### The Cartan Fixed Point Theorems

Lawrence C. Paulson

March 17, 2025

#### Abstract

The Cartan fixed point theorems concern the group of holomorphic automorphisms on a connected open set of  $\mathbb{C}^n$ . Ciolli et al. [1] have formalised the one-dimensional case of these theorems in HOL Light. This entry contains their proofs, ported to Isabelle/HOL. Thus it addresses the authors remark that "it would be important to write a formal proof in a language that can be read by both humans and machines."

## Contents

begin

#### 0.1 First Cartan Theorem

Ported from HOL Light. See Gianni Ciolli, Graziano Gentili, Marco Maggesi. A Certified Proof of the Cartan Fixed Point Theorems. J Automated Reasoning (2011) 47:319–336 DOI 10.1007/s10817-010-9198-6

 $\begin{array}{l} \textbf{lemma deriv-left-inverse:}\\ \textbf{assumes }f\ holomorphic-on\ S\ \textbf{and}\ g\ holomorphic-on\ T\\ \textbf{and}\ open\ S\ \textbf{and}\ open\ T\\ \textbf{and}\ f\ `S\ \subseteq\ T\\ \textbf{and}\ [simp]:\ \bigwedge z.\ z\in S\implies g\ (f\ z)=z\\ \textbf{and}\ w\in S\\ \textbf{shows}\ deriv\ f\ w\ \ast\ deriv\ g\ (f\ w)=1\\ \langle proof \rangle\end{array}$ 

 $\begin{array}{l} \textbf{lemma higher-deriv-comp-lemma:}\\ \textbf{assumes }s: open \ s \ \textbf{and } holf: \ f \ holomorphic-on \ s\\ \textbf{and } z \in s\\ \textbf{and } t: open \ t \ \textbf{and } holg: \ g \ holomorphic-on \ t\\ \textbf{and } fst: \ f \ `s \subseteq t\\ \textbf{and } n: \ i \leq n\\ \textbf{and } dfz: \ deriv \ f \ z = 1 \ \textbf{and } zero: \ \bigwedge i. \ \llbracket 1 < i; \ i \leq n \rrbracket \Longrightarrow (deriv \ \frown i) \ f \ z = 0\\ \textbf{shows } (deriv \ \frown i) \ (g \ o \ f) \ z = (deriv \ \frown i) \ g \ (f \ z)\\ \langle proof \rangle\end{array}$ 

lemma higher-deriv-comp-iter-lemma: assumes s: open s and holf: f holomorphic-on s and fss: f ' s  $\subseteq$  s and z  $\in$  s and [simp]: f z = z and n: i  $\leq$  n and dfz: deriv f z = 1 and zero:  $\land i. [[1 < i; i \leq n]] \implies (deriv \frown i) f z = 0$ shows (deriv  $\frown i$ ) (f $\frown m$ ) z = (deriv  $\frown i$ ) f z  $\langle proof \rangle$ lemma higher-deriv-iter-top-lemma: assumes s: open s and holf: f holomorphic-on s and fss: f ' s  $\subseteq$  s and z  $\in$  s and [simp]: f z = z and dfz [simp]: deriv f z = 1 and n: 1 < n  $\land i. [[1 < i; i < n]] \Longrightarrow (deriv \frown i) f z = 0$ 

shows  $(deriv \ \widehat{n}) (f \ \widehat{m}) z = m * (deriv \ \widehat{n}) f z$  $\langle proof \rangle$ 

Should be proved for n-dimensional vectors of complex numbers

```
theorem first-Cartan-dim-1:

assumes holf: f holomorphic-on s

and open s connected s bounded s

and fss: f ' s \subseteq s

and z \in s and [simp]: f z = z

and dfz [simp]: deriv f z = 1

and w \in s

shows f w = w

\langle proof \rangle
```

Second Cartan Theorem.

```
lemma Cartan-is-linear:

assumes holf: f holomorphic-on s

and open s and connected s

and 0 \in s

and ins: \bigwedge u z. [norm u = 1; z \in s] \Longrightarrow u * z \in s

and feq: \bigwedge u z. [norm u = 1; z \in s] \Longrightarrow f (u * z) = u * f z

shows \exists c. \forall z \in s. f z = c * z

\langle proof \rangle
```

Should be proved for n-dimensional vectors of complex numbers

**theorem** second-Cartan-dim-1: **assumes** holf: f holomorphic-on ball 0 r and holg: g holomorphic-on ball 0 r and [simp]:  $f \ 0 = 0$  and [simp]:  $g \ 0 = 0$ and ballf:  $\bigwedge z. \ z \in ball \ 0 \ r \implies f \ z \in ball \ 0 \ r$ and ballg:  $\bigwedge z. \ z \in ball \ 0 \ r \implies g \ z \in ball \ 0 \ r$ and fg:  $\bigwedge z. \ z \in ball \ 0 \ r \implies f \ (g \ z) = z$  and  $gf: \bigwedge z. \ z \in ball \ 0 \ r \Longrightarrow g \ (f \ z) = z$ and 0 < rshows  $\exists t. \forall z \in ball \ 0 \ r. \ g \ z = exp(i * of-real \ t) * z$  $\langle proof \rangle$ 

 $\mathbf{end}$ 

# Bibliography

[1] G. Ciolli, G. Gentili, and M. Maggesi. A certified proof of the Cartan fixed point theorems. J. Autom. Reason., 47(3):319–336, Oct. 2011.