The Cartan Fixed Point Theorems

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Abstract

The Cartan fixed point theorems concern the group of holomorphic automorphisms on a connected open set of \( \mathbb{C}^n \). Ciolli et al. [1] have formalised the one-dimensional case of these theorems in HOL Light. This entry contains their proofs, ported to Isabelle/HOL. Thus it addresses the authors remark that “it would be important to write a formal proof in a language that can be read by both humans and machines.”
0.1 First Cartan Theorem


lemma deriv-left-inverse:
  assumes \( f \) holomorphic-on \( S \) and \( g \) holomorphic-on \( T \)
  and \( S \) open and \( T \) open
  and \( f \circ S \subseteq T \)
  and \([\text{simp}]: \forall z. z \in S \implies g(fz) = z\)
  and \( w \in S \)
  shows \( \text{deriv } f \, w \ast \text{deriv } g \,(f\,w) = 1 \)

⟨proof⟩

lemma Cauchy-higher-deriv-bound:
  assumes \( holf: f \) holomorphic-on \( \text{ball } z \, r \)
  and \( \text{contf: continuous-on } \text{cball } z \, r \) \( f \)
  and \( 0 < r \) and \( 0 < n \)
  and \( \text{fin: } \forall w. w \in \text{ball } z \, r \implies f\,w \in \text{ball } y \, B0 \)
  shows \( \text{norm } ((\text{deriv } ^\cdot \in n) \, f\,z) \leq (\text{fact } n) \ast B0 / r \ast n \)

⟨proof⟩

lemma higher-deriv-comp-lemma:
  assumes \( s: \) open \( s \) and \( holf: f \) holomorphic-on \( s \)
  and \( z \in s \)
  and \( t: \) open \( t \) and \( holg: g \) holomorphic-on \( t \)
  and \( \text{fst: } f \circ s \subseteq t \)
  and \( n: i \leq n \)
  and \( \text{dfz: } \text{deriv } f\,z = 1 \) and \( \text{zero: } \forall i. [1 < i; i \leq n] \implies (\text{deriv } ^\cdot \in i) \, f\,z = 0 \)
  shows \( (\text{deriv } ^\cdot \in i) \,(g \circ f)\,z = (\text{deriv } ^\cdot \in i) \, g\,(f\,z) \)
\textbf{lemma} higher-deriv-comp-iter-lemma:
\begin{itemize}
  \item \textbf{assumes} \(s: \text{open } s\) and \(holf: f \text{ holomorphic-on } s\)
  \item and \(fss: f' s \subseteq s\)
  \item and \(z \in s\) and \([\text{simp}]: f z = z\)
  \item and \(n: i \leq n\)
  \item and \(dfz: \text{deriv } f z = 1\) and \(\text{zero}: \{1 < i; i \leq n\} \implies (\text{deriv } ^{'^i} i) f z = 0\)
\end{itemize}
\textbf{shows} \((\text{deriv } ^{'^i} i) (f ^{'^m}) z = (\text{deriv } ^{'^i} i) f z\)
\textbf{proof}

\textbf{lemma} higher-deriv-iter-top-lemma:
\begin{itemize}
  \item \textbf{assumes} \(s: \text{open } s\) and \(holf: f \text{ holomorphic-on } s\)
  \item and \(fss: f' s \subseteq s\)
  \item and \(z \in s\) and \([\text{simp}]: f z = z\)
  \item and \(dfz: \text{deriv } f z = 1\)
  \item and \(w \in s\)
\end{itemize}
\textbf{shows} \((\text{deriv } ^{'^i} n) (f ^{'^m}) z = m \ast (\text{deriv } ^{'^n} i) f z\)
\textbf{proof}

\textbf{Theorem} first-Cartan-dim-1:
\begin{itemize}
  \item \textbf{assumes} \(holf: f \text{ holomorphic-on } s\)
  \item and \(open s\) connected \(s\) bounded \(s\)
  \item and \(fss: f' s \subseteq s\)
  \item and \(z \in s\) and \([\text{simp}]: f z = z\)
  \item and \(dfz: \text{deriv } f z = 1\)
  \item and \(w \in s\)
\end{itemize}
\textbf{shows} \(f w = w\)
\textbf{proof}

Second Cartan Theorem.

\textbf{Lemma} Cartan-is-linear:
\begin{itemize}
  \item \textbf{assumes} \(holf: f \text{ holomorphic-on } s\)
  \item and \(open s\) and \(\text{connected } s\)
  \item and \(0 \in s\)
  \item and \(ins: \{u z. \text{norm } u = 1; z \in s\} \implies u \ast z \in s\)
  \item and \(fzg: \{u z. \text{norm } u = 1; z \in s\} \implies f (u \ast z) = u \ast f z\)
\end{itemize}
\textbf{shows} \(\exists c. \forall z \in s. f z = c \ast z\)
\textbf{proof}

Should be proved for n-dimensional vectors of complex numbers

\textbf{Theorem} second-Cartan-dim-1:
\begin{itemize}
  \item \textbf{assumes} \(holf: f \text{ holomorphic-on ball } 0 r\)
  \item and \(holg: g \text{ holomorphic-on ball } 0 r\)
  \item and \([\text{simp}]: f 0 = 0\) and \([\text{simp}]: g 0 = 0\)
  \item and \(ballf: \{z. z \in \text{ball } 0 r \implies f z \in \text{ball } 0 r\}\)
\end{itemize}
and ball: \( \forall z. \, z \in \text{ball} \ 0 \ r \implies g \ z \in \text{ball} \ 0 \ r \)
and fg: \( \forall z. \, z \in \text{ball} \ 0 \ r \implies f \ (g \ z) = z \)
and gf: \( \forall z. \, z \in \text{ball} \ 0 \ r \implies g \ (f \ z) = z \)
and \( 0 < r \)
shows \( \exists t. \, \forall z \in \text{ball} \ 0 \ r. \, g \ z = \exp(i \ \text{of-real} \ t) \ast z \)

(proof)

end
Bibliography