# Cardinality of Multisets

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**Abstract**

This entry provides three lemmas to count the number of multisets of a given size and finite carrier set. The first lemma provides a cardinality formula assuming that the multiset's elements are chosen from the given carrier set. The latter two lemmas provide formulas assuming that the multiset's elements also cover the given carrier set, i.e., each element of the carrier set occurs in the multiset at least once.

The proof of the first lemma uses the argument of the recurrence relation for counting multisets [1]. The proof of the second lemma is straightforward, and the proof of the third lemma is easily obtained using the first cardinality lemma. A challenge for the formalization is the derivation of the required induction rule, which is a special combination of the induction rules for finite sets and natural numbers. The induction rule is derived by defining a suitable inductive predicate and transforming the predicate's induction rule.

## Contents



# 1 Cardinality of Multisets

**theory** *Card-Multisets*
**imports**
*HOL−Library.Multiset*
**begin**

## 1.1 Additions to Multiset Theory

**lemma** *mset-set-set-mset-subseteq*:

*mset-set* (*set-mset* $M$) $\subseteq\#$ $M$
⟨*proof*⟩

**lemma** *size-mset-set-eq-card*:
**assumes** *finite* $A$
**shows** *size* (*mset-set* $A$) = *card* $A$
⟨*proof*⟩

**lemma** *card-set-mset-leq*:
*card* (*set-mset* $M$) $\leq$ *size* $M$
⟨*proof*⟩

## 1.2 Lemma to Enumerate Sets of Multisets

**lemma** *set-of-multisets-eq*:
**assumes** $x \notin A$
**shows** $\{M.$ *set-mset* $M \subseteq$ *insert* $x$ $A \wedge$ *size* $M =$ *Suc* $k\} =$
$\{M.$ *set-mset* $M \subseteq A \wedge$ *size* $M =$ *Suc* $k\} \cup$
$(\lambda M. M + \{\#x\#\})$ ‘ $\{M.$ *set-mset* $M \subseteq$ *insert* $x$ $A \wedge$ *size* $M = k\}$
⟨*proof*⟩

## 1.3 Derivation of Suitable Induction Rule

**context**
**begin**

**private inductive** $R$ :: *'a set* ⇒ *nat* ⇒ *bool*
**where**
*finite* $A \Longrightarrow R$ $A$ *0*
| $R$ {} $k$
| *finite* $A \Longrightarrow x \notin A \Longrightarrow R$ $A$ (*Suc* $k$) $\Longrightarrow R$ (*insert* $x$ $A$) $k \Longrightarrow R$ (*insert* $x$ $A$) (*Suc* $k$)

**private lemma** *R-eq-finite*:
$R$ $A$ $k \longleftrightarrow$ *finite* $A$
⟨*proof*⟩

**lemma** *finite-set-and-nat-induct*[*consumes 1*, *case-names zero empty step*]:
**assumes** *finite* $A$
**assumes** $\bigwedge A.$ *finite* $A \Longrightarrow P$ $A$ *0*
**assumes** $\bigwedge k.$ $P$ {} $k$
**assumes** $\bigwedge A$ $k$ $x.$ *finite* $A \Longrightarrow x \notin A \Longrightarrow P$ $A$ (*Suc* $k$) $\Longrightarrow P$ (*insert* $x$ $A$) $k \Longrightarrow$ $P$ (*insert* $x$ $A$) (*Suc* $k$)
**shows** $P$ $A$ $k$
⟨*proof*⟩

**end**

### 1.4 Finiteness of Sets of Multisets

**lemma** *finite-multisets*:
  **assumes** *finite A*
  **shows** *finite* $\{M.\ set\text{-}mset\ M \subseteq A \wedge size\ M = k\}$
⟨*proof*⟩

### 1.5 Cardinality of Multisets

**lemma** *card-multisets*:
  **assumes** *finite A*
  **shows** *card* $\{M.\ set\text{-}mset\ M \subseteq A \wedge size\ M = k\} = (card\ A + k - 1)\ choose\ k$
⟨*proof*⟩

**lemma** *card-too-small-multisets-covering-set*:
  **assumes** *finite A*
  **assumes** $k < card\ A$
  **shows** *card* $\{M.\ set\text{-}mset\ M = A \wedge size\ M = k\} = 0$
⟨*proof*⟩

**lemma** *card-multisets-covering-set*:
  **assumes** *finite A*
  **assumes** $card\ A \leq k$
  **shows** *card* $\{M.\ set\text{-}mset\ M = A \wedge size\ M = k\} = (k - 1)\ choose\ (k - card\ A)$
⟨*proof*⟩

**end**

## References


[1] Wikipedia. Multiset — wikipedia, the free encyclopedia, 2016. [Online; accessed 23-June-2016].