

Cardinality of Equivalence Relations

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Abstract

This entry provides formulae for counting the number of equivalence relations and partial equivalence relations over a finite carrier set with given cardinality.

To count the number of equivalence relations, we provide bijections between equivalence relations and set partitions [4], and then transfer the main results of the two AFP entries, Cardinality of Set Partitions [1] and Spivey's Generalized Recurrence for Bell Numbers [2], to theorems on equivalence relations. To count the number of partial equivalence relations, we observe that counting partial equivalence relations over a set A is equivalent to counting all equivalence relations over all subsets of the set A . From this observation and the results on equivalence relations, we show that the cardinality of partial equivalence relations over a finite set of cardinality n is equal to the $n + 1$ -th Bell number [3].

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1 Cardinality of Equivalence Relations

```
theory Card-Equiv-Relations
imports
  Card-Partitions.Card-Partitions
  Bell-Numbers-Spivey.Bell-Numbers
begin
```

1.1 Bijection between Equivalence Relations and Set Partitions

1.1.1 Possibly Interesting Theorem for *HOL.Equiv-Relations*

This theorem was historically useful in this theory, but is now after some proof refactoring not needed here anymore. Possibly it is an interesting fact about equivalence relations, though.

```
lemma equiv-quotient-eq-quotient-on-UNIV:
  assumes equiv A R
  shows  $A // R = (UNIV // R) - \{\{\}\}$ 
  <proof>
```

1.1.2 Dedicated Facts for Bijection Proof

```
lemma equiv-relation-of-partition-of:
  assumes equiv A R
  shows  $\{(x, y). \exists X \in A // R. x \in X \wedge y \in X\} = R$ 
  <proof>
```

1.1.3 Bijection Proof

```
lemma bij-betw-partition-of:
  bij-betw  $(\lambda R. A // R) \{R. \text{equiv } A R\} \{P. \text{partition-on } A P\}$ 
  <proof>
```

```
lemma bij-betw-partition-of-equiv-with-k-classes:
  bij-betw  $(\lambda R. A // R) \{R. \text{equiv } A R \wedge \text{card } (A // R) = k\} \{P. \text{partition-on } A P \wedge \text{card } P = k\}$ 
  <proof>
```

1.2 Finiteness of Equivalence Relations

```
lemma finite-equiv:
  assumes finite A
  shows finite  $\{R. \text{equiv } A R\}$ 
  <proof>
```

1.3 Cardinality of Equivalence Relations

```
theorem card-equiv-rel-eq-card-partitions:
```

$\text{card } \{R. \text{equiv } A R\} = \text{card } \{P. \text{partition-on } A P\}$
<proof>

corollary *card-equiv-rel-eq-Bell:*

assumes *finite A*

shows $\text{card } \{R. \text{equiv } A R\} = \text{Bell } (\text{card } A)$

<proof>

corollary *card-equiv-rel-eq-sum-Stirling:*

assumes *finite A*

shows $\text{card } \{R. \text{equiv } A R\} = \text{sum } (\text{Stirling } (\text{card } A)) \{.. \text{card } A\}$

<proof>

theorem *card-equiv-k-classes-eq-card-partitions-k-parts:*

$\text{card } \{R. \text{equiv } A R \wedge \text{card } (A // R) = k\} = \text{card } \{P. \text{partition-on } A P \wedge \text{card } P = k\}$

<proof>

corollary

assumes *finite A*

shows $\text{card } \{R. \text{equiv } A R \wedge \text{card } (A // R) = k\} = \text{Stirling } (\text{card } A) k$

<proof>

end

2 Cardinality of Partial Equivalence Relations

theory *Card-Partial-Equiv-Relations*

imports

Card-Equiv-Relations

begin

2.1 Definition of Partial Equivalence Relation

definition *partial-equiv* :: *'a set* \Rightarrow *('a \times 'a) set* \Rightarrow *bool*

where

$\text{partial-equiv } A R = (R \subseteq A \times A \wedge \text{sym } R \wedge \text{trans } R)$

lemma *partial-equivI:*

assumes $R \subseteq A \times A$ *sym R trans R*

shows *partial-equiv A R*

<proof>

lemma *partial-equiv-iff:*

shows $\text{partial-equiv } A R \longleftrightarrow (\exists A' \subseteq A. \text{equiv } A' R)$

<proof>

2.2 Construction of all Partial Equivalence Relations for a Given Set

definition *all-partial-equivs-on* :: 'a set \Rightarrow (('a \times 'a) set) set

where

```
all-partial-equivs-on A =  
do {  
  k  $\leftarrow$  {0..card A};  
  A'  $\leftarrow$  {A'. A'  $\subseteq$  A  $\wedge$  card A' = k};  
  {R. equiv A' R}  
}
```

lemma *all-partial-equivs-on*:

assumes *finite* A

shows *all-partial-equivs-on* A = {R. *partial-equiv* A R}

<proof>

2.3 Injectivity of the Set Construction

lemma *equiv-inject*:

assumes *equiv* A R *equiv* B R

shows A = B

<proof>

lemma *injectivity*:

assumes (A' \subseteq A \wedge *card* A' = k) \wedge (A'' \subseteq A \wedge *card* A'' = k')

assumes *equiv* A' R \wedge *equiv* A'' R'

assumes R = R'

shows A' = A'' k = k'

<proof>

2.4 Cardinality Theorem of Partial Equivalence Relations

theorem *card-partial-equiv*:

assumes *finite* A

shows *card* {R. *partial-equiv* A R} = *Bell* (*card* A + 1)

<proof>

end

References

- [1] L. Bulwahn. Cardinality of set partitions. *Archive of Formal Proofs*, Dec. 2015. http://www.isa-afp.org/entries/Card_Partitions.shtml, Formal proof development.

- [2] L. Bulwahn. Spivey's generalized recurrence for bell numbers. *Archive of Formal Proofs*, May 2016. http://www.isa-afp.org/entries/Bell_Numbers_Spivey.shtml, Formal proof development.
- [3] N. J. A. Sloane. A000110: Bell or exponential numbers: number of ways to partition a set of n labeled elements. In *The On-Line Encyclopedia of Integer Sequences*. <https://oeis.org/A000110>.
- [4] Wikipedia. Equivalence relation — wikipedia, the free encyclopedia, 2016. [Online; accessed 23-May-2016].