

A framework for establishing Strong Eventual Consistency for Conflict-free Replicated Data types

Victor B. F. Gomes, Martin Kleppmann, Dominic P. Mulligan,
Alastair R. Beresford

August 16, 2018

Abstract

In this work, we focus on the correctness of Conflict-free Replicated Data Types (CRDTs), a class of algorithm that provides strong eventual consistency guarantees for replicated data. We develop a modular and reusable framework for verifying the correctness of CRDT algorithms. We avoid correctness issues that have dogged previous mechanised proofs in this area by including a network model in our formalisation, and proving that our theorems hold in all possible network behaviours. Our axiomatic network model is a standard abstraction that accurately reflects the behaviour of real-world computer networks. Moreover, we identify an abstract convergence theorem, a property of order relations, which provides a formal definition of strong eventual consistency. We then obtain the first machine-checked correctness theorems for three concrete CRDTs: the Replicated Growable Array, the Observed-Remove Set, and an Increment-Decrement Counter.

Contents

1	Introduction	2
2	Technical Lemmas	2
2.1	Kleisli arrow composition	3
2.2	Lemmas about sets	3
2.3	Lemmas about list	3
3	Strong Eventual Consistency	6
3.1	Concurrent operations	6
3.2	Happens-before consistency	7
3.3	Apply operations	9
3.4	Concurrent operations commute	9
3.5	Abstract convergence theorem	10
3.6	Convergence and progress	11
4	Axiomatic network models	12
4.1	Node histories	12
4.2	Asynchronous broadcast networks	15
4.3	Causal networks	16
4.4	Dummy network models	22
5	Replicated Growable Array	22
5.1	Insert and delete operations	22
5.2	Well-definedness of insert and delete	23
5.3	Preservation of element indices	24

5.4	Commutativity of concurrent operations	24
5.5	Alternative definition of insert	25
5.6	Network	27
5.7	Strong eventual consistency	36
6	Increment-Decrement Counter	37
7	Observed-Remove Set	38

1 Introduction

Strong eventual consistency (SEC) is a model that strikes a compromise between strong and eventual consistency [12]. Informally, it guarantees that whenever two nodes have received the same set of messages—possibly in a different order—their view of the shared state is identical, and any conflicting concurrent updates must be merged automatically. Large-scale deployments of SEC algorithms include datacentre-based applications using the Riak distributed database [3], and collaborative editing applications such as Google Docs [5]. Unlike strong consistency models, it is possible to implement SEC in decentralised settings without any central server or leader, and it allows local execution at each node to proceed without waiting for communication with other nodes. However, algorithms for achieving decentralised SEC are currently poorly understood: several such algorithms, published in peer-reviewed venues, were subsequently shown to violate their supposed guarantees [6, 7, 9]. Informal reasoning has repeatedly produced plausible-looking but incorrect algorithms, and there have even been examples of mechanised formal proofs of SEC algorithm correctness later being shown to be flawed. These mechanised proofs failed because, in formalising the algorithm, they made false assumptions about the execution environment.

In this work we use the Isabelle/HOL proof assistant [13] to create a framework for reliably reasoning about the correctness of a particular class of decentralised replication algorithms. We do this by formalising not only the replication algorithms, but also the network in which they execute, allowing us to prove that the algorithm’s assumptions hold in all possible network behaviours. We model the network using the axioms of *asynchronous unreliable causal broadcast*, a well-understood abstraction that is commonly implemented by network protocols, and which can run on almost any computer network, including large-scale networks that delay, reorder, or drop messages, and in which nodes may fail.

We then use this framework to produce machine-checked proofs of correctness for three Conflict-Free Replicated Data Types (CRDTs), a class of replication algorithms that ensure strong eventual consistency [11, 12]. To our knowledge, this is the first machine-checked verification of SEC algorithms that explicitly models the network and reasons about all possible network behaviours. The framework is modular and reusable, making it easy to formulate proofs for new algorithms. We provide the first mechanised proofs of the Replicated Growable Array, the operation-based Observed-Remove Set, and the operation-based counter CRDT.

2 Technical Lemmas

This section contains a list of helper definitions and lemmas about sets, lists and the option monad.

theory

Util

imports

Main

HOL-Library.Monad-Syntax

begin

2.1 Kleisli arrow composition

definition *kleisli* :: ('b ⇒ 'b option) ⇒ ('b ⇒ 'b option) ⇒ ('b ⇒ 'b option) (**infixr** ▷ 65) **where**
f ▷ g ≡ λx. (f x ≫= (λy. g y))

lemma *kleisli-comm-cong*:

assumes x ▷ y = y ▷ x
shows z ▷ x ▷ y = z ▷ y ▷ x
using *assms* **by**(*clarsimp simp add: kleisli-def*)

lemma *kleisli-assoc*:

shows (z ▷ x) ▷ y = z ▷ (x ▷ y)
by(*auto simp add: kleisli-def*)

2.2 Lemmas about sets

lemma *distinct-set-notin* [*dest*]:

assumes *distinct* (x#xs)
shows x ∉ set xs
using *assms* **by**(*induction xs, auto*)

lemma *set-membership-equality-technicalD* [*dest*]:

assumes {x} ∪ (set xs) = {y} ∪ (set ys)
shows x = y ∨ y ∈ set xs
using *assms* **by**(*induction xs, auto*)

lemma *set-equality-technical*:

assumes {x} ∪ (set xs) = {y} ∪ (set ys)
and x ∉ set xs
and y ∉ set ys
and y ∈ set xs
shows {x} ∪ (set xs - {y}) = set ys
using *assms* **by** (*induction xs*) *auto*

lemma *set-elem-nth*:

assumes x ∈ set xs
shows ∃ m. m < length xs ∧ xs ! m = x
using *assms* **by**(*induction xs, simp*) (*meson in-set-conv-nth*)

2.3 Lemmas about list

lemma *list-nil-or-snoc*:

shows xs = [] ∨ (∃ y ys. xs = ys@[y])
by (*induction xs, auto*)

lemma *suffix-eq-distinct-list*:

assumes *distinct* xs
and ys@suf1 = xs
and ys@suf2 = xs
shows suf1 = suf2
using *assms* **by**(*induction xs arbitrary: suf1 suf2 rule: rev-induct, simp*) (*metis append-eq-append-conv*)

lemma *pre-suf-eq-distinct-list*:

assumes *distinct* xs
and ys ≠ []
and pre1@ys@suf1 = xs

```

    and pre2@ys@suf2 = xs
    shows pre1 = pre2  $\wedge$  suf1 = suf2
using assms
apply(induction xs arbitrary: pre1 pre2 ys, simp)
apply(case-tac pre1; case-tac pre2; clarify)
apply(metis suffix-eq-distinct-list append-Nil)
apply(metis Un-iff append-eq-Cons-conv distinct.simps(2) list.set-intros(1) set-append suffix-eq-distinct-list)
apply(metis Un-iff append-eq-Cons-conv distinct.simps(2) list.set-intros(1) set-append suffix-eq-distinct-list)
apply(metis distinct.simps(2) hd-append2 list.sel(1) list.sel(3) list.simps(3) tl-append2)
done

```

```

lemma list-head-unaffected:
  assumes hd (x @ [y, z]) = v
    shows hd (x @ [y ]) = v
  using assms by (metis hd-append list.sel(1))

```

```

lemma list-head-butlast:
  assumes hd xs = v
    and length xs > 1
    shows hd (butlast xs) = v
  using assms by (metis hd-conv-nth length-butlast length-greater-0-conv less-trans nth-butlast zero-less-diff
zero-less-one)

```

```

lemma list-head-length-one:
  assumes hd xs = x
    and length xs = 1
  shows xs = [x]
  using assms by (metis One-nat-def Suc-length-conv hd-Cons-tl length-0-conv list.sel(3))

```

```

lemma list-two-at-end:
  assumes length xs > 1
  shows  $\exists xs' x y. xs = xs' @ [x, y]$ 
  using assms
  apply(induction xs rule: rev-induct, simp)
  apply(case-tac length xs = 1, simp)
  apply(metis append-self-conv2 length-0-conv length-Suc-conv)
  apply(rule-tac x=butlast xs in exI, rule-tac x=last xs in exI, simp)
done

```

```

lemma list-nth-split-technical:
  assumes m < length cs
    and cs  $\neq$  []
    shows  $\exists xs ys. cs = xs@(cs!m)\#ys$ 
  using assms
  apply(induction m arbitrary: cs)
  apply(meson in-set-conv-decomp nth-mem)
  apply(metis in-set-conv-decomp length-list-update set-swap set-update-memI)
done

```

```

lemma list-nth-split:
  assumes m < length cs
    and n < m
    and 1 < length cs
  shows  $\exists xs ys zs. cs = xs@(cs!n)\#ys@(cs!m)\#zs$ 
using assms proof(induction n arbitrary: cs m)
case 0 thus ?case
  apply(case-tac cs; clarsimp)
  apply(rule-tac x=[] in exI, clarsimp)

```

```

    apply(rule list-nth-split-technical, simp, force)
  done
next
case (Suc n)
thus ?case
proof (cases cs)
  case Nil
  then show ?thesis
    using Suc.prem by auto
next
case (Cons a as)
hence  $m-1 < \text{length } as$   $n < m-1$ 
  using Suc by force+
then obtain  $xs\ ys\ zs$  where  $as = xs @ as ! n \# ys @ as ! (m-1) \# zs$ 
  using Suc by force
thus ?thesis
  apply(rule-tac  $x=a\#xs$  in exI)
  using Suc Cons apply force
done
qed
qed

```

lemma *list-split-two-elems*:

```

  assumes distinct cs
    and  $x \in \text{set } cs$ 
    and  $y \in \text{set } cs$ 
    and  $x \neq y$ 
  shows  $\exists pre\ mid\ suf. cs = pre @ x \# mid @ y \# suf \vee cs = pre @ y \# mid @ x \# suf$ 
proof -
  obtain  $xi\ yi$  where  $*$ :  $xi < \text{length } cs \wedge x = cs ! xi$   $yi < \text{length } cs \wedge y = cs ! yi$   $xi \neq yi$ 
    using set-elem-nth linorder-neqE-nat assms by metis
  thus ?thesis
    by (metis list-nth-split One-nat-def less-Suc-eq linorder-neqE-nat not-less-zero)
qed

```

lemma *split-list-unique-prefix*:

```

  assumes  $x \in \text{set } xs$ 
  shows  $\exists pre\ suf. xs = pre @ x \# suf \wedge (\forall y \in \text{set } pre. x \neq y)$ 
using assms proof(induction xs)
  case Nil thus ?case by clarsimp
next
case (Cons y ys)
then show ?case
  proof (cases  $y=x$ )
    case True
    then show ?thesis by force
  next
  case False
  then obtain  $pre\ suf$  where  $ys = pre @ x \# suf \wedge (\forall y \in \text{set } pre. x \neq y)$ 
    using assms Cons by auto
  thus ?thesis
    using split-list-first by force
qed
qed

```

lemma *map-filter-append*:

```

  shows  $List.map-filter\ P\ (xs @ ys) = List.map-filter\ P\ xs @ List.map-filter\ P\ ys$ 
  by(auto simp add: List.map-filter-def)

```

end

3 Strong Eventual Consistency

In this section we formalise the notion of strong eventual consistency. We do not make any assumptions about networks or data structures; instead, we use an abstract model of operations that may be reordered, and we reason about the properties that those operations must satisfy. We then provide concrete implementations of that abstract model in later sections.

theory

Convergence

imports

Util

begin

The *happens-before* relation, as introduced by [8], captures causal dependencies between operations. It can be defined in terms of sending and receiving messages on a network. However, for now, we keep it abstract, our only restriction on the happens-before relation is that it must be a *strict partial order*, that is, it must be irreflexive and transitive, which implies that it is also antisymmetric. We describe the state of a node using an abstract type variable. To model state changes, we assume the existence of an *interpretation* function *interp* which lifts an operation into a *state transformer*—a function that either maps an old state to a new state, or fails.

```
locale happens-before = preorder hb-weak hb
for hb-weak :: 'a ⇒ 'a ⇒ bool (infix ≤ 50)
and hb :: 'a ⇒ 'a ⇒ bool (infix < 50) +
fixes interp :: 'a ⇒ 'b → 'b (<-) [0] 1000
begin
```

3.1 Concurrent operations

We say that two operations x and y are *concurrent*, written $x \parallel y$, whenever one does not happen before the other: $\neg(x < y)$ and $\neg(y < x)$.

definition *concurrent* :: 'a ⇒ 'a ⇒ bool (**infix** ∥ 50) **where**
 $s1 \parallel s2 \equiv \neg(s1 < s2) \wedge \neg(s2 < s1)$

lemma *concurrentI* [intro!]: $\neg(s1 < s2) \implies \neg(s2 < s1) \implies s1 \parallel s2$
by (auto simp: concurrent-def)

lemma *concurrentD1* [dest]: $s1 \parallel s2 \implies \neg(s1 < s2)$
by (auto simp: concurrent-def)

lemma *concurrentD2* [dest]: $s1 \parallel s2 \implies \neg(s2 < s1)$
by (auto simp: concurrent-def)

lemma *concurrent-refl* [intro!, simp]: $s \parallel s$
by (auto simp: concurrent-def)

lemma *concurrent-comm*: $s1 \parallel s2 \longleftrightarrow s2 \parallel s1$
by (auto simp: concurrent-def)

definition *concurrent-set* :: 'a ⇒ 'a list ⇒ bool **where**
 $\text{concurrent-set } x \ xs \equiv \forall y \in \text{set } xs. x \parallel y$

lemma *concurrent-set-empty* [simp, intro!]:
 $\text{concurrent-set } x \ []$

by (auto simp: concurrent-set-def)

lemma *concurrent-set-ConsE* [elim!]:
assumes *concurrent-set a (x#xs)*
and *concurrent-set a xs \implies concurrent x a \implies G*
shows *G*
using *assms* by (auto simp: concurrent-set-def)

lemma *concurrent-set-ConsI* [intro!]:
concurrent-set a xs \implies concurrent a x \implies concurrent-set a (x#xs)
by (auto simp: concurrent-set-def)

lemma *concurrent-set-appendI* [intro!]:
concurrent-set a xs \implies concurrent-set a ys \implies concurrent-set a (xs@ys)
by (auto simp: concurrent-set-def)

lemma *concurrent-set-Cons-Snoc* [simp]:
concurrent-set a (xs@[x]) = concurrent-set a (x#xs)
by (auto simp: concurrent-set-def)

3.2 Happens-before consistency

The purpose of the happens-before relation is to require that some operations must be applied in a particular order, while allowing concurrent operations to be reordered with respect to each other. We assume that each node applies operations in some sequential order (a standard assumption for distributed algorithms), and so we can model the execution history of a node as a list of operations.

inductive *hb-consistent* :: 'a list \Rightarrow bool **where**
 [intro!]: *hb-consistent []* |
 [intro!]: \llbracket *hb-consistent xs*; $\forall x \in \text{set } xs. \neg y \prec x$ $\rrbracket \implies$ *hb-consistent (xs @ [y])*

As a result, whenever two operations x and y appear in a hb-consistent list, and $x \prec y$, then x must appear before y in the list. However, if $x \parallel y$, the operations can appear in the list in either order.

lemma $(x \prec y \vee \text{concurrent } x \ y) = (\neg y \prec x)$
using *less-asym* **by** *blast*

lemma *consistentI* [intro!]:
assumes *hb-consistent (xs @ ys)*
and $\forall x \in \text{set } (xs @ ys). \neg z \prec x$
shows *hb-consistent (xs @ ys @ [z])*
using *assms hb-consistent.intros append-assoc* **by** *metis*

inductive-cases *hb-consistent-elim* [elim]:
hb-consistent []
hb-consistent (xs@[y])
hb-consistent (xs@ys)
hb-consistent (xs@ys@[z])

inductive-cases *hb-consistent-elim-gen*:
hb-consistent zs

lemma *hb-consistent-append-D1* [dest]:
assumes *hb-consistent (xs @ ys)*
shows *hb-consistent xs*
using *assms* **by**(*induction ys arbitrary: xs rule: List.rev-induct*) *auto*

lemma *hb-consistent-append-D2* [*dest*]:
assumes *hb-consistent* (*xs @ ys*)
shows *hb-consistent ys*
using *assms* **by**(*induction ys arbitrary: xs rule: List.rev-induct*) *fastforce*+

lemma *hb-consistent-append-elim-ConsD* [*elim*]:
assumes *hb-consistent* (*y#ys*)
shows *hb-consistent ys*
using *assms hb-consistent-append-D2* **by**(*metis append-Cons append-Nil*)

lemma *hb-consistent-remove1* [*intro*]:
assumes *hb-consistent xs*
shows *hb-consistent* (*remove1 x xs*)
using *assms* **by** (*induction rule: hb-consistent.induct*) (*auto simp: remove1-append*)

lemma *hb-consistent-singleton* [*intro!*]:
shows *hb-consistent* [*x*]
using *hb-consistent.intros* **by** *fastforce*

lemma *hb-consistent-prefix-suffix-exists*:
assumes *hb-consistent ys*
hb-consistent (*xs @ [x]*)
 $\{x\} \cup \text{set } xs = \text{set } ys$
distinct (*x#xs*)
distinct ys
shows $\exists \text{prefix suffix. } ys = \text{prefix} @ x \# \text{suffix} \wedge \text{concurrent-set } x \text{ suffix}$
using *assms* **proof** (*induction arbitrary: xs rule: hb-consistent.induct, simp*)
fix *xs y ys*
assume *IH*: ($\bigwedge xs. \text{hb-consistent } (xs @ [x]) \implies$
 $\{x\} \cup \text{set } xs = \text{set } ys \implies$
 $\text{distinct } (x \# xs) \implies \text{distinct } ys \implies$
 $\exists \text{prefix suffix. } ys = \text{prefix} @ x \# \text{suffix} \wedge \text{concurrent-set } x \text{ suffix}$)
assume *assms*: *hb-consistent ys* $\forall x \in \text{set } ys. \neg \text{hb } y x$
hb-consistent (*xs @ [x]*)
 $\{x\} \cup \text{set } xs = \text{set } (ys @ [y])$
distinct (*x # xs*) *distinct* (*ys @ [y]*)
hence $x = y \vee y \in \text{set } xs$
using *assms* **by** *auto*
moreover {
assume $x = y$
hence $\exists \text{prefix suffix. } ys @ [y] = \text{prefix} @ x \# \text{suffix} \wedge \text{concurrent-set } x \text{ suffix}$
by *force*
}
moreover {
assume *y-in-xs*: $y \in \text{set } xs$
hence $\{x\} \cup (\text{set } xs - \{y\}) = \text{set } ys$
using *assms* **by** (*auto intro: set-equality-technical*)
hence *remove-y-in-xs*: $\{x\} \cup \text{set } (\text{remove1 } y \text{ } xs) = \text{set } ys$
using *assms* **by** *auto*
moreover **have** *hb-consistent* ($(\text{remove1 } y \text{ } xs) @ [x]$)
using *assms hb-consistent-remove1* **by** *force*
moreover **have** *distinct* ($x \# (\text{remove1 } y \text{ } xs)$)
using *assms* **by** *simp*
moreover **have** *distinct ys*
using *assms* **by** *simp*
ultimately obtain *prefix suffix* **where** *ys-split*: $ys = \text{prefix} @ x \# \text{suffix} \wedge \text{concurrent-set } x \text{ suffix}$
using *IH* **by** *force*
moreover {


```

have concurrent x y
  using assms y-in-xs remove-y-in-xs concurrent-def by blast
hence concurrent-set x (suffix@[y])
  using ys-split by clarsimp
}
ultimately have  $\exists$  prefix suffix. ys @ [y] = prefix @ x # suffix  $\wedge$  concurrent-set x suffix
by force
}
ultimately show  $\exists$  prefix suffix. ys @ [y] = prefix @ x # suffix  $\wedge$  concurrent-set x suffix
by auto
qed

```

```

lemma hb-consistent-append [intro!]:
  assumes hb-consistent suffix
    hb-consistent prefix
     $\bigwedge s p. s \in \text{set suffix} \implies p \in \text{set prefix} \implies \neg s \prec p$ 
  shows hb-consistent (prefix @ suffix)
using assms by (induction rule: hb-consistent.induct) force+

```

```

lemma hb-consistent-append-porder:
  assumes hb-consistent (xs @ ys)
    x  $\in$  set xs
    y  $\in$  set ys
  shows  $\neg y \prec x$ 
using assms by (induction ys arbitrary: xs rule: rev-induct) force+

```

3.3 Apply operations

We can now define a function *apply-operations* that composes an arbitrary list of operations into a state transformer. We first map *interp* across the list to obtain a state transformer for each operation, and then collectively compose them using the Kleisli arrow composition combinator.

```

definition apply-operations :: 'a list  $\Rightarrow$  'b  $\rightarrow$  'b where
  apply-operations es  $\equiv$  foldl ( $\triangleright$ ) Some (map interp es)

```

```

lemma apply-operations-empty [simp]: apply-operations [] s = Some s
by(auto simp: apply-operations-def)

```

```

lemma apply-operations-Snoc [simp]:
  apply-operations (xs@[x]) = (apply-operations xs)  $\triangleright$   $\langle$ x $\rangle$ 
by(auto simp add: apply-operations-def kleisli-def)

```

3.4 Concurrent operations commute

We say that two operations *x* and *y* *commute* whenever $\langle x \rangle \triangleright \langle y \rangle = \langle y \rangle \triangleright \langle x \rangle$, i.e. when we can swap the order of the composition of their interpretations without changing the resulting state transformer. For our purposes, requiring that this property holds for *all* pairs of operations is too strong. Rather, the commutation property is only required to hold for operations that are concurrent.

```

definition concurrent-ops-commute :: 'a list  $\Rightarrow$  bool where
  concurrent-ops-commute xs  $\equiv$ 
   $\forall x y. \{x, y\} \subseteq \text{set xs} \longrightarrow \text{concurrent } x y \longrightarrow \langle x \rangle \triangleright \langle y \rangle = \langle y \rangle \triangleright \langle x \rangle$ 

```

```

lemma concurrent-ops-commute-empty [intro!]: concurrent-ops-commute []
by(auto simp: concurrent-ops-commute-def)

```

```

lemma concurrent-ops-commute-singleton [intro!]: concurrent-ops-commute [x]

```

by(*auto simp: concurrent-ops-commute-def*)

lemma *concurrent-ops-commute-appendD* [*dest*]:
assumes *concurrent-ops-commute* (*xs@ys*)
shows *concurrent-ops-commute* *xs*
using *assms* **by** (*auto simp: concurrent-ops-commute-def*)

lemma *concurrent-ops-commute-rearrange*:
concurrent-ops-commute (*xs@x#ys*) = *concurrent-ops-commute* (*xs@ys@[x]*)
by (*clarsimp simp: concurrent-ops-commute-def*)

lemma *concurrent-ops-commute-concurrent-set*:
assumes *concurrent-ops-commute* (*prefix@suffix@[x]*)
concurrent-set *x suffix*
distinct (*prefix @ x # suffix*)
shows *apply-operations* (*prefix @ suffix @ [x]*) = *apply-operations* (*prefix @ x # suffix*)
using *assms* **proof**(*induction suffix arbitrary: rule: rev-induct, force*)
fix *a xs*
assume *IH: concurrent-ops-commute* (*prefix @ xs @ [x]*) \implies
concurrent-set *x xs* \implies *distinct* (*prefix @ x # xs*) \implies
apply-operations (*prefix @ xs @ [x]*) = *apply-operations* (*prefix @ x # xs*)
assume *assms: concurrent-ops-commute* (*prefix @ (xs @ [a]) @ [x]*)
concurrent-set *x (xs @ [a])* *distinct* (*prefix @ x # xs @ [a]*)
hence *ac-comm: <a> ▷ <x> = <x> ▷ <a>*
by (*clarsimp simp: concurrent-ops-commute-def*) *blast*
have *copc: concurrent-ops-commute* (*prefix @ xs @ [x]*)
using *assms* **by** (*clarsimp simp: concurrent-ops-commute-def*) *blast*
have *apply-operations* ((*prefix @ x # xs*) @ [*a*]) = (*apply-operations* (*prefix @ x # xs*)) ▷ <*a*>
by (*simp del: append-assoc*)
also have ... = (*apply-operations* (*prefix @ xs @ [x]*)) ▷ <*a*>
using *IH assms copc* **by** *auto*
also have ... = ((*apply-operations* (*prefix @ xs*)) ▷ <*x*>) ▷ <*a*>
by (*simp add: append-assoc[symmetric] del: append-assoc*)
also have ... = (*apply-operations* (*prefix @ xs*)) ▷ (<*a*> ▷ <*x*>)
using *ac-comm kleisli-comm-cong kleisli-assoc* **by** *simp*
finally show *apply-operations* (*prefix @ (xs @ [a]) @ [x]*) = *apply-operations* (*prefix @ x # xs @ [a]*)
by (*metis Cons-eq-appendI append-assoc apply-operations-Snoc kleisli-assoc*)
qed

3.5 Abstract convergence theorem

We can now state and prove our main theorem, *convergence*. This theorem states that two hb-consistent lists of distinct operations, which are permutations of each other and in which concurrent operations commute, have the same interpretation.

theorem *convergence*:
assumes *set xs = set ys*
concurrent-ops-commute *xs*
concurrent-ops-commute *ys*
distinct *xs*
distinct *ys*
hb-consistent *xs*
hb-consistent *ys*
shows *apply-operations* *xs* = *apply-operations* *ys*
using *assms* **proof**(*induction xs arbitrary: ys rule: rev-induct, simp*)
case *assms: (snoc x xs)*
then obtain *prefix suffix* **where** *ys-split: ys = prefix @ x # suffix* \wedge *concurrent-set* *x suffix*
using *hb-consistent-prefix-suffix-exists* **by** *fastforce*

```

moreover hence *: distinct (prefix @ suffix) hb-consistent xs
  using assms by auto
moreover {
  have hb-consistent prefix hb-consistent suffix
    using ys-split assms hb-consistent-append-D2 hb-consistent-append-elim-ConsD by blast+
  hence hb-consistent (prefix @ suffix)
    by (metis assms(8) hb-consistent-append hb-consistent-append-porder list.set-intros(2) ys-split)
}
moreover have **: concurrent-ops-commute (prefix @ suffix @ [x])
  using assms ys-split by (clarsimp simp: concurrent-ops-commute-def)
moreover hence concurrent-ops-commute (prefix @ suffix)
  by (force simp del: append-assoc simp add: append-assoc[symmetric])
ultimately have apply-operations xs = apply-operations (prefix@suffix)
  using assms by simp (metis Diff-insert-absorb Un-iff * concurrent-ops-commute-appendD set-append)
moreover have apply-operations (prefix@suffix @ [x]) = apply-operations (prefix@x # suffix)
  using ys-split assms ** concurrent-ops-commute-concurrent-set by force
ultimately show ?case
  using ys-split by (force simp: append-assoc[symmetric] simp del: append-assoc)
qed

```

corollary *convergence-ext:*

```

assumes set xs = set ys
  concurrent-ops-commute xs
  concurrent-ops-commute ys
  distinct xs
  distinct ys
  hb-consistent xs
  hb-consistent ys
shows apply-operations xs s = apply-operations ys s
using convergence assms by metis
end

```

3.6 Convergence and progress

Besides convergence, another required property of SEC is *progress*: if a valid operation was issued on one node, then applying that operation on other nodes must also succeed—that is, the execution must not become stuck in an error state. Although the type signature of the interpretation function allows operations to fail, we need to prove that in all *hb-consistent* network behaviours such failure never actually occurs. We capture the combined requirements in the *strong-eventual-consistency* locale, which extends *happens-before*.

```

locale strong-eventual-consistency = happens-before +
  fixes op-history :: 'a list ⇒ bool
    and initial-state :: 'b
  assumes causality: op-history xs ⇒ hb-consistent xs
  assumes distinctness: op-history xs ⇒ distinct xs
  assumes commutativity: op-history xs ⇒ concurrent-ops-commute xs
  assumes no-failure: op-history(xs@[x]) ⇒ apply-operations xs initial-state = Some state ⇒ ⟨x⟩
state ≠ None
  assumes trunc-history: op-history(xs@[x]) ⇒ op-history xs
begin

```

theorem *sec-convergence:*

```

assumes set xs = set ys
  op-history xs
  op-history ys
shows apply-operations xs = apply-operations ys
by (meson assms convergence causality commutativity distinctness)

```

```

theorem sec-progress:
  assumes op-history xs
  shows apply-operations xs initial-state ≠ None
using assms proof(induction xs rule: rev-induct, simp)
  case (snoc x xs)
  have apply-operations xs initial-state ≠ None
    using snoc.IH snoc.prem1 trunc-history kleisli-def bind-def by blast
  moreover have apply-operations (xs @ [x]) = apply-operations xs ▷ ⟨x⟩
    by simp
  ultimately show ?case
    using no-failure snoc.prem1 by (clarsimp simp add: kleisli-def split: bind-splits)
qed

end
end

```

4 Axiomatic network models

In this section we develop a formal definition of an *asynchronous unreliable causal broadcast network*. We choose this model because it satisfies the causal delivery requirements of many operation-based CRDTs [1, 2]. Moreover, it is suitable for use in decentralised settings, as motivated in the introduction, since it does not require waiting for communication with a central server or a quorum of nodes.

```

theory
  Network
imports
  Convergence
begin

```

4.1 Node histories

We model a distributed system as an unbounded number of communicating nodes. We assume nothing about the communication pattern of nodes—we assume only that each node is uniquely identified by a natural number, and that the flow of execution at each node consists of a finite, totally ordered sequence of execution steps (events). We call that sequence of events at node i the *history* of that node. For convenience, we assume that every event or execution step is unique within a node’s history.

```

locale node-histories =
  fixes history :: nat ⇒ 'evt list
  assumes histories-distinct [intro!, simp]: distinct (history i)

```

```

lemma (in node-histories) history-finite:
  shows finite (set (history i))
by auto

```

```

definition (in node-histories) history-order :: 'evt ⇒ nat ⇒ 'evt ⇒ bool (-/ □-/ - [50,1000,50]50)
where
  x □i z ≡ ∃ xs ys zs. xs@x#ys@z#zs = history i

```

```

lemma (in node-histories) node-total-order-trans:
  assumes e1 □i e2
  and e2 □i e3
  shows e1 □i e3

```

proof –

obtain $xs1\ xs2\ ys1\ ys2\ zs1\ zs2$ **where** $*$: $xs1\ @\ e1\ \#\ ys1\ @\ e2\ \#\ zs1 = history\ i$
 $xs2\ @\ e2\ \#\ ys2\ @\ e3\ \#\ zs2 = history\ i$
using *history-order-def* *assms* **by** *auto*
hence $xs1\ @\ e1\ \#\ ys1 = xs2\ \wedge\ zs1 = ys2\ @\ e3\ \#\ zs2$
by(*rule-tac* $xs=history\ i$ **and** $ys=[e2]$ **in** *pre-suf-eq-distinct-list*) *auto*
thus *?thesis*
by(*clarsimp* *simp*: *history-order-def*) (*metis* $*(2)$ *append.assoc* *append-Cons*)
qed

lemma (**in** *node-histories*) *local-order-carrier-closed*:

assumes $e1\ \sqsubset^i\ e2$
shows $\{e1, e2\} \subseteq set\ (history\ i)$
using *assms* **by** (*clarsimp* *simp* *add*: *history-order-def*)
(*metis* *in-set-conv-decomp* *Un-iff* *Un-subset-iff* *insert-subset* *list.simps(15)*
set-append *set-subset-Cons*)**+**

lemma (**in** *node-histories*) *node-total-order-irrefl*:

shows $\neg (e\ \sqsubset^i\ e)$
by(*clarsimp* *simp* *add*: *history-order-def*)
(*metis* *Un-iff* *histories-distinct* *distinct-append* *distinct-set-notin*
list.set-intros(1) *set-append*)

lemma (**in** *node-histories*) *node-total-order-antisym*:

assumes $e1\ \sqsubset^i\ e2$
and $e2\ \sqsubset^i\ e1$
shows *False*
using *assms* *node-total-order-irrefl* *node-total-order-trans* **by** *blast*

lemma (**in** *node-histories*) *node-order-is-total*:

assumes $e1 \in set\ (history\ i)$
and $e2 \in set\ (history\ i)$
and $e1 \neq e2$
shows $e1\ \sqsubset^i\ e2 \vee e2\ \sqsubset^i\ e1$
using *assms* **unfolding** *history-order-def* **by**(*metis* *list-split-two-elems* *histories-distinct*)

definition (**in** *node-histories*) *prefix-of-node-history* :: $'evt\ list \Rightarrow nat \Rightarrow bool$ (**infix** *prefix of 50*) **where**
 $xs\ prefix\ of\ i \equiv \exists\ ys. xs@ys = history\ i$

lemma (**in** *node-histories*) *carriers-head-lt*:

assumes $y\ \#\ ys = history\ i$
shows $\neg (x\ \sqsubset^i\ y)$
using *assms*
apply(*clarsimp* *simp* *add*: *history-order-def*)
apply(*rename-tac* $xs1\ ys1\ zs1$)
apply (*subgoal-tac* $xs1\ @\ x\ \#\ ys1 = [] \wedge zs1 = ys$)
apply *clarsimp*
apply (*rule-tac* $xs=history\ i$ **and** $ys=[y]$ **in** *pre-suf-eq-distinct-list*)
apply *auto*
done

lemma (**in** *node-histories*) *prefix-of-ConsD* [*dest*]:

assumes $x\ \#\ xs\ prefix\ of\ i$
shows $[x]\ prefix\ of\ i$
using *assms* **by**(*auto* *simp*: *prefix-of-node-history-def*)

lemma (**in** *node-histories*) *prefix-of-appendD* [*dest*]:

assumes $xs\ @\ ys\ prefix\ of\ i$

shows xs prefix of i
using *assms* **by**(*auto simp: prefix-of-node-history-def*)

lemma (*in node-histories*) *prefix-distinct*:
assumes xs prefix of i
shows *distinct* xs
using *assms* **by**(*clarsimp simp: prefix-of-node-history-def*) (*metis histories-distinct distinct-append*)

lemma (*in node-histories*) *prefix-to-carriers* [*intro*]:
assumes xs prefix of i
shows $set\ xs \subseteq set\ (history\ i)$
using *assms* **by**(*clarsimp simp: prefix-of-node-history-def*) (*metis Un-iff set-append*)

lemma (*in node-histories*) *prefix-elem-to-carriers*:
assumes xs prefix of i
and $x \in set\ xs$
shows $x \in set\ (history\ i)$
using *assms* **by**(*clarsimp simp: prefix-of-node-history-def*) (*metis Un-iff set-append*)

lemma (*in node-histories*) *local-order-prefix-closed*:
assumes $x \sqsubset^i y$
and xs prefix of i
and $y \in set\ xs$
shows $x \in set\ xs$

proof –
obtain ys **where** $xs @ ys = history\ i$
using *assms* *prefix-of-node-history-def* **by** *blast*
moreover obtain $as\ bs\ cs$ **where** $as @ x \# bs @ y \# cs = history\ i$
using *assms* *history-order-def* **by** *blast*
moreover obtain $pre\ suf$ **where** $*$: $xs = pre @ y \# suf$
using *assms* *split-list* **by** *fastforce*
ultimately have $pre = as @ x \# bs \wedge suf @ ys = cs$
by (*rule-tac xs=history i and ys=[y] in pre-suf-eq-distinct-list*) *auto*
thus *?thesis*
using *assms* $*$ **by** *clarsimp*

qed

lemma (*in node-histories*) *local-order-prefix-closed-last*:
assumes $x \sqsubset^i y$
and $xs@[y]$ prefix of i
shows $x \in set\ xs$

proof –
have $x \in set\ (xs @ [y])$
using *assms* **by** (*force dest: local-order-prefix-closed*)
thus *?thesis*
using *assms* **by**(*force simp add: node-total-order-irrefl prefix-to-carriers*)

qed

lemma (*in node-histories*) *events-before-exist*:
assumes $x \in set\ (history\ i)$
shows $\exists pre. pre @ [x]$ prefix of i

proof –
have $\exists idx. idx < length\ (history\ i) \wedge (history\ i) ! idx = x$
using *assms* **by**(*simp add: set-elem-nth*)
thus *?thesis*
by(*metis append-take-drop-id take-Suc-conv-app-nth prefix-of-node-history-def*)

qed

lemma (in *node-histories*) *events-in-local-order*:
assumes $pre @ [e2]$ *prefix of i*
and $e1 \in set\ pre$
shows $e1 \sqsubset^i e2$
using *assms split-list unfolding history-order-def prefix-of-node-history-def* **by** *fastforce*

4.2 Asynchronous broadcast networks

We define a new locale *network* containing three axioms that define how broadcast and deliver events may interact, with these axioms defining the properties of our network model.

datatype *'msg event*
 $= Broadcast\ 'msg$
 $| Deliver\ 'msg$

locale *network* = *node-histories history* **for** $history :: nat \Rightarrow 'msg\ event\ list +$
fixes $msg-id :: 'msg \Rightarrow 'msgid$

assumes *delivery-has-a-cause*: $\llbracket Deliver\ m \in set\ (history\ i) \rrbracket \Longrightarrow$
 $\exists j. Broadcast\ m \in set\ (history\ j)$
and *deliver-locally*: $\llbracket Broadcast\ m \in set\ (history\ i) \rrbracket \Longrightarrow$
 $Broadcast\ m \sqsubset^i Deliver\ m$
and *msg-id-unique*: $\llbracket Broadcast\ m1 \in set\ (history\ i);$
 $Broadcast\ m2 \in set\ (history\ j);$
 $msg-id\ m1 = msg-id\ m2 \rrbracket \Longrightarrow i = j \wedge m1 = m2$

The axioms can be understood as follows:

delivery-has-a-cause: If some message m was delivered at some node, then there exists some node on which m was broadcast. With this axiom, we assert that messages are not created “out of thin air” by the network itself, and that the only source of messages are the nodes.

deliver-locally: If a node broadcasts some message m , then the same node must subsequently also deliver m to itself. Since m does not actually travel over the network, this local delivery is always possible, even if the network is interrupted. Local delivery may seem redundant, since the effect of the delivery could also be implemented by the broadcast event itself; however, it is standard practice in the description of broadcast protocols that the sender of a message also sends it to itself, since this property simplifies the definition of algorithms built on top of the broadcast abstraction [4].

msg-id-unique: We do not assume that the message type *'msg* has any particular structure; we only assume the existence of a function $msg-id :: 'msg \Rightarrow 'msgid$ that maps every message to some globally unique identifier of type *'msgid*. We assert this uniqueness by stating that if $m1$ and $m2$ are any two messages broadcast by any two nodes, and their *msg-ids* are the same, then they were in fact broadcast by the same node and the two messages are identical. In practice, these globally unique IDs can be implemented using unique node identifiers, sequence numbers or timestamps.

lemma (in *network*) *broadcast-before-delivery*:
assumes $Deliver\ m \in set\ (history\ i)$
shows $\exists j. Broadcast\ m \sqsubset^j Deliver\ m$
using *assms deliver-locally delivery-has-a-cause* **by** *blast*

lemma (in *network*) *broadcasts-unique*:
assumes $i \neq j$
and $Broadcast\ m \in set\ (history\ i)$
shows $Broadcast\ m \notin set\ (history\ j)$

using *assms msg-id-unique* **by** *blast*

Based on the well-known definition by [8], we say that $m1 \prec m2$ if any of the following is true:

1. $m1$ and $m2$ were broadcast by the same node, and $m1$ was broadcast before $m2$.
2. The node that broadcast $m2$ had delivered $m1$ before it broadcast $m2$.
3. There exists some operation $m3$ such that $m1 \prec m3$ and $m3 \prec m2$.

inductive (**in** *network*) *hb* :: '*msg* \Rightarrow '*msg* \Rightarrow *bool* **where**
hb-broadcast: $\llbracket \text{Broadcast } m1 \sqsubset^i \text{Broadcast } m2 \rrbracket \Longrightarrow \text{hb } m1 \ m2 \mid$
hb-deliver: $\llbracket \text{Deliver } m1 \sqsubset^i \text{Broadcast } m2 \rrbracket \Longrightarrow \text{hb } m1 \ m2 \mid$
hb-trans: $\llbracket \text{hb } m1 \ m2; \text{hb } m2 \ m3 \rrbracket \Longrightarrow \text{hb } m1 \ m3$

inductive-cases (**in** *network*) *hb-elim*: *hb* *x* *y*

definition (**in** *network*) *weak-hb* :: '*msg* \Rightarrow '*msg* \Rightarrow *bool* **where**
weak-hb *m1* *m2* $\equiv \text{hb } m1 \ m2 \vee m1 = m2$

locale *causal-network* = *network* +
assumes *causal-delivery*: *Deliver* *m2* \in *set* (*history* *j*) $\Longrightarrow \text{hb } m1 \ m2 \Longrightarrow \text{Deliver } m1 \sqsubset^j \text{Deliver } m2$

lemma (**in** *causal-network*) *causal-broadcast*:
assumes *Deliver* *m2* \in *set* (*history* *j*)
and *Deliver* *m1* \sqsubset^i *Broadcast* *m2*
shows *Deliver* *m1* \sqsubset^j *Deliver* *m2*
using *assms causal-delivery hb.intros(2)* **by** *blast*

lemma (**in** *network*) *hb-broadcast-exists1*:
assumes *hb* *m1* *m2*
shows $\exists i. \text{Broadcast } m1 \in \text{set } (\text{history } i)$
using *assms*
apply(*induction rule: hb.induct*)
apply(*meson insert-subset node-histories.local-order-carrier-closed node-histories-axioms*)
apply(*meson delivery-has-a-cause insert-subset local-order-carrier-closed*)
apply *simp*
done

lemma (**in** *network*) *hb-broadcast-exists2*:
assumes *hb* *m1* *m2*
shows $\exists i. \text{Broadcast } m2 \in \text{set } (\text{history } i)$
using *assms*
apply(*induction rule: hb.induct*)
apply(*meson insert-subset node-histories.local-order-carrier-closed node-histories-axioms*)
apply(*meson delivery-has-a-cause insert-subset local-order-carrier-closed*)
apply *simp*
done

4.3 Causal networks

lemma (**in** *causal-network*) *hb-has-a-reason*:
assumes *hb* *m1* *m2*
and *Broadcast* *m2* \in *set* (*history* *i*)
shows *Deliver* *m1* \in *set* (*history* *i*) $\vee \text{Broadcast } m1 \in \text{set } (\text{history } i)$
using *assms* **apply** (*induction rule: hb.induct*)
apply(*metis insert-subset local-order-carrier-closed network.broadcasts-unique network-axioms*)
apply(*metis insert-subset local-order-carrier-closed network.broadcasts-unique network-axioms*)
using *hb-trans causal-delivery local-order-carrier-closed* **apply** *blast*


```

done

lemma (in causal-network) hb-cross-node-delivery:
  assumes hb m1 m2
    and Broadcast m1 ∈ set (history i)
    and Broadcast m2 ∈ set (history j)
    and i ≠ j
  shows Deliver m1 ∈ set (history j)
  using assms
  apply(induction rule: hb.induct)
    apply(metis broadcasts-unique insert-subset local-order-carrier-closed)
    apply(metis insert-subset local-order-carrier-closed network.broadcasts-unique network-axioms)
  using broadcasts-unique hb.intros(3) hb-has-a-reason apply blast
done

lemma (in causal-network) hb-irrefl:
  assumes hb m1 m2
  shows m1 ≠ m2
  using assms proof(induction rule: hb.induct)
    case (hb-broadcast m1 i m2) thus ?case
      using node-total-order-antisym by blast
  next
    case (hb-deliver m1 i m2) thus ?case
      by(meson causal-broadcast insert-subset local-order-carrier-closed node-total-order-irrefl)
  next
    case (hb-trans m1 m2 m3)
  then obtain i j where Broadcast m3 ∈ set (history i) Broadcast m2 ∈ set (history j)
    using hb-broadcast-exists2 by blast
  then show ?case
    using assms hb-trans by (meson causal-network.causal-delivery causal-network-axioms
      deliver-locally insert-subset network.hb.intros(3) network-axioms
      node-histories.local-order-carrier-closed assms hb-trans
      node-histories-axioms node-total-order-irrefl)
qed

lemma (in causal-network) hb-broadcast-broadcast-order:
  assumes hb m1 m2
    and Broadcast m1 ∈ set (history i)
    and Broadcast m2 ∈ set (history i)
  shows Broadcast m1  $\sqsubset^i$  Broadcast m2
  using assms proof(induction rule: hb.induct)
    case (hb-broadcast m1 i m2) thus ?case
      by(metis insertI1 local-order-carrier-closed network.broadcasts-unique network-axioms subsetCE)
  next
    case (hb-deliver m1 i m2) thus ?case
      by(metis broadcasts-unique insert-subset local-order-carrier-closed
        network.broadcasts-unique insert-subset local-order-carrier-closed
        network.broadcast-before-delivery network-axioms node-total-order-trans)
  next
    case (hb-trans m1 m2 m3)
  then show ?case
  proof (cases Broadcast m2 ∈ set (history i))
    case True thus ?thesis
      using hb-trans node-total-order-trans by blast
  next
    case False hence Deliver m2 ∈ set (history i) m1 ≠ m2 m2 ≠ m3
      using hb-has-a-reason hb-trans by auto
  thus ?thesis
    by(metis hb-trans event.inject(1) hb.intros(1) hb-irrefl network.hb.intros(3) network-axioms)

```

node-order-is-total hb-irrefl)

qed
qed

lemma (in *causal-network*) *hb-antisym*:

assumes *hb x y*
and *hb y x*
shows *False*
using *assms* **proof**(*induction rule: hb.induct*)
fix *m1 i m2*
assume *hb m2 m1* **and** *Broadcast m1* \sqsubset^i *Broadcast m2*
thus *False*
apply – **proof**(*erule hb-elim*)
show $\bigwedge ia. \text{Broadcast } m1 \sqsubset^i \text{Broadcast } m2 \implies \text{Broadcast } m2 \sqsubset^i a \text{Broadcast } m1 \implies \text{False}$
by(*metis broadcasts-unique insert-subset local-order-carrier-closed node-total-order-irrefl node-total-order-trans*)
next
show $\bigwedge ia. \text{Broadcast } m1 \sqsubset^i \text{Broadcast } m2 \implies \text{Deliver } m2 \sqsubset^i a \text{Broadcast } m1 \implies \text{False}$
by(*metis broadcast-before-delivery broadcasts-unique insert-subset local-order-carrier-closed node-total-order-irrefl node-total-order-trans*)
next
show $\bigwedge m2a. \text{Broadcast } m1 \sqsubset^i \text{Broadcast } m2 \implies \text{hb } m2 \ m2a \implies \text{hb } m2a \ m1 \implies \text{False}$
using *assms(1) assms(2) hb.intros(3) hb-irrefl* **by** *blast*
qed
next
fix *m1 i m2*
assume *hb m2 m1*
and *Deliver m1* \sqsubset^i *Broadcast m2*
thus *False*
apply – **proof**(*erule hb-elim*)
show $\bigwedge ia. \text{Deliver } m1 \sqsubset^i \text{Broadcast } m2 \implies \text{Broadcast } m2 \sqsubset^i a \text{Broadcast } m1 \implies \text{False}$
by (*metis broadcast-before-delivery broadcasts-unique insert-subset local-order-carrier-closed node-total-order-irrefl node-total-order-trans*)
next
show $\bigwedge ia. \text{Deliver } m1 \sqsubset^i \text{Broadcast } m2 \implies \text{Deliver } m2 \sqsubset^i a \text{Broadcast } m1 \implies \text{False}$
by (*meson causal-network.causal-delivery causal-network-axioms hb.intros(2) hb.intros(3) insert-subset local-order-carrier-closed node-total-order-irrefl*)
next
show $\bigwedge m2a. \text{Deliver } m1 \sqsubset^i \text{Broadcast } m2 \implies \text{hb } m2 \ m2a \implies \text{hb } m2a \ m1 \implies \text{False}$
by (*meson causal-delivery hb.intros(2) insert-subset local-order-carrier-closed network.hb.intros(3) network-axioms node-total-order-irrefl*)
qed
next
fix *m1 m2 m3*
assume *hb m1 m2 hb m2 m3 hb m3 m1*
and (*hb m2 m1* \implies *False*) (*hb m3 m2* \implies *False*)
thus *False*
using *hb.intros(3)* **by** *blast*
qed

definition (in *network*) *node-deliver-messages* :: '*msg event list* \Rightarrow '*msg list* **where**
node-deliver-messages cs \equiv *List.map-filter* ($\lambda e. \text{case } e \text{ of Deliver } m \Rightarrow \text{Some } m \mid - \Rightarrow \text{None}$) *cs*

lemma (in *network*) *node-deliver-messages-empty* [*simp*]:

shows *node-deliver-messages* [] = []
by(*auto simp add: node-deliver-messages-def List.map-filter-simps*)

lemma (in *network*) *node-deliver-messages-Cons*:

shows *node-deliver-messages* (*x#xs*) = (*node-deliver-messages* [*x*])@(*node-deliver-messages* *xs*)

by(*auto simp add: node-deliver-messages-def map-filter-def*)

lemma (*in network*) *node-deliver-messages-append*:
shows *node-deliver-messages (xs@ys) = (node-deliver-messages xs)@(node-deliver-messages ys)*
by(*auto simp add: node-deliver-messages-def map-filter-def*)

lemma (*in network*) *node-deliver-messages-Broadcast [simp]*:
shows *node-deliver-messages [Broadcast m] = []*
by(*clarsimp simp: node-deliver-messages-def map-filter-def*)

lemma (*in network*) *node-deliver-messages-Deliver [simp]*:
shows *node-deliver-messages [Deliver m] = [m]*
by(*clarsimp simp: node-deliver-messages-def map-filter-def*)

lemma (*in network*) *prefix-msg-in-history*:
assumes *es prefix of i*
and *m ∈ set (node-deliver-messages es)*
shows *Deliver m ∈ set (history i)*
using *assms prefix-to-carriers* **by**(*fastforce simp: node-deliver-messages-def map-filter-def split: event.split-asm*)

lemma (*in network*) *prefix-contains-msg*:
assumes *es prefix of i*
and *m ∈ set (node-deliver-messages es)*
shows *Deliver m ∈ set es*
using *assms* **by**(*auto simp: node-deliver-messages-def map-filter-def split: event.split-asm*)

lemma (*in network*) *node-deliver-messages-distinct*:
assumes *xs prefix of i*
shows *distinct (node-deliver-messages xs)*
using *assms* **proof**(*induction xs rule: rev-induct*)
case Nil **thus** *?case* **by** *simp*
next
case (*snoc x xs*)
{ **fix** *y* **assume** ***: *y ∈ set (node-deliver-messages xs) y ∈ set (node-deliver-messages [x])*
moreover **have** *distinct (xs @ [x])*
using *assms snoc prefix-distinct* **by** *blast*
ultimately **have** *False*
using *assms* **apply**(*case-tac x; clarsimp simp add: map-filter-def node-deliver-messages-def*)
using ** prefix-contains-msg snoc.prems* **by** *blast*
} **thus** *?case*
using *snoc* **by**(*fastforce simp add: node-deliver-messages-append node-deliver-messages-def map-filter-def*)

qed

lemma (*in network*) *drop-last-message*:
assumes *evts prefix of i*
and *node-deliver-messages evts = msgs @ [last-msg]*
shows \exists *pre. pre prefix of i ∧ node-deliver-messages pre = msgs*
proof –
have *Deliver last-msg ∈ set evts*
using *assms network.prefix-contains-msg network-axioms* **by** *force*
then **obtain** *idx* **where** ***: *idx < length evts evts ! idx = Deliver last-msg*
by (*meson set-elem-nth*)
then **obtain** *pre suf* **where** *evts = pre @ (evts ! idx) # suf*
using *id-take-nth-drop* **by** *blast*
hence ****: *evts = pre @ (Deliver last-msg) # suf*
using *assms ** **by** *auto*
moreover **hence** *distinct (node-deliver-messages ([Deliver last-msg] @ suf))*
by (*metis assms(1) assms(2) distinct-singleton node-deliver-messages-Cons node-deliver-messages-Deliver*)

```

    node-deliver-messages-append node-deliver-messages-distinct not-Cons-self2 pre-suf-eq-distinct-list)
ultimately have node-deliver-messages ([Deliver last-msg] @ suf) = [last-msg] @ []
by (metis append-self-conv assms(1) assms(2) node-deliver-messages-Cons node-deliver-messages-Deliver
    node-deliver-messages-append node-deliver-messages-distinct not-Cons-self2 pre-suf-eq-distinct-list)
thus ?thesis
using assms * ** by (metis append1-eq-conv append-Cons append-Nil node-deliver-messages-append
prefix-of-appendD)
qed

```

```

locale network-with-ops = causal-network history fst
for history :: nat  $\Rightarrow$  ('msgid  $\times$  'op) event list +
fixes interp :: 'op  $\Rightarrow$  'state  $\rightarrow$  'state
and initial-state :: 'state

```

context network-with-ops **begin**

```

definition interp-msg :: 'msgid  $\times$  'op  $\Rightarrow$  'state  $\rightarrow$  'state where
    interp-msg msg state  $\equiv$  interp (snd msg) state

```

sublocale hb: happens-before weak-hb hb interp-msg

proof

```

fix x y :: 'msgid  $\times$  'op
show hb x y = (weak-hb x y  $\wedge$   $\neg$  weak-hb y x)
unfolding weak-hb-def using hb-antisym by blast
next
fix x
show weak-hb x x
using weak-hb-def by blast
next
fix x y z
assume weak-hb x y weak-hb y z
thus weak-hb x z
using weak-hb-def by (metis network.hb.intros(3) network-axioms)
qed

```

end

```

definition (in network-with-ops) apply-operations :: ('msgid  $\times$  'op) event list  $\rightarrow$  'state where
    apply-operations es  $\equiv$  hb.apply-operations (node-deliver-messages es) initial-state

```

```

definition (in network-with-ops) node-deliver-ops :: ('msgid  $\times$  'op) event list  $\Rightarrow$  'op list where
    node-deliver-ops cs  $\equiv$  map snd (node-deliver-messages cs)

```

```

lemma (in network-with-ops) apply-operations-empty [simp]:
shows apply-operations [] = Some initial-state
by(auto simp add: apply-operations-def)

```

```

lemma (in network-with-ops) apply-operations-Broadcast [simp]:
shows apply-operations (xs @ [Broadcast m]) = apply-operations xs
by(auto simp add: apply-operations-def node-deliver-messages-def map-filter-def)

```

```

lemma (in network-with-ops) apply-operations-Deliver [simp]:
shows apply-operations (xs @ [Deliver m]) = (apply-operations xs  $\ggg$  interp-msg m)
by(auto simp add: apply-operations-def node-deliver-messages-def map-filter-def kleisli-def)

```

```

lemma (in network-with-ops) hb-consistent-technical:
assumes  $\bigwedge m n. m < \text{length } cs \implies n < m \implies cs ! n \sqsubset^i cs ! m$ 
shows hb.hb-consistent (node-deliver-messages cs)

```

```

using assms proof (induction cs rule: rev-induct)
  case Nil thus ?case
    by(simp add: node-deliver-messages-def hb.hb-consistent.intros(1) map-filter-simps(2))
next
  case (snoc x xs)
  hence *: ( $\bigwedge m n. m < \text{length } xs \implies n < m \implies xs ! n \sqsubset^i xs ! m$ )
    by( $-, \text{erule-tac } x=m \text{ in } \text{meta-allE}, \text{erule-tac } x=n \text{ in } \text{meta-allE}, \text{clarsimp simp add: nth-append}$ )
  then show ?case
  proof (cases x)
    case (Broadcast x1) thus ?thesis
      using snoc * by (simp add: node-deliver-messages-append)
  next
  case (Deliver x2) thus ?thesis
    using snoc * apply(clarsimp simp add: node-deliver-messages-def map-filter-def map-filter-append)
    apply (rename-tac m m1 m2)
    apply (case-tac m; clarsimp)
    apply(drule set-elem-nth, erule exE, erule conjE)
    apply(erule-tac x=length xs in meta-allE)
    apply(clarsimp simp add: nth-append)
    by (metis causal-delivery insert-subset node-histories.local-order-carrier-closed
      node-histories-axioms node-total-order-antisym)
  qed
qed

```

```

corollary (in network-with-ops)
  shows hb.hb-consistent (node-deliver-messages (history i))
  by (metis hb-consistent-technical history-order-def less-one linorder-neqE-nat list-nth-split zero-order(3))

```

```

lemma (in network-with-ops) hb-consistent-prefix:
  assumes xs prefix of i
  shows hb.hb-consistent (node-deliver-messages xs)
using assms proof (clarsimp simp: prefix-of-node-history-def, rule-tac i=i in hb-consistent-technical)
  fix m n ys assume *: xs @ ys = history i m < length xs n < m
  consider (a) xs = [] | (b)  $\exists c. xs = [c]$  | (c) Suc 0 < length (xs)
    by (metis Suc-pred length-Suc-conv length-greater-0-conv zero-less-diff)
  thus xs ! n \sqsubset^i xs ! m
  proof (cases)
    case a thus ?thesis
      using * by clarsimp
  next
  case b thus ?thesis
    using assms * by clarsimp
  next
  case c thus ?thesis
    using assms * apply clarsimp
    apply(drule list-nth-split, assumption, clarsimp simp: c)
    apply (metis append.assoc append.simps(2) history-order-def)
    done
  qed
qed

```

```

locale network-with-constrained-ops = network-with-ops +
  fixes valid-msg :: 'c  $\Rightarrow$  ('a  $\times$  'b)  $\Rightarrow$  bool
  assumes broadcast-only-valid-msgs: pre @ [Broadcast m] prefix of i  $\implies$ 
     $\exists \text{state. apply-operations pre} = \text{Some state} \wedge \text{valid-msg state } m$ 

```

```

lemma (in network-with-constrained-ops) broadcast-is-valid:
  assumes Broadcast m \in set (history i)

```

shows \exists state. valid-msg state m
using *assms broadcast-only-valid-msgs events-before-exist* **by** *blast*

lemma (**in** *network-with-constrained-ops*) *deliver-is-valid*:
assumes *Deliver m \in set (history i)*
shows \exists j pre state. pre @ [Broadcast m] prefix of j \wedge apply-operations pre = Some state \wedge valid-msg state m
using *assms apply (clarsimp dest!: delivery-has-a-cause)*
using *broadcast-only-valid-msgs events-before-exist* **apply** *blast*
done

lemma (**in** *network-with-constrained-ops*) *deliver-in-prefix-is-valid*:
assumes *xs prefix of i*
and *Deliver m \in set xs*
shows \exists state. valid-msg state m
by (*meson assms network-with-constrained-ops.deliver-is-valid network-with-constrained-ops-axioms prefix-elem-to-carriers*)

4.4 Dummy network models

interpretation *trivial-node-histories: node-histories* $\lambda m.$ []
by *standard auto*

interpretation *trivial-network: network* $\lambda m.$ [] *id*
by *standard auto*

interpretation *trivial-causal-network: causal-network* $\lambda m.$ [] *id*
by *standard auto*

interpretation *trivial-network-with-ops: network-with-ops* $\lambda m.$ [] ($\lambda x y.$ Some y) 0
by *standard auto*

interpretation *trivial-network-with-constrained-ops: network-with-constrained-ops* $\lambda m.$ [] ($\lambda x y.$ Some y) 0 $\lambda x y.$ True
by *standard (simp add: trivial-node-histories.prefix-of-node-history-def)*

end

5 Replicated Growable Array

The RGA, introduced by [10], is a replicated ordered list (sequence) datatype that supports *insert* and *delete* operations.

theory
Ordered-List

imports
Util

begin

type-synonym (*'id, 'v*) *elt* = *'id \times 'v \times bool*

5.1 Insert and delete operations

Insertion operations place the new element *after* an existing list element with a given ID, or at the head of the list if no ID is given. Deletion operations refer to the ID of the list element that is to be deleted. However, it is not safe for a deletion operation to completely remove a list element, because then a concurrent insertion after the deleted element would not be able to

locate the insertion position. Instead, the list retains so-called *tombstones*: a deletion operation merely sets a flag on a list element to mark it as deleted, but the element actually remains in the list. A separate garbage collection process can be used to eventually purge tombstones [10], but we do not consider tombstone removal here.

hide-const *insert*

```
fun insert-body :: ('id::{linorder}, 'v) elt list  $\Rightarrow$  ('id, 'v) elt  $\Rightarrow$  ('id, 'v) elt list where
  insert-body [] e = [e] |
  insert-body (x#xs) e =
    (if fst x < fst e then
      e#x#xs
    else x#insert-body xs e)
```

```
fun insert :: ('id::{linorder}, 'v) elt list  $\Rightarrow$  ('id, 'v) elt  $\Rightarrow$  'id option  $\Rightarrow$  ('id, 'v) elt list option where
  insert xs e None = Some (insert-body xs e) |
  insert [] e (Some i) = None |
  insert (x#xs) e (Some i) =
    (if fst x = i then
      Some (x#insert-body xs e)
    else
      insert xs e (Some i)  $\gg$  ( $\lambda$ t. Some (x#t)))
```

```
fun delete :: ('id::{linorder}, 'v) elt list  $\Rightarrow$  'id  $\Rightarrow$  ('id, 'v) elt list option where
  delete [] i = None |
  delete ((i', v, flag)#xs) i =
    (if i' = i then
      Some ((i', v, True)#xs)
    else
      delete xs i  $\gg$  ( $\lambda$ t. Some ((i',v,flag)#t)))
```

5.2 Well-definedness of insert and delete

lemma *insert-no-failure*:

assumes $i = \text{None} \vee (\exists i'. i = \text{Some } i' \wedge i' \in \text{fst } \text{'set } xs)$

shows $\exists xs'. \text{insert } xs \ e \ i = \text{Some } xs'$

using *assms* **by**(*induction rule: insert.induct; force*)

lemma *insert-None-index-neq-None* [*dest*]:

assumes $\text{insert } xs \ e \ i = \text{None}$

shows $i \neq \text{None}$

using *assms* **by**(*cases i, auto*)

lemma *insert-Some-None-index-not-in* [*dest*]:

assumes $\text{insert } xs \ e \ (\text{Some } i) = \text{None}$

shows $i \notin \text{fst } \text{'set } xs$

using *assms* **by**(*induction xs, auto split: if-split-asm bind-splits*)

lemma *index-not-in-insert-Some-None* [*simp*]:

assumes $i \notin \text{fst } \text{'set } xs$

shows $\text{insert } xs \ e \ (\text{Some } i) = \text{None}$

using *assms* **by**(*induction xs, auto*)

lemma *delete-no-failure*:

assumes $i \in \text{fst } \text{'set } xs$

shows $\exists xs'. \text{delete } xs \ i = \text{Some } xs'$

using *assms* **by**(*induction xs; force*)

lemma *delete-None-index-not-in* [*dest*]:
assumes *delete xs i = None*
shows $i \notin \text{fst } ' \text{ set } xs$
using *assms* **by**(*induction xs, auto split: if-split-asm bind-splits simp add: fst-eq-Domain*)

lemma *index-not-in-delete-None* [*simp*]:
assumes $i \notin \text{fst } ' \text{ set } xs$
shows $\text{delete } xs \ i = \text{None}$
using *assms* **by**(*induction xs, auto*)

5.3 Preservation of element indices

lemma *insert-body-preserve-indices* [*simp*]:
shows $\text{fst } ' \text{ set } (\text{insert-body } xs \ e) = \text{fst } ' \text{ set } xs \cup \{\text{fst } e\}$
by(*induction xs, auto simp add: insert-commute*)

lemma *insert-preserve-indices*:
assumes $\exists ys. \text{insert } xs \ e \ i = \text{Some } ys$
shows $\text{fst } ' \text{ set } (\text{the } (\text{insert } xs \ e \ i)) = \text{fst } ' \text{ set } xs \cup \{\text{fst } e\}$
using *assms* **by**(*induction xs; cases i; auto simp add: insert-commute split: bind-splits*)

corollary *insert-preserve-indices'*:
assumes $\text{insert } xs \ e \ i = \text{Some } ys$
shows $\text{fst } ' \text{ set } (\text{the } (\text{insert } xs \ e \ i)) = \text{fst } ' \text{ set } xs \cup \{\text{fst } e\}$
using *assms* *insert-preserve-indices* **by** *blast*

lemma *delete-preserve-indices*:
assumes $\text{delete } xs \ i = \text{Some } ys$
shows $\text{fst } ' \text{ set } xs = \text{fst } ' \text{ set } ys$
using *assms* **by**(*induction xs arbitrary: ys, simp*) (*case-tac a; auto split: if-split-asm bind-splits*)

5.4 Commutativity of concurrent operations

lemma *insert-body-commutes*:
assumes $\text{fst } e1 \neq \text{fst } e2$
shows $\text{insert-body } (\text{insert-body } xs \ e1) \ e2 = \text{insert-body } (\text{insert-body } xs \ e2) \ e1$
using *assms* **by**(*induction xs, auto*)

lemma *insert-insert-body*:
assumes $\text{fst } e1 \neq \text{fst } e2$
and $i2 \neq \text{Some } (\text{fst } e1)$
shows $\text{insert } (\text{insert-body } xs \ e1) \ e2 \ i2 = \text{insert } xs \ e2 \ i2 \gg (\lambda ys. \text{Some } (\text{insert-body } ys \ e1))$
using *assms* **by** (*induction xs; cases i2*) (*auto split: if-split-asm simp add: insert-body-commutes*)

lemma *insert-Nil-None*:
assumes $\text{fst } e1 \neq \text{fst } e2$
and $i \neq \text{fst } e2$
and $i2 \neq \text{Some } (\text{fst } e1)$
shows $\text{insert } [] \ e2 \ i2 \gg (\lambda ys. \text{insert } ys \ e1 \ (\text{Some } i)) = \text{None}$
using *assms* **by** (*cases i2*) *clarsimp+*

lemma *insert-insert-body-commute*:
assumes $i \neq \text{fst } e1$
and $\text{fst } e1 \neq \text{fst } e2$
shows $\text{insert } (\text{insert-body } xs \ e1) \ e2 \ (\text{Some } i) =$
 $\text{insert } xs \ e2 \ (\text{Some } i) \gg (\lambda y. \text{Some } (\text{insert-body } y \ e1))$
using *assms* **by**(*induction xs, auto simp add: insert-body-commutes*)

lemma *insert-commutes*:

assumes $\text{fst } e1 \neq \text{fst } e2$
 $i1 = \text{None} \vee i1 \neq \text{Some } (\text{fst } e2)$
 $i2 = \text{None} \vee i2 \neq \text{Some } (\text{fst } e1)$
shows $\text{insert } xs \ e1 \ i1 \gg (\lambda ys. \text{insert } ys \ e2 \ i2) =$
 $\text{insert } xs \ e2 \ i2 \gg (\lambda ys. \text{insert } ys \ e1 \ i1)$
using *assms* **proof**(*induction rule: insert.induct*)
fix xs **and** $e :: ('a, 'b) \text{elt}$
assume $i2 = \text{None} \vee i2 \neq \text{Some } (\text{fst } e)$ **and** $\text{fst } e \neq \text{fst } e2$
thus $\text{insert } xs \ e \ \text{None} \gg (\lambda ys. \text{insert } ys \ e2 \ i2) = \text{insert } xs \ e2 \ i2 \gg (\lambda ys. \text{insert } ys \ e \ \text{None})$
by(*auto simp add: insert-body-commutes intro: insert-insert-body*)
next
fix i **and** $e :: ('a, 'b) \text{elt}$
assume $\text{fst } e \neq \text{fst } e2$ **and** $i2 = \text{None} \vee i2 \neq \text{Some } (\text{fst } e)$ **and** $\text{Some } i = \text{None} \vee \text{Some } i \neq \text{Some } (\text{fst } e2)$
thus $\text{insert } [] \ e \ (\text{Some } i) \gg (\lambda ys. \text{insert } ys \ e2 \ i2) = \text{insert } [] \ e2 \ i2 \gg (\lambda ys. \text{insert } ys \ e \ (\text{Some } i))$
by (*auto intro: insert-Nil-None[symmetric]*)
next
fix $xs \ i$ **and** $x \ e :: ('a, 'b) \text{elt}$
assume *IH*: $(\text{fst } x \neq i \implies$
 $\text{fst } e \neq \text{fst } e2 \implies$
 $\text{Some } i = \text{None} \vee \text{Some } i \neq \text{Some } (\text{fst } e2) \implies$
 $i2 = \text{None} \vee i2 \neq \text{Some } (\text{fst } e) \implies$
 $\text{insert } xs \ e \ (\text{Some } i) \gg (\lambda ys. \text{insert } ys \ e2 \ i2) = \text{insert } xs \ e2 \ i2 \gg (\lambda ys. \text{insert } ys \ e \ (\text{Some } i)))$
and $\text{fst } e \neq \text{fst } e2$
and $\text{Some } i = \text{None} \vee \text{Some } i \neq \text{Some } (\text{fst } e2)$
and $i2 = \text{None} \vee i2 \neq \text{Some } (\text{fst } e)$
thus $\text{insert } (x \# xs) \ e \ (\text{Some } i) \gg (\lambda ys. \text{insert } ys \ e2 \ i2) = \text{insert } (x \# xs) \ e2 \ i2 \gg (\lambda ys. \text{insert } ys \ e \ (\text{Some } i))$
apply –
apply(*erule disjE, clarsimp, simp, rule conjI*)
apply(*case-tac i2; force simp add: insert-body-commutes insert-insert-body-commute*)
apply(*case-tac i2; clarsimp cong: Option.bind-cong simp add: insert-insert-body split: bind-splits*)
apply *force*
done
qed

lemma *delete-commutes*:

shows $\text{delete } xs \ i1 \gg (\lambda ys. \text{delete } ys \ i2) = \text{delete } xs \ i2 \gg (\lambda ys. \text{delete } ys \ i1)$
by(*induction xs, auto split: bind-splits if-split-asm*)

lemma *insert-body-delete-commute*:

assumes $i2 \neq \text{fst } e$
shows $\text{delete } (\text{insert-body } xs \ e) \ i2 \gg (\lambda t. \text{Some } (x\#t)) =$
 $\text{delete } xs \ i2 \gg (\lambda y. \text{Some } (x\#\text{insert-body } y \ e))$
using *assms* **by** (*induction xs arbitrary: x; cases e, auto split: bind-splits if-split-asm*)

lemma *insert-delete-commute*:

assumes $i2 \neq \text{fst } e$
shows $\text{insert } xs \ e \ i1 \gg (\lambda ys. \text{delete } ys \ i2) = \text{delete } xs \ i2 \gg (\lambda ys. \text{insert } ys \ e \ i1)$
using *assms* **by**(*induction xs; cases e; cases i1, auto split: bind-splits if-split-asm simp add: insert-body-delete-commute*)

5.5 Alternative definition of insert

fun $\text{insert}' :: ('id::\{\text{linorder}\}, 'v) \text{elt list} \Rightarrow ('id, 'v) \text{elt} \Rightarrow 'id \text{option} \rightarrow ('id::\{\text{linorder}\}, 'v) \text{elt list}$
where
 $\text{insert}' [] \ e \quad \text{None} \quad = \text{Some } [e] \mid$

```

insert' [] e (Some i) = None |
insert' (x#xs) e None =
  (if fst x < fst e then
    Some (e#x#xs)
  else
    case insert' xs e None of
      None => None
    | Some t => Some (x#t)) |
insert' (x#xs) e (Some i) =
  (if fst x = i then
    case insert' xs e None of
      None => None
    | Some t => Some (x#t)
  else
    case insert' xs e (Some i) of
      None => None
    | Some t => Some (x#t))

```

lemma [elim!, dest]:

assumes insert' xs e None = None
shows False

using assms **by**(induction xs, auto split: if-split-asm option.split-asm)

lemma insert-body-insert':

shows insert' xs e None = Some (insert-body xs e)

by(induction xs, auto)

lemma insert-insert':

shows insert xs e i = insert' xs e i

by(induction xs; cases e; cases i, auto split: option.split simp add: insert-body-insert')

lemma insert-body-stop-iteration:

assumes fst e > fst x

shows insert-body (x#xs) e = e#x#xs

using assms **by** simp

lemma insert-body-contains-new-elem:

shows $\exists p s. xs = p @ s \wedge \text{insert-body } xs \ e = p @ e \# s$

proof (induction xs)

case Nil thus ?case **by** force

next

case (Cons a xs)

then obtain p s **where** xs = p @ s \wedge insert-body xs e = p @ e # s **by** force

thus ?case

apply clarsimp

apply (rule conjI; clarsimp)

apply force

apply (rule-tac x=a # p **in** exI, force)

done

qed

lemma insert-between-elements:

assumes xs = pre@ref#suf

and distinct (map fst xs)

and $\bigwedge i'. i' \in \text{fst } \text{'set } xs \implies i' < \text{fst } e$

shows insert xs e (Some (fst ref)) = Some (pre @ ref # e # suf)

using assms **by**(induction xs arbitrary: pre ref suf, force) (case-tac pre; case-tac suf; force)

lemma *insert-position-element-technical*:
assumes $\forall x \in \text{set } as. a \neq \text{fst } x$
and *insert-body* $(cs @ ds) e = cs @ e \# ds$
shows *insert* $(as @ (a, aa, b) \# cs @ ds) e (\text{Some } a) = \text{Some } (as @ (a, aa, b) \# cs @ e \# ds)$
using *assms* **by** (*induction as arbitrary: cs ds; clarsimp*)

lemma *split-tuple-list-by-id*:
assumes $(a, b, c) \in \text{set } xs$
and *distinct* $(\text{map } \text{fst } xs)$
shows $\exists \text{pre } \text{suf}. xs = \text{pre} @ (a, b, c) \# \text{suf} \wedge (\forall y \in \text{set } \text{pre}. \text{fst } y \neq a)$
using *assms* **proof** (*induction xs, clarsimp*)
case $(\text{Cons } x \ xs)$
{ **assume** $x \neq (a, b, c)$
hence $(a, b, c) \in \text{set } xs$ *distinct* $(\text{map } \text{fst } xs)$
using *Cons.prem*s **by** *force*+
then obtain *pre* *suf* **where** $xs = \text{pre} @ (a, b, c) \# \text{suf} \wedge (\forall y \in \text{set } \text{pre}. \text{fst } y \neq a)$
using *Cons.IH* **by** *force*
hence *?case*
apply (*rule-tac x=x#pre in exI*)
using *Cons.prem*s(2) **by** *auto*
} **thus** *?case*
by *force*
qed

lemma *insert-preserves-order*:
assumes $i = \text{None} \vee (\exists i'. i = \text{Some } i' \wedge i' \in \text{fst } \text{'set } xs)$
and *distinct* $(\text{map } \text{fst } xs)$
shows $\exists \text{pre } \text{suf}. xs = \text{pre} @ \text{suf} \wedge \text{insert } xs \ e \ i = \text{Some } (\text{pre} @ e \# \text{suf})$
using *assms* **proof** –
{ **assume** $i = \text{None}$
hence *?thesis*
by *clarsimp* (*metis insert-body-contains-new-elem*)
} **moreover** {
assume $\exists i'. i = \text{Some } i' \wedge i' \in \text{fst } \text{'set } xs$
then obtain *j* *v* *b* **where** $i = \text{Some } j \ (j, v, b) \in \text{set } xs$ **by** *force*
moreover then obtain *as* *bs* **where** $xs = as @ (j, v, b) \# bs \ \forall x \in \text{set } as. \text{fst } x \neq j$
using *assms* **by** (*metis split-tuple-list-by-id*)
moreover then obtain *cs* *ds* **where** *insert-body* $bs \ e = cs @ e \# ds \ cs @ ds = bs$
by (*metis insert-body-contains-new-elem*)
ultimately have *?thesis*
by (*rule-tac x=as@(j,v,b)#cs in exI; clarsimp*)(*metis insert-position-element-technical*)
} **ultimately show** *?thesis*
using *assms* **by** *force*
qed
end

5.6 Network

theory

RGA

imports

Network

Ordered-List

begin

datatype $(\text{'id}, \text{'v})$ *operation* =
Insert $(\text{'id}, \text{'v})$ *elt* 'id *option* |
Delete 'id

fun *interpret-ops* :: ('id::linorder, 'v) operation ⇒ ('id, 'v) elt list → ('id, 'v) elt list (⟨-⟩ [0] 1000)
where

interpret-ops (Insert e n) xs = insert xs e n |
interpret-ops (Delete n) xs = delete xs n

definition *element-ids* :: ('id, 'v) elt list ⇒ 'id set **where**
element-ids list ≡ set (map fst list)

definition *valid-rga-msg* :: ('id, 'v) elt list ⇒ 'id × ('id::linorder, 'v) operation ⇒ bool **where**
valid-rga-msg list msg ≡ case msg of
 (i, Insert e None) ⇒ fst e = i |
 (i, Insert e (Some pos)) ⇒ fst e = i ∧ pos ∈ *element-ids* list |
 (i, Delete pos) ⇒ pos ∈ *element-ids* list

locale *rga* = network-with-constrained-ops - *interpret-ops* [] *valid-rga-msg*

definition *indices* :: ('id × ('id, 'v) operation) event list ⇒ 'id list **where**
indices xs ≡
 List.map-filter (λx. case x of Deliver (i, Insert e n) ⇒ Some (fst e) | - ⇒ None) xs

lemma *indices-Nil* [simp]:
shows *indices* [] = []
by(auto simp: *indices-def* map-filter-def)

lemma *indices-append* [simp]:
shows *indices* (xs@ys) = *indices* xs @ *indices* ys
by(auto simp: *indices-def* map-filter-def)

lemma *indices-Broadcast-singleton* [simp]:
shows *indices* [Broadcast b] = []
by(auto simp: *indices-def* map-filter-def)

lemma *indices-Deliver-Insert* [simp]:
shows *indices* [Deliver (i, Insert e n)] = [fst e]
by(auto simp: *indices-def* map-filter-def)

lemma *indices-Deliver-Delete* [simp]:
shows *indices* [Deliver (i, Delete n)] = []
by(auto simp: *indices-def* map-filter-def)

lemma (in *rga*) *idx-in-elem-inserted* [intro]:
assumes Deliver (i, Insert e n) ∈ set xs
shows fst e ∈ set (*indices* xs)
using *assms* **by**(induction xs, auto simp add: *indices-def* map-filter-def)

lemma (in *rga*) *apply-ops-idx-elems*:
assumes es prefix of i
and apply-operations es = Some xs
shows *element-ids* xs = set (*indices* es)
using *assms* **unfolding** *element-ids-def*
proof(induction es arbitrary: xs rule: rev-induct, clarsimp)
case (snoc x xs) **thus** ?case
proof (cases x, clarsimp, blast)
case (Deliver e)
moreover obtain a b **where** e = (a, b) **by** force
ultimately show ?thesis

```

using snoc assms apply (cases b; clarsimp split: bind-splits simp add: interp-msg-def)
apply (metis Un-insert-right append.right-neutral insert-preserve-indices' list.set(1)
      option.sel prefix-of-appendD prod.sel(1) set-append)
by (metis delete-preserve-indices prefix-of-appendD)
qed
qed

lemma (in rga) delete-does-not-change-element-ids:
  assumes es @ [Deliver (i, Delete n)] prefix of j
  and apply-operations es = Some xs1
  and apply-operations (es @ [Deliver (i, Delete n)]) = Some xs2
  shows element-ids xs1 = element-ids xs2
proof -
  have indices es = indices (es @ [Deliver (i, Delete n)])
    by simp
  then show ?thesis
    by (metis (no-types) assms prefix-of-appendD rga.apply-opers-idx-elems rga-axioms)
qed

lemma (in rga) someone-inserted-id:
  assumes es @ [Deliver (i, Insert (k, v, f) n)] prefix of j
  and apply-operations es = Some xs1
  and apply-operations (es @ [Deliver (i, Insert (k, v, f) n)]) = Some xs2
  and a ∈ element-ids xs2
  and a ≠ k
  shows a ∈ element-ids xs1
using assms apply-opers-idx-elems by auto

lemma (in rga) deliver-insert-exists:
  assumes es prefix of j
    and apply-operations es = Some xs
    and a ∈ element-ids xs
  shows ∃ i v f n. Deliver (i, Insert (a, v, f) n) ∈ set es
using assms unfolding element-ids-def
proof(induction es arbitrary: xs rule: rev-induct, clarsimp)
  case (snoc x xs ys) thus ?case
  proof (cases x)
    case (Broadcast e) thus ?thesis
      using snoc by(clarsimp, metis image-eqI prefix-of-appendD prod.sel(1))
  next
    case (Deliver e)
    moreover then obtain xs' where *: apply-operations xs = Some xs'
      using snoc by fastforce
    moreover obtain k v where **: e = (k, v) by force
    ultimately show ?thesis
      using assms snoc proof (cases v)
      case (Insert el -) thus ?thesis
        using snoc Deliver * **
        apply (cases el; cases fst el = a; clarsimp)
        apply (blast, metis (no-types, lifting) element-ids-def prefix-of-appendD set-map snoc.prems(2)
              snoc.prems(3) someone-inserted-id)
      done
    next
    case (Delete -) thus ?thesis
      using snoc Deliver ** apply clarsimp
      apply(drule prefix-of-appendD, clarsimp simp add: bind-eq-Some-conv interp-msg-def)
      apply(metis delete-preserve-indices image-eqI prod.sel(1))
      done

```

qed
 qed
 qed

lemma (in *rga*) *insert-in-apply-set*:
 assumes $es \text{ @ } [Deliver (i, Insert e (Some a))] \text{ prefix of } j$
 and $Deliver (i', Insert e' n) \in \text{set } es$
 and *apply-operations* $es = \text{Some } s$
 shows $\text{fst } e' \in \text{element-ids } s$
 using *assms apply-opers-idx-elems idx-in-elem-inserted prefix-of-appendD* **by** *blast*

lemma (in *rga*) *insert-msg-id*:
 assumes $Broadcast (i, Insert e n) \in \text{set } (\text{history } j)$
 shows $\text{fst } e = i$
proof –
 obtain *state* **where** *1*: *valid-rga-msg state* $(i, Insert e n)$
 using *assms broadcast-is-valid* **by** *blast*
 thus $\text{fst } e = i$
by(*clarsimp simp add: valid-rga-msg-def split: option.split-asm*)
 qed

lemma (in *rga*) *allowed-insert*:
 assumes $Broadcast (i, Insert e n) \in \text{set } (\text{history } j)$
 shows $n = \text{None} \vee (\exists i' e' n'. n = \text{Some } (\text{fst } e') \wedge Deliver (i', Insert e' n') \sqsubset^j Broadcast (i, Insert e n))$
proof –
 obtain *pre* **where** *1*: *pre* $\text{ @ } [Broadcast (i, Insert e n)] \text{ prefix of } j$
 using *assms events-before-exist* **by** *blast*
 from *this* obtain *state* **where** *2*: *apply-operations* $pre = \text{Some } \text{state}$ **and** *3*: *valid-rga-msg state* $(i, Insert e n)$
 using *broadcast-only-valid-msgs* **by** *blast*
 show $n = \text{None} \vee (\exists i' e' n'. n = \text{Some } (\text{fst } e') \wedge Deliver (i', Insert e' n') \sqsubset^j Broadcast (i, Insert e n))$
proof(*cases n*)
 fix *a*
 assume *4*: $n = \text{Some } a$
 hence $a \in \text{element-ids } \text{state}$ **and** *5*: $\text{fst } e = i$
 using *3* **by**(*clarsimp simp add: valid-rga-msg-def*)
 from *this* **have** $\exists i' v' f' n'. Deliver (i', Insert (a, v', f') n') \in \text{set } pre$
 using *deliver-insert-exists 2 1* **by** *blast*
 thus $n = \text{None} \vee (\exists i' e' n'. n = \text{Some } (\text{fst } e') \wedge Deliver (i', Insert e' n') \sqsubset^j Broadcast (i, Insert e n))$
 using *events-in-local-order 1 4 5* **by**(*metis fst-conv*)
 qed *simp*
 qed

lemma (in *rga*) *allowed-delete*:
 assumes $Broadcast (i, Delete x) \in \text{set } (\text{history } j)$
 shows $\exists i' n' v b. Deliver (i', Insert (x, v, b) n') \sqsubset^j Broadcast (i, Delete x)$
proof –
 obtain *pre* **where** *1*: *pre* $\text{ @ } [Broadcast (i, Delete x)] \text{ prefix of } j$
 using *assms events-before-exist* **by** *blast*
 from *this* obtain *state* **where** *2*: *apply-operations* $pre = \text{Some } \text{state}$
and *valid-rga-msg state* $(i, Delete x)$
 using *broadcast-only-valid-msgs* **by** *blast*
 hence $x \in \text{element-ids } \text{state}$
 using *apply-opers-idx-elems* **by**(*simp add: valid-rga-msg-def*)
 hence $\exists i' v' f' n'. Deliver (i', Insert (x, v', f') n') \in \text{set } pre$

using *deliver-insert-exists 1 2* **by** *blast*
thus $\exists i' n' v b. \text{Deliver } (i', \text{Insert } (x, v, b) n') \sqsubset^j \text{Broadcast } (i, \text{Delete } x)$
using *events-in-local-order 1* **by** *blast*
qed

lemma (*in rga*) *insert-id-unique*:
assumes *fst e1 = fst e2*
and *Broadcast (i1, Insert e1 n1) ∈ set (history i)*
and *Broadcast (i2, Insert e2 n2) ∈ set (history j)*
shows *Insert e1 n1 = Insert e2 n2*
using *assms insert-msg-id msg-id-unique Pair-inject fst-conv* **by** *metis*

lemma (*in rga*) *allowed-delete-deliver*:
assumes *Deliver (i, Delete x) ∈ set (history j)*
shows $\exists i' n' v b. \text{Deliver } (i', \text{Insert } (x, v, b) n') \sqsubset^j \text{Deliver } (i, \text{Delete } x)$
using *assms* **by** (*meson allowed-delete bot-least causal-broadcast delivery-has-a-cause insert-subset*)

lemma (*in rga*) *allowed-delete-deliver-in-set*:
assumes (*es@[Deliver (i, Delete m)]*) *prefix of j*
shows $\exists i' n v b. \text{Deliver } (i', \text{Insert } (m, v, b) n) \in \text{set } es$
by(*metis (no-types, lifting) Un-insert-right insert-iff list.simps(15) assms*
local-order-prefix-closed-last rga.allowed-delete-deliver rga-axioms set-append subsetCE prefix-to-carriers)

lemma (*in rga*) *allowed-insert-deliver*:
assumes *Deliver (i, Insert e n) ∈ set (history j)*
shows $n = \text{None} \vee (\exists i' n' n'' v b. n = \text{Some } n' \wedge \text{Deliver } (i', \text{Insert } (n', v, b) n'') \sqsubset^j \text{Deliver } (i, \text{Insert } e n))$

proof –

obtain *ja* **where** *1: Broadcast (i, Insert e n) ∈ set (history ja)*
using *assms delivery-has-a-cause* **by** *blast*
show $n = \text{None} \vee (\exists i' n' n'' v b. n = \text{Some } n' \wedge \text{Deliver } (i', \text{Insert } (n', v, b) n'') \sqsubset^j \text{Deliver } (i, \text{Insert } e n))$

proof(*cases n*)

fix *a*

assume *3: n = Some a*

from this obtain *i' e' n'* **where** *4: Some a = Some (fst e')* **and**

2: Deliver (i', Insert e' n') ⊂^j a Broadcast (i, Insert e (Some a))

using *allowed-insert 1* **by** *blast*

hence *Deliver (i', Insert e' n') ∈ set (history ja)* **and** *Broadcast (i, Insert e (Some a)) ∈ set (history ja)*

using *local-order-carrier-closed* **by** *simp+*

from this obtain *jaa* **where** *Broadcast (i, Insert e (Some a)) ∈ set (history jaa)*

using *delivery-has-a-cause* **by** *simp*

have $\exists i' n' n'' v b. n = \text{Some } n' \wedge \text{Deliver } (i', \text{Insert } (n', v, b) n'') \sqsubset^j \text{Deliver } (i, \text{Insert } e n)$

using *2 3 4* **by**(*metis assms causal-broadcast prod.collapse*)

thus $n = \text{None} \vee (\exists i' n' n'' v b. n = \text{Some } n' \wedge \text{Deliver } (i', \text{Insert } (n', v, b) n'') \sqsubset^j \text{Deliver } (i, \text{Insert } e n))$

by *auto*

qed *simp*

qed

lemma (*in rga*) *allowed-insert-deliver-in-set*:
assumes (*es@[Deliver (i, Insert e m)]*) *prefix of j*
shows $m = \text{None} \vee (\exists i' m' n v b. m = \text{Some } m' \wedge \text{Deliver } (i', \text{Insert } (m', v, b) n) \in \text{set } es)$
by(*metis assms Un-insert-right insert-subset list.simps(15) set-append prefix-to-carriers*
allowed-insert-deliver local-order-prefix-closed-last)

lemma (*in rga*) *Insert-no-failure*:

assumes $es \text{ @ } [Deliver (i, Insert e n)] \text{ prefix of } j$
and *apply-operations* $es = Some s$
shows $\exists ys. insert s e n = Some ys$
by(*metis (no-types, lifting) element-ids-def allowed-insert-deliver-in-set assms fst-conv insert-in-apply-set insert-no-failure set-map*)

lemma (in *rga*) *delete-no-failure*:

assumes $es \text{ @ } [Deliver (i, Delete n)] \text{ prefix of } j$
and *apply-operations* $es = Some s$
shows $\exists ys. delete s n = Some ys$

proof –

obtain $i' na v b$ **where** $1: Deliver (i', Insert (n, v, b) na) \in set es$
using *assms allowed-delete-deliver-in-set* **by** *blast*
also have $fst (n, v, b) \in set (indices es)$
using *assms idx-in-elem-inserted calculation* **by** *blast*
from this *assms* **and** 1 **show** $\exists ys. delete s n = Some ys$
apply –
apply(*rule delete-no-failure*)
apply(*metis apply-opers-idx-elems element-ids-def prefix-of-appendD prod.sel(1) set-map*)
done

qed

lemma (in *rga*) *Insert-equal*:

assumes $fst e1 = fst e2$
and $Broadcast (i1, Insert e1 n1) \in set (history i)$
and $Broadcast (i2, Insert e2 n2) \in set (history j)$
shows $Insert e1 n1 = Insert e2 n2$
using *insert-id-unique assms* **by** *simp*

lemma (in *rga*) *same-insert*:

assumes $fst e1 = fst e2$
and $xs \text{ prefix of } i$
and $(i1, Insert e1 n1) \in set (node-deliver-messages xs)$
and $(i2, Insert e2 n2) \in set (node-deliver-messages xs)$
shows $Insert e1 n1 = Insert e2 n2$

proof –

have $Deliver (i1, Insert e1 n1) \in set (history i)$
using *assms* **by**(*auto simp add: node-deliver-messages-def prefix-msg-in-history*)
from this **obtain** j **where** $1: Broadcast (i1, Insert e1 n1) \in set (history j)$
using *delivery-has-a-cause* **by** *blast*
have $Deliver (i2, Insert e2 n2) \in set (history i)$
using *assms* **by**(*auto simp add: node-deliver-messages-def prefix-msg-in-history*)
from this **obtain** k **where** $2: Broadcast (i2, Insert e2 n2) \in set (history k)$
using *delivery-has-a-cause* **by** *blast*
show $Insert e1 n1 = Insert e2 n2$
by(*rule Insert-equal; force simp add: assms intro: 1 2*)

qed

lemma (in *rga*) *insert-commute-assms*:

assumes $\{Deliver (i, Insert e n), Deliver (i', Insert e' n')\} \subseteq set (history j)$
and $hb.concurrent (i, Insert e n) (i', Insert e' n')$
shows $n = None \vee n \neq Some (fst e')$

using *assms*

apply(*clarsimp simp: hb.concurrent-def*)
apply(*cases e'*)
apply *clarsimp*
apply(*frule delivery-has-a-cause*)
apply(*frule delivery-has-a-cause, clarsimp*)


```

apply(frule allowed-insert)
apply clarsimp
apply(metis Insert-equal delivery-has-a-cause fst-conv hb.intros(2) insert-subset
  local-order-carrier-closed insert-msg-id)
done

lemma subset-reorder:
  assumes  $\{a, b\} \subseteq c$ 
  shows  $\{b, a\} \subseteq c$ 
using assms by simp

lemma (in rga) Insert-Insert-concurrent:
  assumes  $\{Deliver\ (i, Insert\ e\ k),\ Deliver\ (i', Insert\ e'\ (Some\ m))\} \subseteq set\ (history\ j)$ 
  and hb.concurrent  $(i, Insert\ e\ k)\ (i', Insert\ e'\ (Some\ m))$ 
  shows  $fst\ e \neq m$ 
by(metis assms subset-reorder hb.concurrent-comm insert-commute-assms option.simps(3))

lemma (in rga) insert-valid-assms:
  assumes  $Deliver\ (i, Insert\ e\ n) \in set\ (history\ j)$ 
  shows  $n = None \vee n \neq Some\ (fst\ e)$ 
using assms by(meson allowed-insert-deliver hb.concurrent-def hb.less-asm insert-subset
  local-order-carrier-closed rga.insert-commute-assms rga-axioms)

lemma (in rga) Insert-Delete-concurrent:
  assumes  $\{Deliver\ (i, Insert\ e\ n),\ Deliver\ (i', Delete\ n')\} \subseteq set\ (history\ j)$ 
  and hb.concurrent  $(i, Insert\ e\ n)\ (i', Delete\ n')$ 
  shows  $n' \neq fst\ e$ 
by (metis assms Insert-equal allowed-delete delivery-has-a-cause fst-conv hb.concurrent-def
  hb.intros(2) insert-subset local-order-carrier-closed rga.insert-msg-id rga-axioms)

lemma (in rga) concurrent-operations-commute:
  assumes xs prefix of i
  shows hb.concurrent-ops-commute (node-deliver-messages xs)
proof –
  have  $\bigwedge x\ y. \{x, y\} \subseteq set\ (node-deliver-messages\ xs) \implies hb.concurrent\ x\ y \implies interp-msg\ x \triangleright$ 
 $interp-msg\ y = interp-msg\ y \triangleright interp-msg\ x$ 
  proof
    fix x y ii
    assume  $\{x, y\} \subseteq set\ (node-deliver-messages\ xs)$ 
    and C: hb.concurrent x y
    hence X:  $x \in set\ (node-deliver-messages\ xs)$  and Y:  $y \in set\ (node-deliver-messages\ xs)$ 
    by auto
    obtain x1 x2 y1 y2 where  $1: x = (x1, x2)$  and  $2: y = (y1, y2)$ 
    by fastforce
    have  $(interp-msg\ (x1, x2) \triangleright interp-msg\ (y1, y2))\ ii = (interp-msg\ (y1, y2) \triangleright interp-msg\ (x1, x2))$ 
  ii
    proof(cases x2; cases y2)
    fix ix1 ix2 iy1 iy2
    assume X2:  $x2 = Insert\ ix1\ ix2$  and Y2:  $y2 = Insert\ iy1\ iy2$ 
    show  $(interp-msg\ (x1, x2) \triangleright interp-msg\ (y1, y2))\ ii = (interp-msg\ (y1, y2) \triangleright interp-msg\ (x1,$ 
 $x2))\ ii$ 
    proof(cases fst ix1 = fst iy1)
    assume fst ix1 = fst iy1
    hence  $Insert\ ix1\ ix2 = Insert\ iy1\ iy2$ 
    apply(rule same-insert)
    using  $1\ 2\ X\ Y\ X2\ Y2$  assms apply auto
    done
    hence  $ix1 = iy1$  and  $ix2 = iy2$ 

```

```

    by auto
    from this and X2 Y2 show (interp-msg (x1, x2) ▷ interp-msg (y1, y2)) ii = (interp-msg (y1,
y2) ▷ interp-msg (x1, x2)) ii
    by(clarsimp simp add: kleisli-def interp-msg-def)
next
assume NEQ: fst ix1 ≠ fst iy1
have ix2 = None ∨ ix2 ≠ Some (fst iy1)
  apply(rule insert-commute-assms)
  using prefix-msg-in-history[OF assms] X Y X2 Y2 1 2
  apply(clarsimp, blast)
  using C 1 2 X2 Y2 apply blast
done
also have iy2 = None ∨ iy2 ≠ Some (fst ix1)
  apply(rule insert-commute-assms)
  using prefix-msg-in-history[OF assms] X Y X2 Y2 1 2
  apply(clarsimp, blast)
  using 1 2 C X2 Y2 apply blast
done
ultimately have insert ii ix1 ix2 ≫ (λx. insert x iy1 iy2) = insert ii iy1 iy2 ≫ (λx. insert x
ix1 ix2)
  using NEQ insert-commutes by blast
  thus (interp-msg (x1, x2) ▷ interp-msg (y1, y2)) ii = (interp-msg (y1, y2) ▷ interp-msg (x1,
x2)) ii
  by(clarsimp simp add: interp-msg-def X2 Y2 kleisli-def)
qed
next
fix ix1 ix2 yd
assume X2: x2 = Insert ix1 ix2 and Y2: y2 = Delete yd
have hb.concurrent (x1, Insert ix1 ix2) (y1, Delete yd)
  using C X2 Y2 1 2 by simp
also have {Deliver (x1, Insert ix1 ix2), Deliver (y1, Delete yd)} ⊆ set (history i)
  using prefix-msg-in-history assms X2 Y2 X Y 1 2 by blast
ultimately have yd ≠ fst ix1
  apply –
  apply(rule Insert-Delete-concurrent; force)
done
hence insert ii ix1 ix2 ≫ (λx. delete x yd) = delete ii yd ≫ (λx. insert x ix1 ix2)
  by(rule insert-delete-commute)
  thus (interp-msg (x1, x2) ▷ interp-msg (y1, y2)) ii = (interp-msg (y1, y2) ▷ interp-msg (x1,
x2)) ii
  by(clarsimp simp add: interp-msg-def kleisli-def X2 Y2)
next
fix xd iy1 iy2
assume X2: x2 = Delete xd and Y2: y2 = Insert iy1 iy2
have hb.concurrent (x1, Delete xd) (y1, Insert iy1 iy2)
  using C X2 Y2 1 2 by simp
also have {Deliver (x1, Delete xd), Deliver (y1, Insert iy1 iy2)} ⊆ set (history i)
  using prefix-msg-in-history assms X2 Y2 X Y 1 2 by blast
ultimately have xd ≠ fst iy1
  apply –
  apply(rule Insert-Delete-concurrent; force)
done
hence delete ii xd ≫ (λx. insert x iy1 iy2) = insert ii iy1 iy2 ≫ (λx. delete x xd)
  by(rule insert-delete-commute[symmetric])
  thus (interp-msg (x1, x2) ▷ interp-msg (y1, y2)) ii = (interp-msg (y1, y2) ▷ interp-msg (x1,
x2)) ii
  by(clarsimp simp add: interp-msg-def kleisli-def X2 Y2)
next

```

```

fix xd yd
assume X2: x2 = Delete xd and Y2: y2 = Delete yd
have delete ii xd ≫ (λx. delete x yd) = delete ii yd ≫ (λx. delete x xd)
  by(rule delete-commutes)
thus (interp-msg (x1, x2) ▷ interp-msg (y1, y2)) ii = (interp-msg (y1, y2) ▷ interp-msg (x1,
x2)) ii
  by(clarsimp simp add: interp-msg-def kleisli-def X2 Y2)
qed
thus (interp-msg x ▷ interp-msg y) ii = (interp-msg y ▷ interp-msg x) ii
  using 1 2 by auto
qed
thus hb.concurrent-ops-commute (node-deliver-messages xs)
  by(auto simp add: hb.concurrent-ops-commute-def)
qed

```

corollary (**in** *rga*) *concurrent-operations-commute'*:
shows *hb.concurrent-ops-commute (node-deliver-messages (history i))*
by (*meson concurrent-operations-commute append.right-neutral prefix-of-node-history-def*)

lemma (**in** *rga*) *apply-operations-never-fails*:

```

assumes xs prefix of i
shows apply-operations xs ≠ None
using assms proof(induction xs rule: rev-induct)
show apply-operations [] ≠ None
  by clarsimp
next
fix x xs
assume 1: xs prefix of i ⇒ apply-operations xs ≠ None
  and 2: xs @ [x] prefix of i
hence 3: xs prefix of i
  by auto
show apply-operations (xs @ [x]) ≠ None
proof(cases x)
  fix b
  assume x = Broadcast b
  thus apply-operations (xs @ [x]) ≠ None
    using 1 3 by clarsimp
next
fix d
assume 4: x = Deliver d
thus apply-operations (xs @ [x]) ≠ None
proof(cases d; clarify)
  fix a b
  assume 5: x = Deliver (a, b)
show  $\exists y. \text{apply-operations } (xs @ [\text{Deliver } (a, b)]) = \text{Some } y$ 
proof(cases b; clarify)
  fix aa aaa ba x12
  assume 6: b = Insert (aa, aaa, ba) x12
show  $\exists y. \text{apply-operations } (xs @ [\text{Deliver } (a, \text{Insert } (aa, aaa, ba) x12)]) = \text{Some } y$ 
  apply(clarsimp simp add: 1 interp-msg-def split!: bind-splits)
  apply(simp add: 1 3)
  apply(rule rga.Insert-no-failure, rule rga-axioms)
  using 4 5 6 2 apply force+
  done
next
fix x2
assume 6: b = Delete x2
show  $\exists y. \text{apply-operations } (xs @ [\text{Deliver } (a, \text{Delete } x2)]) = \text{Some } y$ 

```

```

    apply (clarsimp simp add: interp-msg-def split!: bind-splits)
    apply (simp add: 1 3)
    apply (rule delete-no-failure)
    using 4 5 6 2 apply force+
  done
qed
qed
qed
qed

lemma (in rga) apply-operations-never-fails':
  shows apply-operations (history i) ≠ None
by (meson apply-operations-never-fails append.right-neutral prefix-of-node-history-def)

corollary (in rga) rga-convergence:
  assumes set (node-deliver-messages xs) = set (node-deliver-messages ys)
    and xs prefix of i
    and ys prefix of j
  shows apply-operations xs = apply-operations ys
using assms by (auto simp add: apply-operations-def intro: hb.convergence-ext
  concurrent-operations-commute node-deliver-messages-distinct hb-consistent-prefix)

```

5.7 Strong eventual consistency

context rga begin

```

sublocale sec: strong-eventual-consistency weak-hb hb interp-msg
  λops. ∃ xs i. xs prefix of i ∧ node-deliver-messages xs = ops []
proof (standard; clarsimp)
  fix xsa i
  assume xsa prefix of i
  thus hb.hb-consistent (node-deliver-messages xsa)
    by (auto simp add: hb-consistent-prefix)
next
  fix xsa i
  assume xsa prefix of i
  thus distinct (node-deliver-messages xsa)
    by (auto simp add: node-deliver-messages-distinct)
next
  fix xsa i
  assume xsa prefix of i
  thus hb.concurrent-ops-commute (node-deliver-messages xsa)
    by (auto simp add: concurrent-operations-commute)
next
  fix xs a b state xsa x
  assume hb.apply-operations xs [] = Some state
    and node-deliver-messages xsa = xs @ [(a, b)]
    and xsa prefix of x
  thus ∃ y. interp-msg (a, b) state = Some y
    by (metis (no-types, lifting) apply-operations-def bind.bind-lunit not-None-eq
      hb.apply-operations-Snoc kleisli-def apply-operations-never-fails interp-msg-def)
next
  fix xs a b xsa x
  assume node-deliver-messages xsa = xs @ [(a, b)]
    and xsa prefix of x
  thus ∃ xsa. (∃ x. xsa prefix of x) ∧ node-deliver-messages xsa = xs
    using drop-last-message by blast
qed

```

end

interpretation *trivial-rga-implementation*: rga $\lambda x.$ []
by(*standard*, *auto simp add: trivial-node-histories.history-order-def*
trivial-node-histories.prefix-of-node-history-def)

end

6 Increment-Decrement Counter

The Increment-Decrement Counter is perhaps the simplest CRDT, and a paradigmatic example of a replicated data structure with commutative operations.

theory

Counter

imports

Network

begin

datatype *operation* = *Increment* | *Decrement*

fun *counter-op* :: *operation* \Rightarrow *int* \rightarrow *int* **where**
counter-op Increment *x* = *Some* (*x* + 1) |
counter-op Decrement *x* = *Some* (*x* - 1)

locale *counter* = *network-with-ops* - *counter-op* 0

lemma (**in** *counter*) *counter-op* *x* \triangleright *counter-op* *y* = *counter-op* *y* \triangleright *counter-op* *x*
by(*case-tac* *x*; *case-tac* *y*; *auto simp add: kleisli-def*)

lemma (**in** *counter*) *concurrent-operations-commute*:

assumes *xs* *prefix of i*

shows *hb.concurrent-ops-commute* (*node-deliver-messages xs*)

using *assms*

apply(*clarsimp simp: hb.concurrent-ops-commute-def*)

apply(*rename-tac a b x y*)

apply(*case-tac b*; *case-tac y*; *force simp add: interp-msg-def kleisli-def*)

done

corollary (**in** *counter*) *counter-convergence*:

assumes *set* (*node-deliver-messages xs*) = *set* (*node-deliver-messages ys*)

and *xs* *prefix of i*

and *ys* *prefix of j*

shows *apply-operations xs* = *apply-operations ys*

using *assms* **by**(*auto simp add: apply-operations-def intro: hb.convergence-ext*

concurrent-operations-commute node-deliver-messages-distinct hb-consistent-prefix)

context *counter* **begin**

sublocale *sec: strong-eventual-consistency weak-hb hb interp-msg*

lops. \exists xs i. xs prefix of i \wedge node-deliver-messages xs = ops 0

apply(*standard*; *clarsimp simp add: hb-consistent-prefix drop-last-message*

node-deliver-messages-distinct concurrent-operations-commute)

apply(*metis (full-types) interp-msg-def counter-op.elims*)

using *drop-last-message* **apply** *blast*

done

end
end

7 Observed-Remove Set

The ORSet is a well-known CRDT for implementing replicated sets, supporting two operations: the *insertion* and *deletion* of an arbitrary element in the shared set.

theory

ORSet

imports

Network

begin

datatype (*'id*, *'a*) *operation* = *Add 'id 'a* | *Rem 'id set 'a*

type-synonym (*'id*, *'a*) *state* = *'a* \Rightarrow *'id set*

definition *op-elem* :: (*'id*, *'a*) *operation* \Rightarrow *'a* **where**

op-elem oper \equiv *case oper of Add i e* \Rightarrow *e* | *Rem is e* \Rightarrow *e*

definition *interpret-op* :: (*'id*, *'a*) *operation* \Rightarrow (*'id*, *'a*) *state* \rightarrow (*'id*, *'a*) *state* ($\langle - \rangle$ [0] 1000) **where**

interpret-op oper state \equiv

let before = *state (op-elem oper)*;

after = *case oper of Add i e* \Rightarrow *before* \cup $\{i\}$ | *Rem is e* \Rightarrow *before* $-$ *is*

in Some (state ((op-elem oper) := after))

definition *valid-behaviours* :: (*'id*, *'a*) *state* \Rightarrow *'id* \times (*'id*, *'a*) *operation* \Rightarrow *bool* **where**

valid-behaviours state msg \equiv

case msg of

(i, Add j e) \Rightarrow *i = j* |

(i, Rem is e) \Rightarrow *is = state e*

locale *orset* = *network-with-constrained-ops - interpret-op* $\lambda x.$ $\{ \}$ *valid-behaviours*

lemma (**in** *orset*) *add-add-commute*:

shows $\langle \text{Add } i1 \ e1 \rangle \triangleright \langle \text{Add } i2 \ e2 \rangle = \langle \text{Add } i2 \ e2 \rangle \triangleright \langle \text{Add } i1 \ e1 \rangle$

by (*auto simp add: interpret-op-def op-elem-def kleisli-def, fastforce*)

lemma (**in** *orset*) *add-rem-commute*:

assumes *i* \notin *is*

shows $\langle \text{Add } i \ e1 \rangle \triangleright \langle \text{Rem } is \ e2 \rangle = \langle \text{Rem } is \ e2 \rangle \triangleright \langle \text{Add } i \ e1 \rangle$

using *assms* **by** (*auto simp add: interpret-op-def kleisli-def op-elem-def, fastforce*)

lemma (**in** *orset*) *apply-operations-never-fails*:

assumes *xs* *prefix of i*

shows *apply-operations xs* \neq *None*

using *assms* **proof** (*induction xs* *rule: rev-induct, clarsimp*)

case (*snoc x xs*) **thus** *?case*

proof (*cases x*)

case (*Broadcast e*) **thus** *?thesis*

using *snoc* **by** *force*

next

case (*Deliver e*) **thus** *?thesis*

using *snoc* **by** (*clarsimp, metis interpret-op-def interp-msg-def bind.bind-lunit prefix-of-appendD*)

qed

qed

lemma (in orset) *add-id-valid*:

assumes *xs* prefix of *j*
and *Deliver* (*i1*, *Add i2 e*) ∈ *set xs*
shows *i1* = *i2*

proof –

have ∃ *s*. *valid-behaviours s* (*i1*, *Add i2 e*)
using *assms deliver-in-prefix-is-valid* **by** *blast*
thus *?thesis*
by (*simp add: valid-behaviours-def*)

qed

definition (in orset) *added-ids* :: ('*id* × ('*id*, '*b*) operation) event list ⇒ '*b* ⇒ '*id* list **where**
added-ids es p ≡ *List.map-filter* (λ*x*. *case x of Deliver* (*i*, *Add j e*) ⇒ *if e = p then Some j else None*
| - ⇒ *None*) *es*

lemma (in orset) [*simp*]:

shows *added-ids* [] *e* = []
by (*auto simp: added-ids-def map-filter-def*)

lemma (in orset) [*simp*]:

shows *added-ids* (*xs @ ys*) *e* = *added-ids xs e @ added-ids ys e*
by (*auto simp: added-ids-def map-filter-append*)

lemma (in orset) *added-ids-Broadcast-collapse* [*simp*]:

shows *added-ids* ([*Broadcast e*]) *e'* = []
by (*auto simp: added-ids-def map-filter-append map-filter-def*)

lemma (in orset) *added-ids-Deliver-Rem-collapse* [*simp*]:

shows *added-ids* ([*Deliver* (*i*, *Rem is e*)]) *e'* = []
by (*auto simp: added-ids-def map-filter-append map-filter-def*)

lemma (in orset) *added-ids-Deliver-Add-diff-collapse* [*simp*]:

shows *e* ≠ *e'* ⇒ *added-ids* ([*Deliver* (*i*, *Add j e*)]) *e'* = []
by (*auto simp: added-ids-def map-filter-append map-filter-def*)

lemma (in orset) *added-ids-Deliver-Add-same-collapse* [*simp*]:

shows *added-ids* ([*Deliver* (*i*, *Add j e*)]) *e* = [*j*]
by (*auto simp: added-ids-def map-filter-append map-filter-def*)

lemma (in orset) *added-id-not-in-set*:

assumes *i1* ∉ *set* (*added-ids* [*Deliver* (*i*, *Add i2 e*)] *e*)
shows *i1* ≠ *i2*
using *assms* **by** *simp*

lemma (in orset) *apply-operations-added-ids*:

assumes *es* prefix of *j*
and *apply-operations es* = *Some f*
shows *f x* ⊆ *set* (*added-ids es x*)
using *assms* **proof** (*induct es arbitrary: f rule: rev-induct, force*)
case (*snoc x xs*) **thus** *?case*
proof (*cases x, force*)
case (*Deliver e*)
moreover obtain *a b* **where** *e* = (*a*, *b*) **by** *force*
ultimately show *?thesis*
using *snoc* **by** (*case-tac b; clarsimp simp: interp-msg-def split: bind-splits,*
force split: if-split-asm simp add: op-elem-def interpret-op-def)

qed

qed

lemma (in orset) *Deliver-added-ids*:
assumes xs prefix of j
and $i \in \text{set } (\text{added-ids } xs \ e)$
shows $\text{Deliver } (i, \text{Add } i \ e) \in \text{set } xs$
using *assms proof* (induct xs rule: rev-induct, clarsimp)
case (snoc $x \ xs$) **thus** ?case
proof (cases x , force)
case (*Deliver* e')
moreover obtain $a \ b$ **where** $e' = (a, b)$ **by** force
ultimately show ?thesis
using snoc **apply** (case-tac b ; clarsimp)
apply (metis added-ids-Deliver-Add-diff-collapse added-ids-Deliver-Add-same-collapse
empty-iff list.set(1) set-ConsD add-id-valid in-set-conv-decomp prefix-of-appendD)
apply force
done
qed
qed

lemma (in orset) *Broadcast-Deliver-prefix-closed*:
assumes $xs \ @ \ [\text{Broadcast } (r, \text{Rem } ix \ e)]$ prefix of j
and $i \in ix$
shows $\text{Deliver } (i, \text{Add } i \ e) \in \text{set } xs$
proof –
obtain y **where** apply-operations $xs = \text{Some } y$
using *assms broadcast-only-valid-msgs* **by** blast
moreover hence $ix = y \ e$
by (metis (mono-tags, lifting) *assms*(1) broadcast-only-valid-msgs operation.case(2) option.simps(1)
valid-behaviours-def case-prodD)
ultimately show ?thesis
using *assms Deliver-added-ids apply-operations-added-ids* **by** blast
qed

lemma (in orset) *Broadcast-Deliver-prefix-closed2*:
assumes xs prefix of j
and $\text{Broadcast } (r, \text{Rem } ix \ e) \in \text{set } xs$
and $i \in ix$
shows $\text{Deliver } (i, \text{Add } i \ e) \in \text{set } xs$
using *assms Broadcast-Deliver-prefix-closed* **by** (induction xs rule: rev-induct; force)

lemma (in orset) *concurrent-add-remove-independent-technical*:
assumes $i \in is$
and xs prefix of j
and $(i, \text{Add } i \ e) \in \text{set } (\text{node-deliver-messages } xs)$ **and** $(ir, \text{Rem } is \ e) \in \text{set } (\text{node-deliver-messages } xs)$
shows $hb \ (i, \text{Add } i \ e) \ (ir, \text{Rem } is \ e)$
proof –
obtain $pre \ k$ **where** $pre @ [\text{Broadcast } (ir, \text{Rem } is \ e)]$ prefix of k
using *assms delivery-has-a-cause events-before-exist prefix-msg-in-history* **by** blast
moreover hence $\text{Deliver } (i, \text{Add } i \ e) \in \text{set } pre$
using *Broadcast-Deliver-prefix-closed* *assms*(1) **by** auto
ultimately show ?thesis
using $hb.intros(2)$ *events-in-local-order* **by** blast
qed

lemma (in orset) *Deliver-Add-same-id-same-message*:
assumes $\text{Deliver } (i, \text{Add } i \ e1) \in \text{set } (\text{history } j)$ **and** $\text{Deliver } (i, \text{Add } i \ e2) \in \text{set } (\text{history } j)$
shows $e1 = e2$

proof –

obtain $pre1\ pre2\ k1\ k2$ **where** $*$: $pre1@[Broadcast\ (i,\ Add\ i\ e1)]$ *prefix of* $k1$ $pre2@[Broadcast\ (i,\ Add\ i\ e2)]$ *prefix of* $k2$

using *assms delivery-has-a-cause events-before-exist* **by** *meson*

moreover hence $Broadcast\ (i,\ Add\ i\ e1) \in set\ (history\ k1)$ $Broadcast\ (i,\ Add\ i\ e2) \in set\ (history\ k2)$

using *node-histories.prefix-to-carriers node-histories-axioms* **by** *force+*

ultimately show *?thesis*

using *msg-id-unique* **by** *fastforce*

qed

lemma (**in** *orset*) *ids-imply-messages-same*:

assumes $i \in is$

and xs *prefix of* j

and $(i,\ Add\ i\ e1) \in set\ (node-deliver-messages\ xs)$ **and** $(ir,\ Rem\ is\ e2) \in set\ (node-deliver-messages\ xs)$

shows $e1 = e2$

proof –

obtain $pre\ k$ **where** $pre@[Broadcast\ (ir,\ Rem\ is\ e2)]$ *prefix of* k

using *assms delivery-has-a-cause events-before-exist prefix-msg-in-history* **by** *blast*

moreover hence $Deliver\ (i,\ Add\ i\ e2) \in set\ pre$

using *Broadcast-Deliver-prefix-closed assms(1)* **by** *blast*

moreover have $Deliver\ (i,\ Add\ i\ e1) \in set\ (history\ j)$

using *assms(2) assms(3) prefix-msg-in-history* **by** *blast*

ultimately show *?thesis*

by (*metis fst-conv msg-id-unique network.delivery-has-a-cause network-axioms operation.inject(1) prefix-elem-to-carriers prefix-of-appendD prod.inject*)

qed

corollary (**in** *orset*) *concurrent-add-remove-independent*:

assumes $\neg hb\ (i,\ Add\ i\ e1)\ (ir,\ Rem\ is\ e2)$ **and** $\neg hb\ (ir,\ Rem\ is\ e2)\ (i,\ Add\ i\ e1)$

and xs *prefix of* j

and $(i,\ Add\ i\ e1) \in set\ (node-deliver-messages\ xs)$ **and** $(ir,\ Rem\ is\ e2) \in set\ (node-deliver-messages\ xs)$

shows $i \notin is$

using *assms ids-imply-messages-same concurrent-add-remove-independent-technical* **by** *fastforce*

lemma (**in** *orset*) *rem-rem-commute*:

shows $\langle Rem\ i1\ e1 \rangle \triangleright \langle Rem\ i2\ e2 \rangle = \langle Rem\ i2\ e2 \rangle \triangleright \langle Rem\ i1\ e1 \rangle$

by(*unfold interpret-op-def op-elem-def kleisli-def, fastforce*)

lemma (**in** *orset*) *concurrent-operations-commute*:

assumes xs *prefix of* i

shows $hb.concurrent-ops-commute\ (node-deliver-messages\ xs)$

proof –

{ **fix** $a\ b\ x\ y$

assume $(a,\ b) \in set\ (node-deliver-messages\ xs)$

$(x,\ y) \in set\ (node-deliver-messages\ xs)$

$hb.concurrent\ (a,\ b)\ (x,\ y)$

hence $interp-msg\ (a,\ b) \triangleright interp-msg\ (x,\ y) = interp-msg\ (x,\ y) \triangleright interp-msg\ (a,\ b)$

apply(*unfold interp-msg-def, case-tac b; case-tac y; simp add: add-add-commute rem-rem-commute hb.concurrent-def*)

apply (*metis add-id-valid add-rem-commute assms concurrent-add-remove-independent hb.concurrentD1 hb.concurrentD2 prefix-contains-msg*)+

done

} **thus** *?thesis*

by(*fastforce simp: hb.concurrent-ops-commute-def*)

qed

```

theorem (in orset) convergence:
  assumes set (node-deliver-messages xs) = set (node-deliver-messages ys)
    and xs prefix of i and ys prefix of j
  shows apply-operations xs = apply-operations ys
using assms by (auto simp add: apply-operations-def intro: hb.convergence-ext concurrent-operations-commute
  node-deliver-messages-distinct hb-consistent-prefix)

context orset begin

sublocale sec: strong-eventual-consistency weak-hb hb interp-msg
  λops.∃ xs i. xs prefix of i ∧ node-deliver-messages xs = ops λx.{}
  apply(standard; clarsimp simp add: hb-consistent-prefix node-deliver-messages-distinct
  concurrent-operations-commute)
  apply(metis (no-types, lifting) apply-operations-def bind.bind-lunit not-None-eq
  hb.apply-operations-Snoc kleisli-def apply-operations-never-fails interp-msg-def)
  using drop-last-message apply blast
done

end
end

```

References

- [1] P. S. Almeida, A. Shoker, and C. Baquero. Efficient state-based CRDTs by delta-mutation. In *International Conference on Networked Systems (NETYS)*, May 2015.
- [2] C. Baquero, P. S. Almeida, and A. Shoker. Making operation-based CRDTs operation-based. In *14th IFIP International Conference on Distributed Applications and Interoperable Systems (DAIS)*, pages 126–140, June 2014.
- [3] R. Brown, S. Cribbs, C. Meiklejohn, and S. Elliott. Riak DT map: a composable, convergent replicated dictionary. In *1st Workshop on Principles and Practice of Eventual Consistency (PaPEC)*, Apr. 2014.
- [4] C. Cachin, R. Guerraoui, and L. Rodrigues. *Introduction to Reliable and Secure Distributed Programming*. Springer, second edition, Feb. 2011.
- [5] J. Day-Richter. What’s different about the new Google Docs: Making collaboration fast, Sept. 2010.
- [6] A. Imine, P. Molli, G. Oster, and M. Rusinowitch. Proving correctness of transformation functions in real-time groupware. In *8th European Conference on Computer-Supported Cooperative Work (ECSCW)*, pages 277–293, Sept. 2003.
- [7] A. Imine, M. Rusinowitch, G. Oster, and P. Molli. Formal design and verification of operational transformation algorithms for copies convergence. *Theoretical Computer Science*, 351(2):167–183, Feb. 2006.
- [8] L. Lamport. Time, clocks, and the ordering of events in a distributed system. *Communications of the ACM*, 21(7):558–565, July 1978.
- [9] G. Oster, P. Urso, P. Molli, and A. Imine. Proving correctness of transformation functions in collaborative editing systems. Technical Report RR-5795, Dec. 2005.

- [10] H.-G. Roh, M. Jeon, J.-S. Kim, and J. Lee. Replicated abstract data types: Building blocks for collaborative applications. *Journal of Parallel and Distributed Computing*, 71(3):354–368, 2011.
- [11] M. Shapiro, N. Preguiça, C. Baquero, and M. Zawirski. A comprehensive study of convergent and commutative replicated data types. Technical Report 7506, INRIA, 2011.
- [12] M. Shapiro, N. Preguiça, C. Baquero, and M. Zawirski. Conflict-free replicated data types. In *13th International Symposium on Stabilization, Safety, and Security of Distributed Systems (SSS)*, pages 386–400, Oct. 2011.
- [13] M. Wenzel, L. C. Paulson, and T. Nipkow. The Isabelle framework. In *Theorem Proving in Higher Order Logics, 21st International Conference, TPHOLs 2008, Montreal, Canada, August 18-21, 2008. Proceedings*, pages 33–38, 2008.