Büchi Complementation

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Abstract

This entry provides a verified implementation of rank-based Büchi Complementation [1]. The verification is done in three steps:

1. Definition of odd rankings and proof that an automaton rejects a word iff there exists an odd ranking for it.
2. Definition of the complement automaton and proof that it accepts exactly those words for which there is an odd ranking.
3. Verified implementation of the complement automaton using the Isabelle Collections Framework.

Contents

1 Alternating Function Iteration

theory *Alternate*
imports *Main*
begin

primrec *alternate* :: $('a \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'a) \Rightarrow nat \Rightarrow ('a \Rightarrow 'a)$ **where**
alternate f g 0 = *id* | *alternate f g* (*Suc k*) = *alternate g f k* ∘ *f*

lemma *alternate-Suc*[*simp*]: *alternate f g* (*Suc k*) = (*if even k then f else g*) ∘ *alternate f g k*
⟨*proof*⟩

declare *alternate.simps*(*2*)[*simp del*]

lemma *alternate-antimono*:
assumes $\bigwedge x.\ f\ x \leq x \bigwedge x.\ g\ x \leq x$
shows *antimono* (*alternate f g*)
⟨*proof*⟩

end

2 Run Graphs

theory *Graph*

imports *Transition-Systems-and-Automata.NBA*
begin

type-synonym *'state node* = *nat* × *'state*

abbreviation *ginitial A* ≡ {*0*} × *initial A*
abbreviation *gaccepting A* ≡ *accepting A* ∘ *snd*

global-interpretation *graph*: *transition-system-initial*
const
λ *u* (*k*, *p*). *w* !! *k* ∈ *alphabet A* ∧ *u* ∈ {*Suc k*} × *transition A* (*w* !! *k*) *p* ∩ *V*
λ *v*. *v* ∈ *ginitial A* ∩ *V*
for *A w V*
defines
gpath = *graph.path* **and** *grun* = *graph.run* **and**
greachable = *graph.reachable* **and** *gnodes* = *graph.nodes*
⟨*proof*⟩

We disable rules that are degenerate due to *execute* = (λ*x* -. *x*).

declare *graph.reachable.execute*[*rule del*]
declare *graph.nodes.execute*[*rule del*]

abbreviation *gtarget* ≡ *graph.target*
abbreviation *gstates* ≡ *graph.states*
abbreviation *gtrace* ≡ *graph.trace*

abbreviation *gsuccessors* :: (*'label*, *'state*) *nba* ⇒ *'label stream* ⇒
'state node set ⇒ *'state node* ⇒ *'state node set* **where**
gsuccessors A w V ≡ *graph.successors TYPE*(*'label*) *w A V*

abbreviation *gusuccessors A w* ≡ *gsuccessors A w UNIV*
abbreviation *gupath A w* ≡ *gpath A w UNIV*
abbreviation *gurun A w* ≡ *grun A w UNIV*
abbreviation *gureachable A w* ≡ *greachable A w UNIV*
abbreviation *gunodes A w* ≡ *gnodes A w UNIV*

lemma *gtarget-alt-def*: *gtarget r v* = *last* (*v* # *r*) ⟨*proof*⟩
lemma *gstates-alt-def*: *gstates r v* = *r* ⟨*proof*⟩
lemma *gtrace-alt-def*: *gtrace r v* = *r* ⟨*proof*⟩

lemma *gpath-elim*[*elim?*]:
assumes *gpath A w V s v*
obtains *r k p*
where *s* = [*Suc k* ..< *Suc k* + *length r*] || *r v* = (*k*, *p*)
⟨*proof*⟩

lemma *gpath-path*[*symmetric*]: *path A* (*stake* (*length r*) (*sdrop k w*) || *r*) *p* ⟷
gpath A w UNIV ([*Suc k* ..< *Suc k* + *length r*] || *r*) (*k*, *p*)
⟨*proof*⟩

lemma *grun-elim*[*elim?*]:
assumes *grun A w V s v*
obtains *r k p*
where *s* = *fromN* (*Suc k*) ||| *r v* = (*k*, *p*)
⟨*proof*⟩

lemma *run-grun*:
assumes *run A* (*sdrop k w* ||| *r*) *p*
shows *gurun A w* (*fromN* (*Suc k*) ||| *r*) (*k*, *p*)
⟨*proof*⟩

lemma *grun-run*:
assumes *grun A w V* (*fromN* (*Suc k*) ||| *r*) (*k*, *p*)
shows *run A* (*sdrop k w* ||| *r*) *p*
⟨*proof*⟩

lemma *greachable-reachable*:
fixes *l q k p*
defines *u* ≡ (*l*, *q*)
defines *v* ≡ (*k*, *p*)
assumes *u* ∈ *greachable A w V v*
shows *q* ∈ *reachable A p*
⟨*proof*⟩

lemma *gnodes-nodes*: *gnodes A w V* ⊆ *UNIV* × *nodes A*
⟨*proof*⟩

lemma *gpath-subset*:
assumes *gpath A w V r v*
assumes *set* (*gstates r v*) ⊆ *U*
shows *gpath A w U r v*
⟨*proof*⟩
lemma *grun-subset*:
assumes *grun A w V r v*
assumes *sset* (*gtrace r v*) ⊆ *U*
shows *grun A w U r v*
⟨*proof*⟩

lemma *greachable-subset*: *greachable A w V v* ⊆ *insert v V*
⟨*proof*⟩

lemma *gtrace-infinite*:
assumes *grun A w V r v*
shows *infinite* (*sset* (*gtrace r v*))
⟨*proof*⟩

lemma *infinite-greachable-gtrace*:
assumes *grun A w V r v*

assumes $u \in sset\ (gtrace\ r\ v)$
shows *infinite* $(greachable\ A\ w\ V\ u)$
⟨*proof*⟩

lemma *finite-nodes-gsuccessors*:
assumes *finite* $(nodes\ A)$
assumes $v \in gunodes\ A\ w$
shows *finite* $(gusuccessors\ A\ w\ v)$
⟨*proof*⟩

end

3 Rankings

theory *Ranking*
imports
Alternate
Graph
begin

3.1 Rankings

type-synonym $'state\ ranking = 'state\ node \Rightarrow nat$

definition *ranking* :: $('label, 'state)\ nba \Rightarrow 'label\ stream \Rightarrow 'state\ ranking \Rightarrow bool$
where
$ranking\ A\ w\ f \equiv$
$(\forall\ v \in gunodes\ A\ w.\ f\ v \leq 2 * card\ (nodes\ A)) \wedge$
$(\forall\ v \in gunodes\ A\ w.\ \forall\ u \in gusuccessors\ A\ w\ v.\ f\ u \leq f\ v) \wedge$
$(\forall\ v \in gunodes\ A\ w.\ gaccepting\ A\ v \longrightarrow even\ (f\ v)) \wedge$
$(\forall\ v \in gunodes\ A\ w.\ \forall\ r\ k.\ gurun\ A\ w\ r\ v \longrightarrow smap\ f\ (gtrace\ r\ v) = sconst\ k \longrightarrow odd\ k)$

3.2 Ranking Implies Word not in Language

lemma *ranking-stuck*:
assumes *ranking* $A\ w\ f$
assumes $v \in gunodes\ A\ w\ gurun\ A\ w\ r\ v$
obtains $n\ k$
where $smap\ f\ (gtrace\ (sdrop\ n\ r)\ (gtarget\ (stake\ n\ r)\ v)) = sconst\ k$
⟨*proof*⟩

lemma *ranking-stuck-odd*:
assumes *ranking* $A\ w\ f$
assumes $v \in gunodes\ A\ w\ gurun\ A\ w\ r\ v$
obtains n
where $Ball\ (sset\ (smap\ f\ (gtrace\ (sdrop\ n\ r)\ (gtarget\ (stake\ n\ r)\ v))))\ odd$
⟨*proof*⟩

lemma *ranking-language*:
assumes *ranking A w f*
shows *w* ∉ *language A*
⟨*proof*⟩

3.3 Word not in Language Implies Ranking

3.3.1 Removal of Endangered Nodes

definition *clean* :: ('*label*, '*state*) *nba* ⇒ '*label stream* ⇒ '*state node set* ⇒ '*state node set* **where**
clean A w V ≡ {*v* ∈ *V*. *infinite* (*greachable A w V v*)}

lemma *clean-decreasing*: *clean A w V* ⊆ *V* ⟨*proof*⟩
lemma *clean-successors*:
assumes *v* ∈ *V u* ∈ *gusuccessors A w v*
shows *u* ∈ *clean A w V* ⟹ *v* ∈ *clean A w V*
⟨*proof*⟩

3.3.2 Removal of Safe Nodes

definition *prune* :: ('*label*, '*state*) *nba* ⇒ '*label stream* ⇒ '*state node set* ⇒ '*state node set* **where**
prune A w V ≡ {*v* ∈ *V*. ∃ *u* ∈ *greachable A w V v*. *gaccepting A u*}

lemma *prune-decreasing*: *prune A w V* ⊆ *V* ⟨*proof*⟩
lemma *prune-successors*:
assumes *v* ∈ *V u* ∈ *gusuccessors A w v*
shows *u* ∈ *prune A w V* ⟹ *v* ∈ *prune A w V*
⟨*proof*⟩

3.3.3 Run Graph Interation

definition *graph* :: ('*label*, '*state*) *nba* ⇒ '*label stream* ⇒ *nat* ⇒ '*state node set* **where**
graph A w k ≡ *alternate* (*clean A w*) (*prune A w*) *k* (*gunodes A w*)

abbreviation *level A w k l* ≡ {*v* ∈ *graph A w k*. *fst v* = *l*}

lemma *graph-0*[*simp*]: *graph A w 0* = *gunodes A w* ⟨*proof*⟩
lemma *graph-Suc*[*simp*]: *graph A w* (*Suc k*) = (*if even k then clean A w else prune A w*) (*graph A w k*)
⟨*proof*⟩

lemma *graph-antimono*: *antimono* (*graph A w*)
⟨*proof*⟩
lemma *graph-nodes*: *graph A w k* ⊆ *gunodes A w* ⟨*proof*⟩
lemma *graph-successors*:
assumes *v* ∈ *gunodes A w u* ∈ *gusuccessors A w v*
shows *u* ∈ *graph A w k* ⟹ *v* ∈ *graph A w k*

⟨*proof*⟩

lemma *graph-level-finite*:
assumes *finite* (*nodes A*)
shows *finite* (*level A w k l*)
⟨*proof*⟩

lemma *find-safe*:
assumes $w \notin$ *language A*
assumes $V \neq \{\}$ $V \subseteq$ *gunodes A w*
assumes $\bigwedge v.\ v \in V \Longrightarrow$ *gsuccessors A w V v* $\neq \{\}$
obtains *v*
where $v \in V\ \forall\ u \in$ *greachable A w V v.* $\neg$ *gaccepting A u*
⟨*proof*⟩

lemma *remove-run*:
assumes *finite* (*nodes A*) $w \notin$ *language A*
assumes $V \subseteq$ *gunodes A w clean A w V* $\neq \{\}$
obtains *v r*
where
grun A w V r v
sset (*gtrace r v*) $\subseteq$ *clean A w V*
sset (*gtrace r v*) $\subseteq -$ *prune A w* (*clean A w V*)
⟨*proof*⟩

lemma *level-bounded*:
assumes *finite* (*nodes A*) $w \notin$ *language A*
obtains *n*
where $\bigwedge l.\ l \geq n \Longrightarrow$ *card* (*level A w* (*2* $*$ *k*) *l*) $\leq$ *card* (*nodes A*) $-$ *k*
⟨*proof*⟩
lemma *graph-empty*:
assumes *finite* (*nodes A*) $w \notin$ *language A*
shows *graph A w* (*Suc* (*2* $*$ *card* (*nodes A*))) $= \{\}$
⟨*proof*⟩
lemma *graph-le*:
assumes *finite* (*nodes A*) $w \notin$ *language A*
assumes $v \in$ *graph A w k*
shows $k \leq$ *2* $*$ *card* (*nodes A*)
⟨*proof*⟩

3.4 Node Ranks

definition *rank* :: (*'label*, *'state*) *nba* $\Rightarrow$ *'label stream* $\Rightarrow$ *'state node* $\Rightarrow$ *nat* **where**
rank A w v $\equiv$ *GREATEST k. v* $\in$ *graph A w k*

lemma *rank-member*:
assumes *finite* (*nodes A*) $w \notin$ *language A* $v \in$ *gunodes A w*
shows $v \in$ *graph A w* (*rank A w v*)
⟨*proof*⟩

lemma *rank-removed*:
assumes *finite* (*nodes A*) $w \notin$ *language A*
shows $v \notin$ *graph A w* (*Suc* (*rank A w v*))
⟨*proof*⟩
lemma *rank-le*:
assumes *finite* (*nodes A*) $w \notin$ *language A*
assumes $v \in$ *gunodes A w u* $\in$ *gusuccessors A w v*
shows *rank A w u* $\leq$ *rank A w v*
⟨*proof*⟩

lemma *language-ranking*:
assumes *finite* (*nodes A*) $w \notin$ *language A*
shows *ranking A w* (*rank A w*)
⟨*proof*⟩

3.5 Correctness Theorem

theorem *language-ranking-iff*:
assumes *finite* (*nodes A*)
shows $w \notin$ *language A* $\longleftrightarrow$ ($\exists$ *f*. *ranking A w f*)
⟨*proof*⟩

end

4 Complementation

theory *Complementation*
imports
Transition-Systems-and-Automata.Maps
Ranking
begin

4.1 Level Rankings and Complementation States

type-synonym *'state lr* = *'state* $\rightharpoonup$ *nat*

definition *lr-succ* :: (*'label*, *'state*) *nba* $\Rightarrow$ *'label* $\Rightarrow$ *'state lr* $\Rightarrow$ *'state lr set* **where**
lr-succ A a f $\equiv$ {*g*.
dom g = $\bigcup$ (*transition A a* ‘ *dom f*) $\wedge$
($\forall$ *p* $\in$ *dom f*. $\forall$ *q* $\in$ *transition A a p*. *the* (*g q*) $\leq$ *the* (*f p*)) $\wedge$
($\forall$ *q* $\in$ *dom g*. *accepting A q* $\longrightarrow$ *even* (*the* (*g q*)))}

type-synonym *'state st* = *'state set*

definition *st-succ* :: (*'label*, *'state*) *nba* $\Rightarrow$ *'label* $\Rightarrow$ *'state lr* $\Rightarrow$ *'state st* $\Rightarrow$ *'state st* **where**
st-succ A a g P $\equiv$ {*q* $\in$ *if P* = {} *then dom g else* $\bigcup$ (*transition A a* ‘ *P*). *even* (*the* (*g q*))}

type-synonym *'state cs = 'state lr × 'state st*

definition *complement-succ* :: *('label, 'state) nba ⇒ 'label ⇒ 'state cs ⇒ 'state cs set* **where**
complement-succ A a ≡ λ (f, P). {(g, st-succ A a g P) |g. g ∈ lr-succ A a f}

definition *complement* :: *('label, 'state) nba ⇒ ('label, 'state cs) nba* **where**
complement A ≡ nba
(alphabet A)
*({const (Some (2 * card (nodes A))) |` initial A} × {{}})*
(complement-succ A)
(λ (f, P). P = {})

lemma *dom-nodes*:
assumes *fP ∈ nodes (complement A)*
shows *dom (fst fP) ⊆ nodes A*
⟨*proof*⟩
lemma *ran-nodes*:
assumes *fP ∈ nodes (complement A)*
shows *ran (fst fP) ⊆ {0 .. 2 * card (nodes A)}*
⟨*proof*⟩
lemma *states-nodes*:
assumes *fP ∈ nodes (complement A)*
shows *snd fP ⊆ nodes A*
⟨*proof*⟩

theorem *complement-finite*:
assumes *finite (nodes A)*
shows *finite (nodes (complement A))*
⟨*proof*⟩

lemma *complement-trace-snth*:
assumes *run (complement A) (w ||| r) p*
defines *m ≡ p ## trace (w ||| r) p*
obtains
fst (m !! Suc k) ∈ lr-succ A (w !! k) (fst (m !! k))
snd (m !! Suc k) = st-succ A (w !! k) (fst (m !! Suc k)) (snd (m !! k))
⟨*proof*⟩

4.2 Word in Complement Language Implies Ranking

lemma *complement-ranking*:
assumes *w ∈ language (complement A)*
obtains *f*
where *ranking A w f*
⟨*proof*⟩

4.3 Ranking Implies Word in Complement Language

definition *reach* **where**

reach A w i ≡ {*target r p* |*r p*. *path A r p* ∧ *p* ∈ *initial A* ∧ *map fst r* = *stake i w*}

lemma *reach-0*[*simp*]: *reach A w 0* = *initial A* ⟨*proof*⟩
lemma *reach-Suc-empty*:
assumes *w* !! *n* ∉ *alphabet A*
shows *reach A w* (*Suc n*) = {}
⟨*proof*⟩
lemma *reach-Suc-succ*:
assumes *w* !! *n* ∈ *alphabet A*
shows *reach A w* (*Suc n*) = ⋃ (*transition A* (*w* !! *n*) ‘ *reach A w n*)
⟨*proof*⟩
lemma *reach-Suc*[*simp*]: *reach A w* (*Suc n*) = (*if w* !! *n* ∈ *alphabet A*
then ⋃ (*transition A* (*w* !! *n*) ‘ *reach A w n*) *else* {})
⟨*proof*⟩
lemma *reach-nodes*: *reach A w i* ⊆ *nodes A* ⟨*proof*⟩
lemma *reach-gunodes*: {*i*} × *reach A w i* ⊆ *gunodes A w*
⟨*proof*⟩

lemma *ranking-complement*:
assumes *finite* (*nodes A*) *w* ∈ *streams* (*alphabet A*) *ranking A w f*
shows *w* ∈ *language* (*complement A*)
⟨*proof*⟩

4.4 Correctness Theorem

theorem *complement-language*:
assumes *finite* (*nodes A*)
shows *language* (*complement A*) = *streams* (*alphabet A*) − *language A*
⟨*proof*⟩

end

5 Complementation Implementation

theory *Complementation-Implement*
imports
Transition-Systems-and-Automata.NBA-Implement
Complementation
begin

unbundle *lattice-syntax*

type-synonym *item* = *nat* × *bool*
type-synonym *'state items* = *'state* ⇀ *item*

type-synonym *state* = (*nat* × *item*) *list*
abbreviation *item-rel* ≡ *nat-rel* ×$_r$ *bool-rel*
abbreviation *state-rel* ≡ ⟨*nat-rel*, *item-rel*⟩ *list-map-rel*

abbreviation *pred A a q* ≡ {*p*. *q* ∈ *transition A a p*}

5.1 Phase 1

definition *cs-lr* :: *'state items* ⇒ *'state lr* **where**
cs-lr f ≡ *map-option fst* ∘ *f*
definition *cs-st* :: *'state items* ⇒ *'state st* **where**
cs-st f ≡ *f* −‘ *Some* ‘ *snd* −‘ {*True*}
abbreviation *cs-abs* :: *'state items* ⇒ *'state cs* **where**
cs-abs f ≡ (*cs-lr f*, *cs-st f*)
definition *cs-rep* :: *'state cs* ⇒ *'state items* **where**
cs-rep ≡ λ (*g*, *P*) *p*. *map-option* (λ *k*. (*k*, *p* ∈ *P*)) (*g p*)

lemma *cs-abs-rep*[*simp*]: *cs-rep* (*cs-abs f*) = *f*
⟨*proof*⟩
lemma *cs-rep-lr*[*simp*]: *cs-lr* (*cs-rep* (*g*, *P*)) = *g*
⟨*proof*⟩
lemma *cs-rep-st*[*simp*]: *cs-st* (*cs-rep* (*g*, *P*)) = *P* ∩ *dom g*
⟨*proof*⟩

lemma *cs-lr-dom*[*simp*]: *dom* (*cs-lr f*) = *dom f* ⟨*proof*⟩
lemma *cs-lr-apply*[*simp*]:
assumes *p* ∈ *dom f*
shows *the* (*cs-lr f p*) = *fst* (*the* (*f p*))
⟨*proof*⟩

lemma *cs-rep-dom*[*simp*]: *dom* (*cs-rep* (*g*, *P*)) = *dom g* ⟨*proof*⟩
lemma *cs-rep-apply*[*simp*]:
assumes *p* ∈ *dom f*
shows *fst* (*the* (*cs-rep* (*f*, *P*) *p*)) = *the* (*f p*)
⟨*proof*⟩

abbreviation *cs-rel* :: (*'state items* × *'state cs*) *set* **where**
cs-rel ≡ *br cs-abs top*

lemma *cs-rel-inv-single-valued*: *single-valued* (*cs-rel*$^{-1}$)
⟨*proof*⟩

definition *refresh-1* :: *'state items* ⇒ *'state items* **where**
refresh-1 f ≡ *if True* ∈ *snd* ‘ *ran f then f else map-option* (*apsnd top*) ∘ *f*
definition *ranks-1* ::
(*'label*, *'state*) *nba* ⇒ *'label* ⇒ *'state items* ⇒ *'state items set* **where**
ranks-1 A a f ≡ {*g*.
dom g = ⋃((*transition A a*) ‘ (*dom f*)) ∧
(∀ *p* ∈ *dom f*. ∀ *q* ∈ *transition A a p*. *fst* (*the* (*g q*)) ≤ *fst* (*the* (*f p*))) ∧
(∀ *q* ∈ *dom g*. *accepting A q* ⟶ *even* (*fst* (*the* (*g q*)))) ∧
cs-st g = {*q* ∈ ⋃((*transition A a*) ‘ (*cs-st f*)). *even* (*fst* (*the* (*g q*)))}}
definition *complement-succ-1* ::

(*'label*, *'state*) *nba* ⇒ *'label* ⇒ *'state items* ⇒ *'state items set* **where**
complement-succ-1 A a = *ranks-1 A a* ∘ *refresh-1*
definition *complement-1* :: (*'label*, *'state*) *nba* ⇒ (*'label*, *'state items*) *nba* **where**
complement-1 A ≡ *nba*
(*alphabet A*)
({*const* (*Some* (2 ∗ *card* (*nodes A*), *False*)) |‘ *initial A*})
(*complement-succ-1 A*)
(λ *f*. *cs-st f* = {})

lemma *refresh-1-dom*[*simp*]: *dom* (*refresh-1 f*) = *dom f* ⟨*proof*⟩
lemma *refresh-1-apply*[*simp*]: *fst* (*the* (*refresh-1 f p*)) = *fst* (*the* (*f p*))
⟨*proof*⟩
lemma *refresh-1-cs-st*[*simp*]: *cs-st* (*refresh-1 f*) = (*if cs-st f* = {} *then dom f else cs-st f*)
⟨*proof*⟩

lemma *complement-succ-1-abs*:
assumes *g* ∈ *complement-succ-1 A a f*
shows *cs-abs g* ∈ *complement-succ A a* (*cs-abs f*)
⟨*proof*⟩
lemma *complement-succ-1-rep*:
assumes *P* ⊆ *dom f* (*g*, *Q*) ∈ *complement-succ A a* (*f*, *P*)
shows *cs-rep* (*g*, *Q*) ∈ *complement-succ-1 A a* (*cs-rep* (*f*, *P*))
⟨*proof*⟩

lemma *complement-succ-1-refine*: (*complement-succ-1*, *complement-succ*) ∈
Id → *Id* → *cs-rel* → ⟨*cs-rel*⟩ *set-rel*
⟨*proof*⟩
lemma *complement-1-refine*: (*complement-1*, *complement*) ∈ ⟨*Id*, *Id*⟩ *nba-rel* → ⟨*Id*, *cs-rel*⟩ *nba-rel*
⟨*proof*⟩

5.2 Phase 2

definition *ranks-2* :: (*'label*, *'state*) *nba* ⇒ *'label* ⇒ *'state items* ⇒ *'state items set* **where**
ranks-2 A a f ≡ {*g*.
dom g = ⋃((*transition A a*) ‘ (*dom f*)) ∧
(∀ *q l d*. *g q* = *Some* (*l*, *d*) ⟶
l ≤ ⨅ (*fst* ‘ *Some* −‘ *f* ‘ *pred A a q*) ∧
(*d* ⟷ ⨆ (*snd* ‘ *Some* −‘ *f* ‘ *pred A a q*) ∧ *even l*) ∧
(*accepting A q* ⟶ *even l*))}
definition *complement-succ-2* ::
(*'label*, *'state*) *nba* ⇒ *'label* ⇒ *'state items* ⇒ *'state items set* **where**
complement-succ-2 A a ≡ *ranks-2 A a* ∘ *refresh-1*
definition *complement-2* :: (*'label*, *'state*) *nba* ⇒ (*'label*, *'state items*) *nba* **where**
complement-2 A ≡ *nba*
(*alphabet A*)
({*const* (*Some* (2 ∗ *card* (*nodes A*), *False*)) |‘ *initial A*})

(*complement-succ-2 A*)
(λ *f*. *True* ∉ *snd* ‘ *ran f*)

lemma *ranks-2-refine*: *ranks-2* = *ranks-1*
⟨*proof*⟩

lemma *complement-2-refine*: (*complement-2*, *complement-1*) ∈ ⟨*Id*, *Id*⟩ *nba-rel* → ⟨*Id*, *Id*⟩ *nba-rel*
⟨*proof*⟩

5.3 Phase 3

definition *bounds-3* :: (′*label*, ′*state*) *nba* ⇒ ′*label* ⇒ ′*state items* ⇒ ′*state items* **where**
bounds-3 A a f ≡ λ *q*. *let S* = *Some* −‘ *f* ‘ *pred A a q in*
if S = {} *then None else Some* (⨅ (*fst* ‘ *S*), ⨆ (*snd* ‘ *S*))
definition *items-3* :: (′*label*, ′*state*) *nba* ⇒ ′*state* ⇒ *item* ⇒ *item set* **where**
items-3 A p ≡ λ (*k*, *c*). {(*l*, *c* ∧ *even l*) |*l*. *l* ≤ *k* ∧ (*accepting A p* ⟶ *even l*)}
definition *get-3* :: (′*label*, ′*state*) *nba* ⇒ ′*state items* ⇒ (′*state* ⇀ *item set*) **where**
get-3 A f ≡ λ *p*. *map-option* (*items-3 A p*) (*f p*)
definition *complement-succ-3* ::
(′*label*, ′*state*) *nba* ⇒ ′*label* ⇒ ′*state items* ⇒ ′*state items set* **where**
complement-succ-3 A a ≡ *expand-map* ∘ *get-3 A* ∘ *bounds-3 A a* ∘ *refresh-1*
definition *complement-3* :: (′*label*, ′*state*) *nba* ⇒ (′*label*, ′*state items*) *nba* **where**
complement-3 A ≡ *nba*
(*alphabet A*)
({(*Some* ∘ (*const* (2 ∗ *card* (*nodes A*), *False*))) |‘ *initial A*})
(*complement-succ-3 A*)
(λ *f*. ∀ (*p*, *k*, *c*) ∈ *map-to-set f*. ¬ *c*)

lemma *bounds-3-dom*[*simp*]: *dom* (*bounds-3 A a f*) = ⋃ ((*transition A a*) ‘ (*dom f*))
⟨*proof*⟩

lemma *items-3-nonempty*[*intro*!, *simp*]: *items-3 A p s* ≠ {} ⟨*proof*⟩
lemma *items-3-finite*[*intro*!, *simp*]: *finite* (*items-3 A p s*)
⟨*proof*⟩

lemma *get-3-dom*[*simp*]: *dom* (*get-3 A f*) = *dom f* ⟨*proof*⟩
lemma *get-3-finite*[*intro*, *simp*]: *S* ∈ *ran* (*get-3 A f*) ⟹ *finite S*
⟨*proof*⟩
lemma *get-3-update*[*simp*]: *get-3 A* (*f* (*p* ↦ *s*)) = (*get-3 A f*) (*p* ↦ *items-3 A p s*)
⟨*proof*⟩

lemma *expand-map-get-bounds-3*: *expand-map* ∘ *get-3 A* ∘ *bounds-3 A a* = *ranks-2 A a*
⟨*proof*⟩

lemma *complement-succ-3-refine*: *complement-succ-3* = *complement-succ-2*
⟨*proof*⟩
lemma *complement-initial-3-refine*: {*const* (*Some* (2 ∗ *card* (*nodes A*), *False*)) |‘ *initial A*} =
{(*Some* ∘ (*const* (2 ∗ *card* (*nodes A*), *False*))) |‘ *initial A*}
⟨*proof*⟩
lemma *complement-accepting-3-refine*: *True* ∉ *snd* ‘ *ran f* ⟷ (∀ (*p*, *k*, *c*) ∈ *map-to-set f*. ¬ *c*)
⟨*proof*⟩

lemma *complement-3-refine*: (*complement-3*, *complement-2*) ∈ ⟨*Id*, *Id*⟩ *nba-rel* → ⟨*Id*, *Id*⟩ *nba-rel*
⟨*proof*⟩

5.4 Phase 4

definition *items-4* :: (′*label*, ′*state*) *nba* ⇒ ′*state* ⇒ *item* ⇒ *item set* **where**
items-4 A p ≡ λ (*k*, *c*). {(*l*, *c* ∧ *even l*) |*l*. *k* ≤ *Suc l* ∧ *l* ≤ *k* ∧ (*accepting A p* ⟶ *even l*)}
definition *get-4* :: (′*label*, ′*state*) *nba* ⇒ ′*state items* ⇒ (′*state* ⇀ *item set*) **where**
get-4 A f ≡ λ *p*. *map-option* (*items-4 A p*) (*f p*)
definition *complement-succ-4* ::
(′*label*, ′*state*) *nba* ⇒ ′*label* ⇒ ′*state items* ⇒ ′*state items set* **where**
complement-succ-4 A a ≡ *expand-map* ∘ *get-4 A* ∘ *bounds-3 A a* ∘ *refresh-1*
definition *complement-4* :: (′*label*, ′*state*) *nba* ⇒ (′*label*, ′*state items*) *nba* **where**
complement-4 A ≡ *nba*
(*alphabet A*)
({(*Some* ∘ (*const* (2 ∗ *card* (*nodes A*), *False*))) |‘ *initial A*})
(*complement-succ-4 A*)
(λ *f*. ∀ (*p*, *k*, *c*) ∈ *map-to-set f*. ¬ *c*)

lemma *get-4-dom*[*simp*]: *dom* (*get-4 A f*) = *dom f* ⟨*proof*⟩

definition *R* :: ′*state items rel* **where**
R ≡ {(*f*, *g*).
dom f = *dom g* ∧
(∀ *p* ∈ *dom f*. *fst* (*the* (*f p*)) ≤ *fst* (*the* (*g p*))) ∧
(∀ *p* ∈ *dom f*. *snd* (*the* (*f p*)) ⟷ *snd* (*the* (*g p*)))}

lemma *bounds-R*:
assumes (*f*, *g*) ∈ *R*
assumes *bounds-3 A a* (*refresh-1 f*) *p* = *Some* (*n*, *e*)
assumes *bounds-3 A a* (*refresh-1 g*) *p* = *Some* (*k*, *c*)
shows *n* ≤ *k e* ⟷ *c*
⟨*proof*⟩

lemma *complement-4-language-1*: *language* (*complement-3 A*) ⊆ *language* (*complement-4*

A)
⟨*proof*⟩
lemma *complement-4-less*: *complement-4 A* ≤ *complement-3 A*
⟨*proof*⟩
lemma *complement-4-language-2*: *language* (*complement-4 A*) ⊆ *language* (*complement-3 A*)
⟨*proof*⟩
lemma *complement-4-language*: *language* (*complement-3 A*) = *language* (*complement-4 A*)
⟨*proof*⟩

lemma *complement-4-finite*[*simp*]:
assumes *finite* (*nodes A*)
shows *finite* (*nodes* (*complement-4 A*))
⟨*proof*⟩
lemma *complement-4-correct*:
assumes *finite* (*nodes A*)
shows *language* (*complement-4 A*) = *streams* (*alphabet A*) − *language A*
⟨*proof*⟩

5.5 Phase 5

definition *refresh-5* :: *'state items* ⇒ *'state items nres* **where**
refresh-5 f ≡ *if* ∃ (*p*, *k*, *c*) ∈ *map-to-set f*. *c*
then RETURN f
else do
{
ASSUME (*finite* (*dom f*));
FOREACH (*map-to-set f*) (λ (*p*, *k*, *c*) *m*. *do*
{
ASSERT (*p* ∉ *dom m*);
RETURN (*m* (*p* ↦ (*k*, *True*)))
}
) *Map.empty*
}
definition *merge-5* :: *item* ⇒ *item option* ⇒ *item* **where**
merge-5 ≡ λ (*k*, *c*). λ *None* ⇒ (*k*, *c*) | *Some* (*l*, *d*) ⇒ (*k* ⊓ *l*, *c* ⊔ *d*)
definition *bounds-5* :: (*'label*, *'state*) *nba* ⇒ *'label* ⇒ *'state items* ⇒ *'state items nres* **where**
bounds-5 A a f ≡ *do*
{
ASSUME (*finite* (*dom f*));
ASSUME (∀ *p*. *finite* (*transition A a p*));
FOREACH (*map-to-set f*) (λ (*p*, *s*) *m*.
FOREACH (*transition A a p*) (λ *q f*.
RETURN (*f* (*q* ↦ *merge-5 s* (*f q*))))
m)
Map.empty
}

definition *items-5* :: (*'label*, *'state*) *nba* ⇒ *'state* ⇒ *item* ⇒ *item set* **where**
items-5 A p ≡ λ (*k*, *c*). *do*
{
let values = *if accepting A p then Set.filter even* {*k* − *1* .. *k*} *else* {*k* − *1* .. *k*};
let item = λ *l*. (*l*, *c* ∧ *even l*);
item ‘ *values*
}
definition *get-5* :: (*'label*, *'state*) *nba* ⇒ *'state items* ⇒ (*'state* ⇀ *item set*) **where**
get-5 A f ≡ λ *p*. *map-option* (*items-5 A p*) (*f p*)
definition *expand-5* :: (*'a* ⇀ *'b set*) ⇒ (*'a* ⇀ *'b*) *set nres* **where**
expand-5 f ≡ *FOREACH* (*map-to-set f*) (λ (*x*, *S*) *X*. *do* {
ASSERT (∀ *g* ∈ *X*. *x* ∉ *dom g*);
ASSERT (∀ *a* ∈ *S*. ∀ *b* ∈ *S*. *a* ≠ *b* ⟶ (λ *y*. (λ *g*. *g* (*x* ↦ *y*)) ‘ *X*) *a* ∩ (λ *y*. (λ *g*. *g* (*x* ↦ *y*)) ‘ *X*) *b* = {});
RETURN (⋃ *y* ∈ *S*. (λ *g*. *g* (*x* ↦ *y*)) ‘ *X*)
}) {*Map.empty*}
definition *complement-succ-5* ::
(*'label*, *'state*) *nba* ⇒ *'label* ⇒ *'state items* ⇒ *'state items set nres* **where**
complement-succ-5 A a f ≡ *do*
{
f ← *refresh-5 f*;
f ← *bounds-5 A a f*;
ASSUME (*finite* (*dom f*));
expand-5 (*get-5 A f*)
}

lemma *bounds-3-empty*: *bounds-3 A a Map.empty* = *Map.empty*
⟨*proof*⟩
lemma *bounds-3-update*: *bounds-3 A a* (*f* (*p* ↦ *s*)) =
override-on (*bounds-3 A a f*) (*Some* ∘ *merge-5 s* ∘ *bounds-3 A a* (*f* (*p* := *None*))) (*transition A a p*)
⟨*proof*⟩

lemma *refresh-5-refine*: (*refresh-5*, λ *f*. *RETURN* (*refresh-1 f*)) ∈ *Id* → ⟨*Id*⟩ *nres-rel*
⟨*proof*⟩
lemma *bounds-5-refine*: (*bounds-5 A a*, λ *f*. *RETURN* (*bounds-3 A a f*)) ∈ *Id* → ⟨*Id*⟩ *nres-rel*
⟨*proof*⟩
lemma *items-5-refine*: *items-5* = *items-4*
⟨*proof*⟩
lemma *get-5-refine*: *get-5* = *get-4*
⟨*proof*⟩
lemma *expand-5-refine*: (*expand-5 f*, *ASSERT* (*finite* (*dom f*)) ≫ *RETURN* (*expand-map f*)) ∈ ⟨*Id*⟩ *nres-rel*
⟨*proof*⟩

lemma *complement-succ-5-refine*: (*complement-succ-5*, *RETURN* ∘∘∘ *complement-succ-4*) ∈
Id → *Id* → *Id* → ⟨*Id*⟩ *nres-rel*
⟨*proof*⟩

5.6 Phase 6

definition *expand-map-get-6* :: ('*label*, '*state*) *nba* ⇒ '*state items* ⇒ '*state items set nres* **where**
expand-map-get-6 A f ≡ *FOREACH* (*map-to-set f*) (λ (*k*, *v*) *X*. *do* {
ASSERT (∀ *g* ∈ *X*. *k* ∉ *dom g*);
ASSERT (∀ *a* ∈ (*items-5 A k v*). ∀ *b* ∈ (*items-5 A k v*). *a* ≠ *b* ⟶ (λ *y*. (λ *g*. *g* (*k* ↦ *y*)) ' *X*) *a* ∩ (λ *y*. (λ *g*. *g* (*k* ↦ *y*)) ' *X*) *b* = {});
RETURN (⋃ *y* ∈ *items-5 A k v*. (λ *g*. *g* (*k* ↦ *y*)) ' *X*)
}) {*Map.empty*}

lemma *expand-map-get-6-refine*: (*expand-map-get-6*, *expand-5* ∘∘ *get-5*) ∈ *Id* → *Id* → ⟨*Id*⟩ *nres-rel*
⟨*proof*⟩

definition *complement-succ-6* ::
('*label*, '*state*) *nba* ⇒ '*label* ⇒ '*state items* ⇒ '*state items set nres* **where**
complement-succ-6 A a f ≡ *do*
{
f ← *refresh-5 f*;
f ← *bounds-5 A a f*;
ASSUME (*finite* (*dom f*));
expand-map-get-6 A f
}

lemma *complement-succ-6-refine*:
(*complement-succ-6*, *complement-succ-5*) ∈ *Id* → *Id* → *Id* → ⟨*Id*⟩ *nres-rel*
⟨*proof*⟩

5.7 Phase 7

interpretation *autoref-syn* ⟨*proof*⟩

context
fixes *fi f*
assumes *fi*[*autoref-rules*]: (*fi*, *f*) ∈ *state-rel*
begin

private lemma [*simp*]: *finite* (*dom f*)
⟨*proof*⟩

schematic-goal *refresh-7*: (*?f* :: *?'a*, *refresh-5 f*) ∈ *?R*
⟨*proof*⟩

end

concrete-definition *refresh-7* **uses** *refresh-7*

lemma *refresh-7-refine*: $(\lambda\ f.\ RETURN\ (refresh\text{-}7\ f),\ refresh\text{-}5) \in state\text{-}rel \to \langle state\text{-}rel\rangle\ nres\text{-}rel$
⟨*proof*⟩

context
fixes $A :: ('label,\ nat)\ nba$
fixes *succi a fi f*
assumes *succi*[*autoref-rules*]: $(succi,\ transition\ A\ a) \in nat\text{-}rel \to \langle nat\text{-}rel\rangle\ list\text{-}set\text{-}rel$
assumes *fi*[*autoref-rules*]: $(fi,\ f) \in state\text{-}rel$
begin

private lemma [*simp*]: *finite* (*transition A a p*)
⟨*proof*⟩ **lemma** [*simp*]: *finite* (*dom f*) ⟨*proof*⟩ **lemma** [*autoref-op-pat*]: $transition\ A\ a \equiv OP\ (transition\ A\ a)$ ⟨*proof*⟩ **lemma** [*autoref-rules*]: $(min,\ min) \in nat\text{-}rel \to nat\text{-}rel \to nat\text{-}rel$ ⟨*proof*⟩

schematic-goal *bounds-7*:
notes *ty-REL*[**where** $R = \langle nat\text{-}rel,\ item\text{-}rel\rangle\ dflt\text{-}ahm\text{-}rel$, *autoref-tyrel*]
shows $(?f :: ?'a,\ bounds\text{-}5\ A\ a\ f) \in ?R$
⟨*proof*⟩

end

concrete-definition *bounds-7* **uses** *bounds-7*

lemma *bounds-7-refine*: $(si,\ transition\ A\ a) \in nat\text{-}rel \to \langle nat\text{-}rel\rangle\ list\text{-}set\text{-}rel \Longrightarrow$
$(\lambda\ p.\ RETURN\ (bounds\text{-}7\ si\ p),\ bounds\text{-}5\ A\ a) \in$
$state\text{-}rel \to \langle\langle nat\text{-}rel,\ item\text{-}rel\rangle\ dflt\text{-}ahm\text{-}rel\rangle\ nres\text{-}rel$
⟨*proof*⟩

context
fixes $A :: ('label,\ nat)\ nba$
fixes *acci*
assumes [*autoref-rules*]: $(acci,\ accepting\ A) \in nat\text{-}rel \to bool\text{-}rel$
begin

private lemma [*autoref-op-pat*]: $accepting\ A \equiv OP\ (accepting\ A)$ ⟨*proof*⟩ **lemma** [*autoref-rules*]: $((dvd),\ (dvd)) \in nat\text{-}rel \to nat\text{-}rel \to bool\text{-}rel$ ⟨*proof*⟩ **lemma** [*autoref-rules*]: $(\lambda\ k\ l.\ upt\ k\ (Suc\ l),\ atLeastAtMost) \in$
$nat\text{-}rel \to nat\text{-}rel \to \langle nat\text{-}rel\rangle\ list\text{-}set\text{-}rel$
⟨*proof*⟩

schematic-goal *items-7*: $(?f :: ?'a,\ items\text{-}5\ A) \in ?R$
⟨*proof*⟩

end

concrete-definition *items-7* **uses** *items-7*

context
fixes $A :: ('label, nat)\ nba$
fixes *ai*
fixes *fi f*
assumes *ai*: $(ai, accepting\ A) \in nat\text{-}rel \to bool\text{-}rel$
assumes *fi*[*autoref-rules*]: $(fi, f) \in \langle nat\text{-}rel, item\text{-}rel\rangle\ dflt\text{-}ahm\text{-}rel$
begin

private lemma [*simp*]: *finite* (*dom f*)
⟨*proof*⟩ **lemma** [*simp*]:
assumes $\bigwedge\ m.\ m \in S \Longrightarrow x \notin dom\ m$
shows *inj-on* $(\lambda\ m.\ m\ (x \mapsto y))\ S$
⟨*proof*⟩ **lemmas** [*simp*] = *op-map-update-def*[*abs-def*]

private lemma [*autoref-op-pat*]: *items-5 A* ≡ *OP* (*items-5 A*) ⟨*proof*⟩ **lemmas** [*autoref-rules*] = *items-7.refine*[*OF ai*]

schematic-goal *expand-map-get-7*: (*?f*, *expand-map-get-6 A f*) ∈
⟨⟨*state-rel*⟩ *list-set-rel*⟩ *nres-rel*
⟨*proof*⟩

end

concrete-definition *expand-map-get-7* **uses** *expand-map-get-7*

lemma *expand-map-get-7-refine*:
assumes $(ai, accepting\ A) \in nat\text{-}rel \to bool\text{-}rel$
shows $(\lambda\ fi.\ RETURN\ (expand\text{-}map\text{-}get\text{-}7\ ai\ fi),$
$\lambda\ f.\ ASSUME\ (finite\ (dom\ f)) \gg expand\text{-}map\text{-}get\text{-}6\ A\ f) \in$
$\langle nat\text{-}rel, item\text{-}rel\rangle\ dflt\text{-}ahm\text{-}rel \to \langle\langle state\text{-}rel\rangle\ list\text{-}set\text{-}rel\rangle\ nres\text{-}rel$
⟨*proof*⟩

context
fixes $A :: ('label, nat)\ nba$
fixes $a :: 'label$
fixes $p :: nat\ items$
fixes *Ai*
fixes *ai*
fixes *pi*
assumes *Ai*: $(Ai, A) \in \langle Id, Id\rangle\ nbai\text{-}nba\text{-}rel$
assumes *ai*: $(ai, a) \in Id$
assumes *pi*[*autoref-rules*]: $(pi, p) \in state\text{-}rel$
begin

private lemmas *succi* = *nbai-nba-param*(4)[*THEN fun-relD*, *OF Ai*, *THEN fun-relD*, *OF ai*]
private lemmas *acceptingi* = *nbai-nba-param*(5)[*THEN fun-relD*, *OF Ai*]

private lemma [*autoref-op-pat*]: (λ *g*. *ASSUME* (*finite* (*dom g*)) ≫ *expand-map-get-6 A g*) ≡
OP (λ *g*. *ASSUME* (*finite* (*dom g*)) ≫ *expand-map-get-6 A g*) ⟨*proof*⟩
lemma [*autoref-op-pat*]: *bounds-5 A a* ≡ *OP* (*bounds-5 A a*) ⟨*proof*⟩ **lemmas** [*autoref-rules*] =
refresh-7-refine
bounds-7-refine[*OF succi*]
expand-map-get-7-refine[*OF acceptingi*]

schematic-goal *complement-succ-7*: (*?f* :: *?'a*, *complement-succ-6 A a p*) ∈ *?R*
⟨*proof*⟩

end

concrete-definition *complement-succ-7* **uses** *complement-succ-7*

lemma *complement-succ-7-refine*:
(*RETURN* ∘∘∘ *complement-succ-7*, *complement-succ-6*) ∈
⟨*Id*, *Id*⟩ *nbai-nba-rel* → *Id* → *state-rel* →
⟨⟨*state-rel*⟩ *list-set-rel*⟩ *nres-rel*
⟨*proof*⟩

context
fixes *A* :: ('*label*, *nat*) *nba*
fixes *Ai*
fixes *n ni* :: *nat*
assumes *Ai*: (*Ai*, *A*) ∈ ⟨*Id*, *Id*⟩ *nbai-nba-rel*
assumes *ni*[*autoref-rules*]: (*ni*, *n*) ∈ *Id*
begin

private lemma [*autoref-op-pat*]: *initial A* ≡ *OP* (*initial A*) ⟨*proof*⟩ **lemmas** [*autoref-rules*] = *nbai-nba-param*(3)[*THEN fun-relD*, *OF Ai*]

schematic-goal *complement-initial-7*:
(*?f*, {(*Some* ∘ (*const* (2 ∗ *n*, *False*))) |‘ *initial A*}) ∈ ⟨*state-rel*⟩ *list-set-rel*
⟨*proof*⟩

end

concrete-definition *complement-initial-7* **uses** *complement-initial-7*

schematic-goal *complement-accepting-7*: (*?f*, λ *f*. ∀ (*p*, *k*, *c*) ∈ *map-to-set f*. ¬ *c*) ∈
state-rel → *bool-rel*

⟨*proof*⟩

concrete-definition *complement-accepting-7* **uses** *complement-accepting-7*

definition *complement-7* :: ('*label*, *nat*) *nbai* ⇒ *nat* ⇒ ('*label*, *state*) *nbai* **where**
complement-7 Ai ni ≡ *nbai*
(*alphabeti Ai*)
(*complement-initial-7 Ai ni*)
(*complement-succ-7 Ai*)
(*complement-accepting-7*)

lemma *complement-7-refine*[*autoref-rules*]:
assumes (*Ai*, *A*) ∈ ⟨*Id*, *Id*⟩ *nbai-nba-rel*
assumes (*ni*,
(*OP card* ::: ⟨*Id*⟩ *ahs-rel bhc* → *nat-rel*) $
((*OP nodes* ::: ⟨*Id*, *Id*⟩ *nbai-nba-rel* → ⟨*Id*⟩ *ahs-rel bhc*) $ *A*)) ∈ *nat-rel*
shows (*complement-7 Ai ni*, (*OP complement-4* :::
⟨*Id*, *Id*⟩ *nbai-nba-rel* → ⟨*Id*, *state-rel*⟩ *nbai-nba-rel*) $ *A*) ∈ ⟨*Id*, *state-rel*⟩
nbai-nba-rel
⟨*proof*⟩

end

6 Boolean Formulae

theory *Formula*
imports *Main*
begin

datatype '*a formula* =
False |
True |
Variable '*a* |
Negation '*a formula* |
Conjunction '*a formula* '*a formula* |
Disjunction '*a formula* '*a formula*

primrec *satisfies* :: '*a set* ⇒ '*a formula* ⇒ *bool* **where**
satisfies A False ⟷ *HOL.False* |
satisfies A True ⟷ *HOL.True* |
satisfies A (*Variable a*) ⟷ *a* ∈ *A* |
satisfies A (*Negation x*) ⟷ ¬ *satisfies A x* |
satisfies A (*Conjunction x y*) ⟷ *satisfies A x* ∧ *satisfies A y* |
satisfies A (*Disjunction x y*) ⟷ *satisfies A x* ∨ *satisfies A y*

end

7 Final Instantiation of Algorithms Related to Complementation

theory *Complementation-Final*
imports
Complementation-Implement
Formula
Transition-Systems-and-Automata.NBA-Translate
Transition-Systems-and-Automata.NGBA-Algorithms
HOL−Library.Multiset
begin

7.1 Syntax

no-syntax -*do-let* :: [*pttrn*, *'a*] ⇒ *do-bind* (‹(‹*indent=2 notation=‹infix do let››let - =/ -)› [*1000*, *13*] *13*)
syntax -*do-let* :: [*pttrn*, *'a*] ⇒ *do-bind* (‹(‹*indent=2 notation=‹infix do let››let - =/ -)› *13*)

7.2 Hashcodes on Complement States

definition *hci k* ≡ *uint32-of-nat k* ∗ *1103515245* + *12345*
definition *hc* ≡ λ (*p*, *q*, *b*). *hci p* + *hci q* ∗ *31* + (*if b then 1 else 0*)
definition *list-hash xs* ≡ *fold* (*xor* ∘ *hc*) *xs 0*

lemma *list-hash-eq*:
assumes *distinct xs distinct ys set xs* = *set ys*
shows *list-hash xs* = *list-hash ys*
⟨*proof*⟩

definition *state-hash* :: *nat* ⇒ *Complementation-Implement.state* ⇒ *nat* **where**
state-hash n p ≡ *nat-of-hashcode* (*list-hash p*) *mod n*

lemma *state-hash-bounded-hashcode*[*autoref-ga-rules*]: *is-bounded-hashcode state-rel*
(*gen-equals* (*Gen-Map.gen-ball* (*foldli* ∘ *list-map-to-list*)) (*list-map-lookup* (=))
(*prod-eq* (=) (⟷))) *state-hash*
⟨*proof*⟩

7.3 Complementation

schematic-goal *complement-impl*:
assumes [*simp*]: *finite* (*NBA.nodes A*)
assumes [*autoref-rules*]: (*Ai*, *A*) ∈ ⟨*Id*, *nat-rel*⟩ *nbai-nba-rel*
shows (*?f* :: *?'c*, *op-translate* (*complement-4 A*)) ∈ *?R*
⟨*proof*⟩
concrete-definition *complement-impl* **uses** *complement-impl*

theorem *complement-impl-correct*:
assumes *finite* (*NBA.nodes A*)

assumes $(Ai,\ A) \in \langle Id,\ nat\text{-}rel\rangle\ nbai\text{-}nba\text{-}rel$
shows $NBA.language\ (nbae\text{-}nba\ (nbaei\text{-}nbae\ (complement\text{-}impl\ Ai))) =$
$streams\ (nba.alphabet\ A) - NBA.language\ A$
⟨*proof*⟩

7.4 Language Subset

definition [*simp*]: $op\text{-}language\text{-}subset\ A\ B \equiv NBA.language\ A \subseteq NBA.language\ B$

lemmas [*autoref-op-pat*] = *op-language-subset-def*[*symmetric*]

schematic-goal *language-subset-impl*:
assumes [*simp*]: *finite* $(NBA.nodes\ B)$
assumes [*autoref-rules*]: $(Ai,\ A) \in \langle Id,\ nat\text{-}rel\rangle\ nbai\text{-}nba\text{-}rel$
assumes [*autoref-rules*]: $(Bi,\ B) \in \langle Id,\ nat\text{-}rel\rangle\ nbai\text{-}nba\text{-}rel$
shows (*?f* :: *?'c*, *do* {
$let\ AB' = intersect'\ A\ (complement\text{-}4\ B)$;
$ASSERT\ (finite\ (NGBA.nodes\ AB'))$;
$RETURN\ (NGBA.language\ AB' = \{\})$
}) ∈ *?R*
⟨*proof*⟩
concrete-definition *language-subset-impl* **uses** *language-subset-impl*
lemma *language-subset-impl-refine*[*autoref-rules*]:
assumes $SIDE\text{-}PRECOND\ (finite\ (NBA.nodes\ A))$
assumes $SIDE\text{-}PRECOND\ (finite\ (NBA.nodes\ B))$
assumes $SIDE\text{-}PRECOND\ (nba.alphabet\ A \subseteq nba.alphabet\ B)$
assumes $(Ai,\ A) \in \langle Id,\ nat\text{-}rel\rangle\ nbai\text{-}nba\text{-}rel$
assumes $(Bi,\ B) \in \langle Id,\ nat\text{-}rel\rangle\ nbai\text{-}nba\text{-}rel$
shows $(language\text{-}subset\text{-}impl\ Ai\ Bi,\ (OP\ op\text{-}language\text{-}subset :::$
$\langle Id,\ nat\text{-}rel\rangle\ nbai\text{-}nba\text{-}rel \to \langle Id,\ nat\text{-}rel\rangle\ nbai\text{-}nba\text{-}rel \to bool\text{-}rel)\ \$\ A\ \$\ B) \in$
bool-rel
⟨*proof*⟩

7.5 Language Equality

definition [*simp*]: $op\text{-}language\text{-}equal\ A\ B \equiv NBA.language\ A = NBA.language\ B$

lemmas [*autoref-op-pat*] = *op-language-equal-def*[*symmetric*]

schematic-goal *language-equal-impl*:
assumes [*simp*]: *finite* $(NBA.nodes\ A)$
assumes [*simp*]: *finite* $(NBA.nodes\ B)$
assumes [*simp*]: $nba.alphabet\ A = nba.alphabet\ B$
assumes [*autoref-rules*]: $(Ai,\ A) \in \langle Id,\ nat\text{-}rel\rangle\ nbai\text{-}nba\text{-}rel$
assumes [*autoref-rules*]: $(Bi,\ B) \in \langle Id,\ nat\text{-}rel\rangle\ nbai\text{-}nba\text{-}rel$
shows $(?f :: ?'c,\ NBA.language\ A \subseteq NBA.language\ B \wedge NBA.language\ B \subseteq$
$NBA.language\ A) \in ?R$
⟨*proof*⟩

concrete-definition *language-equal-impl* **uses** *language-equal-impl*
lemma *language-equal-impl-refine*[*autoref-rules*]:
assumes *SIDE-PRECOND* (*finite* (*NBA.nodes A*))
assumes *SIDE-PRECOND* (*finite* (*NBA.nodes B*))
assumes *SIDE-PRECOND* (*nba.alphabet A* = *nba.alphabet B*)
assumes (*Ai*, *A*) ∈ ⟨*Id*, *nat-rel*⟩ *nbai-nba-rel*
assumes (*Bi*, *B*) ∈ ⟨*Id*, *nat-rel*⟩ *nbai-nba-rel*
shows (*language-equal-impl Ai Bi*, (*OP op-language-equal* :::
⟨*Id*, *nat-rel*⟩ *nbai-nba-rel* → ⟨*Id*, *nat-rel*⟩ *nbai-nba-rel* → *bool-rel*) $ *A* $ *B*) ∈ *bool-rel*
⟨*proof*⟩

schematic-goal *product-impl*:
assumes [*simp*]: *finite* (*NBA.nodes B*)
assumes [*autoref-rules*]: (*Ai*, *A*) ∈ ⟨*Id*, *nat-rel*⟩ *nbai-nba-rel*
assumes [*autoref-rules*]: (*Bi*, *B*) ∈ ⟨*Id*, *nat-rel*⟩ *nbai-nba-rel*
shows (*?f* :: *?'c*, *do* {
let AB' = *intersect A* (*complement-4 B*);
ASSERT (*finite* (*NBA.nodes AB'*));
op-translate AB'
}) ∈ *?R*
⟨*proof*⟩
concrete-definition *product-impl* **uses** *product-impl*

export-code
Set.empty Set.insert Set.member
Inf :: *'a set set* ⇒ *'a set Sup* :: *'a set set* ⇒ *'a set image Pow set*
nat-of-integer integer-of-nat
Variable Negation Conjunction Disjunction satisfies map-formula
nbaei alphabetei initialei transitionei acceptingei
nbae-nba-impl complement-impl language-equal-impl product-impl
in *SML* **module-name** *Complementation* **file-prefix** *Complementation*

end

8 Build and test exported program with MLton

theory *Complementation-Build*
imports *Complementation-Final*
begin

external-file ‹*code/Autool.mlb*›
external-file ‹*code/Prelude.sml*›
external-file ‹*code/Autool.sml*›

compile-generated-files
‹*code/Complementation.ML*› (**in** *Complementation-Final*)
external-files

```
    ‹code/Autool.mlb›
    ‹code/Prelude.sml›
    ‹code/Autool.sml›
  export-files ‹code/Complementation.sml› and ‹code/Autool› (exe)
  where ‹fn dir =>
    let
      val exec = Generated-Files.execute (dir + Path.basic code);
      val - = exec ‹Prepare› mv Complementation.ML Complementation.sml;
    val - = exec ‹Compilation› (verbatim ‹$ISABELLE-MLTON $ISABELLE-MLTON-OPTIONS
› ^
            -profile time -default-type intinf Autool.mlb);
      val - = exec ‹Test› ./Autool help;
    in () end›

end
```

References

[1] O. Kupferman and M. Y. Vardi. Weak alternating automata are not that weak. *ACM Trans. Comput. Logic*, 2(3):408–429, July 2001.