The Cartan Fixed Point Theorems

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December 14, 2021

Abstract

The Cartan fixed point theorems concern the group of holomorphic automorphisms on a connected open set of $\mathbb{C}^n$. Ciolli et al. [1] have formalised the one-dimensional case of these theorems in HOL Light. This entry contains their proofs, ported to Isabelle/HOL. Thus it addresses the authors remark that "it would be important to write a formal proof in a language that can be read by both humans and machines."

Contents

theory *Cartan*
imports *HOL−Complex-Analysis.Complex-Analysis*

begin

0.1 First Cartan Theorem

Ported from HOL Light. See Gianni Ciolli, Graziano Gentili, Marco Maggesi. A Certified Proof of the Cartan Fixed Point Theorems. J Automated Reasoning (2011) 47:319–336 DOI 10.1007/s10817-010-9198-6

lemma *deriv-left-inverse*:
  assumes *f holomorphic-on S* **and** *g holomorphic-on T*
    and *open S* **and** *open T*
    and f ‘ $S \subseteq T$
    and [*simp*]: $\bigwedge z.\ z \in S \Longrightarrow g\ (f\ z) = z$
    and $w \in S$
  shows *deriv f w* ∗ *deriv g* (*f w*) = *1*
⟨*proof*⟩

lemma *Cauchy-higher-deriv-bound*:
  assumes *holf*: *f holomorphic-on* (*ball z r*)
    and *contf*: *continuous-on* (*cball z r*) *f*
    and $0 < r$ **and** $0 < n$
    and *fin* : $\bigwedge w.\ w \in$ *ball z r* $\Longrightarrow$ *f w* $\in$ *ball y B0*
  shows *norm* ((*deriv* ^^ *n*) *f z*) ≤ (*fact n*) ∗ *B0* / *r*^*n*
⟨*proof*⟩

lemma *higher-deriv-comp-lemma*:
  assumes *s*: *open s* **and** *holf*: *f holomorphic-on s*
    and $z \in s$
    and *t*: *open t* **and** *holg*: *g holomorphic-on t*
    and *fst*: f ‘ $s \subseteq t$
    and *n*: $i \le n$
    and *dfz*: *deriv f z* = *1* **and** *zero*: $\bigwedge i.\ [\![1 < i;\ i \le n]\!] \Longrightarrow$ (*deriv* ^^ *i*) *f z* = *0*
  shows (*deriv* ^^ *i*) (*g o f*) *z* = (*deriv* ^^ *i*) *g* (*f z*)
⟨*proof*⟩

lemma *higher-deriv-comp-iter-lemma*:
  assumes *s*: *open s* **and** *holf*: *f holomorphic-on s*
    and *fss*: *f ‘ s* $\subseteq$ *s*
    and $z \in s$ **and** [*simp*]: $f\ z = z$
    and *n*: $i \leq n$
    and *dfz*: *deriv f z* = *1* **and** *zero*: $\bigwedge i.\ [\![1 < i;\ i \leq n]\!] \Longrightarrow (deriv\ \widehat{}\widehat{}\ i)\ f\ z = 0$
  shows $(deriv\ \widehat{}\widehat{}\ i)\ (f\widehat{}\widehat{}m)\ z = (deriv\ \widehat{}\widehat{}\ i)\ f\ z$
⟨*proof*⟩

lemma *higher-deriv-iter-top-lemma*:
  assumes *s*: *open s* **and** *holf*: *f holomorphic-on s*
    and *fss*: *f ‘ s* $\subseteq$ *s*
    and $z \in s$ **and** [*simp*]: $f\ z = z$
    and *dfz* [*simp*]: *deriv f z* = *1*
    and *n*: $1 < n$ $\bigwedge i.\ [\![1 < i;\ i < n]\!] \Longrightarrow (deriv\ \widehat{}\widehat{}\ i)\ f\ z = 0$
  shows $(deriv\ \widehat{}\widehat{}\ n)\ (f\ \widehat{}\widehat{}\ m)\ z = m * (deriv\ \widehat{}\widehat{}\ n)\ f\ z$
⟨*proof*⟩

Should be proved for n-dimensional vectors of complex numbers

theorem *first-Cartan-dim-1*:
  assumes *holf*: *f holomorphic-on s*
    and *open s connected s bounded s*
    and *fss*: *f ‘ s* $\subseteq$ *s*
    and $z \in s$ **and** [*simp*]: $f\ z = z$
    and *dfz* [*simp*]: *deriv f z* = *1*
    and $w \in s$
  shows $f\ w = w$
⟨*proof*⟩

Second Cartan Theorem.

lemma *Cartan-is-linear*:
  assumes *holf*: *f holomorphic-on s*
    and *open s* **and** *connected s*
    and $0 \in s$
    and *ins*: $\bigwedge u\ z.\ [\![norm\ u = 1;\ z \in s]\!] \Longrightarrow u * z \in s$
    and *feq*: $\bigwedge u\ z.\ [\![norm\ u = 1;\ z \in s]\!] \Longrightarrow f\ (u * z) = u * f\ z$
  shows $\exists\, c.\ \forall\, z \in s.\ f\ z = c * z$
⟨*proof*⟩

Should be proved for n-dimensional vectors of complex numbers

theorem *second-Cartan-dim-1*:
  assumes *holf*: *f holomorphic-on ball 0 r*
    and *holg*: *g holomorphic-on ball 0 r*
    and [*simp*]: $f\ 0 = 0$ **and** [*simp*]: $g\ 0 = 0$
    and *ballf*: $\bigwedge z.\ z \in ball\ 0\ r \Longrightarrow f\ z \in ball\ 0\ r$
    and *ballg*: $\bigwedge z.\ z \in ball\ 0\ r \Longrightarrow g\ z \in ball\ 0\ r$
    and *fg*: $\bigwedge z.\ z \in ball\ 0\ r \Longrightarrow f\ (g\ z) = z$

and *gf*: $\bigwedge z.\ z \in$ *ball 0 r* $\Longrightarrow g\ (f\ z) = z$
and $0 < r$
shows $\exists\, t.\ \forall\, z \in$ *ball 0 r*. $g\ z = exp(\mathrm{i} * of\text{-}real\ t) * z$
⟨*proof*⟩

end

Bibliography

[1] G. Ciolli, G. Gentili, and M. Maggesi. A certified proof of the Cartan fixed point theorems. *J. Autom. Reason.*, 47(3):319–336, Oct. 2011.