

Cardinality of Equivalence Relations

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Abstract

This entry provides formulae for counting the number of equivalence relations and partial equivalence relations over a finite carrier set with given cardinality.

To count the number of equivalence relations, we provide bijections between equivalence relations and set partitions [4], and then transfer the main results of the two AFP entries, Cardinality of Set Partitions [1] and Spivey’s Generalized Recurrence for Bell Numbers [2], to theorems on equivalence relations. To count the number of partial equivalence relations, we observe that counting partial equivalence relations over a set A is equivalent to counting all equivalence relations over all subsets of the set A . From this observation and the results on equivalence relations, we show that the cardinality of partial equivalence relations over a finite set of cardinality n is equal to the $n + 1$ -th Bell number [3].

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1 Cardinality of Equivalence Relations

```
theory Card-Equiv-Relations
imports
  Card-Partitions.Card-Partitions
  Bell-Numbers-Spivey.Bell-Numbers
begin
```

1.1 Bijection between Equivalence Relations and Set Partitions

1.1.1 Possibly Interesting Theorem for *HOL.Equiv-Relations*

This theorem was historically useful in this theory, but is now after some proof refactoring not needed here anymore. Possibly it is an interesting fact about equivalence relations, though.

```
lemma equiv-quotient-eq-quotient-on-UNIV:
  assumes equiv A R
  shows A // R = (UNIV // R) - {{}}
proof
  show UNIV // R - {{}}  $\subseteq$  A // R
  proof
    fix X
    assume X  $\in$  UNIV // R - {{}}
    from this obtain x where X = R “ {x} and X  $\neq$  {}
      by (auto elim!: quotientE)
    from this have x  $\in$  A
      using  $\langle$ equiv A R $\rangle$  equiv-class-eq-iff by fastforce
    from this  $\langle$ X = R “ {x} $\rangle$  show X  $\in$  A // R
      by (auto intro!: quotientI)
  qed
next
  show A // R  $\subseteq$  UNIV // R - {{}}
  proof
    fix X
    assume X  $\in$  A // R
    from this have X  $\neq$  {}
      using  $\langle$ equiv A R $\rangle$  in-quotient-imp-non-empty by auto
    moreover from  $\langle$ X  $\in$  A // R $\rangle$  have X  $\in$  UNIV // R
      by (metis UNIV-I assms proj-Eps proj-preserves)
    ultimately show X  $\in$  UNIV // R - {{}} by simp
  qed
qed
```

1.1.2 Dedicated Facts for Bijection Proof

```
lemma equiv-relation-of-partition-of:
  assumes equiv A R
  shows  $\{(x, y). \exists X \in A // R. x \in X \wedge y \in X\} = R$ 
```

proof
show $\{(x, y). \exists X \in A // R. x \in X \wedge y \in X\} \subseteq R$
proof
fix xy
assume $xy \in \{(x, y). \exists X \in A // R. x \in X \wedge y \in X\}$
from this obtain $x y$ **and** X **where** $xy = (x, y)$
and $X \in A // R$ **and** $x \in X y \in X$
by *auto*
from $\langle X \in A // R \rangle$ **obtain** z **where** $X = R \text{ `` } \{z\}$
by (*auto elim: quotientE*)
show $xy \in R$
using $\langle xy = (x, y) \rangle \langle X = R \text{ `` } \{z\} \rangle \langle x \in X \rangle \langle y \in X \rangle \langle \text{equiv } A R \rangle$
by (*simp add: equiv-class-eq-iff*)
qed
next
show $R \subseteq \{(x, y). \exists X \in A // R. x \in X \wedge y \in X\}$
proof
fix xy
assume $xy \in R$
obtain $x y$ **where** $xy = (x, y)$ **by** *fastforce*
from $\langle xy \in R \rangle$ **have** $(x, y) \in R$
using $\langle xy = (x, y) \rangle$ **by** *simp*
have $R \text{ `` } \{x\} \in A // R$
using $\langle \text{equiv } A R \rangle \langle (x, y) \in R \rangle$
by (*simp add: equiv-class-eq-iff quotientI*)
moreover have $x \in R \text{ `` } \{x\}$
using $\langle (x, y) \in R \rangle \langle \text{equiv } A R \rangle$
by (*meson equiv-class-eq-iff equiv-class-self*)
moreover have $y \in R \text{ `` } \{x\}$
using $\langle (x, y) \in R \rangle \langle \text{equiv } A R \rangle$ **by** *simp*
ultimately have $(x, y) \in \{(x, y). \exists X \in A // R. x \in X \wedge y \in X\}$
by *auto*
from this show $xy \in \{(x, y). \exists X \in A // R. x \in X \wedge y \in X\}$
using $\langle xy = (x, y) \rangle$ **by** *simp*
qed
qed

1.1.3 Bijection Proof

lemma *bij-betw-partition-of*:

bij-betw $(\lambda R. A // R) \{R. \text{equiv } A R\} \{P. \text{partition-on } A P\}$

proof (*rule bij-betw-byWitness* **where** $f' = \lambda P. \{(x, y). \exists X \in P. x \in X \wedge y \in X\}$)

show $\forall R \in \{R. \text{equiv } A R\}. \{(x, y). \exists X \in A // R. x \in X \wedge y \in X\} = R$

by (*simp add: equiv-relation-of-partition-of*)

show $\forall P \in \{P. \text{partition-on } A P\}. A // \{(x, y). \exists X \in P. x \in X \wedge y \in X\} = P$

by (*simp add: partition-on-eq-quotient*)

show $(\lambda R. A // R) \text{ `` } \{R. \text{equiv } A R\} \subseteq \{P. \text{partition-on } A P\}$

using *partition-on-quotient* **by** *auto*

show $(\lambda P. \{(x, y). \exists X \in P. x \in X \wedge y \in X\}) \text{ `` } \{P. \text{partition-on } A P\} \subseteq \{R.$

equiv A R
using *equiv-partition-on by auto*
qed

lemma *bij-betw-partition-of-equiv-with-k-classes:*

bij-betw $(\lambda R. A // R) \{R. \text{equiv } A R \wedge \text{card } (A // R) = k\} \{P. \text{partition-on } A P \wedge \text{card } P = k\}$

proof (*rule* *bij-betw-byWitness*[**where** $f' = \lambda P. \{(x, y). \exists X \in P. x \in X \wedge y \in X\}$])

show $\forall R \in \{R. \text{equiv } A R \wedge \text{card } (A // R) = k\}. \{(x, y). \exists X \in A // R. x \in X \wedge y \in X\} = R$

by (*auto simp add: equiv-relation-of-partition-of*)

show $\forall P \in \{P. \text{partition-on } A P \wedge \text{card } P = k\}. A // \{(x, y). \exists X \in P. x \in X \wedge y \in X\} = P$

by (*auto simp add: partition-on-eq-quotient*)

show $(\lambda R. A // R) \{R. \text{equiv } A R \wedge \text{card } (A // R) = k\} \subseteq \{P. \text{partition-on } A P \wedge \text{card } P = k\}$

using *partition-on-quotient by auto*

show $(\lambda P. \{(x, y). \exists X \in P. x \in X \wedge y \in X\}) \{P. \text{partition-on } A P \wedge \text{card } P = k\} \subseteq \{R. \text{equiv } A R \wedge \text{card } (A // R) = k\}$

using *equiv-partition-on by (auto simp add: partition-on-eq-quotient)*

qed

1.2 Finiteness of Equivalence Relations

lemma *finite-equiv:*

assumes *finite A*

shows *finite* $\{R. \text{equiv } A R\}$

proof –

have *bij-betw* $(\lambda R. A // R) \{R. \text{equiv } A R\} \{P. \text{partition-on } A P\}$

by (*rule* *bij-betw-partition-of*)

from *this* **show** *finite* $\{R. \text{equiv } A R\}$

using $\langle \text{finite } A \rangle$ *finitely-many-partition-on by (simp add: bij-betw-finite)*

qed

1.3 Cardinality of Equivalence Relations

theorem *card-equiv-rel-eq-card-partitions:*

card $\{R. \text{equiv } A R\} = \text{card } \{P. \text{partition-on } A P\}$

proof –

have *bij-betw* $(\lambda R. A // R) \{R. \text{equiv } A R\} \{P. \text{partition-on } A P\}$

by (*rule* *bij-betw-partition-of*)

from *this* **show** *card* $\{R. \text{equiv } A R\} = \text{card } \{P. \text{partition-on } A P\}$

by (*rule* *bij-betw-same-card*)

qed

corollary *card-equiv-rel-eq-Bell:*

assumes *finite A*

shows *card* $\{R. \text{equiv } A R\} = \text{Bell } (\text{card } A)$

using *assms Bell-altdef card-equiv-rel-eq-card-partitions by force*

corollary *card-equiv-rel-eq-sum-Stirling*:

assumes *finite A*

shows $\text{card } \{R. \text{equiv } A R\} = \text{sum } (\text{Stirling } (\text{card } A)) \{.. \text{card } A\}$

using *assms card-equiv-rel-eq-Bell Bell-Stirling-eq* **by** *simp*

theorem *card-equiv-k-classes-eq-card-partitions-k-parts*:

$\text{card } \{R. \text{equiv } A R \wedge \text{card } (A // R) = k\} = \text{card } \{P. \text{partition-on } A P \wedge \text{card } P = k\}$

proof –

have *bij-betw* $(\lambda R. A // R) \{R. \text{equiv } A R \wedge \text{card } (A // R) = k\} \{P. \text{partition-on } A P \wedge \text{card } P = k\}$

by (*rule bij-betw-partition-of-equiv-with-k-classes*)

from *this* **show** $\text{card } \{R. \text{equiv } A R \wedge \text{card } (A // R) = k\} = \text{card } \{P. \text{partition-on } A P \wedge \text{card } P = k\}$

by (*rule bij-betw-same-card*)

qed

corollary

assumes *finite A*

shows $\text{card } \{R. \text{equiv } A R \wedge \text{card } (A // R) = k\} = \text{Stirling } (\text{card } A) k$

using *card-equiv-k-classes-eq-card-partitions-k-parts*

card-partition-on[OF <finite A>] **by** *metis*

end

2 Cardinality of Partial Equivalence Relations

theory *Card-Partial-Equiv-Relations*

imports

Card-Equiv-Relations

begin

2.1 Definition of Partial Equivalence Relation

definition *partial-equiv* :: *'a set* \Rightarrow *('a* \times *'a) set* \Rightarrow *bool*

where

partial-equiv *A R* = $(R \subseteq A \times A \wedge \text{sym } R \wedge \text{trans } R)$

lemma *partial-equivI*:

assumes $R \subseteq A \times A$ *sym R trans R*

shows *partial-equiv A R*

using *assms unfolding partial-equiv-def* **by** *auto*

lemma *partial-equiv-iff*:

shows *partial-equiv A R* \longleftrightarrow $(\exists A' \subseteq A. \text{equiv } A' R)$

proof

assume *partial-equiv A R*

from *<partial-equiv A R>* **have** $R \text{ “ } A \subseteq A$

unfolding *partial-equiv-def* **by** *blast*

```

moreover have equiv (R “ A) R
proof (rule equivI)
  from ⟨partial-equiv A R⟩ show sym R
    unfolding partial-equiv-def by blast
  from ⟨partial-equiv A R⟩ show trans R
    unfolding partial-equiv-def by blast
  show refl-on (R “ A) R
proof (rule refl-onI)
  show  $R \subseteq R \text{ “ } A \times R \text{ “ } A$ 
  proof
    fix p
    assume  $p \in R$ 
    obtain x y where  $p = (x, y)$  by fastforce
    moreover have  $x \in R \text{ “ } A$ 
      using ⟨p ∈ R⟩ ⟨ $p = (x, y)$ ⟩ ⟨partial-equiv A R⟩
        partial-equiv-def sym-def by fastforce
    moreover have  $y \in R \text{ “ } A$ 
      using ⟨p ∈ R⟩ ⟨ $p = (x, y)$ ⟩ ⟨ $R \text{ “ } A \subseteq A$ ⟩ ⟨ $x \in R \text{ “ } A$ ⟩ by blast
    ultimately show  $p \in R \text{ “ } A \times R \text{ “ } A$  by auto
  qed
next
  fix y
  assume  $y \in R \text{ “ } A$ 
  from this obtain x where  $(x, y) \in R$  by auto
  from ⟨ $(x, y) \in R$ ⟩ have  $(y, x) \in R$ 
    using ⟨sym R⟩ by (meson symE)
  from ⟨ $(x, y) \in R$ ⟩ ⟨ $(y, x) \in R$ ⟩ show  $(y, y) \in R$ 
    using ⟨trans R⟩ by (meson transE)
  qed
qed
ultimately show  $\exists A' \subseteq A. \text{equiv } A' R$  by blast
next
assume  $\exists A' \subseteq A. \text{equiv } A' R$ 
from this obtain A' where  $A' \subseteq A$  and equiv A' R by blast
show partial-equiv A R
proof (rule partial-equivI)
  from ⟨equiv A' R⟩ ⟨ $A' \subseteq A$ ⟩ show  $R \subseteq A \times A$ 
    using equiv-class-eq-iff by fastforce
  from ⟨equiv A' R⟩ show sym R
    using equivE by blast
  from ⟨equiv A' R⟩ show trans R
    using equivE by blast
  qed
qed

```

2.2 Construction of all Partial Equivalence Relations for a Given Set

definition *all-partial-equivs-on* :: '*a* set \Rightarrow ((*a* × '*a*) set) set

where

```
all-partial-equivs-on A =  
do {  
  k ← {0..card A};  
  A' ← {A'. A' ⊆ A ∧ card A' = k};  
  {R. equiv A' R}  
}
```

lemma *all-partial-equivs-on*:

assumes *finite A*

shows *all-partial-equivs-on A = {R. partial-equiv A R}*

proof

show *all-partial-equivs-on A ⊆ {R. partial-equiv A R}*

proof

fix *R*

assume *R ∈ all-partial-equivs-on A*

from this obtain A' where A' ⊆ A and equiv A' R

unfolding all-partial-equivs-on-def by auto

from this have partial-equiv A R

using partial-equiv-iff by blast

from this show R ∈ {R. partial-equiv A R} ..

qed

next

show *{R. partial-equiv A R} ⊆ all-partial-equivs-on A*

proof

fix *R*

assume *R ∈ {R. partial-equiv A R}*

from this obtain A' where A' ⊆ A and equiv A' R

using partial-equiv-iff by (metis mem-Collect-eq)

moreover have card A' ∈ {0..card A}

using ⟨A' ⊆ A⟩ ⟨finite A⟩ by (simp add: card-mono)

ultimately show R ∈ all-partial-equivs-on A

unfolding all-partial-equivs-on-def

by (auto simp del: atLeastAtMost-iff)

qed

qed

2.3 Injectivity of the Set Construction

lemma *equiv-inject*:

assumes *equiv A R equiv B R*

shows *A = B*

proof –

from assms have R ⊆ A × A R ⊆ B × B by (simp add: equiv-type)+

moreover from assms have $\forall x \in A. (x, x) \in R \ \forall x \in B. (x, x) \in R$

by (simp add: eq-equiv-class)+

ultimately show ?thesis

using mem-Sigma-iff subset-antisym subset-eq by blast

qed

lemma *injectivity*:
assumes $(A' \subseteq A \wedge \text{card } A' = k) \wedge (A'' \subseteq A \wedge \text{card } A'' = k')$
assumes $\text{equiv } A' R \wedge \text{equiv } A'' R'$
assumes $R = R'$
shows $A' = A'' \wedge k = k'$
proof –
from $\langle R = R' \rangle$ *assms(2)* **show** $A' = A''$
using *equiv-inject* **by** *fast*
from *this* *assms(1)* **show** $k = k'$ **by** *simp*
qed

2.4 Cardinality Theorem of Partial Equivalence Relations

theorem *card-partial-equiv*:
assumes *finite* A
shows $\text{card } \{R. \text{partial-equiv } A R\} = \text{Bell } (\text{card } A + 1)$
proof –
let $?expr = \text{do } \{$
 $k \leftarrow \{0.. \text{card } A\};$
 $A' \leftarrow \{A'. A' \subseteq A \wedge \text{card } A' = k\};$
 $\{R. \text{equiv } A' R\}$
 $\}$
have $\text{card } \{R. \text{partial-equiv } A R\} = \text{card } (\text{all-partial-equivs-on } A)$
using $\langle \text{finite } A \rangle$ **by** *(simp add: all-partial-equivs-on)*
also have $\text{card } (\text{all-partial-equivs-on } A) = \text{card } ?expr$
unfolding *all-partial-equivs-on-def ..*
also have $\text{card } ?expr = (\sum k = 0.. \text{card } A. (\text{card } A \text{ choose } k) * \text{Bell } k)$
proof –
let $?S \gg= ?comp = ?expr$
 $\{$
fix k
assume $k: k \in \{.. \text{card } A\}$
let $?expr = ?comp$ k
let $?S \gg= ?comp = ?expr$
have *finite* $?S$ **using** $\langle \text{finite } A \rangle$ **by** *simp*
moreover $\{$
fix A'
assume $A': A' \in \{A'. A' \subseteq A \wedge \text{card } A' = k\}$
from *this* **have** $A' \subseteq A$ **and** $\text{card } A' = k$ **by** *auto*
let $?expr = ?comp$ A'
have *finite* A'
using $\langle \text{finite } A \rangle \langle A' \subseteq A \rangle$ **by** *(simp add: finite-subset)*
have $\text{card } ?expr = \text{Bell } k$
using $\langle \text{finite } A \rangle \langle \text{finite } A' \rangle \langle A' \subseteq A \rangle \langle \text{card } A' = k \rangle$
by *(simp add: card-equiv-rel-eq-Bell)*
moreover have *finite* $?expr$
using $\langle \text{finite } A' \rangle$ **by** *(simp add: finite-equiv)*
ultimately have *finite* $?expr \wedge \text{card } ?expr = \text{Bell } k$ **by** *blast*


```

} note inner = this
moreover have disjoint-family-on ?comp ?S
  by (injectivity-solver rule: injectivity(1))
moreover have card ?S = card A choose k
  using ⟨finite A⟩ by (simp add: n-subsets)
ultimately have card ?expr = (card A choose k) * Bell k (is - = ?formula)
  by (subst card-bind-constant) auto
moreover have finite ?expr
  using ⟨finite ?S⟩ inner by (auto intro!: finite-bind)
ultimately have finite ?expr ∧ card ?expr = ?formula by blast
}
moreover have finite ?S by simp
moreover have disjoint-family-on ?comp ?S
  by (injectivity-solver rule: injectivity(2))
ultimately show card ?expr = (∑ k = 0..card A. (card A choose k) * Bell k)
  by (subst card-bind) auto
qed
also have ... = (∑ k ≤ card A. (card A choose k) * Bell k)
  by (auto intro: sum.cong)
also have ... = Bell (card A + 1)
  using Bell-recursive-eq by simp
finally show ?thesis .
qed
end

```

References

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