

NP-hardness of CVP and SVP

Katharina Kreuzer

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Abstract

This article formalizes the NP-hardness proofs of the Closest Vector Problem (CVP) and the Shortest Vector Problem (SVP) in maximum norm as well as the CVP in any p -norm for $p \geq 1$. CVP and SVP are two fundamental problems in lattice theory. Lattices are a discrete, additive subgroup of \mathbb{R}^n and are used for lattice-based cryptography. The CVP asks to find the nearest lattice vector to a target. The SVP asks to find the shortest non-zero lattice vector.

This entry formalizes the basic properties of lattices, the reduction from CVP to Subset Sum in both maximum and p -norm for $1 \leq p < \infty$ and the reduction of SVP to Partition using the Bounded Homogeneous Linear Equations problem (BHLE) as an intermediate step. The formalization uncovered a number of problems with the existing proofs in the literature [7] and [6].

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1 Introduction

Two problems form the very basis for cryptography on lattices, namely the closest vector problem (CVP) and the shortest vector problem (SVP). Given a finite set of basis vectors in \mathbb{R}^n , the set of all linear combinations with integer coefficients forms a lattice. In optimization form, SVP asks for the shortest vector in the lattice and CVP asks for the lattice vector closest to some given target vector, both with respect to some given norm. The proof of NP-hardness of these problems in ℓ_∞ goes back to an early technical report by van Emde-Boas [7]. Indeed, for other norms (especially for the Euclidean norm), NP-hardness of the SVP could only be shown using a randomized reduction [1]. For CVP, NP-hardness has been shown in any p -norm for $p \geq 1$. One exemplary proof can be found in the book by Micciancio and Goldwasser [6, Chapter 3, Thm 3.1].

The CVP and SVP are not only used in cryptography, but in various other areas as well. For example, they closely relate to integer programs (see Lenstra's paper [5]) or finding irreducible factors of polynomials as in Lenstra's paper [3]. For example, the LLL-algorithm by Lenstra, Lenstra and Lovász [4] gives a polynomial-time algorithm for lattice basis reduction which solves integer linear programs in fixed dimensions. Using this reduced basis, one can find good approximations to CVP using Babai's algorithm [2] for certain approximation factors. Still, for arbitrary dimensions, the problem remains NP-hard.

theory *Reduction*

imports

Main

begin

2 Reduction Function

This definition was taken from the developments at <https://github.com/wimmers/poly-reductions> "Karp21/Reductions.thy". TODO: When this repo comes into the AFP, link to original definition.

```
definition is_reduction :: "('a  $\Rightarrow$  'b)  $\Rightarrow$  'a set  $\Rightarrow$  'b set  $\Rightarrow$  bool" where  
  "is_reduction f A B  $\equiv \forall a. a \in A \longleftrightarrow f a \in B$  "  
end  
theory Lattice_int
```

imports

"Jordan_Normal_Form.Matrix"

"Jordan_Normal_Form.VS_Connect"

```
"BenOr_Kozen_Reif.More_Matrix"
begin
```

3 Basics on Lattices

Connect the type `vec` to records of rings, commutative rings, fields and modules in order to use properties of modules such as `lin_indpt` (linear independence).

```
lemma dim_carrier: "dim_vec z = dim_col A  $\implies$  A *_v z  $\in$  carrier_vec (dim_row A)"
  <proof>
```

Concrete type conversion `int vec` to `real vec`

```
definition real_of_int_vec :: "int vec  $\Rightarrow$  real vec" where
  "real_of_int_vec v = map_vec real_of_int v"
```

```
definition real_to_int_vec :: "real vec  $\Rightarrow$  int vec" where
  "real_to_int_vec v = map_vec floor v"
```

```
lemma dim_vec_real_of_int_vec[simp]: "dim_vec (real_of_int_vec v) = dim_vec v"
  <proof>
```

```
lemma real_of_int_vec_nth[simp, intro]:
  "i < dim_vec v  $\implies$  (real_of_int_vec v) $ i = real_of_int (v $ i)"
  <proof>
```

```
lemma real_of_int_vec_vec:
  "real_of_int_vec (vec n f) = vec n (real_of_int  $\circ$  f)"
  <proof>
```

Concrete type conversion `int mat` to `real mat`

```
definition real_of_int_mat :: "int mat  $\Rightarrow$  real mat" where
  "real_of_int_mat A = map_mat real_of_int A"
```

```
definition real_to_int_mat :: "real mat  $\Rightarrow$  int mat" where
  "real_to_int_mat A = map_mat floor A"
```

```
lemma dim_col_real_of_int_mat[simp]: "dim_col (real_of_int_mat A) = dim_col A"
  <proof>
```

```
lemma dim_row_real_of_int_mat[simp]: "dim_row (real_of_int_mat A) = dim_row A"
  <proof>
```

```

lemma real_of_int_mat_nth[simp, intro]:
  "i < dim_row A  $\implies$  j < dim_col A  $\implies$  (real_of_int_mat A) $$ (i,j) = real_of_int
  (A $$ (i,j))"
<proof>

```

```

lemma real_of_int_mat_mat:
  "real_of_int_mat (mat n m f) = mat n m (real_of_int  $\circ$  f)"
<proof>

```

```

lemma real_of_int_mat_cols:
  "cols (real_of_int_mat A) = map real_of_int_vec (cols A)"
<proof>

```

```

lemma distinct_cols_real_of_int_mat:
  "distinct (cols A) = distinct (cols (real_of_int_mat A))"
<proof>

```

Algebraic lattices are discrete additive subgroups of \mathbb{R}^n . Lattices can be represented by a basis, multiple bases can represent the same lattice.

```

type_synonym int_lattice = "int vec set"

```

Linear independence

```

consts is_indep :: "'a  $\Rightarrow$  bool"

```

overloading

```

  is_indep_real  $\equiv$  "is_indep :: real mat  $\Rightarrow$  bool"
  is_indep_int  $\equiv$  "is_indep :: int mat  $\Rightarrow$  bool"
begin
definition is_indep_real :: "real mat  $\Rightarrow$  bool" where
  "is_indep A = ( $\forall z ::$  real vec. (A  $\ast_v$  z = 0_v (dim_row A)  $\wedge$ 
    dim_vec z = dim_col A)  $\longrightarrow$  z = 0_v (dim_vec z))"
definition is_indep_int :: "int mat  $\Rightarrow$  bool" where
  "is_indep A = ( $\forall z ::$  real vec. ((real_of_int_mat A)  $\ast_v$  z = 0_v (dim_row
  A)  $\wedge$ 
    dim_vec z = dim_col A)  $\longrightarrow$  z = 0_v (dim_vec z))"
end

```

Definition of lattices

```

definition is_lattice :: "int_lattice  $\Rightarrow$  bool" where
  "is_lattice L  $\equiv$  ( $\exists B ::$  (int mat).
    L = {B  $\ast_v$  z | z :: int vec. dim_vec z = dim_col B}
     $\wedge$  is_indep B)"

```

The lattice generated by the column vectors of a matrix. This matrix does not need to be linearly independent. Make certain that the output is indeed a lattice and not the entire space.

```

definition gen_lattice :: "int mat  $\Rightarrow$  int vec set" where

```

"gen_lattice A = {A *_v z | z::int vec. dim_vec z = dim_col A}"

Theorems for lattices. The aim is to have a change of basis theorem for lattice bases. The theorems are mostly taken from the respective theorems for the type ('a,'k::finite) vec and adapted to the type 'a vec.

lemma finsum_vec_carrier:
assumes "f ' A ⊆ carrier_vec nr" "finite A"
shows "finsum_vec TYPE('a::ring) nr f A ∈ carrier_vec nr"
 ⟨proof⟩

lemma dim_vec_finsum_vec:
assumes "f ' A ⊆ carrier_vec nr" "finite A"
shows "dim_vec (finsum_vec TYPE('a::ring) nr f A) = nr"
 ⟨proof⟩

Matrix-Vector-Multiplication as an element in the column span.

lemma mat_mult_as_col_span:
assumes "(A :: 'a :: comm_ring mat) ∈ carrier_mat nr nc"
and "(z :: 'a vec) ∈ carrier_vec nc"
shows "A *_v z = finsum_vec TYPE('a) nr (λi. z\$*i* ·_v col A i) {0..*nc*}"
 (is "?left = ?right")
 ⟨proof⟩

A matrix is identical to the matrix generated by its indices.

lemma mat_index:
 "B = mat (dim_row B) (dim_col B) (λ(i,j). B \$\$ (i,j))"
 ⟨proof⟩

Lemmas about the columns of a matrix.

lemma col_unit_vec:
assumes "i < dim_col (B:: 'a :: {monoid_mult, semiring_0, zero_neq_one} mat)"
shows "B *_v (unit_vec (dim_col B) i) = col B i" (is "?left = ?right")
 ⟨proof⟩

lemma is_indep_not_null:
assumes "is_indep (B:: real mat)" "i < dim_col B"
shows "col B i ≠ 0_v (dim_row B)"
 ⟨proof⟩

lemma col_in_gen_lattice:
assumes "i < dim_col B"
shows "col B i ∈ gen_lattice B"
 ⟨proof⟩

4 Change of Basis in a Lattice

Definition of the column span.

definition `span` :: `("('a :: semiring_0) mat \Rightarrow 'a vec set" where
"span B = {B *_v z | z:: 'a vec. dim_vec z = dim_col B}"`

Definition of the column dimension of a matrix

definition `dim` :: `"int mat \Rightarrow nat" where
"dim B = (if \exists b. is_indep (real_of_int_mat b) \wedge span (real_of_int_mat b) = span (real_of_int_mat B)
then dim_col (SOME b. is_indep (real_of_int_mat b) \wedge
span (real_of_int_mat b) = span (real_of_int_mat B)) else 0)"`

Abbreviation of the set of columns

definition `set_cols[simp]`: `"set_cols A = set (cols A)"`

For the change of basis, we need to be able to delete and insert a column vector in the column span.

definition `insert_col`:
`"insert_col A c = mat_of_cols (dim_row A) (c # cols A)"`

definition `delete_col`:
`"delete_col A c = mat_of_cols (dim_row A) (filter (λ x. \neg (x = c)) (cols A))"`

lemma `set_cols_col`:
`"set_cols A = col A ' {0.. \dim _col A}"`
`<proof>`

lemma `set_cols_subset_col`:
`assumes "set_cols A \subseteq set_cols B"`
`"i < \dim _col A"`
`obtains j where "col A i = col B j"`
`<proof>`

Monotonicity of the span.

lemma `span_mono`:
`assumes "set_cols (A :: 'a::comm_ring mat) \subseteq set_cols B" "dim_row A = dim_row B"`
`"distinct (cols A)"`
`shows "span A \subseteq span B"`
`<proof>`

Monotonicity of linear independence

lemma `is_indep_mono`:
`assumes "set_cols B \subseteq set_cols A" "is_indep A" "distinct (cols B)" "dim_row A = dim_row B"`
`shows "is_indep (B::real mat)"`
`<proof>`

Linear independent vectors are distinct.

```

lemma is_indep_distinct:
assumes "is_indep (A::real mat)"
shows "distinct (cols A)"
⟨proof⟩

```

Column vectors are in the column span.

```

lemma span_base:
assumes "a :: 'a :: {monoid_mult,semiring_0,zero_neq_one} vec) ∈ set_cols S"
shows "a ∈ span S"
⟨proof⟩

```

Representing vectors in the column span of a linear independent matrix. The representation function returns the coefficients needed to generate the vector as an element in the column span.

Representation function and its lemmas were adapted from “HOL/Modules.thy”

```

definition representation :: "real mat ⇒ real vec ⇒ real vec ⇒ real"
where "representation B v =
  (if is_indep B ∧ v ∈ (span B) then
    SOME f. (∀v. f v ≠ 0 ⇒ v ∈ set_cols B) ∧
    B *v (vec (dim_col B) (λi. f (col B i))) = v
  else (λb. 0))"

```

If the column vectors form a basis of the column span, the representation function is unique.

```

lemma unique_representation:
assumes basis: "is_indep (basis::real mat)"
and in_basis: "∧v. f v ≠ 0 ⇒ v ∈ set_cols basis" "∧v. g v ≠
0 ⇒ v ∈ set_cols basis"
and eq: "basis *v (vec (dim_col basis) (λi. f (col basis i))) =
basis *v (vec (dim_col basis) (λi. g (col basis i)))"
shows "f = g"
⟨proof⟩

```

More properties on the representation function.

```

lemma
shows representation_ne_zero: "∧b. representation basis v b ≠ 0 ⇒
b ∈ set_cols basis"
and sum_nonzero_representation_eq:
  "is_indep basis ⇒ v ∈ span basis ⇒
  basis *v (vec (dim_col basis) (λi. representation basis v (col basis
i))) = v"
⟨proof⟩

```

```

lemma representation_eqI:

```



```

assumes basis: "is_indep basis" and b: "v ∈ span basis"
  and ne_zero: "∧b. f b ≠ 0 ⇒ b ∈ set_cols basis"
  and eq: "basis *v (vec (dim_col basis) (λi. f (col basis i))) = v"
shows "representation basis v = f"
⟨proof⟩

```

Representation function when extending the column space.

```

lemma representation_extend:
  assumes basis: "is_indep basis" and v: "v ∈ span basis'"
  and basis': "set_cols basis' ⊆ set_cols basis" and d: "dim_row basis
= dim_row basis'"
  and dc: "distinct (cols basis)"
shows "representation basis v = representation basis' v"
⟨proof⟩

```

The representation of the i -th column vector is the i -th unit vector.

```

lemma representation_basis:
  assumes basis: "is_indep basis" and b: "b ∈ set_cols basis"
shows "representation basis b = (λv. if v = b then 1 else 0)"
⟨proof⟩

```

The spanning and independent set of vectors is a basis.

```

lemma spanning_subset_independent:
assumes BA: "set_cols B ⊆ set_cols A" and iA: "is_indep (A::real mat)"

  and AsB: "set_cols A ⊆ span B"
  and d: "distinct (cols B)" and dr: "dim_row A = dim_row B"
shows "set_cols A = set_cols B"
⟨proof⟩

```

```

lemma nth_lin_combo:
assumes "i < dim_row A" "dim_col A = dim_vec b"
shows "(A *v b) $ i = (∑ j=0..

```

Insert and delete with the span.

```

lemma dim_col_insert_col:
  "dim_col (insert_col S a) = dim_col S + 1"
⟨proof⟩

```

```

lemma dim_col_delete_col:
assumes "a ∈ set_cols S" "distinct (cols S)" "dim_vec a = dim_row S"

shows "dim_col (delete_col S a) = dim_col S - 1"
⟨proof⟩

```

```

lemma dim_row_delete_col:

```

"dim_row (delete_col T b) = dim_row T"
 ⟨proof⟩

lemma span_insert1:
 assumes "dim_vec (a :: 'a :: comm_ring vec) = dim_row S"
 shows "span (insert_col S a) = {x. ∃k y. x = y + k ·_v a ∧ y ∈ span S
 ∧ dim_vec x = dim_row S}"
 ⟨proof⟩

lemma span_insert2:
 assumes "dim_vec a = dim_row S"
 shows "span (insert_col S (a :: 'a :: comm_ring vec)) =
 {x. ∃k. (x - k ·_v a) ∈ span S ∧ dim_vec x = dim_row S}"
 ⟨proof⟩

lemma insert_delete_span:
 assumes "(a :: 'a :: comm_ring vec) ∈ set_cols A" "distinct (cols A)"
 "dim_vec a = dim_row A"
 shows "span (insert_col (delete_col A a) a) = span (A)"
 ⟨proof⟩

Deleting a generating element.

lemma span_breakdown:
 assumes bS: "(b :: 'a :: comm_ring vec) ∈ set_cols S"
 and aS: "a ∈ span S"
 and b: "dim_vec b = dim_row S"
 and d: "distinct (cols S)"
 shows "∃k. a - k ·_v b ∈ span (delete_col S b)"
 ⟨proof⟩

lemma uminus_scalar_mult:
 "(-k) ·_v a = - (k ·_v (a :: 'a :: comm_ring_1 vec))"
 ⟨proof⟩

lemma span_scale: "(x :: 'a :: field vec) ∈ span S ⇒ c ·_v x ∈ span S"
 ⟨proof⟩

lemma span_diff:
 assumes "(x :: 'a :: ring vec) ∈ span S" "y ∈ span S"
 "x ∈ carrier_vec (dim_row S)" "y ∈ carrier_vec (dim_row S)"
 shows "x - y ∈ span S"
 ⟨proof⟩

Adding a generating element.

lemma in_span_insert:

```

assumes a: "(a :: 'a :: field vec) ∈ span (insert_col S b)"
  and na: "a ∉ span S"
  and br: "dim_vec b = dim_row S"
  and bnotS: "b ∉ set_cols S"
  and d: "distinct (cols S)"
shows "b ∈ span (insert_col S a)"
⟨proof⟩

```

```

lemma delete_not_in_set_cols:
assumes "b ∉ set_cols T"
shows "delete_col T b = T"
⟨proof⟩

```

```

lemma delete_col_span:
assumes "(a :: 'a :: comm_ring vec) ∈ span (delete_col T b)" "distinct
(cols T)"
shows "a ∈ span T"
⟨proof⟩

```

Changing a generating element.

```

lemma in_span_delete_field:
assumes aT: "(a :: 'a :: field vec) ∈ span T" and aTb: "a ∉ span (delete_col
T b)"
  and dr: "dim_vec b = dim_row T" and d: "distinct (cols T)"
shows "b ∈ span (insert_col (delete_col T b) a)"
⟨proof⟩

```

Simplifications on multiplication.

```

lemma mat_of_cols_mult_vCons:
assumes "dim_vec (x :: 'a :: comm_ring vec) = dim_row S" "dim_vec zs =
dim_col S"
shows "mat_of_cols (dim_row S) (x # cols S) *_v vCons z0 zs = z0 *_v x +
S *_v zs"
⟨proof⟩

```

```

lemma mult_mat_vec_ring:
  assumes m: "(A :: 'a :: comm_ring mat) ∈ carrier_mat nr nc" and v: "v
∈ carrier_vec nc"
  shows "A *_v (k *_v v) = k *_v (A *_v v)" (is "?l = ?r")
⟨proof⟩

```

Adding a span element to the generating set does not change the span.

```

lemma span_redundant:
assumes "(x :: 'a :: comm_ring vec) ∈ span S"
shows "span (insert_col S x) = span S"
⟨proof⟩

```

Transitivity of inserting elements.

```
lemma span_trans:
  assumes "(x :: 'a :: comm_ring vec) ∈ span S" "y ∈ span (insert_col S x)"
  shows "y ∈ span S"
  <proof>
```

More on inserting, deleting and the set of columns.

```
lemma delet_col_not_in_set_cols:
  assumes "dim_vec b = dim_row T"
  shows "b ∉ set_cols (delete_col T b)"
  <proof>
```

```
lemma dim_col_distinct:
  assumes "distinct (cols S)"
  shows "card (set_cols S) = dim_col S"
  <proof>
```

```
lemma set_cols_mono:
  assumes "set_cols S ⊆ set_cols T" "distinct (cols S)" "distinct (cols T)"
  shows "dim_col S ≤ dim_col T"
  <proof>
```

```
lemma set_cols_insert_col:
  assumes "dim_vec b = dim_row U"
  shows "set_cols (insert_col U b) = set_cols U ∪ {b}"
  <proof>
```

```
lemma set_cols_real_of_int_mat:
  "set_cols (real_of_int_mat S) = real_of_int_vec ' (set_cols S)"
  <proof>
```

```
lemma real_of_int_mat_span:
  "real_of_int_vec ' (span S) ⊆ span (real_of_int_mat S)"
  <proof>
```

```
lemma real_of_int_vec_ex:
  assumes "x ∈ real_of_int_vec ' A"
  shows "∃ y. x = real_of_int_vec y"
  <proof>
```

```
lemma real_of_int_vec_obtain:
  assumes "x ∈ real_of_int_vec ' A"
  obtains y where "x = real_of_int_vec y"
  <proof>
```

```
lemma real_of_int_vec_inj:
```

```
"inj real_of_int_vec"
⟨proof⟩
```

```
lemma real_of_int_mat_mat_of_cols:
  assumes "∀i<length cs. cs ! i ∈ carrier_vec nr"
  shows "real_of_int_mat (mat_of_cols nr cs) = mat_of_cols nr (map real_of_int_vec cs)"
  ⟨proof⟩
```

```
lemma real_of_int_mat_delete_col:
  "delete_col (real_of_int_mat T) (real_of_int_vec b) = real_of_int_mat (delete_col T b)"
  ⟨proof⟩
```

Lemma to exchange a subset of columns from one basis to another. Pay attention that this only works over the reals, not over the integers since we need to do divisions. The `exchange_lemma` was adapted from “HOL/Vector_Spaces.thy”. Note that the connection “Jordan_Normal_Form/VS_Connect” between the type $(\text{'a}, \text{'k}::\text{finite}) \text{vec}$ and 'a vec does not include the theorem `exchange_lemma`. due to some problems in lifting/transfer.

```
lemma exchange_lemma:
  assumes i: "is_indep (S::real mat)"
    and sp: "set_cols S ⊆ span T"
    and d: "distinct (cols T)"
    and dr: "dim_row S = dim_row T"
  shows "∃t'. dim_col t' = dim_col T ∧ set_cols S ⊆ set_cols t' ∧
        set_cols t' ⊆ set_cols S ∪ set_cols T ∧
        distinct (cols t')"
```

```
⟨proof⟩
```

A linearly independent set has smaller or equal cardinality than the span it lies in.

```
lemma independent_span_bound:
  assumes i: "is_indep (S :: int mat)" and d: "distinct (cols T)"
    and sp: "set_cols S ⊆ span T"
    and dr: "dim_row S = dim_row T"
  shows "dim_col S ≤ dim_col T"
  ⟨proof⟩
```

When two bases B and B' generate the same lattice, both have the same length because the change of basis theorem allows us to convert one basis in the other.

```
lemma gen_lattice_in_span:
  assumes "gen_lattice B = gen_lattice B'"
  shows "set_cols B ⊆ Lattice_int.span B'"
  ⟨proof⟩
```

```

lemma basis_exchange:
  assumes gen_eq: "gen_lattice B = gen_lattice B'"
    and "is_indep B" and "is_indep B'"
  shows "dim_col B = dim_col B'"
<proof>

```

Basis matrix of a lattice

```

definition basis_of :: "int vec set  $\Rightarrow$  int mat" where
  "basis_of L = (SOME B. L = gen_lattice B  $\wedge$  is_indep B)"

```

```

definition dim_lattice :: "int_lattice  $\Rightarrow$  nat" where
  "dim_lattice L = (THE x. x = dim_col (basis_of L))"

```

```

lemma dim_lattice_gen_lattice:
  assumes "is_indep B"
  shows "dim_lattice (gen_lattice B) = dim_col B"
<proof>

```

A lattice generated by a linearly independent matrix is indeed a lattice.

```

lemma is_lattice_gen_lattice:
  assumes "is_indep A"
  shows "is_lattice (gen_lattice A)"
<proof>

```

```

end
theory Partition

```

```

imports
  Main
  Reduction
begin

```

5 Partition Problem

The Partition Problem is a widely known NP-hard problem. TODO: Reduction proof to SAT

```

definition is_partition :: "int list  $\Rightarrow$  nat set  $\Rightarrow$  bool" where
  "is_partition a I = (I  $\subseteq$  {0.. $\text{length a}$ }  $\wedge$  ( $\sum_{i \in I} a ! i$ ) = ( $\sum_{i \in (\{0.. $\text{length a}\} - I)} a ! i$ ))"$ 
```

```

definition partition_problem :: "(int list) set " where
  "partition_problem = {a.  $\exists I. I \subseteq \{0.. $\text{length a}\} \wedge$ 
    ( $\sum_{i \in I} a ! i$ ) = ( $\sum_{i \in (\{0.. $\text{length a}\} - I)} a ! i$ )}"$$ 
```

```

definition partition_problem_nonzero :: "(int list) set " where
  "partition_problem_nonzero = {a.  $\exists I. I \subseteq \{0..<length\ a\} \wedge length\ a > 0 \wedge$ 
     $(\sum_{i \in I}. a\ !\ i) = (\sum_{i \in (\{0..<length\ a\} - I)}. a\ !\ i)}$ "

```

```

end
theory Subset_Sum

```

```

imports
  Main
  Partition
  "Jordan_Normal_Form.Matrix"
begin

```

6 Subset Sum Problem

The Subset Sum Problem is a common NP-hard Problem.

```

definition subset_sum :: "(int vec) * int) set" where
  "subset_sum  $\equiv \{(as,s). (\exists xs::int\ vec. (\forall i<dim\_vec\ xs. xs\ \$i \in \{0,1\}) \wedge xs \cdot as = s \wedge dim\_vec\ xs = dim\_vec\ as)\}$ "

```

```

definition subset_sum_nonzero :: "(int vec) * int) set" where
  "subset_sum_nonzero  $\equiv \{(as,s). (\exists xs::int\ vec. (\forall i<dim\_vec\ xs. xs\ \$i \in \{0,1\}) \wedge xs \cdot as = s \wedge dim\_vec\ xs = dim\_vec\ as) \wedge dim\_vec\ as \neq 0\}$ "

```

```

definition subset_sum_list :: "(int list) * int) set" where
  "subset_sum_list  $\equiv \{(as,s). (\exists xs::int\ list. (\forall i<length\ xs. xs\ !i \in \{0,1\}) \wedge (\sum_{i<length\ as. as\ !i * xs\ !i) = s \wedge length\ xs = length\ as)\}$ "

```

Reduction Subset Sum to Partition.

```

definition reduce_subset_sum_partition :: "(int list)  $\Rightarrow$  ((int list) * int)"
where
  "reduce_subset_sum_partition  $\equiv (\lambda a. (a, (if\ sum\_list\ a\ mod\ 2 = 0\ then\ sum\_list\ a\ div\ 2\ else\ (\sum_{i<length\ a. |a\ !i|) + 1)))$ "

```

Well-definedness of reduction function

```

lemma sum_list_map_2:
  assumes "length a = length xs"
  shows "(sum_list (map2 (*) a xs)) =  $(\sum_{i<length\ a. a\ !i * xs\ !i}$ "

```

<proof>

```
lemma ex_01_list:  
  assumes "I ⊆ {0..<n>}"  
  shows "∃ xs :: int list. (∀ i ∈ I. xs ! i = 1) ∧ (∀ i ∈ {0..<n>} - I. xs ! i  
= 0) ∧ length xs = n"  
<proof>
```

```
lemma sum_list_even:  
  assumes "a ∈ partition_problem"  
  shows "sum_list a mod 2 = 0"  
<proof>
```

```
lemma well_defined_reduction_subset_sum:  
  assumes "a ∈ partition_problem"  
  shows "reduce_subset_sum_partition a ∈ subset_sum_list"  
<proof>
```

NP-hardness of reduction function

```
lemma NP_hardness_reduction_subset_sum:  
  assumes "reduce_subset_sum_partition a ∈ subset_sum_list"  
  shows "a ∈ partition_problem"  
<proof>
```

The Subset Sum on lists is NP-hard.

```
lemma "is_reduction reduce_subset_sum_partition partition_problem subset_sum_list"  
<proof>
```

Subset Sum on vectors is the same as on lists

```
lemma subset_sum_vec_list:  
"(as,s) ∈ subset_sum ↔ (list_of_vec as,s) ∈ subset_sum_list"  
<proof>
```

```
end  
theory CVP_p
```

```
imports  
  Main  
  Reduction  
  Lattice_int  
  Subset_Sum  
begin
```

7 CVP in ℓ_p for $p \geq 1$

This file provides the reduction proof of Subset Sum to CVP in ℓ_p . Proof can be easily instantiated for any $p \geq 1$ using the locale variables.


```

definition pth_p_norm_vec :: "nat  $\Rightarrow$  ('a::{abs, power, comm_monoid_add})
vec  $\Rightarrow$  'a" where
  "pth_p_norm_vec p v = ( $\sum$  i<dim_vec v. |v[i]|p)"

```

```

locale fixed_p =
fixes p::nat
assumes p_def: "p $\geq$ 1"
begin

```

```

definition "p_norm_vec  $\equiv$  pth_p_norm_vec p"

```

The CVP in ℓ_p

```

definition is_closest_vec :: "int_lattice  $\Rightarrow$  int vec  $\Rightarrow$  int vec  $\Rightarrow$  bool"
where
  "is_closest_vec L b v  $\equiv$  (is_lattice L)  $\wedge$ 
  ( $\forall$  x $\in$ L. p_norm_vec (x - b)  $\geq$  p_norm_vec (v - b)  $\wedge$  v $\in$ L)"

```

The decision problem associated with solving CVP exactly.

```

definition gap_cvp :: "(int_lattice  $\times$  int vec  $\times$  real) set" where
  "gap_cvp  $\equiv$  {(L, b, r). (is_lattice L)  $\wedge$  ( $\exists$  v $\in$ L. of_int (p_norm_vec
(v - b))  $\leq$  rp)}"

```

Reduction function for Subset Sum to CVP in euclidean norm

```

definition gen_basis_p :: "int vec  $\Rightarrow$  int mat" where
  "gen_basis_p as = mat (dim_vec as + 1) (dim_vec as) ( $\lambda$  (i, j). if i
= 0 then as[j]
  else (if i = j + 1 then 2 else 0))"

```

```

definition gen_t_p :: "int vec  $\Rightarrow$  int  $\Rightarrow$  int vec" where
  "gen_t_p as s = vec (dim_vec as + 1) (( $\lambda$  i. 1)(0:= s))"

```

```

definition reduce_cvp_subset_sum_p ::
  "((int vec) * int)  $\Rightarrow$  (int_lattice * (int vec) * real)" where
  "reduce_cvp_subset_sum_p  $\equiv$  ( $\lambda$  (as,s).
  (gen_lattice (gen_basis_p as), gen_t_p as s, root p (dim_vec as)))"

```

Lemmas for Proof

```

lemma vec_lambda_eq[intro]: "( $\forall$  i<n. a i = b i)  $\longrightarrow$  vec n a = vec n b"
<proof>

```

```

lemma eq_fun_applic: assumes "x = y" shows "f x = f y"
<proof>

```

```

lemma sum_if_zero:

```

```

    assumes "finite A" "i∈A"
    shows "(∑ j∈A. (if i = j then a j else 0)) = a i"
  <proof>

```

```

lemma set_compr_elem:
  assumes "finite A" "a∈A"
  shows "{f i | i. i∈A} = {f a} ∪ {f i | i. i∈A-{a}}"
  <proof>

```

```

lemma Bx_rewrite:
  assumes x_dim: "dim_vec as = dim_vec x"
  shows "(gen_basis_p as) *v x =
    vec (dim_vec as + 1) (λ i. if i = 0 then (x · as)
    else (2 * x$(i-1)))"
    (is "?init_vec = ?goal_vec")
  <proof>

```

```

lemma Bx_s_rewrite:
  assumes x_dim: "dim_vec as = dim_vec x"
  shows "(gen_basis_p as) *v x - (gen_t_p as s) =
    vec (dim_vec as + 1) (λ i. if i = 0 then (x · as - s) else (2 *
    x$(i-1) - 1))"
    (is "?init_vec = ?goal_vec")
  <proof>

```

```

lemma p_norm_vec_Bx_s:
  assumes x_dim: "dim_vec as = dim_vec x"
  shows "p_norm_vec ((gen_basis_p as) *v x - (gen_t_p as s)) =
    |x · as - s|^p + (∑ i=1..<dim_vec as +1. |2*x$(i-1)-1|^p)"
  <proof>

```

gen_basis_p actually generates a basis which spans the *int_lattice* (by definition) and is linearly independent.

```

lemma is_indep_gen_basis_p:
  "is_indep (gen_basis_p as)"
  <proof>

```

Well-definedness of the reduction function.

```

lemma well_defined_reduction:
  assumes "(as, s) ∈ subset_sum"
  shows "reduce_cvp_subset_sum_p (as, s) ∈ gap_cvp"
  <proof>

```

NP-hardness of reduction function.

```

lemma NP_hardness_reduction:

```

```

    assumes "reduce_cvp_subset_sum_p (as, s) ∈ gap_cvp"
    shows "(as, s) ∈ subset_sum"
  <proof>

CVP is NP-hard in  $p$ -norm.

lemma "is_reduction reduce_cvp_subset_sum_p subset_sum gap_cvp"
  <proof>

end

end
theory infnorm

imports
  Main
  "Jordan_Normal_Form.Matrix"
  "LLL_Basis_Reduction.Norms"
begin

```

8 Maximum Norm

The ℓ_∞ norm on vectors is exactly the maximum of the absolute value of all entries.

```

lemma linf_norm_vec_Max:
  "||v||∞ = Max (insert 0 {vi | i. i < dim_vec v})"
  <proof>

```

```

end
theory CVP_vec

```

```

imports
  Main
  Reduction
  Lattice_int
  Subset_Sum
  infnorm
begin

```

9 CVP in ℓ_∞

The closest vector problem.

```

definition is_closest_vec :: "int_lattice ⇒ int vec ⇒ int vec ⇒ bool"
where
  "is_closest_vec L b v ≡ (is_lattice L) ∧
    (∀ x ∈ L. linf_norm_vec (x - b) ≥ linf_norm_vec (v - b) ∧ v ∈ L)"

```

The decision problem associated with solving CVP exactly.

```
definition gap_cvp :: "(int_lattice × int vec × int) set" where
  "gap_cvp ≡ {(L, b, r). (is_lattice L) ∧ (∃ v ∈ L. linf_norm_vec (v -
b) ≤ r)}"
```

Reduction function for CVP to subset sum

```
definition gen_basis :: "int vec ⇒ int mat" where
  "gen_basis as = mat (dim_vec as + 2) (dim_vec as) (λ (i, j). if i ∈
{0,1} then as$j
  else (if i = j + 2 then 2 else 0))"
```

```
definition gen_t :: "int vec ⇒ int ⇒ int vec" where
  "gen_t as s = vec (dim_vec as + 2) ((λ i. 1)(0:= s+1, 1:= s-1))"
```

```
definition reduce_cvp_subset_sum ::
  "((int vec) * int) ⇒ (int_lattice * (int vec) * int)" where
  "reduce_cvp_subset_sum ≡ (λ (as,s).
  (gen_lattice (gen_basis as), gen_t as s, (1::int)))"
```

Lemmas for Proof

```
lemma vec_lambda_eq[intro]: "(∀ i < n. a i = b i) → vec n a = vec n b"
<proof>
```

```
lemma eq_fun_applic: assumes "x = y" shows "f x = f y"
<proof>
```

```
lemma sum_if_zero:
  assumes "finite A" "i ∈ A"
  shows "(∑ j ∈ A. (if i = j then a j else 0)) = a i"
<proof>
```

```
lemma set_compr_elem:
  assumes "finite A" "a ∈ A"
  shows "{f i | i. i ∈ A} = {f a} ∪ {f i | i. i ∈ A - {a}}"
<proof>
```

```
lemma Bx_rewrite:
  assumes x_dim: "dim_vec as = dim_vec x"
  shows "(gen_basis as) *v x =
  vec (dim_vec as + 2) (λ i. if i ∈ {0,1} then (x · as)
  else (2 * x$(i-2)))"
  (is "?init_vec = ?goal_vec")
<proof>
```

```
lemma Bx_s_rewrite:
```

```

assumes x_dim: "dim_vec as = dim_vec x"
shows "(gen_basis as) *v x - (gen_t as s) =
  vec (dim_vec as + 2) (λ i. if i = 0 then (x · as - s - 1) else (
    if i = 1 then (x · as - s + 1) else (2 * x$(i-2) - 1)))"
  (is "?init_vec = ?goal_vec")
⟨proof⟩

```

```

lemma linf_norm_vec_Bx_s:
assumes x_dim: "dim_vec as = dim_vec x"
shows "linf_norm_vec ((gen_basis as) *v x - (gen_t as s)) =
  Max (insert 0 ({ |x · as - s - 1|} ∪ { |x · as - s + 1|} ∪
    { |2*x$(i-2)-1| | i. 1<i ∧ i<dim_vec as+2 })))"
⟨proof⟩

```

gen_basis actually generates a basis which spans the *int_lattice* (by definition) and is linearly independent.

```

lemma is_indep_gen_basis:
  "is_indep (gen_basis as)"
⟨proof⟩

```

The CVP is NP-hard in ℓ_∞ .

```

lemma well_defined_reduction:
assumes "(as, s) ∈ subset_sum"
shows "reduce_cvp_subset_sum (as, s) ∈ gap_cvp"
⟨proof⟩

```

NP-hardness part of reduction.

```

lemma NP_hardness_reduction:
assumes "reduce_cvp_subset_sum (as, s) ∈ gap_cvp"
shows "(as, s) ∈ subset_sum"
⟨proof⟩

```

The CVP is NP-hard in ℓ_∞ .

```

lemma "is_reduction reduce_cvp_subset_sum subset_sum gap_cvp"
⟨proof⟩

```

```

end
theory Digits_int
  imports
    Complex_Main
begin

```

10 Representation of Integers in Different Number Systems

This file is an adaption of *Van_der_Waerden/Digits.thy* for representation of the Integers (instead of the natural numbers) in different number systems. The main difference is the integer input in the function *digit*.

First, we look at some useful lemmas for splitting sums.

```
lemma split_sum_first_elt_less: assumes "n < m"
  shows "( $\sum_{i \in \{n..<m\}} f i$ ) = f n + ( $\sum_{i \in \{Suc n ..<m\}} f i$ )"
  <proof>
```

```
lemma split_sum_mid_less: assumes "i < (n::nat)"
  shows "( $\sum_{j < n} f j$ ) = ( $\sum_{j < i} f j$ ) + ( $\sum_{j=i..<n} f j$ )"
  <proof>
```

In order to use representation of numbers in a basis *base* and to calculate the conversion to and from integers, we introduce the following locale.

```
locale digits =
  fixes base :: int
  assumes base_pos: "base > 0"
begin
```

Conversion from basis *base* to integers: *from_digits n d*

```
n:          nat      length of representation in basis base
d:          nat  $\Rightarrow$  nat  function of digits in basis base where d i is the
                               i-th digit in basis base
output:     nat      natural number corresponding to
                                $d(n-1) \dots d(0)$  as integer
```

```
fun from_digits :: "nat  $\Rightarrow$  (nat  $\Rightarrow$  nat)  $\Rightarrow$  nat" where
  "from_digits 0 d = 0"
| "from_digits (Suc n) d = d 0 + nat base * from_digits n (d  $\circ$  Suc)"
```

Alternative definition using sum:

```
lemma from_digits_altdef: "from_digits n d = ( $\sum_{i < n} d i * nat base ^ i$ )"
  <proof>
```

```
lemma int_from_digits:
  "int (from_digits n d) = ( $\sum_{i < n} int (d i) * base ^ i$ )"
  <proof>
```

Digit in basis *base* of some integer number: *digit x i*

```
x:          int  integer
i:          nat  index
output:     int  i-th digit of representation in basis base of x
```

```

fun digit :: "int  $\Rightarrow$  nat  $\Rightarrow$  int" where
  "digit x 0 = |x| mod base"
| "digit x (Suc i) = digit (|x| div base) i"

```

Alternative definition using divisor and modulo:

```

lemma digit_altdef: "digit x i = ( |x| div (base ^ i)) mod base"
<proof>

```

Every digit must be smaller than the base.

```

lemma digit_less_base: "digit x i < base"
<proof>

```

A representation in basis *base* of length *n* must be less than base^n .

```

lemma from_digits_less:
  assumes " $\forall i < n. d\ i < \text{base}$ "
  shows "from_digits n d < base ^ n"
<proof>

```

Lemmas for *mod* and *div* in number systems of basis *base*:

```

lemma mod_base: assumes " $\bigwedge i. i < n \implies d\ i < \text{base}$ " "n>0"
  shows "from_digits n d mod base = d 0"
<proof>

```

```

lemma mod_base_i:
  assumes " $\bigwedge i. i < n \implies (d\ i :: \text{nat}) < \text{base}$ " "n>0" "i < n"
  shows " $(\sum_{j=i..<n}. d\ j * \text{base} ^ (j-i)) \text{ mod } \text{base} = d\ i$ "
<proof>

```

```

lemma div_base_i:
  assumes " $\bigwedge i. i < n \implies (d\ i :: \text{nat}) < \text{base}$ " "n>0" "i < n"
  shows "from_digits n d div (base ^ i) =  $(\sum_{j=i..<n}. d\ j * \text{base} ^ (j-i))$ "
<proof>

```

Conversions are inverse to each other.

```

lemma digit_from_digits:
  assumes " $\bigwedge j. j < n \implies d\ j < \text{base}$ " "n>0" "i < n"
  shows "digit (from_digits n d) i = d i"
<proof>

```

```

lemma div_distrib: assumes "i < n"
  shows " $(a * \text{base} ^ n + b) \text{ div } \text{base} ^ i \text{ mod } \text{base} = b \text{ div } \text{base} ^ i \text{ mod } \text{base}$ "
<proof>

```

```

lemma(in digits) digits_eq_0:
  assumes "x = 0"
  shows "digit x i = 0"

```

```

<proof>
end

```

```

lemma split_digits_eq_zero:
  assumes "a + base * b = 0" "|a| < base" "(base::int) > 2"
  shows "a = 0 ∧ b = 0"
<proof>

```

```

lemma representation_in_basis_eq_zero:
  assumes "(∑ i < n. c i * base^i) = 0" "(base::int) > 2" "∧ i. i < n ⇒
|c i| < base" "i < n"
  shows "c i = 0"
<proof>

```

```

end
theory Additional_Lemmas

```

```

imports
  Main
  infnorm
  Partition
  Lattice_int
  Digits_int
begin

```

11 Additional Lemmas

Lemma about length and concat.

```

lemma length_concat_map_5:
  "length (concat (map (λi. [f1 i, f2 i, f3 i, f4 i, f5 i]) xs)) = length
xs * 5"
<proof>

```

Lemma about splitting the index of sums.

```

lemma sum_split_idx_prod:
  "(∑ i=0..<k*1::nat. f i) = (∑ i=0..<k. (∑ j=0..<1. f (i*1+j)))"
<proof>

```

Helping lemma to split sums.

```

lemma lt_M:
  assumes "M > (∑ i < (n::nat). |b i|::int)"
  "∀ i < n. |x i| ≤ 1"
  shows "|(∑ i < n. x i * b i)| < M"
<proof>

```



```

lemma split_sum:
  "( $\sum_{i < n} x_i * (a_i + M * b_i) :: int$ ) = ( $\sum_{i < n} x_i * a_i$ ) +
  M * ( $\sum_{i < n} x_i * b_i$ )"
<proof>

```

```

lemma split_eq_system:
  assumes "M > ( $\sum_{i < n} |a_i| :: int$ )"
    "∀ i < n. |x_i| ≤ 1"
    "( $\sum_{i < n} x_i * (a_i + M * b_i) = 0$ "
  shows "( $\sum_{i < n} x_i * a_i = 0 \wedge \sum_{i < n} x_i * b_i = 0$ "
<proof>

```

Lemmas about splitting into 4 or 5 cases.

Split into 4 modulo classes

```

lemma lt_4_split: "(i :: nat) < 4 → i = 0 ∨ i = 1 ∨ i = 2 ∨ i = 3"
<proof>

```

```

lemma mod_exhaust_less_4_int: "(i :: int) mod 4 = 0 ∨ i mod 4 = 1 ∨ i
mod 4 = 2 ∨ i mod 4 = 3"
<proof>

```

```

lemma mod_4_choices:
  assumes "i mod 4 = 0 → P i"
    "i mod 4 = 1 → P i"
    "i mod 4 = 2 → P i"
    "i mod 4 = 3 → P i"
  shows "P (i :: nat)"
<proof>

```

```

lemma mod_4_if_split:
  assumes "i mod 4 = 0 → P = P0 i"
    "i mod 4 = 1 → P = P1 i"
    "i mod 4 = 2 → P = P2 i"
    "i mod 4 = 3 → P = P3 i"
  shows "P = (if i mod 4 = 0 then P0 i else
    (if i mod 4 = 1 then P1 i else
    (if i mod 4 = 2 then P2 i else P3 (i :: nat))))" (is "?P i")
<proof>

```

Split into 5 modulo classes

```

lemma lt_5_split: "(i :: nat) < 5 → i = 0 ∨ i = 1 ∨ i = 2 ∨ i = 3 ∨
i = 4"
<proof>

```

```

lemma mod_exhaust_less_5_int:

```

```

    "(i::int) mod 5 = 0 ∨ i mod 5 = 1 ∨ i mod 5 = 2 ∨ i mod 5 = 3 ∨ i
mod 5 = 4"
⟨proof⟩

```

```

lemma mod_exhaust_less_5:
    "(i::nat) mod 5 = 0 ∨ i mod 5 = 1 ∨ i mod 5 = 2 ∨ i mod 5 = 3 ∨ i
mod 5 = 4"
⟨proof⟩

```

```

lemma mod_5_choices:
    assumes "i mod 5 = 0 → P i"
            "i mod 5 = 1 → P i"
            "i mod 5 = 2 → P i"
            "i mod 5 = 3 → P i"
            "i mod 5 = 4 → P i"
    shows "P (i::nat)"
⟨proof⟩

```

```

lemma mod_5_if_split:
    assumes "i mod 5 = 0 → P = P0 i"
            "i mod 5 = 1 → P = P1 i"
            "i mod 5 = 2 → P = P2 i"
            "i mod 5 = 3 → P = P3 i"
            "i mod 5 = 4 → P = P4 i"
    shows "P = (if i mod 5 = 0 then P0 i else
                (if i mod 5 = 1 then P1 i else
                 (if i mod 5 = 2 then P2 i else
                  (if i mod 5 = 3 then P3 i else
                   P4 (i::nat))))))" (is "?P i")
⟨proof⟩

```

Representation of natural number in interval using lower bound.

```

lemma split_lower_plus_diff:
    assumes "s ∈ {n..<m::nat}"
    obtains j where "s = n+j" and "j<m-n"
⟨proof⟩

```

```

end
theory BHLE

imports
    Main
    Additional_Lemmas
begin

```

12 Bounded Homogeneous Linear Equation Problem

definition *bhle* :: "(int vec * int) set" where
 "bhle $\equiv \{(a,k). \exists x. a \cdot x = 0 \wedge \dim_vec\ x = \dim_vec\ a \wedge \dim_vec\ a > 0 \wedge x \neq 0_v (\dim_vec\ x) \wedge \|x\|_\infty \leq k\}$ "

Reduction of bounded homogeneous linear equation to partition problem

For the reduction function, one defines five-tuples for every element in *a*. The last tuple plays an important role. It consists only of four elements in order to add constraints to the other tuples. These values are formed in a way such that they form all constraints needed for the reductions. Note, that some indices have been changed with respect to [7] to enable better indexing in the vectors/lists.

definition *b1* :: "nat \Rightarrow int \Rightarrow int \Rightarrow int" where
 "b1 *i* *M* *a* = *a* + *M* * (5^(4*i-4) + 5^(4*i-3) + 5^(4*i-1))"

definition *b2* :: "nat \Rightarrow int \Rightarrow int" where
 "b2 *i* *M* = *M* * (5^(4*i-3) + 5^(4*i))"

definition *b2_last* :: "nat \Rightarrow int \Rightarrow int" where
 "b2_last *i* *M* = *M* * (5^(4*i-3) + 1)"

definition *b3* :: "nat \Rightarrow int \Rightarrow int" where
 "b3 *i* *M* = *M* * (5^(4*i-4) + 5^(4*i-2))"

definition *b4* :: "nat \Rightarrow int \Rightarrow int \Rightarrow int" where
 "b4 *i* *M* *a* = *a* + *M* * (5^(4*i-2) + 5^(4*i-1) + 5^(4*i))"

definition *b4_last* :: "nat \Rightarrow int \Rightarrow int \Rightarrow int" where
 "b4_last *i* *M* *a* = *a* + *M* * (5^(4*i-2) + 5^(4*i-1) + 1)"

definition *b5* :: "nat \Rightarrow int \Rightarrow int" where
 "b5 *i* *M* = *M* * (5^(4*i-1))"

Change order of indices in [7] such that *b3* is in last place and can be omitted in the last entry. This ensures that the weight of the solution is 1 or -1, essential for the proof of NP-hardnes.

definition *b_list* :: "int list \Rightarrow nat \Rightarrow int \Rightarrow int list" where
 "b_list *as* *i* *M* = [b1 (i+1) *M* (as!i), b2 (i+1) *M*, b4 (i+1) *M* (as!i), b5 (i+1) *M*, b3 (i+1) *M*]"

definition *b_list_last* :: "int list \Rightarrow nat \Rightarrow int \Rightarrow int list" where
 "b_list_last *as* *n* *M* = [b1 *n* *M* (last *as*), b2_last *n* *M*, b4_last *n* *M* (last *as*), b5 *n* *M*]"

```

definition gen_bhle :: "int list  $\Rightarrow$  int vec" where
"gen_bhle as = (let M = 2*( $\sum$  i<length as. |as!i|)+1; n = length as in
  vec_of_list (concat
    (map ( $\lambda$ i. b_list as i M) [0.. $n$ -1])
    @ (if n>0 then b_list_last as n M else [])))"

```

The reduction function.

```

definition reduce_bhle_partition:: "(int list)  $\Rightarrow$  ((int vec) * int)" where
"reduce_bhle_partition  $\equiv$  ( $\lambda$  a. (gen_bhle a, 1))"

```

Lemmas for proof

```

lemma dim_vec_gen_bhle:
  assumes "as  $\neq$  []"
  shows "dim_vec (gen_bhle as) = 5 * (length as) - 1"
<proof>

```

```

lemma dim_vec_gen_bhle_empty:
  "dim_vec (gen_bhle []) = 0"
<proof>

```

Lemmas about length

```

lemma length_b_list:
  "length (b_list a i M) = 5" <proof>

```

```

lemma length_b_list_last:
  "length (b_list_last a n M) = 4" <proof>

```

```

lemma length_concat_map_b_list:
  "length (concat (map ( $\lambda$ i. b_list as i M) [0.. $k$ ])) = 5 * k"
<proof>

```

Last values of gen_bhle

```

lemma gen_bhle_last0:
  assumes "length as > 0"
  shows "(gen_bhle as) $ ((length as - 1) * 5) =
    b1 (length as) (2*( $\sum$  i<length as. |as!i|)+1) (last as)"
<proof>

```

```

lemma gen_bhle_last1:
  assumes "length as > 0"
  shows "(gen_bhle as) $ ((length as - 1) * 5 + 1) =
    b2_last (length as) (2*( $\sum$  i<length as. |as!i|)+1)"
<proof>

```

The third entry of the last tuple is omitted, thus we skip one lemma

```

lemma gen_bhle_last3:
  assumes "length as > 0"

```

shows "(gen_bhle as) \$ ((length as -1) * 5 + 2) =
 b4_last (length as) (2*($\sum_{i < \text{length as}} |\text{as}!i|$)+1) (last as)"
 <proof>

lemma gen_bhle_last4:
 assumes "length as > 0"
 shows "(gen_bhle as) \$ ((length as-1) * 5 + 3) =
 b5 (length as) (2*($\sum_{i < \text{length as}} |\text{as}!i|$)+1)"
 <proof>

Up to last values of gen_bhle

lemma b_list_nth:
 assumes "i < length as-1" "j < 5"
 shows "concat (map ($\lambda i. \text{b_list as } i \text{ } M$) [0.. $\text{length as} - 1$]) ! (i *
 5 + j) =
 b_list as i M ! j"
 <proof>

lemma b_list_nth0:
 assumes "i < length as-1"
 shows "concat (map ($\lambda i. \text{b_list as } i \text{ } M$) [0.. $\text{length as} - 1$]) ! (i *
 5) =
 b_list as i M ! 0"
 <proof>

lemma gen_bhle_0:
 assumes "i < length as-1"
 shows "(gen_bhle as) \$ (i * 5) =
 b1 (i+1) (2*($\sum_{i < \text{length as}} |\text{as}!i|$)+1) (as!i)"
 <proof>

lemma gen_bhle_1:
 assumes "i < length as-1"
 shows "(gen_bhle as) \$ (i * 5 + 1) =
 b2 (i+1) (2*($\sum_{i < \text{length as}} |\text{as}!i|$)+1)"
 <proof>

lemma gen_bhle_4:
 assumes "i < length as-1"
 shows "(gen_bhle as) \$ (i * 5 + 4) =
 b3 (i+1) (2*($\sum_{i < \text{length as}} |\text{as}!i|$)+1)"
 <proof>

lemma gen_bhle_2:
 assumes "i < length as-1"
 shows "(gen_bhle as) \$ (i * 5 + 2) =
 b4 (i+1) (2*($\sum_{i < \text{length as}} |\text{as}!i|$)+1) (as!i)"
 <proof>

```

lemma gen_bhle_3:
  assumes "i < length as-1"
  shows "(gen_bhle as) $ (i * 5 + 3) =
    b5 (i+1) (2*( $\sum_{i < \text{length } as. |as!i|$ )+1)"
<proof>

```

Well-definedness of reduction function

```

lemma well_defined_reduction_subset_sum:
  assumes "a  $\in$  partition_problem_nonzero"
  shows "reduce_bhle_partition a  $\in$  bhle"
<proof>

```

NP-hardness of reduction function

```

lemma NP_hardness_reduction_subset_sum:
  assumes "reduce_bhle_partition a  $\in$  bhle"
  shows "a  $\in$  partition_problem_nonzero"
<proof>

```

The Gap-SVP is NP-hard.

```

lemma "is_reduction reduce_bhle_partition partition_problem_nonzero bhle"
<proof>

```

end

theory SVP_vec

imports

BHLE

begin

13 SVP in ℓ_∞

The reduction of SVP to BHLE in l_∞ norm

The shortest vector problem.

```

definition is_shortest_vec :: "int_lattice  $\Rightarrow$  int vec  $\Rightarrow$  bool" where
  "is_shortest_vec L v  $\equiv$  (is_lattice L)  $\wedge$  ( $\forall x \in L. \|x\|_\infty \geq \|v\|_\infty \wedge v \in L$ )
"

```

The decision problem associated with solving SVP exactly.

```

definition gap_svp :: "(int_lattice  $\times$  int) set" where
  "gap_svp  $\equiv$  {(L, r). (is_lattice L)  $\wedge$  (dim_lattice L  $\geq$  2)  $\wedge$ 
    ( $\exists v \in L. \|v\|_\infty \leq r \wedge v \neq 0_v$  (dim_vec v))}"

```

Generate a lattice to solve SVP for reduction.

Here, the factor K'' from the paper [7] was changed to be $2 \cdot \mathbf{k} \cdot (k+1) \cdot \sum_i |\mathbf{a}_i|$ in order for the proofs to finish.

```

definition gen_svp_basis :: "int vec  $\Rightarrow$  int  $\Rightarrow$  int mat" where
  "gen_svp_basis a k = mat (dim_vec a + 1) (dim_vec a + 1)
    ( $\lambda$  (i, j). if (i < dim_vec a)  $\wedge$  (j < dim_vec a) then (if i = j then
1 else 0)
      else (if (i < dim_vec a)  $\wedge$  (j  $\geq$  dim_vec a) then 0
      else (if (i  $\geq$  dim_vec a)  $\wedge$  (j < dim_vec a) then (k+1) * (a $ j)
      else 2*k*(k+1)* ( $\sum$  i  $\in$  {0 ..< dim_vec a}. |a $ i| + 1 )))"

```

Reduction SVP to bounded homogeneous linear equation problem in ℓ_∞ norm.

```

definition reduce_svp_bhle :: "(int vec  $\times$  int)  $\Rightarrow$  (int_lattice  $\times$  int)"
where
  "reduce_svp_bhle  $\equiv$  ( $\lambda$  (a, k). (gen_lattice (gen_svp_basis a k), k))"

```

Lemmas for proof

```

lemma gen_svp_basis_mult:
  assumes "dim_vec z = dim_vec a + 1"
  shows "(gen_svp_basis a k) *_v z = vec (dim_vec a + 1)
    ( $\lambda$ i. if i < dim_vec a then z$i else (k+1) * ( $\sum$  i  $\in$  {0 ..< dim_vec
a}. z $ i * a $ i) +
      (2*k*(k+1)* ( $\sum$  i  $\in$  {0 ..< dim_vec a}. |a $ i| + 1) * (z$(dim_vec
a))))"
  <proof>

```

```

lemma gen_svp_basis_mult_real:
  assumes "dim_vec z = dim_vec a + 1"
  shows "real_of_int_mat (gen_svp_basis a k) *_v z = vec (dim_vec a + 1)
    ( $\lambda$ i. if i < dim_vec a then z$i else (k+1) * ( $\sum$  i  $\in$  {0 ..< dim_vec
a}. z $ i * a $ i) +
      (2*k*(k+1)* ( $\sum$  i  $\in$  {0 ..< dim_vec a}. |a $ i| + 1) * (z$(dim_vec
a))))"
  <proof>

```

```

lemma gen_svp_basis_mult_last:
  assumes "dim_vec z = dim_vec a + 1"
  shows "((gen_svp_basis a k) *_v z) $ (dim_vec a) =
    (k+1) * ( $\sum$  i  $\in$  {0 ..< dim_vec a}. z $ i * a $ i) +
    (2*k*(k+1)* ( $\sum$  i  $\in$  {0 ..< dim_vec a}. |a $ i| + 1) * (z$(dim_vec
a)))"
  <proof>

```

The set generated by `gen_svp_basis` is indeed linearly independent.

```

lemma is_indep_gen_svp_basis:
  assumes "k>0"
  shows "is_indep (gen_svp_basis a k)"
  <proof>

```

lemma insert_set_comprehension:
 "insert (f x) {f i | i. i < (x::nat)} = {f i | i. i < x+1}"
 <proof>

lemma bhle_k_pos:
 assumes "(a,k) ∈ bhle"
 shows "k > 0"
 <proof>

lemma svp_k_pos:
 assumes "reduce_svp_bhle (a, k) ∈ gap_svp"
 shows "k > 0"
 <proof>

lemma svp_dim_vec_a:
 assumes "reduce_svp_bhle (a, k) ∈ gap_svp"
 shows "dim_vec a > 0"
 <proof>

lemma bhle_dim_vec_a:
 assumes "(a, k) ∈ bhle"
 shows "dim_vec a > 0"
 <proof>

lemma first_lt_second:
 assumes "k > 0" and z_le_k: " $\bigwedge i. i < \dim_vec\ a \implies |z\ \$\ i| \leq k$ "
 shows " $2 * |(k + 1) * (\sum i = 0..<\dim_vec\ a. z\ \$\ i * a\ \$\ i)|$
 < $(2 * k * (k + 1) * (\sum i = 0..<\dim_vec\ a. |a\ \$\ i|) + 1|::int)$ "
 <proof>

Well-definedness of reduction function

lemma well_defined_reduction_svp:
 assumes "(a,k) ∈ bhle"
 shows "reduce_svp_bhle (a,k) ∈ gap_svp"
 <proof>

NP-hardness of reduction function

lemma NP_hardness_reduction_svp:
 assumes "reduce_svp_bhle (a,k) ∈ gap_svp"
 shows "(a,k) ∈ bhle"
 <proof>

The SVP is NP-hard in ℓ_∞ .

lemma "is_reduction reduce_svp_bhle bhle gap_svp"
 <proof>

end

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