langle x, ct: True\rangle \Longrightarrow P \Longrightarrow V\langle x, ct: P\rangle$
unfolding *LABEL-simps* **by** *simp-all*

lemma *LABELs-to-concl*:
 $C\langle IC, CT, OC: True\rangle \Longrightarrow C\langle IC, CT, OC: P\rangle \Longrightarrow P$
 $V\langle x, ct: True\rangle \Longrightarrow V\langle x, ct: P\rangle \Longrightarrow P$
unfolding *LABEL-simps* .

end

ML-file $\langle labeled\ hoare\ tac.ML\rangle$

method-setup *labeled-vcg* = \langle
 $Scan.succeed\ (fn\ ctxt \Rightarrow SIMPLE\METHOD'\ (Labeled\ Hoare.hoare\ tac\ ctxt\ (K$
 $all\ tac)))\rangle$
verification condition generator

method-setup *labeled-vcg-simp* = \langle
 $Scan.succeed\ (fn\ ctxt \Rightarrow SIMPLE\METHOD'\ (Labeled\ Hoare.hoare\ tac\ ctxt\ (asm\ full\ simp\ tac$
 $ctxt)))\rangle$
verification condition generator

method-setup *casified-vcg* = \langle
 $Scan.lift\ (Casify.casify\ options\ casify\ defs) \gg$
 $(fn\ opt \Rightarrow fn\ ctxt \Rightarrow Util.SIMPLE\METHOD\CASES\ ($
 $HEADGOAL\ (Labeled\ Hoare.hoare\ tac\ ctxt\ (K\ all\ tac))$
 $THEN\ CONTEXT\ Casify.casify\ tac\ opt))$
 \rangle

method-setup *casified-vcg-simp* = \langle
 $Scan.lift\ (Casify.casify\ options\ casify\ defs) \gg$
 $(fn\ opt \Rightarrow fn\ ctxt \Rightarrow Util.SIMPLE\METHOD\CASES\ ($
 $HEADGOAL\ (Labeled\ Hoare.hoare\ tac\ ctxt\ (asm\ full\ simp\ tac\ ctxt))$
 $THEN\ CONTEXT\ Casify.casify\ tac\ opt))$
 \rangle

end

theory *Labeled-Hoare-Examples*

imports

Labeled-Hoare

HOL-Hoare.Arith2

begin

4.4.1 Multiplication by successive addition

lemma *multiply-by-add*: *VARs* $m\ s\ a\ b$

$\{a=A \wedge b=B\}$

$m := 0; s := 0;$

WHILE $m \neq a$

INV $\{s=m*b \wedge a=A \wedge b=B\}$

DO $s := s+b; m := m+(1::nat)$ *OD*

$\{s = A*B\}$

by *vcg-simp*

lemma *VARs* $M\ N\ P :: int$

$\{m=M \wedge n=N\}$

IF $M < 0$ *THEN* $M := -M; N := -N$ *ELSE SKIP FI*;

$P := 0;$

WHILE $0 < M$

INV $\{0 \leq M \wedge (\exists p. p = (if\ m < 0\ then\ -m\ else\ m) \wedge p*N = m*n \wedge P = (p-M)*N)\}$

DO $P := P+N; M := M - 1$ *OD*

$\{P = m*n\}$

proof *casified-vcg-simp*

case *while*

{ case *postcondition*

then show *?case* **by** *auto*

next

case *invariant*

{ case *basic*

then show *?case* **by** (*auto simp: int-distrib*)

}

}

qed

4.4.2 Euclid's algorithm for GCD

lemma *Euclid-GCD*: *VARs* $a\ b$

$\{0 < A \wedge 0 < B\}$

$a := A; b := B;$

WHILE $a \neq b$

INV $\{0 < a \wedge 0 < b \wedge gcd\ A\ B = gcd\ a\ b\}$

DO IF $a < b$ *THEN* $b := b-a$ *ELSE* $a := a-b$ *FI OD*

$\{a = gcd\ A\ B\}$

proof *casified-vcg-simp*

```

case while
{ case postcondition
  then show ?case by (auto elim: gcd-nnn)
next
case invariant
{ case cond
  { case vc
    then show ?case
      by (simp-all add: linorder-not-less gcd-diff-l gcd-diff-r less-imp-le)
  }
}
}
qed

```

4.4.3 Dijkstra's extension of Euclid's algorithm for simultaneous GCD and SCM

From E.W. Disjkstra. Selected Writings on Computing, p 98 (EWD474), where it is given without the invariant. Instead of defining scm explicitly we have used the theorem $\text{scm } x \ y = x*y/\text{gcd } x \ y$ and avoided division by mupltiplying with gcd x y.

```

lemmas distribs =
  diff-mult-distrib diff-mult-distrib2 add-mult-distrib add-mult-distrib2

```

```

lemma gcd-scm: VARS a b x y
{0 < a ∧ 0 < b ∧ a = A ∧ b = B ∧ x = B ∧ y = A}
WHILE a ≠ b
INV {0 < a ∧ 0 < b ∧ gcd A B = gcd a b ∧ 2 * A * B = a * x + b * y}
DO IF a < b THEN (b := b - a; x := x + y) ELSE (a := a - b; y := y + x) FI OD
{a = gcd A B ∧ 2 * A * B = a * (x + y)}

```

```

proof casified-vcg
case while {
  case precondition then show ?case by simp
next
case invariant
case cond
case vc
then show ?case
  by (simp add: distribs gcd-diff-r linorder-not-less gcd-diff-l)
next
case postcondition then show ?case
  by (simp add: distribs gcd-nnn)
}
qed

```

4.4.4 Power by iterated squaring and multiplication

```

lemma power-by-mult: VARS a b c

```



```

{a=A ∧ b=B}
c := (1::nat);
WHILE b ≠ 0
INV {A^B = c * a^b}
DO WHILE b mod 2 = 0
  INV {A^B = c * a^b}
  DO a := a*a; b := b div 2 OD;
  c := c*a; b := b - 1
OD
{c = A^B}
proof casified-vcg-simp
  case while
  case invariant
  case while
  case postcondition
  then show ?case by (cases b) simp-all
qed

```

4.4.5 Factorial

```

lemma factorial: VARS a b
{a=A}
b := 1;
WHILE a ≠ 0
INV {fac A = b * fac a}
DO b := b*a; a := a - 1 OD
{b = fac A}
apply vcg-simp
apply(clarsimp split: nat-diff-split)
done

```

```

lemma VARS i f
{True}
i := (1::nat); f := 1;
WHILE i ≤ n INV {f = fac(i - 1) ∧ 1 ≤ i ∧ i ≤ n+1}
DO f := f*i; i := i+1 OD
{f = fac n}
proof casified-vcg-simp
  case while
  { case invariant
    { case basic
      then show ?case by (induct i) simp-all
    }
  }
  next
  case postcondition
  then have i = Suc n by simp
  then show ?case by simp
}
qed

```

4.4.6 Quicksort

The ‘partition’ procedure for quicksort. ‘A’ is the array to be sorted (modelled as a list). Elements of A must be of class order to infer at the end that the elements between u and l are equal to pivot.

Ambiguity warnings of parser are due to := being used both for assignment and list update.

lemma *Partition:*

```

fixes pivot
defines leq  $\equiv \lambda A i. \forall k. k < i \longrightarrow A!k \leq pivot$ 
defines geq  $\equiv \lambda A i. \forall k. i < k \wedge k < length\ A \longrightarrow pivot \leq A!k$ 
shows
  VARS A u l
  {0 < length(A::('a::order)list)}
  l := 0; u := length A - Suc 0;
  WHILE l  $\leq$  u
    INV {leq A l  $\wedge$  geq A u  $\wedge$  u < length A  $\wedge$  l  $\leq$  length A}
    DO WHILE l < length A  $\wedge$  A!l  $\leq$  pivot
      INV {leq A l  $\wedge$  geq A u  $\wedge$  u < length A  $\wedge$  l  $\leq$  length A}
      DO l := l+1 OD;
      WHILE 0 < u  $\wedge$  pivot  $\leq$  A!u
        INV {leq A l  $\wedge$  geq A u  $\wedge$  u < length A  $\wedge$  l  $\leq$  length A}
        DO u := u - 1 OD;
      IF l  $\leq$  u THEN A := A[l := A!u, u := A!l] ELSE SKIP FI
    OD
  {leq A u  $\wedge$  ( $\forall k. u < k \wedge k < l \longrightarrow A!k = pivot$ )  $\wedge$  geq A l}
unfolding leq-def geq-def
proof casified-vcg-simp
  case basic
  then show ?case by auto
next
  case while
  { case postcondition
    then show ?case by (force simp: nth-list-update)
  }
next
  case invariant
  { case while
    { case invariant
      { case basic
        then show ?case by (blast elim!: less-SucE intro: Suc-leI)
      }
    }
  }
next
  case whilea
  { case invariant
    { case basic
      have lem:  $\bigwedge m\ n. m - Suc\ 0 < n \implies m < Suc\ n$  by linarith
      from basic show ?case by (blast elim!: less-SucE intro: less-imp-diff-less

```

```
dest: lem)
  }
}
}
}
qed
```

end

References

- [1] D. Greenaway, J. Andronick, and G. Klein. Bridging the gap: Automatic verified abstraction of C. In *Interactive Theorem Proving*, Lecture Notes in Computer Science, pages 99–115. Springer, Jan. 2012.