Cardinality of Set Partitions

Lukas Bulwahn

March 17, 2025

Abstract

The theory's main theorem states that the cardinality of set partitions of size k on a carrier set of size n is expressed by Stirling numbers of the second kind. In Isabelle, Stirling numbers of the second kind are defined in the AFP entry 'Discrete Summation' [1] through their well-known recurrence relation. The main theorem relates them to the alternative definition as cardinality of set partitions. The proof follows the simple and short explanation in Richard P. Stanley's 'Enumerative Combinatorics: Volume 1' [2] and Wikipedia [3], and unravels the full details and implicit reasoning steps of these explanations.

Contents

1 Set Partitions

theory *Set-Partition*
imports
HOL−Library.Disjoint-Sets

HOL−Library.FuncSet
begin

1.1 Useful Additions to Main Theories

lemma *set-eqI′*:
assumes $\bigwedge x.\ x \in A \Longrightarrow x \in B$
assumes $\bigwedge x.\ x \in B \Longrightarrow x \in A$
shows $A = B$
using *assms* **by** *auto*

lemma *comp-image*:
$((`)\ f \circ (`)\ g) = (`)\ (f\ o\ g)$
by *rule auto*

1.2 Introduction and Elimination Rules

The definition of *partition-on* is in *HOL−Library.Disjoint-Sets*.

lemma *partition-onI*:
assumes $\bigwedge p.\ p \in P \Longrightarrow p \neq \{\}$
assumes $\bigcup P = A$
assumes $\bigwedge p\ p'.\ p \in P \Longrightarrow p' \in P \Longrightarrow p \neq p' \Longrightarrow p \cap p' = \{\}$
shows *partition-on A P*
using *assms* **unfolding** *partition-on-def disjoint-def* **by** *blast*

lemma *partition-onE*:
assumes *partition-on A P*
obtains $\bigwedge p.\ p \in P \Longrightarrow p \neq \{\}$
$\bigcup P = A$
$\bigwedge p\ p'.\ p \in P \Longrightarrow p' \in P \Longrightarrow p \neq p' \Longrightarrow p \cap p' = \{\}$
using *assms* **unfolding** *partition-on-def disjoint-def* **by** *blast*

1.3 Basic Facts on Set Partitions

lemma *partition-onD4*: *partition-on A P* $\Longrightarrow p \in P \Longrightarrow q \in P \Longrightarrow x \in p \Longrightarrow x \in q \Longrightarrow p = q$
by (*auto simp*: *partition-on-def disjoint-def*)

lemma *partition-subset-imp-notin*:
assumes *partition-on A P X* $\in P$
assumes $X' \subset X$
shows $X' \notin P$
proof
assume $X' \in P$
from ‹$X' \in P$› ‹*partition-on A P*› **have** $X' \neq \{\}$
using *partition-onD3* **by** *blast*
moreover from ‹$X' \in P$› ‹$X \in P$› ‹*partition-on A P*› ‹$X' \subset X$› **have** *disjnt X X′*
by (*metis disjnt-def disjointD inf.strict-order-iff partition-onD2*)

moreover note ‹$X' \subset X$›
ultimately show *False*
by (*meson all-not-in-conv disjnt-iff psubsetD*)
qed

lemma *partition-on-Diff*:
assumes *P*: *partition-on A P* **shows** $Q \subseteq P \Longrightarrow$ *partition-on* $(A - \bigcup Q)$ $(P - Q)$
using *P P*[*THEN partition-onD4*] **by** (*auto simp*: *partition-on-def disjoint-def*)

lemma *partition-on-UN*:
assumes *A*: *partition-on A B* **and** *B*: $\bigwedge b.\ b \in B \Longrightarrow$ *partition-on b* $(P\ b)$
shows *partition-on A* $(\bigcup b{\in}B.\ P\ b)$
proof (*rule partition-onI*)
show $\bigcup(\bigcup b{\in}B.\ P\ b) = A$
using *B*[*THEN partition-onD1*] *A*[*THEN partition-onD1*] **by** *blast*
next
show $p \neq \{\}$ **if** $p \in (\bigcup b{\in}B.\ P\ b)$ **for** *p*
using *B*[*THEN partition-onD3*] *that* **by** *auto*
next
fix *p q* **assume** $p \in (\bigcup i{\in}B.\ P\ i)$ $q \in (\bigcup i{\in}B.\ P\ i)$ **and** $p \neq q$
then obtain *i j* **where** *i*: $p \in P\ i\ i \in B$ **and** *j*: $q \in P\ j\ j \in B$
by *auto*
show $p \cap q = \{\}$
proof *cases*
assume $i = j$ **then show** *?thesis*
using *i j* ‹$p \neq q$› *B*[*THEN partition-onD2*, *of i*] **by** (*simp add*: *disjointD*)
next
assume $i \neq j$
then have *disjnt i j*
using *i j A*[*THEN partition-onD2*] **by** (*auto simp*: *pairwise-def*)
moreover have $p \subseteq i$ $q \subseteq j$
using *B*[*THEN partition-onD1*, *of i*, *symmetric*] *B*[*THEN partition-onD1*, *of j*, *symmetric*] *i j* **by** *auto*
ultimately show *?thesis*
by (*auto simp*: *disjnt-def*)
qed
qed

lemma *partition-on-nonemptyI*:
assumes *partition-on A P*
assumes $A \neq \{\}$
shows $P \neq \{\}$
using *assms* **by** (*auto elim*: *partition-onE*)

lemma *partition-on-disjoint*:
assumes *partition-on A P*
assumes *partition-on B Q*
assumes $A \cap B = \{\}$

shows $P \cap Q = \{\}$
using *assms* **by** (*fastforce elim*: *partition-onE*)

lemma *partition-on-eq-implies-eq-carrier*:
assumes *partition-on A Q*
assumes *partition-on B Q*
shows $A = B$
using *assms* **by** (*fastforce elim*: *partition-onE*)

lemma *partition-on-insert*:
assumes *partition-on A P*
assumes *disjnt A X X* $\neq \{\}$
assumes $A \cup X = A'$
shows *partition-on A'* (*insert X P*)
using *assms* **by** (*auto simp*: *partition-on-def disjoint-def disjnt-def*)

An alternative formulation of ⟦*partition-on ?A ?P*; *disjnt ?A ?X*; *?X* $\neq \{\}$; *?A* $\cup$ *?X* = *?A'*⟧ $\Longrightarrow$ *partition-on ?A'* (*insert ?X ?P*)

lemma *partition-on-insert'*:
assumes *partition-on* $(A - X)$ *P*
assumes $X \subseteq A$ *X* $\neq \{\}$
shows *partition-on A* (*insert X P*)
proof −
have *disjnt* $(A - X)$ *X* **by** (*simp add*: *disjnt-iff*)
from *assms*(*1*) *this assms*(*3*) **have** *partition-on* $((A - X) \cup X)$ (*insert X P*)
by (*auto intro*: *partition-on-insert*)
from *this* ‹$X \subseteq A$› **show** *?thesis*
by (*metis Diff-partition sup-commute*)
qed

lemma *partition-on-insert-singleton*:
assumes *partition-on A P a* $\notin$ *A insert a A* = *A'*
shows *partition-on A'* (*insert* $\{a\}$ *P*)
using *assms* **by** (*auto simp*: *partition-on-def disjoint-def disjnt-def*)

lemma *partition-on-remove-singleton*:
assumes *partition-on A P X* $\in$ *P A* − *X* = *A'*
shows *partition-on A'* $(P - \{X\})$
using *assms partition-on-Diff* **by** (*metis Diff-cancel Diff-subset cSup-singleton insert-subset*)

1.4 The Unique Part Containing an Element in a Set Partition

lemma *partition-on-partition-on-unique*:
assumes *partition-on A P*
assumes $x \in A$
shows $\exists!X.\ x \in X \wedge X \in P$
proof −

```
  from ‹partition-on A P› have ⋃P = A
    by (auto elim: partition-onE)
  from this ‹x ∈ A› obtain X where X: x ∈ X ∧ X ∈ P by blast
  {
    fix Y
    assume x ∈ Y ∧ Y ∈ P
    from this have X = Y
      using X ‹partition-on A P› by (meson partition-onE disjoint-iff-not-equal)
  }
  from this X show ?thesis by auto
qed

lemma partition-on-the-part-mem:
  assumes partition-on A P
  assumes x ∈ A
  shows (THE X. x ∈ X ∧ X ∈ P) ∈ P
proof −
  from ‹x ∈ A› have ∃!X. x ∈ X ∧ X ∈ P
    using ‹partition-on A P› by (simp add: partition-on-partition-on-unique)
  from this show (THE X. x ∈ X ∧ X ∈ P) ∈ P
    by (metis (no-types, lifting) theI)
qed

lemma partition-on-in-the-unique-part:
  assumes partition-on A P
  assumes x ∈ A
  shows x ∈ (THE X. x ∈ X ∧ X ∈ P)
proof −
  from assms have ∃!X. x ∈ X ∧ X ∈ P
    by (simp add: partition-on-partition-on-unique)
  from this show ?thesis
    by (metis (mono-tags, lifting) theI')
qed

lemma partition-on-the-part-eq:
  assumes partition-on A P
  assumes x ∈ X X ∈ P
  shows (THE X. x ∈ X ∧ X ∈ P) = X
proof −
  from ‹x ∈ X› ‹X ∈ P› have x ∈ A
    using ‹partition-on A P› by (auto elim: partition-onE)
  from this have ∃!X. x ∈ X ∧ X ∈ P
    using ‹partition-on A P› by (simp add: partition-on-partition-on-unique)
  from ‹x ∈ X› ‹X ∈ P› this show (THE X. x ∈ X ∧ X ∈ P) = X
    by (auto intro!: the1-equality)
qed


lemma the-unique-part-alternative-def:
```

assumes *partition-on A P*
assumes $x \in A$
shows $(THE\ X.\ x \in X \wedge X \in P) = \{y.\ \exists X{\in}P.\ x \in X \wedge y \in X\}$
proof
show $(THE\ X.\ x \in X \wedge X \in P) \subseteq \{y.\ \exists X{\in}P.\ x \in X \wedge y \in X\}$
proof
fix y
assume $y \in (THE\ X.\ x \in X \wedge X \in P)$
moreover from ‹$x \in A$› **have** $x \in (THE\ X.\ x \in X \wedge X \in P)$
using ‹*partition-on A P*› *partition-on-in-the-unique-part* **by** *force*
moreover from ‹$x \in A$› **have** $(THE\ X.\ x \in X \wedge X \in P) \in P$
using ‹*partition-on A P*› *partition-on-the-part-mem* **by** *force*
ultimately show $y \in \{y.\ \exists X{\in}P.\ x \in X \wedge y \in X\}$ **by** *auto*
qed
next
show $\{y.\ \exists X{\in}P.\ x \in X \wedge y \in X\} \subseteq (THE\ X.\ x \in X \wedge X \in P)$
proof
fix y
assume $y \in \{y.\ \exists X{\in}P.\ x \in X \wedge y \in X\}$
from *this* **obtain** X **where** $x \in X$ **and** $y \in X$ **and** $X \in P$ **by** *auto*
from ‹$x \in X$› ‹$X \in P$› **have** $(THE\ X.\ x \in X \wedge X \in P) = X$
using ‹*partition-on A P*› *partition-on-the-part-eq* **by** *force*
from *this* ‹$y \in X$› **show** $y \in (THE\ X.\ x \in X \wedge X \in P)$ **by** *simp*
qed
qed

lemma *partition-on-all-in-part-eq-part*:
assumes *partition-on A P*
assumes $X' \in P$
shows $\{x \in A.\ (THE\ X.\ x \in X \wedge X \in P) = X'\} = X'$
proof
show $\{x \in A.\ (THE\ X.\ x \in X \wedge X \in P) = X'\} \subseteq X'$
using *assms(1) partition-on-in-the-unique-part* **by** *force*
next
show $X' \subseteq \{x \in A.\ (THE\ X.\ x \in X \wedge X \in P) = X'\}$
proof
fix x
assume $x \in X'$
from ‹$x \in X'$› ‹$X' \in P$› **have** $x \in A$
using ‹*partition-on A P*› **by** (*auto elim*: *partition-onE*)
moreover from ‹$x \in X'$› ‹$X' \in P$› **have** $(THE\ X.\ x \in X \wedge X \in P) = X'$
using ‹*partition-on A P*› *partition-on-the-part-eq* **by** *fastforce*
ultimately show $x \in \{x \in A.\ (THE\ X.\ x \in X \wedge X \in P) = X'\}$ **by** *auto*
qed
qed

lemma *partition-on-part-characteristic*:
assumes *partition-on A P*
assumes $X \in P\ x \in X$

shows $X = \{y.\ \exists X{\in}P.\ x \in X \wedge y \in X\}$
proof −
from ‹$x \in X$› ‹$X \in P$› **have** $x \in A$
using ‹*partition-on A P*› *partition-onE* **by** *blast*
from ‹$x \in X$› ‹$X \in P$› **have** $X = (THE\ X.\ x \in X \wedge X \in P)$
using ‹*partition-on A P*› **by** (*simp add*: *partition-on-the-part-eq*)
also from ‹$x \in A$› **have** $(THE\ X.\ x \in X \wedge X \in P) = \{y.\ \exists X{\in}P.\ x \in X \wedge y \in X\}$
using ‹*partition-on A P*› *the-unique-part-alternative-def* **by** *force*
finally show *?thesis* **.**
qed

lemma *partition-on-no-partition-outside-carrier*:
assumes *partition-on A P*
assumes $x \notin A$
shows $\{y.\ \exists X{\in}P.\ x \in X \wedge y \in X\} = \{\}$
using *assms* **unfolding** *partition-on-def* **by** *auto*

1.5 Cardinality of Parts in a Set Partition

lemma *partition-on-le-set-elements*:
assumes *finite A*
assumes *partition-on A P*
shows *card P* $\leq$ *card A*
using *assms*
proof (*induct A arbitrary*: *P*)
case *empty*
from *this* **show** *card P* $\leq$ *card* $\{\}$ **by** (*simp add*: *partition-on-empty*)
next
case (*insert a A*)
show *?case*
proof (*cases* $\{a\} \in P$)
case *True*
have *prop-partition-on*: $\forall\, p{\in}P.\ p \neq \{\} \bigcup P = insert\ a\ A$
$\forall\, p{\in}P.\ \forall\, p'{\in}P.\ p \neq p' \longrightarrow p \cap p' = \{\}$
using ‹*partition-on* (*insert a A*) *P*› **by** (*fastforce elim*: *partition-onE*)+
from *this*(*2*, *3*) ‹$a \notin A$› ‹$\{a\} \in P$› **have** *A-eq*: $A = \bigcup (P - \{\{a\}\})$
by *auto* (*metis Int-iff UnionI empty-iff insert-iff*)
from *prop-partition-on A-eq* **have** *partition-on*: *partition-on A* $(P - \{\{a\}\})$
by (*intro partition-onI*) *auto*
from *insert.hyps*(*3*) *this* **have** *card* $(P - \{\{a\}\}) \leq$ *card A* **by** *simp*
from *this insert*(*1*, *2*, *4*) ‹$\{a\} \in P$› **show** *?thesis*
using *finite-elements*[*OF* ‹*finite A*› *partition-on*] **by** *simp*
next
case *False*
from ‹*partition-on* (*insert a A*) *P*› **obtain** *p* **where** *p-def*: $p \in P\ a \in p$
by (*blast elim*: *partition-onE*)
from ‹*partition-on* (*insert a A*) *P*› *p-def* **have** *a-notmem*: $\forall\, p'{\in}\ P - \{p\}.\ a \notin p'$

by (*blast elim*: *partition-onE*)
from ‹*partition-on* (*insert a A*) *P*› *p-def* **have** $p - \{a\} \notin P$
unfolding *partition-on-def disjoint-def*
by (*metis Diff-insert-absorb Diff-subset inf.orderE mk-disjoint-insert*)
let *?P'* = *insert* $(p - \{a\})$ $(P - \{p\})$
have *partition-on A ?P'*
proof (*rule partition-onI*)
from ‹*partition-on* (*insert a A*) *P*› **have** $\forall p{\in}P.\ p \neq \{\}$ **by** (*auto elim*: *partition-onE*)
from *this p-def* ‹$\{a\} \notin P$› **show** $\bigwedge p'.\ p'{\in}$*insert* $(p - \{a\})$ $(P - \{p\}) \Longrightarrow p' \neq \{\}$
by (*simp*; *metis* (*no-types*) *Diff-eq-empty-iff subset-singletonD*)
next
from ‹*partition-on* (*insert a A*) *P*› **have** $\bigcup P$ = *insert a A* **by** (*auto elim*: *partition-onE*)
from *p-def this* ‹$a \notin A$› *a-notmem* **show** $\bigcup$(*insert* $(p - \{a\})$ $(P - \{p\}))$ = *A* **by** *auto*
next
show $\bigwedge pa\ pa'.\ pa{\in}$*insert* $(p - \{a\})$ $(P - \{p\}) \Longrightarrow pa'{\in}$*insert* $(p - \{a\})$ $(P - \{p\}) \Longrightarrow pa \neq pa' \Longrightarrow pa \cap pa' = \{\}$
using ‹*partition-on* (*insert a A*) *P*› *p-def a-notmem*
unfolding *partition-on-def disjoint-def*
by (*metis disjoint-iff-not-equal insert-Diff insert-iff*)
qed
have *finite P* **using** ‹*finite A*› ‹*partition-on A ?P'*› *finite-elements* **by** *fastforce*
have *card P* = *Suc* (*card* $(P - \{p\})$)
using *p-def* ‹*finite P*› *card.remove* **by** *fastforce*
also have ... = *card ?P'* **using** ‹$p - \{a\} \notin P$› ‹*finite P*› **by** *simp*
also have ... $\leq$ *card A* **using** ‹*partition-on A ?P'*› *insert.hyps*(*3*) **by** *simp*
also have ... $\leq$ *card* (*insert a A*) **by** (*simp add*: *card-insert-le* ‹*finite A*›)
finally show *?thesis* **.**
qed
qed

1.6 Operations on Set Partitions

lemma *partition-on-union*:
assumes $A \cap B = \{\}$
assumes *partition-on A P*
assumes *partition-on B Q*
shows *partition-on* $(A \cup B)$ $(P \cup Q)$
proof (*rule partition-onI*)
fix *X*
assume $X \in P \cup Q$
from *this* ‹*partition-on A P*› ‹*partition-on B Q*› **show** $X \neq \{\}$
by (*auto elim*: *partition-onE*)
next
show $\bigcup(P \cup Q) = A \cup B$
using ‹*partition-on A P*› ‹*partition-on B Q*› **by** (*auto elim*: *partition-onE*)

next
fix X Y
assume $X \in P \cup Q$ $Y \in P \cup Q$ $X \neq Y$
from *this assms* **show** $X \cap Y = \{\}$
by (*elim UnE partition-onE*) *auto*
qed

lemma *partition-on-split1*:
assumes *partition-on* A $(P \cup Q)$
shows *partition-on* $(\bigcup P)$ P
proof (*rule partition-onI*)
fix p
assume $p \in P$
from *this assms* **show** $p \neq \{\}$
using *Un-iff partition-onE* **by** *auto*
next
show $\bigcup P = \bigcup P$ **..**
next
fix p p'
assume *a*: $p \in P$ $p' \in P$ $p \neq p'$
from *this assms* **show** $p \cap p' = \{\}$
using *partition-onE subsetCE sup-ge1* **by** *blast*
qed

lemma *partition-on-split2*:
assumes *partition-on* A $(P \cup Q)$
shows *partition-on* $(\bigcup Q)$ Q
using *assms partition-on-split1 sup-commute* **by** *metis*

lemma *partition-on-intersect-on-elements*:
assumes *partition-on* $(A \cup C)$ P
assumes $\forall X \in P. \exists x. x \in X \cap C$
shows *partition-on* C $((\lambda X.\ X \cap C)\ `\ P)$
proof (*rule partition-onI*)
fix p
assume $p \in (\lambda X.\ X \cap C)\ `\ P$
from *this assms* **show** $p \neq \{\}$ **by** *auto*
next
have $\bigcup P = A \cup C$
using *assms* **by** (*auto elim*: *partition-onE*)
from *this* **show** $\bigcup((\lambda X.\ X \cap C)\ `\ P) = C$ **by** *auto*
next
fix p p'
assume $p \in (\lambda X.\ X \cap C)\ `\ P$ $p' \in (\lambda X.\ X \cap C)\ `\ P$ $p \neq p'$
from *this assms*(*1*) **show** $p \cap p' = \{\}$
by (*blast elim*: *partition-onE*)
qed

lemma *partition-on-insert-elements*:

assumes $A \cap B = \{\}$
assumes *partition-on B P*
assumes $f \in A \rightarrow_E P$
shows *partition-on* $(A \cup B)$ $((\lambda X.\ X \cup \{x \in A.\ f\,x = X\})$ ` $P)$ (**is** *partition-on - ?P*)
proof (*rule partition-onI*)
fix X
assume $X \in ?P$
from *this* ‹*partition-on B P*› **show** $X \neq \{\}$
by (*auto elim*: *partition-onE*)
next
show $\bigcup ?P = A \cup B$
using ‹*partition-on B P*› ‹$f \in A \rightarrow_E P$› **by** (*auto elim*: *partition-onE*)
next
fix $X\ Y$
assume $X \in ?P\ Y \in ?P\ X \neq Y$
from ‹$X \in ?P$› **obtain** X' **where** X': $X = X' \cup \{x \in A.\ f\,x = X'\}$ $X' \in P$ **by** *auto*
from ‹$Y \in ?P$› **obtain** Y' **where** Y': $Y = Y' \cup \{x \in A.\ f\,x = Y'\}$ $Y' \in P$ **by** *auto*
from ‹$X \neq Y$› $X'\ Y'$ **have** $X' \neq Y'$ **by** *auto*
from *this* $X'\ Y'$ **have** $X' \cap Y' = \{\}$
using ‹*partition-on B P*› **by** (*auto elim*!: *partition-onE*)
from $X'\ Y'$ **have** $X' \subseteq B\ Y' \subseteq B$
using ‹*partition-on B P*› **by** (*auto elim*!: *partition-onE*)
from *this* ‹$X' \cap Y' = \{\}$› $X'\ Y'$ ‹$X' \neq Y'$› **show** $X \cap Y = \{\}$
using ‹$A \cap B = \{\}$› **by** *auto*
qed

lemma *partition-on-map*:
assumes *inj-on f A*
assumes *partition-on A P*
shows *partition-on* $(f$ ` $A)$ $((`)\ f$ ` $P)$
proof −
{
fix $X\ Y$
assume $X \in P\ Y \in P\ f$ ` $X \neq f$ ` Y
moreover from *assms* **have** $\forall\, p{\in}P.\ \forall\, p'{\in}P.\ p \neq p' \longrightarrow p \cap p' = \{\}$ **and** *inj-on f* $(\bigcup P)$
by (*auto elim*!: *partition-onE*)
ultimately have f ` $X \cap f$ ` $Y = \{\}$
unfolding *inj-on-def* **by** *auto* (*metis IntI empty-iff rev-image-eqI*)+
}
from *assms this* **show** *partition-on* $(f$ ` $A)$ $((`)\ f$ ` $P)$
by (*auto intro*!: *partition-onI elim*!: *partition-onE*)
qed

lemma *set-of-partition-on-map*:
assumes *inj-on f A*

```
  shows (`) ((`) f) ` {P. partition-on A P} = {P. partition-on (f ` A) P}
proof (rule set-eqI')
  fix x
  assume x ∈ (`) ((`) f) ` {P. partition-on A P}
  from this ‹inj-on f A› show x ∈ {P. partition-on (f ` A) P}
    by (auto intro: partition-on-map)
next
  fix P
  assume P ∈ {P. partition-on (f ` A) P}
  from this have partition-on (f ` A) P by auto
  from this have mem: ⋀X x. X ∈ P ⟹ x ∈ X ⟹ x ∈ f ` A
    by (auto elim!: partition-onE)
  have (`) (f ∘ the-inv-into A f) ` P = (`) f ` (`) (the-inv-into A f) ` P
    by (simp add: image-image cong: image-cong-simp)
  moreover have P = (`) (f ∘ the-inv-into A f) ` P
  proof (rule set-eqI')
    fix X
    assume X: X ∈ P
    moreover from X mem have in-range: ∀ x∈X. x ∈ f ` A by auto
    moreover have X = (f ∘ the-inv-into A f) ` X
    proof (rule set-eqI')
      fix x
      assume x ∈ X
      show x ∈ (f ∘ the-inv-into A f) ` X
      proof (rule image-eqI)
        from in-range ‹x ∈ X› assms show x = (f ∘ the-inv-into A f) x
          by (auto simp add: f-the-inv-into-f[of f])
        from ‹x ∈ X› show x ∈ X by assumption
      qed
    next
      fix x
      assume x ∈ (f ∘ the-inv-into A f) ` X
      from this obtain x' where x': x' ∈ X ∧ x = f (the-inv-into A f x') by auto
      from in-range x' have f: f (the-inv-into A f x') ∈ X
        by (subst f-the-inv-into-f[of f]) (auto intro: ‹inj-on f A›)
      from x' ‹X ∈ P› f show x ∈ X by auto
    qed
    ultimately show X ∈ (`) (f ∘ the-inv-into A f) ` P by auto
  next
    fix X
    assume X ∈ (`) (f ∘ the-inv-into A f) ` P
    moreover
    {
      fix Y
      assume Y ∈ P
      from this ‹inj-on f A› mem have ∀ x∈Y. f (the-inv-into A f x) = x
        by (auto simp add: f-the-inv-into-f)
      from this have (f ∘ the-inv-into A f) ` Y = Y by force
    }
```

ultimately show $X \in P$ **by** *auto*
qed
ultimately have P: $P = (`)\ f\ `\ (`)\ (\textit{the-inv-into}\ A\ f)\ `\ P$ **by** *simp*
have A-*eq*: $A = \textit{the-inv-into}\ A\ f\ `\ f\ `\ A$ **by** (*simp add*: *assms*)
from ‹*inj-on f A*› **have** *inj-on* (*the-inv-into A f*) (*f ` A*)
using ‹*partition-on* (*f ` A*) *P*› **by** (*simp add*: *inj-on-the-inv-into*)
from *this* **have** $(`)\ (\textit{the-inv-into}\ A\ f)\ `\ P \in \{P.\ \textit{partition-on}\ A\ P\}$
using ‹*partition-on* (*f ` A*) *P*› **by** (*subst A-eq*, *auto intro*!: *partition-on-map*)
from *P this* **show** $P \in (`)\ ((`)\ f)\ `\ \{P.\ \textit{partition-on}\ A\ P\}$ **by** (*rule image-eqI*)
qed

end

2 Combinatorial Basics

theory *Injectivity-Solver*
imports
HOL−Library.Disjoint-Sets
HOL−Library.Monad-Syntax
HOL−Eisbach.Eisbach
begin

2.1 Preliminaries

These lemmas shall be added to the Disjoint Set theory.

2.1.1 Injectivity and Disjoint Families

lemma *inj-on-impl-disjoint-family-on-singleton*:
assumes *inj-on f A*
shows *disjoint-family-on* ($\lambda x.\ \{f\ x\}$) *A*
using *assms disjoint-family-on-def inj-on-contraD* **by** *fastforce*

2.1.2 Cardinality Theorems for Set.bind

lemma *card-bind*:
assumes *finite S*
assumes $\forall X \in S.$ *finite* (*f X*)
assumes *disjoint-family-on f S*
shows *card* ($S \gg\!= f$) = ($\sum x{\in}S.$ *card* (*f x*))
proof −
have *card* ($S \gg\!= f$) = *card* ($\bigcup (f\ `\ S)$)
by (*simp add*: *bind-UNION*)
also have *card* ($\bigcup (f\ `\ S)$) = ($\sum x{\in}S.$ *card* (*f x*))
using *assms* **unfolding** *disjoint-family-on-def* **by** (*simp add*: *card-UN-disjoint*)
finally show *?thesis* **.**
qed

lemma *card-bind-constant*:
assumes *finite S*
assumes $\forall X \in S.$ *finite* $(f\ X)$
assumes *disjoint-family-on f S*
assumes $\bigwedge x.\ x \in S \Longrightarrow$ *card* $(f\ x) = k$
shows *card* $(S \gg\!= f) =$ *card* $S * k$
using *assms* **by** (*simp add*: *card-bind*)

lemma *card-bind-singleton*:
assumes *finite S*
assumes *inj-on f S*
shows *card* $(S \gg\!= (\lambda x.\ \{f\ x\})) =$ *card S*
using *assms* **by** (*auto simp add*: *card-bind-constant inj-on-impl-disjoint-family-on-singleton*)

2.2 Third Version of Injectivity Solver

Here, we provide a third version of the injectivity solver. The original first version was provided in the AFP entry 'Spivey's Generalized Recurrence for Bell Numbers'. From that method, I derived a second version in the AFP entry 'Cardinality of Equivalence Relations'. At roughly the same time, Makarius improved the injectivity solver in the development version of the first AFP entry. This third version now includes the improvements of the second version and Makarius improvements to the first, and it further extends the method to handle the new cases in the cardinality proof of this AFP entry.

As the implementation of the injectivity solver only evolves in the development branch of the AFP, the submissions of the three AFP entries that employ the injectivity solver, have to create clones of the injectivity solver for the identified and needed method adjustments. Ultimately, these three clones should only remain in the stable branches of the AFP from Isabelle2016 to Isabelle2017 to work with their corresponding release versions. In the development version, I have now consolidated the three versions here. In the next step, I will move this version of the injectivity solver in the *HOL−Library.Disjoint-Sets* and it will hopefully only evolve further there.

lemma *disjoint-family-onI*:
assumes $\bigwedge i\ j.\ i \in I \wedge j \in I \Longrightarrow i \neq j \Longrightarrow (A\ i) \cap (A\ j) = \{\}$
shows *disjoint-family-on A I*
using *assms* **unfolding** *disjoint-family-on-def* **by** *auto*

lemma *disjoint-bind*: $\bigwedge S\ T\ f\ g.\ (\bigwedge s\ t.\ S\ s \wedge T\ t \Longrightarrow f\ s \cap g\ t = \{\}) \Longrightarrow (\{s.\ S\ s\} \gg\!= f) \cap (\{t.\ T\ t\} \gg\!= g) = \{\}$
by *fastforce*

lemma *disjoint-bind'*: $\bigwedge S\ T\ f\ g.\ (\bigwedge s\ t.\ s \in S \wedge t \in T \Longrightarrow f\ s \cap g\ t = \{\}) \Longrightarrow (S \gg\!= f) \cap (T \gg\!= g) = \{\}$
by *fastforce*

lemma *injectivity-solver-CollectE*:
  assumes $a \in \{x.\ P\ x\} \wedge a' \in \{x.\ P'\ x\}$
  assumes $(P\ a \wedge P'\ a') \Longrightarrow W$
  shows W
using *assms* **by** *auto*

lemma *injectivity-solver-prep-assms-Collect*:
  assumes $x \in \{x.\ P\ x\}$
  shows $P\ x \wedge P\ x$
using *assms* **by** *simp*

lemma *injectivity-solver-prep-assms*: $x \in A \Longrightarrow x \in A \wedge x \in A$
  by *simp*

lemma *disjoint-terminal-singleton*: $\bigwedge s\ t\ X\ Y.\ s \neq t \Longrightarrow (X = Y \Longrightarrow s = t) \Longrightarrow \{X\} \cap \{Y\} = \{\}$
by *auto*

lemma *disjoint-terminal-Collect*:
  assumes $s \neq t$
  assumes $\bigwedge x\ x'.\ S\ x \wedge T\ x' \Longrightarrow x = x' \Longrightarrow s = t$
  shows $\{x.\ S\ x\} \cap \{x.\ T\ x\} = \{\}$
using *assms* **by** *auto*

lemma *disjoint-terminal*:
  $s \neq t \Longrightarrow (\bigwedge x\ x'.\ x \in S \wedge x' \in T \Longrightarrow x = x' \Longrightarrow s = t) \Longrightarrow S \cap T = \{\}$
by *blast*

lemma *elim-singleton*:
  assumes $x \in \{s\} \wedge x' \in \{t\}$
  obtains $x = s \wedge x' = t$
using *assms* **by** *blast*

method *injectivity-solver* **uses** *rule* =
  insert method-facts,
  use nothing **in** ‹
   ((*drule injectivity-solver-prep-assms-Collect* | *drule injectivity-solver-prep-assms*)+)*?*;
    rule disjoint-family-onI;
    ((*rule disjoint-bind* | *rule disjoint-bind'*)+)*?*;
    (*erule elim-singleton*)*?*;
     (*erule disjoint-terminal-singleton* | *erule disjoint-terminal-Collect* | *erule disjoint-terminal*);
    (*elim injectivity-solver-CollectE*)*?*;
    rule rule;
    assumption+
  ›

end

3 Cardinality of Set Partitions

theory *Card-Partitions*
imports
HOL−Combinatorics.Stirling
Set-Partition
Injectivity-Solver
begin

lemma *set-partition-on-insert-with-fixed-card-eq*:
assumes *finite A*
assumes $a \notin A$
shows $\{P.\ \textit{partition-on}\ (\textit{insert}\ a\ A)\ P \wedge \textit{card}\ P = \textit{Suc}\ k\} = (\textit{do}\ \{$
$P <- \{P.\ \textit{partition-on}\ A\ P \wedge \textit{card}\ P = \textit{Suc}\ k\};$
$p <- P;$
$\{\textit{insert}\ (\textit{insert}\ a\ p)\ (P - \{p\})\}$
$\})$
$\cup\ (\textit{do}\ \{$
$P <- \{P.\ \textit{partition-on}\ A\ P \wedge \textit{card}\ P = k\};$
$\{\textit{insert}\ \{a\}\ P\}$
$\})\ (\textbf{is}\ ?S = ?T)$
proof
show $?S \subseteq ?T$
proof
fix P
assume $P \in \{P.\ \textit{partition-on}\ (\textit{insert}\ a\ A)\ P \wedge \textit{card}\ P = \textit{Suc}\ k\}$
from *this* **have** $\textit{partition-on}\ (\textit{insert}\ a\ A)\ P$ **and** $\textit{card}\ P = \textit{Suc}\ k$ **by** *auto*
show $P \in ?T$
proof *cases*
assume $\{a\} \in P$
have $\textit{partition-on}\ A\ (P - \{\{a\}\})$
using ‹$\{a\} \in P$› ‹$\textit{partition-on}\ (\textit{insert}\ a\ A)\ P$›[*THEN partition-on-Diff*, *of* $\{\{a\}\}$] ‹$a \notin A$›
by *auto*
moreover from ‹$\{a\} \in P$› ‹$\textit{card}\ P = \textit{Suc}\ k$› **have** $\textit{card}\ (P - \{\{a\}\}) = k$
by (*subst card-Diff-singleton*) (*auto intro*: *card-ge-0-finite*)
moreover from ‹$\{a\} \in P$› **have** $P = \textit{insert}\ \{a\}\ (P - \{\{a\}\})$ **by** *auto*
ultimately have $P \in \{P.\ \textit{partition-on}\ A\ P \wedge \textit{card}\ P = k\} \gg= (\lambda P.\ \{\textit{insert}\ \{a\}\ P\})$
by *auto*
from *this* **show** *?thesis* **by** *auto*
next
assume $\{a\} \notin P$
let $?p' = (\textit{THE}\ X.\ a \in X \wedge X \in P)$
let $?p = (\textit{THE}\ X.\ a \in X \wedge X \in P) - \{a\}$
let $?P' = \textit{insert}\ ?p\ (P - \{?p'\})$
from ‹$\textit{partition-on}\ (\textit{insert}\ a\ A)\ P$› **have** $a \in (\textit{THE}\ X.\ a \in X \wedge X \in P)$
using *partition-on-in-the-unique-part* **by** *fastforce*
from ‹$\textit{partition-on}\ (\textit{insert}\ a\ A)\ P$› **have** $(\textit{THE}\ X.\ a \in X \wedge X \in P) \in P$

using *partition-on-the-part-mem* **by** *fastforce*
from *this* ‹*partition-on* (*insert a A*) *P*› **have** (*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*) − {*a*} ∉ *P*
using *partition-subset-imp-notin* ‹*a* ∈ (*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*)› **by** *blast*
have (*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*) ≠ {*a*}
using ‹(*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*) ∈ *P*› ‹{*a*} ∉ *P*› **by** *auto*
from ‹*partition-on* (*insert a A*) *P*› **have** (*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*) ⊆ *insert a A*
using ‹(*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*) ∈ *P*› *partition-onD1* **by** *fastforce*
note *facts-on-the-part-of* = ‹*a* ∈ (*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*)› ‹(*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*) ∈ *P*›
‹(*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*) − {*a*} ∉ *P*› ‹(*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*) ≠ {*a*}›
‹(*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*) ⊆ *insert a A*›
from ‹*partition-on* (*insert a A*) *P*› ‹*finite A*› **have** *finite P*
by (*meson finite.insertI finite-elements*)
from ‹*partition-on* (*insert a A*) *P*› ‹*a* ∉ *A*› **have** *partition-on* (*A* − *?p*) (*P* − {*?p′*})
using *facts-on-the-part-of* **by** (*auto intro*: *partition-on-remove-singleton*)
from *this* **have** *partition-on A ?P′*
using *facts-on-the-part-of* **by** (*auto intro*: *partition-on-insert simp add*: *disjnt-iff*)
moreover have *card ?P′* = *Suc k*
proof −
from ‹*card P* = *Suc k*› **have** *card* (*P* − {*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*}) = *k*
using ‹*finite P*› ‹(*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*) ∈ *P*› **by** *simp*
from *this* **show** *?thesis*
using ‹*finite P*› ‹(*THE X*. *a* ∈ *X* ∧ *X* ∈ *P*) − {*a*} ∉ *P*› **by** (*simp add*: *card-insert-if*)
qed
moreover have *?p* ∈ *?P′* **by** *auto*
moreover have *P* = *insert* (*insert a ?p*) (*?P′* − {*?p*})
using *facts-on-the-part-of* **by** (*auto simp add*: *insert-absorb*)
ultimately have *P* ∈ {*P*. *partition-on A P* ∧ *card P* = *Suc k*} ⤜ (λ*P*. *P* ⤜ (λ*p*. {*insert* (*insert a p*) (*P* − {*p*})}))
by *auto*
then show *?thesis* **by** *auto*
qed
qed
next
show *?T* ⊆ *?S*
proof
fix *P*
assume *P* ∈ *?T* (**is** - ∈ *?subexpr1* ∪ *?subexpr2*)
from *this* **show** *P* ∈ *?S*
proof
assume *P* ∈ *?subexpr1*
from *this* **obtain** *p P′* **where** *P* = *insert* (*insert a p*) (*P′* − {*p*})
and *partition-on A P′* **and** *card P′* = *Suc k* **and** *p* ∈ *P′* **by** *auto*
from ‹*p* ∈ *P′*› ‹*partition-on A P′*› **have** *partition-on* (*A* − *p*) (*P′* − {*p*})

```
    by (simp add: partition-on-remove-singleton)
  from ‹partition-on A P'› ‹finite A› have finite P
    using ‹P = -› finite-elements by auto
  from ‹partition-on A P'› ‹a ∉ A› have insert a p ∉ P' − {p}
    using partition-onD1 by fastforce
  from ‹P = -› this ‹card P' = Suc k› ‹finite P› ‹p ∈ P'›
  have card P = Suc k by auto
  moreover have partition-on (insert a A) P
    using ‹partition-on (A − p) (P' − {p})› ‹a ∉ A› ‹p ∈ P'› ‹partition-on A
P'› ‹P = -›
    by (auto intro!: partition-on-insert dest: partition-onD1 simp add: disjnt-iff)
  ultimately show P ∈ ?S by auto
 next
  assume P ∈ ?subexpr2
  from this obtain P' where P = insert {a} P' and partition-on A P' and
card P' = k by auto
  from ‹partition-on A P'› ‹finite A› have finite P
    using ‹P = insert {a} P'› finite-elements by auto
  from ‹partition-on A P'› ‹a ∉ A› have {a} ∉ P'
    using partition-onD1 by fastforce
  from ‹P = insert {a} P'› ‹card P' = k› this ‹finite P› have card P = Suc k
by auto
  moreover from ‹partition-on A P'› ‹a ∉ A› have partition-on (insert a A)
P
    using ‹P = insert {a} P'› by (simp add: partition-on-insert-singleton)
  ultimately show P ∈ ?S by auto
 qed
qed
qed

lemma injectivity-subexpr1:
 assumes a ∉ A
 assumes X ∈ P ∧ X' ∈ P'
 assumes insert (insert a X) (P − {X}) = insert (insert a X') (P' − {X'})
 assumes (partition-on A P ∧ card P = Suc k') ∧ (partition-on A P' ∧ card P'
= Suc k')
 shows P = P' and X = X'
proof −
 from assms(1, 2, 4) have a ∉ X a ∉ X'
   using partition-onD1 by auto
 from assms(1, 4) have insert a X ∉ P insert a X' ∉ P'
   using partition-onD1 by auto
 from assms(1, 3, 4) have insert a X = insert a X'
   by (metis Diff-iff insertE insertI1 mem-simps(9) partition-onD1)
 from this ‹a ∉ X'› ‹a ∉ X› show X = X'
   by (meson insert-ident)
 from assms(2, 3) show P = P'
   using ‹insert a X = insert a X'› ‹insert a X ∉ P› ‹insert a X' ∉ P'›
   by (metis insert-Diff insert-absorb insert-commute insert-ident)
```

qed

lemma *injectivity-subexpr2*:
assumes $a \notin A$
assumes *insert* $\{a\}$ P = *insert* $\{a\}$ P'
assumes (*partition-on* A $P \wedge$ *card* $P = k'$) $\wedge$ *partition-on* A $P' \wedge$ *card* $P' = k'$
shows $P = P'$
proof −
from *assms*(*1*, *3*) **have** $\{a\} \notin P$ **and** $\{a\} \notin P'$
using *partition-onD1* **by** *auto*
from ‹$\{a\} \notin P$› **have** P = *insert* $\{a\}$ $P - \{\{a\}\}$ **by** *simp*
also from ‹*insert* $\{a\}$ P = *insert* $\{a\}$ P'› **have** ... = *insert* $\{a\}$ $P' - \{\{a\}\}$ **by** *simp*
also from ‹$\{a\} \notin P'$› **have** ... = P' **by** *simp*
finally show *?thesis* **.**
qed

theorem *card-partition-on*:
assumes *finite* A
shows *card* $\{P.$ *partition-on* A $P \wedge$ *card* $P = k\}$ = *Stirling* (*card* A) k
using *assms*
proof (*induct* A *arbitrary*: k)
case *empty*
have *eq*: $\{P.$ $P = \{\} \wedge$ *card* $P = 0\} = \{\{\}\}$ **by** *auto*
show *?case* **by** (*cases* k) (*auto simp add*: *partition-on-empty eq*)
next
case (*insert* a A)
from *this* **show** *?case*
proof (*cases* k)
case *0*
from *insert*(*1*) **have** *empty*: $\{P.$ *partition-on* (*insert* a A) $P \wedge$ *card* $P = 0\} = \{\}$
unfolding *partition-on-def* **by** (*auto simp add*: *card-eq-0-iff finite-UnionD*)
from *0 insert* **show** *?thesis* **by** (*auto simp add*: *empty*)
next
case (*Suc* k')
let *?subexpr1* = *do* {
P <− $\{P.$ *partition-on* A $P \wedge$ *card* $P =$ *Suc* $k'\}$;
p <− P;
$\{$*insert* (*insert* a p) ($P - \{p\}$)$\}$
}
let *?subexpr2* = *do* {
P <− $\{P.$ *partition-on* A $P \wedge$ *card* $P = k'\}$;
$\{$*insert* $\{a\}$ $P\}$
}
let *?expr* = *?subexpr1* ∪ *?subexpr2*
have *card* $\{P.$ *partition-on* (*insert* a A) $P \wedge$ *card* $P = k\}$ = *card* *?expr*
using ‹*finite* A› ‹$a \notin A$› ‹k = *Suc* k'› **by** (*simp add*: *set-partition-on-insert-with-fixed-card-eq*)
also have *card* *?expr* = *Stirling* (*card* A) k' + *Stirling* (*card* A) (*Suc* k') ∗ *Suc*

```
k'
  proof −
    have finite ?subexpr1 ∧ card ?subexpr1 = Stirling (card A) (Suc k') * Suc k'
    proof −
      from ‹finite A› have finite {P. partition-on A P ∧ card P = Suc k'}
        by (simp add: finitely-many-partition-on)
       moreover have ∀ X∈{P. partition-on A P ∧ card P = Suc k'}. finite (X
>>= (λp. {insert (insert a p) (X − {p})}))
        using finite-elements ‹finite A› finite-bind
        by (metis (no-types, lifting) finite.emptyI finite-insert mem-Collect-eq)
      moreover have disjoint-family-on (λP. P >>= (λp. {insert (insert a p) (P
− {p})})) {P. partition-on A P ∧ card P = Suc k'}
        by (injectivity-solver rule: injectivity-subexpr1(1)[OF ‹a ∉ A›])
      moreover have card (P >>= (λp. {insert (insert a p) (P − {p})})) = Suc
k'
        if P ∈ {P. partition-on A P ∧ card P = Suc k'} for P
      proof −
        from that ‹finite A› have finite P
          using finite-elements by blast
        moreover have inj-on (λp. insert (insert a p) (P − {p})) P
          using that injectivity-subexpr1(2)[OF ‹a ∉ A›] by (simp add: inj-onI)
        moreover from that have card P = Suc k' by simp
        ultimately show ?thesis by (simp add: card-bind-singleton)
      qed
      ultimately have card ?subexpr1 = card {P. partition-on A P ∧ card P =
Suc k'} * Suc k'
        by (subst card-bind-constant) simp+
      from this have card ?subexpr1 = Stirling (card A) (Suc k') * Suc k'
        using insert.hyps(3) by simp
      moreover have finite ?subexpr1
        using ‹finite {P. partition-on A P ∧ card P = Suc k'}›
        ‹∀ X∈{P. partition-on A P ∧ card P = Suc k'}. finite (X >>= (λp. {insert
(insert a p) (X − {p})}))›
        by (auto intro: finite-bind)
      ultimately show ?thesis by blast
    qed
    moreover have finite ?subexpr2 ∧ card ?subexpr2 = Stirling (card A) k'
    proof −
      from ‹finite A› have finite {P. partition-on A P ∧ card P = k'}
        by (simp add: finitely-many-partition-on)
      moreover have  inj-on (insert {a}) {P. partition-on A P ∧ card P = k'}
        using injectivity-subexpr2[OF ‹a ∉ A›] by (simp add: inj-on-def)
      ultimately have card ?subexpr2 = card {P. partition-on A P ∧ card P =
k'}
        by (simp add: card-bind-singleton)
      also have ... = Stirling (card A) k'
        using insert.hyps(3) .
      finally have card ?subexpr2 = Stirling (card A) k' .
      moreover have finite ?subexpr2
```

```
      by (simp add: ‹finite {P. partition-on A P ∧ card P = k'}› finite-bind)
    ultimately show ?thesis by blast
  qed
  moreover have ?subexpr1 ∩ ?subexpr2 = {}
  proof −
    have ∀ P∈?subexpr1. {a} ∉ P
      using insert.hyps(2) by (force elim!: partition-onE)
    moreover have ∀ P∈?subexpr2. {a} ∈ P by auto
    ultimately show ?subexpr1 ∩ ?subexpr2 = {} by blast
  qed
  ultimately show ?thesis
    by (simp add: card-Un-disjoint)
 qed
 also have ... = Stirling (card (insert a A)) k
  using insert(1, 2) ‹k = Suc k'› by simp
 finally show ?thesis .
 qed
qed

theorem card-partition-on-at-most-size:
 assumes finite A
 shows card {P. partition-on A P ∧ card P ≤ k} = (∑j≤k. Stirling (card A) j)
proof −
 have card {P. partition-on A P ∧ card P ≤ k} = card (⋃j≤k. {P. partition-on
A P ∧ card P = j})
   by (rule arg-cong[where f=card]) auto
 also have ... = (∑j≤k. card {P. partition-on A P ∧ card P = j})
  by (subst card-UN-disjoint) (auto simp add: ‹finite A› finitely-many-partition-on)
 also have (∑j≤k. card {P. partition-on A P ∧ card P = j}) = (∑j≤k. Stirling
(card A) j)
   using ‹finite A› by (simp add: card-partition-on)
 finally show ?thesis .
qed

theorem partition-on-size1:
 assumes finite A
 shows {P. partition-on A P ∧ (∀ X∈P. card X = 1)} = {(λa. {a}) ‘ A}
proof
 show {P. partition-on A P ∧ (∀ X∈P. card X = 1)} ⊆ {(λa. {a}) ‘ A}
 proof
   fix P
   assume P: P ∈ {P. partition-on A P ∧ (∀ X∈P. card X = 1)}
   have P = (λa. {a}) ‘ A
   proof
     show P ⊆ (λa. {a}) ‘ A
     proof
       fix X
       assume X ∈ P
       from P this obtain x where X = {x}
```

```
        by (auto simp add: card-Suc-eq)
      from this ‹X ∈ P› have x ∈ A
        using P unfolding partition-on-def by blast
      from this ‹X = {x}› show X ∈(λa. {a}) ‘ A by auto
    qed
  next
    show (λa. {a}) ‘ A ⊆ P
    proof
      fix X
      assume X ∈ (λa. {a}) ‘ A
      from this obtain x where X: X = {x} x ∈ A by auto
      have ⋃P = A
        using P unfolding partition-on-def by blast
      from this ‹x ∈ A› obtain X' where x ∈ X' and X' ∈ P
        using UnionE by blast
      from ‹X' ∈ P› have card X' = 1
        using P unfolding partition-on-def by auto
      from this ‹x ∈ X'› have X' = {x}
        using card-1-singletonE by blast
      from this X(1) ‹X' ∈ P› show X ∈ P by auto
    qed
  qed
  from this show P ∈ {(λa. {a}) ‘ A} by auto
 qed
next
 show {(λa. {a}) ‘ A} ⊆ {P. partition-on A P ∧ (∀X∈P. card X = 1)}
 proof
  fix P
  assume P ∈ {(λa. {a}) ‘ A}
  from this have P: P = (λa. {a}) ‘ A by auto
  from this have partition-on A P by (auto intro: partition-onI)
  from P this show P ∈ {P. partition-on A P ∧ (∀X∈P. card X = 1)} by auto
 qed
qed

theorem card-partition-on-size1:
 assumes finite A
 shows card {P. partition-on A P ∧ (∀X∈P. card X = 1)} = 1
using assms partition-on-size1 by fastforce

lemma card-partition-on-size1-eq-1:
 assumes finite A
 assumes card A ≤ k
 shows card {P. partition-on A P ∧ card P ≤ k ∧ (∀X∈P. card X = 1)} = 1
proof −
 {
  fix P
  assume partition-on A P ∀X∈P. card X = 1
  from this have P ∈ {P. partition-on A P ∧ (∀X∈P. card X = 1)} by simp
```

```
    from this have P ∈ {(λa. {a}) ‘ A}
      using partition-on-size1 ‹finite A› by auto
    from this have P = (λa. {a}) ‘ A by auto
    moreover from this have card P = card A
      by (auto intro: card-image)
  }
  from this have {P. partition-on A P ∧ card P ≤ k ∧ (∀X∈P. card X = 1)} =
{P. partition-on A P ∧ (∀X∈P. card X = 1)}
    using ‹card A ≤ k› by auto
  from this show ?thesis
    using ‹finite A› by (simp only: card-partition-on-size1)
qed

lemma card-partition-on-size1-eq-0:
  assumes finite A
  assumes k < card A
  shows card {P. partition-on A P ∧ card P ≤ k ∧ (∀X∈P. card X = 1)} = 0
proof −
  {
    fix P
    assume partition-on A P ∀X∈P. card X = 1
    from this have P ∈ {P. partition-on A P ∧ (∀X∈P. card X = 1)} by simp
    from this have P ∈ {(λa. {a}) ‘ A}
      using partition-on-size1 ‹finite A› by auto
    from this have P = (λa. {a}) ‘ A by auto
    from this have card P = card A
      by (auto intro: card-image)
  }
  from this assms(2) have {P. partition-on A P ∧ card P ≤ k ∧ (∀X∈P. card X
= 1)} = {}
    using Collect-empty-eq leD by fastforce
  from this show ?thesis by (simp only: card.empty)
qed

end
```

References

[1] F. Haftmann. Discrete summation. *Archive of Formal Proofs*, Apr. 2014. http://isa-afp.org/entries/Discrete_Summation.shtml, Formal proof development.

[2] R. P. Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge University Press, second edition, 2012.

[3] Wikipedia. Stirling numbers of the second kind — wikipedia, the free encyclopedia, 2015. https://en.wikipedia.org/w/index.php?

title=Stirling_numbers_of_the_second_kind&oldid=693800357, [Online; accessed 12-December-2015].