

Cardinality of Number Partitions

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Abstract

This entry provides a basic library for number partitions, defines the two-argument partition function through its recurrence relation and relates this partition function to the cardinality of number partitions. The main proof shows that the recursively-defined partition function with arguments n and k equals the cardinality of number partitions of n with exactly k parts. The combinatorial proof follows the proof sketch of Theorem 2.4.1 in Mazur’s textbook “Combinatorics: A Guided Tour” [2]. This entry can serve as starting point for various more intrinsic properties about number partitions, the partition function and related recurrence relations.

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1 Additions to Isabelle's Main Theories

```
theory Additions-to-Main
imports HOL-Library.Multiset
begin
```

1.1 Addition to Finite-Set Theory

```
lemma bound-domain-and-range-impl-finitely-many-functions:
  finite {f::nat⇒nat. (∀i. f i ≤ n) ∧ (∀i≥m. f i = 0)}
⟨proof⟩
```

1.2 Addition to Set-Interval Theory

```
lemma sum-atMost-remove-nat:
  assumes k ≤ (n :: nat)
  shows (∑ i≤n. f i) = f k + (∑ i∈{..n}-{k}. f i)
⟨proof⟩
```

1.3 Additions to Multiset Theory

```
lemma set-mset-Abs-multiset:
  assumes f ∈ multiset
  shows set-mset (Abs-multiset f) = {x. f x > 0}
⟨proof⟩
```

```
lemma sum-mset-sum-count:
  sum-mset M = (∑ i∈set-mset M. count M i * i)
⟨proof⟩
```

```
lemma sum-mset-eq-sum-on-supersets:
  assumes finite A set-mset M ⊆ A
  shows (∑ i∈set-mset M. count M i * i) = (∑ i∈A. count M i * i)
⟨proof⟩
```

end

2 Number Partitions

```
theory Number-Partition
imports Additions-to-Main
begin
```

2.1 Number Partitions as $\text{nat} \Rightarrow \text{nat}$ Functions

```
definition partitions :: (nat ⇒ nat) ⇒ nat ⇒ bool (infix partitions 50)
where
```

```
  partitions n = ((∀i. p i ≠ 0 ⟶ 1 ≤ i ∧ i ≤ n) ∧ (∑ i≤n. p i * i) = n)
```

lemma *partitionsI*:

assumes $\bigwedge i. p\ i \neq 0 \implies 1 \leq i \wedge i \leq n$

assumes $(\sum_{i \leq n}. p\ i * i) = n$

shows *p partitions n*

<proof>

lemma *partitionsE*:

assumes *p partitions n*

obtains $\bigwedge i. p\ i \neq 0 \implies 1 \leq i \wedge i \leq n \ (\sum_{i \leq n}. p\ i * i) = n$

<proof>

lemma *partitions-zero*:

p partitions 0 $\longleftrightarrow p = (\lambda i. 0)$

<proof>

lemma *partitions-one*:

p partitions (Suc 0) $\longleftrightarrow p = (\lambda i. 0)(1 := 1)$

<proof>

2.2 Bounds and Finiteness of Number Partitions

lemma *partitions-imp-finite-elements*:

assumes *p partitions n*

shows *finite {i. 0 < p i}*

<proof>

lemma *partitions-imp-multiset*:

assumes *p partitions n*

shows *p ∈ multiset*

<proof>

lemma *partitions-bounds*:

assumes *p partitions n*

shows *p i ≤ n*

<proof>

lemma *partitions-parts-bounded*:

assumes *p partitions n*

shows *sum p {..n} ≤ n*

<proof>

lemma *finite-partitions*:

finite {p. p partitions n}

<proof>

lemma *finite-partitions-k-parts*:

finite {p. p partitions n ∧ sum p {..n} = k}

<proof>

lemma *partitions-remaining-Max-part*:
assumes p partitions n
assumes $0 < p\ k$
shows $\forall i. n - k < i \wedge i \neq k \longrightarrow p\ i = 0$
 \langle *proof* \rangle

2.3 Operations of Number Partitions

lemma *partitions-remove1-bounds*:
assumes partitions: p partitions n
assumes $gr0$: $0 < p\ k$
assumes neq : $(p(k := p\ k - 1))\ i \neq 0$
shows $1 \leq i \wedge i \leq n - k$
 \langle *proof* \rangle

lemma *partitions-remove1*:
assumes partitions: p partitions n
assumes $gr0$: $0 < p\ k$
shows $p(k := p\ k - 1)$ partitions $(n - k)$
 \langle *proof* \rangle

lemma *partitions-insert1*:
assumes p : p partitions n
assumes $k > 0$
shows $(p(k := p\ k + 1))$ partitions $(n + k)$
 \langle *proof* \rangle

lemma *count-remove1*:
assumes p partitions n
assumes $0 < p\ k$
shows $(\sum_{i \leq n - k}. (p(k := p\ k - 1))\ i) = (\sum_{i \leq n}. p\ i) - 1$
 \langle *proof* \rangle

lemma *count-insert1*:
assumes p partitions n
shows $sum\ (p(k := p\ k + 1))\ \{..n + k\} = (\sum_{i \leq n}. p\ i) + 1$
 \langle *proof* \rangle

lemma *partitions-decrease1*:
assumes p : p partitions m
assumes sum : $sum\ p\ \{..m\} = k$
assumes $p\ 1 = 0$
shows $(\lambda i. p\ (i + 1))$ partitions $m - k$
 \langle *proof* \rangle

lemma *partitions-increase1*:
assumes partitions: p partitions $m - k$
assumes k : $sum\ p\ \{..m - k\} = k$
shows $(\lambda i. p\ (i - 1))$ partitions m

<proof>

lemma *count-decrease1*:

assumes *p*: *p partitions m*

assumes *sum*: $\text{sum } p \{..m\} = k$

assumes $p \ 1 = 0$

shows $\text{sum } (\lambda i. p \ (i + 1)) \{..m - k\} = k$

<proof>

lemma *count-increase1*:

assumes *partitions*: *p partitions m - k*

assumes *k*: $\text{sum } p \{..m - k\} = k$

shows $(\sum_{i \leq m. p \ (i - 1)}) = k$

<proof>

2.4 Number Partitions as Multisets on Natural Numbers

definition *number-partition* :: *nat* \Rightarrow *nat multiset* \Rightarrow *bool*

where

number-partition *n N* = (*sum-mset N* = *n* \wedge $0 \notin \# N$)

2.4.1 Relationship to Definition on Functions

lemma *count-partitions-iff*:

count N partitions n \longleftrightarrow *number-partition n N*

<proof>

lemma *partitions-iff-Abs-multiset*:

p partitions n \longleftrightarrow *finite* $\{x. 0 < p \ x\} \wedge$ *number-partition n (Abs-multiset p)*

<proof>

lemma *size-nat-multiset-eq*:

fixes *N* :: *nat multiset*

assumes *number-partition n N*

shows $\text{size } N = \text{sum } (\text{count } N) \{..n\}$

<proof>

end

3 Cardinality of Number Partitions

theory *Card-Number-Partitions*

imports *Number-Partition*

begin

3.1 The Partition Function

fun *Partition* :: *nat* \Rightarrow *nat* \Rightarrow *nat*

where

$Partition\ 0\ 0 = 1$
| $Partition\ 0\ (Suc\ k) = 0$
| $Partition\ (Suc\ m)\ 0 = 0$
| $Partition\ (Suc\ m)\ (Suc\ k) = Partition\ m\ k + Partition\ (m - k)\ (Suc\ k)$

lemma *Partition-less*:

assumes $m < k$
shows $Partition\ m\ k = 0$
(*proof*)

lemma *Partition-sum-Partition-diff*:

assumes $k \leq m$
shows $Partition\ m\ k = (\sum_{i \leq k}. Partition\ (m - k)\ i)$
(*proof*)

lemma *Partition-parts1*:

$Partition\ (Suc\ m)\ (Suc\ 0) = 1$
(*proof*)

lemma *Partition-diag*:

$Partition\ (Suc\ m)\ (Suc\ m) = 1$
(*proof*)

lemma *Partition-diag1*:

$Partition\ (Suc\ (Suc\ m))\ (Suc\ m) = 1$
(*proof*)

lemma *Partition-parts2*:

shows $Partition\ m\ 2 = m\ div\ 2$
(*proof*)

3.2 Cardinality of Number Partitions

lemma *set-rewrite1*:

$\{p. p\ partitions\ Suc\ m \wedge sum\ p\ \{..Suc\ m\} = Suc\ k \wedge p\ 1 \neq 0\}$
 $= (\lambda p. p(1 := p\ 1 + 1))\ \{p. p\ partitions\ m \wedge sum\ p\ \{..m\} = k\}$ (**is** $?S = ?T$)
(*proof*)

lemma *set-rewrite2*:

$\{p. p\ partitions\ m \wedge sum\ p\ \{..m\} = k \wedge p\ 1 = 0\}$
 $= (\lambda p. (\lambda i. p\ (i - 1)))\ \{p. p\ partitions\ (m - k) \wedge sum\ p\ \{..m - k\} = k\}$
(**is** $?S = ?T$)
(*proof*)

theorem *card-partitions-k-parts*:

$card\ \{p. p\ partitions\ n \wedge (\sum_{i \leq n}. p\ i) = k\} = Partition\ n\ k$
(*proof*)

theorem *card-partitions:*

$\text{card } \{p. p \text{ partitions } n\} = (\sum k \leq n. \text{Partition } n k)$
(proof)

lemma *card-partitions-atmost-k-parts:*

$\text{card } \{p. p \text{ partitions } n \wedge \text{sum } p \{..n\} \leq k\} = \text{Partition } (n + k) k$
(proof)

3.3 Cardinality of Number Partitions as Multisets of Natural Numbers

lemma *bij-betw-multiset-number-partition-with-size:*

$\text{bij-betw count } \{N. \text{number-partition } n N \wedge \text{size } N = k\} \{p. p \text{ partitions } n \wedge \text{sum } p \{..n\} = k\}$
(proof)

lemma *bij-betw-multiset-number-partition-with-atmost-size:*

$\text{bij-betw count } \{N. \text{number-partition } n N \wedge \text{size } N \leq k\} \{p. p \text{ partitions } n \wedge \text{sum } p \{..n\} \leq k\}$
(proof)

theorem *card-number-partitions-with-atmost-k-parts:*

shows $\text{card } \{N. \text{number-partition } n N \wedge \text{size } N \leq x\} = \text{Partition } (n + x) x$
(proof)

theorem *card-partitions-with-k-parts:*

$\text{card } \{N. \text{number-partition } n N \wedge \text{size } N = k\} = \text{Partition } n k$
(proof)

3.4 Cardinality of Number Partitions with only 1-parts

lemma *number-partition1-eq-replicate-mset:*

$\{N. (\forall n. n \in \# N \longrightarrow n = 1) \wedge \text{number-partition } n N\} = \{\text{replicate-mset } n 1\}$
(proof)

lemma *card-number-partitions-with-only-parts-1-eq-1:*

assumes $n \leq x$
shows $\text{card } \{N. (\forall n. n \in \# N \longrightarrow n = 1) \wedge \text{number-partition } n N \wedge \text{size } N \leq x\} = 1$ (is card ?N = -)
(proof)

lemma *card-number-partitions-with-only-parts-1-eq-0:*

assumes $x < n$
shows $\text{card } \{N. (\forall n. n \in \# N \longrightarrow n = 1) \wedge \text{number-partition } n N \wedge \text{size } N \leq x\} = 0$ (is card ?N = -)
(proof)

end

References

- [1] M. Junker. Diskrete Algebraische Strukturen, 2010. German lecture notes from Mathematisches Institut Albert-Ludwigs-Universität Freiburg.
- [2] D. R. Mazur. *Combinatorics: a guided tour*. MAA textbooks. Mathematical Association of America, 2010.