Abstract
This entry provides formulae for counting the number of equivalence relations and partial equivalence relations over a finite carrier set with given cardinality.

To count the number of equivalence relations, we provide bijections between equivalence relations and set partitions [4], and then transfer the main results of the two AFP entries, Cardinality of Set Partitions [1] and Spivey’s Generalized Recurrence for Bell Numbers [2], to theorems on equivalence relations. To count the number of partial equivalence relations, we observe that counting partial equivalence relations over a set $A$ is equivalent to counting all equivalence relations over all subsets of the set $A$. From this observation and the results on equivalence relations, we show that the cardinality of partial equivalence relations over a finite set of cardinality $n$ is equal to the $n + 1$-th Bell number [3].
1 Cardinality of Equivalence Relations

theory Card-Equiv-Relations
imports
  Card-Partitions.Card-Partitions
  Bell-Numbers-Spivey.Bell-Numbers
begin

1.1 Bijection between Equivalence Relations and Set Partitions

1.1.1 Possibly Interesting Theorem for HOL.Equiv-Relations

This theorem was historically useful in this theory, but is now after some proof refactoring not needed here anymore. Possibly it is an interesting fact about equivalence relations, though.

lemma equiv-quotient-eq-quotient-on-UNIV:
  assumes equiv A R
  shows A // R = (UNIV // R) − {{}}
⟨proof⟩

1.1.2 Dedicated Facts for Bijection Proof

lemma equiv-relation-of-partition-of:
  assumes equiv A R
  shows \{(x, y). \exists X \in A // R. x \in X \land y \in X\} = R
⟨proof⟩

1.1.3 Bijection Proof

lemma bij-betw-partition-of:
  bij_betw (λR. A // R) {R. equiv A R} {P. partition-on A P}
⟨proof⟩

lemma bij-betw-partition-of-equiv-with-k-classes:
  bij_betw (λR. A // R) {R. equiv A R \land card (A // R) = k} {P. partition-on A P \land card P = k}
⟨proof⟩

1.2 Finiteness of Equivalence Relations

lemma finite-equiv:
  assumes finite A
  shows finite \{R. equiv A R\}
⟨proof⟩

1.3 Cardinality of Equivalence Relations

theorem card-equiv-rel-eq-card-partitions:
\begin{verbatim}
  \text{card} \{R. \text{equiv} A R\} = \text{card} \{P. \text{partition-on} A P\}
\end{verbatim}

**corollary** \text{card-equiv-rel-eq-Bell}:
- \textbf{assumes} finite \(A\)
- \textbf{shows} \(\text{card} \{R. \text{equiv} A R\} = \text{Bell} (\text{card} A)\)

**corollary** \text{card-equiv-rel-eq-sum-Stirling}:
- \textbf{assumes} finite \(A\)
- \textbf{shows} \(\text{card} \{R. \text{equiv} A R\} = \text{sum} (\text{Stirling} (\text{card} A)) \{..\text{card} A\}\)

**theorem** \text{card-equiv-k-classes-eq-card-partitions-k-parts}:
- \(\text{card} \{R. \text{equiv} A R \land \text{card} (A \mathbin{/}\ R) = k\} = \text{card} \{P. \text{partition-on} A P \land \text{card} P = k\}\)

**corollary**
- \textbf{assumes} finite \(A\)
- \textbf{shows} \(\text{card} \{R. \text{equiv} A R \land \text{card} (A \mathbin{/}\ R) = k\} = \text{Stirling} (\text{card} A) k\)

\end{verbatim}

2 Cardinality of Partial Equivalence Relations

**theory** \text{Card-Partial-Equiv-Relations}

**imports**
- \text{Card-Equiv-Relations}

**begin**

2.1 Definition of Partial Equivalence Relation

**definition** \text{partial-equiv} :: \\
'\text{a set} \Rightarrow (\text{'}\text{a} \times \text{'}\text{a}) \Rightarrow \text{bool}
**where**
\text{partial-equiv} A R = (R \subseteq A \times A \land \text{sym} R \land \text{trans} R)

**lemma** \text{partial-equiv-f}:
- \textbf{assumes} \(R \subseteq A \times A\ \text{sym} R\ \text{trans} R\)
- \textbf{shows} \text{partial-equiv} A R

**lemma** \text{partial-equiv-iff}:
- \textbf{shows} \text{partial-equiv} A R \iff (\exists A' \subseteq A.\ \text{equiv} A' R)

end
2.2 Construction of all Partial Equivalence Relations for a Given Set

definition all-partial-equivs-on :: 'a set ⇒ (('a × 'a) set) set
where
  all-partial-equivs-on A =
  do
    k ← {0..card A};
    A' ← {A'. A' ⊆ A ∧ card A' = k};
    {R. equiv A' R}
  
lemma all-partial-equivs-on:
  assumes finite A
  shows all-partial-equivs-on A = {R. partial-equiv A R}
  ⟨proof⟩

2.3 Injectivity of the Set Construction

lemma equiv-inject:
  assumes equiv A R equiv B R
  shows A = B
  ⟨proof⟩

lemma injectivity:
  assumes (A' ⊆ A ∧ card A' = k) ∧ (A'' ⊆ A ∧ card A'' = k')
  assumes equiv A' R ∧ equiv A'' R'
  assumes R = R'
  shows A' = A'' k = k'
  ⟨proof⟩

2.4 Cardinality Theorem of Partial Equivalence Relations

theorem card-partial-equiv:
  assumes finite A
  shows card {R. partial-equiv A R} = Bell (card A + 1)
  ⟨proof⟩

end

References

