The HOL-CSP Refinement Toolkit

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Abstract

Recently, a modern version of Roscoes and Brookes [3] Failure-Divergence Semantics for CSP has been formalized in Isabelle [10].

We use this formal development called HOL-CSP2.0 to analyse a family of refinement notions, comprising classic and new ones. This analysis enables to derive a number of properties that allow to deepen the understanding of these notions, in particular with respect to specification decomposition principles for the case of infinite sets of events. The established relations between the refinement relations help to clarify some obscure points in the CSP literature, but also provide a weapon for shorter refinement proofs. Furthermore, we provide a framework for state-normalisation allowing to formally reason on parameterised process architectures.

As a result, we have a modern environment for formal proofs of concurrent systems that allow for the combination of general infinite processes with locally finite ones in a logically safe way. We demonstrate these verification-techniques for classical, generalised examples: The CopyBuffer for arbitrary data and the Dijkstra's Dining Philosopher Problem of arbitrary size.

If you consider to cite this work, please refer to [11].

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Chapter 1

Context

1.1 Introduction

Communicating Sequential Processes CSP is a language to specify and verify patterns of interaction of concurrent systems. Together with CCS and LOTOS, it belongs to the family of *process algebras*. CSP's rich theory comprises denotational, operational and algebraic semantic facets and has influenced programming languages such as Limbo, Crystal, Clojure and most notably Golang [5]. CSP has been applied in industry as a tool for specifying and verifying the concurrent aspects of hardware systems, such as the T9000 transputer [1].

The theory of CSP, in particular the denotational Failure/Divergence Denotational Semantics, has been initially proposed in the book by Tony Hoare [6], but evolved substantially since [2, 3, 8].

Verification of CSP properties has been centered around the notion of *process refinement orderings*, most notably $\text{-}\sqsubseteq_{FD}\text{-}$ and $\text{-}\sqsubseteq\text{-}$. The latter turns the denotational domain of CSP into a Scott cpo [9], which yields semantics for the fixed point operator $\mu x.\ f(x)$ provided that f is continuous with respect to $\text{-}\sqsubseteq\text{-}$. Since it is possible to express deadlock-freeness and livelock-freeness as a refinement problem, the verification of properties has been reduced traditionally to a model-checking problem for a finite set of events A.

We are interested in verification techniques for arbitrary event sets A or arbitrarily parameterized processes. Such processes can be used to model dense-timed processes, processes with dynamic thread creation, and processes with unbounded thread-local variables and buffers. Events may even be higher-order objects such as functions or again processes, paving the way for the modeling of re-programmable compute servers or dynamic distributed computing architectures. However, this adds substantial complexity to the process theory: when it comes to study the interplay of different denotational models, refinement-orderings, and side-conditions for continuity, paper-and-pencil proofs easily reach their limits of precision.

Several attempts have been undertaken to develop the formal theory of CSP in an interactive proof system, mostly in Isabelle/HOL [4, 12, 7]. This work is based on the most recent instance in this line, HOL-CSP 2.0, which has been published as AFP submission [10] and whose development is hosted at https://gitlri.lri.fr/burkhart.wolff/hol-csp2.0.

The present AFP Module is an add-on on this work and develops some support for

1. advanced induction schemes (mutual fixed-point Induction, K-induction),
2. bridge-Lemmas between the classical refinement relations in the FD-semantics, which allow for reduced refinement proof complexity in certain cases, and
3. a theory of explicit state normalisation which allows for proofs over certain communicating networks of arbitrary size.

1.2 The Global Architecture of CSP_RefTk

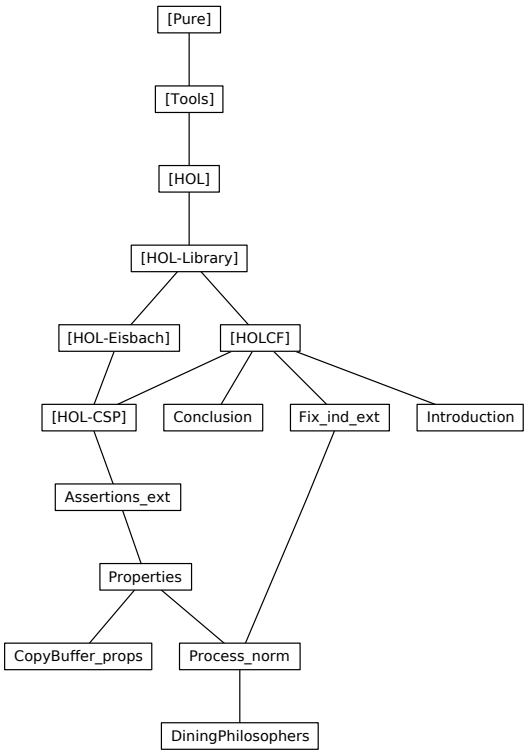

Figure 1.1: The overall architecture: HOLCF, HOL-CSP, and CSP_RefTk

The global architecture of CSP_RefTk is shown in Figure 1.1. The entire package resides on:

1. `HOL-Eisbach` from the Isabelle/HOL distribution,
2. `HOLCF` from the Isabelle/HOL distribution, and
3. `HOL-CSP` 2.0 from the Isabelle Archive of Formal Proofs.

   The theories `Assertion_ext` and `Fixind_ext` are extensions of the corresponding theories in `HOL-CSP`.

theory *Properties*
imports *HOL−CSP.Assertions*
begin

1.3 Non-terminating Runs

definition *non-terminating* :: *'a process* ⇒ *bool*
where *non-terminating P* ≡ *RUN UNIV* ⊑$_T$ *P*

corollary *non-terminating-refine-DF*: *non-terminating P* = *DF UNIV* ⊑$_T$ *P*
and *non-terminating-refine-CHAOS*: *non-terminating P* = *CHAOS UNIV* ⊑$_T$ *P*
by (*simp-all add*: *DF-all-tickfree-traces1 RUN-all-tickfree-traces1 CHAOS-all-tickfree-traces1*

non-terminating-def trace-refine-def)

lemma *non-terminating-is-right*: *non-terminating* (*P*::*'a process*) ⟷ (∀ *s*∈$\mathcal{T}$ *P*. *tickFree s*)
apply (*rule iffI*)
by (*auto simp add*:*non-terminating-def trace-refine-def tickFree-def RUN-all-tickfree-traces1*)[*1*]
(*auto simp add*:*non-terminating-def trace-refine-def RUN-all-tickfree-traces2*)

lemma *nonterminating-implies-div-free*: *non-terminating P* ⟹ $\mathcal{D}$ *P* = {}
unfolding *non-terminating-is-right*
by (*metis NT-ND equals0I front-tickFree-charn process-charn tickFree-Cons tickFree-append*)

lemma *non-terminating-implies-F*: *non-terminating P* ⟹ *CHAOS UNIV* ⊑$_F$ *P*

apply(*auto simp add*: *non-terminating-is-right failure-refine-def*)
using *CHAOS-has-all-tickFree-failures F-T* **by** *blast*

corollary *non-terminating-F*: *non-terminating P = CHAOS UNIV* $\sqsubseteq_F$ *P*
by (*auto simp add*:*non-terminating-implies-F non-terminating-refine-CHAOS leF-imp-leT*)

corollary *non-terminating-FD*: *non-terminating P = CHAOS UNIV* $\sqsubseteq_{FD}$ *P*
unfolding *non-terminating-F*
using *div-free-CHAOS nonterminating-implies-div-free leFD-imp-leF leF-leD-imp-leFD divergence-refine-def non-terminating-F*
by *fastforce*

1.4 Lifelock Freeness

thm *lifelock-free-def*

corollary *lifelock-free-is-non-terminating*: *lifelock-free P = non-terminating P*
unfolding *lifelock-free-def non-terminating-FD* **by** *rule*

lemma *div-free-divergence-refine*:
$\mathcal{D}$ *P* = {} $\longleftrightarrow$ $CHAOS_{SKIP}$ *UNIV* $\sqsubseteq_D$ *P*
$\mathcal{D}$ *P* = {} $\longleftrightarrow$ *CHAOS UNIV* $\sqsubseteq_D$ *P*
$\mathcal{D}$ *P* = {} $\longleftrightarrow$ *RUN UNIV* $\sqsubseteq_D$ *P*
$\mathcal{D}$ *P* = {} $\longleftrightarrow$ DF_{SKIP} *UNIV* $\sqsubseteq_D$ *P*
$\mathcal{D}$ *P* = {} $\longleftrightarrow$ *DF UNIV* $\sqsubseteq_D$ *P*
by (*simp-all add*: $div\text{-}free\text{-}CHAOS_{SKIP}$ *div-free-CHAOS div-free-RUN div-free-DF* $div\text{-}free\text{-}DF_{SKIP}$ *divergence-refine-def*)

definition *lifelock-free-v2* :: *'a process* $\Rightarrow$ *bool*
where *lifelock-free-v2 P* $\equiv$ $CHAOS_{SKIP}$ *UNIV* $\sqsubseteq_{FD}$ *P*

lemma *div-free-is-lifelock-free-v2*: *lifelock-free-v2 P* $\longleftrightarrow$ $\mathcal{D}$ *P* = {}
using $CHAOS_{SKIP}$*-has-all-failures-Un leFD-imp-leD leF-leD-imp-leFD div-free-divergence-refine*(*1*) *lifelock-free-v2-def*
by *blast*

lemma *lifelock-free-is-lifelock-free-v2*: *lifelock-free P* $\Longrightarrow$ *lifelock-free-v2 P*
by (*simp add*: *leFD-imp-leD div-free-divergence-refine*(*2*) *div-free-is-lifelock-free-v2 lifelock-free-def*)

corollary *deadlock-free-v2-is-lifelock-free-v2*: *deadlock-free-v2 P* $\Longrightarrow$ *lifelock-free-v2 P*
by (*simp add*: *deadlock-free-v2-implies-div-free div-free-is-lifelock-free-v2*)

1.5 New laws

lemma *non-terminating-Seq*:
non-terminating P $\Longrightarrow$ (*P* **;** *Q*) = *P*
apply(*auto simp add*: *non-terminating-is-right Process-eq-spec D-Seq F-Seq F-T is-processT7*)
using *process-charn* **apply** *blast*
using *process-charn* **apply** *blast*
using *F-T is-processT5-S2a* **apply** *fastforce*
using *D-T front-tickFree-Nil* **by** *blast*

lemma *non-terminating-Sync*:
non-terminating P $\Longrightarrow$ *lifelock-free-v2 Q* $\Longrightarrow$ *non-terminating* (*P* $[\![A]\!]$ *Q*)
apply(*auto simp add*: *non-terminating-is-right div-free-is-lifelock-free-v2 T-Sync*)

apply (*metis equals0D ftf-Sync1 ftf-Sync21 insertI1 tickFree-def*)
apply (*meson NT-ND is-processT7-S tickFree-append*)
by (*metis D-T empty-iff ftf-Sync1 ftf-Sync21 insertI1 tickFree-def*)

lemmas *non-terminating-Par* = *non-terminating-Sync*[**where** *A* = ‹*UNIV*›]
and *non-terminating-Inter* = *non-terminating-Sync*[**where** *A* = ‹{}›]

end

Chapter 2

Normalisation of Deterministic CSP Processes

theory *Process-norm*

imports *Properties HOL−CSP.Induction-ext*

begin

2.1 Deterministic normal-forms with explicit state

abbreviation *P-dnorm* $\tau\ \upsilon \equiv (\mu\ X.\ (\lambda\ s.\ \square\ e \in (\tau\ s) \rightarrow X\ (\upsilon\ s\ e)))$

notation *P-dnorm* $(P_{norm}[\![\text{-},\text{-}]\!]\ 60)$

lemma *dnorm-cont*[*simp*]:
fixes τ::$'\sigma$::*type* $\Rightarrow$ $'$*event*::*type set* **and** υ::$'\sigma \Rightarrow$ $'$*event* $\Rightarrow$ $'\sigma$
shows *cont* $(\lambda X.\ (\lambda s.\ \square\ e \in (\tau\ s) \rightarrow X\ (\upsilon\ s\ e)))$ (**is** *cont ?f*)
proof −
have *cont* $(\lambda X.\ ?f\ X\ s)$ **for** s **by** (*simp add*:*cont-fun*)
then show *?thesis* **by** *simp*
qed

2.2 Interleaving product lemma

lemma *dnorm-inter*:
fixes τ_1 ::$'\sigma_1$::*type* $\Rightarrow$ $'$*event*::*type set* **and** τ_2::$'\sigma_2$::*type* $\Rightarrow$ $'$*event set*
and υ_1 ::$'\sigma_1 \Rightarrow$ $'$*event* $\Rightarrow$ $'\sigma_1$ **and** υ_2::$'\sigma_2 \Rightarrow$ $'$*event* $\Rightarrow$ $'\sigma_2$
defines *P*: $P \equiv P_{norm}[\![\tau_1,\upsilon_1]\!]$ (**is** $P \equiv$ *fix*·$(\Lambda\ X.\ ?P\ X)$)
defines *Q*: $Q \equiv P_{norm}[\![\tau_2,\upsilon_2]\!]$ (**is** $Q \equiv$ *fix*·$(\Lambda\ X.\ ?Q\ X)$)

assumes *indep*: $\langle \forall\ s_1\ s_2.\ \tau_1\ s_1 \cap \tau_2\ s_2 = \{\} \rangle$

defines *Tr*: $\tau \equiv (\lambda(s_1,s_2).\ \tau_1\ s_1 \cup \tau_2\ s_2)$

defines *Up*: $\upsilon \equiv (\lambda(s_1,s_2)\ e.\ \mathit{if}\ e \in \tau_1\ s_1\ \mathit{then}\ (\upsilon_1\ s_1\ e,s_2)$
$\mathit{else\ if}\ e \in \tau_2\ s_2\ \mathit{then}\ (s_1,\ \upsilon_2\ s_2\ e)\ \mathit{else}\ (s_1,s_2))$
defines *S*: $S \equiv P_{norm}[\![\tau,\upsilon]\!]$ (**is** $S \equiv \mathit{fix}\cdot(\Lambda\ X.\ ?S\ X)$)

shows $(P\ s_1\ |||\ Q\ s_2) = S\ (s_1,s_2)$

proof −
have *P-rec*: $P = ?P\ P$ **using** *fix-eq*[*of* $(\Lambda\ X.\ ?P\ X)$] P **by** *simp*
have *Q-rec*: $Q = ?Q\ Q$ **using** *fix-eq*[*of* $(\Lambda\ X.\ ?Q\ X)$] Q **by** *simp*
have *S-rec*: $S = ?S\ S$ **using** *fix-eq*[*of* $(\Lambda\ X.\ ?S\ X)$] S **by** *simp*
have *dir1*: $\forall\ s_1\ s_2.\ (P\ s_1\ |||\ Q\ s_2) \sqsubseteq_{FD} S\ (s_1,\ s_2)$
proof(*subst P*, *subst Q*,
induct rule:*parallel-fix-ind-inc*[*of* $\lambda x\ y.\ \forall\ s_1\ s_2.\ (x\ s_1\ |||\ y\ s_2) \sqsubseteq_{FD} S\ (s_1,s_2)$])
case *admissibility*
then show *?case*
by (*intro adm-all le-FD-adm*) (*simp-all add*: *cont2cont-fun monofunI*)
next
case (*base-fst y*)
then show *?case* **by** (*metis app-strict BOT-leFD Sync-BOT Sync-commute*)
next
case (*base-snd x*)
then show *?case* **by** (*simp add*: *Sync-BOT*)
next
case (*step x*)
then show *?case* (**is** $\forall\ s_1\ s_2.\ ?C\ s_1\ s_2$)
proof(*intro allI*)
fix $s_1\ s_2$
show $?C\ s_1\ s_2$
apply *simp*
apply (*subst Mprefix-Sync-distr-indep*[**where** $S = \{\}$, *simplified*])
apply (*subst S-rec*, *simp add*: *Tr Up Mprefix-Un-distrib*)
apply (*intro mono-Det-FD mono-Mprefix-FD*)
using *step*(*3*)[*simplified*] *indep* **apply** *simp*
using *step*(*2*)[*simplified*] *indep* **by** *fastforce*
qed
qed
have *dir2*: $\forall\ s_1\ s_2.\ \ S\ (s_1,\ s_2) \sqsubseteq_{FD} (P\ s_1\ |||\ Q\ s_2)$
proof(*subst S*, *induct rule*:*fix-ind-k*[*of* $\lambda x.\ \forall\ s_1\ s_2.\ x\ (s_1,s_2) \sqsubseteq_{FD} (P\ s_1\ |||\ Q\ s_2)\ 1$])
case *admissibility*
show *?case* **by** (*intro adm-all le-FD-adm*) (*simp-all add*: *cont-fun monofunI*)

next
case *base-k-steps*
then show *?case* **by** *simp*
next
case *step*
then show *?case* (**is** $\forall\ s_1\ s_2.\ ?C\ s_1\ s_2$)
proof(*intro allI*)

fix s_1 s_2
have *P-rec-sym*:*Mprefix* (τ_1 s_1) ($\lambda e.$ P (υ_1 s_1 e)) = P s_1 **using** *P-rec* **by** *metis*
have *Q-rec-sym*:*Mprefix* (τ_2 s_2) ($\lambda e.$ Q (υ_2 s_2 e)) = Q s_2 **using** *Q-rec* **by** *metis*
show *?C* s_1 s_2
apply (*simp add*: *Tr Up Mprefix-Un-distrib*)
apply (*subst P-rec*, *subst Q-rec*, *subst Mprefix-Sync-distr-indep*[**where** S={}, *simplified*])
apply (*intro mono-Det-FD mono-Mprefix-FD*)
apply (*subst Q-rec-sym*, *simp add*:*step*[*simplified*])
apply (*subst P-rec-sym*) **using** *step*[*simplified*] *indep* **by** *fastforce*
qed
qed
from *dir1 dir2* **show** *?thesis* **using** *FD-antisym* **by** *blast*
qed

2.3 Synchronous product lemma

lemma *dnorm-par*:
fixes τ_1 ::'σ_1::*type* ⇒ '*event*::*type set* **and** τ_2::'σ_2::*type* ⇒ '*event set*
and υ_1 ::'σ_1 ⇒ '*event* ⇒ 'σ_1 **and** υ_2::'σ_2 ⇒ '*event* ⇒ 'σ_2
defines *P*: $P \equiv P_{norm}[\![\tau_1,\upsilon_1]\!]$ (**is** $P \equiv$ *fix*·(Λ *X*. *?P X*))
defines *Q*: $Q \equiv P_{norm}[\![\tau_2,\upsilon_2]\!]$ (**is** $Q \equiv$ *fix*·(Λ *X*. *?Q X*))

defines *Tr*: $\tau \equiv (\lambda(s_1,s_2).\ \tau_1\ s_1 \cap \tau_2\ s_2)$
defines *Up*: $\upsilon \equiv (\lambda(s_1,s_2)\ e.\ (\upsilon_1\ s_1\ e, \upsilon_2\ s_2\ e))$
defines *S*: $S \equiv P_{norm}[\![\tau,\upsilon]\!]$ (**is** $S \equiv$ *fix*·(Λ *X*. *?S X*))

shows (P s_1 || Q s_2) = S (s_1,s_2)

proof −
have *P-rec*: P = *?P P* **using** *fix-eq*[*of* (Λ *X*. *?P X*)] P **by** *simp*
have *Q-rec*: Q = *?Q Q* **using** *fix-eq*[*of* (Λ *X*. *?Q X*)] Q **by** *simp*
have *S-rec*: S = *?S S* **using** *fix-eq*[*of* (Λ *X*. *?S X*)] S **by** *simp*
have *dir1*: ∀ s_1 s_2. (P s_1 || Q s_2) $\sqsubseteq_{FD}$ S (s_1, s_2)
proof(*subst P*, *subst Q*,
induct rule:*parallel-fix-ind*[*of* $\lambda x\ y.$ ∀ s_1 s_2. (x s_1 || y s_2) $\sqsubseteq_{FD}$ S (s_1,s_2)])
case *adm*:*1*
then show *?case*
by (*intro adm-all le-FD-adm*) (*simp-all add*: *cont2cont-fun monofunI*)
next
case *base*:*2*
then show *?case* **by** (*simp add*: *Sync-BOT*)
next
case *step*:(*3 x y*)
then show *?case* (**is** ∀ s_1 s_2. *?C* s_1 s_2)
proof(*intro allI*)
fix s_1 s_2

```
      show ?C s1 s2
        apply(simp)
        apply(subst Mprefix-Sync-distr-subset[where S=UNIV, simplified])
        apply(subst S-rec, simp add: Tr Up Mprefix-Un-distrib)
        by (simp add: step)
    qed
  qed
 have dir2: ∀ s1 s2. S (s1, s2) ⊑FD (P s1 || Q s2)
  proof(subst S, induct rule:fix-ind-k[of λx. ∀ s1 s2. x (s1,s2) ⊑FD (P s1 || Q
s2) 1])
   case admissibility
   show ?case by (intro adm-all le-FD-adm) (simp-all add: cont-fun monofunI)

  next
   case base-k-steps
   then show ?case by simp
  next
   case step
   then show ?case (is ∀ s1 s2. ?C s1 s2)
   proof(intro allI)
    fix s1 s2
     have P-rec-sym:Mprefix (τ1 s1) (λe. P (υ1 s1 e)) = P s1 using P-rec by
metis
     have Q-rec-sym:Mprefix (τ2 s2) (λe. Q (υ2 s2 e)) = Q s2 using Q-rec by
metis
    show ?C s1 s2
      apply(simp add: Tr Up)
        apply(subst P-rec, subst Q-rec, subst Mprefix-Sync-distr-subset[where
S=UNIV, simplified])
      apply(rule mono-Mprefix-FD)
      using step by auto
   qed
  qed
 from dir1 dir2 show ?thesis using FD-antisym by blast
qed
```

2.4 Consequences

```
inductive-set ℜ for      τ ::'σ::type ⇒ 'event::type set
                  and υ ::'σ ⇒ 'event ⇒ 'σ
                  and σ0 ::'σ
 where rbase: σ0 ∈ ℜ τ υ σ0
    | rstep: s ∈ ℜ τ υ σ0 ⟹ e ∈ τ s ⟹ υ s e ∈ ℜ τ υ σ0
```

— Deadlock freeness

```
lemma deadlock-free-dnorm- :
 fixes τ ::'σ::type ⇒ 'event::type set
```

and $\upsilon :: 'a \Rightarrow 'event \Rightarrow 'a$
and $\sigma_0 :: 'a$
assumes *non-reachable-sink*: $\forall\, s \in \mathfrak{R}\ \tau\ \upsilon\ \sigma_0.\ \tau\ s \neq \{\}$
defines *P*: $P \equiv P_{norm}[\![\tau,\upsilon]\!]$ (**is** $P \equiv$ *fix*·$(\Lambda\ X.\ ?P\ X)$)
shows $s \in \mathfrak{R}\ \tau\ \upsilon\ \sigma_0 \Longrightarrow$ *deadlock-free-v2* $(P\ s)$
proof(*unfold deadlock-free-v2-FD* DF_{SKIP}*-def*, *induct arbitrary*:*s rule*:*fix-ind*)
show *adm* $(\lambda a.\ \forall\, x.\ x \in \mathfrak{R}\ \tau\ \upsilon\ \sigma_0 \longrightarrow a \sqsubseteq_{FD} P\ x)$ **by** (*simp add*: *monofun-def*)
next
fix $s :: 'a$
show $s \in \mathfrak{R}\ \tau\ \upsilon\ \sigma_0 \Longrightarrow \bot \sqsubseteq_{FD} P\ s$ **by** *simp*
next
fix $s :: 'a$ **and** $x ::$ *'event process*
have *P-rec*: $P = ?P\ P$ **using** *fix-eq*[*of* $(\Lambda\ X.\ ?P\ X)$] P **by** *simp*
assume *1* : $\bigwedge s.\ s \in \mathfrak{R}\ \tau\ \upsilon\ \sigma_0 \Longrightarrow x \sqsubseteq_{FD} P\ s$
and *2* : $s \in \mathfrak{R}\ \tau\ \upsilon\ \sigma_0$
from *1 2* **show** $(\Lambda\ x.\ (\sqcap xa{\in}UNIV \rightarrow x) \sqcap SKIP)\cdot x \sqsubseteq_{FD} P\ s$
apply (*subst P-rec*, *rule-tac trans-FD*[*rotated*, *OF Mprefix-refines-Mndetprefix-FD*])
apply *simp*
apply (*rule mono-Ndet-FD-left*)
apply (*rule trans-FD*[*OF mono-Mndetprefix-FD-set*[*of* ‹τ s› ‹$UNIV$›]
mono-Mndetprefix-FD[*rule-format*, *OF 1*]])
using *non-reachable-sink*[*rule-format*, *OF 2*] **apply** *assumption*
by *blast* (*meson* $\mathfrak{R}$.*rstep*)
qed

lemmas *deadlock-free-dnorm* = *deadlock-free-dnorm-*[*rotated*, *OF rbase*, *rule-format*]

end

Chapter 3

Examples

3.1 CopyBuffer Refinement over an infinite alphabet

theory *CopyBuffer-props*
imports *HOL−CSP.CopyBuffer Properties*
begin

3.1.1 The Copy-Buffer vs. reference processes

lemma *DF-COPY*: $(DF\ (range\ left \cup range\ right)) \sqsubseteq_{FD} COPY$
apply(*simp add*: *DF-def*,*rule fix-ind2*, *simp-all*)
proof −
show *adm* $(\lambda a.\ a \sqsubseteq_{FD} COPY)$ **by** (*rule le-FD-adm*, *simp-all add*: *monofunI*)
next
have *1*: $(\sqcap xa \in range\ left \cup range\ right \rightarrow \bot) \sqsubseteq_{FD} (\sqcap xa \in range\ left \rightarrow \bot)$ **by** *simp*
have *2*: $(\sqcap xa \in range\ left \rightarrow \bot) \sqsubseteq_{FD} (left?x \rightarrow \bot)$
unfolding *read-def*
by (*meson BOT-leFD mono-Mndetprefix-FD Mprefix-refines-Mndetprefix-FD trans-FD*)
show $(\sqcap x \in range\ left \cup range\ right \rightarrow \bot) \sqsubseteq_{FD} COPY$
by (*metis* (*mono-tags*, *lifting*) *1 2 BOT-leFD COPY-rec mono-read-FD trans-FD*)
next
fix *P*::*'a channel process*
assume $*$: $P \sqsubseteq_{FD} COPY$ **and** $**$: $(\sqcap x \in range\ left \cup range\ right \rightarrow P) \sqsubseteq_{FD} COPY$
have *1*: $(\sqcap xa \in range\ left \cup range\ right \rightarrow P) \sqsubseteq_{FD} (\sqcap xa \in range\ right \rightarrow P)$ **by** *force*
have *2*: $(\sqcap xa \in range\ right \rightarrow P) \sqsubseteq_{FD} (right!x \rightarrow P)$ **for** *x*
by (*metis mono-Mndetprefix-FD-set*[*of* {*right x*} *range right*] *Mprefix-refines-Mndetprefix-FD empty-not-insert insert-subset rangeI sup-bot-left sup-ge1 trans-FD write-def*)

from *1 2* **have** *ab*: $(\sqcap xa \in$ *range left* $\cup$ *range right* $\rightarrow$ *P*$) \sqsubseteq_{FD} ($*right*!x* $\rightarrow$ *P*$)$
for *x*
using *trans-FD* **by** *blast*
hence *3*: $($*left*?x $\rightarrow (\sqcap xa \in$ *range left* $\cup$ *range right* $\rightarrow$ *P*$)) \sqsubseteq_{FD} ($*left*?x $\rightarrow ($*right*!x $\rightarrow$ *P*$))$ **by** *simp*
have *4*: $\bigwedge X. (\sqcap xa \in$ *range left* $\cup$ *range right* $\rightarrow X) \sqsubseteq_{FD} (\sqcap xa \in$ *range left* $\rightarrow X)$ **by** *simp*
have *5*: $\bigwedge X. (\sqcap xa \in$ *range left* $\rightarrow X) \sqsubseteq_{FD} ($*left*?x $\rightarrow X)$ **by** (*simp add*: *read-def K-record-comp*)
from *3 4*[*of* $(\sqcap xa \in$ *range left* $\cup$ *range right* $\rightarrow$ *P*$)$]
5 [*of* $(\sqcap xa \in$ *range left* $\cup$ *range right* $\rightarrow$ *P*$)$]
have *6*:$(\sqcap xa \in$ *range left* $\cup$ *range right* $\rightarrow$
$(\sqcap xa \in$ *range left* $\cup$ *range right* $\rightarrow$ *P*$)) \sqsubseteq_{FD} ($*left*?x $\rightarrow ($*right*!x $\rightarrow$ *P*$))$
using *trans-FD* **by** *blast*
from $*$ $**$ **have** *7*: $($*left*?x $\rightarrow ($*right*!x $\rightarrow$ *P*$)) \sqsubseteq_{FD} ($*left*?x $\rightarrow ($*right*!x $\rightarrow$ *COPY*$))$ **by** *fastforce*
show $(\sqcap x \in$*range left* $\cup$ *range right* $\rightarrow \sqcap x \in$*range left* $\cup$ *range right* $\rightarrow$ *P*$) \sqsubseteq_{FD}$ *COPY*
by (*metis* (*mono-tags*, *lifting*) *6 7 COPY-rec trans-FD*)
qed

3.1.2 ... and abstract consequences

corollary *df-COPY*: *deadlock-free COPY*
and *lf-COPY*: *lifelock-free COPY*
apply (*meson DF-COPY DF-Univ-freeness UNIV-not-empty image-is-empty sup-eq-bot-iff*)
by (*meson CHAOS-DF-refine-FD DF-COPY DF-Univ-freeness UNIV-not-empty deadlock-free-def*
image-is-empty lifelock-free-def sup-eq-bot-iff trans-FD)

corollary *df-v2-COPY*: *deadlock-free-v2 COPY*
and *lf-v2-COPY*: *lifelock-free-v2 COPY*
and *nt-COPY*: *non-terminating COPY*
apply (*simp add*: *df-COPY deadlock-free-is-deadlock-free-v2*)
apply (*simp add*: *lf-COPY lifelock-free-is-lifelock-free-v2*)
using *lf-COPY lifelock-free-is-non-terminating* **by** *blast*

lemma *DF-SYSTEM*: *DF UNIV* $\sqsubseteq_{FD}$ *SYSTEM*
using *DF-subset*[*of* (*range left* $\cup$ *range right*) *UNIV*, *simplified*]
DF-COPY impl-refines-spec trans-FD **by** *blast*

corollary *df-SYSTEM*: *deadlock-free SYSTEM*
and *lf-SYSTEM*: *lifelock-free SYSTEM*
apply (*simp add*: *DF-SYSTEM deadlock-free-def*)
using *CHAOS-DF-refine-FD DF-SYSTEM lifelock-free-def trans-FD* **by** *blast*

corollary *df-v2-SYSTEM*: *deadlock-free-v2 SYSTEM*
and *lf-v2-SYSTEM*: *lifelock-free-v2 SYSTEM*
and *nt-SYSTEM*: *non-terminating SYSTEM*
apply (*simp add*: *df-SYSTEM deadlock-free-is-deadlock-free-v2*)
apply (*simp add*: *lf-SYSTEM lifelock-free-is-lifelock-free-v2*)
using *lf-SYSTEM lifelock-free-is-non-terminating* **by** *blast*

end

3.2 Generalized Dining Philosophers

theory *DiningPhilosophers*
imports *Process-norm*
begin

3.2.1 Preliminary lemmas for proof automation

lemma *Suc-mod*: $n > 1 \Longrightarrow i \neq Suc\ i\ mod\ n$
by (*metis One-nat-def mod-Suc mod-if mod-mod-trivial n-not-Suc-n*)

lemmas *suc-mods* = *Suc-mod Suc-mod*[*symmetric*]

lemma *l-suc*: $n > 1 \Longrightarrow \neg\ n \leq Suc\ 0$
by *simp*

lemma *minus-suc*: $n > 0 \Longrightarrow n - Suc\ 0 \neq n$
by *linarith*

lemma *numeral-4-eq-4*:*4* = *Suc* (*Suc* (*Suc* (*Suc 0*)))
by *simp*

lemma *numeral-5-eq-5*:*5* = *Suc* (*Suc* (*Suc* (*Suc* (*Suc 0*))))
by *simp*

3.2.2 The dining processes definition

locale *DiningPhilosophers* =

fixes *N*::*nat*
assumes *N-g1*[*simp*] : $N > 1$

begin

datatype *dining-event* = *picks* (*phil*:*nat*) (*fork*:*nat*)
| *putsdown* (*phil*:*nat*) (*fork*:*nat*)

definition *RPHIL*:: *nat* ⇒ *dining-event process*
where *RPHIL i* = (*μ X*. (*picks i i* → (*picks i* (*i*−*1*) → (*putsdown i* (*i*−*1*) → (*putsdown i i* → *X*)))))

definition *LPHIL0*:: *dining-event process*
where *LPHIL0* = (*μ X*. (*picks 0* (*N*−*1*) → (*picks 0 0* → (*putsdown 0 0* → (*putsdown 0* (*N*−*1*) → *X*)))))

definition *FORK* :: *nat* ⇒ *dining-event process*
where *FORK i* = (*μ X*. (*picks i i* → (*putsdown i i* → *X*))
□ (*picks* ((*i*+*1*) *mod N*) *i* → (*putsdown* ((*i*+*1*) *mod N*) *i* → *X*)))

abbreviation *foldPHILs n* ≡ *fold* (*λ i P*. *P* ||| *RPHIL i*) [*1*..< *n*] (*LPHIL0*)
abbreviation *foldFORKs n* ≡ *fold* (*λ i P*. *P* ||| *FORK i*) [*1*..< *n*] (*FORK 0*)

abbreviation *PHILs* ≡ *foldPHILs N*
abbreviation *FORKs* ≡ *foldFORKs N*

corollary *FORKs-def2*: *FORKs* = *fold* (*λ i P*. *P* ||| *FORK i*) [*0*..< *N*] *SKIP*
using *N-g1* **by** (*subst* (*2*) *upt-rec*, *auto*) (*metis* (*no-types*, *lifting*) *Inter-commute Inter-SKIP*)

corollary *N* = *3* ⟹ *PHILs* = (*LPHIL0* ||| *RPHIL 1* ||| *RPHIL 2*)
by (*subst upt-rec*, *auto simp add*:*numeral-2-eq-2*)+

definition *DINING* :: *dining-event process*
where *DINING* = (*FORKs* || *PHILs*)

Unfolding rules

lemma *RPHIL-rec*:
RPHIL i = (*picks i i* → (*picks i* (*i*−*1*) → (*putsdown i* (*i*−*1*) → (*putsdown i i* → *RPHIL i*))))
by (*simp add*:*RPHIL-def write0-def*, *subst fix-eq*, *simp*)

lemma *LPHIL0-rec*:
LPHIL0 = (*picks 0* (*N*−*1*) → (*picks 0 0* → (*putsdown 0 0* → (*putsdown 0* (*N*−*1*) → *LPHIL0*))))
by (*simp add*:*LPHIL0-def write0-def*, *subst fix-eq*, *simp*)

lemma *FORK-rec*: *FORK i* = ((*picks i i* → (*putsdown i i* → (*FORK i*)))
□ (*picks* ((*i*+*1*) *mod N*) *i* → (*putsdown* ((*i*+*1*) *mod N*) *i* → (*FORK i*))))
by (*simp add*:*FORK-def write0-def*, *subst fix-eq*, *simp*)

3.2.3 Translation into normal form

lemma *N-pos*[*simp*]: *N* > *0*
using *N-g1 neq0-conv* **by** *blast*

lemmas *N-pos-simps*[*simp*] = *suc-mods*[*OF N-g1*] *l-suc*[*OF N-g1*] *minus-suc*[*OF N-pos*]

The one-fork process

type-synonym id_{fork} = *nat*

type-synonym σ_{fork} = *nat*

definition *fork-transitions*:: $id_{fork} \Rightarrow \sigma_{fork} \Rightarrow$ *dining-event set* (Tr_f)
where Tr_f *i s* = (*if s* = *0* *then* {*picks i i*} ∪ {*picks* ((*i*+*1*) *mod N*) *i*}
else if s = *1* *then* {*putsdown i i*}
else if s = *2* *then* {*putsdown* ((*i*+*1*) *mod N*) *i*}
else {})
declare *Un-insert-right*[*simp del*] *Un-insert-left*[*simp del*]

lemma *ev-id*$_{fork}$*x*[*simp*]: *e* ∈ Tr_f *i s* ⟹ *fork e* = *i*
by (*auto simp add*:*fork-transitions-def split*:*if-splits*)

definition σ_{fork}*-update*:: $id_{fork} \Rightarrow \sigma_{fork} \Rightarrow$ *dining-event* $\Rightarrow \sigma_{fork}$ (Up_f)
where Up_f *i s e* = (*if e* = (*picks i i*) *then 1*
else if e = (*picks* ((*i*+*1*) *mod N*) *i*) *then 2*
else *0*)

definition $FORK_{norm}$:: $id_{fork} \Rightarrow \sigma_{fork} \Rightarrow$ *dining-event process*
where $FORK_{norm}$ *i* = $P_{norm}[\![Tr_f\ i\ , Up_f\ i]\!]$

lemma $FORK_{norm}$*-rec*: $FORK_{norm}$ *i* = (λ *s*. □ *e* ∈ (Tr_f *i s*) → $FORK_{norm}$ *i* (Up_f *i s e*))
using *fix-eq*[*of* Λ *X*. (λ*s*. *Mprefix* (Tr_f *i s*) (λ*e*. *X* (Up_f *i s e*)))] $FORK_{norm}$*-def*
by *simp*

lemma *FORK-refines-FORK*$_{norm}$: $FORK_{norm}$ *i 0* ⊑$_{FD}$ *FORK i*
proof(*unfold* $FORK_{norm}$*-def*,
induct rule:*fix-ind-k-skip*[**where** *k*=*2* **and** *f*=Λ *x*.(λ*s*. *Mprefix* (Tr_f *i s*) (λ*e*. *x* (Up_f *i s e*)))])
case *lower-bound*
then show *?case* **by** *simp*
next
case *admissibility*
then show *?case* **by** (*simp add*: *cont-fun monofunI*)
next
case *base-k-steps*
then show *?case* (**is** ∀ *j*<*2*. (*iterate j*·*?f*·⊥) *0* ⊑$_{FD}$ *FORK i*)
proof −
have *less-2*:⋀*j*. (*j*::*nat*) < *2* = (*j* = *0* ∨ *j* = *1*) **by** *linarith*
moreover have (*iterate 0*·*?f*·⊥) *0* ⊑$_{FD}$ *FORK i* **by** *simp*
moreover have (*iterate 1*·*?f*·⊥) *0* ⊑$_{FD}$ *FORK i*

```
    by (subst FORK-rec)
       (simp add: write0-def fork-transitions-def Mprefix-Un-distrib)
   ultimately show ?thesis by simp
 qed
next
 case (step x)
 then show ?case (is (iterate 2·?f·x) 0 ⊑FD FORK i)
   by (subst FORK-rec)
      (simp add: write0-def numeral-2-eq-2 fork-transitions-def σfork-update-def
Mprefix-Un-distrib)
qed



lemma FORKnorm-refines-FORK: FORK i ⊑FD FORKnorm i 0
proof(unfold FORK-def,
     induct rule:fix-ind-k-skip[where k=1])
 show (1::nat) ≤ 1 by simp
next
 show adm (λa. a ⊑FD FORKnorm i 0) by (simp add: monofunI)
next
 case base-k-steps
 show ?case by simp
next
 case (step x)
 then show ?case (is iterate 1·?f·x ⊑FD FORKnorm i 0)
   apply (subst FORKnorm-rec)
  apply (simp add: write0-def fork-transitions-def σfork-update-def Mprefix-Un-distrib)
   by (subst (1 2) FORKnorm-rec)
      (simp add: fork-transitions-def σfork-update-def Mprefix-Un-distrib)
qed

lemma FORKnorm-is-FORK: FORK i = FORKnorm i 0
 using FORK-refines-FORKnorm FORKnorm-refines-FORK FD-antisym by blast
```

The all-forks process in normal form

```
type-synonym σforks = nat list

definition forks-transitions:: nat ⇒ σforks ⇒ dining-event set (TrF)
 where TrF n fs = (⋃ i<n. Trf i (fs!i))

lemma forks-transitions-take: TrF n fs = TrF n (take n fs)
 by (simp add:forks-transitions-def)

definition σforks-update:: σforks ⇒ dining-event ⇒ σforks (UpF)
 where UpF fs e = (let i=(fork e) in fs[i:=(Upf i (fs!i) e)])

lemma forks-update-take: take n (UpF fs e) = UpF (take n fs) e
 unfolding σforks-update-def
```

by (*metis nat-less-le nat-neq-iff nth-take order-refl take-update-cancel take-update-swap*)

definition $FORKs_{norm}$:: *nat* ⇒ σ_{forks} ⇒ *dining-event process*
where $FORKs_{norm}$ *n* = $P_{norm}[\![Tr_F\ n\ , Up_F]\!]$

lemma $FORKs_{norm}$*-rec*: $FORKs_{norm}$ *n* = (λ *fs*. □ *e* ∈ (Tr_F *n fs*) → $FORKs_{norm}$ *n* (Up_F *fs e*))
using *fix-eq*[*of* Λ *X*. (λ*fs*. *Mprefix* (Tr_F *n fs*) (λ*e*. *X* (Up_F *fs e*)))] $FORKs_{norm}$*-def*
by *simp*

lemma $FORKs_{norm}$*-0*: $FORKs_{norm}$ *0 fs* = *STOP*
by (*subst* $FORKs_{norm}$*-rec*, *simp add*:*forks-transitions-def Mprefix-STOP*)

lemma $FORKs_{norm}$*-1-dir1*: *length fs* > *0* ⟹ $FORKs_{norm}$ *1 fs* $\sqsubseteq_{FD}$ ($FORK_{norm}$ *0* (*fs*!*0*))
proof(*unfold* $FORKs_{norm}$*-def*,
induct arbitrary:*fs rule*:*fix-ind-k*[**where** *k*=*1*
and *f*=Λ *x*. (λ*fs*. *Mprefix* (Tr_F *1 fs*) (λ*e*. *x* (Up_F *fs e*)))])
case *admissibility*
then show *?case* **by** (*simp add*: *cont-fun monofunI*)
next
case *base-k-steps*
then show *?case* **by** *simp*
next
case (*step X*)
hence (⋃ *i*<*Suc 0*. Tr_f *i* (*fs* ! *i*)) = Tr_f *0* (*fs* ! *0*) **by** *auto*
with *step* **show** *?case*
apply (*subst* $FORK_{norm}$*-rec*, *simp add*:σ_{forks}*-update-def forks-transitions-def*)

apply (*intro mono-Mprefix-FD*, *safe*)
by (*metis ev-id*$_{fork}$*x step.prems list-update-nonempty nth-list-update-eq*)
qed

lemma $FORKs_{norm}$*-1-dir2*: *length fs* > *0* ⟹ ($FORK_{norm}$ *0* (*fs*!*0*)) $\sqsubseteq_{FD}$ $FORKs_{norm}$ *1 fs*
proof(*unfold* $FORK_{norm}$*-def*,
induct arbitrary:*fs rule*:*fix-ind-k*[**where** *k*=*1*
and *f*=Λ *x*. (λ*s*. *Mprefix* (Tr_f *0 s*) (λ*e*. *x* (Up_f *0 s e*)))])
case *admissibility*
then show *?case* **by** (*simp add*: *cont-fun monofunI*)
next
case *base-k-steps*
then show *?case* **by** *simp*
next
case (*step X*)

```
  have (⋃ i<Suc 0. Tr_f i (fs ! i)) = Tr_f 0 (fs ! 0) by auto
  with step show ?case
   apply (subst FORKs_norm-rec, simp add: σ_forks-update-def forks-transitions-def)

    apply (intro mono-Mprefix-FD, safe)
    by (metis ev-id_fork x step.prems list-update-nonempty nth-list-update-eq)
qed

lemma FORKs_norm-1: length fs > 0 ⟹ (FORK_norm 0 (fs!0)) = FORKs_norm 1
fs
  using FORKs_norm-1-dir1 FORKs_norm-1-dir2 FD-antisym by blast


lemma FORKs_norm-unfold:
0 < n ⟹ length fs = Suc n ⟹
                                  FORKs_norm (Suc n) fs = (FORKs_norm n (butlast
fs)|||(FORK_norm n (fs!n)))
proof(rule FD-antisym)
  show 0 < n ⟹ length fs = Suc n ⟹
                                FORKs_norm (Suc n) fs ⊑_FD (FORKs_norm n (butlast
fs)|||FORK_norm n (fs!n))
  proof(subst FORKs_norm-def,
        induct arbitrary:fs
              rule:fix-ind-k[where k=1
                          and f=Λ x. (λfs. Mprefix (Tr_F (Suc n) fs) (λe. x (Up_F fs
e)))])
    case admissibility
    then show ?case by (simp add: cont-fun monofunI)
  next
    case base-k-steps
    then show ?case by simp
  next
    case (step X)
    have indep:∀ s_1 s_2. Tr_F n s_1 ∩ Tr_f n s_2 = {}
      by (auto simp add:forks-transitions-def fork-transitions-def)
    from step show ?case
      apply (auto simp add:indep dnorm-inter FORKs_norm-def FORK_norm-def)
      apply (subst fix-eq, simp add:forks-transitions-def Un-commute lessThan-Suc
nth-butlast)
    proof(rule mono-Mprefix-FD, safe, goal-cases)
      case (1 e)
      from 1(4) have a:fork e = n
        by (auto simp add:fork-transitions-def split:if-splits)
      show ?case
        using 1(1)[rule-format, of (Up_F fs e)]
        apply (simp add: 1 a butlast-list-update σ_forks-update-def)
        by (metis 1(4) ev-id_fork x lessThan-iff less-not-refl)
    next
      case (2 e m)
```

```
      hence a:e ∉ Tr_f n (fs ! n)
        using ev-id_fork x by fastforce
      hence c:Up_F fs e ! n = fs ! n
        by (metis 2(4) ev-id_fork x σ_forks-update-def nth-list-update-neq)
      have d:Up_F (butlast fs) e = butlast (Up_F fs e)
        apply(simp add:σ_forks-update-def)
       by (metis butlast-conv-take σ_forks-update-def forks-update-take length-list-update)
      from 2 a show ?case
        using 2(1)[rule-format, of (Up_F fs e)] c d σ_forks-update-def by auto
    qed
  qed
next
  have indep:∀ s_1 s_2. Tr_F n s_1 ∩ Tr_f n s_2 = {}
    by (auto simp add: forks-transitions-def fork-transitions-def)
  show 0 < n ⟹ length fs = Suc n ⟹
                          (FORKs_norm n (butlast fs)|||FORK_norm n (fs!n)) ⊑_FD
FORKs_norm (Suc n) fs
   apply (subst FORKs_norm-def, auto simp add:indep dnorm-inter FORK_norm-def)
  proof(rule fix-ind[where
        P=λa. 0 < n ⟶ (∀ x. length x = Suc n ⟶ a (butlast x, x ! n) ⊑_FD
FORKs_norm (Suc n) x),
        rule-format], simp-all, goal-cases)
    case base:1
    then show ?case by (simp add: cont-fun monofunI)
  next
    case step:(2 X fs)
    then show ?case
      apply (unfold FORKs_norm-def, subst fix-eq, simp add:forks-transitions-def
                                                  Un-commute lessThan-Suc nth-butlast)
    proof(rule mono-Mprefix-FD, safe, goal-cases)
      case (1 e)
      from 1(6) have a:fork e = n
        by (auto simp add:fork-transitions-def split:if-splits)
      show ?case
        using 1(1)[rule-format, of (Up_F fs e)]
        apply (simp add: 1 a butlast-list-update σ_forks-update-def)
        using a ev-id_fork x by blast
    next
      case (2 e m)
      have a:Up_F (butlast fs) e = butlast (Up_F fs e)
        apply(simp add:σ_forks-update-def)
       by (metis butlast-conv-take σ_forks-update-def forks-update-take length-list-update)
      from 2 show ?case
        using 2(1)[rule-format, of (Up_F fs e)] a σ_forks-update-def by auto
    qed
  qed
qed

lemma ft: 0 < n ⟹ FORKs_norm n (replicate n 0) = foldFORKs n
```

proof (*induct n*, *simp*)
case (*Suc n*)
then show *?case*
apply (*auto simp add*: $FORKs_{norm}$*-unfold* $FORK_{norm}$*-is-FORK*)
apply (*metis Suc-le-D butlast-snoc replicate-Suc replicate-append-same*)
by (*metis* $FORKs_{norm}$*-1 One-nat-def leI length-replicate less-Suc0 nth-replicate replicate-Suc*)
qed

corollary $FORKs\text{-}is\text{-}FORKs_{norm}$: $FORKs_{norm}\ N\ (replicate\ N\ 0) = FORKs$
using *ft N-pos* **by** *simp*

The one-philosopher process in normal form:

type-synonym *phil-id* = *nat*
type-synonym *phil-state* = *nat*

definition *rphil-transitions*:: *phil-id* ⇒ *phil-state* ⇒ *dining-event set* (Tr_{rp})
where $Tr_{rp}\ i\ s = (\ if\ \ s = 0\ \ then\ \{picks\ i\ i\}$
else if $s = 1$ *then* $\{picks\ i\ (i-1)\}$
else if $s = 2$ *then* $\{putsdown\ i\ (i-1)\}$
else if $s = 3$ *then* $\{putsdown\ i\ i\}$
else $\{\})$

definition *lphil0-transitions*:: *phil-state* ⇒ *dining-event set* (Tr_{lp})
where $Tr_{lp}\ s = (\ if\ s = 0\ \ then\ \{picks\ 0\ (N-1)\}$
else if $s = 1$ *then* $\{picks\ 0\ 0\}$
else if $s = 2$ *then* $\{putsdown\ 0\ 0\}$
else if $s = 3$ *then* $\{putsdown\ 0\ (N-1)\}$
else $\{\})$

corollary *rphil-phil*: $e \in Tr_{rp}\ i\ s \Longrightarrow phil\ e = i$
and *lphil0-phil*: $e \in Tr_{lp}\ s \Longrightarrow phil\ e = 0$
by (*simp-all add*:*rphil-transitions-def lphil0-transitions-def split*:*if-splits*)

definition *rphil-state-update*:: $id_{fork} \Rightarrow \sigma_{fork} \Rightarrow$ *dining-event* $\Rightarrow \sigma_{fork}$ (Up_{rp})
where $Up_{rp}\ i\ s\ e = (\ if\ e = (picks\ i\ i)\ \ then\ 1$
else if $e = (picks\ i\ (i-1))$ *then 2*
else if $e = (putsdown\ i\ (i-1))$ *then 3*
else *0*)

definition *lphil0-state-update*:: $\sigma_{fork} \Rightarrow$ *dining-event* $\Rightarrow \sigma_{fork}$ (Up_{lp})
where $Up_{lp}\ s\ e = (\ if\ e = (picks\ 0\ (N-1))\ \ then\ 1$
else if $e = (picks\ 0\ 0)$ *then 2*
else if $e = (putsdown\ 0\ 0)$ *then 3*
else *0*)

definition $RPHIL_{norm}$:: $id_{fork} \Rightarrow \sigma_{fork} \Rightarrow$ *dining-event process*
where $RPHIL_{norm}\ i = P_{norm}[\![Tr_{rp}\ i, Up_{rp}\ i]\!]$

definition $LPHIL0_{norm}$:: $\sigma_{fork} \Rightarrow$ *dining-event process*
where $LPHIL0_{norm} = P_{norm}[\![Tr_{lp}, Up_{lp}]\!]$

lemma $RPHIL_{norm}$*-rec*: $RPHIL_{norm}\ i = (\lambda\ s.\ \Box\ e \in (Tr_{rp}\ i\ s) \rightarrow RPHIL_{norm}\ i\ (Up_{rp}\ i\ s\ e))$
using *fix-eq*[*of* $\Lambda\ X.\ (\lambda s.\ Mprefix\ (Tr_{rp}\ i\ s)\ (\lambda e.\ X\ (Up_{rp}\ i\ s\ e)))$] $RPHIL_{norm}$*-def*
by *simp*

lemma $LPHIL0_{norm}$*-rec*: $LPHIL0_{norm} = (\lambda\ s.\ \Box\ e \in (Tr_{lp}\ s) \rightarrow LPHIL0_{norm}\ (Up_{lp}\ s\ e))$
using *fix-eq*[*of* $\Lambda\ X.\ (\lambda s.\ Mprefix\ (Tr_{lp}\ s)\ (\lambda e.\ X\ (Up_{lp}\ s\ e)))$] $LPHIL0_{norm}$*-def*
by *simp*

lemma $RPHIL$*-refines-*$RPHIL_{norm}$:
assumes *i-pos*: $i > 0$
shows $RPHIL_{norm}\ i\ 0 \sqsubseteq_{FD} RPHIL\ i$
proof(*unfold* $RPHIL_{norm}$*-def*,
induct rule:*fix-ind-k-skip*[**where** k=*4* **and** f=$\Lambda\ x.\ (\lambda s.\ Mprefix\ (Tr_{rp}\ i\ s)\ (\lambda e.\ x\ (Up_{rp}\ i\ s\ e)))$])
case *lower-bound*
then show *?case* (**is** *1* $\leq$ *4*) **by** *simp*
next
case *admissibility*
then show *?case* (**is** *adm* $(\lambda a.\ a\ 0 \sqsubseteq_{FD} RPHIL\ i)$)
by (*simp add*: *cont-fun monofunI*)
next
case *base-k-steps*
then show *?case* (**is** $\forall j{<}4.$ (*iterate j*·*?f*·$\bot$) *0* $\sqsubseteq_{FD}$ *RPHIL i*)
proof −
have *less-2*:$\bigwedge j.$ (*j*::*nat*) < *4* = (*j* = *0* ∨ *j* = *1* ∨ *j* = *2* ∨ *j* = *3*) **by** *linarith*
moreover have (*iterate 0*·*?f*·$\bot$) *0* $\sqsubseteq_{FD}$ *RPHIL i* **by** *simp*
moreover have (*iterate 1*·*?f*·$\bot$) *0* $\sqsubseteq_{FD}$ *RPHIL i*
by (*subst RPHIL-rec*) (*simp add*: *write0-def rphil-transitions-def*)
moreover have (*iterate 2*·*?f*·$\bot$) *0* $\sqsubseteq_{FD}$ *RPHIL i*
by (*subst RPHIL-rec*)
(*simp add*: *numeral-2-eq-2 write0-def rphil-transitions-def rphil-state-update-def*)
moreover have (*iterate 3*·*?f*·$\bot$) *0* $\sqsubseteq_{FD}$ *RPHIL i*
by (*subst RPHIL-rec*) (*simp add*: *numeral-3-eq-3 write0-def rphil-transitions-def rphil-state-update-def minus-suc*[*OF i-pos*])
ultimately show *?thesis* **by** *simp*
qed
next
case (*step x*)
then show *?case* (**is** (*iterate 4*·*?f*·*x*) *0* $\sqsubseteq_{FD}$ *RPHIL i*)
apply (*subst RPHIL-rec*)
apply (*simp add*: *write0-def numeral-4-eq-4 rphil-transitions-def rphil-state-update-def*)
apply (*rule mono-Mprefix-FD*, *auto simp*:*minus-suc*[*OF i-pos*])+

using *minus-suc*[*OF i-pos*] **by** *auto*
qed

lemma *LPHIL0-refines-LPHIL0*$_{norm}$: *LPHIL0*$_{norm}$ *0* $\sqsubseteq_{FD}$ *LPHIL0*
proof(*unfold LPHIL0*$_{norm}$*-def*,
induct rule:*fix-ind-k-skip*[**where** *k*=*4* **and** *f*=Λ *x*. (λ*s*. *Mprefix* (*Tr*$_{lp}$ *s*) (λ*e*. *x* (*Up*$_{lp}$ *s e*)))])
show (*1*::*nat*) $\leq$ *4* **by** *simp*
next
show *adm* (λ*a*. *a 0* $\sqsubseteq_{FD}$ *LPHIL0*) **by** (*simp add*: *cont-fun monofunI*)
next
case *base-k-steps*
show *?case* (**is** $\forall$ *j*<*4*. (*iterate j*·*?f*·$\bot$) *0* $\sqsubseteq_{FD}$ *LPHIL0*)
proof −
have *less-2*:$\bigwedge$*j*. (*j*::*nat*) < *4* = (*j* = *0* $\vee$ *j* = *1* $\vee$ *j* = *2* $\vee$ *j* = *3*) **by** *linarith*
moreover have (*iterate 0*·*?f*·$\bot$) *0* $\sqsubseteq_{FD}$ *LPHIL0* **by** *simp*
moreover have (*iterate 1*·*?f*·$\bot$) *0* $\sqsubseteq_{FD}$ *LPHIL0*
by (*subst LPHIL0-rec*) (*simp add*: *write0-def lphil0-transitions-def*)
moreover have (*iterate 2*·*?f*·$\bot$) *0* $\sqsubseteq_{FD}$ *LPHIL0*
by (*subst LPHIL0-rec*) (*simp add*: *numeral-2-eq-2 write0-def lphil0-transitions-def*
lphil0-state-update-def)
moreover have (*iterate 3*·*?f*·$\bot$) *0* $\sqsubseteq_{FD}$ *LPHIL0*
by (*subst LPHIL0-rec*) (*simp add*: *numeral-3-eq-3 write0-def lphil0-transitions-def*
lphil0-state-update-def)
ultimately show *?thesis* **by** *simp*
qed
next
case (*step x*)
then show *?case* (**is** (*iterate 4*·*?f*·*x*) *0* $\sqsubseteq_{FD}$ *LPHIL0*)
by (*subst LPHIL0-rec*) (*simp add*: *write0-def numeral-4-eq-4 lphil0-transitions-def*
lphil0-state-update-def)
qed

lemma *RPHIL*$_{norm}$*-refines-RPHIL*:
assumes *i-pos*: *i* > *0*
shows *RPHIL i* $\sqsubseteq_{FD}$ *RPHIL*$_{norm}$ *i 0*
proof(*unfold RPHIL-def*, *induct rule*:*fix-ind-k-skip*[**where** *k*=*1*])
case *lower-bound*
then show *?case* (**is** *1* $\leq$ *1*) **by** *simp*
next
case *admissibility*
then show *?case* **by** (*simp add*: *monofunI*)
next
case *base-k-steps*
then show *?case* **by** *simp*

next
  case (*step x*)**then show** *?case*
   apply (*subst* $RPHIL_{norm}$*-rec*, *simp add*: *write0-def rphil-transitions-def rphil-state-update-def*)

    apply (*rule mono-Mprefix-FD*, *simp*)
   apply (*subst* $RPHIL_{norm}$*-rec*, *simp add*: *write0-def rphil-transitions-def rphil-state-update-def*)
    apply (*rule mono-Mprefix-FD*, *simp add*:*minus-suc*[*OF i-pos*])
   apply (*subst* $RPHIL_{norm}$*-rec*, *simp add*: *write0-def rphil-transitions-def rphil-state-update-def*)
    apply (*rule mono-Mprefix-FD*, *simp add*:*minus-suc*[*OF i-pos*])
   apply (*subst* $RPHIL_{norm}$*-rec*, *simp add*: *write0-def rphil-transitions-def rphil-state-update-def*)
    apply (*rule mono-Mprefix-FD*, *simp add*:*minus-suc*[*OF i-pos*])
    using *minus-suc*[*OF i-pos*] **by** *auto*
qed

lemma $LPHIL0_{norm}$*-refines-LPHIL0*: *LPHIL0* $\sqsubseteq_{FD}$ $LPHIL0_{norm}$ *0*
proof(*unfold LPHIL0-def*,
    induct rule:*fix-ind-k-skip*[**where** *k=1*])
  show (*1*::*nat*) $\leq$ *1* **by** *simp*
next
  show *adm* ($\lambda a.\ a \sqsubseteq_{FD}$ $LPHIL0_{norm}$ *0*) **by** (*simp add*: *monofunI*)
next
  case *base-k-steps*
  show *?case* **by** *simp*
next
  case (*step x*)
  then show *?case* (**is** *iterate 1·?f·x* $\sqsubseteq_{FD}$ $LPHIL0_{norm}$ *0*)
      apply (*subst* $LPHIL0_{norm}$*-rec*, *simp add*: *write0-def lphil0-transitions-def lphil0-state-update-def*)
    apply (*rule mono-Mprefix-FD*, *simp*)
      apply (*subst* $LPHIL0_{norm}$*-rec*, *simp add*: *write0-def lphil0-transitions-def lphil0-state-update-def*)
    apply (*rule mono-Mprefix-FD*, *simp*)
      apply (*subst* $LPHIL0_{norm}$*-rec*, *simp add*: *write0-def lphil0-transitions-def lphil0-state-update-def*)
    apply (*rule mono-Mprefix-FD*, *simp*)
        by (*subst* $LPHIL0_{norm}$*-rec*, *simp add*: *write0-def lphil0-transitions-def lphil0-state-update-def*)
qed

lemma $RPHIL_{norm}$*-is-RPHIL*: $i > 0 \Longrightarrow$ *RPHIL i* = $RPHIL_{norm}$ *i 0*
  using *RPHIL-refines-*$RPHIL_{norm}$ $RPHIL_{norm}$*-refines-RPHIL FD-antisym* **by** *blast*

lemma $LPHIL0_{norm}$*-is-LPHIL0*: *LPHIL0* = $LPHIL0_{norm}$ *0*
  using *LPHIL0-refines-*$LPHIL0_{norm}$ $LPHIL0_{norm}$*-refines-LPHIL0 FD-antisym* **by** *blast*

3.2.4 The normal form for the global philosopher network

type-synonym $\sigma_{phils} = nat\ list$

definition *phils-transitions*:: $nat \Rightarrow \sigma_{phils} \Rightarrow$ *dining-event set* (Tr_P)
where $Tr_P\ n\ ps = Tr_{lp}\ (ps!0) \cup (\bigcup i{\in}\{1..<n\}.\ Tr_{rp}\ i\ (ps!i))$

corollary *phils-phil*: $0 < n \Longrightarrow e \in Tr_P\ n\ s \Longrightarrow phil\ e < n$
by (*auto simp add:phils-transitions-def lphil0-phil rphil-phil*)

lemma *phils-transitions-take*: $0 < n \Longrightarrow Tr_P\ n\ ps = Tr_P\ n\ (take\ n\ ps)$
by (*auto simp add:phils-transitions-def*)

definition σ_{phils}*-update*:: $\sigma_{phils} \Rightarrow$ *dining-event* $\Rightarrow \sigma_{phils}$ (Up_P)
where $Up_P\ ps\ e = (let\ i{=}(phil\ e)\ in\ if\ i = 0\ then\ ps[i{:=}(Up_{lp}\ (ps!i)\ e)]$
$else\quad ps[i{:=}(Up_{rp}\ i\ (ps!i)\ e)])$

lemma *phils-update-take*: $take\ n\ (Up_P\ ps\ e) = Up_P\ (take\ n\ ps)\ e$
by (*cases e*) (*simp-all add*: σ_{phils}*-update-def lphil0-state-update-def*
rphil-state-update-def take-update-swap)

definition $PHILs_{norm}$:: $nat \Rightarrow \sigma_{phils} \Rightarrow$ *dining-event process*
where $PHILs_{norm}\ n = P_{norm}[\![Tr_P\ n, Up_P]\!]$

lemma $PHILs_{norm}$*-rec*: $PHILs_{norm}\ n = (\lambda\ ps.\ \Box\ e \in (Tr_P\ n\ ps) \rightarrow PHILs_{norm}\ n\ (Up_P\ ps\ e))$
using *fix-eq*[*of* $\Lambda\ X.\ (\lambda ps.\ Mprefix\ (Tr_P\ n\ ps)\ (\lambda e.\ X\ (Up_P\ ps\ e)))$] $PHILs_{norm}$*-def*
by *simp*

lemma $PHILs_{norm}$*-1-dir1*: $length\ ps > 0 \Longrightarrow PHILs_{norm}\ 1\ ps \sqsubseteq_{FD} (LPHIL0_{norm}\ (ps!0))$
proof(*unfold* $PHILs_{norm}$*-def*,
induct arbitrary:ps
rule:fix-ind-k[**where** *k=1*
and $f{=}\Lambda\ x.\ (\lambda ps.\ Mprefix\ (Tr_P\ 1\ ps)\ (\lambda e.\ x\ (Up_P\ ps\ e)))$])
case *admissibility*
then show *?case* **by** (*simp add*: *cont-fun monofunI*)
next
case *base-k-steps*
then show *?case* **by** *simp*
next
case (*step X*)
then show *?case*
apply (*subst* $LPHIL0_{norm}$*-rec*, *simp add:*σ_{phils}*-update-def phils-transitions-def*)

proof (*intro mono-Mprefix-FD*, *safe*, *goal-cases*)
case (*1 e*)
with *1*(*2*) **show** *?case*
using *1*(*1*)[*rule-format*, *of ps*[*0* := Up_{lp} (*ps* ! *0*) *e*]]
by (*simp add:lphil0-transitions-def split:if-splits*)

```
    qed
qed

lemma PHILs_norm-1-dir2: length ps > 0 ⟹ (LPHIL0_norm (ps!0)) ⊑_FD PHILs_norm
1 ps
proof(unfold LPHIL0_norm-def,
      induct arbitrary:ps rule:fix-ind-k[where k=1
                                  and f=Λ x. (λs. Mprefix (Tr_lp s) (λe. x (Up_lp s
e)))])
  case admissibility
  then show ?case by (simp add: cont-fun monofunI)
next
  case base-k-steps
  then show ?case by simp
next
  case (step X)
  then show ?case
   apply (subst PHILs_norm-rec, simp add:σ_phils-update-def phils-transitions-def)
   proof (intro mono-Mprefix-FD, safe, goal-cases)
     case (1 e)
     with 1(2) show ?case
       using 1(1)[rule-format, of ps[0 := Up_lp (ps ! 0) e]]
       by (simp add:lphil0-transitions-def split:if-splits)
   qed
qed

lemma PHILs_norm-1: length ps > 0 ⟹ PHILs_norm 1 ps = (LPHIL0_norm (ps!0))
  using PHILs_norm-1-dir1 PHILs_norm-1-dir2 FD-antisym by blast

lemma PHILs_norm-unfold:
  assumes n-pos:0 < n
  shows length ps = Suc n ⟹
                                  PHILs_norm (Suc n) ps = (PHILs_norm n (butlast
ps)|||(RPHIL_norm n (ps!n)))
proof(rule FD-antisym)
  show length ps = Suc n ⟹ PHILs_norm (Suc n) ps ⊑_FD (PHILs_norm n (butlast
ps)|||RPHIL_norm n (ps!n))
  proof(subst PHILs_norm-def,
        induct arbitrary:ps
              rule:fix-ind-k[where k=1
                          and f=Λ x. (λps. Mprefix (Tr_P (Suc n) ps) (λe. x (Up_P
ps e)))])
    case admissibility
    then show ?case by (simp add: cont-fun monofunI)
  next
    case base-k-steps
    then show ?case by simp
  next
    case (step X)
```

```
    have indep:∀ s1 s2. Tr_P n s1 ∩ Tr_rp n s2 = {}
      using phils-phil rphil-phil n-pos by blast
    from step have tra:(Tr_P (Suc n) ps) =(Tr_P n (butlast ps) ∪ Tr_rp n (ps ! n))
      by (auto simp add:n-pos phils-transitions-def nth-butlast Suc-leI
                    atLeastLessThanSuc Un-commute Un-assoc)
    from step show ?case
      apply (auto simp add:indep dnorm-inter PHILs_norm-def RPHIL_norm-def)
      apply (subst fix-eq, auto simp add:tra)
    proof(rule mono-Mprefix-FD, safe, goal-cases)
      case (1 e)
      hence c:Up_P ps e ! n = ps ! n
        using 1(3) phils-phil σ_phils-update-def step n-pos
        by (cases phil e, auto) (metis exists-least-iff nth-list-update-neq)
      have d:Up_P (butlast ps) e = butlast (Up_P ps e)
        by (cases phil e, auto simp add:σ_phils-update-def butlast-list-update
                                        lphil0-state-update-def rphil-state-update-def)
      have e:length (Up_P ps e) = Suc n
        by (metis (full-types) step(2) length-list-update σ_phils-update-def)
      from 1 show ?case
        using 1(1)[rule-format, of (Up_P ps e)] c d e by auto
    next
      case (2 e)
      have e:length (Up_P ps e) = Suc n
        by (metis (full-types) step(2) length-list-update σ_phils-update-def)
      from 2 show ?case
        using 2(1)[rule-format, of (Up_P ps e), OF e] n-pos
        apply(auto simp add: butlast-list-update rphil-phil σ_phils-update-def)
        by (meson disjoint-iff-not-equal indep)
    qed
  qed
next
  have indep:∀ s1 s2. Tr_P n s1 ∩ Tr_rp n s2 = {}
    using phils-phil rphil-phil using n-pos by blast

  show length ps = Suc n ⟹ (PHILs_norm n (butlast ps)|||RPHIL_norm n (ps!n))
⊑_FD PHILs_norm (Suc n) ps
   apply (subst PHILs_norm-def, auto simp add:indep dnorm-inter RPHIL_norm-def)
  proof(rule fix-ind[where
        P=λa. ∀ x. length x = Suc n ⟶ a (butlast x, x ! n) ⊑_FD PHILs_norm (Suc
n) x, rule-format],
        simp-all, goal-cases base step)
    case base
    then show ?case by (simp add: cont-fun monofunI)
  next
    case (step X ps)
    hence tra:(Tr_P (Suc n) ps) =(Tr_P n (butlast ps) ∪ Tr_rp n (ps ! n))
      by (auto simp add:n-pos phils-transitions-def nth-butlast
                    Suc-leI atLeastLessThanSuc Un-commute Un-assoc)
    from step show ?case
```

apply (*auto simp add*:*indep dnorm-inter* $PHILs_{norm}$*-def* $RPHIL_{norm}$*-def*)
apply (*subst fix-eq*, *auto simp add*:*tra*)
proof(*rule mono-Mprefix-FD*, *safe*, *goal-cases*)
case (*1 e*)
hence *c*:Up_P *ps e* ! *n* = *ps* ! *n*
using *1*(*3*) *phils-phil* σ_{phils}*-update-def step n-pos*
by (*cases phil e*, *auto*) (*metis exists-least-iff nth-list-update-neq*)
have *d*:Up_P (*butlast ps*) *e* = *butlast* (Up_P *ps e*)
by (*cases phil e*, *auto simp add*:σ_{phils}*-update-def butlast-list-update lphil0-state-update-def rphil-state-update-def*)
have *e*:*length* (Up_P *ps e*) = *Suc n*
by (*metis* (*full-types*) *step*(*3*) *length-list-update* σ_{phils}*-update-def*)
from *1* **show** *?case*
using *1*(*2*)[*rule-format*, *of* (Up_P *ps e*), *OF e*] *c d* **by** *auto*
next
case (*2 e*)
have *e*:*length* (Up_P *ps e*) = *Suc n*
by (*metis* (*full-types*) *2*(*3*) *length-list-update* σ_{phils}*-update-def*)
from *2* **show** *?case*
using *2*(*2*)[*rule-format*, *of* (Up_P *ps e*), *OF e*] *n-pos*
apply(*auto simp add*: *butlast-list-update rphil-phil* σ_{phils}*-update-def*)
by (*meson disjoint-iff-not-equal indep*)
qed
qed
qed

lemma *pt*: *0* < *n* ⟹ $PHILs_{norm}$ *n* (*replicate n 0*) = *foldPHILs n*
proof (*induct n*, *simp*)
case (*Suc n*)
then show *?case*
apply (*auto simp add*: $PHILs_{norm}$*-unfold* $LPHIL0_{norm}$*-is-LPHIL0*)
apply (*metis Suc-le-eq butlast.simps*(*2*) *butlast-snoc* $RPHIL_{norm}$*-is-RPHIL nat-neq-iff replicate-append-same replicate-empty*)
by (*metis One-nat-def leI length-replicate less-Suc0* $PHILs_{norm}$*-1 nth-Cons-0 replicate-Suc*)
qed

corollary $PHILs\text{-}is\text{-}PHILs_{norm}$: $PHILs_{norm}$ *N* (*replicate N 0*) = *PHILs*
using *pt N-pos* **by** *simp*

3.2.5 The complete process system under normal form

definition *dining-transitions*:: *nat* ⇒ σ_{phils} × σ_{forks} ⇒ *dining-event set* (Tr_D)
where Tr_D *n* = (λ(*ps*,*fs*). (Tr_P *n ps*) ∩ (Tr_F *n fs*))

definition *dining-state-update*::
σ_{phils} × σ_{forks} ⇒ *dining-event* ⇒ σ_{phils} × σ_{forks} (Up_D)
where Up_D = (λ(*ps*,*fs*) *e*. (Up_P *ps e*, Up_F *fs e*))

definition $DINING_{norm}$:: *nat* ⇒ $\sigma_{phils} \times \sigma_{forks}$ ⇒ *dining-event process*
where $DINING_{norm}\ n = P_{norm}[\![Tr_D\ n,\ Up_D]\!]$

lemma *ltsDining-rec*: $DINING_{norm}\ n = (\lambda\ s.\ \square\ e \in (Tr_D\ n\ s) \rightarrow DINING_{norm}\ n\ (Up_D\ s\ e))$
using *fix-eq*[*of* Λ *X*. (λ*s*. *Mprefix* (*Tr*$_D$ *n s*) (λ*e*. *X* (*Up*$_D$ *s e*)))] $DINING_{norm}$-*def*
by *simp*

lemma *DINING-is-DINING*$_{norm}$: *DINING* = $DINING_{norm}$ *N* (*replicate N 0*, *replicate N 0*)
proof −
have $DINING_{norm}$ *N* (*replicate N 0*, *replicate N 0*) =
($PHILs_{norm}$ *N* (*replicate N 0*) || $FORKs_{norm}$ *N* (*replicate N 0*))
unfolding $DINING_{norm}$-*def* $PHILs_{norm}$-*def* $FORKs_{norm}$-*def dining-transitions-def*
dining-state-update-def dnorm-par **by** *simp*
thus *?thesis*
using *PHILs-is-PHILs*$_{norm}$ *FORKs-is-FORKs*$_{norm}$ *DINING-def*
by (*simp add*: *Par-commute*)
qed

3.2.6 And finally: Philosophers may dine ! Always !

corollary *lphil-states*:Up_{lp} *r e* = *0* ∨ Up_{lp} *r e* = *1* ∨ Up_{lp} *r e* = *2* ∨ Up_{lp} *r e* = *3*
and *rphil-states*:Up_{rp} *i r e* = *0* ∨ Up_{rp} *i r e* = *1* ∨ Up_{rp} *i r e* = *2* ∨ Up_{rp} *i r e* = *3*
unfolding *lphil0-state-update-def rphil-state-update-def* **by** *auto*

lemma *dining-events*:
e ∈ Tr_D *N s* ⟹
(∃ *i*∈{*1*..<*N*}. *e* = *picks i i* ∨ *e* = *picks i* (*i*−*1*) ∨ *e* = *putsdown i i* ∨ *e* = *putsdown i* (*i*−*1*))
∨ (*e* = *picks 0 0* ∨ *e* = *picks 0* (*N*−*1*) ∨ *e* = *putsdown 0 0* ∨ *e* = *putsdown 0* (*N*−*1*))
by (*auto simp add*:*dining-transitions-def phils-transitions-def rphil-transitions-def*
lphil0-transitions-def split:*prod.splits if-splits*)

definition *inv-dining ps fs* ≡
(∀ *i*. *Suc i* < *N* ⟶ ((*fs*![*Suc i*] = *1*) ⟷ *ps*![*Suc i*] ≠ *0*)) ∧ (*fs*![*N*−*1*] = *2* ⟷ *ps*![*0*] ≠ *0*)
∧ (∀ *i* < *N* − *1*. *fs*![*i*] = *2* ⟷ *ps*![*Suc i*] = *2*) ∧ (*fs*![*0*] = *1* ⟷ *ps*![*0*] = *2*)
∧ (∀ *i* < *N*. *fs*![*i*] = *0* ∨ *fs*![*i*] = *1* ∨ *fs*![*i*] = *2*)
∧ (∀ *i* < *N*. *ps*![*i*] = *0* ∨ *ps*![*i*] = *1* ∨ *ps*![*i*] = *2* ∨ *ps*![*i*] = *3*)
∧ *length fs* = *N* ∧ *length ps* = *N*

lemma *inv-DINING*: $s \in \mathfrak{R}\ (Tr_D\ N)\ Up_D\ (replicate\ N\ 0,\ replicate\ N\ 0) \Longrightarrow$
inv-dining (*fst s*) (*snd s*)
proof(*induct rule*:$\mathfrak{R}$.*induct*)
case *rbase*
show *?case*
by (*simp add*: *inv-dining-def*)
next
case (*rstep s e*)
from *rstep*(*2*,*3*) **show** *?case*
apply(*auto simp add*:*dining-transitions-def phils-transitions-def forks-transitions-def*
lphil0-transitions-def rphil-transitions-def fork-transitions-def
lphil0-state-update-def rphil-state-update-def σ_{fork}*-update-def*
dining-state-update-def σ_{phils}*-update-def* σ_{forks}*-update-def*
split:*if-splits prod.split*)
unfolding *inv-dining-def*
proof(*goal-cases*)
case (*1 ps fs*)
then show *?case*
by (*simp add*:*nth-list-update*) *force*
next
case (*2 ps fs*)
then show *?case*
by (*auto simp add*:*nth-list-update*)
next
case (*3 ps fs*)
then show *?case*
using *N-pos-simps*(*3*) **by** *force*
next
case (*4 ps fs*)
then show *?case*
by (*simp add*:*nth-list-update*) *force*
next
case (*5 ps fs*)
then show *?case*
using *N-g1* **by** *linarith*
next
case (*6 ps fs*)
then show *?case*
by (*auto simp add*:*nth-list-update*)
next
case (*7 ps fs i*)
then show *?case*
apply (*simp add*:*nth-list-update*, *intro impI conjI*, *simp-all*)
by *auto*[*1*] (*metis N-pos Suc-pred less-antisym*, *metis zero-neq-numeral*)
next
case (*8 ps fs i*)
then show *?case*
apply (*simp add*:*nth-list-update*, *intro impI conjI allI*, *simp-all*)
by (*metis 8*(*1*) *zero-neq-one*)+

```
  next
    case (9 ps fs i)
    then show ?case
      apply (simp add:nth-list-update, intro impI conjI allI, simp-all)
      by (metis N-pos Suc-pred less-antisym) (metis n-not-Suc-n numeral-2-eq-2)
  next
    case (10 ps fs i)
    then show ?case
      apply (simp add:nth-list-update, intro impI conjI allI, simp-all)
      by (metis 10(1) 10(5) One-nat-def n-not-Suc-n numeral-2-eq-2)+
  qed
qed

lemma inv-implies-DF:inv-dining ps fs ⟹ Tr_D N (ps, fs) ≠ {}
  unfolding inv-dining-def
  apply(simp add:dining-transitions-def phils-transitions-def forks-transitions-def
                 lphil0-transitions-def
             split: if-splits prod.splits)
proof(elim conjE, intro conjI impI, goal-cases)
  case 1
  hence putsdown 0 (N − Suc 0) ∈ (⋃i<N. Tr_f i (fs ! i))
     by (auto simp add:fork-transitions-def)
  then show ?case
    by blast
next
  case 2
  hence putsdown 0 0 ∈ (⋃i<N. Tr_f i (fs ! i))
     by (auto simp add:fork-transitions-def)
  then show ?case
    by (simp add:fork-transitions-def) force
next
  case 3
  hence a:fs!0 = 0 ⟹ picks 0 0 ∈ (⋃i<N. Tr_f i (fs ! i))
     by (auto simp add:fork-transitions-def)
  from 3 have b1:ps!1 = 2 ⟹ putsdown 1 0 ∈ (⋃x∈{Suc 0..<N}. Tr_rp x (ps !
x))
    using N-g1 by (auto simp add:rphil-transitions-def)
  from 3 have b2:fs!0 = 2 ⟹ putsdown 1 0 ∈ Tr_f 0 (fs ! 0)
    using N-g1 by (auto simp add:fork-transitions-def) fastforce
  from 3 have c:fs!0 ≠ 0 ⟹ ps!1 = 2
    by (metis N-pos N-pos-simps(3) One-nat-def diff-is-0-eq neq0-conv)
  from 3 have d:fs!0 ≠ 0 ⟹ fs!0 = 2
    using N-pos by meson
  then show ?case
    apply(cases fs!0 = 0)
    using a apply (simp add: fork-transitions-def Un-insert-left)
    using b1[OF c] b2[OF d] N-pos by blast
next
  case 4
```

```
  then show ?case
    using 4(5)[rule-format, of 0, OF N-pos] apply(elim disjE)
  proof(goal-cases)
    case 41:1
    then show ?case
    using 4(5)[rule-format, of 1, OF N-g1] apply(elim disjE)
    proof(goal-cases)
      case 411:1
      from 411 have a0: ps!1 = 0
        by (metis N-g1 One-nat-def neq0-conv)
      from 411 have a1: picks 1 1 ∈ (⋃ i<N. Tr_f i (fs ! i))
        apply (auto simp add:fork-transitions-def)
        by (metis (mono-tags, lifting) N-g1 Int-Collect One-nat-def lessThan-iff)
      from 411 have a2: ps!1 = 0 ⟹ picks 1 1 ∈ (⋃ i∈{Suc 0..<N}. Tr_rp i (ps !
i))
        apply (auto simp add:rphil-transitions-def)
        using N-g1 by linarith
      from 411 show ?case
        using a0 a1 a2 by blast
    next
      case 412:2
      hence ps!1 = 1 ∨ ps!1 = 3
        by (metis N-g1 One-nat-def less-numeral-extra(3) zero-less-diff)
      with 412 show ?case
      proof(elim disjE, goal-cases)
        case 4121:1
        from 4121 have b1: picks 1 0 ∈ (⋃ i<N. Tr_f i (fs ! i))
          apply (auto simp add:fork-transitions-def)
            by (metis (full-types) Int-Collect N-g1 N-pos One-nat-def lessThan-iff
mod-less)
        from 4121 have b2: picks 1 0 ∈ (⋃ i∈{Suc 0..<N}. Tr_rp i (ps ! i))
          apply (auto simp add:rphil-transitions-def)
          using N-g1 by linarith
        from 4121 show ?case
          using b1 b2 by blast
      next
        case 4122:2
        from 4122 have b3: putsdown 1 1 ∈ (⋃ i<N. Tr_f i (fs ! i))
          apply (auto simp add:fork-transitions-def)
          using N-g1 by linarith
        from 4122 have b4: putsdown 1 1 ∈ (⋃ i∈{Suc 0..<N}. Tr_rp i (ps ! i))
          apply (auto simp add:rphil-transitions-def)
          using N-g1 by linarith
        then show ?case
          using b3 b4 by blast
      qed
    next
      case 413:3
      then show ?case
```

```
    proof(cases N = 2)
      case True
      with 413 show ?thesis by simp
    next
      case False
      from False 413 have c0: ps!2 = 2
        by (metis N-g1 Suc-1 Suc-diff-1 nat-neq-iff not-gr0 zero-less-diff)
      from False 413 have c1: putsdown 2 1 ∈ (⋃i<N. Tr_f i (fs ! i))
        apply (auto simp add:fork-transitions-def)
        using N-g1 apply linarith
        using N-g1 by auto
      from False 413 have c2: ps!2 = 2 ⟹ putsdown 2 1 ∈ (⋃i∈{Suc 0..<N}.
Tr_rp i (ps ! i))
        apply (auto simp add:rphil-transitions-def)
        using N-g1 by linarith
      from 413 False show ?thesis
        using c0 c1 c2 by blast
    qed
  qed
 next
  case 42:2
  then show ?case by blast
 next
  case 43:3
  from 43 have d0: ps!1 = 2
    by (metis One-nat-def gr0I)
  from 43 have d1: putsdown 1 0 ∈ (⋃i<N. Tr_f i (fs ! i))
    by (auto simp add:fork-transitions-def)
  from 43 have d2: ps!1 = 2 ⟹ putsdown 1 0 ∈ (⋃i∈{Suc 0..<N}. Tr_rp i (ps
! i))
    apply (auto simp add:rphil-transitions-def)
    using N-g1 by linarith
  from 43 show ?case
    using d0 d1 d2 by blast
 qed
next
 case 5
 then show ?case
  using 5(6)[rule-format, of 0] by simp
qed

corollary DF-DINING: deadlock-free-v2 DINING
 unfolding DINING-is-DINING_norm DINING_norm-def
 using inv-DINING inv-implies-DF by (subst deadlock-free-dnorm) auto

end

end
```

Chapter 4

Conclusion

We presented a formalisation of the most comprehensive semantic model for CSP, a 'classical' language for the specification and analysis of concurrent systems studied in a rich body of literature. For this purpose, we ported [12] to a modern version of Isabelle, restructured the proofs, and extended the resulting theory of the language substantially. The result HOL-CSP 2 has been submitted to the Isabelle AFP [10], thus a fairly sustainable format accessible to other researchers and tools.

We developed a novel set of deadlock - and livelock inference proof principles based on classical and denotational characterizations. In particular, we formally investigated the relations between different refinement notions in the presence of deadlock - and livelock; an area where traditional CSP literature skates over the nitty-gritty details. Finally, we demonstrated how to exploit these results for deadlock/livelock analysis of protocols.

We put a large body of abstract CSP laws and induction principles together to form concrete verification technologies for generalized classical problems, which have been considered so far from the perspective of data-independence or structural parametricity. The underlying novel principle of "trading rich structure against rich state" allows one to convert processes into classical transition systems for which established invariant techniques become applicable.

Future applications of HOL-CSP 2 could comprise a combination with model checkers, where our theory with its derived rules can be used to certify the output of a model-checker over CSP. In our experience, labelled transition systems generated by model checkers may be used to steer inductions or to construct the normalized processes $P_{norm}[\![\tau,\upsilon]\!]$ automatically, thus combining efficient finite reasoning over finite sub-systems with globally infinite systems in a logically safe way.

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