Cardinality of Number Partitions

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Abstract

This entry provides a basic library for number partitions, defines the two-argument partition function through its recurrence relation and relates this partition function to the cardinality of number partitions. The main proof shows that the recursively-defined partition function with arguments n and k equals the cardinality of number partitions of n with exactly k parts. The combinatorial proof follows the proof sketch of Theorem 2.4.1 in Mazur's textbook "Combinatorics: A Guided Tour" [2]. This entry can serve as starting point for various more intrinsic properties about number partitions, the partition function and related recurrence relations.

Contents

1 Additions to Isabelle's Main Theories

theory *Additions-to-Main*
imports *HOL−Library.Multiset*
begin

1.1 Addition to Finite-Set Theory

lemma *bound-domain-and-range-impl-finitely-many-functions*:
$\mathit{finite}\ \{f{::}\mathit{nat}{\Rightarrow}\mathit{nat}.\ (\forall\, i.\ f\ i \le n) \wedge (\forall\, i{\ge}m.\ f\ i = 0)\}$
⟨*proof*⟩

1.2 Addition to Set-Interval Theory

lemma *sum-atMost-remove-nat*:
assumes $k \le (n :: \mathit{nat})$
shows $(\sum i{\le}n.\ f\ i) = f\ k + (\sum i{\in}\{..n\}{-}\{k\}.\ f\ i)$
⟨*proof*⟩

1.3 Additions to Multiset Theory

lemma *set-mset-Abs-multiset*:
assumes $\mathit{finite}\ \{x.\ f\ x > 0\}$
shows $\mathit{set\text{-}mset}\ (\mathit{Abs\text{-}multiset}\ f) = \{x.\ f\ x > 0\}$
⟨*proof*⟩

lemma *sum-mset-sum-count*:
$\mathit{sum\text{-}mset}\ M = (\sum i{\in}\mathit{set\text{-}mset}\ M.\ \mathit{count}\ M\ i * i)$
⟨*proof*⟩

lemma *sum-mset-eq-sum-on-supersets*:
assumes $\mathit{finite}\ A\ \ \mathit{set\text{-}mset}\ M \subseteq A$
shows $(\sum i{\in}\mathit{set\text{-}mset}\ M.\ \mathit{count}\ M\ i * i) = (\sum i{\in}A.\ \mathit{count}\ M\ i * i)$
⟨*proof*⟩

end

2 Number Partitions

theory *Number-Partition*
imports *Additions-to-Main*
begin

2.1 Number Partitions as $nat \Rightarrow nat$ Functions

definition *partitions* :: $(\mathit{nat} \Rightarrow \mathit{nat}) \Rightarrow \mathit{nat} \Rightarrow \mathit{bool}$ (**infix** ‹*partitions*› *50*)
where
$p\ \mathit{partitions}\ n = ((\forall\, i.\ p\ i \ne 0 \longrightarrow 1 \le i \wedge i \le n) \wedge (\sum i{\le}n.\ p\ i * i) = n)$

lemma *partitionsI*:
  assumes $\bigwedge i.\ p\ i \neq 0 \Longrightarrow 1 \leq i \wedge i \leq n$
  assumes $(\sum i{\leq}n.\ p\ i * i) = n$
  shows *p partitions n*
$\langle proof \rangle$

lemma *partitionsE*:
  assumes *p partitions n*
  obtains $\bigwedge i.\ p\ i \neq 0 \Longrightarrow 1 \leq i \wedge i \leq n\ (\sum i{\leq}n.\ p\ i * i) = n$
$\langle proof \rangle$

lemma *partitions-zero*:
  $p\ partitions\ 0 \longleftrightarrow p = (\lambda i.\ 0)$
$\langle proof \rangle$

lemma *partitions-one*:
  $p\ partitions\ (Suc\ 0) \longleftrightarrow p = (\lambda i.\ 0)(1 := 1)$
$\langle proof \rangle$

2.2 Bounds and Finiteness of Number Partitions

lemma *partitions-imp-finite-elements*:
  assumes *p partitions n*
  shows $finite\ \{i.\ 0 < p\ i\}$
$\langle proof \rangle$

lemma *partitions-bounds*:
  assumes *p partitions n*
  shows $p\ i \leq n$
$\langle proof \rangle$

lemma *partitions-parts-bounded*:
  assumes *p partitions n*
  shows $sum\ p\ \{..n\} \leq n$
$\langle proof \rangle$

lemma *finite-partitions*:
  $finite\ \{p.\ p\ partitions\ n\}$
$\langle proof \rangle$

lemma *finite-partitions-k-parts*:
  $finite\ \{p.\ p\ partitions\ n \wedge sum\ p\ \{..n\} = k\}$
$\langle proof \rangle$

lemma *partitions-remaining-Max-part*:
  assumes *p partitions n*
  assumes $0 < p\ k$
  shows $\forall i.\ n - k < i \wedge i \neq k \longrightarrow p\ i = 0$
$\langle proof \rangle$

2.3 Operations of Number Partitions

lemma *partitions-remove1-bounds*:
  assumes *partitions*: p *partitions* n
  assumes *gr0*: $0 < p\ k$
  assumes *neq*: $(p(k := p\ k - 1))\ i \neq 0$
  shows $1 \leq i \wedge i \leq n - k$
⟨*proof*⟩

lemma *partitions-remove1*:
  assumes *partitions*: p *partitions* n
  assumes *gr0*: $0 < p\ k$
  shows $p(k := p\ k - 1)$ *partitions* $(n - k)$
⟨*proof*⟩

lemma *partitions-insert1*:
  assumes *p*: p *partitions* n
  assumes $k > 0$
  shows $(p(k := p\ k + 1))$ *partitions* $(n + k)$
⟨*proof*⟩

lemma *count-remove1*:
  assumes p *partitions* n
  assumes $0 < p\ k$
  shows $(\sum i{\leq}n - k.\ (p(k := p\ k - 1))\ i) = (\sum i{\leq}n.\ p\ i) - 1$
⟨*proof*⟩

lemma *count-insert1*:
  assumes p *partitions* n
  shows *sum* $(p(k := p\ k + 1))\ \{..n + k\} = (\sum i{\leq}n.\ p\ i) + 1$
⟨*proof*⟩

lemma *partitions-decrease1*:
  assumes *p*: p *partitions* m
  assumes *sum*: *sum* $p\ \{..m\} = k$
  assumes $p\ 1 = 0$
  shows $(\lambda i.\ p\ (i + 1))$ *partitions* $m - k$
⟨*proof*⟩

lemma *partitions-increase1*:
  assumes *partitions*: p *partitions* $m - k$
  assumes *k*: *sum* $p\ \{..m - k\} = k$
  shows $(\lambda i.\ p\ (i - 1))$ *partitions* m
⟨*proof*⟩

lemma *count-decrease1*:
  assumes *p*: p *partitions* m
  assumes *sum*: *sum* $p\ \{..m\} = k$
  assumes $p\ 1 = 0$
  shows *sum* $(\lambda i.\ p\ (i + 1))\ \{..m - k\} = k$

⟨*proof*⟩

lemma *count-increase1*:
  assumes *partitions*: *p partitions* $m - k$
  assumes *k*: *sum p* $\{..m - k\} = k$
  shows $(\sum i \leq m.\ p\ (i - 1)) = k$
⟨*proof*⟩

2.4 Number Partitions as Multisets on Natural Numbers

definition *number-partition* :: *nat* ⇒ *nat multiset* ⇒ *bool*
where
  number-partition n N = (*sum-mset N* = *n* ∧ *0* ∉# *N*)

2.4.1 Relationship to Definition on Functions

lemma *count-partitions-iff*:
  count N partitions n ⟷ *number-partition n N*
⟨*proof*⟩

lemma *partitions-iff-Abs-multiset*:
  p partitions n ⟷ *finite* $\{x.\ 0 < p\ x\}$ ∧ *number-partition n* (*Abs-multiset p*)
⟨*proof*⟩

lemma *size-nat-multiset-eq*:
  fixes *N* :: *nat multiset*
  assumes *number-partition n N*
  shows *size N* = *sum* (*count N*) $\{..n\}$
⟨*proof*⟩

end

3 Cardinality of Number Partitions

theory *Card-Number-Partitions*
imports *Number-Partition*
begin

3.1 The Partition Function

fun *Partition* :: *nat* ⇒ *nat* ⇒ *nat*
where
  Partition 0 0 = *1*
| *Partition 0* (*Suc k*) = *0*
| *Partition* (*Suc m*) *0* = *0*
| *Partition* (*Suc m*) (*Suc k*) = *Partition m k* + *Partition* $(m - k)$ (*Suc k*)

lemma *Partition-less*:

assumes $m < k$
shows *Partition* $m\ k = 0$
⟨*proof*⟩

lemma *Partition-sum-Partition-diff*:
assumes $k \le m$
shows *Partition* $m\ k = (\sum i{\le}k.\ Partition\ (m - k)\ i)$
⟨*proof*⟩

lemma *Partition-parts1*:
Partition $(Suc\ m)\ (Suc\ 0) = 1$
⟨*proof*⟩

lemma *Partition-diag*:
Partition $(Suc\ m)\ (Suc\ m) = 1$
⟨*proof*⟩

lemma *Partition-diag1*:
Partition $(Suc\ (Suc\ m))\ (Suc\ m) = 1$
⟨*proof*⟩

lemma *Partition-parts2*:
shows *Partition* $m\ 2 = m\ div\ 2$
⟨*proof*⟩

3.2 Cardinality of Number Partitions

lemma *set-rewrite1*:
$\{p.\ p\ partitions\ Suc\ m \wedge sum\ p\ \{..Suc\ m\} = Suc\ k \wedge p\ 1 \neq 0\}$
$= (\lambda p.\ p(1 := p\ 1 + 1))$ ‘ $\{p.\ p\ partitions\ m \wedge sum\ p\ \{..m\} = k\}$ (**is** $?S =$ $?T$)
⟨*proof*⟩

lemma *set-rewrite2*:
$\{p.\ p\ partitions\ m \wedge sum\ p\ \{..m\} = k \wedge p\ 1 = 0\}$
$= (\lambda p.\ (\lambda i.\ p\ (i - 1)))$ ‘ $\{p.\ p\ partitions\ (m - k) \wedge sum\ p\ \{..m - k\} = k\}$
(**is** $?S = ?T$)
⟨*proof*⟩

theorem *card-partitions-k-parts*:
$card\ \{p.\ p\ partitions\ n \wedge (\sum i{\le}n.\ p\ i) = k\} = Partition\ n\ k$
⟨*proof*⟩

theorem *card-partitions*:
$card\ \{p.\ p\ partitions\ n\} = (\sum k{\le}n.\ Partition\ n\ k)$
⟨*proof*⟩

lemma *card-partitions-atmost-k-parts*:
$card\ \{p.\ p\ partitions\ n \wedge sum\ p\ \{..n\} \le k\} = Partition\ (n + k)\ k$

$\langle proof \rangle$

3.3 Cardinality of Number Partitions as Multisets of Natural Numbers

lemma *bij-betw-multiset-number-partition-with-size*:
  $bij\text{-}betw\ count\ \{N.\ number\text{-}partition\ n\ N \wedge size\ N = k\}\ \{p.\ p\ partitions\ n \wedge sum\ p\ \{..n\} = k\}$
$\langle proof \rangle$

lemma *bij-betw-multiset-number-partition-with-atmost-size*:
  $bij\text{-}betw\ count\ \{N.\ number\text{-}partition\ n\ N \wedge size\ N \leq k\}\ \{p.\ p\ partitions\ n \wedge sum\ p\ \{..n\} \leq k\}$
$\langle proof \rangle$

theorem *card-number-partitions-with-atmost-k-parts*:
  shows $card\ \{N.\ number\text{-}partition\ n\ N \wedge size\ N \leq x\} = Partition\ (n + x)\ x$
$\langle proof \rangle$

theorem *card-partitions-with-k-parts*:
  $card\ \{N.\ number\text{-}partition\ n\ N \wedge size\ N = k\} = Partition\ n\ k$
$\langle proof \rangle$

3.4 Cardinality of Number Partitions with only 1-parts

lemma *number-partition1-eq-replicate-mset*:
  $\{N.\ (\forall n.\ n \in\#\ N \longrightarrow n = 1) \wedge number\text{-}partition\ n\ N\} = \{replicate\text{-}mset\ n\ 1\}$
$\langle proof \rangle$

lemma *card-number-partitions-with-only-parts-1-eq-1*:
  assumes $n \leq x$
  shows $card\ \{N.\ (\forall n.\ n \in\#\ N \longrightarrow n = 1) \wedge number\text{-}partition\ n\ N \wedge size\ N \leq x\} = 1$ (**is** $card\ ?N = -$)
$\langle proof \rangle$

lemma *card-number-partitions-with-only-parts-1-eq-0*:
  assumes $x < n$
  shows $card\ \{N.\ (\forall n.\ n \in\#\ N \longrightarrow n = 1) \wedge number\text{-}partition\ n\ N \wedge size\ N \leq x\} = 0$ (**is** $card\ ?N = -$)
$\langle proof \rangle$

end

References

[1] M. Junker. Diskrete Algebraische Strukturen, 2010. German lecture notes from Mathematisches Institut Albert-Ludwigs-Universität Freiburg.

[2] D. R. Mazur. *Combinatorics: a guided tour.* MAA textbooks. Mathematical Association of America, 2010.