

# Cardinality of Multisets

Lukas Bulwahn

April 19, 2020

## Abstract

This entry provides three lemmas to count the number of multisets of a given size and finite carrier set. The first lemma provides a cardinality formula assuming that the multiset's elements are chosen from the given carrier set. The latter two lemmas provide formulas assuming that the multiset's elements also cover the given carrier set, i.e., each element of the carrier set occurs in the multiset at least once.

The proof of the first lemma uses the argument of the recurrence relation for counting multisets [1]. The proof of the second lemma is straightforward, and the proof of the third lemma is easily obtained using the first cardinality lemma. A challenge for the formalization is the derivation of the required induction rule, which is a special combination of the induction rules for finite sets and natural numbers. The induction rule is derived by defining a suitable inductive predicate and transforming the predicate's induction rule.

## Contents

<b>1</b>	<b>Cardinality of Multisets</b>	<b>1</b>
1.1	Additions to Multiset Theory . . . . .	1
1.2	Lemma to Enumerate Sets of Multisets . . . . .	2
1.3	Derivation of Suitable Induction Rule . . . . .	3
1.4	Finiteness of Sets of Multisets . . . . .	4
1.5	Cardinality of Multisets . . . . .	4

## 1 Cardinality of Multisets

```
theory Card-Multisets
imports
  HOL-Library.Multiset
begin
```

### 1.1 Additions to Multiset Theory

```
lemma mset-set-set-mset-subseteq:
```

```

  mset-set (set-mset M)  $\subseteq$  # M
proof (induct M)
  case empty
  show ?case by simp
next
  case (add x M)
  from this show ?case
  proof (cases x  $\in$  # M)
    assume x  $\in$  # M
    from this have mset-set (set-mset (M + {#x#})) = mset-set (set-mset M)
      by (simp add: insert-absorb)
    from this add.hyps show ?thesis
      using subset-mset.order.trans by fastforce
  next
  assume  $\neg$  x  $\in$  # M
  from this add.hyps have {#x#} + mset-set (set-mset M)  $\subseteq$  # M + {#x#}
    by (simp add: insert-subset-eq-iff)
  from this ( $\neg$  x  $\in$  # M) show ?thesis by simp
qed
qed

```

**lemma** *size-mset-set-eq-card*:  
 assumes *finite A*  
 shows  $\text{size (mset-set A)} = \text{card A}$   
 using *assms* by (induct A) auto

**lemma** *card-set-mset-leq*:  
 $\text{card (set-mset M)} \leq \text{size M}$   
 by (induct M) (auto simp add: card-insert-le-m1)

## 1.2 Lemma to Enumerate Sets of Multisets

**lemma** *set-of-multisets-eq*:  
 assumes  $x \notin A$   
 shows  $\{M. \text{set-mset } M \subseteq \text{insert } x A \wedge \text{size } M = \text{Suc } k\} =$   
 $\{M. \text{set-mset } M \subseteq A \wedge \text{size } M = \text{Suc } k\} \cup$   
 $(\lambda M. M + \{#x\}) \cdot \{M. \text{set-mset } M \subseteq \text{insert } x A \wedge \text{size } M = k\}$   
**proof** –  
 from ( $x \notin A$ ) have  $\{M. \text{set-mset } M \subseteq \text{insert } x A \wedge \text{size } M = \text{Suc } k\} =$   
 $\{M. \text{set-mset } M \subseteq A \wedge \text{size } M = \text{Suc } k\} \cup$   
 $\{M. \text{set-mset } M \subseteq \text{insert } x A \wedge \text{size } M = \text{Suc } k \wedge x \in \# M\}$   
 by auto  
 moreover have  $\{M. \text{set-mset } M \subseteq \text{insert } x A \wedge \text{size } M = \text{Suc } k \wedge x \in \# M\} =$   
 $(\lambda M. M + \{#x\}) \cdot \{M. \text{set-mset } M \subseteq \text{insert } x A \wedge \text{size } M = k\}$  (is ?S =  
 ?T)  
**proof**  
 show ?S  $\subseteq$  ?T  
**proof**  
 fix M

```

assume  $M \in ?S$ 
from this have  $M = M - \{\#x\# \} + \{\#x\# \}$  by auto
moreover have  $M - \{\#x\# \} \in \{M. \text{set-mset } M \subseteq \text{insert } x \ A \wedge \text{size } M =$ 
 $k\}$ 
proof –
  have  $\text{set-mset } (M - \{\#x\# \} + \{\#x\# \}) \subseteq \text{insert } x \ A$ 
  using  $\langle M \in ?S \rangle$  by force
  moreover have  $\text{size } (M - \{\#x\# \} + \{\#x\# \}) = \text{Suc } k \wedge x \in \# \ M -$ 
 $\{\#x\# \} + \{\#x\# \}$ 
  using  $\langle M \in ?S \rangle$  by force
  ultimately show  $?thesis$  by force
qed
  ultimately show  $M \in ?T$  by auto
qed
next
  show  $?T \subseteq ?S$  by force
qed
ultimately show  $?thesis$  by auto
qed

```

### 1.3 Derivation of Suitable Induction Rule

**context**

**begin**

**private inductive**  $R :: 'a \text{ set} \Rightarrow \text{nat} \Rightarrow \text{bool}$

**where**

$\text{finite } A \Longrightarrow R \ A \ 0$

$| R \ \{\} \ k$

$| \text{finite } A \Longrightarrow x \notin A \Longrightarrow R \ A \ (\text{Suc } k) \Longrightarrow R \ (\text{insert } x \ A) \ k \Longrightarrow R \ (\text{insert } x \ A)$   
 $(\text{Suc } k)$

**private lemma**  $R\text{-eq-finite}$ :

$R \ A \ k \longleftrightarrow \text{finite } A$

**proof**

**assume**  $R \ A \ k$

**from this show**  $\text{finite } A$  **by cases auto**

**next**

**assume**  $\text{finite } A$

**from this show**  $R \ A \ k$

**proof**  $(\text{induct } A)$

**case empty**

**from this show**  $?case$  **by**  $(\text{rule } R.\text{intros}(2))$

**next**

**case insert**

**from this show**  $?case$

**proof**  $(\text{induct } k)$

**case 0**

**from this show**  $?case$

```

      by (intro R.intros(1) finite.insertI)
    next
      case Suc
      from this show ?case
        by (metis R.simps Zero-neq-Suc diff-Suc-1)
      qed
    qed
  qed

```

**lemma** *finite-set-and-nat-induct*[consumes 1, case-names zero empty step]:

```

  assumes finite A
  assumes  $\bigwedge A. \text{finite } A \implies P A 0$ 
  assumes  $\bigwedge k. P \{ \} k$ 
  assumes  $\bigwedge A k x. \text{finite } A \implies x \notin A \implies P A (\text{Suc } k) \implies P (\text{insert } x A) k \implies P (\text{insert } x A) (\text{Suc } k)$ 
  shows  $P A k$ 
  proof -
    from  $\langle \text{finite } A \rangle$  have  $R A k$  by (subst R.eq-finite)
    from this assms(2-4) show ?thesis by (induct A k) auto
  qed
end

```

## 1.4 Finiteness of Sets of Multisets

**lemma** *finite-multisets*:

```

  assumes finite A
  shows  $\text{finite } \{M. \text{set-mset } M \subseteq A \wedge \text{size } M = k\}$ 
  using assms
  proof (induct A k rule: finite-set-and-nat-induct)
    case zero
    from this show ?case by auto
  next
    case empty
    from this show ?case by auto
  next
    case (step A k x)
    from this show ?case
      using set-of-multisets-eq[OF  $\langle x \notin A \rangle$ ] by simp
  qed

```

## 1.5 Cardinality of Multisets

**lemma** *card-multisets*:

```

  assumes finite A
  shows  $\text{card } \{M. \text{set-mset } M \subseteq A \wedge \text{size } M = k\} = (\text{card } A + k - 1) \text{ choose } k$ 
  using assms
  proof (induct A k rule: finite-set-and-nat-induct)
    case (zero A)
    assume finite (A :: 'a set)

```

**have**  $\{M. \text{set-mset } M \subseteq A \wedge \text{size } M = 0\} = \{\{\#\}\}$  **by** *auto*  
**from this show**  $\text{card } \{M. \text{set-mset } M \subseteq A \wedge \text{size } M = 0\} = \text{card } A + 0 - 1$   
*choose 0*  
**by** *simp*  
**next**  
**case** (*empty k*)  
**show**  $\text{card } \{M. \text{set-mset } M \subseteq \{\} \wedge \text{size } M = k\} = \text{card } \{\} + k - 1$  *choose k*  
**by** (*cases k*) (*auto simp add: binomial-eq-0*)  
**next**  
**case** (*step A k x*)  
**let**  $?S_1 = \{M. \text{set-mset } M \subseteq A \wedge \text{size } M = \text{Suc } k\}$   
**and**  $?S_2 = \{M. \text{set-mset } M \subseteq \text{insert } x \ A \wedge \text{size } M = k\}$   
**assume** *hyps1*:  $\text{card } ?S_1 = \text{card } A + \text{Suc } k - 1$  *choose Suc k*  
**assume** *hyps2*:  $\text{card } ?S_2 = \text{card } (\text{insert } x \ A) + k - 1$  *choose k*  
**have** *finite-sets*: *finite*  $?S_1$  *finite*  $((\lambda M. M + \{\#x\}) \text{ ` } ?S_2)$   
**using**  $\langle \text{finite } A \rangle$  **by** (*auto simp add: finite-multisets*)  
**have** *inj*: *inj-on*  $(\lambda M. M + \{\#x\}) \text{ ` } ?S_2$  **by** (*rule inj-onI*) *auto*  
**have**  $\text{card } \{M. \text{set-mset } M \subseteq \text{insert } x \ A \wedge \text{size } M = \text{Suc } k\} =$   
 $\text{card } (?S_1 \cup (\lambda M. M + \{\#x\}) \text{ ` } ?S_2)$   
**using** *set-of-multisets-eq*  $\langle x \notin A \rangle$  **by** *fastforce*  
**also have**  $\dots = \text{card } ?S_1 + \text{card } ((\lambda M. M + \{\#x\}) \text{ ` } ?S_2)$   
**using** *finite-sets*  $\langle x \notin A \rangle$  **by** (*subst card-Un-disjoint*) *auto*  
**also have**  $\dots = \text{card } ?S_1 + \text{card } ?S_2$   
**using** *inj* **by** (*auto intro: card-image*)  
**also have**  $\dots = \text{card } A + \text{Suc } k - 1$  *choose Suc k* +  $(\text{card } (\text{insert } x \ A) + k -$   
 $1)$  *choose k*)  
**using** *hyps1 hyps2* **by** *simp*  
**also have**  $\dots = \text{card } (\text{insert } x \ A) + \text{Suc } k - 1$  *choose Suc k*  
**using**  $\langle x \notin A \rangle \langle \text{finite } A \rangle$  **by** *simp*  
**finally show** *?case* .  
**qed**

**lemma** *card-too-small-multisets-covering-set*:

**assumes** *finite A*  
**assumes**  $k < \text{card } A$   
**shows**  $\text{card } \{M. \text{set-mset } M = A \wedge \text{size } M = k\} = 0$   
**proof** –  
**from**  $\langle k < \text{card } A \rangle$  **have** *eq*:  $\{M. \text{set-mset } M = A \wedge \text{size } M = k\} = \{\}$   
**using** *card-set-mset-leq Collect-empty-eq leD* **by** *auto*  
**from this show** *?thesis* **by** (*metis card-empty*)  
**qed**

**lemma** *card-multisets-covering-set*:

**assumes** *finite A*  
**assumes**  $\text{card } A \leq k$   
**shows**  $\text{card } \{M. \text{set-mset } M = A \wedge \text{size } M = k\} = (k - 1)$  *choose*  $(k - \text{card } A)$   
**proof** –  
**have**  $\{M. \text{set-mset } M = A \wedge \text{size } M = k\} = (\lambda M. M + \text{mset-set } A) \text{ ` }$

```

    { $M$ . set-mset  $M \subseteq A \wedge \text{size } M = k - \text{card } A$ } (is  $?S = ?f \text{ ' } ?T$ )
  proof
    show  $?S \subseteq ?f \text{ ' } ?T$ 
    proof
      fix  $M$ 
      assume  $M \in ?S$ 
      from this have  $M = M - \text{mset-set } A + \text{mset-set } A$ 
        by (auto simp add: mset-set-set-mset-subseteq subset-mset.diff-add)
      moreover from  $\langle M \in ?S \rangle$  have  $M - \text{mset-set } A \in ?T$ 
        by (auto simp add: mset-set-set-mset-subseteq size-Diff-submset size-mset-set-eq-card
in-diffD)
      ultimately show  $M \in ?f \text{ ' } ?T$  by auto
    qed
  next
    from  $\langle \text{finite } A \rangle \langle \text{card } A \leq k \rangle$  show  $?f \text{ ' } ?T \subseteq ?S$ 
      by (auto simp add: size-mset-set-eq-card)+
    qed
    moreover have inj-on  $?f \text{ ' } ?T$  by (rule inj-onI) auto
    ultimately have  $\text{card } ?S = \text{card } ?T$  by (simp add: card-image)
    also have  $\dots = \text{card } A + (k - \text{card } A) - 1$  choose  $(k - \text{card } A)$ 
      using  $\langle \text{finite } A \rangle$  by (simp only: card-multisets)
    also have  $\dots = (k - 1)$  choose  $(k - \text{card } A)$ 
      using  $\langle \text{card } A \leq k \rangle$  by auto
    finally show ?thesis .
  qed
end

```

## References

- [1] Wikipedia. Multiset — wikipedia, the free encyclopedia, 2016. [Online; accessed 23-June-2016].