

Cardinality of Set Partitions

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Abstract

The theory's main theorem states that the cardinality of set partitions of size k on a carrier set of size n is expressed by Stirling numbers of the second kind. In Isabelle, Stirling numbers of the second kind are defined in the AFP entry 'Discrete Summation' [1] through their well-known recurrence relation. The main theorem relates them to the alternative definition as cardinality of set partitions. The proof follows the simple and short explanation in Richard P. Stanley's 'Enumerative Combinatorics: Volume 1' [2] and Wikipedia [3], and unravels the full details and implicit reasoning steps of these explanations.

Contents

1	Set Partitions	1
1.1	Useful Additions to Main Theories	2
1.2	Introduction and Elimination Rules	2
1.3	Basic Facts on Set Partitions	2
1.4	The Unique Part Containing an Element in a Set Partition	4
1.5	Cardinality of Parts in a Set Partition	5
1.6	Operations on Set Partitions	5
2	Combinatorial Basics	6
2.1	Preliminaries	6
2.1.1	Injectivity and Disjoint Families	6
2.1.2	Cardinality Theorems for Set.bind	6
2.2	Third Version of Injectivity Solver	7
3	Cardinality of Set Partitions	8

1 Set Partitions

```
theory Set-Partition
imports
  HOL-Library.Disjoint-Sets
```

HOL-Library.FuncSet
begin

1.1 Useful Additions to Main Theories

lemma *set-eqI'*:
 assumes $\bigwedge x. x \in A \implies x \in B$
 assumes $\bigwedge x. x \in B \implies x \in A$
 shows $A = B$
<proof>

lemma *comp-image*:
 $((\cdot) f \circ (\cdot) g) = (\cdot) (f \circ g)$
<proof>

1.2 Introduction and Elimination Rules

The definition of *partition-on* is in *HOL-Library.Disjoint-Sets*.

lemma *partition-onI*:
 assumes $\bigwedge p. p \in P \implies p \neq \{\}$
 assumes $\bigcup P = A$
 assumes $\bigwedge p p'. p \in P \implies p' \in P \implies p \neq p' \implies p \cap p' = \{\}$
 shows *partition-on* $A P$
<proof>

lemma *partition-onE*:
 assumes *partition-on* $A P$
 obtains $\bigwedge p. p \in P \implies p \neq \{\}$
 $\bigcup P = A$
 $\bigwedge p p'. p \in P \implies p' \in P \implies p \neq p' \implies p \cap p' = \{\}$
<proof>

1.3 Basic Facts on Set Partitions

lemma *partition-onD4*: *partition-on* $A P \implies p \in P \implies q \in P \implies x \in p \implies x \in q \implies p = q$
<proof>

lemma *partition-subset-imp-notin*:
 assumes *partition-on* $A P X \in P$
 assumes $X' \subset X$
 shows $X' \notin P$
<proof>

lemma *partition-on-Diff*:
 assumes P : *partition-on* $A P$ **shows** $Q \subseteq P \implies \text{partition-on } (A - \bigcup Q) (P - Q)$
<proof>

lemma *partition-on-UN*:
assumes A : *partition-on* A B **and** B : $\bigwedge b. b \in B \implies \text{partition-on } b (P \ b)$
shows *partition-on* A $(\bigcup_{b \in B. P \ b})$
 $\langle \text{proof} \rangle$

lemma *partition-on-notemptyI*:
assumes *partition-on* A P
assumes $A \neq \{\}$
shows $P \neq \{\}$
 $\langle \text{proof} \rangle$

lemma *partition-on-disjoint*:
assumes *partition-on* A P
assumes *partition-on* B Q
assumes $A \cap B = \{\}$
shows $P \cap Q = \{\}$
 $\langle \text{proof} \rangle$

lemma *partition-on-eq-implies-eq-carrier*:
assumes *partition-on* A Q
assumes *partition-on* B Q
shows $A = B$
 $\langle \text{proof} \rangle$

lemma *partition-on-insert*:
assumes *partition-on* A P
assumes *disjnt* A X $X \neq \{\}$
assumes $A \cup X = A'$
shows *partition-on* A' $(\text{insert } X \ P)$
 $\langle \text{proof} \rangle$

An alternative formulation of $\llbracket \text{partition-on } ?A \ ?P; \text{disjnt } ?A \ ?X; ?X \neq \{\}; ?A \cup ?X = ?A \rrbracket \implies \text{partition-on } ?A' (\text{insert } ?X \ ?P)$

lemma *partition-on-insert'*:
assumes *partition-on* $(A - X)$ P
assumes $X \subseteq A$ $X \neq \{\}$
shows *partition-on* A $(\text{insert } X \ P)$
 $\langle \text{proof} \rangle$

lemma *partition-on-insert-singleton*:
assumes *partition-on* A P $a \notin A$ $\text{insert } a \ A = A'$
shows *partition-on* A' $(\text{insert } \{a\} \ P)$
 $\langle \text{proof} \rangle$

lemma *partition-on-remove-singleton*:
assumes *partition-on* A P $X \in P$ $A - X = A'$
shows *partition-on* A' $(P - \{X\})$
 $\langle \text{proof} \rangle$

1.4 The Unique Part Containing an Element in a Set Partition

lemma *partition-on-partition-on-unique:*
assumes *partition-on A P*
assumes $x \in A$
shows $\exists!X. x \in X \wedge X \in P$
<proof>

lemma *partition-on-the-part-mem:*
assumes *partition-on A P*
assumes $x \in A$
shows $(THE X. x \in X \wedge X \in P) \in P$
<proof>

lemma *partition-on-in-the-unique-part:*
assumes *partition-on A P*
assumes $x \in A$
shows $x \in (THE X. x \in X \wedge X \in P)$
<proof>

lemma *partition-on-the-part-eq:*
assumes *partition-on A P*
assumes $x \in X \wedge X \in P$
shows $(THE X. x \in X \wedge X \in P) = X$
<proof>

lemma *the-unique-part-alternative-def:*
assumes *partition-on A P*
assumes $x \in A$
shows $(THE X. x \in X \wedge X \in P) = \{y. \exists X \in P. x \in X \wedge y \in X\}$
<proof>

lemma *partition-on-all-in-part-eq-part:*
assumes *partition-on A P*
assumes $X' \in P$
shows $\{x \in A. (THE X. x \in X \wedge X \in P) = X'\} = X'$
<proof>

lemma *partition-on-part-characteristic:*
assumes *partition-on A P*
assumes $X \in P \wedge x \in X$
shows $X = \{y. \exists X \in P. x \in X \wedge y \in X\}$
<proof>

lemma *partition-on-no-partition-outside-carrier:*
assumes *partition-on A P*
assumes $x \notin A$
shows $\{y. \exists X \in P. x \in X \wedge y \in X\} = \{\}$

<proof>

1.5 Cardinality of Parts in a Set Partition

lemma *partition-on-le-set-elements:*

assumes *finite A*

assumes *partition-on A P*

shows $\text{card } P \leq \text{card } A$

<proof>

1.6 Operations on Set Partitions

lemma *partition-on-union:*

assumes $A \cap B = \{\}$

assumes *partition-on A P*

assumes *partition-on B Q*

shows *partition-on (A \cup B) (P \cup Q)*

<proof>

lemma *partition-on-split1:*

assumes *partition-on A (P \cup Q)*

shows *partition-on (\bigcup P) P*

<proof>

lemma *partition-on-split2:*

assumes *partition-on A (P \cup Q)*

shows *partition-on (\bigcup Q) Q*

<proof>

lemma *partition-on-intersect-on-elements:*

assumes *partition-on (A \cup C) P*

assumes $\forall X \in P. \exists x. x \in X \cap C$

shows *partition-on C (($\lambda X. X \cap C$) ' P)*

<proof>

lemma *partition-on-insert-elements:*

assumes $A \cap B = \{\}$

assumes *partition-on B P*

assumes $f \in A \rightarrow_E P$

shows *partition-on (A \cup B) (($\lambda X. X \cup \{x \in A. f x = X\}$) ' P) (is partition-on - ?P)*

<proof>

lemma *partition-on-map:*

assumes *inj-on f A*

assumes *partition-on A P*

shows *partition-on (f ' A) ((\cdot) ' P)*

<proof>

lemma *set-of-partition-on-map:*

```

assumes inj-on f A
shows  $(\cdot) ((\cdot) f) \cdot \{P. \text{partition-on } A P\} = \{P. \text{partition-on } (f \cdot A) P\}$ 
<proof>

end

```

2 Combinatorial Basics

```

theory Injectivity-Solver
imports
  HOL-Library.Disjoint-Sets
  HOL-Library.Monad-Syntax
  HOL-Eisbach.Eisbach
begin

```

2.1 Preliminaries

These lemmas shall be added to the Disjoint Set theory.

2.1.1 Injectivity and Disjoint Families

```

lemma inj-on-impl-disjoint-family-on-singleton:
assumes inj-on f A
shows disjoint-family-on  $(\lambda x. \{f x\}) A$ 
<proof>

```

2.1.2 Cardinality Theorems for Set.bind

```

lemma card-bind:
assumes finite S
assumes  $\forall X \in S. \text{finite } (f X)$ 
assumes disjoint-family-on f S
shows  $\text{card } (S \gg= f) = (\sum x \in S. \text{card } (f x))$ 
<proof>

```

```

lemma card-bind-constant:
assumes finite S
assumes  $\forall X \in S. \text{finite } (f X)$ 
assumes disjoint-family-on f S
assumes  $\bigwedge x. x \in S \implies \text{card } (f x) = k$ 
shows  $\text{card } (S \gg= f) = \text{card } S * k$ 
<proof>

```

```

lemma card-bind-singleton:
assumes finite S
assumes inj-on f S
shows  $\text{card } (S \gg= (\lambda x. \{f x\})) = \text{card } S$ 
<proof>

```

2.2 Third Version of Injectivity Solver

Here, we provide a third version of the injectivity solver. The original first version was provided in the AFP entry ‘Spivey’s Generalized Recurrence for Bell Numbers’. From that method, I derived a second version in the AFP entry ‘Cardinality of Equivalence Relations’. At roughly the same time, Makarius improved the injectivity solver in the development version of the first AFP entry. This third version now includes the improvements of the second version and Makarius improvements to the first, and it further extends the method to handle the new cases in the cardinality proof of this AFP entry.

As the implementation of the injectivity solver only evolves in the development branch of the AFP, the submissions of the three AFP entries that employ the injectivity solver, have to create clones of the injectivity solver for the identified and needed method adjustments. Ultimately, these three clones should only remain in the stable branches of the AFP from Isabelle2016 to Isabelle2017 to work with their corresponding release versions. In the development version, I have now consolidated the three versions here. In the next step, I will move this version of the injectivity solver in the *HOL–Library.Disjoint-Sets* and it will hopefully only evolve further there.

lemma *disjoint-family-onI*:

assumes $\bigwedge i j. i \in I \wedge j \in I \implies i \neq j \implies (A\ i) \cap (A\ j) = \{\}$

shows *disjoint-family-on A I*

<proof>

lemma *disjoint-bind*: $\bigwedge S\ T\ f\ g. (\bigwedge s\ t. S\ s \wedge T\ t \implies f\ s \cap g\ t = \{\}) \implies (\{s. S\ s\} \ggg f) \cap (\{t. T\ t\} \ggg g) = \{\}$

<proof>

lemma *disjoint-bind'*: $\bigwedge S\ T\ f\ g. (\bigwedge s\ t. s \in S \wedge t \in T \implies f\ s \cap g\ t = \{\}) \implies (S \ggg f) \cap (T \ggg g) = \{\}$

<proof>

lemma *injectivity-solver-CollectE*:

assumes $a \in \{x. P\ x\} \wedge a' \in \{x. P'\ x\}$

assumes $(P\ a \wedge P'\ a') \implies W$

shows W

<proof>

lemma *injectivity-solver-prep-assms-Collect*:

assumes $x \in \{x. P\ x\}$

shows $P\ x \wedge P\ x$

<proof>

lemma *injectivity-solver-prep-assms*: $x \in A \implies x \in A \wedge x \in A$

<proof>

lemma *disjoint-terminal-singleton*: $\bigwedge s t X Y. s \neq t \implies (X = Y \implies s = t) \implies \{X\} \cap \{Y\} = \{\}$
 <proof>

lemma *disjoint-terminal-Collect*:

assumes $s \neq t$
assumes $\bigwedge x x'. S x \wedge T x' \implies x = x' \implies s = t$
shows $\{x. S x\} \cap \{x. T x\} = \{\}$
 <proof>

lemma *disjoint-terminal*:

$s \neq t \implies (\bigwedge x x'. x \in S \wedge x' \in T \implies x = x' \implies s = t) \implies S \cap T = \{\}$
 <proof>

lemma *elim-singleton*:

assumes $x \in \{s\} \wedge x' \in \{t\}$
obtains $x = s \wedge x' = t$
 <proof>

method *injectivity-solver* **uses** *rule* =

insert method-facts,
use nothing in <
 ((*drule injectivity-solver-prep-assms-Collect* | *drule injectivity-solver-prep-assms*)+)?;
rule disjoint-family-onI;
 ((*rule disjoint-bind* | *rule disjoint-bind'*)+)?;
 (*erule elim-singleton*)?;
 (*erule disjoint-terminal-singleton* | *erule disjoint-terminal-Collect* | *erule disjoint-terminal*);
 (*elim injectivity-solver-CollectE*)?;
rule rule;
assumption+
 >

end

3 Cardinality of Set Partitions

theory *Card-Partitions*

imports

HOL-Combinatorics.Stirling
Set-Partition
Injectivity-Solver

begin

lemma *set-partition-on-insert-with-fixed-card-eq*:

assumes *finite A*
assumes $a \notin A$
shows $\{P. \text{partition-on } (\text{insert } a \ A) \ P \wedge \text{card } P = \text{Suc } k\} = (\text{do } \{$
 $P \leftarrow \{P. \text{partition-on } A \ P \wedge \text{card } P = \text{Suc } k\};$

$p <- P;$
 $\{insert (insert a p) (P - \{p\})\}$
 $\}$
 $\cup (do \{$
 $P <- \{P. partition-on A P \wedge card P = k\};$
 $\{insert \{a\} P\}$
 $\}) (is ?S = ?T)$
 $\langle proof \rangle$

lemma *injectivity-subexpr1:*

assumes $a \notin A$
assumes $X \in P \wedge X' \in P'$
assumes $insert (insert a X) (P - \{X\}) = insert (insert a X') (P' - \{X'\})$
assumes $(partition-on A P \wedge card P = Suc k') \wedge (partition-on A P' \wedge card P' = Suc k')$
shows $P = P'$ **and** $X = X'$
 $\langle proof \rangle$

lemma *injectivity-subexpr2:*

assumes $a \notin A$
assumes $insert \{a\} P = insert \{a\} P'$
assumes $(partition-on A P \wedge card P = k') \wedge (partition-on A P' \wedge card P' = k')$
shows $P = P'$
 $\langle proof \rangle$

theorem *card-partition-on:*

assumes *finite* A
shows $card \{P. partition-on A P \wedge card P = k\} = Stirling (card A) k$
 $\langle proof \rangle$

theorem *card-partition-on-at-most-size:*

assumes *finite* A
shows $card \{P. partition-on A P \wedge card P \leq k\} = (\sum j \leq k. Stirling (card A) j)$
 $\langle proof \rangle$

theorem *partition-on-size1:*

assumes *finite* A
shows $\{P. partition-on A P \wedge (\forall X \in P. card X = 1)\} = \{(\lambda a. \{a\}) ' A\}$
 $\langle proof \rangle$

theorem *card-partition-on-size1:*

assumes *finite* A
shows $card \{P. partition-on A P \wedge (\forall X \in P. card X = 1)\} = 1$
 $\langle proof \rangle$

lemma *card-partition-on-size1-eq-1:*

assumes *finite* A
assumes $card A \leq k$
shows $card \{P. partition-on A P \wedge card P \leq k \wedge (\forall X \in P. card X = 1)\} = 1$

<proof>

lemma *card-partition-on-size1-eq-0:*

assumes *finite A*

assumes $k < \text{card } A$

shows $\text{card } \{P. \text{partition-on } A \ P \wedge \text{card } P \leq k \wedge (\forall X \in P. \text{card } X = 1)\} = 0$
<proof>

end

References

- [1] F. Haftmann. Discrete summation. *Archive of Formal Proofs*, Apr. 2014. http://isa-afp.org/entries/Discrete_Summation.shtml, Formal proof development.
- [2] R. P. Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge University Press, second edition, 2012.
- [3] Wikipedia. Stirling numbers of the second kind — wikipedia, the free encyclopedia, 2015. https://en.wikipedia.org/w/index.php?title=Stirling_numbers_of_the_second_kind&oldid=693800357, [Online; accessed 12-December-2015].