

A framework for establishing Strong Eventual Consistency for Conflict-free Replicated Data types

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Abstract

In this work, we focus on the correctness of Conflict-free Replicated Data Types (CRDTs), a class of algorithm that provides strong eventual consistency guarantees for replicated data. We develop a modular and reusable framework for verifying the correctness of CRDT algorithms. We avoid correctness issues that have dogged previous mechanised proofs in this area by including a network model in our formalisation, and proving that our theorems hold in all possible network behaviours. Our axiomatic network model is a standard abstraction that accurately reflects the behaviour of real-world computer networks. Moreover, we identify an abstract convergence theorem, a property of order relations, which provides a formal definition of strong eventual consistency. We then obtain the first machine-checked correctness theorems for three concrete CRDTs: the Replicated Growable Array, the Observed-Remove Set, and an Increment-Decrement Counter.

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1 Introduction

Strong eventual consistency (SEC) is a model that strikes a compromise between strong and eventual consistency [12]. Informally, it guarantees that whenever two nodes have received the same set of messages—possibly in a different order—their view of the shared state is identical, and any conflicting concurrent updates must be merged automatically. Large-scale deployments of SEC algorithms include datacentre-based applications using the Riak distributed database [3], and collaborative editing applications such as Google Docs [5]. Unlike strong consistency models, it is possible to implement SEC in decentralised settings without any central server or leader, and it allows local execution at each node to proceed without waiting for communication with other nodes. However, algorithms for achieving decentralised SEC are currently poorly understood: several such algorithms, published in peer-reviewed venues, were subsequently shown to violate their supposed guarantees [6, 7, 9]. Informal reasoning has repeatedly produced plausible-looking but incorrect algorithms, and there have even been examples of mechanised formal proofs of SEC algorithm correctness later being shown to be flawed. These mechanised proofs failed because, in formalising the algorithm, they made false assumptions about the execution environment.

In this work we use the Isabelle/HOL proof assistant [13] to create a framework for reliably reasoning about the correctness of a particular class of decentralised replication algorithms. We do this by formalising not only the replication algorithms, but also the network in which they execute, allowing us to prove that the algorithm’s assumptions hold in all possible network behaviours. We model the network using the axioms of *asynchronous unreliable causal broadcast*, a well-understood abstraction that is commonly implemented by network protocols, and which can run on almost any computer network, including large-scale networks that delay, reorder, or drop messages, and in which nodes may fail.

We then use this framework to produce machine-checked proofs of correctness for three Conflict-Free Replicated Data Types (CRDTs), a class of replication algorithms that ensure strong eventual consistency [11, 12]. To our knowledge, this is the first machine-checked verification of SEC algorithms that explicitly models the network and reasons about all possible network behaviours. The framework is modular and reusable, making it easy to formulate proofs for new algorithms. We provide the first mechanised proofs of the Replicated Growable Array, the operation-based Observed-Remove Set, and the operation-based counter CRDT.

2 Technical Lemmas

This section contains a list of helper definitions and lemmas about sets, lists and the option monad.

theory

Util

imports

Main

HOL-Library.Monad-Syntax

begin

2.1 Kleisli arrow composition

definition *kleisli* :: ('b ⇒ 'b option) ⇒ ('b ⇒ 'b option) ⇒ ('b ⇒ 'b option) (**infixr** ▷ 65) **where**
f ▷ g ≡ λx. (f x ≫= (λy. g y))

lemma *kleisli-comm-cong*:

assumes x ▷ y = y ▷ x
shows z ▷ x ▷ y = z ▷ y ▷ x
using *assms* **by**(*clarsimp simp add: kleisli-def*)

lemma *kleisli-assoc*:

shows (z ▷ x) ▷ y = z ▷ (x ▷ y)
by(*auto simp add: kleisli-def*)

2.2 Lemmas about sets

lemma *distinct-set-notin* [*dest*]:

assumes *distinct* (x#xs)
shows x ∉ set xs
using *assms* **by**(*induction xs, auto*)

lemma *set-membership-equality-technicalD* [*dest*]:

assumes {x} ∪ (set xs) = {y} ∪ (set ys)
shows x = y ∨ y ∈ set xs
using *assms* **by**(*induction xs, auto*)

lemma *set-equality-technical*:

assumes {x} ∪ (set xs) = {y} ∪ (set ys)
and x ∉ set xs
and y ∉ set ys
and y ∈ set xs
shows {x} ∪ (set xs - {y}) = set ys
using *assms* **by** (*induction xs*) *auto*

lemma *set-elem-nth*:

assumes x ∈ set xs
shows ∃ m. m < length xs ∧ xs ! m = x
using *assms* **by**(*induction xs, simp*) (*meson in-set-conv-nth*)

2.3 Lemmas about list

lemma *list-nil-or-snoc*:

shows xs = [] ∨ (∃ y ys. xs = ys@[y])
by (*induction xs, auto*)

lemma *suffix-eq-distinct-list*:

assumes *distinct* xs
and ys@suf1 = xs
and ys@suf2 = xs
shows suf1 = suf2
using *assms* **by**(*induction xs arbitrary: suf1 suf2 rule: rev-induct, simp*) (*metis append-eq-append-conv*)

lemma *pre-suf-eq-distinct-list*:

assumes *distinct* xs
and ys ≠ []
and pre1@ys@suf1 = xs

```

    and pre2@ys@suf2 = xs
    shows pre1 = pre2  $\wedge$  suf1 = suf2
using assms
apply(induction xs arbitrary: pre1 pre2 ys, simp)
apply(case-tac pre1; case-tac pre2; clarify)
apply(metis suffix-eq-distinct-list append-Nil)
apply(metis Un-iff append-eq-Cons-conv distinct.simps(2) list.set-intros(1) set-append suffix-eq-distinct-list)
apply(metis Un-iff append-eq-Cons-conv distinct.simps(2) list.set-intros(1) set-append suffix-eq-distinct-list)
apply(metis distinct.simps(2) hd-append2 list.sel(1) list.sel(3) list.simps(3) tl-append2)
done

```

```

lemma list-head-unaaffected:
  assumes hd (x @ [y, z]) = v
  shows hd (x @ [y ]) = v
using assms by (metis hd-append list.sel(1))

```

```

lemma list-head-butlast:
  assumes hd xs = v
  and length xs > 1
  shows hd (butlast xs) = v
using assms by (metis hd-conv-nth length-butlast length-greater-0-conv less-trans nth-butlast zero-less-diff
zero-less-one)

```

```

lemma list-head-length-one:
  assumes hd xs = x
  and length xs = 1
  shows xs = [x]
using assms by (metis One-nat-def Suc-length-conv hd-Cons-tl length-0-conv list.sel(3))

```

```

lemma list-two-at-end:
  assumes length xs > 1
  shows  $\exists$  xs' x y. xs = xs' @ [x, y]
using assms
apply(induction xs rule: rev-induct, simp)
apply(case-tac length xs = 1, simp)
apply(metis append-self-conv2 length-0-conv length-Suc-conv)
apply(rule-tac x=butlast xs in exI, rule-tac x=last xs in exI, simp)
done

```

```

lemma list-nth-split-technical:
  assumes m < length cs
  and cs  $\neq$  []
  shows  $\exists$  xs ys. cs = xs@(cs!m)#ys
using assms
apply(induction m arbitrary: cs)
apply(meson in-set-conv-decomp nth-mem)
apply(metis in-set-conv-decomp length-list-update set-swap set-update-memI)
done

```

```

lemma list-nth-split:
  assumes m < length cs
  and n < m
  and 1 < length cs
  shows  $\exists$  xs ys zs. cs = xs@(cs!n)#ys@(cs!m)#zs
using assms proof(induction n arbitrary: cs m)
case 0 thus ?case
  apply(case-tac cs; clarsimp)
  apply(rule-tac x=[] in exI, clarsimp)

```

```

    apply(rule list-nth-split-technical, simp, force)
  done
next
case (Suc n)
thus ?case
proof (cases cs)
  case Nil
  then show ?thesis
    using Suc.prem by auto
next
case (Cons a as)
hence  $m-1 < \text{length } as$   $n < m-1$ 
  using Suc by force+
then obtain  $xs\ ys\ zs$  where  $as = xs @ as ! n \# ys @ as ! (m-1) \# zs$ 
  using Suc by force
thus ?thesis
  apply(rule-tac  $x=a\#xs$  in exI)
  using Suc Cons apply force
done
qed
qed

lemma list-split-two-elems:
  assumes distinct cs
    and  $x \in \text{set } cs$ 
    and  $y \in \text{set } cs$ 
    and  $x \neq y$ 
  shows  $\exists pre\ mid\ suf. cs = pre @ x \# mid @ y \# suf \vee cs = pre @ y \# mid @ x \# suf$ 
proof -
  obtain  $xi\ yi$  where  $*$ :  $xi < \text{length } cs \wedge x = cs ! xi$   $yi < \text{length } cs \wedge y = cs ! yi$   $xi \neq yi$ 
  using set-elem-nth linorder-neqE-nat assms by metis
  thus ?thesis
    by (metis list-nth-split One-nat-def less-Suc-eq linorder-neqE-nat not-less-zero)
qed

lemma split-list-unique-prefix:
  assumes  $x \in \text{set } xs$ 
  shows  $\exists pre\ suf. xs = pre @ x \# suf \wedge (\forall y \in \text{set } pre. x \neq y)$ 
using assms proof(induction xs)
  case Nil thus ?case by clarsimp
next
case (Cons y ys)
then show ?case
  proof (cases  $y=x$ )
    case True
    then show ?thesis by force
  next
    case False
    then obtain  $pre\ suf$  where  $ys = pre @ x \# suf \wedge (\forall y \in \text{set } pre. x \neq y)$ 
      using assms Cons by auto
    thus ?thesis
      using split-list-first by force
  qed
qed

lemma map-filter-append:
  shows  $List.map-filter P (xs @ ys) = List.map-filter P xs @ List.map-filter P ys$ 
by(auto simp add: List.map-filter-def)

```

end

3 Strong Eventual Consistency

In this section we formalise the notion of strong eventual consistency. We do not make any assumptions about networks or data structures; instead, we use an abstract model of operations that may be reordered, and we reason about the properties that those operations must satisfy. We then provide concrete implementations of that abstract model in later sections.

theory

Convergence

imports

Util

begin

The *happens-before* relation, as introduced by [8], captures causal dependencies between operations. It can be defined in terms of sending and receiving messages on a network. However, for now, we keep it abstract, our only restriction on the happens-before relation is that it must be a *strict partial order*, that is, it must be irreflexive and transitive, which implies that it is also antisymmetric. We describe the state of a node using an abstract type variable. To model state changes, we assume the existence of an *interpretation* function *interp* which lifts an operation into a *state transformer*—a function that either maps an old state to a new state, or fails.

```

locale happens-before = preorder hb-weak hb
  for hb-weak :: 'a ⇒ 'a ⇒ bool (infix ≤ 50)
  and hb :: 'a ⇒ 'a ⇒ bool (infix < 50) +
  fixes interp :: 'a ⇒ 'b → 'b (<-) [0] 1000
begin

```

3.1 Concurrent operations

We say that two operations x and y are *concurrent*, written $x \parallel y$, whenever one does not happen before the other: $\neg(x < y)$ and $\neg(y < x)$.

definition *concurrent* :: 'a ⇒ 'a ⇒ bool (**infix** || 50) **where**
 $s1 \parallel s2 \equiv \neg(s1 < s2) \wedge \neg(s2 < s1)$

lemma *concurrentI* [intro!]: $\neg(s1 < s2) \implies \neg(s2 < s1) \implies s1 \parallel s2$
by (auto simp: concurrent-def)

lemma *concurrentD1* [dest]: $s1 \parallel s2 \implies \neg(s1 < s2)$
by (auto simp: concurrent-def)

lemma *concurrentD2* [dest]: $s1 \parallel s2 \implies \neg(s2 < s1)$
by (auto simp: concurrent-def)

lemma *concurrent-refl* [intro!, simp]: $s \parallel s$
by (auto simp: concurrent-def)

lemma *concurrent-comm*: $s1 \parallel s2 \longleftrightarrow s2 \parallel s1$
by (auto simp: concurrent-def)

definition *concurrent-set* :: 'a ⇒ 'a list ⇒ bool **where**
 $\text{concurrent-set } x \ xs \equiv \forall y \in \text{set } xs. x \parallel y$

lemma *concurrent-set-empty* [simp, intro!]:
 $\text{concurrent-set } x \ []$

by (auto simp: concurrent-set-def)

lemma *concurrent-set-ConsE* [elim]:
assumes *concurrent-set a (x#xs)*
and *concurrent-set a xs \implies concurrent x a \implies G*
shows *G*
using *assms* by (auto simp: concurrent-set-def)

lemma *concurrent-set-ConsI* [intro]:
concurrent-set a xs \implies concurrent a x \implies concurrent-set a (x#xs)
by (auto simp: concurrent-set-def)

lemma *concurrent-set-appendI* [intro]:
concurrent-set a xs \implies concurrent-set a ys \implies concurrent-set a (xs@ys)
by (auto simp: concurrent-set-def)

lemma *concurrent-set-Cons-Snoc* [simp]:
concurrent-set a (xs@[x]) = concurrent-set a (x#xs)
by (auto simp: concurrent-set-def)

3.2 Happens-before consistency

The purpose of the happens-before relation is to require that some operations must be applied in a particular order, while allowing concurrent operations to be reordered with respect to each other. We assume that each node applies operations in some sequential order (a standard assumption for distributed algorithms), and so we can model the execution history of a node as a list of operations.

inductive *hb-consistent* :: 'a list \Rightarrow bool **where**
[*intro!*]: *hb-consistent []* |
[*intro!*]: [*hb-consistent xs*; $\forall x \in \text{set } xs. \neg y \prec x$] \implies *hb-consistent (xs @ [y])*

As a result, whenever two operations x and y appear in a hb-consistent list, and $x \prec y$, then x must appear before y in the list. However, if $x \parallel y$, the operations can appear in the list in either order.

lemma $(x \prec y \vee \text{concurrent } x \ y) = (\neg y \prec x)$
using *less-asym* **by** *blast*

lemma *consistentI* [intro]:
assumes *hb-consistent (xs @ ys)*
and $\forall x \in \text{set } (xs @ ys). \neg z \prec x$
shows *hb-consistent (xs @ ys @ [z])*
using *assms hb-consistent.intros append-assoc* **by** *metis*

inductive-cases *hb-consistent-elim* [elim]:
hb-consistent []
hb-consistent (xs@[y])
hb-consistent (xs@ys)
hb-consistent (xs@ys@[z])

inductive-cases *hb-consistent-elim-gen*:
hb-consistent zs

lemma *hb-consistent-append-D1* [*dest*]:
assumes *hb-consistent (xs @ ys)*
shows *hb-consistent xs*
using *assms* **by** (*induction ys arbitrary: xs rule: List.rev-induct*) *auto*

```

lemma hb-consistent-append-D2 [dest]:
  assumes hb-consistent (xs @ ys)
  shows hb-consistent ys
  using assms by (induction ys arbitrary: xs rule: List.rev-induct) fastforce+

lemma hb-consistent-append-elim-ConsD [elim]:
  assumes hb-consistent (y#ys)
  shows hb-consistent ys
  using assms hb-consistent-append-D2 by (metis append-Cons append-Nil)

lemma hb-consistent-remove1 [intro]:
  assumes hb-consistent xs
  shows hb-consistent (remove1 x xs)
  using assms by (induction rule: hb-consistent.induct) (auto simp: remove1-append)

lemma hb-consistent-singleton [intro!]:
  shows hb-consistent [x]
  using hb-consistent.intros by fastforce

lemma hb-consistent-prefix-suffix-exists:
  assumes hb-consistent ys
    hb-consistent (xs @ [x])
     $\{x\} \cup \text{set } xs = \text{set } ys$ 
    distinct (x#xs)
    distinct ys
  shows  $\exists \text{prefix suffix. } ys = \text{prefix} @ x \# \text{suffix} \wedge \text{concurrent-set } x \text{ suffix}$ 
using assms proof (induction arbitrary: xs rule: hb-consistent.induct, simp)
fix xs y ys
assume IH:  $(\bigwedge xs. \text{hb-consistent } (xs @ [x]) \implies$ 
   $\{x\} \cup \text{set } xs = \text{set } ys \implies$ 
   $\text{distinct } (x \# xs) \implies \text{distinct } ys \implies$ 
   $\exists \text{prefix suffix. } ys = \text{prefix} @ x \# \text{suffix} \wedge \text{concurrent-set } x \text{ suffix})$ 
assume assms: hb-consistent ys  $\forall x \in \text{set } ys. \neg \text{hb } y x$ 
  hb-consistent (xs @ [x])
   $\{x\} \cup \text{set } xs = \text{set } (ys @ [y])$ 
  distinct (x # xs) distinct (ys @ [y])
hence  $x = y \vee y \in \text{set } xs$ 
using assms by auto
moreover {
  assume  $x = y$ 
  hence  $\exists \text{prefix suffix. } ys @ [y] = \text{prefix} @ x \# \text{suffix} \wedge \text{concurrent-set } x \text{ suffix}$ 
  by force
}
moreover {
  assume y-in-xs:  $y \in \text{set } xs$ 
  hence  $\{x\} \cup (\text{set } xs - \{y\}) = \text{set } ys$ 
  using assms by (auto intro: set-equality-technical)
  hence remove-y-in-xs:  $\{x\} \cup \text{set } (\text{remove1 } y \text{ } xs) = \text{set } ys$ 
  using assms by auto
  moreover have hb-consistent  $((\text{remove1 } y \text{ } xs) @ [x])$ 
  using assms hb-consistent-remove1 by force
  moreover have distinct  $(x \# (\text{remove1 } y \text{ } xs))$ 
  using assms by simp
  moreover have distinct ys
  using assms by simp
  ultimately obtain prefix suffix where ys-split:  $ys = \text{prefix} @ x \# \text{suffix} \wedge \text{concurrent-set } x \text{ suffix}$ 
  using IH by force
moreover {

```



```

have concurrent x y
  using assms y-in-xs remove-y-in-xs concurrent-def by blast
hence concurrent-set x (suffix@[y])
  using ys-split by clarsimp
}
ultimately have  $\exists$  prefix suffix. ys @ [y] = prefix @ x # suffix  $\wedge$  concurrent-set x suffix
by force
}
ultimately show  $\exists$  prefix suffix. ys @ [y] = prefix @ x # suffix  $\wedge$  concurrent-set x suffix
by auto
qed

```

```

lemma hb-consistent-append [intro!]:
  assumes hb-consistent suffix
    hb-consistent prefix
     $\bigwedge s p. s \in \text{set suffix} \implies p \in \text{set prefix} \implies \neg s < p$ 
  shows hb-consistent (prefix @ suffix)
using assms by (induction rule: hb-consistent.induct) force+

```

```

lemma hb-consistent-append-porder:
  assumes hb-consistent (xs @ ys)
    x  $\in$  set xs
    y  $\in$  set ys
  shows  $\neg y < x$ 
using assms by (induction ys arbitrary: xs rule: rev-induct) force+

```

3.3 Apply operations

We can now define a function *apply-operations* that composes an arbitrary list of operations into a state transformer. We first map *interp* across the list to obtain a state transformer for each operation, and then collectively compose them using the Kleisli arrow composition combinator.

```

definition apply-operations :: 'a list  $\Rightarrow$  'b  $\rightarrow$  'b where
  apply-operations es  $\equiv$  foldl ( $\triangleright$ ) Some (map interp es)

```

```

lemma apply-operations-empty [simp]: apply-operations [] s = Some s
by(auto simp: apply-operations-def)

```

```

lemma apply-operations-Snoc [simp]:
  apply-operations (xs@[x]) = (apply-operations xs)  $\triangleright$   $\langle$ x $\rangle$ 
by(auto simp add: apply-operations-def kleisli-def)

```

3.4 Concurrent operations commute

We say that two operations x and y *commute* whenever $\langle x \rangle \triangleright \langle y \rangle = \langle y \rangle \triangleright \langle x \rangle$, i.e. when we can swap the order of the composition of their interpretations without changing the resulting state transformer. For our purposes, requiring that this property holds for *all* pairs of operations is too strong. Rather, the commutation property is only required to hold for operations that are concurrent.

```

definition concurrent-ops-commute :: 'a list  $\Rightarrow$  bool where
  concurrent-ops-commute xs  $\equiv$ 
     $\forall x y. \{x, y\} \subseteq \text{set xs} \longrightarrow \text{concurrent } x y \longrightarrow \langle x \rangle \triangleright \langle y \rangle = \langle y \rangle \triangleright \langle x \rangle$ 

```

```

lemma concurrent-ops-commute-empty [intro!]: concurrent-ops-commute []
by(auto simp: concurrent-ops-commute-def)

```

```

lemma concurrent-ops-commute-singleton [intro!]: concurrent-ops-commute [x]

```

by(auto simp: concurrent-ops-commute-def)

lemma *concurrent-ops-commute-appendD* [dest]:
assumes *concurrent-ops-commute* (xs@ys)
shows *concurrent-ops-commute* xs
using *assms* **by** (auto simp: concurrent-ops-commute-def)

lemma *concurrent-ops-commute-rearrange*:
concurrent-ops-commute (xs@x#ys) = *concurrent-ops-commute* (xs@ys@[x])
by (clarsimp simp: concurrent-ops-commute-def)

lemma *concurrent-ops-commute-concurrent-set*:
assumes *concurrent-ops-commute* (prefix@suffix@[x])
concurrent-set x suffix
distinct (prefix @ x # suffix)
shows *apply-operations* (prefix @ suffix @ [x]) = *apply-operations* (prefix @ x # suffix)
using *assms* **proof**(*induction* suffix arbitrary: rule: rev-induct, force)
fix a xs
assume IH: *concurrent-ops-commute* (prefix @ xs @ [x]) \implies
concurrent-set x xs \implies *distinct* (prefix @ x # xs) \implies
apply-operations (prefix @ xs @ [x]) = *apply-operations* (prefix @ x # xs)
assume *assms*: *concurrent-ops-commute* (prefix @ (xs @ [a]) @ [x])
concurrent-set x (xs @ [a]) *distinct* (prefix @ x # xs @ [a])
hence *ac-comm*: $\langle a \rangle \triangleright \langle x \rangle = \langle x \rangle \triangleright \langle a \rangle$
by (clarsimp simp: concurrent-ops-commute-def) blast
have *copc*: *concurrent-ops-commute* (prefix @ xs @ [x])
using *assms* **by** (clarsimp simp: concurrent-ops-commute-def) blast
have *apply-operations* ((prefix @ x # xs) @ [a]) = (*apply-operations* (prefix @ x # xs)) $\triangleright \langle a \rangle$
by (simp del: append-assoc)
also have ... = (*apply-operations* (prefix @ xs @ [x])) $\triangleright \langle a \rangle$
using IH *assms* *copc* **by** auto
also have ... = ((*apply-operations* (prefix @ xs)) $\triangleright \langle x \rangle$) $\triangleright \langle a \rangle$
by (simp add: append-assoc[symmetric] del: append-assoc)
also have ... = (*apply-operations* (prefix @ xs)) $\triangleright (\langle a \rangle \triangleright \langle x \rangle)$
using *ac-comm* *kleisli-comm-cong* *kleisli-assoc* **by** simp
finally show *apply-operations* (prefix @ (xs @ [a]) @ [x]) = *apply-operations* (prefix @ x # xs @ [a])
by (metis Cons-eq-appendI append-assoc *apply-operations-Snoc* *kleisli-assoc*)
qed

3.5 Abstract convergence theorem

We can now state and prove our main theorem, *convergence*. This theorem states that two hb-consistent lists of distinct operations, which are permutations of each other and in which concurrent operations commute, have the same interpretation.

theorem *convergence*:
assumes *set* xs = *set* ys
concurrent-ops-commute xs
concurrent-ops-commute ys
distinct xs
distinct ys
hb-consistent xs
hb-consistent ys
shows *apply-operations* xs = *apply-operations* ys
using *assms* **proof**(*induction* xs arbitrary: ys rule: rev-induct, simp)
case *assms*: (snoc x xs)
then obtain prefix suffix **where** *ys-split*: ys = prefix @ x # suffix \wedge *concurrent-set* x suffix
using *hb-consistent-prefix-suffix-exists* **by** fastforce

```

moreover hence *: distinct (prefix @ suffix) hb-consistent xs
  using assms by auto
moreover {
  have hb-consistent prefix hb-consistent suffix
    using ys-split assms hb-consistent-append-D2 hb-consistent-append-elim-ConsD by blast+
  hence hb-consistent (prefix @ suffix)
    by (metis assms(8) hb-consistent-append hb-consistent-append-porder list.set-intros(2) ys-split)
}
moreover have **: concurrent-ops-commute (prefix @ suffix @ [x])
  using assms ys-split by (clarsimp simp: concurrent-ops-commute-def)
moreover hence concurrent-ops-commute (prefix @ suffix)
  by (force simp del: append-assoc simp add: append-assoc[symmetric])
ultimately have apply-operations xs = apply-operations (prefix@suffix)
  using assms by simp (metis Diff-insert-absorb Un-iff * concurrent-ops-commute-appendD set-append)
moreover have apply-operations (prefix@suffix @ [x]) = apply-operations (prefix@x # suffix)
  using ys-split assms ** concurrent-ops-commute-concurrent-set by force
ultimately show ?case
  using ys-split by (force simp: append-assoc[symmetric] simp del: append-assoc)
qed

```

corollary *convergence-ext:*

```

assumes set xs = set ys
  concurrent-ops-commute xs
  concurrent-ops-commute ys
  distinct xs
  distinct ys
  hb-consistent xs
  hb-consistent ys
shows apply-operations xs s = apply-operations ys s
using convergence assms by metis
end

```

3.6 Convergence and progress

Besides convergence, another required property of SEC is *progress*: if a valid operation was issued on one node, then applying that operation on other nodes must also succeed—that is, the execution must not become stuck in an error state. Although the type signature of the interpretation function allows operations to fail, we need to prove that in all *hb-consistent* network behaviours such failure never actually occurs. We capture the combined requirements in the *strong-eventual-consistency* locale, which extends *happens-before*.

```

locale strong-eventual-consistency = happens-before +
  fixes op-history :: 'a list ⇒ bool
  and initial-state :: 'b
  assumes causality: op-history xs ⇒ hb-consistent xs
  assumes distinctness: op-history xs ⇒ distinct xs
  assumes commutativity: op-history xs ⇒ concurrent-ops-commute xs
  assumes no-failure: op-history(xs@[x]) ⇒ apply-operations xs initial-state = Some state ⇒ ⟨x⟩
state ≠ None
  assumes trunc-history: op-history(xs@[x]) ⇒ op-history xs
begin

```

theorem *sec-convergence:*

```

assumes set xs = set ys
  op-history xs
  op-history ys
shows apply-operations xs = apply-operations ys
by (meson assms convergence causality commutativity distinctness)

```

```

theorem sec-progress:
  assumes op-history xs
  shows apply-operations xs initial-state ≠ None
using assms proof(induction xs rule: rev-induct, simp)
  case (snoc x xs)
  have apply-operations xs initial-state ≠ None
    using snoc.IH snoc.prem1 trunc-history kleisli-def bind-def by blast
  moreover have apply-operations (xs @ [x]) = apply-operations xs ▷ ⟨x⟩
    by simp
  ultimately show ?case
    using no-failure snoc.prem1 by (clarsimp simp add: kleisli-def split: bind-splits)
qed

end
end

```

4 Axiomatic network models

In this section we develop a formal definition of an *asynchronous unreliable causal broadcast network*. We choose this model because it satisfies the causal delivery requirements of many operation-based CRDTs [1, 2]. Moreover, it is suitable for use in decentralised settings, as motivated in the introduction, since it does not require waiting for communication with a central server or a quorum of nodes.

```

theory
  Network
imports
  Convergence
begin

```

4.1 Node histories

We model a distributed system as an unbounded number of communicating nodes. We assume nothing about the communication pattern of nodes—we assume only that each node is uniquely identified by a natural number, and that the flow of execution at each node consists of a finite, totally ordered sequence of execution steps (events). We call that sequence of events at node i the *history* of that node. For convenience, we assume that every event or execution step is unique within a node’s history.

```

locale node-histories =
  fixes history :: nat ⇒ 'evt list
  assumes histories-distinct [intro!, simp]: distinct (history i)

```

```

lemma (in node-histories) history-finite:
  shows finite (set (history i))
by auto

```

```

definition (in node-histories) history-order :: 'evt ⇒ nat ⇒ 'evt ⇒ bool (-/ □-/ - [50,1000,50]50)
where
   $x \sqsubset^i z \equiv \exists xs \ ys \ zs. xs @ x \# ys @ z \# zs = \text{history } i$ 

```

```

lemma (in node-histories) node-total-order-trans:
  assumes  $e1 \sqsubset^i e2$ 
  and  $e2 \sqsubset^i e3$ 
  shows  $e1 \sqsubset^i e3$ 

```

proof –

obtain $xs1\ xs2\ ys1\ ys2\ zs1\ zs2$ **where** $*$: $xs1\ @\ e1\ \#\ ys1\ @\ e2\ \#\ zs1 = history\ i$
 $xs2\ @\ e2\ \#\ ys2\ @\ e3\ \#\ zs2 = history\ i$
using *history-order-def* *assms* **by** *auto*
hence $xs1\ @\ e1\ \#\ ys1 = xs2\ \wedge\ zs1 = ys2\ @\ e3\ \#\ zs2$
by(*rule-tac* $xs=history\ i$ **and** $ys=[e2]$ **in** *pre-suf-eq-distinct-list*) *auto*
thus *?thesis*
by(*clarsimp* *simp*: *history-order-def*) (*metis* $*(2)$ *append.assoc* *append-Cons*)
qed

lemma (**in** *node-histories*) *local-order-carrier-closed*:

assumes $e1\ \sqsubset^i\ e2$
shows $\{e1, e2\} \subseteq set\ (history\ i)$
using *assms* **by** (*clarsimp* *simp* *add*: *history-order-def*)
(*metis* *in-set-conv-decomp* *Un-iff* *Un-subset-iff* *insert-subset* *list.simps(15)*
set-append *set-subset-Cons*)**+**

lemma (**in** *node-histories*) *node-total-order-irrefl*:

shows $\neg (e\ \sqsubset^i\ e)$
by(*clarsimp* *simp* *add*: *history-order-def*)
(*metis* *Un-iff* *histories-distinct* *distinct-append* *distinct-set-notin*
list.set-intros(1) *set-append*)

lemma (**in** *node-histories*) *node-total-order-antisym*:

assumes $e1\ \sqsubset^i\ e2$
and $e2\ \sqsubset^i\ e1$
shows *False*
using *assms* *node-total-order-irrefl* *node-total-order-trans* **by** *blast*

lemma (**in** *node-histories*) *node-order-is-total*:

assumes $e1 \in set\ (history\ i)$
and $e2 \in set\ (history\ i)$
and $e1 \neq e2$
shows $e1\ \sqsubset^i\ e2 \vee e2\ \sqsubset^i\ e1$
using *assms* **unfolding** *history-order-def* **by**(*metis* *list-split-two-elems* *histories-distinct*)

definition (**in** *node-histories*) *prefix-of-node-history* :: $'evt\ list \Rightarrow nat \Rightarrow bool$ (**infix** *prefix* of 50) **where**
 $xs\ prefix\ of\ i \equiv \exists\ ys.\ xs@ys = history\ i$

lemma (**in** *node-histories*) *carriers-head-lt*:

assumes $y\ \#\ ys = history\ i$
shows $\neg (x\ \sqsubset^i\ y)$
using *assms*
apply(*clarsimp* *simp* *add*: *history-order-def*)
apply(*rename-tac* $xs1\ ys1\ zs1$)
apply (*subgoal-tac* $xs1\ @\ x\ \#\ ys1 = [] \wedge zs1 = ys$)
apply *clarsimp*
apply (*rule-tac* $xs=history\ i$ **and** $ys=[y]$ **in** *pre-suf-eq-distinct-list*)
apply *auto*
done

lemma (**in** *node-histories*) *prefix-of-ConsD* [*dest*]:

assumes $x\ \#\ xs\ prefix\ of\ i$
shows $[x]\ prefix\ of\ i$
using *assms* **by**(*auto* *simp*: *prefix-of-node-history-def*)

lemma (**in** *node-histories*) *prefix-of-appendD* [*dest*]:

assumes $xs\ @\ ys\ prefix\ of\ i$

shows xs prefix of i
using *assms* **by**(*auto simp: prefix-of-node-history-def*)

lemma (in *node-histories*) *prefix-distinct*:
assumes xs prefix of i
shows *distinct* xs
using *assms* **by**(*clarsimp simp: prefix-of-node-history-def*) (*metis histories-distinct distinct-append*)

lemma (in *node-histories*) *prefix-to-carriers* [*intro*]:
assumes xs prefix of i
shows $set\ xs \subseteq set\ (history\ i)$
using *assms* **by**(*clarsimp simp: prefix-of-node-history-def*) (*metis Un-iff set-append*)

lemma (in *node-histories*) *prefix-elem-to-carriers*:
assumes xs prefix of i
and $x \in set\ xs$
shows $x \in set\ (history\ i)$
using *assms* **by**(*clarsimp simp: prefix-of-node-history-def*) (*metis Un-iff set-append*)

lemma (in *node-histories*) *local-order-prefix-closed*:
assumes $x \sqsubset^i y$
and xs prefix of i
and $y \in set\ xs$
shows $x \in set\ xs$

proof –
obtain ys **where** $xs @ ys = history\ i$
using *assms* *prefix-of-node-history-def* **by** *blast*
moreover obtain $as\ bs\ cs$ **where** $as @ x \# bs @ y \# cs = history\ i$
using *assms* *history-order-def* **by** *blast*
moreover obtain $pre\ suf$ **where** $*$: $xs = pre @ y \# suf$
using *assms* *split-list* **by** *fastforce*
ultimately have $pre = as @ x \# bs \wedge suf @ ys = cs$
by (*rule-tac* $xs=history\ i$ **and** $ys=[y]$ **in** *pre-suf-eq-distinct-list*) *auto*
thus *?thesis*
using *assms* $*$ **by** *clarsimp*

qed

lemma (in *node-histories*) *local-order-prefix-closed-last*:
assumes $x \sqsubset^i y$
and $xs@[y]$ prefix of i
shows $x \in set\ xs$

proof –
have $x \in set\ (xs @ [y])$
using *assms* **by** (*force dest: local-order-prefix-closed*)
thus *?thesis*
using *assms* **by**(*force simp add: node-total-order-irrefl prefix-to-carriers*)

qed

lemma (in *node-histories*) *events-before-exist*:
assumes $x \in set\ (history\ i)$
shows $\exists pre. pre @ [x]$ prefix of i

proof –
have $\exists idx. idx < length\ (history\ i) \wedge (history\ i) ! idx = x$
using *assms* **by**(*simp add: set-elem-nth*)
thus *?thesis*
by(*metis append-take-drop-id take-Suc-conv-app-nth prefix-of-node-history-def*)

qed

lemma (in *node-histories*) *events-in-local-order*:
assumes $pre \ @ \ [e2] \ \text{prefix of } i$
and $e1 \in \text{set } pre$
shows $e1 \sqsubset^i e2$
using *assms split-list unfolding history-order-def prefix-of-node-history-def* **by** *fastforce*

4.2 Asynchronous broadcast networks

We define a new locale *network* containing three axioms that define how broadcast and deliver events may interact, with these axioms defining the properties of our network model.

datatype *'msg event*
 $= \text{Broadcast } 'msg$
 $| \text{Deliver } 'msg$

locale *network* = *node-histories history* **for** $history :: \text{nat} \Rightarrow 'msg \ \text{event list} +$
fixes $msg-id :: 'msg \Rightarrow 'msgid$

assumes *delivery-has-a-cause*: $\llbracket \text{Deliver } m \in \text{set } (history \ i) \rrbracket \Longrightarrow$
 $\exists j. \text{Broadcast } m \in \text{set } (history \ j)$
and *deliver-locally*: $\llbracket \text{Broadcast } m \in \text{set } (history \ i) \rrbracket \Longrightarrow$
 $\text{Broadcast } m \sqsubset^i \text{Deliver } m$
and *msg-id-unique*: $\llbracket \text{Broadcast } m1 \in \text{set } (history \ i);$
 $\text{Broadcast } m2 \in \text{set } (history \ j);$
 $msg-id \ m1 = msg-id \ m2 \rrbracket \Longrightarrow i = j \wedge m1 = m2$

The axioms can be understood as follows:

delivery-has-a-cause: If some message m was delivered at some node, then there exists some node on which m was broadcast. With this axiom, we assert that messages are not created “out of thin air” by the network itself, and that the only source of messages are the nodes.

deliver-locally: If a node broadcasts some message m , then the same node must subsequently also deliver m to itself. Since m does not actually travel over the network, this local delivery is always possible, even if the network is interrupted. Local delivery may seem redundant, since the effect of the delivery could also be implemented by the broadcast event itself; however, it is standard practice in the description of broadcast protocols that the sender of a message also sends it to itself, since this property simplifies the definition of algorithms built on top of the broadcast abstraction [4].

msg-id-unique: We do not assume that the message type *'msg* has any particular structure; we only assume the existence of a function $msg-id :: 'msg \Rightarrow 'msgid$ that maps every message to some globally unique identifier of type *'msgid*. We assert this uniqueness by stating that if $m1$ and $m2$ are any two messages broadcast by any two nodes, and their *msg-ids* are the same, then they were in fact broadcast by the same node and the two messages are identical. In practice, these globally unique IDs can be implemented using unique node identifiers, sequence numbers or timestamps.

lemma (in *network*) *broadcast-before-delivery*:
assumes $\text{Deliver } m \in \text{set } (history \ i)$
shows $\exists j. \text{Broadcast } m \sqsubset^j \text{Deliver } m$
using *assms deliver-locally delivery-has-a-cause* **by** *blast*

lemma (in *network*) *broadcasts-unique*:
assumes $i \neq j$
and $\text{Broadcast } m \in \text{set } (history \ i)$
shows $\text{Broadcast } m \notin \text{set } (history \ j)$

using *assms msg-id-unique* **by** *blast*

Based on the well-known definition by [8], we say that $m1 \prec m2$ if any of the following is true:

1. $m1$ and $m2$ were broadcast by the same node, and $m1$ was broadcast before $m2$.
2. The node that broadcast $m2$ had delivered $m1$ before it broadcast $m2$.
3. There exists some operation $m3$ such that $m1 \prec m3$ and $m3 \prec m2$.

inductive (**in** *network*) *hb* :: '*msg* \Rightarrow '*msg* \Rightarrow *bool* **where**
hb-broadcast: $\llbracket \text{Broadcast } m1 \sqsubset^i \text{Broadcast } m2 \rrbracket \Longrightarrow \text{hb } m1 \ m2 \mid$
hb-deliver: $\llbracket \text{Deliver } m1 \sqsubset^i \text{Broadcast } m2 \rrbracket \Longrightarrow \text{hb } m1 \ m2 \mid$
hb-trans: $\llbracket \text{hb } m1 \ m2; \text{hb } m2 \ m3 \rrbracket \Longrightarrow \text{hb } m1 \ m3$

inductive-cases (**in** *network*) *hb-elim*: *hb* *x* *y*

definition (**in** *network*) *weak-hb* :: '*msg* \Rightarrow '*msg* \Rightarrow *bool* **where**
weak-hb *m1* *m2* $\equiv \text{hb } m1 \ m2 \vee m1 = m2$

locale *causal-network* = *network* +
assumes *causal-delivery*: *Deliver* *m2* \in *set* (*history* *j*) $\Longrightarrow \text{hb } m1 \ m2 \Longrightarrow \text{Deliver } m1 \sqsubset^j \text{Deliver } m2$

lemma (**in** *causal-network*) *causal-broadcast*:
assumes *Deliver* *m2* \in *set* (*history* *j*)
and *Deliver* *m1* \sqsubset^i *Broadcast* *m2*
shows *Deliver* *m1* \sqsubset^j *Deliver* *m2*
using *assms causal-delivery hb.intros(2)* **by** *blast*

lemma (**in** *network*) *hb-broadcast-exists1*:
assumes *hb* *m1* *m2*
shows $\exists i. \text{Broadcast } m1 \in \text{set } (\text{history } i)$
using *assms*
apply(*induction rule: hb.induct*)
apply(*meson insert-subset node-histories.local-order-carrier-closed node-histories-axioms*)
apply(*meson delivery-has-a-cause insert-subset local-order-carrier-closed*)
apply *simp*
done

lemma (**in** *network*) *hb-broadcast-exists2*:
assumes *hb* *m1* *m2*
shows $\exists i. \text{Broadcast } m2 \in \text{set } (\text{history } i)$
using *assms*
apply(*induction rule: hb.induct*)
apply(*meson insert-subset node-histories.local-order-carrier-closed node-histories-axioms*)
apply(*meson delivery-has-a-cause insert-subset local-order-carrier-closed*)
apply *simp*
done

4.3 Causal networks

lemma (**in** *causal-network*) *hb-has-a-reason*:
assumes *hb* *m1* *m2*
and *Broadcast* *m2* \in *set* (*history* *i*)
shows *Deliver* *m1* \in *set* (*history* *i*) $\vee \text{Broadcast } m1 \in \text{set } (\text{history } i)$
using *assms* **apply** (*induction rule: hb.induct*)
apply(*metis insert-subset local-order-carrier-closed network.broadcasts-unique network-axioms*)
apply(*metis insert-subset local-order-carrier-closed network.broadcasts-unique network-axioms*)
using *hb-trans causal-delivery local-order-carrier-closed* **apply** *blast*


```

done

lemma (in causal-network) hb-cross-node-delivery:
  assumes hb m1 m2
    and Broadcast m1 ∈ set (history i)
    and Broadcast m2 ∈ set (history j)
    and i ≠ j
  shows Deliver m1 ∈ set (history j)
  using assms
  apply(induction rule: hb.induct)
    apply(metis broadcasts-unique insert-subset local-order-carrier-closed)
    apply(metis insert-subset local-order-carrier-closed network.broadcasts-unique network-axioms)
  using broadcasts-unique hb.intros(3) hb-has-a-reason apply blast
done

lemma (in causal-network) hb-irrefl:
  assumes hb m1 m2
  shows m1 ≠ m2
using assms proof(induction rule: hb.induct)
  case (hb-broadcast m1 i m2) thus ?case
    using node-total-order-antisym by blast
next
  case (hb-deliver m1 i m2) thus ?case
    by(meson causal-broadcast insert-subset local-order-carrier-closed node-total-order-irrefl)
next
  case (hb-trans m1 m2 m3)
  then obtain i j where Broadcast m3 ∈ set (history i) Broadcast m2 ∈ set (history j)
    using hb-broadcast-exists2 by blast
  then show ?case
    using assms hb-trans by (meson causal-network.causal-delivery causal-network-axioms
      deliver-locally insert-subset network.hb.intros(3) network-axioms
      node-histories.local-order-carrier-closed assms hb-trans
      node-histories-axioms node-total-order-irrefl)
qed

lemma (in causal-network) hb-broadcast-broadcast-order:
  assumes hb m1 m2
    and Broadcast m1 ∈ set (history i)
    and Broadcast m2 ∈ set (history i)
  shows Broadcast m1  $\sqsubset^i$  Broadcast m2
using assms proof(induction rule: hb.induct)
  case (hb-broadcast m1 i m2) thus ?case
    by(metis insertI1 local-order-carrier-closed network.broadcasts-unique network-axioms subsetCE)
next
  case (hb-deliver m1 i m2) thus ?case
    by(metis broadcasts-unique insert-subset local-order-carrier-closed
      network.broadcast-before-delivery network-axioms node-total-order-trans)
next
  case (hb-trans m1 m2 m3)
  then show ?case
  proof (cases Broadcast m2 ∈ set (history i))
    case True thus ?thesis
      using hb-trans node-total-order-trans by blast
  next
    case False hence Deliver m2 ∈ set (history i) m1 ≠ m2 m2 ≠ m3
      using hb-has-a-reason hb-trans by auto
  thus ?thesis
    by(metis hb-trans event.inject(1) hb.intros(1) hb-irrefl network.hb.intros(3) network-axioms node-order-is-total

```

hb-irrefl)
qed
qed

lemma (in *causal-network*) *hb-antisym*:

assumes *hb x y*
and *hb y x*
shows *False*
using *assms* **proof**(*induction rule: hb.induct*)
fix *m1 i m2*
assume *hb m2 m1* **and** *Broadcast m1* \sqsubset^i *Broadcast m2*
thus *False*
apply – **proof**(*erule hb-elim*)
show $\bigwedge ia. \text{Broadcast } m1 \sqsubset^i \text{Broadcast } m2 \implies \text{Broadcast } m2 \sqsubset^i a \text{Broadcast } m1 \implies \text{False}$
by(*metis broadcasts-unique insert-subset local-order-carrier-closed node-total-order-irrefl node-total-order-trans*)
next
show $\bigwedge ia. \text{Broadcast } m1 \sqsubset^i \text{Broadcast } m2 \implies \text{Deliver } m2 \sqsubset^i a \text{Broadcast } m1 \implies \text{False}$
by(*metis broadcast-before-delivery broadcasts-unique insert-subset local-order-carrier-closed node-total-order-irrefl node-total-order-trans*)
next
show $\bigwedge m2a. \text{Broadcast } m1 \sqsubset^i \text{Broadcast } m2 \implies \text{hb } m2 \ m2a \implies \text{hb } m2a \ m1 \implies \text{False}$
using *assms(1) assms(2) hb.intros(3) hb-irrefl* **by** *blast*
qed
next
fix *m1 i m2*
assume *hb m2 m1*
and *Deliver m1* \sqsubset^i *Broadcast m2*
thus *False*
apply – **proof**(*erule hb-elim*)
show $\bigwedge ia. \text{Deliver } m1 \sqsubset^i \text{Broadcast } m2 \implies \text{Broadcast } m2 \sqsubset^i a \text{Broadcast } m1 \implies \text{False}$
by (*metis broadcast-before-delivery broadcasts-unique insert-subset local-order-carrier-closed node-total-order-irrefl node-total-order-trans*)
next
show $\bigwedge ia. \text{Deliver } m1 \sqsubset^i \text{Broadcast } m2 \implies \text{Deliver } m2 \sqsubset^i a \text{Broadcast } m1 \implies \text{False}$
by (*meson causal-network.causal-delivery causal-network-axioms hb.intros(2) hb.intros(3) insert-subset local-order-carrier-closed node-total-order-irrefl*)
next
show $\bigwedge m2a. \text{Deliver } m1 \sqsubset^i \text{Broadcast } m2 \implies \text{hb } m2 \ m2a \implies \text{hb } m2a \ m1 \implies \text{False}$
by (*meson causal-delivery hb.intros(2) insert-subset local-order-carrier-closed network.hb.intros(3) network-axioms node-total-order-irrefl*)
qed
next
fix *m1 m2 m3*
assume *hb m1 m2 hb m2 m3 hb m3 m1*
and (*hb m2 m1* \implies *False*) (*hb m3 m2* \implies *False*)
thus *False*
using *hb.intros(3)* **by** *blast*
qed

definition (in *network*) *node-deliver-messages* :: '*msg event list* \Rightarrow '*msg list* **where**
node-deliver-messages cs \equiv *List.map-filter* ($\lambda e. \text{case } e \text{ of Deliver } m \Rightarrow \text{Some } m \mid - \Rightarrow \text{None}$) *cs*

lemma (in *network*) *node-deliver-messages-empty* [*simp*]:

shows *node-deliver-messages* [] = []
by(*auto simp add: node-deliver-messages-def List.map-filter-simps*)

lemma (in *network*) *node-deliver-messages-Cons*:

shows *node-deliver-messages* (*x#xs*) = (*node-deliver-messages* [*x*])@(*node-deliver-messages* *xs*)

```

by(auto simp add: node-deliver-messages-def map-filter-def)

lemma (in network) node-deliver-messages-append:
  shows node-deliver-messages (xs@ys) = (node-deliver-messages xs)@(node-deliver-messages ys)
  by(auto simp add: node-deliver-messages-def map-filter-def)

lemma (in network) node-deliver-messages-Broadcast [simp]:
  shows node-deliver-messages [Broadcast m] = []
  by(clarsimp simp: node-deliver-messages-def map-filter-def)

lemma (in network) node-deliver-messages-Deliver [simp]:
  shows node-deliver-messages [Deliver m] = [m]
  by(clarsimp simp: node-deliver-messages-def map-filter-def)

lemma (in network) prefix-msg-in-history:
  assumes es prefix of i
    and m ∈ set (node-deliver-messages es)
  shows Deliver m ∈ set (history i)
using assms prefix-to-carriers by(fastforce simp: node-deliver-messages-def map-filter-def split: event.split-asm)

lemma (in network) prefix-contains-msg:
  assumes es prefix of i
    and m ∈ set (node-deliver-messages es)
  shows Deliver m ∈ set es
using assms by(auto simp: node-deliver-messages-def map-filter-def split: event.split-asm)

lemma (in network) node-deliver-messages-distinct:
  assumes xs prefix of i
  shows distinct (node-deliver-messages xs)
using assms proof(induction xs rule: rev-induct)
  case Nil thus ?case by simp
next
  case (snoc x xs)
  { fix y assume *: y ∈ set (node-deliver-messages xs) y ∈ set (node-deliver-messages [x])
    moreover have distinct (xs @ [x])
      using assms snoc prefix-distinct by blast
    ultimately have False
      using assms apply(case-tac x; clarsimp simp add: map-filter-def node-deliver-messages-def)
      using * prefix-contains-msg snoc.prem1s by blast
  } thus ?case
  using snoc by(fastforce simp add: node-deliver-messages-append node-deliver-messages-def map-filter-def)
qed

lemma (in network) drop-last-message:
  assumes evts prefix of i
  and node-deliver-messages evts = msgs @ [last-msg]
  shows ∃ pre. pre prefix of i ∧ node-deliver-messages pre = msgs
proof -
  have Deliver last-msg ∈ set evts
    using assms network.prefix-contains-msg network-axioms by force
  then obtain idx where *: idx < length evts evts ! idx = Deliver last-msg
    by (meson set-elem-nth)
  then obtain pre suf where evts = pre @ (evts ! idx) # suf
    using id-take-nth-drop by blast
  hence **: evts = pre @ (Deliver last-msg) # suf
    using assms * by auto
  moreover hence distinct (node-deliver-messages ([Deliver last-msg] @ suf))
  by (metis assms(1) assms(2) distinct-singleton node-deliver-messages-Cons node-deliver-messages-Deliver

```

```

    node-deliver-messages-append node-deliver-messages-distinct not-Cons-self2 pre-suf-eq-distinct-list)
ultimately have node-deliver-messages ([Deliver last-msg] @ suf) = [last-msg] @ []
by (metis append-self-conv assms(1) assms(2) node-deliver-messages-Cons node-deliver-messages-Deliver
    node-deliver-messages-append node-deliver-messages-distinct not-Cons-self2 pre-suf-eq-distinct-list)
thus ?thesis
using assms * ** by (metis append1-eq-conv append-Cons append-Nil node-deliver-messages-append
prefix-of-appendD)
qed

```

```

locale network-with-ops = causal-network history fst
for history :: nat  $\Rightarrow$  ('msgid  $\times$  'op) event list +
fixes interp :: 'op  $\Rightarrow$  'state  $\rightarrow$  'state
and initial-state :: 'state

```

context network-with-ops **begin**

```

definition interp-msg :: 'msgid  $\times$  'op  $\Rightarrow$  'state  $\rightarrow$  'state where
  interp-msg msg state  $\equiv$  interp (snd msg) state

```

sublocale hb: happens-before weak-hb hb interp-msg

proof

```

fix x y :: 'msgid  $\times$  'op
show hb x y = (weak-hb x y  $\wedge$   $\neg$  weak-hb y x)
unfolding weak-hb-def using hb-antisym by blast
next
fix x
show weak-hb x x
using weak-hb-def by blast
next
fix x y z
assume weak-hb x y weak-hb y z
thus weak-hb x z
using weak-hb-def by (metis network.hb.intros(3) network-axioms)
qed

```

end

```

definition (in network-with-ops) apply-operations :: ('msgid  $\times$  'op) event list  $\rightarrow$  'state where
  apply-operations es  $\equiv$  hb.apply-operations (node-deliver-messages es) initial-state

```

```

definition (in network-with-ops) node-deliver-ops :: ('msgid  $\times$  'op) event list  $\Rightarrow$  'op list where
  node-deliver-ops cs  $\equiv$  map snd (node-deliver-messages cs)

```

```

lemma (in network-with-ops) apply-operations-empty [simp]:
shows apply-operations [] = Some initial-state
by(auto simp add: apply-operations-def)

```

```

lemma (in network-with-ops) apply-operations-Broadcast [simp]:
shows apply-operations (xs @ [Broadcast m]) = apply-operations xs
by(auto simp add: apply-operations-def node-deliver-messages-def map-filter-def)

```

```

lemma (in network-with-ops) apply-operations-Deliver [simp]:
shows apply-operations (xs @ [Deliver m]) = (apply-operations xs  $\gg$  interp-msg m)
by(auto simp add: apply-operations-def node-deliver-messages-def map-filter-def kleisli-def)

```

```

lemma (in network-with-ops) hb-consistent-technical:
assumes  $\bigwedge m n. m < \text{length } cs \implies n < m \implies cs ! n \sqsubset^i cs ! m$ 
shows hb.hb-consistent (node-deliver-messages cs)

```

```

using assms proof (induction cs rule: rev-induct)
  case Nil thus ?case
    by(simp add: node-deliver-messages-def hb.hb-consistent.intros(1) map-filter-simps(2))
next
  case (snoc x xs)
  hence *: ( $\bigwedge m n. m < \text{length } xs \implies n < m \implies xs ! n \sqsubset^i xs ! m$ )
    by( $-, \text{erule-tac } x=m \text{ in meta-allE}, \text{erule-tac } x=n \text{ in meta-allE}, \text{clarsimp simp add: nth-append}$ )
  then show ?case
  proof (cases x)
    case (Broadcast x1) thus ?thesis
      using snoc * by (simp add: node-deliver-messages-append)
  next
  case (Deliver x2) thus ?thesis
    using snoc * [[simproc del: defined-all]]
    apply (clarsimp simp add: node-deliver-messages-def map-filter-def map-filter-append)
    apply (rename-tac m m1 m2)
    apply (case-tac m; clarsimp)
    apply (drule set-elem-nth, erule exE, erule conjE)
    apply (erule-tac x=length xs in meta-allE)
    apply (clarsimp simp add: nth-append)
    apply (metis causal-delivery insert-subset local-order-carrier-closed
      node-total-order-antisym)
    done
  qed
qed

corollary (in network-with-ops)
  shows hb.hb-consistent (node-deliver-messages (history i))
  by (metis hb-consistent-technical history-order-def less-one linorder-neqE-nat list-nth-split zero-order(3))

lemma (in network-with-ops) hb-consistent-prefix:
  assumes xs prefix of i
  shows hb.hb-consistent (node-deliver-messages xs)
using assms proof (clarsimp simp: prefix-of-node-history-def, rule-tac i=i in hb-consistent-technical)
  fix m n ys assume *: xs @ ys = history i m < length xs n < m
  consider (a) xs = [] | (b)  $\exists c. xs = [c]$  | (c) Suc 0 < length (xs)
    by (metis Suc-pred length-Suc-conv length-greater-0-conv zero-less-diff)
  thus xs ! n  $\sqsubset^i$  xs ! m
  proof (cases)
    case a thus ?thesis
      using * by clarsimp
  next
    case b thus ?thesis
      using assms * by clarsimp
  next
    case c thus ?thesis
      using assms * apply clarsimp
      apply(drule list-nth-split, assumption, clarsimp simp: c)
      apply (metis append.assoc append.simps(2) history-order-def)
      done
  qed
qed

locale network-with-constrained-ops = network-with-ops +
  fixes valid-msg :: 'c  $\Rightarrow$  ('a  $\times$  'b)  $\Rightarrow$  bool
  assumes broadcast-only-valid-msgs: pre @ [Broadcast m] prefix of i  $\implies$ 
     $\exists \text{state. apply-operations pre} = \text{Some state} \wedge \text{valid-msg state } m$ 

```

lemma (in *network-with-constrained-ops*) *broadcast-is-valid*:
assumes *Broadcast m ∈ set (history i)*
shows \exists *state. valid-msg state m*
using *assms broadcast-only-valid-msgs events-before-exist* **by** *blast*

lemma (in *network-with-constrained-ops*) *deliver-is-valid*:
assumes *Deliver m ∈ set (history i)*
shows $\exists j$ *pre state. pre @ [Broadcast m] prefix of j ∧ apply-operations pre = Some state ∧ valid-msg state m*
using *assms apply (clarsimp dest!: delivery-has-a-cause)*
using *broadcast-only-valid-msgs events-before-exist* **apply** *blast*
done

lemma (in *network-with-constrained-ops*) *deliver-in-prefix-is-valid*:
assumes *xs prefix of i*
and *Deliver m ∈ set xs*
shows \exists *state. valid-msg state m*
by (*meson assms network-with-constrained-ops.deliver-is-valid network-with-constrained-ops-axioms prefix-elem-to-carriers*)

4.4 Dummy network models

interpretation *trivial-node-histories: node-histories* $\lambda m. []$
by *standard auto*

interpretation *trivial-network: network* $\lambda m. []$ *id*
by *standard auto*

interpretation *trivial-causal-network: causal-network* $\lambda m. []$ *id*
by *standard auto*

interpretation *trivial-network-with-ops: network-with-ops* $\lambda m. []$ $(\lambda x y. \text{Some } y)$ *0*
by *standard auto*

interpretation *trivial-network-with-constrained-ops: network-with-constrained-ops* $\lambda m. []$ $(\lambda x y. \text{Some } y)$ *0* $\lambda x y. \text{True}$
by *standard (simp add: trivial-node-histories.prefix-of-node-history-def)*

end

5 Replicated Growable Array

The RGA, introduced by [10], is a replicated ordered list (sequence) datatype that supports *insert* and *delete* operations.

theory
Ordered-List

imports
Util

begin

type-synonym $(\text{'id}, \text{'v}) \text{elt} = \text{'id} \times \text{'v} \times \text{bool}$

5.1 Insert and delete operations

Insertion operations place the new element *after* an existing list element with a given ID, or at the head of the list if no ID is given. Deletion operations refer to the ID of the list element

that is to be deleted. However, it is not safe for a deletion operation to completely remove a list element, because then a concurrent insertion after the deleted element would not be able to locate the insertion position. Instead, the list retains so-called *tombstones*: a deletion operation merely sets a flag on a list element to mark it as deleted, but the element actually remains in the list. A separate garbage collection process can be used to eventually purge tombstones [10], but we do not consider tombstone removal here.

hide-const *insert*

fun *insert-body* :: ('id::{linorder}, 'v) elt list \Rightarrow ('id, 'v) elt \Rightarrow ('id, 'v) elt list **where**

insert-body [] e = [e] |
insert-body (x#xs) e =
 (if fst x < fst e then
 e#x#xs
 else x#*insert-body* xs e)

fun *insert* :: ('id::{linorder}, 'v) elt list \Rightarrow ('id, 'v) elt \Rightarrow 'id option \Rightarrow ('id, 'v) elt list option **where**

insert xs e None = Some (*insert-body* xs e) |
insert [] e (Some i) = None |
insert (x#xs) e (Some i) =
 (if fst x = i then
 Some (x#*insert-body* xs e)
 else
 insert xs e (Some i) \ggg (λt . Some (x#t)))

fun *delete* :: ('id::{linorder}, 'v) elt list \Rightarrow 'id \Rightarrow ('id, 'v) elt list option **where**

delete [] i = None |
delete ((i', v, flag)#xs) i =
 (if i' = i then
 Some ((i', v, True)#xs)
 else
 delete xs i \ggg (λt . Some ((i',v,flag)#t)))

5.2 Well-definedness of insert and delete

lemma *insert-no-failure*:

assumes $i = \text{None} \vee (\exists i'. i = \text{Some } i' \wedge i' \in \text{fst } \text{'set } xs)$

shows $\exists xs'. \text{insert } xs \ e \ i = \text{Some } xs'$

using *assms* **by**(*induction rule: insert.induct; force*)

lemma *insert-None-index-neq-None* [*dest*]:

assumes *insert* xs e i = None

shows $i \neq \text{None}$

using *assms* **by**(*cases i, auto*)

lemma *insert-Some-None-index-not-in* [*dest*]:

assumes *insert* xs e (Some i) = None

shows $i \notin \text{fst } \text{'set } xs$

using *assms* **by**(*induction xs, auto split: if-split-asm bind-splits*)

lemma *index-not-in-insert-Some-None* [*simp*]:

assumes $i \notin \text{fst } \text{'set } xs$

shows *insert* xs e (Some i) = None

using *assms* **by**(*induction xs, auto*)

lemma *delete-no-failure*:

assumes $i \in \text{fst } \text{'set } xs$

shows $\exists xs'. \text{delete } xs \ i = \text{Some } xs'$

using *assms* **by**(*induction xs; force*)

lemma *delete-None-index-not-in* [*dest*]:

assumes *delete xs i = None*

shows $i \notin \text{fst } ' \text{ set } xs$

using *assms* **by**(*induction xs, auto split: if-split-asm bind-splits simp add: fst-eq-Domain*)

lemma *index-not-in-delete-None* [*simp*]:

assumes $i \notin \text{fst } ' \text{ set } xs$

shows *delete xs i = None*

using *assms* **by**(*induction xs, auto*)

5.3 Preservation of element indices

lemma *insert-body-preserve-indices* [*simp*]:

shows $\text{fst } ' \text{ set } (\text{insert-body } xs \ e) = \text{fst } ' \text{ set } xs \cup \{\text{fst } e\}$

by(*induction xs, auto simp add: insert-commute*)

lemma *insert-preserve-indices*:

assumes $\exists ys. \text{insert } xs \ e \ i = \text{Some } ys$

shows $\text{fst } ' \text{ set } (\text{the } (\text{insert } xs \ e \ i)) = \text{fst } ' \text{ set } xs \cup \{\text{fst } e\}$

using *assms* **by**(*induction xs; cases i; auto simp add: insert-commute split: bind-splits*)

corollary *insert-preserve-indices'*:

assumes $\text{insert } xs \ e \ i = \text{Some } ys$

shows $\text{fst } ' \text{ set } (\text{the } (\text{insert } xs \ e \ i)) = \text{fst } ' \text{ set } xs \cup \{\text{fst } e\}$

using *assms* *insert-preserve-indices* **by** *blast*

lemma *delete-preserve-indices*:

assumes $\text{delete } xs \ i = \text{Some } ys$

shows $\text{fst } ' \text{ set } xs = \text{fst } ' \text{ set } ys$

using *assms* **by**(*induction xs arbitrary: ys, simp*) (*case-tac a; auto split: if-split-asm bind-splits*)

5.4 Commutativity of concurrent operations

lemma *insert-body-commutes*:

assumes $\text{fst } e1 \neq \text{fst } e2$

shows $\text{insert-body } (\text{insert-body } xs \ e1) \ e2 = \text{insert-body } (\text{insert-body } xs \ e2) \ e1$

using *assms* **by**(*induction xs, auto*)

lemma *insert-insert-body*:

assumes $\text{fst } e1 \neq \text{fst } e2$

and $i2 \neq \text{Some } (\text{fst } e1)$

shows $\text{insert } (\text{insert-body } xs \ e1) \ e2 \ i2 = \text{insert } xs \ e2 \ i2 \ggg (\lambda ys. \text{Some } (\text{insert-body } ys \ e1))$

using *assms* **by** (*induction xs; cases i2*) (*auto split: if-split-asm simp add: insert-body-commutes*)

lemma *insert-Nil-None*:

assumes $\text{fst } e1 \neq \text{fst } e2$

and $i \neq \text{fst } e2$

and $i2 \neq \text{Some } (\text{fst } e1)$

shows $\text{insert } [] \ e2 \ i2 \ggg (\lambda ys. \text{insert } ys \ e1 \ (\text{Some } i)) = \text{None}$

using *assms* **by** (*cases i2*) *clarsimp+*

lemma *insert-insert-body-commute*:

assumes $i \neq \text{fst } e1$

and $\text{fst } e1 \neq \text{fst } e2$

shows $\text{insert } (\text{insert-body } xs \ e1) \ e2 \ (\text{Some } i) =$

$\text{insert } xs \ e2 \ (\text{Some } i) \ggg (\lambda y. \text{Some } (\text{insert-body } y \ e1))$

using *assms* **by**(*induction xs, auto simp add: insert-body-commutes*)

lemma *insert-commutes*:

assumes $\text{fst } e1 \neq \text{fst } e2$

$i1 = \text{None} \vee i1 \neq \text{Some } (\text{fst } e2)$

$i2 = \text{None} \vee i2 \neq \text{Some } (\text{fst } e1)$

shows $\text{insert } xs \ e1 \ i1 \ggg (\lambda ys. \text{insert } ys \ e2 \ i2) =$

$\text{insert } xs \ e2 \ i2 \ggg (\lambda ys. \text{insert } ys \ e1 \ i1)$

using *assms* **proof**(*induction rule: insert.induct*)

fix *xs* **and** $e :: ('a, 'b) \text{elt}$

assume $i2 = \text{None} \vee i2 \neq \text{Some } (\text{fst } e)$ **and** $\text{fst } e \neq \text{fst } e2$

thus $\text{insert } xs \ e \ \text{None} \ggg (\lambda ys. \text{insert } ys \ e2 \ i2) = \text{insert } xs \ e2 \ i2 \ggg (\lambda ys. \text{insert } ys \ e \ \text{None})$

by(*auto simp add: insert-body-commutes intro: insert-insert-body*)

next

fix *i* **and** $e :: ('a, 'b) \text{elt}$

assume $\text{fst } e \neq \text{fst } e2$ **and** $i2 = \text{None} \vee i2 \neq \text{Some } (\text{fst } e)$ **and** $\text{Some } i = \text{None} \vee \text{Some } i \neq \text{Some } (\text{fst } e2)$

thus $\text{insert } [] \ e \ (\text{Some } i) \ggg (\lambda ys. \text{insert } ys \ e2 \ i2) = \text{insert } [] \ e2 \ i2 \ggg (\lambda ys. \text{insert } ys \ e \ (\text{Some } i))$

by (*auto intro: insert-Nil-None[symmetric]*)

next

fix *xs i* **and** $x \ e :: ('a, 'b) \text{elt}$

assume *IH*: $(\text{fst } x \neq i \implies$

$\text{fst } e \neq \text{fst } e2 \implies$

$\text{Some } i = \text{None} \vee \text{Some } i \neq \text{Some } (\text{fst } e2) \implies$

$i2 = \text{None} \vee i2 \neq \text{Some } (\text{fst } e) \implies$

$\text{insert } xs \ e \ (\text{Some } i) \ggg (\lambda ys. \text{insert } ys \ e2 \ i2) = \text{insert } xs \ e2 \ i2 \ggg (\lambda ys. \text{insert } ys \ e \ (\text{Some } i)))$

and $\text{fst } e \neq \text{fst } e2$

and $\text{Some } i = \text{None} \vee \text{Some } i \neq \text{Some } (\text{fst } e2)$

and $i2 = \text{None} \vee i2 \neq \text{Some } (\text{fst } e)$

thus $\text{insert } (x \# xs) \ e \ (\text{Some } i) \ggg (\lambda ys. \text{insert } ys \ e2 \ i2) = \text{insert } (x \# xs) \ e2 \ i2 \ggg (\lambda ys. \text{insert } ys \ e \ (\text{Some } i))$

apply –

apply(*erule disjE, clarsimp, simp, rule conjI*)

apply(*case-tac i2; force simp add: insert-body-commutes insert-insert-body-commute*)

apply(*case-tac i2; clarsimp cong: Option.bind-cong simp add: insert-insert-body split: bind-splits*)

apply *force*

done

qed

lemma *delete-commutes*:

shows $\text{delete } xs \ i1 \ggg (\lambda ys. \text{delete } ys \ i2) = \text{delete } xs \ i2 \ggg (\lambda ys. \text{delete } ys \ i1)$

by(*induction xs, auto split: bind-splits if-split-asm*)

lemma *insert-body-delete-commute*:

assumes $i2 \neq \text{fst } e$

shows $\text{delete } (\text{insert-body } xs \ e) \ i2 \ggg (\lambda t. \text{Some } (x\#t)) =$

$\text{delete } xs \ i2 \ggg (\lambda y. \text{Some } (x\#\text{insert-body } y \ e))$

using *assms* **by** (*induction xs arbitrary: x; cases e, auto split: bind-splits if-split-asm*)

lemma *insert-delete-commute*:

assumes $i2 \neq \text{fst } e$

shows $\text{insert } xs \ e \ i1 \ggg (\lambda ys. \text{delete } ys \ i2) = \text{delete } xs \ i2 \ggg (\lambda ys. \text{insert } ys \ e \ i1)$

using *assms* **by**(*induction xs; cases e; cases i1, auto split: bind-splits if-split-asm simp add: insert-body-delete-commute*)

5.5 Alternative definition of insert

fun *insert'* :: ('id::{linorder}, 'v) elt list \Rightarrow ('id, 'v) elt \Rightarrow 'id option \rightarrow ('id::{linorder}, 'v) elt list
where

```

insert' [] e None = Some [e] |
insert' [] e (Some i) = None |
insert' (x#xs) e None =
  (if fst x < fst e then
    Some (e#x#xs)
  else
    case insert' xs e None of
      None  $\Rightarrow$  None
    | Some t  $\Rightarrow$  Some (x#t)) |
insert' (x#xs) e (Some i) =
  (if fst x = i then
    case insert' xs e None of
      None  $\Rightarrow$  None
    | Some t  $\Rightarrow$  Some (x#t)
  else
    case insert' xs e (Some i) of
      None  $\Rightarrow$  None
    | Some t  $\Rightarrow$  Some (x#t))

```

lemma [*elim!*, *dest*]:

assumes *insert'* xs e None = None
shows False

using *assms* **by** (*induction* xs, *auto split: if-split-asm option.split-asm*)

lemma *insert-body-insert'*:

shows *insert'* xs e None = Some (*insert-body* xs e)

by (*induction* xs, *auto*)

lemma *insert-insert'*:

shows *insert* xs e i = *insert'* xs e i

by (*induction* xs; *cases* e; *cases* i, *auto split: option.split simp add: insert-body-insert'*)

lemma *insert-body-stop-iteration*:

assumes fst e > fst x

shows *insert-body* (x#xs) e = e#x#xs

using *assms* **by** *simp*

lemma *insert-body-contains-new-elem*:

shows $\exists p s. xs = p @ s \wedge \text{insert-body } xs \ e = p @ e \# s$

proof (*induction* xs)

case Nil **thus** ?*case* **by** *force*

next

case (Cons a xs)

then obtain p s **where** xs = p @ s \wedge *insert-body* xs e = p @ e # s **by** *force*

thus ?*case*

apply *clarsimp*

apply (*rule conjI*; *clarsimp*)

apply *force*

apply (*rule-tac* x=a # p **in** *exI*, *force*)

done

qed

lemma *insert-between-elements*:

assumes xs = pre@ref#suf

and *distinct* (*map fst xs*)
and $\bigwedge i'. i' \in \text{fst } \text{' set } xs \implies i' < \text{fst } e$
shows *insert xs e* (*Some* (*fst ref*)) = *Some* (*pre @ ref # e # suf*)
using *assms* **by** (*induction xs arbitrary: pre ref suf, force*) (*case-tac pre; case-tac suf; force*)

lemma *insert-position-element-technical*:

assumes $\forall x \in \text{set } as. a \neq \text{fst } x$
and *insert-body* (*cs @ ds*) *e* = *cs @ e # ds*
shows *insert* (*as @ (a, aa, b) # cs @ ds*) *e* (*Some a*) = *Some* (*as @ (a, aa, b) # cs @ e # ds*)
using *assms* **by** (*induction as arbitrary: cs ds; clarsimp*)

lemma *split-tuple-list-by-id*:

assumes (*a, b, c*) $\in \text{set } xs$
and *distinct* (*map fst xs*)
shows $\exists \text{pre suf}. xs = \text{pre} @ (a, b, c) \# \text{suf} \wedge (\forall y \in \text{set } \text{pre}. \text{fst } y \neq a)$
using *assms* **proof** (*induction xs, clarsimp*)
case (*Cons x xs*)
{ **assume** $x \neq (a, b, c)$
hence (*a, b, c*) $\in \text{set } xs$ *distinct* (*map fst xs*)
using *Cons.prem*s **by** *force*+
then obtain *pre suf* **where** $xs = \text{pre} @ (a, b, c) \# \text{suf} \wedge (\forall y \in \text{set } \text{pre}. \text{fst } y \neq a)$
using *Cons.IH* **by** *force*
hence *?case*
apply (*rule-tac x=x#pre in exI*)
using *Cons.prem*s(2) **by** *auto*
} **thus** *?case*
by *force*
qed

lemma *insert-preserves-order*:

assumes $i = \text{None} \vee (\exists i'. i = \text{Some } i' \wedge i' \in \text{fst } \text{' set } xs)$
and *distinct* (*map fst xs*)
shows $\exists \text{pre suf}. xs = \text{pre} @ \text{suf} \wedge \text{insert } xs \ e \ i = \text{Some} (\text{pre} @ e \# \text{suf})$
using *assms* **proof** –
{ **assume** $i = \text{None}$
hence *?thesis*
by *clarsimp* (*metis insert-body-contains-new-elem*)
} **moreover** {
assume $\exists i'. i = \text{Some } i' \wedge i' \in \text{fst } \text{' set } xs$
then obtain *j v b* **where** $i = \text{Some } j$ (*j, v, b*) $\in \text{set } xs$ **by** *force*
moreover then obtain *as bs* **where** $xs = \text{as} @ (j, v, b) \# bs \ \forall x \in \text{set } as. \text{fst } x \neq j$
using *assms* **by** (*metis split-tuple-list-by-id*)
moreover then obtain *cs ds* **where** *insert-body* *bs e* = *cs @ e # ds* *cs @ ds* = *bs*
by (*metis insert-body-contains-new-elem*)
ultimately have *?thesis*
by (*rule-tac x=as@(j,v,b)#cs in exI; clarsimp*)(*metis insert-position-element-technical*)
} **ultimately show** *?thesis*
using *assms* **by** *force*
qed
end

5.6 Network

theory

RGA

imports

Network

Ordered-List

begin

datatype ('id, 'v) operation =
 Insert ('id, 'v) elt 'id option |
 Delete 'id

fun interpret-ops :: ('id::linorder, 'v) operation \Rightarrow ('id, 'v) elt list \rightarrow ('id, 'v) elt list ($\langle - \rangle$ [0] 1000)
where

 interpret-ops (Insert e n) xs = insert xs e n |
 interpret-ops (Delete n) xs = delete xs n

definition element-ids :: ('id, 'v) elt list \Rightarrow 'id set **where**
 element-ids list \equiv set (map fst list)

definition valid-rga-msg :: ('id, 'v) elt list \Rightarrow 'id \times ('id::linorder, 'v) operation \Rightarrow bool **where**
 valid-rga-msg list msg \equiv case msg of

 (i, Insert e None) \Rightarrow fst e = i |
 (i, Insert e (Some pos)) \Rightarrow fst e = i \wedge pos \in element-ids list |
 (i, Delete pos) \Rightarrow pos \in element-ids list

locale rga = network-with-constrained-ops - interpret-ops [] valid-rga-msg

definition indices :: ('id \times ('id, 'v) operation) event list \Rightarrow 'id list **where**
 indices xs \equiv

 List.map-filter (λx . case x of Deliver (i, Insert e n) \Rightarrow Some (fst e) | - \Rightarrow None) xs

lemma indices-Nil [simp]:

shows indices [] = []

by(auto simp: indices-def map-filter-def)

lemma indices-append [simp]:

shows indices (xs@ys) = indices xs @ indices ys

by(auto simp: indices-def map-filter-def)

lemma indices-Broadcast-singleton [simp]:

shows indices [Broadcast b] = []

by(auto simp: indices-def map-filter-def)

lemma indices-Deliver-Insert [simp]:

shows indices [Deliver (i, Insert e n)] = [fst e]

by(auto simp: indices-def map-filter-def)

lemma indices-Deliver-Delete [simp]:

shows indices [Deliver (i, Delete n)] = []

by(auto simp: indices-def map-filter-def)

lemma (in rga) idx-in-elem-inserted [intro]:

assumes Deliver (i, Insert e n) \in set xs

shows fst e \in set (indices xs)

using assms **by**(induction xs, auto simp add: indices-def map-filter-def)

lemma (in rga) apply-ops-idx-elems:

assumes es prefix of i

and apply-operations es = Some xs

shows element-ids xs = set (indices es)

using assms **unfolding** element-ids-def

proof(induction es arbitrary: xs rule: rev-induct, clarsimp)

```

case (snoc x xs) thus ?case
proof (cases x, clarsimp, blast)
  case (Deliver e)
  moreover obtain a b where e = (a, b) by force
  ultimately show ?thesis
    using snoc assms apply (cases b; clarsimp split: bind-splits simp add: interp-msg-def)
    apply (metis Un-insert-right append.right-neutral insert-preserve-indices' list.set(1)
      option.sel prefix-of-appendD prod.sel(1) set-append)
    by (metis delete-preserve-indices prefix-of-appendD)
qed
qed

lemma (in rga) delete-does-not-change-element-ids:
  assumes es @ [Deliver (i, Delete n)] prefix of j
  and apply-operations es = Some xs1
  and apply-operations (es @ [Deliver (i, Delete n)]) = Some xs2
  shows element-ids xs1 = element-ids xs2
proof -
  have indices es = indices (es @ [Deliver (i, Delete n)])
    by simp
  then show ?thesis
    by (metis (no-types) assms prefix-of-appendD rga.apply-opers-idx-elems rga-axioms)
qed

lemma (in rga) someone-inserted-id:
  assumes es @ [Deliver (i, Insert (k, v, f) n)] prefix of j
  and apply-operations es = Some xs1
  and apply-operations (es @ [Deliver (i, Insert (k, v, f) n)]) = Some xs2
  and a ∈ element-ids xs2
  and a ≠ k
  shows a ∈ element-ids xs1
using assms apply-opers-idx-elems by auto

lemma (in rga) deliver-insert-exists:
  assumes es prefix of j
    and apply-operations es = Some xs
    and a ∈ element-ids xs
  shows ∃ i v f n. Deliver (i, Insert (a, v, f) n) ∈ set es
using assms unfolding element-ids-def
proof (induction es arbitrary: xs rule: rev-induct, clarsimp)
  case (snoc x xs ys) thus ?case
  proof (cases x)
    case (Broadcast e) thus ?thesis
      using snoc by (clarsimp, metis image-eqI prefix-of-appendD prod.sel(1))
  next
    case (Deliver e)
    moreover then obtain xs' where *: apply-operations xs = Some xs'
      using snoc by fastforce
    moreover obtain k v where **: e = (k, v) by force
    ultimately show ?thesis
      using assms snoc proof (cases v)
      case (Insert el -) thus ?thesis
        using snoc Deliver * **
        apply (cases el; cases fst el = a; clarsimp)
        apply (blast, metis (no-types, lifting) element-ids-def prefix-of-appendD set-map snoc.prem(2)
          snoc.prem(3) someone-inserted-id)
      done
  next

```

```

case (Delete -) thus ?thesis
  using snoc Deliver ** apply clarsimp
  apply(drule prefix-of-appendD, clarsimp simp add: bind-eq-Some-conv interp-msg-def)
  apply(metis delete-preserve-indices image-eqI prod.sel(1))
done
qed
qed
qed

```

```

lemma (in rga) insert-in-apply-set:
  assumes es @ [Deliver (i, Insert e (Some a))] prefix of j
  and Deliver (i', Insert e' n) ∈ set es
  and apply-operations es = Some s
  shows fst e' ∈ element-ids s
using assms apply-opers-idx-elems idx-in-elem-inserted prefix-of-appendD by blast

```

```

lemma (in rga) insert-msg-id:
  assumes Broadcast (i, Insert e n) ∈ set (history j)
  shows fst e = i
proof -
  obtain state where 1: valid-rga-msg state (i, Insert e n)
  using assms broadcast-is-valid by blast
  thus fst e = i
  by(clarsimp simp add: valid-rga-msg-def split: option.split-asm)
qed

```

```

lemma (in rga) allowed-insert:
  assumes Broadcast (i, Insert e n) ∈ set (history j)
  shows n = None ∨ (∃ i' e' n'. n = Some (fst e') ∧ Deliver (i', Insert e' n') ⊑j Broadcast (i, Insert e n))
proof -
  obtain pre where 1: pre @ [Broadcast (i, Insert e n)] prefix of j
  using assms events-before-exist by blast
  from this obtain state where 2: apply-operations pre = Some state and 3: valid-rga-msg state (i, Insert e n)
  using broadcast-only-valid-msgs by blast
  show n = None ∨ (∃ i' e' n'. n = Some (fst e') ∧ Deliver (i', Insert e' n') ⊑j Broadcast (i, Insert e n))
  proof(cases n)
    fix a
    assume 4: n = Some a
    hence a ∈ element-ids state and 5: fst e = i
    using 3 by(clarsimp simp add: valid-rga-msg-def)+
    from this have ∃ i' v' f' n'. Deliver (i', Insert (a, v', f') n') ∈ set pre
    using deliver-insert-exists 2 1 by blast
    thus n = None ∨ (∃ i' e' n'. n = Some (fst e') ∧ Deliver (i', Insert e' n') ⊑j Broadcast (i, Insert e n))
    using events-in-local-order 1 4 5 by(metis fst-conv)
  qed simp
qed

```

```

lemma (in rga) allowed-delete:
  assumes Broadcast (i, Delete x) ∈ set (history j)
  shows ∃ i' n' v b. Deliver (i', Insert (x, v, b) n') ⊑j Broadcast (i, Delete x)
proof -
  obtain pre where 1: pre @ [Broadcast (i, Delete x)] prefix of j
  using assms events-before-exist by blast
  from this obtain state where 2: apply-operations pre = Some state

```

and *valid-rga-msg state* (i , *Delete* x)
using *broadcast-only-valid-msgs* **by** *blast*
hence $x \in \text{element-ids state}$
using *apply-opers-idx-elems* **by** (*simp add: valid-rga-msg-def*)
hence $\exists i' v' f' n'. \text{Deliver } (i', \text{Insert } (x, v', f') n') \in \text{set pre}$
using *deliver-insert-exists 1 2* **by** *blast*
thus $\exists i' n' v b. \text{Deliver } (i', \text{Insert } (x, v, b) n') \sqsubset^j \text{Broadcast } (i, \text{Delete } x)$
using *events-in-local-order 1* **by** *blast*
qed

lemma (*in rga*) *insert-id-unique*:
assumes $\text{fst } e1 = \text{fst } e2$
and $\text{Broadcast } (i1, \text{Insert } e1 n1) \in \text{set } (\text{history } i)$
and $\text{Broadcast } (i2, \text{Insert } e2 n2) \in \text{set } (\text{history } j)$
shows $\text{Insert } e1 n1 = \text{Insert } e2 n2$
using *assms insert-msg-id msg-id-unique Pair-inject fst-conv* **by** *metis*

lemma (*in rga*) *allowed-delete-deliver*:
assumes $\text{Deliver } (i, \text{Delete } x) \in \text{set } (\text{history } j)$
shows $\exists i' n' v b. \text{Deliver } (i', \text{Insert } (x, v, b) n') \sqsubset^j \text{Deliver } (i, \text{Delete } x)$
using *assms* **by** (*meson allowed-delete bot-least causal-broadcast delivery-has-a-cause insert-subset*)

lemma (*in rga*) *allowed-delete-deliver-in-set*:
assumes (*es@[Deliver (i, Delete m)]*) *prefix of j*
shows $\exists i' n v b. \text{Deliver } (i', \text{Insert } (m, v, b) n) \in \text{set } es$
by(*metis (no-types, lifting) Un-insert-right insert-iff list.simps(15) assms*
local-order-prefix-closed-last rga.allowed-delete-deliver rga-axioms set-append subsetCE prefix-to-carriers)

lemma (*in rga*) *allowed-insert-deliver*:
assumes $\text{Deliver } (i, \text{Insert } e n) \in \text{set } (\text{history } j)$
shows $n = \text{None} \vee (\exists i' n' n'' v b. n = \text{Some } n' \wedge \text{Deliver } (i', \text{Insert } (n', v, b) n'') \sqsubset^j \text{Deliver } (i, \text{Insert } e n))$

proof –

obtain ja **where** $1: \text{Broadcast } (i, \text{Insert } e n) \in \text{set } (\text{history } ja)$
using *assms delivery-has-a-cause* **by** *blast*
show $n = \text{None} \vee (\exists i' n' n'' v b. n = \text{Some } n' \wedge \text{Deliver } (i', \text{Insert } (n', v, b) n'') \sqsubset^j \text{Deliver } (i, \text{Insert } e n))$
proof(*cases n*)
fix a
assume $3: n = \text{Some } a$
from this obtain $i' e' n'$ **where** $4: \text{Some } a = \text{Some } (\text{fst } e')$ **and**
 $2: \text{Deliver } (i', \text{Insert } e' n') \sqsubset^j a \text{Broadcast } (i, \text{Insert } e (\text{Some } a))$
using *allowed-insert 1* **by** *blast*
hence $\text{Deliver } (i', \text{Insert } e' n') \in \text{set } (\text{history } ja)$ **and** $\text{Broadcast } (i, \text{Insert } e (\text{Some } a)) \in \text{set } (\text{history } ja)$
using *local-order-carrier-closed* **by** *simp+*
from this obtain jaa **where** $\text{Broadcast } (i, \text{Insert } e (\text{Some } a)) \in \text{set } (\text{history } jaa)$
using *delivery-has-a-cause* **by** *simp*
have $\exists i' n' n'' v b. n = \text{Some } n' \wedge \text{Deliver } (i', \text{Insert } (n', v, b) n'') \sqsubset^j \text{Deliver } (i, \text{Insert } e n)$
using $2\ 3\ 4$ **by**(*metis assms causal-broadcast prod.collapse*)
thus $n = \text{None} \vee (\exists i' n' n'' v b. n = \text{Some } n' \wedge \text{Deliver } (i', \text{Insert } (n', v, b) n'') \sqsubset^j \text{Deliver } (i, \text{Insert } e n))$
by *auto*
qed *simp*
qed

lemma (*in rga*) *allowed-insert-deliver-in-set*:
assumes (*es@[Deliver (i, Insert e m)]*) *prefix of j*

shows $m = \text{None} \vee (\exists i' m' n v b. m = \text{Some } m' \wedge \text{Deliver } (i', \text{Insert } (m', v, b) n) \in \text{set } es)$
by(metis assms Un-insert-right insert-subset list.simps(15) set-append prefix-to-carriers
 allowed-insert-deliver local-order-prefix-closed-last)

lemma (in rga) *Insert-no-failure:*

assumes $es @ [\text{Deliver } (i, \text{Insert } e n)]$ prefix of j

and apply-operations $es = \text{Some } s$

shows $\exists ys. \text{insert } s e n = \text{Some } ys$

by(metis (no-types, lifting) element-ids-def allowed-insert-deliver-in-set assms fst-conv
 insert-in-apply-set insert-no-failure set-map)

lemma (in rga) *delete-no-failure:*

assumes $es @ [\text{Deliver } (i, \text{Delete } n)]$ prefix of j

and apply-operations $es = \text{Some } s$

shows $\exists ys. \text{delete } s n = \text{Some } ys$

proof –

obtain $i' na v b$ **where** $1: \text{Deliver } (i', \text{Insert } (n, v, b) na) \in \text{set } es$

using assms allowed-delete-deliver-in-set **by** blast

also have $\text{fst } (n, v, b) \in \text{set } (\text{indices } es)$

using assms idx-in-elem-inserted calculation **by** blast

from this assms **and** 1 **show** $\exists ys. \text{delete } s n = \text{Some } ys$

apply –

apply(rule delete-no-failure)

apply(metis apply-opers-idx-elems element-ids-def prefix-of-appendD prod.sel(1) set-map)

done

qed

lemma (in rga) *Insert-equal:*

assumes $\text{fst } e1 = \text{fst } e2$

and Broadcast $(i1, \text{Insert } e1 n1) \in \text{set } (\text{history } i)$

and Broadcast $(i2, \text{Insert } e2 n2) \in \text{set } (\text{history } j)$

shows $\text{Insert } e1 n1 = \text{Insert } e2 n2$

using insert-id-unique assms **by** simp

lemma (in rga) *same-insert:*

assumes $\text{fst } e1 = \text{fst } e2$

and xs prefix of i

and $(i1, \text{Insert } e1 n1) \in \text{set } (\text{node-deliver-messages } xs)$

and $(i2, \text{Insert } e2 n2) \in \text{set } (\text{node-deliver-messages } xs)$

shows $\text{Insert } e1 n1 = \text{Insert } e2 n2$

proof –

have $\text{Deliver } (i1, \text{Insert } e1 n1) \in \text{set } (\text{history } i)$

using assms **by**(auto simp add: node-deliver-messages-def prefix-msg-in-history)

from this **obtain** j **where** $1: \text{Broadcast } (i1, \text{Insert } e1 n1) \in \text{set } (\text{history } j)$

using delivery-has-a-cause **by** blast

have $\text{Deliver } (i2, \text{Insert } e2 n2) \in \text{set } (\text{history } i)$

using assms **by**(auto simp add: node-deliver-messages-def prefix-msg-in-history)

from this **obtain** k **where** $2: \text{Broadcast } (i2, \text{Insert } e2 n2) \in \text{set } (\text{history } k)$

using delivery-has-a-cause **by** blast

show $\text{Insert } e1 n1 = \text{Insert } e2 n2$

by(rule Insert-equal; force simp add: assms intro: 1 2)

qed

lemma (in rga) *insert-commute-assms:*

assumes $\{\text{Deliver } (i, \text{Insert } e n), \text{Deliver } (i', \text{Insert } e' n')\} \subseteq \text{set } (\text{history } j)$

and hb.concurrent $(i, \text{Insert } e n) (i', \text{Insert } e' n')$

shows $n = \text{None} \vee n \neq \text{Some } (\text{fst } e')$

using assms


```

apply(clarsimp simp: hb.concurrent-def)
apply(cases e')
apply clarsimp
apply(frule delivery-has-a-cause)
apply(frule delivery-has-a-cause, clarsimp)
apply(frule allowed-insert)
apply clarsimp
apply(metis Insert-equal delivery-has-a-cause fst-conv hb.intros(2) insert-subset
  local-order-carrier-closed insert-msg-id)
done

```

```

lemma subset-reorder:
  assumes  $\{a, b\} \subseteq c$ 
  shows  $\{b, a\} \subseteq c$ 
using assms by simp

```

```

lemma (in rga) Insert-Insert-concurrent:
  assumes  $\{Deliver\ (i, Insert\ e\ k),\ Deliver\ (i', Insert\ e'\ (Some\ m))\} \subseteq set\ (history\ j)$ 
  and hb.concurrent  $(i, Insert\ e\ k)\ (i', Insert\ e'\ (Some\ m))$ 
  shows  $fst\ e \neq m$ 
by(metis assms subset-reorder hb.concurrent-comm insert-commute-assms option.simps(3))

```

```

lemma (in rga) insert-valid-assms:
  assumes  $Deliver\ (i, Insert\ e\ n) \in set\ (history\ j)$ 
  shows  $n = None \vee n \neq Some\ (fst\ e)$ 
using assms by(meson allowed-insert-deliver hb.concurrent-def hb.less-asymp insert-subset
  local-order-carrier-closed rga.insert-commute-assms rga-axioms)

```

```

lemma (in rga) Insert-Delete-concurrent:
  assumes  $\{Deliver\ (i, Insert\ e\ n),\ Deliver\ (i', Delete\ n')\} \subseteq set\ (history\ j)$ 
  and hb.concurrent  $(i, Insert\ e\ n)\ (i', Delete\ n')$ 
  shows  $n' \neq fst\ e$ 
by (metis assms Insert-equal allowed-delete delivery-has-a-cause fst-conv hb.concurrent-def
  hb.intros(2) insert-subset local-order-carrier-closed rga.insert-msg-id rga-axioms)

```

```

lemma (in rga) concurrent-operations-commute:
  assumes xs prefix of i
  shows hb.concurrent-ops-commute (node-deliver-messages xs)

```

proof –

```

  have  $\bigwedge x\ y.\ \{x, y\} \subseteq set\ (node-deliver-messages\ xs) \implies hb.concurrent\ x\ y \implies interp-msg\ x \triangleright$ 
   $interp-msg\ y = interp-msg\ y \triangleright interp-msg\ x$ 

```

proof

```

  fix x y ii

```

```

  assume  $\{x, y\} \subseteq set\ (node-deliver-messages\ xs)$ 

```

```

  and C: hb.concurrent x y

```

```

  hence X: x ∈ set (node-deliver-messages xs) and Y: y ∈ set (node-deliver-messages xs)

```

```

  by auto

```

```

  obtain x1 x2 y1 y2 where  $1: x = (x1, x2)$  and  $2: y = (y1, y2)$ 

```

```

  by fastforce

```

```

  have  $(interp-msg\ (x1, x2) \triangleright interp-msg\ (y1, y2))\ ii = (interp-msg\ (y1, y2) \triangleright interp-msg\ (x1, x2))$ 

```

ii

```

  proof(cases x2; cases y2)

```

```

    fix ix1 ix2 iy1 iy2

```

```

    assume X2: x2 = Insert ix1 ix2 and Y2: y2 = Insert iy1 iy2

```

```

    show  $(interp-msg\ (x1, x2) \triangleright interp-msg\ (y1, y2))\ ii = (interp-msg\ (y1, y2) \triangleright interp-msg\ (x1,$ 

```

$x2))\ ii$

```

    proof(cases fst ix1 = fst iy1)

```

```

      assume  $fst\ ix1 = fst\ iy1$ 

```

```

hence Insert ix1 ix2 = Insert iy1 iy2
  apply(rule same-insert)
  using 1 2 X Y X2 Y2 assms apply auto
  done
hence ix1 = iy1 and ix2 = iy2
  by auto
from this and X2 Y2 show (interp-msg (x1, x2) ▷ interp-msg (y1, y2)) ii = (interp-msg (y1,
y2) ▷ interp-msg (x1, x2)) ii
  by(clarsimp simp add: kleisli-def interp-msg-def)
next
assume NEQ: fst ix1 ≠ fst iy1
have ix2 = None ∨ ix2 ≠ Some (fst iy1)
  apply(rule insert-commute-assms)
  using prefix-msg-in-history[OF assms] X Y X2 Y2 1 2
  apply(clarsimp, blast)
  using C 1 2 X2 Y2 apply blast
  done
also have iy2 = None ∨ iy2 ≠ Some (fst ix1)
  apply(rule insert-commute-assms)
  using prefix-msg-in-history[OF assms] X Y X2 Y2 1 2
  apply(clarsimp, blast)
  using 1 2 C X2 Y2 apply blast
  done
ultimately have insert ii ix1 ix2 ≍ (λx. insert x iy1 iy2) = insert ii iy1 iy2 ≍ (λx. insert x
ix1 ix2)
  using NEQ insert-commutes by blast
  thus (interp-msg (x1, x2) ▷ interp-msg (y1, y2)) ii = (interp-msg (y1, y2) ▷ interp-msg (x1,
x2)) ii
  by(clarsimp simp add: interp-msg-def X2 Y2 kleisli-def)
qed
next
fix ix1 ix2 yd
assume X2: x2 = Insert ix1 ix2 and Y2: y2 = Delete yd
have hb.concurrent (x1, Insert ix1 ix2) (y1, Delete yd)
  using C X2 Y2 1 2 by simp
also have {Deliver (x1, Insert ix1 ix2), Deliver (y1, Delete yd)} ⊆ set (history i)
  using prefix-msg-in-history assms X2 Y2 X Y 1 2 by blast
ultimately have yd ≠ fst ix1
  apply –
  apply(rule Insert-Delete-concurrent; force)
  done
hence insert ii ix1 ix2 ≍ (λx. delete x yd) = delete ii yd ≍ (λx. insert x ix1 ix2)
  by(rule insert-delete-commute)
thus (interp-msg (x1, x2) ▷ interp-msg (y1, y2)) ii = (interp-msg (y1, y2) ▷ interp-msg (x1,
x2)) ii
  by(clarsimp simp add: interp-msg-def kleisli-def X2 Y2)
next
fix xd iy1 iy2
assume X2: x2 = Delete xd and Y2: y2 = Insert iy1 iy2
have hb.concurrent (x1, Delete xd) (y1, Insert iy1 iy2)
  using C X2 Y2 1 2 by simp
also have {Deliver (x1, Delete xd), Deliver (y1, Insert iy1 iy2)} ⊆ set (history i)
  using prefix-msg-in-history assms X2 Y2 X Y 1 2 by blast
ultimately have xd ≠ fst iy1
  apply –
  apply(rule Insert-Delete-concurrent; force)
  done
hence delete ii xd ≍ (λx. insert x iy1 iy2) = insert ii iy1 iy2 ≍ (λx. delete x xd)

```

```

    by(rule insert-delete-commute[symmetric])
  thus (interp-msg (x1, x2) ▷ interp-msg (y1, y2)) ii = (interp-msg (y1, y2) ▷ interp-msg (x1,
x2)) ii
    by(clarsimp simp add: interp-msg-def kleisli-def X2 Y2)
next
  fix xd yd
  assume X2: x2 = Delete xd and Y2: y2 = Delete yd
  have delete ii xd ≫ (λx. delete x yd) = delete ii yd ≫ (λx. delete x xd)
    by(rule delete-commutes)
  thus (interp-msg (x1, x2) ▷ interp-msg (y1, y2)) ii = (interp-msg (y1, y2) ▷ interp-msg (x1,
x2)) ii
    by(clarsimp simp add: interp-msg-def kleisli-def X2 Y2)
qed
  thus (interp-msg x ▷ interp-msg y) ii = (interp-msg y ▷ interp-msg x) ii
    using 1 2 by auto
qed
  thus hb.concurrent-ops-commute (node-deliver-messages xs)
    by(auto simp add: hb.concurrent-ops-commute-def)
qed

```

corollary (in rga) *concurrent-operations-commute'*:
 shows hb.concurrent-ops-commute (node-deliver-messages (history i))
 by (meson concurrent-operations-commute append.right-neutral prefix-of-node-history-def)

lemma (in rga) *apply-operations-never-fails*:
 assumes xs prefix of i
 shows apply-operations xs ≠ None
 using assms **proof**(induction xs rule: rev-induct)
 show apply-operations [] ≠ None
 by clarsimp
next
 fix x xs
 assume 1: xs prefix of i ⇒ apply-operations xs ≠ None
 and 2: xs @ [x] prefix of i
 hence 3: xs prefix of i
 by auto
 show apply-operations (xs @ [x]) ≠ None
proof(cases x)
 fix b
 assume x = Broadcast b
 thus apply-operations (xs @ [x]) ≠ None
 using 1 3 by clarsimp
next
 fix d
 assume 4: x = Deliver d
 thus apply-operations (xs @ [x]) ≠ None
proof(cases d; clarify)
 fix a b
 assume 5: x = Deliver (a, b)
 show ∃ y. apply-operations (xs @ [Deliver (a, b)]) = Some y
proof(cases b; clarify)
 fix aa aaa ba x12
 assume 6: b = Insert (aa, aaa, ba) x12
 show ∃ y. apply-operations (xs @ [Deliver (a, Insert (aa, aaa, ba) x12)]) = Some y
 apply(clarsimp simp add: 1 interp-msg-def split!: bind-splits)
 apply(simp add: 1 3)
 apply(rule rga.Insert-no-failure, rule rga-axioms)
 using 4 5 6 2 apply force+

```

    done
  next
  fix x2
  assume 6: b = Delete x2
  show  $\exists y. \text{apply-operations } (xs @ [\text{Deliver } (a, \text{Delete } x2)]) = \text{Some } y$ 
    apply (clarsimp simp add: interp-msg-def split!: bind-splits)
    apply (simp add: 1 3)
    apply (rule delete-no-failure)
    using 4 5 6 2 apply force+
  done
qed
qed
qed
qed

lemma (in rga) apply-operations-never-fails':
  shows apply-operations (history i)  $\neq$  None
by (meson apply-operations-never-fails append.right-neutral prefix-of-node-history-def)

corollary (in rga) rga-convergence:
  assumes set (node-deliver-messages xs) = set (node-deliver-messages ys)
  and xs prefix of i
  and ys prefix of j
  shows apply-operations xs = apply-operations ys
using assms by (auto simp add: apply-operations-def intro: hb.convergence-ext
  concurrent-operations-commute node-deliver-messages-distinct hb-consistent-prefix)

```

5.7 Strong eventual consistency

context rga begin

```

sublocale sec: strong-eventual-consistency weak-hb hb interp-msg
   $\lambda ops. \exists xs i. xs \text{ prefix of } i \wedge \text{node-deliver-messages } xs = ops []$ 
proof (standard;clarsimp)
  fix xsa i
  assume xsa prefix of i
  thus hb.hb-consistent (node-deliver-messages xsa)
    by (auto simp add: hb-consistent-prefix)
next
  fix xsa i
  assume xsa prefix of i
  thus distinct (node-deliver-messages xsa)
    by (auto simp add: node-deliver-messages-distinct)
next
  fix xsa i
  assume xsa prefix of i
  thus hb.concurrent-ops-commute (node-deliver-messages xsa)
    by (auto simp add: concurrent-operations-commute)
next
  fix xs a b state xsa x
  assume hb.apply-operations xs [] = Some state
  and node-deliver-messages xsa = xs @ [(a, b)]
  and xsa prefix of x
  thus  $\exists y. \text{interp-msg } (a, b) \text{ state} = \text{Some } y$ 
    by (metis (no-types, lifting) apply-operations-def bind.bind-lunit not-None-eq
      hb.apply-operations-Snoc kleisli-def apply-operations-never-fails interp-msg-def)
next
  fix xs a b xsa x

```

```

assume node-deliver-messages  $xs = xs @ [(a, b)]$ 
and xs prefix of x
thus  $\exists xs. (\exists x. xs \text{ prefix of } x) \wedge \text{node-deliver-messages } xs = xs$ 
using drop-last-message by blast
qed

end

```

```

interpretation trivial-rga-implementation: rga  $\lambda x. []$ 
by(standard, auto simp add: trivial-node-histories.history-order-def
trivial-node-histories.prefix-of-node-history-def)

end

```

6 Increment-Decrement Counter

The Increment-Decrement Counter is perhaps the simplest CRDT, and a paradigmatic example of a replicated data structure with commutative operations.

```

theory
  Counter
imports
  Network
begin

datatype operation = Increment | Decrement

fun counter-op :: operation  $\Rightarrow$  int  $\rightarrow$  int where
  counter-op Increment  $x = \text{Some } (x + 1)$  |
  counter-op Decrement  $x = \text{Some } (x - 1)$ 

locale counter = network-with-ops - counter-op 0

lemma (in counter) counter-op  $x \triangleright \text{counter-op } y = \text{counter-op } y \triangleright \text{counter-op } x$ 
by(case-tac  $x$ ; case-tac  $y$ ; auto simp add: kleisli-def)

lemma (in counter) concurrent-operations-commute:
assumes xs prefix of i
shows hb.concurrent-ops-commute (node-deliver-messages xs)
using assms
apply(clarsimp simp: hb.concurrent-ops-commute-def)
apply(rename-tac  $a$   $b$   $x$   $y$ )
apply(case-tac  $b$ ; case-tac  $y$ ; force simp add: interp-msg-def kleisli-def)
done

corollary (in counter) counter-convergence:
assumes set (node-deliver-messages xs) = set (node-deliver-messages ys)
and xs prefix of i
and ys prefix of j
shows apply-operations xs = apply-operations ys
using assms by(auto simp add: apply-operations-def intro: hb.convergence-ext
concurrent-operations-commute node-deliver-messages-distinct hb-consistent-prefix)

context counter begin

sublocale sec: strong-eventual-consistency weak-hb hb interp-msg
  lops.  $\exists xs$   $i. xs \text{ prefix of } i \wedge \text{node-deliver-messages } xs = ops$  0
apply(standard;clarsimp simp add: hb-consistent-prefix drop-last-message)

```

```

    node-deliver-messages-distinct concurrent-operations-commute)
  apply(metis (full-types) interp-msg-def counter-op.elims)
  using drop-last-message apply blast
done

```

```

end
end

```

7 Observed-Remove Set

The ORSet is a well-known CRDT for implementing replicated sets, supporting two operations: the *insertion* and *deletion* of an arbitrary element in the shared set.

theory

ORSet

imports

Network

begin

datatype (*'id*, *'a*) *operation* = *Add 'id 'a* | *Rem 'id set 'a*

type-synonym (*'id*, *'a*) *state* = *'a* \Rightarrow *'id set*

definition *op-elem* :: (*'id*, *'a*) *operation* \Rightarrow *'a* **where**

op-elem oper \equiv *case oper of Add i e* \Rightarrow *e* | *Rem is e* \Rightarrow *e*

definition *interpret-op* :: (*'id*, *'a*) *operation* \Rightarrow (*'id*, *'a*) *state* \rightarrow (*'id*, *'a*) *state* ($\langle _ \rangle [0] 1000$) **where**

interpret-op oper state \equiv

let before = *state (op-elem oper)*;

after = *case oper of Add i e* \Rightarrow *before* \cup $\{i\}$ | *Rem is e* \Rightarrow *before* $-$ *is*

in Some (state ((op-elem oper) := after))

definition *valid-behaviours* :: (*'id*, *'a*) *state* \Rightarrow *'id* \times (*'id*, *'a*) *operation* \Rightarrow *bool* **where**

valid-behaviours state msg \equiv

case msg of

(i, Add j e) \Rightarrow *i = j* |

(i, Rem is e) \Rightarrow *is = state e*

locale *orset* = *network-with-constrained-ops - interpret-op* $\lambda x.$ $\{ \}$ *valid-behaviours*

lemma (**in** *orset*) *add-add-commute*:

shows $\langle \text{Add } i1 \ e1 \rangle \triangleright \langle \text{Add } i2 \ e2 \rangle = \langle \text{Add } i2 \ e2 \rangle \triangleright \langle \text{Add } i1 \ e1 \rangle$

by(*auto simp add: interpret-op-def op-elem-def kleisli-def, fastforce*)

lemma (**in** *orset*) *add-rem-commute*:

assumes *i* \notin *is*

shows $\langle \text{Add } i \ e1 \rangle \triangleright \langle \text{Rem } is \ e2 \rangle = \langle \text{Rem } is \ e2 \rangle \triangleright \langle \text{Add } i \ e1 \rangle$

using *assms* **by**(*auto simp add: interpret-op-def kleisli-def op-elem-def, fastforce*)

lemma (**in** *orset*) *apply-operations-never-fails*:

assumes *xs* *prefix of i*

shows *apply-operations xs* \neq *None*

using *assms* **proof**(*induction xs* *rule: rev-induct, clarsimp*)

case (*snoc x xs*) **thus** *?case*

proof (*cases x*)

case (*Broadcast e*) **thus** *?thesis*

using *snoc* **by** *force*

next

```

    case (Deliver e) thus ?thesis
      using snoc by (clarsimp, metis interpret-op-def interp-msg-def bind.bind-lunit prefix-of-appendD)
qed
qed

```

```

lemma (in orset) add-id-valid:
  assumes xs prefix of j
    and Deliver (i1, Add i2 e) ∈ set xs
  shows i1 = i2

```

```

proof -
  have ∃ s. valid-behaviours s (i1, Add i2 e)
    using assms deliver-in-prefix-is-valid by blast
  thus ?thesis
    by (simp add: valid-behaviours-def)
qed

```

```

definition (in orset) added-ids :: ('id × ('id, 'b) operation) event list ⇒ 'b ⇒ 'id list where
  added-ids es p ≡ List.map-filter (λx. case x of Deliver (i, Add j e) ⇒ if e = p then Some j else None
  | - ⇒ None) es

```

```

lemma (in orset) [simp]:
  shows added-ids [] e = []
  by (auto simp: added-ids-def map-filter-def)

```

```

lemma (in orset) [simp]:
  shows added-ids (xs @ ys) e = added-ids xs e @ added-ids ys e
  by (auto simp: added-ids-def map-filter-append)

```

```

lemma (in orset) added-ids-Broadcast-collapse [simp]:
  shows added-ids ([Broadcast e]) e' = []
  by (auto simp: added-ids-def map-filter-append map-filter-def)

```

```

lemma (in orset) added-ids-Deliver-Rem-collapse [simp]:
  shows added-ids ([Deliver (i, Rem is e)]) e' = []
  by (auto simp: added-ids-def map-filter-append map-filter-def)

```

```

lemma (in orset) added-ids-Deliver-Add-diff-collapse [simp]:
  shows e ≠ e' ⇒ added-ids ([Deliver (i, Add j e)]) e' = []
  by (auto simp: added-ids-def map-filter-append map-filter-def)

```

```

lemma (in orset) added-ids-Deliver-Add-same-collapse [simp]:
  shows added-ids ([Deliver (i, Add j e)]) e = [j]
  by (auto simp: added-ids-def map-filter-append map-filter-def)

```

```

lemma (in orset) added-id-not-in-set:
  assumes i1 ∉ set (added-ids [Deliver (i, Add i2 e)] e)
  shows i1 ≠ i2
  using assms by simp

```

```

lemma (in orset) apply-operations-added-ids:
  assumes es prefix of j
    and apply-operations es = Some f
  shows f x ⊆ set (added-ids es x)
using assms proof (induct es arbitrary: f rule: rev-induct, force)
  case (snoc x xs) thus ?case
  proof (cases x, force)
    case (Deliver e)
    moreover obtain a b where e = (a, b) by force

```

```

ultimately show ?thesis
  using snoc by(case-tac b; clarsimp simp: interp-msg-def split: bind-splits,
    force split: if-split-asm simp add: op-elem-def interpret-op-def)
qed
qed

lemma (in orset) Deliver-added-ids:
  assumes xs prefix of j
  and i ∈ set (added-ids xs e)
  shows Deliver (i, Add i e) ∈ set xs
using assms proof (induct xs rule: rev-induct, clarsimp)
  case (snoc x xs) thus ?case
  proof (cases x, force)
    case (Deliver e')
    moreover obtain a b where e' = (a, b) by force
    ultimately show ?thesis
      using snoc apply (case-tac b; clarsimp)
      apply (metis added-ids-Deliver-Add-diff-collapse added-ids-Deliver-Add-same-collapse
        empty-iff list.set(1) set-ConsD add-id-valid in-set-conv-decomp prefix-of-appendD)
      apply force
    done
  qed
qed

lemma (in orset) Broadcast-Deliver-prefix-closed:
  assumes xs @ [Broadcast (r, Rem ix e)] prefix of j
  and i ∈ ix
  shows Deliver (i, Add i e) ∈ set xs
proof -
  obtain y where apply-operations xs = Some y
  using assms broadcast-only-valid-msgs by blast
  moreover hence ix = y e
  by (metis (mono-tags, lifting) assms(1) broadcast-only-valid-msgs operation.case(2) option.simps(1)
    valid-behaviours-def case-prodD)
  ultimately show ?thesis
    using assms Deliver-added-ids apply-operations-added-ids by blast
qed

lemma (in orset) Broadcast-Deliver-prefix-closed2:
  assumes xs prefix of j
  and Broadcast (r, Rem ix e) ∈ set xs
  and i ∈ ix
  shows Deliver (i, Add i e) ∈ set xs
using assms Broadcast-Deliver-prefix-closed by (induction xs rule: rev-induct; force)

lemma (in orset) concurrent-add-remove-independent-technical:
  assumes i ∈ is
  and xs prefix of j
  and (i, Add i e) ∈ set (node-deliver-messages xs) and (ir, Rem is e) ∈ set (node-deliver-messages
xs)
  shows hb (i, Add i e) (ir, Rem is e)
proof -
  obtain pre k where pre@[Broadcast (ir, Rem is e)] prefix of k
  using assms delivery-has-a-cause events-before-exist prefix-msg-in-history by blast
  moreover hence Deliver (i, Add i e) ∈ set pre
  using Broadcast-Deliver-prefix-closed assms(1) by auto
  ultimately show ?thesis
    using hb.intros(2) events-in-local-order by blast

```


qed

lemma (in orset) *Deliver-Add-same-id-same-message*:

assumes $Deliver (i, Add\ i\ e1) \in set\ (history\ j)$ **and** $Deliver (i, Add\ i\ e2) \in set\ (history\ j)$
shows $e1 = e2$

proof –

obtain $pre1\ pre2\ k1\ k2$ **where** $*$: $pre1@[Broadcast\ (i, Add\ i\ e1)]$ prefix of $k1$ $pre2@[Broadcast\ (i, Add\ i\ e2)]$ prefix of $k2$

using *assms delivery-has-a-cause events-before-exist* **by** *meson*

moreover hence $Broadcast\ (i, Add\ i\ e1) \in set\ (history\ k1)$ $Broadcast\ (i, Add\ i\ e2) \in set\ (history\ k2)$

using *node-histories.prefix-to-carriers node-histories-axioms* **by** *force+*

ultimately show *?thesis*

using *msg-id-unique* **by** *fastforce*

qed

lemma (in orset) *ids-imply-messages-same*:

assumes $i \in is$

and xs prefix of j

and $(i, Add\ i\ e1) \in set\ (node-deliver-messages\ xs)$ **and** $(ir, Rem\ is\ e2) \in set\ (node-deliver-messages\ xs)$

shows $e1 = e2$

proof –

obtain $pre\ k$ **where** $pre@[Broadcast\ (ir, Rem\ is\ e2)]$ prefix of k

using *assms delivery-has-a-cause events-before-exist prefix-msg-in-history* **by** *blast*

moreover hence $Deliver\ (i, Add\ i\ e2) \in set\ pre$

using *Broadcast-Deliver-prefix-closed assms(1)* **by** *blast*

moreover have $Deliver\ (i, Add\ i\ e1) \in set\ (history\ j)$

using *assms(2) assms(3) prefix-msg-in-history* **by** *blast*

ultimately show *?thesis*

by (*metis fst-conv msg-id-unique network.delivery-has-a-cause network-axioms operation.inject(1) prefix-elem-to-carriers prefix-of-appendD prod.inject*)

qed

corollary (in orset) *concurrent-add-remove-independent*:

assumes $\neg hb\ (i, Add\ i\ e1)\ (ir, Rem\ is\ e2)$ **and** $\neg hb\ (ir, Rem\ is\ e2)\ (i, Add\ i\ e1)$

and xs prefix of j

and $(i, Add\ i\ e1) \in set\ (node-deliver-messages\ xs)$ **and** $(ir, Rem\ is\ e2) \in set\ (node-deliver-messages\ xs)$

shows $i \notin is$

using *assms ids-imply-messages-same concurrent-add-remove-independent-technical* **by** *fastforce*

lemma (in orset) *rem-rem-commute*:

shows $\langle Rem\ i1\ e1 \rangle \triangleright \langle Rem\ i2\ e2 \rangle = \langle Rem\ i2\ e2 \rangle \triangleright \langle Rem\ i1\ e1 \rangle$

by(*unfold interpret-op-def op-elem-def kleisli-def, fastforce*)

lemma (in orset) *concurrent-operations-commute*:

assumes xs prefix of i

shows $hb.concurrent-ops-commute\ (node-deliver-messages\ xs)$

proof –

{ **fix** $a\ b\ x\ y$

assume $(a, b) \in set\ (node-deliver-messages\ xs)$

$(x, y) \in set\ (node-deliver-messages\ xs)$

$hb.concurrent\ (a, b)\ (x, y)$

hence $interp-msg\ (a, b) \triangleright interp-msg\ (x, y) = interp-msg\ (x, y) \triangleright interp-msg\ (a, b)$

apply(*unfold interp-msg-def, case-tac b; case-tac y; simp add: add-add-commute rem-rem-commute hb.concurrent-def*)

apply (*metis add-id-valid add-rem-commute assms concurrent-add-remove-independent hb.concurrentD1*)

```

hb.concurrentD2 prefix-contains-msg)+
  done
} thus ?thesis
by(fastforce simp: hb.concurrent-ops-commute-def)
qed

```

theorem (in *orset*) *convergence*:

```

assumes set (node-deliver-messages xs) = set (node-deliver-messages ys)
and xs prefix of i and ys prefix of j
shows apply-operations xs = apply-operations ys
using assms by(auto simp add: apply-operations-def intro: hb.convergence-ext concurrent-operations-commute
node-deliver-messages-distinct hb-consistent-prefix)

```

context *orset* **begin**

```

sublocale sec: strong-eventual-consistency weak-hb hb interp-msg
λops.∃ xs i. xs prefix of i ∧ node-deliver-messages xs = ops λx.{ }
apply(standard; clarsimp simp add: hb-consistent-prefix node-deliver-messages-distinct
concurrent-operations-commute)
apply(metis (no-types, lifting) apply-operations-def bind.bind-lunit not-None-eq
hb.apply-operations-Snoc kleisli-def apply-operations-never-fails interp-msg-def)
using drop-last-message apply blast
done

```

```

end
end

```

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