# Buffon's Needle Problem

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**Abstract**

In the 18th century, Georges-Louis Leclerc, Comte de Buffon posed and later solved the following problem [1, 2], which is often called the first problem ever solved in geometric probability: Given a floor divided into vertical strips of the same width, what is the probability that a needle thrown onto the floor randomly will cross two strips?

This entry formally defines the problem in the case where the needle's position is chosen uniformly at random in a single strip around the origin (which is equivalent to larger arrangements due to symmetry). It then provides proofs of the simple solution in the case where the needle's length is no greater than the width of the strips and the more complicated solution in the opposite case.

## Contents

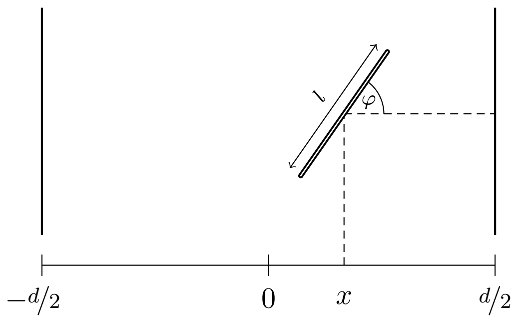


Figure 1: A sketch of the situation in Buffon's needle experiment. There is a needle of length $l$ with its centre at a certain $x$ coordinate, angled at an angle $\varphi$ off the horizontal axis. The two vertical lines are a distance of $d$ apart, each being $d/2$ away from the origin.

# 1 Buffon's Needle Problem

**theory** *Buffons-Needle*
  **imports** *HOL−Probability.Probability*
**begin**

## 1.1 Auxiliary material

**lemma** *sin-le-zero'*: $sin\ x \leq 0$ **if** $x \geq -pi\ x \leq 0$ **for** $x$
  ⟨*proof*⟩

## 1.2 Problem definition

Consider a needle of length $l$ whose centre has the $x$-coordinate $x$. The following then defines the set of all $x$-coordinates that the needle covers (i.e. the projection of the needle onto the $x$-axis.)

**definition** *needle* :: *real* ⇒ *real* ⇒ *real* ⇒ *real set* **where**
  *needle l x φ = closed-segment (x − l / 2 ∗ sin φ) (x + l / 2 ∗ sin φ)*

Buffon's Needle problem is then this: Assuming the needle's $x$ position is chosen uniformly at random in a strip of width $d$ centred at the origin, what is the probability that the needle crosses at least one of the left/right boundaries of that strip (located at $x = \pm\frac{1}{2}d$)?
We will show that, if we let $x := l/d$, the probability of this is

$$\mathcal{P}_{l,d} = \begin{cases} 2/\pi \cdot x & \text{if } l \leq d \\ 2/\pi \cdot (x - \sqrt{x^2 - 1} + \arccos(1/x)) & \text{if } l \geq d \end{cases}$$

A plot of this function can be found in Figure 2.

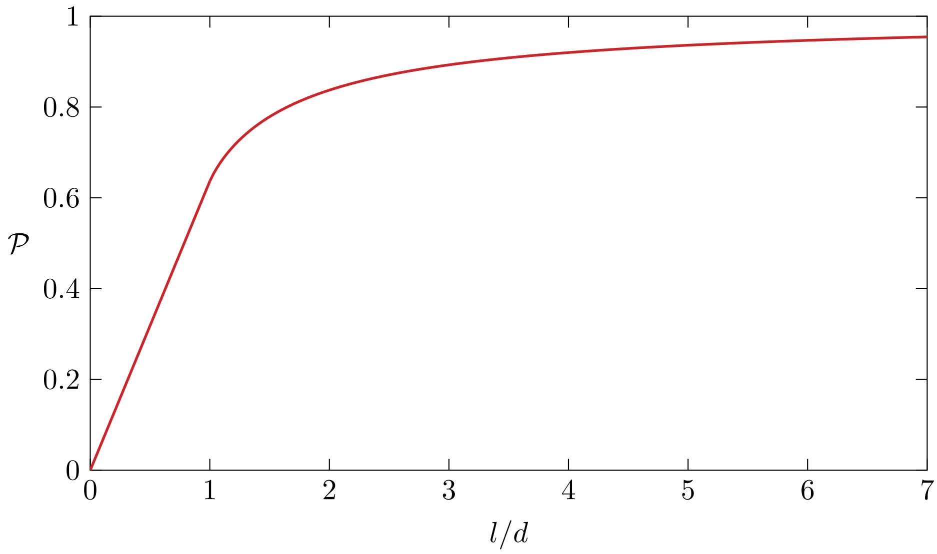


Figure 2: The probability $\mathcal{P}$ of the needle hitting one of the lines, as a function of the quotient $l/d$ (where $l$ is the length of the needle and $d$ the horizontal distance between the lines).

**locale** *Buffon* =
  **fixes** *d l* :: *real*
  **assumes** *d*: *d* > *0* **and** *l*: *l* > *0*
**begin**

**definition** *Buffon* :: (*real* × *real*) *measure* **where**
  *Buffon* = *uniform-measure lborel* ({−*d*/*2*..*d*/*2*} × {−*pi*..*pi*})

**lemma** *space-Buffon* [*simp*]: *space Buffon* = *UNIV*
  ⟨*proof*⟩

**definition** *Buffon-set* :: (*real* × *real*) *set* **where**
  *Buffon-set* = {(*x*,*φ*) ∈ {−*d*/*2*..*d*/*2*} × {−*pi*..*pi*}. *needle l x φ* ∩ {−*d*/*2*, *d*/*2*} ≠ {}}

## 1.3 Derivation of the solution

The following form is a bit easier to handle.

**lemma** *Buffon-set-altdef1*:
  *Buffon-set* =
    {(*x*,*φ*) ∈ {−*d*/*2*..*d*/*2*} × {−*pi*..*pi*}.
      *let a* = *x* − *l* / *2* ∗ *sin φ*; *b* = *x* + *l* / *2* ∗ *sin φ*
      *in min a b* + *d*/*2* ≤ *0* ∧ *max a b* + *d*/*2* ≥ *0* ∨ *min a b* − *d*/*2* ≤ *0* ∧ *max a b* − *d*/*2* ≥ *0*}

⟨*proof*⟩

**lemma** *Buffon-set-altdef2*:
*Buffon-set* = {(*x*,*φ*) ∈ {−*d*/*2*..*d*/*2*} × {−*pi*..*pi*}. *abs x* ≥ *d* / *2* − *abs* (*sin φ*) ∗ *l* / *2*}
⟨*proof*⟩

By using the symmetry inherent in the problem, we can reduce the problem to the following set, which corresponds to one quadrant of the original set:

**definition** *Buffon-set′* :: (*real* × *real*) *set* **where**
*Buffon-set′* = {(*x*,*φ*) ∈ {*0*..*d*/*2*} × {*0*..*pi*}. *x* ≥ *d* / *2* − *sin φ* ∗ *l* / *2*}

**lemma** *closed-buffon-set* [*simp*, *intro*, *measurable*]: *closed Buffon-set*
⟨*proof*⟩

**lemma** *closed-buffon-set′* [*simp*, *intro*, *measurable*]: *closed Buffon-set′*
⟨*proof*⟩

**lemma** *measurable-buffon-set* [*measurable*]: *Buffon-set* ∈ *sets borel*
⟨*proof*⟩

**lemma** *measurable-buffon-set′* [*measurable*]: *Buffon-set′* ∈ *sets borel*
⟨*proof*⟩

**sublocale** *prob-space Buffon*
⟨*proof*⟩

**lemma** *buffon-prob-aux*:
*emeasure Buffon* {(*x*,*φ*). *needle l x φ* ∩ {−*d*/*2*, *d*/*2*} ≠ {}} =
*emeasure lborel Buffon-set* / *ennreal* (*2* ∗ *d* ∗ *pi*)
⟨*proof*⟩

**lemma** *emeasure-buffon-set-conv-buffon-set′*:
*emeasure lborel Buffon-set* = *4* ∗ *emeasure lborel Buffon-set′*
⟨*proof*⟩

It only remains now to compute the measure of *Buffon-set′*. We first reduce this problem to a relatively simple integral:

**lemma** *emeasure-buffon-set′*:
*emeasure lborel Buffon-set′* =
*ennreal* (*integral* {*0*..*pi*} (*λx*. *min* (*d* / *2*) (*sin x* ∗ *l* / *2*)))
(**is** *emeasure lborel ?A* = -)
⟨*proof*⟩

We now have to distinguish two cases: The first and easier one is that where the length of the needle, *l*, is less than or equal to the strip width, *d*:

**context**

**assumes** *l-le-d*: $l \leq d$
**begin**

**lemma** *emeasure-buffon-set'-short*: *emeasure lborel Buffon-set'* = *ennreal l*
⟨*proof*⟩

**lemma** *emeasure-buffon-set-short*: *emeasure lborel Buffon-set* = *4* ∗ *ennreal l*
⟨*proof*⟩

**lemma** *prob-short-aux*:
*Buffon* {(*x*, φ). *needle l x* φ ∩ {− *d* / *2*, *d* / *2*} ≠ {}} = *ennreal* (*2* ∗ *l* / (*d* ∗ *pi*))
⟨*proof*⟩

**lemma** *prob-short*: $\mathcal{P}$((*x*,φ) *in Buffon*. *needle l x* φ ∩ {−*d*/*2*, *d*/*2*} ≠ {}) = *2* ∗ *l* / (*d* ∗ *pi*)
⟨*proof*⟩

**end**

The other case where the needle is at least as long as the strip width is more complicated:

**context**
**assumes** *l-ge-d*: $l \geq d$
**begin**

**lemma** *emeasure-buffon-set'-long*:
**shows** *l* ∗ (*1* − *sqrt* (*1* − (*d* / *l*)$^2$)) + *arccos* (*d* / *l*) ∗ *d* ≥ *0*
**and** *emeasure lborel Buffon-set'* =
*ennreal* (*l* ∗ (*1* − *sqrt* (*1* − (*d* / *l*)$^2$)) + *arccos* (*d* / *l*) ∗ *d*)
⟨*proof*⟩

**lemma** *emeasure-set-long*: *emeasure lborel Buffon-set* =
*4* ∗ *ennreal* (*l* ∗ (*1* − *sqrt* (*1* − (*d* / *l*)$^2$)) + *arccos* (*d* / *l*) ∗ *d*)
⟨*proof*⟩

**lemma** *prob-long-aux*:
**shows** *2* / *pi* ∗ ((*l* / *d*) − *sqrt* ((*l* / *d*)$^2$ − *1*) + *arccos* (*d* / *l*)) ≥ *0*
**and** *Buffon* {(*x*, φ). *needle l x* φ ∩ {− *d* / *2*, *d* / *2*} ≠ {}} =
*ennreal* (*2* / *pi* ∗ ((*l* / *d*) − *sqrt* ((*l* / *d*)$^2$ − *1*) + *arccos* (*d* / *l*)))
⟨*proof*⟩

**lemma** *prob-long*:
$\mathcal{P}$((*x*,φ) *in Buffon*. *needle l x* φ ∩ {−*d*/*2*, *d*/*2*} ≠ {}) =
*2* / *pi* ∗ ((*l* / *d*) − *sqrt* ((*l* / *d*)$^2$ − *1*) + *arccos* (*d* / *l*))
⟨*proof*⟩

**end**

```
theorem prob-eq:
  defines x ≡ l / d
  shows   𝒫((x,φ) in Buffon. needle l x φ ∩ {−d/2, d/2} ≠ {}) =
          (if l ≤ d then
             2 / pi ∗ x
           else
             2 / pi ∗ (x − sqrt (x² − 1) + arccos (1 / x)))
  ⟨proof⟩

end

end
```

## References


[1] J. F. Ramaley. Buffon's Noodle Problem. *The American Mathematical Monthly*, 76(8):916–918, 1969.

[2] E. W. Weisstein. MathWorld – Buffon's Needle Problem. http://mathworld.wolfram.com/BuffonsNeedleProblem.html.