Buffon's Needle Problem

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Abstract

In the 18th century, Georges-Louis Leclerc, Comte de Buffon posed and later solved the following problem [1, 2], which is often called the first problem ever solved in geometric probability: Given a floor divided into vertical strips of the same width, what is the probability that a needle thrown onto the floor randomly will cross two strips?

This entry formally defines the problem in the case where the needle's position is chosen uniformly at random in a single strip around the origin (which is equivalent to larger arrangements due to symmetry). It then provides proofs of the simple solution in the case where the needle's length is no greater than the width of the strips and the more complicated solution in the opposite case.

Contents

1 Buffon's Needle Problem

theory *Buffons-Needle*
  imports *Probability*
begin

1.1 Auxiliary material

lemma *sin-le-zero'*: *sin* $x \leq 0$ **if** $x \geq -pi$ $x \leq 0$ **for** x
  ⟨*proof*⟩

lemma *emeasure-Un'*:
  assumes $A \in$ *sets* M $B \in$ *sets* M $A \cap B \in$ *null-sets* M
  shows *emeasure* M $(A \cup B) =$ *emeasure* M A $+$ *emeasure* M B
⟨*proof*⟩

lemma *singleton-null-set-lborel* [*simp*,*intro*]: $\{x\} \in$ *null-sets lborel*
  ⟨*proof*⟩

lemma *continuous-on-min* [*continuous-intros*]:
  fixes f g :: *'a::topological-space* ⇒ *'b::linorder-topology*
  shows *continuous-on* A $f \Longrightarrow$ *continuous-on* A $g \Longrightarrow$ *continuous-on* A $(\lambda x.$ *min* $(f\ x)$ $(g\ x))$
  ⟨*proof*⟩

lemma *integral-shift*:
  fixes f :: *real* ⇒ *'a::euclidean-space*
  assumes *cont*: *continuous-on* $\{a + c..b + c\}$ f
  shows *integral* $\{a..b\}$ $(f \circ (\lambda x.\ x + c)) =$ *integral* $\{a + c..b + c\}$ f
⟨*proof*⟩

lemma *arcsin-le-iff*:
  assumes $x \geq -1$ $x \leq 1$ $y \geq -pi/2$ $y \leq pi/2$
  shows *arcsin* $x \leq y \longleftrightarrow x \leq$ *sin* y
⟨*proof*⟩

lemma *le-arcsin-iff*:
  assumes $x \geq -1$ $x \leq 1$ $y \geq -pi/2$ $y \leq pi/2$
  shows *arcsin* $x \geq y \longleftrightarrow x \geq$ *sin* y
⟨*proof*⟩

1.2 Problem definition

Consider a needle of length l whose centre has the x-coordinate x. The following then defines the set of all x-coordinates that the needle covers (i.e. the projection of the needle onto the x-axis.)

definition *needle* :: *real* ⇒ *real* ⇒ *real* ⇒ *real set* **where**
  needle l x φ = *closed-segment* $(x - l\ /\ 2 * sin\ \varphi)$ $(x + l\ /\ 2 * sin\ \varphi)$

Buffon's Needle problem is then this: Assuming the needle's x position is chosen uniformly at random in a strip of width d centred at the origin, what is the probability that the needle crosses at least one of the left/right boundaries of that strip (located at $x = \pm\frac{1}{2}d$)?

definition *buffon* :: *real* ⇒ *real* ⇒ *bool measure* **where**
  buffon l d =
    do {
      (*x*, *φ*) ← *uniform-measure lborel* ({−*d*/*2*..*d*/*2*} × {−*pi*..*pi*});
      return (*count-space UNIV*) (*needle l x φ* ∩ {−*d*/*2*, *d*/*2*} ≠ {})
    }

1.3 Derivation of the solution

The following form is a bit easier to handle.

lemma *buffon-altdef*:
  buffon l d =
    do {
      (*x*, *φ*) ← *uniform-measure lborel* ({−*d*/*2*..*d*/*2*} × {−*pi*..*pi*});
      return (*count-space UNIV*)
        (*let a* = *x* − *l* / *2* ∗ *sin φ*; *b* = *x* + *l* / *2* ∗ *sin φ*
         in min a b + *d*/*2* ≤ *0* ∧ *max a b* + *d*/*2* ≥ *0* ∨ *min a b* − *d*/*2* ≤ *0* ∧ *max a b* − *d*/*2* ≥ *0*)
    }
⟨*proof*⟩

It is obvious that the problem boils down to determining the measure of the following set:

definition *buffon-set* :: *real* ⇒ *real* ⇒ (*real* × *real*) *set* **where**
  buffon-set l d = {(*x*,*φ*) ∈ {−*d*/*2*..*d*/*2*} × {−*pi*..*pi*}. *abs x* ≥ *d* / *2* − *abs* (*sin φ*) ∗ *l* / *2*}

By using the symmetry inherent in the problem, we can reduce the problem to the following set, which corresponds to one quadrant of the original set:

definition *buffon-set′* :: *real* ⇒ *real* ⇒ (*real* × *real*) *set* **where**
  buffon-set′ l d = {(*x*,*φ*) ∈ {*0*..*d*/*2*} × {*0*..*pi*}. *x* ≥ *d* / *2* − *sin φ* ∗ *l* / *2*}

lemma *closed-buffon-set* [*simp*, *intro*, *measurable*]: *closed* (*buffon-set l d*)
⟨*proof*⟩

lemma *closed-buffon-set′* [*simp*, *intro*, *measurable*]: *closed* (*buffon-set′ l d*)
⟨*proof*⟩

lemma *measurable-buffon-set* [*measurable*]: *buffon-set l d* ∈ *sets borel*
  ⟨*proof*⟩

lemma *measurable-buffon-set′* [*measurable*]: *buffon-set′ l d* ∈ *sets borel*
  ⟨*proof*⟩

context
  fixes *d l* :: *real*
  assumes *d*: $d > 0$ **and** *l*: $l > 0$
begin

lemma *buffon-altdef'*:
  buffon l d = *distr* (*uniform-measure lborel* ($\{-d/2..d/2\} \times \{-pi..pi\}$))
    (*count-space UNIV*) ($\lambda z.\ z \in$ *buffon-set l d*)
⟨*proof*⟩

lemma *buffon-prob-aux*:
  emeasure (*buffon l d*) {*True*} = *emeasure lborel* (*buffon-set l d*) / *ennreal* ($2 * d$
$* pi$)
⟨*proof*⟩

lemma *emeasure-buffon-set-conv-buffon-set'*:
  emeasure lborel (*buffon-set l d*) = 4 * *emeasure lborel* (*buffon-set' l d*)
⟨*proof*⟩

It only remains now to compute the measure of *buffon-set'*. We first reduce this problem to a relatively simple integral:

lemma *emeasure-buffon-set'*:
  emeasure lborel (*buffon-set' l d*) =
    ennreal (*integral* $\{0..pi\}$ ($\lambda x.$ *min* (d / 2) (*sin* $x * l$ / 2)))
  (**is** *emeasure lborel ?A* = -)
⟨*proof*⟩

We now have to distinguish two cases: The first and easier one is that where the length of the needle, l, is less than or equal to the strip width, d:

context
  assumes *l-le-d*: $l \leq d$
begin

lemma *emeasure-buffon-set'-short*: *emeasure lborel* (*buffon-set' l d*) = *ennreal l*
⟨*proof*⟩

lemma *emeasure-buffon-set-short*: *emeasure lborel* (*buffon-set l d*) = 4 * *ennreal l*
  ⟨*proof*⟩

theorem *buffon-short*: *emeasure* (*buffon l d*) {*True*} = *ennreal* ($2 * l$ / ($d * pi$))
⟨*proof*⟩

end

The other case where the needle is at least as long as the strip width is more complicated:

context

assumes *l-ge-d*: $l \geq d$
begin

lemma *emeasure-buffon-set'-long*:
emeasure lborel (*buffon-set'* l d) =
ennreal (l ∗ (*1* − *sqrt* (*1* − (d / l)^2)) + *arccos* (d / l) ∗ d)
⟨*proof*⟩

lemma *emeasure-buffon-set-long*: *emeasure lborel* (*buffon-set* l d) =
4 ∗ *ennreal* (l ∗ (*1* − *sqrt* (*1* − (d / l)^2)) + *arccos* (d / l) ∗ d)
⟨*proof*⟩

theorem *buffon-long*:
emeasure (*buffon* l d) {*True*} =
ennreal (*2* / *pi* ∗ ((l / d) − *sqrt* ((l / d)^2 − *1*) + *arccos* (d / l)))
⟨*proof*⟩

end
end

end

References

[1] J. F. Ramaley. Buffon's Noodle Problem. *The American Mathematical Monthly*, 76(8):916–918, 1969.

[2] E. W. Weisstein. MathWorld – Buffon's Needle Problem. http://mathworld.wolfram.com/BuffonsNeedleProblem.html.