Cardinality of Equivalence Relations

Lukas Bulwahn

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Abstract

This entry provides formulae for counting the number of equivalence relations and partial equivalence relations over a finite carrier set with given cardinality.

To count the number of equivalence relations, we provide bijections between equivalence relations and set partitions [4], and then transfer the main results of the two AFP entries, Cardinality of Set Partitions [1] and Spivey's Generalized Recurrence for Bell Numbers [2], to theorems on equivalence relations. To count the number of partial equivalence relations, we observe that counting partial equivalence relations over a set A is equivalent to counting all equivalence relations over all subsets of the set A. From this observation and the results on equivalence relations, we show that the cardinality of partial equivalence relations over a finite set of cardinality n is equal to the $n+1$-th Bell number [3].

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1 Cardinality of Equivalence Relations

theory *Card-Equiv-Relations*
imports
  Card-Partitions.Card-Partitions
  Bell-Numbers-Spivey.Bell-Numbers
begin

1.1 Bijection between Equivalence Relations and Set Partitions

1.1.1 Possibly Interesting Theorem for *HOL.Equiv-Relations*

This theorem was historically useful in this theory, but is now after some proof refactoring not needed here anymore. Possibly it is an interesting fact about equivalence relations, though.

lemma *equiv-quotient-eq-quotient-on-UNIV*:
  assumes $equiv\ A\ R$
  shows $A\ //\ R = (UNIV\ //\ R) - \{\{\}\}$
$\langle proof \rangle$

1.1.2 Dedicated Facts for Bijection Proof

lemma *equiv-relation-of-partition-of*:
  assumes $equiv\ A\ R$
  shows $\{(x, y).\ \exists X \in A\ //\ R.\ x \in X \wedge y \in X\} = R$
$\langle proof \rangle$

1.1.3 Bijection Proof

lemma *bij-betw-partition-of*:
  $bij\text{-}betw\ (\lambda R.\ A\ //\ R)\ \{R.\ equiv\ A\ R\}\ \{P.\ partition\text{-}on\ A\ P\}$
$\langle proof \rangle$

lemma *bij-betw-partition-of-equiv-with-k-classes*:
  $bij\text{-}betw\ (\lambda R.\ A\ //\ R)\ \{R.\ equiv\ A\ R \wedge card\ (A\ //\ R) = k\}\ \{P.\ partition\text{-}on\ A\ P \wedge card\ P = k\}$
$\langle proof \rangle$

1.2 Finiteness of Equivalence Relations

lemma *finite-equiv*:
  assumes $finite\ A$
  shows $finite\ \{R.\ equiv\ A\ R\}$
$\langle proof \rangle$

1.3 Cardinality of Equivalence Relations

theorem *card-equiv-rel-eq-card-partitions*:

card {*R*. *equiv* *A* *R*} = *card* {*P*. *partition-on* *A* *P*}
⟨*proof*⟩

corollary *card-equiv-rel-eq-Bell*:
assumes *finite A*
shows *card* {*R*. *equiv* *A* *R*} = *Bell* (*card* *A*)
⟨*proof*⟩

corollary *card-equiv-rel-eq-sum-Stirling*:
assumes *finite A*
shows *card* {*R*. *equiv* *A* *R*} = *sum* (*Stirling* (*card* *A*)) {..*card* *A*}
⟨*proof*⟩

theorem *card-equiv-k-classes-eq-card-partitions-k-parts*:
card {*R*. *equiv* *A* *R* ∧ *card* (*A* // *R*) = *k*} = *card* {*P*. *partition-on* *A* *P* ∧ *card* *P* = *k*}
⟨*proof*⟩

corollary
assumes *finite A*
shows *card* {*R*. *equiv* *A* *R* ∧ *card* (*A* // *R*) = *k*} = *Stirling* (*card* *A*) *k*
⟨*proof*⟩

end

2 Cardinality of Partial Equivalence Relations

theory *Card-Partial-Equiv-Relations*
imports
Card-Equiv-Relations
begin

2.1 Definition of Partial Equivalence Relation

definition *partial-equiv* :: *'a set* ⇒ (*'a* × *'a*) *set* ⇒ *bool*
where
partial-equiv *A* *R* = (*R* ⊆ *A* × *A* ∧ *sym* *R* ∧ *trans* *R*)

lemma *partial-equivI*:
assumes *R* ⊆ *A* × *A* *sym* *R* *trans* *R*
shows *partial-equiv* *A* *R*
⟨*proof*⟩

lemma *partial-equiv-iff*:
shows *partial-equiv* *A* *R* ⟷ (∃ *A'* ⊆ *A*. *equiv* *A'* *R*)
⟨*proof*⟩

2.2 Construction of all Partial Equivalence Relations for a Given Set

definition *all-partial-equivs-on* :: $'a\ set \Rightarrow (('a \times 'a)\ set)\ set$
where
  $all\text{-}partial\text{-}equivs\text{-}on\ A =$
    $do\ \{$
      $k \leftarrow \{0..card\ A\};$
      $A' \leftarrow \{A'.\ A' \subseteq A \wedge card\ A' = k\};$
      $\{R.\ equiv\ A'\ R\}$
    $\}$

lemma *all-partial-equivs-on*:
  assumes *finite A*
  shows $all\text{-}partial\text{-}equivs\text{-}on\ A = \{R.\ partial\text{-}equiv\ A\ R\}$
$\langle proof \rangle$

2.3 Injectivity of the Set Construction

lemma *equiv-inject*:
  assumes $equiv\ A\ R$ $equiv\ B\ R$
  shows $A = B$
$\langle proof \rangle$

lemma *injectivity*:
  assumes $(A' \subseteq A \wedge card\ A' = k) \wedge (A'' \subseteq A \wedge card\ A'' = k')$
  assumes $equiv\ A'\ R \wedge equiv\ A''\ R'$
  assumes $R = R'$
  shows $A' = A''$ $k = k'$
$\langle proof \rangle$

2.4 Cardinality Theorem of Partial Equivalence Relations

theorem *card-partial-equiv*:
  assumes *finite A*
  shows $card\ \{R.\ partial\text{-}equiv\ A\ R\} = Bell\ (card\ A + 1)$
$\langle proof \rangle$

end

References

[1] L. Bulwahn. Cardinality of set partitions. *Archive of Formal Proofs*, Dec. 2015. http://www.isa-afp.org/entries/Card_Partitions.shtml, Formal proof development.

[2] L. Bulwahn. Spivey's generalized recurrence for bell numbers. *Archive of Formal Proofs*, May 2016. http://www.isa-afp.org/entries/Bell_Numbers_Spivey.shtml, Formal proof development.

[3] N. J. A. Sloane. A000110: Bell or exponential numbers: number of ways to partition a set of n labeled elements. In *The On-Line Encyclopedia of Integer Sequences*. https://oeis.org/A000110.

[4] Wikipedia. Equivalence relation — wikipedia, the free encyclopedia, 2016. [Online; accessed 23-May-2016].