

Putting the ‘K’ into Bird’s derivation of Knuth-Morris-Pratt string matching

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February 6, 2026

Abstract

Richard Bird and collaborators have proposed a derivation of an intricate cyclic program that implements the Morris-Pratt string matching algorithm. Here we provide a proof of total correctness for Bird’s derivation and complete it by adding Knuth’s optimisation.

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1 Introduction

We formalize a derivation of the string-matching algorithm of [Knuth et al. \(1977\)](#) (KMP) due to [Bird \(2010, Chapter 17\)](#). The central novelty of this approach is its use of a circular data structure to simultaneously compute and represent the failure function; see [Figure 1](#) for the final program. This is challenging to model in a logic of total functions, as we discuss below, which leads us to employ the venerable machinery of domain theory.

```

module KMP where

-- For testing
import Data.List ( isInfixOf )
import qualified Test.QuickCheck as QC

-- Bird's Morris-Pratt string matcher, without the 'K' optimisation
-- Chapter 17, "Pearls of Functional Algorithm Design", 2010.

data Tree a = Null
            | Node [a] (Tree a) {- ! -} (Tree a) -- remains correct with strict right subtrees

matches :: Eq a => [a] -> [a] -> [Integer]
matches ws = map fst . filter (ok . snd) . scanl step (0, root)
  where
    ok (Node vs _l _r) = null vs
    step (n, t) x = (n + 1, op t x)

    op Null _x = root
    op (Node [] l _r) x = op l x
    op (Node (v : _vs) l r) x = if x == v then r else op l x

    root = grep Null ws

    grep l [] = Node [] l Null
    grep l vvs@(v : vs) = Node vvs l (grep (op l v) vs)

-- matches [1,2,3,1,2] [1,2,1,2,3,1,2,3,1,2]

-- Our KMP (= MP with the 'K' optimisation)

kmatches :: Eq a => [a] -> [a] -> [Integer]
kmatches ws = map fst . filter (ok . snd) . scanl step (0, root)
  where
    ok (Node vs _l _r) = null vs
    step (n, t) x = (n + 1, op t x)

    op Null _x = root
    op (Node [] l _r) x = op l x
    op (Node (v : _vs) l r) x = if x == v then r else op l x

    root = grep Null ws

    next _x Null = Null
    next _x t@(Node [] _l _r) = t
    next x t@(Node (v : _vs) l _r) = if x == v then l else t

    grep l [] = Node [] l Null
    grep l vvs@(v : vs) = Node vvs (next v l) (grep (op l v) vs)

prop_matches :: [Bool] -> [Bool] -> Bool
prop_matches as bs = (as 'isInfixOf' bs) == (as 'matches' bs /= [])

prop_kmatches :: [Bool] -> [Bool] -> Bool
prop_kmatches as bs = (as 'matches' bs) == (as 'kmatches' bs)

tests :: IO ()
tests =
  do QC.quickCheck prop_matches
     QC.quickCheck prop_kmatches

```

Figure 1: Bird's KMP as a Haskell program.

Our development completes Bird’s derivation of the Morris-Pratt (MP) algorithm with proofs that each derivation step preserves productivity, yielding total correctness; in other words, we show that this circular program is extensionally equal to its specification. We also add what we call the ‘K’ optimisation to yield the full KMP algorithm (§4.2). Our analysis inspired a Prolog implementation (§6) that some may find more perspicuous.

Here we focus on the formalities of this style of program refinement and defer further background on string matching to two excellent monographs: [Gusfield \(1997, §2.3\)](#) and [Crochemore and Rytter \(2002, §2.1\)](#). Both provide traditional presentations of the problem, the KMP algorithm and correctness proofs and complexity results. We discuss related work in §5.

1.1 Formal setting

Bird does not make his formal context explicit. The program requires non-strict datatypes and sharing to obtain the expected complexity, which implies that he is working in a lazy (call-by-need) language. For reasons we observe during our development in §4, some of Bird’s definitions are difficult to make directly in Isabelle/HOL (a logic of total functions over types denoting sets) using the existing mechanisms.

We therefore adopt domain theory as mechanised by HOLCF ([Müller et al. 1999](#)). This logic provides a relatively straightforward if awkward way to reason about non-strict (call-by-name) programs at the cost of being too abstract to express sharing.

Bird’s derivation implicitly appeals to the fold/unfold framework of [Burstall and Darlington \(1977\)](#), which guarantees the preservation of partial correctness: informally, if the implementation terminates then it yields a value that coincides with the specification, or implementation \sqsubseteq specification in domain-theoretic terms. These rules come with side conditions that would ensure that productivity is preserved – that the implementation and specification are moreover extensionally equal – but Bird does not establish them. We note that it is easy to lose productivity through subtle uses of cyclic data structures (see §4.6 in particular), and that this derivation does not use well-known structured recursion patterns like *map* or *foldr* that mitigate these issues.

We attempt to avoid the confusions that can arise when transforming programs with named expressions (definitions or declarations) by making each step in the derivation completely self-contained: specifically, all definitions that change or depend on a definition that changes are redefined at each step. Briefly this avoids the conflation of equations with definitions; for instance, $f = f$ holds for all functions but makes for a poor definition. The issues become more subtle in the presence of recursion modelled as least fixed points, where satisfying a fixed-point equation $Ff = f$ does not always imply the desired equality $f = \text{lfp } F$. [Tullsen \(2002\)](#) provides a fuller discussion. As our main interest is the introduction of the circular data structure (§4.2), we choose to work with datatypes that simplify other aspects of this story. Specifically we use strict lists (§3) as they allow us to adapt many definitions and lemmas about HOL’s lists and localise (the many!) definedness conditions. We also impose strong conditions on equality (§2.2) for similar reasons, and, less critically, assume products behave pleasantly (§4.1). Again [Tullsen \(2002\)](#) discusses how these may violate Haskell expectations.

We suggest the reader skip the next two sections and proceed to the derivation which begins in §4.

2 Extra HOLCF

lemma *lfp-fusion*:

assumes $g \cdot \perp = \perp$

assumes $g \circ f = h \circ g$

shows $g \cdot (\text{fix } f) = \text{fix } h$

proof(*induct rule: parallel-fix-ind*)

case 2 show $g \cdot \perp = \perp$ **by fact**

case $(\exists x y)$

from $\langle g \cdot x = y \rangle \langle g \circ f = h \circ g \rangle$ **show** $g \cdot (f \cdot x) = h \cdot y$

by (*simp add: cfun-eq-iff*)

qed *simp*

lemma *predE*:

obtains (*strict*) $p \cdot \perp = \perp \mid (FF) p = (\Lambda x. FF) \mid (TT) p = (\Lambda x. TT)$

using *flat-codom* [**where** $f=p$ **and** $x=\perp$] **by** (*cases p* \perp ; *force simp: cfun-eq-iff*)

lemma *retraction-cfcomp-strict*:

assumes $f \circ g = ID$
shows $f \cdot \perp = \perp$
using *assms retraction-strict* **by** (*clarsimp simp: cfun-eq-iff*)

lemma *match-Pair-csplit*[*simp*]: $match\text{-}Pair \cdot x \cdot k = k \cdot (cfst \cdot x) \cdot (csnd \cdot x)$
by (*cases x*) *simp*

lemmas *oo-assoc = assoc-oo* — Normalize name

lemma *If-cancel*[*simp*]: $(If\ b\ then\ x\ else\ x) = seq \cdot b \cdot x$
by (*cases b*) *simp-all*

lemma *seq-below*[*iff*]: $seq \cdot x \cdot y \sqsubseteq y$
by (*simp add: seq-conv-if*)

lemma *seq-strict-distr*: $f \cdot \perp = \perp \implies seq \cdot x \cdot (f \cdot y) = f \cdot (seq \cdot x \cdot y)$
by (*cases x = \perp; clarsimp*)

lemma *strictify-below*[*iff*]: $strictify \cdot f \sqsubseteq f$
unfolding *strictify-def* **by** (*clarsimp simp: cfun-below-iff*)

lemma *If-distr*:
 $\llbracket f \perp = \perp; cont\ f \rrbracket \implies f\ (If\ b\ then\ t\ else\ e) = (If\ b\ then\ f\ t\ else\ f\ e)$
 $\llbracket cont\ t'; cont\ e' \rrbracket \implies (If\ b\ then\ t'\ else\ e')\ x = (If\ b\ then\ t'\ x\ else\ e'\ x)$
 $(If\ b\ then\ t''' \ else\ e''') \cdot x = (If\ b\ then\ t''' \cdot x\ else\ e''' \cdot x)$
 $\llbracket g \perp = \perp; cont\ g \rrbracket \implies g\ (If\ b\ then\ t'' \ else\ e'')\ y = (If\ b\ then\ g\ t''\ y\ else\ g\ e''\ y)$
by (*case-tac [!] b*) *simp-all*

lemma *If2-split-asm*: $P\ (If2\ Q\ x\ y) \longleftrightarrow \neg(Q = \perp \wedge \neg P \perp \vee Q = TT \wedge \neg P\ x \vee Q = FF \wedge \neg P\ y)$
by (*cases Q*) (*simp-all add: If2-def*)

lemmas *If2-splits = split-If2 If2-split-asm*

lemma *If2-cont*[*simp, cont2cont*]:
assumes *cont i*
assumes *cont t*
assumes *cont e*
shows *cont* $(\lambda x. If2\ (i\ x)\ (t\ x)\ (e\ x))$
using *assms unfolding If2-def* **by** *simp*

lemma *If-else-FF*[*simp*]: $(If\ b\ then\ t\ else\ FF) = (b\ andalso\ t)$
by (*cases b*) *simp-all*

lemma *If-then-TT*[*simp*]: $(If\ b\ then\ TT\ else\ e) = (b\ orelse\ e)$
by (*cases b*) *simp-all*

lemma *If-cong*:
assumes $b = b'$
assumes $b = TT \implies t = t'$
assumes $b = FF \implies e = e'$
shows $(If\ b\ then\ t\ else\ e) = (If\ b'\ then\ t'\ else\ e')$
using *assms* **by** (*cases b*) *simp-all*

lemma *If-tr*: $(If\ b\ then\ t\ else\ e) = ((b\ andalso\ t)\ orelse\ (neg \cdot b\ andalso\ e))$
by (*cases b*) *simp-all*

lemma *If-andalso*:
shows $If\ p\ andalso\ q\ then\ t\ else\ e = If\ p\ then\ If\ q\ then\ t\ else\ e\ else\ e$

by (cases p) simp-all

lemma *If-else-absorb*:

assumes $c = \perp \implies e = \perp$

assumes $c = TT \implies e = t$

shows *If c then t else e = e*

using *assms* by (cases c; clarsimp)

lemma *andalso-cong*: $\llbracket P = P'; P' = TT \implies Q = Q' \rrbracket \implies (P \text{ andalso } Q) = (P' \text{ andalso } Q')$

by (cases P) simp-all

lemma *andalso-weaken-left*:

assumes $P = TT \implies Q = TT$

assumes $P = FF \implies Q \neq \perp$

assumes $P = \perp \implies Q \neq FF$

shows $P = (Q \text{ andalso } P)$

using *assms* by (cases P; cases Q; simp)

lemma *orelse-cong*: $\llbracket P = P'; P' = FF \implies Q = Q' \rrbracket \implies (P \text{ orelse } Q) = (P' \text{ orelse } Q')$

by (cases P) simp-all

lemma *orelse-conv*[simp]:

$((x \text{ orelse } y) = TT) \longleftrightarrow (x = TT \vee (x = FF \wedge y = TT))$

$((x \text{ orelse } y) = \perp) \longleftrightarrow (x = \perp \vee (x = FF \wedge y = \perp))$

by (cases x; cases y; simp)+

lemma *csplit-cfun2*: $\text{cont } F \implies (\Lambda x. F x) = (\Lambda (x, y). F (x, y))$

by (clarsimp simp: cfun-eq-iff prod-cont-iff)

lemma *csplit-cfun3*: $\text{cont } F \implies (\Lambda x. F x) = (\Lambda (x, y, z). F (x, y, z))$

by (clarsimp simp: cfun-eq-iff prod-cont-iff)

definition *convol* :: $('a::\text{cpo} \rightarrow 'b::\text{cpo}) \rightarrow ('a \rightarrow 'c::\text{cpo}) \rightarrow 'a \rightarrow 'b \times 'c$ **where**

$\text{convol} = (\Lambda f g x. (f \cdot x, g \cdot x))$

abbreviation *convol-syn* :: $('a::\text{cpo} \rightarrow 'b::\text{cpo}) \Rightarrow ('a \rightarrow 'c::\text{cpo}) \Rightarrow 'a \rightarrow 'b \times 'c$ (**infix** $\langle \&\& \rangle$ 65) **where**

$f \&\& g \equiv \text{convol} \cdot f \cdot g$

lemma *convol-strict*[simp]:

$\text{convol} \cdot \perp \cdot \perp = \perp$

unfolding *convol-def* by simp

lemma *convol-simp*[simp]: $(f \&\& g) \cdot x = (f \cdot x, g \cdot x)$

unfolding *convol-def* by simp

definition *map-prod* :: $('a::\text{cpo} \rightarrow 'c::\text{cpo}) \rightarrow ('b::\text{cpo} \rightarrow 'd) \rightarrow 'a \times 'b \rightarrow 'c \times 'd$ **where**

$\text{map-prod} = (\Lambda f g (x, y). (f \cdot x, g \cdot y))$

abbreviation *map-prod-syn* :: $('a \rightarrow 'c) \Rightarrow ('b \rightarrow 'd) \Rightarrow 'a \times 'b \rightarrow 'c \times 'd$ (**infix** $\langle ** \rangle$ 65) **where**

$f ** g \equiv \text{map-prod} \cdot f \cdot g$

lemma *map-prod-cfcomp*[simp]: $(f ** m) \text{ oo } (g ** n) = (f \text{ oo } g) ** (m \text{ oo } n)$

unfolding *map-prod-def* by (clarsimp simp: cfun-eq-iff)

lemma *map-prod-ID*[simp]: $ID ** ID = ID$

unfolding *map-prod-def* by (clarsimp simp: cfun-eq-iff)

lemma *map-prod-app*[simp]: $(f ** g) \cdot x = (f \cdot (\text{cfst} \cdot x), g \cdot (\text{csnd} \cdot x))$

unfolding *map-prod-def* **by** (*cases x*) (*clarsimp simp: cfun-eq-iff*)

lemma *map-prod-cfst[simp]*: $cfst \circ (f ** g) = f \circ cfst$
by (*clarsimp simp: cfun-eq-iff*)

lemma *map-prod-csnd[simp]*: $csnd \circ (f ** g) = g \circ csnd$
by (*clarsimp simp: cfun-eq-iff*)

2.1 Extra HOLCF Prelude.

lemma *eq-strict[simp]*: $eq(\perp :: 'a::Eq-strict) = \perp$
by (*simp add: cfun-eq-iff*)

lemma *Integer-le-both-plus-1[simp]*:
 fixes $m :: Integer$
 shows $le \cdot (m + 1) \cdot (n + 1) = le \cdot m \cdot n$
by (*cases m; cases n; simp add: one-Integer-def*)

lemma *plus-eq-MkI-conv*:
 $l + n = MkI \cdot m \iff (\exists l' n'. l = MkI \cdot l' \wedge n = MkI \cdot n' \wedge m = l' + n')$
by (*cases l, simp*) (*cases n, auto*)

lemma *lt-defined*:
 fixes $x :: Integer$
 shows
 $lt \cdot x \cdot y = TT \implies (x \neq \perp \wedge y \neq \perp)$
 $lt \cdot x \cdot y = FF \implies (x \neq \perp \wedge y \neq \perp)$
by (*cases x; cases y; clarsimp*)⁺

lemma *le-defined*:
 fixes $x :: Integer$
 shows
 $le \cdot x \cdot y = TT \implies (x \neq \perp \wedge y \neq \perp)$
 $le \cdot x \cdot y = FF \implies (x \neq \perp \wedge y \neq \perp)$
by (*cases x; cases y; clarsimp*)⁺

Induction on *Integer*, following the setup for the *int* type.

definition *Integer-ge-less-than* $:: int \Rightarrow (Integer \times Integer) \text{ set}$
 where *Integer-ge-less-than* $d = \{(MkI \cdot z', MkI \cdot z) \mid z \ z'. d \leq z' \wedge z' < z\}$

lemma *wf-Integer-ge-less-than*: *wf* (*Integer-ge-less-than d*)

proof(*rule wf-subset*)

show *Integer-ge-less-than d* $\subseteq \text{measure } (\lambda z. \text{nat } (if\ z = \perp\ \text{then } d\ \text{else } (THE\ z'.\ z = MkI \cdot z') - d))$
 unfolding *Integer-ge-less-than-def* **by** *clarsimp*
qed *simp*

2.2 Element equality

To avoid many extraneous headaches that take us far away from the interesting parts of our derivation, we assume that the elements of the pattern and text are drawn from a *pcpo* where, if the *eq* function on this type is given defined arguments, then its result is defined and coincides with (=).

Note this effectively restricts us to *flat* element types; see Paulson (1987, §4.12) for a discussion.

class *Eq-def* = *Eq-eq* +
 assumes *eq-defined*: $\llbracket x \neq \perp; y \neq \perp \rrbracket \implies eq \cdot x \cdot y \neq \perp$
begin

lemma *eq-bottom-iff[simp]*: $(eq \cdot x \cdot y = \perp) \iff (x = \perp \vee y = \perp)$
using *eq-defined* **by** *auto*

lemma *eq-defined-reflD*[simp]:

$(eq \cdot a \cdot a = TT) \longleftrightarrow a \neq \perp$

$(TT = eq \cdot a \cdot a) \longleftrightarrow a \neq \perp$

$a \neq \perp \implies eq \cdot a \cdot a = TT$

using *eq-refl* **by** *auto*

lemma *eq-FF*[simp]:

$(FF = eq \cdot xs \cdot ys) \longleftrightarrow (xs \neq \perp \wedge ys \neq \perp \wedge xs \neq ys)$

$(eq \cdot xs \cdot ys = FF) \longleftrightarrow (xs \neq \perp \wedge ys \neq \perp \wedge xs \neq ys)$

by (*metis* (*mono-tags*, *opaque-lifting*) *Exh-tr dist-eq-tr*(5) *eq-TT-dest eq-bottom-iff eq-self-neq-FF*)⁺

lemma *eq-TT*[simp]:

$(TT = eq \cdot xs \cdot ys) \longleftrightarrow (xs \neq \perp \wedge ys \neq \perp \wedge xs = ys)$

$(eq \cdot xs \cdot ys = TT) \longleftrightarrow (xs \neq \perp \wedge ys \neq \perp \wedge xs = ys)$

by (*auto simp: local.eq-TT-dest*)

end

instance *Integer* :: *Eq-def* **by** *standard simp*

2.3 Recursive let bindings

Title: HOL/HOLCF/ex/Letrec.thy

Author: Brian Huffman

See §4.9 for an example use.

definition

$CLetrec :: ('a :: pcpo \rightarrow 'a \times 'b :: pcpo) \rightarrow 'b$ **where**

$CLetrec = (\Lambda F. prod.snd (F \cdot (\mu x. prod.fst (F \cdot x))))$

nonterminal *recbinds* **and** *recbindt* **and** *recbind*

syntax

-recbind :: *logic* \Rightarrow *logic* \Rightarrow *recbind* ($\langle \langle \langle \text{indent}=2 \text{ notation}=\langle \text{mixfix Letrec binding} \rangle \rangle \rangle \text{ - } / \text{ -} \rangle 10$)

:: *recbind* \Rightarrow *recbindt* ($\langle \langle \rangle \rangle$)

-recbindt :: *recbind* \Rightarrow *recbindt* \Rightarrow *recbindt* ($\langle \langle \langle \rangle \rangle / \langle \rangle \rangle$)

:: *recbindt* \Rightarrow *recbinds* ($\langle \langle \rangle \rangle$)

-recbinds :: *recbindt* \Rightarrow *recbinds* \Rightarrow *recbinds* ($\langle \langle \langle \rangle \rangle / \langle \rangle \rangle$)

-Letrec :: *recbinds* \Rightarrow *logic* \Rightarrow *logic* ($\langle \langle \langle \text{notation}=\langle \text{mixfix Letrec expression} \rangle \rangle \rangle \text{ Letrec } (-) / \text{ in } (-) \rangle 10$)

syntax-consts

-recbind -recbindt -recbinds -Letrec == *CLetrec*

translations

$(recbindt) x = a, (y, ys) = (b, bs) == (recbindt) (x, y, ys) = (a, b, bs)$

$(recbindt) x = a, y = b == (recbindt) (x, y) = (a, b)$

translations

-Letrec (-recbinds b bs) e == *-Letrec b (-Letrec bs e)*

Letrec xs = a in (e, es) == *CONST CLetrec \cdot (\Lambda xs. (a, e, es))*

Letrec xs = a in e == *CONST CLetrec \cdot (\Lambda xs. (a, e))*

3 Strict lists

Head- and tail-strict lists. Many technical Isabelle details are lifted from *HOLCF-Prelude.Data-List*; names follow HOL, prefixed with *s*.

domain $'a$ *slist* ($\langle[:\cdot:] \rangle$) =
 $snil \langle[:\cdot:] \rangle$
 $| scon\ (shead :: 'a) \ (stail :: 'a\ slist) \ (\mathbf{infixr} \ \langle:\#\rangle \ 65)$

lemma *scons-strict*[*simp*]: $scons.\perp = \perp$
by (*clarsimp simp: cfun-eq-iff*)

lemma *shead-bottom-iff*[*simp*]: $(shead.\ xs = \perp) \longleftrightarrow (xs = \perp \vee xs = [::])$
by (*cases xs simp-all*)

lemma *stail-bottom-iff*[*simp*]: $(stail.\ xs = \perp) \longleftrightarrow (xs = \perp \vee xs = [::])$
by (*cases xs simp-all*)

lemma *match-snil-match-scons-slist-case*: $match-snil.\ xs.k1 \ +\ +\ + \ match-scons.\ xs.k2 = slist-case.k1.k2.\ xs$
by (*cases xs simp-all*)

lemma *slist-bottom'*: $slist-case.\perp.\perp.\ xs = \perp$
by (*cases xs; simp*)

lemma *slist-bottom*[*simp*]: $slist-case.\perp.\perp = \perp$
by (*simp add: cfun-eq-iff slist-bottom'*)

lemma *slist-case-distr*:
 $f.\perp = \perp \implies f.\ (slist-case.g.h.\ xs) = slist-case.(f.g).\ (\Lambda x\ xs.\ f.(h.x.\ xs)).\ xs$
 $slist-case.g'.h'.\ xs.z = slist-case.(g'.z).\ (\Lambda x\ xs.\ h'.x.\ xs.z).\ xs$
by (*case-tac [!] xs simp-all*)

lemma *slist-case-cong*:
assumes $xs = xs'$
assumes $xs' = [::] \implies n = n'$
assumes $\bigwedge y\ ys.\ \llbracket xs' = y :\# ys; y \neq \perp; ys \neq \perp \rrbracket \implies c\ y\ ys = c'\ y\ ys$
assumes $cont\ (\lambda(x, y).\ c\ x\ y)$
assumes $cont\ (\lambda(x, y).\ c'\ x\ y)$
shows $slist-case.n.\ (\Lambda x\ xs.\ c\ x\ xs).\ xs = slist-case.n'.(\Lambda x\ xs.\ c'\ x\ xs).\ xs'$
using *assms* **by** (*cases xs; cases xs'; clarsimp simp: prod-cont-iff*)

Section syntax for *scons* ala Haskell.

syntax
 $-scons-section :: 'a \rightarrow [:'a:] \rightarrow [:'a:] \ (\langle'(\#\') \rangle)$
 $-scons-section-left :: 'a \Rightarrow [:'a:] \rightarrow [:'a:] \ (\langle'(-\#\') \rangle)$

syntax-consts
 $-scons-section-left == scon\$

translations
 $(x:\#) == (CONST\ Rep-cfun)\ (CONST\ scon)\ x$

abbreviation *scons-section-right* $:: [:'a:] \Rightarrow 'a \rightarrow [:'a:] \ (\langle'(\#\ -') \rangle)$ **where**
 $(:\#xs) \equiv \Lambda x.\ x :\# xs$

syntax
 $-strict-list :: args \Rightarrow [:'a:] \ (\langle[:(-):] \rangle)$

syntax-consts
 $-strict-list == scon\$

translations
 $[:x, xs:] == x :\# [xs:]$

$[:x:] == x :# [::]$

Class instances.

instantiation $slist :: (Eq) Eq-strict$

begin

fixrec $eq-slist :: [:'a:] \rightarrow [:'a:] \rightarrow tr$ **where**

$eq-slist.[::].[::] = TT$

$\llbracket [x \neq \perp; xs \neq \perp] \rrbracket \implies eq-slist.(x :# xs).[::] = FF$

$\llbracket [y \neq \perp; ys \neq \perp] \rrbracket \implies eq-slist.[::].(y :# ys) = FF$

$\llbracket [x \neq \perp; xs \neq \perp; y \neq \perp; ys \neq \perp] \rrbracket \implies eq-slist.(x :# xs).(y :# ys) = (eq.x.y \text{ andalso } eq-slist.xs.ys)$

instance proof

fix $xs :: [:'a:]$

show $eq.xs.\perp = \perp$

by ($cases\ xs$) ($subst\ eq-slist.unfold; simp$) $+$

show $eq.\perp.xs = \perp$

by ($cases\ xs$) ($subst\ eq-slist.unfold; simp$) $+$

qed

end

instance $slist :: (Eq-sym) Eq-sym$

proof

fix $xs\ ys :: [:'a:]$

show $eq.xs.ys = eq.ys.xs$

proof ($induct\ xs\ arbitrary: ys$)

case $snil$

show $?case$ **by** ($cases\ ys; simp$)

next

case $scons$

then show $?case$ **by** ($cases\ ys; simp\ add: eq-sym$)

qed $simp-all$

qed

instance $slist :: (Eq-equiv) Eq-equiv$

proof

fix $xs\ ys\ zs :: [:'a:]$

show $eq.xs.xs \neq FF$

by ($induct\ xs$) $simp-all$

assume $eq.xs.ys = TT$ **and** $eq.ys.zs = TT$ **then show** $eq.xs.zs = TT$

proof ($induct\ xs\ arbitrary: ys\ zs$)

case ($snil\ ys\ zs$) **then show** $?case$ **by** ($cases\ ys, simp-all$)

next

case ($scons\ x\ xs\ ys\ zs$) **with** $eq-trans$ **show** $?case$

by ($cases\ ys; cases\ zs$) $auto$

qed $simp-all$

qed

instance $slist :: (Eq-eq) Eq-eq$

proof

fix $xs\ ys :: [:'a:]$

show $eq.xs.xs \neq FF$

by ($induct\ xs$) $simp-all$

assume $eq.xs.ys = TT$ **then show** $xs = ys$

proof ($induct\ xs\ arbitrary: ys$)

case ($snil\ ys$) **then show** $?case$ **by** ($cases\ ys$) $simp-all$

next

case (*scons* *x xs ys*) **then show** ?*case* **by** (*cases ys*) *auto*
qed simp
qed

instance *slist* :: (*Eq-def*) *Eq-def*

proof

fix *xs ys* :: [*'a*:]
assume *xs* $\neq \perp$ **and** *ys* $\neq \perp$
then show *eq.xs.ys* $\neq \perp$
proof(*induct xs arbitrary: ys*)
case (*snil ys*) **then show** ?*case* **by** (*cases ys*) *simp-all*
next
case (*scons a xs*) **then show** ?*case* **by** (*cases ys*) *simp-all*
qed simp
qed

lemma *slist-eq-TT-snil*[*simp*]:

fixes *xs* :: [*'a*::*Eq*:]
shows (*eq.xs.[::]* = *TT*) \longleftrightarrow (*xs* = [*::*])
(*eq.[::].xs* = *TT*) \longleftrightarrow (*xs* = [*::*])
by (*cases xs; simp*)**+**

lemma *slist-eq-FF-snil*[*simp*]:

fixes *xs* :: [*'a*::*Eq*:]
shows (*eq.xs.[::]* = *FF*) \longleftrightarrow ($\exists y ys. y \neq \perp \wedge ys \neq \perp \wedge xs = y :\# ys$)
(*eq.[::].xs* = *FF*) \longleftrightarrow ($\exists y ys. y \neq \perp \wedge ys \neq \perp \wedge xs = y :\# ys$)
by (*cases xs; force*)**+**

3.1 Some of the usual reasoning infrastructure

inductive *slistmem* :: *'a* \Rightarrow [*'a*:] \Rightarrow *bool* **where**

$\llbracket x \neq \perp; xs \neq \perp \rrbracket \Longrightarrow$ *slistmem* *x* (*x* :# *xs*)
 $\llbracket \text{slistmem } x \text{ } xs; y \neq \perp \rrbracket \Longrightarrow$ *slistmem* *x* (*y* :# *xs*)

lemma *slistmem-bottom1*[*iff*]:

fixes *x* :: *'a*
shows \neg *slistmem* *x* \perp
by rule (*induct x* \perp :: [*'a*:] *rule: slistmem.induct; fastforce*)

lemma *slistmem-bottom2*[*iff*]:

fixes *xs* :: [*'a*:]
shows \neg *slistmem* \perp *xs*
by rule (*induct* \perp :: [*'a*:] *xs rule: slistmem.induct; fastforce*)

lemma *slistmem-nil*[*iff*]:

shows \neg *slistmem* *x* [*::*]
by (*fastforce elim: slistmem.cases*)

lemma *slistmem-scons*[*simp*]:

shows *slistmem* *x* (*y* :# *ys*) \longleftrightarrow (*x* = *y* \wedge *x* $\neq \perp$ \wedge *ys* $\neq \perp$) \vee (*slistmem* *x* *ys* \wedge *y* $\neq \perp$)
proof –
have *x* = *y* \vee *slistmem* *x* *ys* **if** *slistmem* *x* (*y* :# *ys*)
using that by (*induct x y* :# *ys arbitrary: y ys rule: slistmem.induct; force*)
then show ?*thesis* **by** (*auto elim: slistmem.cases intro: slistmem.intros*)
qed

definition *sset* :: [*'a*:] \Rightarrow *'a* *set* **where**

sset *xs* = {*x*. *slistmem* *x* *xs*}

lemma *sset-simp*[*simp*]:
shows $sset \perp = \{\}$
and $sset [::] = \{\}$
and $\llbracket x \neq \perp; xs \neq \perp \rrbracket \implies sset (x :\# xs) = insert\ x (sset\ xs)$
unfolding *sset-def* **by** (*auto elim: slistmem.cases intro: slistmem.intros*)

lemma *sset-defined*[*simp*]:
assumes $x \in sset\ xs$
shows $x \neq \perp$
using *assms sset-def* **by force**

lemma *sset-below*:
assumes $y \in sset\ ys$
assumes $xs \sqsubseteq ys$
assumes $xs \neq \perp$
obtains x **where** $x \in sset\ xs$ **and** $x \sqsubseteq y$
using *assms*
proof(*induct ys arbitrary: xs*)
case (*scons y ys xs*) **then show** *?case* **by** (*cases xs*) *auto*
qed *simp-all*

3.2 Some of the usual operations

A variety of functions on lists. Drawn from Bird (1987), *HOL.List* and *HOLCF-Prelude.Data-List*. The definitions vary because, for instance, the strictness of some of those in *HOLCF-Prelude.Data-List* correspond neither to those in Haskell nor Bird's expectations (specifically *stails*, *inits*, *sscanl*).

fixrec *snull* :: $[:'a:] \rightarrow tr$ **where**
 $snull.[::] = TT$
 $\llbracket x \neq \perp; xs \neq \perp \rrbracket \implies snull.(x :\# xs) = FF$

lemma *snull-strict*[*simp*]: $snull.\perp = \perp$
by *fixrec-simp*

lemma *snull-bottom-iff*[*simp*]: $(snull.xs = \perp) \longleftrightarrow (xs = \perp)$
by (*cases xs*) *simp-all*

lemma *snull-FF-conv*: $(snull.xxs = FF) \longleftrightarrow (\exists x\ xs. xxs \neq \perp \wedge xxs = x :\# xs)$
by (*cases xxs*) *simp-all*

lemma *snull-TT-conv*[*simp*]: $(snull.xs = TT) \longleftrightarrow (xs = [::])$
by (*cases xs*) *simp-all*

lemma *snull-eq-snil*: $snull.xs = eq.xs.[::]$
by (*cases xs*) *simp-all*

fixrec *smap* :: $('a \rightarrow 'b) \rightarrow [:'a:] \rightarrow [:'b:]$ **where**
 $smap.f.[::] = [::]$
 $\llbracket x \neq \perp; xs \neq \perp \rrbracket \implies smap.f.(x :\# xs) = f.x :\# smap.f.xs$

lemma *smap-strict*[*simp*]: $smap.f.\perp = \perp$
by *fixrec-simp*

lemma *smap-bottom-iff*[*simp*]: $(smap.f.xs = \perp) \longleftrightarrow (xs = \perp \vee (\exists x \in sset\ xs. f.x = \perp))$
by (*induct xs*) *simp-all*

lemma *smap-is-snil-conv*[*simp*]:
 $(smap.f.xs = [::]) \longleftrightarrow (xs = [::])$

$([::] = \text{smap} \cdot f \cdot xs) \longleftrightarrow (xs = [::])$

by $(\text{cases } xs; \text{simp})+$

lemma $\text{smap-strict-scons}[simp]$:

assumes $f \cdot \perp = \perp$

shows $\text{smap} \cdot f \cdot (x :\# xs) = f \cdot x :\# \text{smap} \cdot f \cdot xs$

using assms **by** $(\text{cases } x :\# xs = \perp; \text{fastforce})$

lemma $\text{smap-ID}'$: $\text{smap} \cdot \text{ID} \cdot xs = xs$

by $(\text{induct } xs) \text{simp-all}$

lemma $\text{smap-ID}[simp]$: $\text{smap} \cdot \text{ID} = \text{ID}$

by $(\text{clarsimp } \text{simp}; \text{cfun-eq-iff } \text{smap-ID}')$

lemma smap-cong :

assumes $xs = xs'$

assumes $\bigwedge x. x \in \text{sset } xs \implies f \cdot x = f' \cdot x$

shows $\text{smap} \cdot f \cdot xs = \text{smap} \cdot f' \cdot xs'$

using assms **by** $(\text{induct } xs \text{arbitrary}; xs') \text{auto}$

lemma $\text{smap-smap}'[simp]$:

assumes $f \cdot \perp = \perp$

shows $\text{smap} \cdot f \cdot (\text{smap} \cdot g \cdot xs) = \text{smap} \cdot (f \text{ oo } g) \cdot xs$

using assms **by** $(\text{induct } xs) \text{simp-all}$

lemma $\text{smap-smap}[simp]$:

assumes $f \cdot \perp = \perp$

shows $\text{smap} \cdot f \text{ oo } \text{smap} \cdot g = \text{smap} \cdot (f \text{ oo } g)$

using assms **by** $(\text{clarsimp } \text{simp}; \text{cfun-eq-iff})$

lemma $\text{sset-smap}[simp]$:

assumes $\bigwedge x. x \in \text{sset } xs \implies f \cdot x \neq \perp$

shows $\text{sset } (\text{smap} \cdot f \cdot xs) = \{ f \cdot x \mid x. x \in \text{sset } xs \}$

using assms **by** $(\text{induct } xs) \text{auto}$

lemma shead-smap-distr :

assumes $f \cdot \perp = \perp$

assumes $\bigwedge x. x \in \text{sset } xs \implies f \cdot x \neq \perp$

shows $\text{shead} \cdot (\text{smap} \cdot f \cdot xs) = f \cdot (\text{shead} \cdot xs)$

using assms **by** $(\text{induct } xs) \text{simp-all}$

fixrec $\text{sappend} :: [:'a:] \rightarrow [:'a:] \rightarrow [:'a:]$ **where**

$\text{sappend} \cdot [::] \cdot ys = ys$

$| \llbracket x \neq \perp; xs \neq \perp \rrbracket \implies \text{sappend} \cdot (x :\# xs) \cdot ys = x :\# \text{sappend} \cdot xs \cdot ys$

abbreviation $\text{sappend-syn} :: 'a \text{ slist} \Rightarrow 'a \text{ slist} \Rightarrow 'a \text{ slist}$ (**infixr** $\langle :@ \rangle$ 65) **where**

$xs :@ ys \equiv \text{sappend} \cdot xs \cdot ys$

lemma $\text{sappend-strict}[simp]$: $\text{sappend} \cdot \perp = \perp$

by fixrec-simp

lemma $\text{sappend-strict2}[simp]$: $xs :@ \perp = \perp$

by $(\text{induct } xs) \text{simp-all}$

lemma $\text{sappend-bottom-iff}[simp]$: $(xs :@ ys = \perp) \longleftrightarrow (xs = \perp \vee ys = \perp)$

by $(\text{induct } xs) \text{simp-all}$

lemma $\text{sappend-scons}[simp]$: $(x :\# xs) :@ ys = x :\# xs :@ ys$

by (*cases* $x : \# xs = \perp$; *fastforce*)

lemma *sappend-assoc*[*simp*]: $(xs : @ ys) : @ zs = xs : @ (ys : @ zs)$
by (*induct* xs) *simp-all*

lemma *sappend-snil-id-left*[*simp*]: $sappend \cdot [::] = ID$
by (*simp add: cfun-eq-iff*)

lemma *sappend-snil-id-right*[*iff*]: $xs : @ [::] = xs$
by (*induct* xs) *simp-all*

lemma *snil-append-iff*[*iff*]: $xs : @ ys = [::] \longleftrightarrow xs = [::] \wedge ys = [::]$
by (*induct* xs) *simp-all*

lemma *smap-sappend*[*simp*]: $smap \cdot f \cdot (xs : @ ys) = smap \cdot f \cdot xs : @ smap \cdot f \cdot ys$
by (*induct* xs ; *cases* $ys = \perp$; *simp*)

lemma *stail-sappend*: $stail \cdot (xs : @ ys) = (case\ xs\ of\ [::] \Rightarrow stail \cdot ys \mid z : \# zs \Rightarrow zs : @ ys)$
by (*induct* xs) *simp-all*

lemma *stail-append2*[*simp*]: $xs \neq [::] \Longrightarrow stail \cdot (xs : @ ys) = stail \cdot xs : @ ys$
by (*induct* xs) *simp-all*

lemma *slist-case-snoc*:
 $g \cdot \perp \cdot \perp = \perp \Longrightarrow slist\ case \cdot f \cdot g \cdot (xs : @ [x:]) = g \cdot (shead \cdot (xs : @ [x:])) \cdot (stail \cdot (xs : @ [x:]))$
by (*cases* $x = \perp$; *cases* xs ; *clarsimp*)

fixrec *sall* :: $('a \rightarrow tr) \rightarrow [:'a:] \rightarrow tr$ **where**
 $sall \cdot p \cdot [::] = TT$
 $\mid [x \neq \perp; xs \neq \perp] \Longrightarrow sall \cdot p \cdot (x : \# xs) = (p \cdot x\ andalso\ sall \cdot p \cdot xs)$

lemma *sall-strict*[*simp*]: $sall \cdot p \cdot \perp = \perp$
by *fixrec-simp*

lemma *sall-const-TT*[*simp*]:
assumes $xs \neq \perp$
shows $sall \cdot (\Lambda x. TT) \cdot xs = TT$
using *assms* **by** (*induct* xs) *simp-all*

lemma *sall-const-TT-conv*[*simp*]: $(sall \cdot (\Lambda x. TT) \cdot xs = TT) \longleftrightarrow (xs \neq \perp)$
by *auto*

lemma *sall-TT*[*simp*]: $(sall \cdot p \cdot xs = TT) \longleftrightarrow (xs \neq \perp \wedge (\forall x \in sset\ xs. p \cdot x = TT))$
by (*induct* xs) *simp-all*

fixrec *sfilter* :: $('a \rightarrow tr) \rightarrow [:'a:] \rightarrow [:'a:]$ **where**
 $sfilter \cdot p \cdot [::] = [::]$
 $\mid [x \neq \perp; xs \neq \perp] \Longrightarrow sfilter \cdot p \cdot (x : \# xs) = If\ p \cdot x\ then\ x : \# sfilter \cdot p \cdot xs\ else\ sfilter \cdot p \cdot xs$

lemma *sfilter-strict*[*simp*]: $sfilter \cdot p \cdot \perp = \perp$
by *fixrec-simp*

lemma *sfilter-bottom-iff*[*simp*]: $(sfilter \cdot p \cdot xs = \perp) \longleftrightarrow (xs = \perp \vee (\exists x \in sset\ xs. p \cdot x = \perp))$
by (*induct* xs) (*use* *trE* **in** *auto*)

lemma *sset-sfilter*[*simp*]:
assumes $\bigwedge x. x \in sset\ xs \Longrightarrow p \cdot x \neq \perp$
shows $sset\ (sfilter \cdot p \cdot xs) = \{x \mid x \in sset\ xs \wedge p \cdot x = TT\}$

using *assms* **by** (*induct xs*) (*fastforce simp: If2-def[symmetric] split: If2-splits*)⁺

lemma *sfilter-strict-scons*[*simp*]:

assumes $p \cdot \perp = \perp$

shows $sfilter \cdot p \cdot (x : \# xs) = \text{If } p \cdot x \text{ then } x : \# sfilter \cdot p \cdot xs \text{ else } sfilter \cdot p \cdot xs$

using *assms* **by** (*cases x = ⊥*; *cases xs = ⊥*; *simp*)

lemma *sfilter-scons-let*:

assumes $p \cdot \perp = \perp$

shows $sfilter \cdot p \cdot (x : \# xs) = (\text{let } xs' = sfilter \cdot p \cdot xs \text{ in } \text{If } p \cdot x \text{ then } x : \# xs' \text{ else } xs')$

unfolding *Let-def* **using** *assms* **by** *simp*

lemma *sfilter-sappend*[*simp*]: $sfilter \cdot p \cdot (xs : @ ys) = sfilter \cdot p \cdot xs : @ sfilter \cdot p \cdot ys$

by (*cases ys*; *clarsimp*) (*induct xs*; *fastforce simp: If2-def[symmetric] split: If2-splits*)

lemma *sfilter-const-FF*[*simp*]:

assumes $xs \neq \perp$

shows $sfilter \cdot (\Lambda x. FF) \cdot xs = [::]$

using *assms* **by** (*induct xs*) *simp-all*

lemma *sfilter-const-FF-conv*[*simp*]: $(sfilter \cdot (\Lambda x. FF) \cdot xs = [::]) \longleftrightarrow (xs \neq \perp)$

by *auto*

lemma *sfilter-const-TT*[*simp*]: $sfilter \cdot (\Lambda x. TT) \cdot xs = xs$

by (*induct xs*) *simp-all*

lemma *sfilter-cong*:

assumes $xs = xs'$

assumes $\bigwedge x. x \in \text{sset } xs \implies p \cdot x = p' \cdot x$

shows $sfilter \cdot p \cdot xs = sfilter \cdot p' \cdot xs'$

using *assms* **by** (*induct xs arbitrary: xs'*) *auto*

lemma *sfilter-snil-conv*[*simp*]: $sfilter \cdot p \cdot xs = [::] \longleftrightarrow \text{sall} \cdot (\text{neg } oo p) \cdot xs = TT$

by (*induct xs*; *force simp: If2-def[symmetric] split: If2-splits*)

lemma *sfilter-sfilter'*: $sfilter \cdot p \cdot (sfilter \cdot q \cdot xs) = sfilter \cdot (\Lambda x. q \cdot x \text{ andalso } p \cdot x) \cdot xs$

proof(*induct xs*)

case (*scons x xs*) **from** *scons*(1, 2) **show** ?*case*

by (*cases sfilter \cdot q \cdot xs = ⊥*)

(*simp-all add: If-distr If-andalso scons*(3)[*symmetric*] *del: sfilter-bottom-iff*)

qed *simp-all*

lemma *sfilter-sfilter*: $sfilter \cdot p \text{ } oo \text{ } sfilter \cdot q = sfilter \cdot (\Lambda x. q \cdot x \text{ andalso } p \cdot x)$

by (*clarsimp simp: cfun-eq-iff sfilter-sfilter'*)

lemma *sfilter-smap'*:

assumes $p \cdot \perp = \perp$

shows $sfilter \cdot p \cdot (\text{smap} \cdot f \cdot xs) = \text{smap} \cdot f \cdot (sfilter \cdot (p \text{ } oo \text{ } f) \cdot xs)$

using *assms* **by** (*induct xs*; *simp add: If2-def[symmetric] split: If2-splits*) (*metis slist.con-rews*(2) *smap.simps*(2) *smap-strict*)

lemma *sfilter-smap*:

assumes $p \cdot \perp = \perp$

shows $sfilter \cdot p \text{ } oo \text{ } \text{smap} \cdot f = \text{smap} \cdot f \text{ } oo \text{ } sfilter \cdot (p \text{ } oo \text{ } f)$

using *assms* **by** (*clarsimp simp: cfun-eq-iff sfilter-smap'*)

fixrec *sfoldl* :: (*'a*::*pcpo* → *'b*::*domain* → *'a*) → *'a* → [*'b*:] → *'a* **where**

sfoldl \cdot *f* \cdot *z* \cdot [::] = *z*

| $\llbracket x \neq \perp; xs \neq \perp \rrbracket \implies \mathit{sfoldl}.f.z.(x \# xs) = \mathit{sfoldl}.f.(f.z.x).xs$

lemma $\mathit{sfoldl}\text{-strict}[simp]$: $\mathit{sfoldl}.f.z.\perp = \perp$
by $\mathit{fixrec}\text{-simp}$

lemma $\mathit{sfoldl}\text{-strict}\text{-}f[simp]$:
 assumes $f.\perp = \perp$
 shows $\mathit{sfoldl}.f.\perp.xs = \perp$
using assms **by** ($\mathit{induct}\ xs$) $\mathit{simp}\text{-all}$

lemma $\mathit{sfoldl}\text{-cong}$:
 assumes $xs = xs'$
 assumes $z = z'$
 assumes $\bigwedge x z. x \in \mathit{sset}\ xs \implies f.z.x = f'.z.x$
 shows $\mathit{sfoldl}.f.z.xs = \mathit{sfoldl}.f'.z'.xs'$
using assms **by** ($\mathit{induct}\ xs$ arbitrary: $xs' z z'$) auto

lemma $\mathit{sfoldl}\text{-sappend}[simp]$:
 assumes $f.\perp = \perp$
 shows $\mathit{sfoldl}.f.z.(xs \text{:@} ys) = \mathit{sfoldl}.f.(\mathit{sfoldl}.f.z.xs).ys$
using assms **by** ($\mathit{cases}\ ys = \perp, \mathit{force}$) ($\mathit{induct}\ xs$ arbitrary: z ; simp)

fixrec $\mathit{sfoldr} :: ('b \rightarrow 'a::\mathit{pcpo} \rightarrow 'a) \rightarrow 'a \rightarrow [:'b:] \rightarrow 'a$ **where**
 $\mathit{sfoldr}.f.z.[:] = z$
| $\llbracket x \neq \perp; xs \neq \perp \rrbracket \implies \mathit{sfoldr}.f.z.(x \# xs) = f.x.(\mathit{sfoldr}.f.z.xs)$

lemma $\mathit{sfoldr}\text{-strict}[simp]$: $\mathit{sfoldr}.f.z.\perp = \perp$
by $\mathit{fixrec}\text{-simp}$

fixrec $\mathit{sconcat} :: [:'a:] \rightarrow [:'a:]$ **where**
 $\mathit{sconcat}.[:] = [:]$
| $\llbracket x \neq \perp; xs \neq \perp \rrbracket \implies \mathit{sconcat}.(x \# xs) = x \text{:@} \mathit{sconcat}.xs$

lemma $\mathit{sconcat}\text{-strict}[simp]$: $\mathit{sconcat}.\perp = \perp$
by $\mathit{fixrec}\text{-simp}$

lemma $\mathit{sconcat}\text{-scons}[simp]$:
 shows $\mathit{sconcat}.(x \# xs) = x \text{:@} \mathit{sconcat}.xs$
by ($\mathit{cases}\ x = \perp, \mathit{force}$) ($\mathit{induct}\ xs$; $\mathit{fastforce}$)

lemma $\mathit{sconcat}\text{-sfoldl}\text{-aux}$: $\mathit{sfoldl}.sappend.z.xs = z \text{:@} \mathit{sconcat}.xs$
by ($\mathit{induct}\ xs$ arbitrary: z) $\mathit{simp}\text{-all}$

lemma $\mathit{sconcat}\text{-sfoldl}$: $\mathit{sconcat} = \mathit{sfoldl}.sappend.[:]$
by ($\mathit{clarsimp}\ \mathit{simp}$: $\mathit{cfun}\text{-eq}\text{-iff}\ \mathit{sconcat}\text{-sfoldl}\text{-aux}$)

lemma $\mathit{sconcat}\text{-sappend}[simp]$: $\mathit{sconcat}.(xs \text{:@} ys) = \mathit{sconcat}.xs \text{:@} \mathit{sconcat}.ys$
by ($\mathit{induct}\ xs$) $\mathit{simp}\text{-all}$

fixrec $\mathit{slength} :: [:'a:] \rightarrow \mathit{Integer}$
where
 $\mathit{slength}.[:] = 0$
| $\llbracket x \neq \perp; xs \neq \perp \rrbracket \implies \mathit{slength}.(x \# xs) = \mathit{slength}.xs + 1$

lemma $\mathit{slength}\text{-strict}[simp]$: $\mathit{slength}.\perp = \perp$
by $\mathit{fixrec}\text{-simp}$

lemma $\mathit{slength}\text{-bottom}\text{-iff}[simp]$: $(\mathit{slength}.xs = \perp) \longleftrightarrow (xs = \perp)$

by (induct xs) force+

lemma *length-ge-0*: $\text{length}\cdot xs = \text{MkI}\cdot n \implies n \geq 0$

by (induct xs arbitrary: n) (simp add: one-Integer-def plus-eq-MkI-conv; force)+

lemma *lengthE*:

shows $\llbracket xs \neq \perp; \bigwedge n. \llbracket \text{length}\cdot xs = \text{MkI}\cdot n; 0 \leq n \rrbracket \implies Q \rrbracket \implies Q$

by (meson Integer.exhaust length-bottom-iff length-ge-0)

lemma *length-0-conv[simp]*:

$(\text{length}\cdot xs = 0) \longleftrightarrow (xs = [::])$

$(\text{length}\cdot xs = \text{MkI}\cdot 0) \longleftrightarrow (xs = [::])$

$\text{eq}\cdot 0\cdot(\text{length}\cdot xs) = \text{snul}\cdot xs$

$\text{eq}\cdot(\text{length}\cdot xs)\cdot 0 = \text{snul}\cdot xs$

by (induct xs) (auto simp: one-Integer-def elim: lengthE)

lemma *le-length-0[simp]*: $(\text{le}\cdot 0\cdot(\text{length}\cdot xs) = \text{TT}) \longleftrightarrow (xs \neq \perp)$

by (cases length.xs) (auto simp: length-ge-0 zero-Integer-def)

lemma *lt-length-0[simp]*:

$xs \neq \perp \implies \text{lt}\cdot(\text{length}\cdot xs)\cdot 0 = \text{FF}$

$xs \neq \perp \implies \text{lt}\cdot(\text{length}\cdot xs)\cdot(\text{length}\cdot xs + 1) = \text{TT}$

unfolding zero-Integer-def one-Integer-def by (auto elim: lengthE)

lemma *length-smap[simp]*:

assumes $\bigwedge x. x \neq \perp \implies f\cdot x \neq \perp$

shows $\text{length}\cdot(\text{smap}\cdot f\cdot xs) = \text{length}\cdot xs$

using assms by (induct xs) simp-all

lemma *length-sappend[simp]*: $\text{length}\cdot(xs :@ ys) = \text{length}\cdot xs + \text{length}\cdot ys$

by (cases ys = \perp , force) (induct xs; force simp: ac-simps)

lemma *length-sfoldl-aux*: $\text{sfoldl}\cdot(\bigwedge i -. i + 1)\cdot z\cdot xs = z + \text{length}\cdot xs$

by (induct xs arbitrary: z) (simp-all add: ac-simps)

lemma *length-sfoldl*: $\text{length} = \text{sfoldl}\cdot(\bigwedge i -. i + 1)\cdot 0$

by (clarsimp simp: cfun-eq-iff length-sfoldl-aux)

lemma *le-length-plus*:

assumes $xs \neq \perp$

assumes $n \neq \perp$

shows $\text{le}\cdot n\cdot(\text{length}\cdot xs + n) = \text{TT}$

using assms by (cases n; force elim: lengthE)

fixrec *srev* :: $[:'a:] \rightarrow [:'a:]$ **where**

$\text{srev}\cdot [::] = [::]$

| $\llbracket x \neq \perp; xs \neq \perp \rrbracket \implies \text{srev}\cdot(x :# xs) = \text{srev}\cdot xs :@ [x:]$

lemma *srev-strict[simp]*: $\text{srev}\cdot \perp = \perp$

by fixrec-simp

lemma *srev-bottom-iff[simp]*: $(\text{srev}\cdot xs = \perp) \longleftrightarrow (xs = \perp)$

by (induct xs) simp-all

lemma *srev-scons[simp]*: $\text{srev}\cdot(x :# xs) = \text{srev}\cdot xs :@ [x:]$

by (cases x = \perp , clarsimp) (induct xs; force)

lemma *srev-sappend[simp]*: $\text{srev}\cdot(xs :@ ys) = \text{srev}\cdot ys :@ \text{srev}\cdot xs$

by (induct xs) simp-all

lemma srev-srev-ident[simp]: srev.(srev.xs) = xs

by (induct xs) auto

lemma srev-cases[case-names bottom snil ssnoc]:

assumes xs = \perp \implies P

assumes xs = [::] \implies P

assumes $\bigwedge y ys. \llbracket y \neq \perp; ys \neq \perp; xs = ys :@ [y:] \rrbracket \implies P$

shows P

using assms by (metis slist.exhaust srev.simps(1) srev-scons srev-srev-ident srev-strict)

lemma srev-induct[case-names bottom snil ssnoc]:

assumes P \perp

assumes P [::]

assumes $\bigwedge x xs. \llbracket x \neq \perp; xs \neq \perp; P xs \rrbracket \implies P (xs :@ [x:])$

shows P xs

proof –

have P (srev.(srev.xs)) by (rule slist.induct[where x=srev.xs]; simp add: assms)

then show ?thesis by simp

qed

lemma sfldr-conv-sfoldl:

assumes $\bigwedge x. f \cdot x \cdot \perp = \perp$ — f must be strict in the accumulator.

shows sfldr.f.z.xs = sfoldl.($\bigwedge acc x. f \cdot x \cdot acc$).z.(srev.xs)

using assms by (induct xs arbitrary: z) simp-all

fixrec stake :: Integer \rightarrow [$'a$:] \rightarrow [$'a$:] **where** — Note: strict in both parameters.

stake. \perp . \perp = \perp

| $i \neq \perp \implies$ stake.i.[::] = [::]

| $\llbracket x \neq \perp; xs \neq \perp \rrbracket \implies$ stake.i.(x :# xs) = If le.i.0 then [::] else x :# stake.(i - 1).xs

lemma stake-strict[simp]:

stake. \perp = \perp

stake.i. \perp = \perp

by fixrec-simp+

lemma stake-bottom-iff[simp]: (stake.i.xs = \perp) \longleftrightarrow (i = $\perp \vee$ xs = \perp)

by (induct xs arbitrary: i; clarsimp; case-tac i; clarsimp)

lemma stake-0[simp]:

xs $\neq \perp \implies$ stake.0.xs = [::]

xs $\neq \perp \implies$ stake.(MkI.0).xs = [::]

stake.0.xs \sqsubseteq [::]

by (cases xs; simp add: zero-Integer-def)+

lemma stake-scons[simp]: le.1.i = TT \implies stake.i.(x :# xs) = x :# stake.(i - 1).xs

by (cases i; cases x = \perp ; cases xs = \perp ;

simp add: zero-Integer-def one-Integer-def split: if-splits)

lemma take-MkI-scons[simp]:

0 < n \implies stake.(MkI.n).(x :# xs) = x :# stake.(MkI.(n - 1)).xs

by (cases x = \perp ; cases xs = \perp ; simp add: zero-Integer-def one-Integer-def)

lemma stake-numeral-scons[simp]:

xs $\neq \perp \implies$ stake.1.(x :# xs) = [x:]

stake.(numeral (Num.Bit0 k)).(x :# xs) = x :# stake.(numeral (Num.BitM k)).xs

stake.(numeral (Num.Bit1 k)).(x :# xs) = x :# stake.(numeral (Num.Bit0 k)).xs

by (cases $x = \perp$; cases xs ; simp add: zero-Integer-def one-Integer-def numeral-Integer-eq)+

lemma *stake-all*:

assumes $le \cdot (\text{slength} \cdot xs) \cdot i = TT$

shows $\text{stake} \cdot i \cdot xs = xs$

using *assms*

proof(induct xs arbitrary: i)

case (scons $x xs i$) then show ?case

by (cases i ; clarsimp simp: If2-def[symmetric] zero-Integer-def one-Integer-def split: If2-splits if-splits elim!: slengthE)

qed (simp-all add: le-defined)

lemma *stake-all-triv*[simp]: $\text{stake} \cdot (\text{slength} \cdot xs) \cdot xs = xs$

by (cases $xs = \perp$) (auto simp: stake-all)

lemma *stake-append*[simp]: $\text{stake} \cdot i \cdot (xs :@ ys) = \text{stake} \cdot i \cdot xs :@ \text{stake} \cdot (i - \text{slength} \cdot xs) \cdot ys$

proof(induct xs arbitrary: i)

case (snil i) then show ?case by (cases i ; simp add: zero-Integer-def)

next

case (scons $x xs i$) then show ?case

by (cases i ; cases ys ; clarsimp simp: If2-def[symmetric] zero-Integer-def one-Integer-def split: If2-splits elim!: slengthE)

qed simp-all

fixrec *sdrop* :: $Integer \rightarrow [:'a:] \rightarrow [:'a:]$ **where** — Note: strict in both parameters.

[simp del]: $\text{sdrop} \cdot i \cdot xs = \text{If } le \cdot i \cdot 0 \text{ then } xs \text{ else } (\text{case } xs \text{ of } [::] \Rightarrow [::] \mid y :\# ys \Rightarrow \text{sdrop} \cdot (i - 1) \cdot ys)$

lemma *sdrop-strict*[simp]:

$\text{sdrop} \cdot \perp = \perp$

$\text{sdrop} \cdot i \cdot \perp = \perp$

by *fixrec-simp*+

lemma *sdrop-bottom-iff*[simp]: $(\text{sdrop} \cdot i \cdot xs = \perp) \longleftrightarrow (i = \perp \vee xs = \perp)$

proof(induct xs arbitrary: i)

case (snil i) then show ?case by (subst *sdrop.unfold*) (cases i ; simp)

next

case (scons $x xs i$) then show ?case by (subst *sdrop.unfold*) fastforce

qed simp

lemma *sdrop-snil*[simp]:

assumes $i \neq \perp$

shows $\text{sdrop} \cdot i \cdot [::] = [::]$

using *assms* by (subst *sdrop.unfold*; fastforce)

lemma *sdrop-snil-conv*[simp]: $(\text{sdrop} \cdot i \cdot [::] = [::]) \longleftrightarrow (i \neq \perp)$

by (subst *sdrop.unfold*; fastforce)

lemma *sdrop-0*[simp]:

$\text{sdrop} \cdot 0 \cdot xs = xs$

$\text{sdrop} \cdot (\text{MkI} \cdot 0) \cdot xs = xs$

by (subst *sdrop.simps*, simp add: zero-Integer-def)+

lemma *sdrop-pos*:

$le \cdot i \cdot 0 = FF \implies \text{sdrop} \cdot i \cdot xs = (\text{case } xs \text{ of } [::] \Rightarrow [::] \mid y :\# ys \Rightarrow \text{sdrop} \cdot (i - 1) \cdot ys)$

by (subst *sdrop.simps*, simp)

lemma *sdrop-neg*:

$le \cdot i \cdot 0 = TT \implies \text{sdrop} \cdot i \cdot xs = xs$

by (subst sdrop.simps, simp)

lemma sdrop-numeral-scons[simp]:

$x \neq \perp \implies \text{sdrop} \cdot 1 \cdot (x :\# xs) = xs$

$x \neq \perp \implies \text{sdrop} \cdot (\text{numeral } (\text{Num.Bit0 } k)) \cdot (x :\# xs) = \text{sdrop} \cdot (\text{numeral } (\text{Num.BitM } k)) \cdot xs$

$x \neq \perp \implies \text{sdrop} \cdot (\text{numeral } (\text{Num.Bit1 } k)) \cdot (x :\# xs) = \text{sdrop} \cdot (\text{numeral } (\text{Num.Bit0 } k)) \cdot xs$

by (subst sdrop.simps,

simp add: zero-Integer-def one-Integer-def numeral-Integer-eq; cases xs; simp)+

lemma sdrop-sappend[simp]:

$\text{sdrop} \cdot i \cdot (xs :@ ys) = \text{sdrop} \cdot i \cdot xs :@ \text{sdrop} \cdot (i - \text{slength} \cdot xs) \cdot ys$

proof(induct xs arbitrary: i)

case (snil i) **then show** ?case **by** (cases i; simp add: zero-Integer-def)

next

case (scons x xs i) **then show** ?case

by (cases ys = \perp ; cases le·i·0; cases i;

clarsimp simp: zero-Integer-def one-Integer-def sdrop-neg sdrop-pos add.commute diff-diff-add

split: if-splits elim!: slengthE)

qed simp

lemma sdrop-all:

assumes le·(slength·xs)·i = TT

shows sdrop·i·xs = [::]

using assms

proof(induct xs arbitrary: i)

case (scons x xs i) **then show** ?case

by (subst sdrop.unfold; cases i;

clarsimp simp: If2-def[symmetric] zero-Integer-def one-Integer-def split: If2-splits if-splits elim!: slengthE)

qed (simp-all add: le-defined)

lemma slength-sdrop[simp]:

$\text{slength} \cdot (\text{sdrop} \cdot i \cdot xs) = \text{If } \text{le} \cdot i \cdot 0 \text{ then } \text{slength} \cdot xs \text{ else If } \text{le} \cdot (\text{slength} \cdot xs) \cdot i \text{ then } 0 \text{ else } \text{slength} \cdot xs - i$

proof(induct xs arbitrary: i)

case (snil i) **then show** ?case **by** (cases i; simp add: zero-Integer-def)

next

case (scons x xs i) **then show** ?case

by (subst sdrop.unfold; cases i; clarsimp simp: zero-Integer-def one-Integer-def elim!: slengthE)

qed simp

lemma sdrop-not-snilD:

assumes sdrop·(MkI·i)·xs \neq [::]

assumes xs $\neq \perp$

shows lt·(MkI·i)·(slength·xs) = TT \wedge xs \neq [::]

using assms

proof(induct xs arbitrary: i)

case (scons x xs i) **then show** ?case

by (subst (asm) (2) sdrop.unfold, clarsimp simp: zero-Integer-def one-Integer-def not-le sdrop-all elim!: slengthE)

qed simp-all

lemma sdrop-sappend-same:

assumes xs $\neq \perp$

shows sdrop·(slength·xs)·(xs :@ ys) = ys

using assms

proof(induct xs arbitrary: ys)

case (scons x xs ys) **then show** ?case

by (cases ys = \perp ; subst sdrop.unfold; clarsimp simp: zero-Integer-def one-Integer-def elim!: slengthE)

qed simp-all

fixrec *sscanl* :: ('a → 'b → 'a) → 'a → [:'b:] → [:'a:] **where**
sscanl·f·z·[::] = z :# [::]
| [[x ≠ ⊥; xs ≠ ⊥]] ⇒ *sscanl*·f·z·(x :# xs) = z :# *sscanl*·f·(f·z·x)·xs

lemma *sscanl-strict*[*simp*]:
sscanl·f·⊥·xs = ⊥
sscanl·f·z·⊥ = ⊥
by (*cases xs*) *fixrec-simp*+

lemma *sscanl-cong*:
assumes *xs* = *xs'*
assumes *z* = *z'*
assumes $\bigwedge x z. x \in \text{sset } xs \implies f \cdot z \cdot x = f' \cdot z \cdot x$
shows *sscanl*·f·z·xs = *sscanl*·f'·z'·xs'
using *assms* **by** (*induct xs arbitrary: xs' z z'*) *auto*

lemma *sscanl-lfp-fusion'*:
assumes *g*·⊥ = ⊥
assumes *: $\bigwedge \text{acc } x. x \neq \perp \implies g \cdot (f \cdot \text{acc} \cdot x) = f' \cdot (g \cdot \text{acc}) \cdot x$
shows *smap*·*g*·(*sscanl*·f·z·xs) = *sscanl*·f'·(g·z)·xs
using *assms* **by** (*induct xs arbitrary: z*) *simp-all*

lemma *sscanl-lfp-fusion*:
assumes *g*·⊥ = ⊥
assumes *: $\bigwedge \text{acc } x. x \neq \perp \implies g \cdot (f \cdot \text{acc} \cdot x) = f' \cdot (g \cdot \text{acc}) \cdot x$
shows *smap*·*g* oo *sscanl*·f·z = *sscanl*·f'·(g·z)
using *assms* **by** (*clarsimp simp: cfun-eq-iff sscanl-lfp-fusion'*)

lemma *sscanl-ww-fusion'*: — Worker/wrapper (Gammie 2011; Gill and Hutton 2009) specialised to *sscanl*
fixes *wrap* :: 'b → 'a
fixes *unwrap* :: 'a → 'b
fixes *z* :: 'a
fixes *f* :: 'a → 'c → 'a
fixes *f'* :: 'b → 'c → 'b
assumes *ww*: *wrap* oo *unwrap* = *ID*
assumes *wb*: $\bigwedge z x. x \neq \perp \implies \text{unwrap} \cdot (f \cdot (\text{wrap} \cdot z) \cdot x) = f' \cdot (\text{unwrap} \cdot (\text{wrap} \cdot z)) \cdot x$
shows *sscanl*·f·z·xs = *smap*·*wrap*·(*sscanl*·f'·(unwrap·z)·xs)
using *assms*
by (*induct xs arbitrary: z*) (*simp add: cfun-eq-iff retraction-cfcomp-strict |metis*)+

lemma *sscanl-ww-fusion*: — Worker/wrapper (Gammie 2011; Gill and Hutton 2009) specialised to *sscanl*
fixes *wrap* :: 'b → 'a
fixes *unwrap* :: 'a → 'b
fixes *z* :: 'a
fixes *f* :: 'a → 'c → 'a
fixes *f'* :: 'b → 'c → 'b
assumes *ww*: *wrap* oo *unwrap* = *ID*
assumes *wb*: $\bigwedge z x. x \neq \perp \implies \text{unwrap} \cdot (f \cdot (\text{wrap} \cdot z) \cdot x) = f' \cdot (\text{unwrap} \cdot (\text{wrap} \cdot z)) \cdot x$
shows *sscanl*·f·z = *smap*·*wrap* oo *sscanl*·f'·(unwrap·z)
using *assms* **by** (*clarsimp simp: cfun-eq-iff sscanl-ww-fusion'*)

fixrec *sinits* :: [:'a:] → [:[:'a:]:] **where**
sinits·[::] = [::] :# [::]
| [[x ≠ ⊥; xs ≠ ⊥]] ⇒ *sinits*·(x :# xs) = [::] :# *smap*·(*scons*·x)·(*sinits*·xs)

lemma *sinits-strict*[*simp*]: *sinits*·⊥ = ⊥
by *fixrec-simp*

lemma *sinits-bottom-iff*[simp]: $(\text{sinits}\cdot xs = \perp) \longleftrightarrow (xs = \perp)$

by (*induct xs*) *simp-all*

lemma *sinits-not-snil*[iff]: $\text{sinits}\cdot xs \neq [::]$

by (*cases xs*) *simp-all*

lemma *sinits-empty-bottom*[simp]: $(\text{sset}(\text{sinits}\cdot xs) = \{\}) \longleftrightarrow (xs = \perp)$

by (*cases xs*) *simp-all*

lemma *sinits-scons*[simp]: $\text{sinits}\cdot(x \# xs) = [::] \# \text{smap}\cdot(x \#)\cdot(\text{sinits}\cdot xs)$

by (*cases x = \perp , force*) (*induct xs; force*)

lemma *sinits-length*[simp]: $\text{slength}\cdot(\text{sinits}\cdot xs) = \text{slength}\cdot xs + 1$

by (*induct xs*) *simp-all*

lemma *sinits-snoc*[simp]: $\text{sinits}\cdot(xs \text{:}@ [x:]) = \text{sinits}\cdot xs \text{:}@ [xs \text{:}@ [x:]]$

by (*induct xs*) *simp-all*

lemma *sinits-foldr'*: — Bird (1987, p30)

shows $\text{sinits}\cdot xs = \text{sfoldr}\cdot(\Lambda x xs. [::] \text{:}@ \text{smap}\cdot(x \#)\cdot xs)\cdot[::]\cdot xs$

by (*induct xs*) *simp-all*

lemma *sinits-sscanl'*:

shows $\text{smap}\cdot(\text{sfoldl}\cdot f\cdot z)\cdot(\text{sinits}\cdot xs) = \text{sscanl}\cdot f\cdot z\cdot xs$

by (*induct xs arbitrary: z*) (*simp-all cong: smap-cong add: oo-def eta-cfun*)

lemma *sinits-sscanl*: — Bird (1987, Lemma 5), Bird (2010, p118 “the scan lemma”)

shows $\text{smap}\cdot(\text{sfoldl}\cdot f\cdot z) \text{ oo } \text{sinits} = \text{sscanl}\cdot f\cdot z$

by (*simp add: sinits-sscanl' cfun-eq-iff*)

lemma *sinits-all*[simp]: $(xs \in \text{sset}(\text{sinits}\cdot xs)) \longleftrightarrow (xs \neq \perp)$

by (*induct xs*) *simp-all*

fixrec *stails* :: $[:'a:] \rightarrow [:::'a:]$ **where**

$\text{stails}\cdot[::] = [::] \# [::]$

$| [x \neq \perp; xs \neq \perp] \Longrightarrow \text{stails}\cdot(x \# xs) = (x \# xs) \# \text{stails}\cdot xs$

lemma *stails-strict*[simp]: $\text{stails}\cdot\perp = \perp$

by *fixrec-simp*

lemma *stails-bottom-iff*[simp]: $(\text{stails}\cdot xs = \perp) \longleftrightarrow (xs = \perp)$

by (*induct xs*) *simp-all*

lemma *stails-not-snil*[iff]: $\text{stails}\cdot xs \neq [::]$

by (*cases xs*) *simp-all*

lemma *stails-scons*[simp]: $\text{stails}\cdot(x \# xs) = (x \# xs) \# \text{stails}\cdot xs$

by (*induct xs*) (*cases x = \perp ; simp*)⁺

lemma *stails-slength*[simp]: $\text{slength}\cdot(\text{stails}\cdot xs) = \text{slength}\cdot xs + 1$

by (*induct xs*) *simp-all*

lemma *stails-snoc*[simp]:

shows $\text{stails}\cdot(xs \text{:}@ [x:]) = \text{smap}\cdot(\Lambda ys. ys \text{:}@ [x:])\cdot(\text{stails}\cdot xs) \text{:}@ [:::]$

by (*induct xs*) *simp-all*

lemma *stails-sfoldl'*:

shows $\text{stails}\cdot xs = \text{sfoldl}\cdot(\Lambda xs x. \text{smap}\cdot(\Lambda ys. ys \text{:}@ [x:])\cdot xs \text{:}@ [:::])\cdot[::]\cdot xs$

by (induct xs rule: srev-induct) simp-all

lemma *stails-sfoldl*:

shows $stails = sfoldl \cdot (\Lambda xs x. smap \cdot (\Lambda ys. ys :@ [x:])) \cdot xs :@ [:::] \cdot [:::]$

by (clarsimp simp: cfun-eq-iff tails-sfoldl')

lemma *stails-all*[simp]: $(xs \in sset (stails \cdot xs)) \longleftrightarrow (xs \neq \perp)$

by (cases xs) simp-all

fixrec *selem* :: 'a::Eq-def \rightarrow [:'a:] \rightarrow tr **where**

$selem \cdot x \cdot [::] = FF$

| $\llbracket y \neq \perp; ys \neq \perp \rrbracket \implies selem \cdot x \cdot (y :# ys) = (eq \cdot x \cdot y \text{ orelse } selem \cdot x \cdot ys)$

lemma *selem-strict*[simp]: $selem \cdot x \cdot \perp = \perp$

by fixrec-simp

lemma *selem-bottom-iff*[simp]: $(selem \cdot x \cdot xs = \perp) \longleftrightarrow (xs = \perp \vee (xs \neq [::] \wedge x = \perp))$

by (induct xs) auto

lemma *selem-sappend*[simp]:

assumes $ys \neq \perp$

shows $selem \cdot x \cdot (xs :@ ys) = (selem \cdot x \cdot xs \text{ orelse } selem \cdot x \cdot ys)$

using assms by (induct xs) simp-all

lemma *elem-TT*[simp]: $(selem \cdot x \cdot xs = TT) \longleftrightarrow (x \in sset xs)$

by (induct xs; auto) (metis sset-defined)+

lemma *elem-FF*[simp]: $(selem \cdot x \cdot xs = FF) \longleftrightarrow (xs = [::] \vee (x \neq \perp \wedge xs \neq \perp \wedge x \notin sset xs))$

by (induct xs) auto

lemma *selem-snil-stails*[iff]:

assumes $xs \neq \perp$

shows $selem \cdot [::] \cdot (stails \cdot xs) = TT$

using assms by (induct xs) simp-all

fixrec *sconcatMap* :: ('a \rightarrow [:'b:]) \rightarrow [:'a:] \rightarrow [:'b:] **where**

[simp del]: $sconcatMap \cdot f = sconcat \text{ oo } smap \cdot f$

lemma *sconcatMap-strict*[simp]: $sconcatMap \cdot f \cdot \perp = \perp$

by fixrec-simp

lemma *sconcatMap-snil*[simp]: $sconcatMap \cdot f \cdot [::] = [::]$

by fixrec-simp

lemma *sconcatMap-scons*[simp]: $x \neq \perp \implies sconcatMap \cdot f \cdot (x :# xs) = f \cdot x :@ sconcatMap \cdot f \cdot xs$

by (cases xs = \perp ; simp add: sconcatMap.unfold)

lemma *sconcatMap-bottom-iff*[simp]: $(sconcatMap \cdot f \cdot xs = \perp) \longleftrightarrow (xs = \perp \vee (\exists x \in sset xs. f \cdot x = \perp))$

by (induct xs) simp-all

lemma *sconcatMap-sappend*[simp]: $sconcatMap \cdot f \cdot (xs :@ ys) = sconcatMap \cdot f \cdot xs :@ sconcatMap \cdot f \cdot ys$

by (induct xs) simp-all

lemma *sconcatMap-monad-laws*:

$sconcatMap \cdot (\Lambda x. [x:]) \cdot xs = xs$

$sconcatMap \cdot g \cdot (sconcatMap \cdot f \cdot xs) = sconcatMap \cdot (\Lambda x. sconcatMap \cdot g \cdot (f \cdot x)) \cdot xs$

by (induct xs) simp-all

fixrec *supto* :: *Integer* → *Integer* → [*Integer*:] **where**
[simp del]: *supto*·*i*·*j* = *If* *le*·*i*·*j* *then* *i* :# *supto*·(*i*+1)·*j* *else* [::]

lemma *upto-strict*[*simp*]:

supto·⊥ = ⊥
supto·*m*·⊥ = ⊥

by *fixrec-simp*+

lemma *supto-is-snil-conv*[*simp*]:

(*supto*·(*MkI*·*i*)·(*MkI*·*j*) = [::]) ↔ (*j* < *i*)
([::] = *supto*·(*MkI*·*i*)·(*MkI*·*j*)) ↔ (*j* < *i*)

by (*subst* *supto*·*unfold*; *simp*)+

lemma *supto-simp*[*simp*]:

j < *i* ⇒ *supto*·(*MkI*·*i*)·(*MkI*·*j*) = [::]
i ≤ *j* ⇒ *supto*·(*MkI*·*i*)·(*MkI*·*j*) = *MkI*·*i* :# *supto*·(*MkI*·*i*+1)·(*MkI*·*j*)
supto·0·0 = [0:]

by (*subst* *supto*·*simps*, *simp*)+

lemma *supto-defined*[*simp*]: *supto*·(*MkI*·*i*)·(*MkI*·*j*) ≠ ⊥ (**is** ?*P* *i* *j*)

proof (*cases* *j* - *i*)

fix *d*

assume *j* - *i* = *int* *d*

then show ?*P* *i* *j*

proof (*induct* *d* *arbitrary*: *i* *j*)

case (*Suc* *d* *i* *j*)

then have *j* - (*i* + 1) = *int* *d* **and** *le*: *i* ≤ *j* **by** *simp-all*

from *Suc*(1)[*OF* *this*(1)] **have** *IH*: ?*P* (*i*+1) *j* .

then show ?*case* **using** *le* **by** (*simp* *add*: *one-Integer-def*)

qed (*simp* *add*: *one-Integer-def*)

next

fix *d*

assume *j* - *i* = - *int* *d*

then have *j* ≤ *i* **by** *auto*

moreover

{ **assume** *j* = *i* **then have** ?*P* *i* *j* **by** (*simp* *add*: *one-Integer-def*) }

moreover

{ **assume** *j* < *i* **then have** ?*P* *i* *j* **by** (*simp* *add*: *one-Integer-def*) }

ultimately show ?*thesis* **by** *arith*

qed

lemma *supto-bottom-iff*[*simp*]:

(*supto*·*i*·*j* = ⊥) ↔ (*i* = ⊥ ∨ *j* = ⊥)

by (*cases* *i*; *simp*; *cases* *j*; *simp*)

lemma *supto-snoc*[*simp*]:

i ≤ *j* ⇒ *supto*·(*MkI*·*i*)·(*MkI*·*j*) = *supto*·(*MkI*·*i*)·(*MkI*·*j*-1) :@ [*MkI*·*j*:]

proof(*induct* *nat*(*j* - *i*) *arbitrary*: *i* *j*)

case 0 **then show** ?*case* **by** (*simp* *add*: *one-Integer-def*)

next

case (*Suc* *k* *i* *j*)

then have *k* = *nat* (*j* - (*i* + 1)) *i* < *j* **by** *linarith*+

from *this*(2) *Suc*·*hyps*(1)[*OF* *this*(1)] *Suc*(2,3) **show** ?*case* **by** (*simp* *add*: *one-Integer-def*)

qed

lemma *slength-supto*[*simp*]: *slength*·(*supto*·(*MkI*·*i*)·(*MkI*·*j*)) = *MkI*·(*if* *j* < *i* *then* 0 *else* *j* - *i* + 1) (**is** ?*P* *i* *j*)

proof (*cases* *j* - *i*)

fix *d*

```

assume  $j - i = \text{int } d$ 
then show  $?P \ i \ j$ 
proof (induct  $d$  arbitrary:  $i \ j$ )
  case (Suc  $d \ i \ j$ )
  then have  $j - (i + 1) = \text{int } d$  and  $le: i \leq j$  by simp-all
  from Suc(1)[OF this(1)] have IH:  $?P \ (i+1) \ j$  .
  then show  $?case$  using  $le$  by (simp add: one-Integer-def)
qed (simp add: one-Integer-def)
next
fix  $d$ 
assume  $j - i = - \text{int } d$ 
then have  $j \leq i$  by auto
moreover
{ assume  $j = i$  then have  $?P \ i \ j$  by (simp add: one-Integer-def) }
moreover
{ assume  $j < i$  then have  $?P \ i \ j$  by (simp add: one-Integer-def) }
ultimately show  $?thesis$  by arith
qed

```

```

lemma sset-supto[simp]:
  sset (supto.(MkI.i).(MkI.j)) = {MkI.k | k.  $i \leq k \wedge k \leq j$ } (is sset (?u i j) = ?R i j)
proof (cases  $j - i$ )
  case (nonneg  $k$ )
  then show  $?thesis$ 
  proof (induct  $k$  arbitrary:  $i \ j$ )
    case (Suc  $k$ )
    then have *:  $j - (i + 1) = \text{int } k$  by simp
    from Suc(1)[OF *] have IH: sset (?u (i+1) j) = ?R (i+1) j .
    from * have  $i \leq j$  by simp
    then have sset (?u i j) = sset (MkI.i :# ?u (i+1) j) by (simp add: one-Integer-def)
    also have ... = insert (MkI.i) (?R (i+1) j) by (simp add: IH)
    also have ... = ?R i j using  $\langle i \leq j \rangle$  by auto
    finally show  $?case$  .
  qed (force simp: one-Integer-def)
qed simp

```

```

lemma supto-split1: — From HOL.List
  assumes  $i \leq j$ 
  assumes  $j \leq k$ 
  shows supto.(MkI.i).(MkI.k) = supto.(MkI.i).(MkI.(j - 1)) :@ supto.(MkI.j).(MkI.k)
using assms
proof (induct  $j$  rule: int-ge-induct)
  case (step 1) with supto-simp(2) supto-snoc show  $?case$  by (clarsimp simp: one-Integer-def)
qed simp

```

```

lemma supto-split2: — From HOL.List
  assumes  $i \leq j$ 
  assumes  $j \leq k$ 
  shows supto.(MkI.i).(MkI.k) = supto.(MkI.i).(MkI.j) :@ supto.(MkI.(j + 1)).(MkI.k)
proof(cases  $j + 1 \leq k$ )
  case True with assms show  $?thesis$ 
  by (subst supto-split1[where  $j=j + 1$  and  $k=k$ ]; clarsimp simp: one-Integer-def)
next
  case False with assms show  $?thesis$  by (clarsimp simp: one-Integer-def not-le)
qed

```

```

lemma supto-split3: — From HOL.List
  assumes  $i \leq j$ 

```

assumes $j \leq k$
shows $\text{supto} \cdot (\text{MkI} \cdot i) \cdot (\text{MkI} \cdot k) = \text{supto} \cdot (\text{MkI} \cdot i) \cdot (\text{MkI} \cdot (j - 1)) :@ \text{MkI} \cdot j :# \text{supto} \cdot (\text{MkI} \cdot (j + 1)) \cdot (\text{MkI} \cdot k)$
using *assms supto-simp(2) supto-split1* **by** (*metis one-Integer-def plus-MkI-MkI*)

lemma *sinits-stake'*:

shows $\text{sinits} \cdot xs = \text{smap} \cdot (\Lambda i. \text{stake} \cdot i \cdot xs) \cdot (\text{supto} \cdot 0 \cdot (\text{slength} \cdot xs))$
proof(*induct xs rule: srev-induct*)
case (*ssnoc x xs*) **then show** ?*case*
apply (*clarsimp simp: zero-Integer-def one-Integer-def stake-all*
simp del: supto-simp
elim!: slengthE)
apply (*rule arg-cong, rule smap-cong[OF refl]*)
apply *clarsimp*
done
qed *simp-all*

lemma *stails-sdrop'*:

shows $\text{stails} \cdot xs = \text{smap} \cdot (\Lambda i. \text{sdrop} \cdot i \cdot xs) \cdot (\text{supto} \cdot 0 \cdot (\text{slength} \cdot xs))$
proof(*induct xs rule: srev-induct*)
case (*ssnoc x xs*) **then show** ?*case*
apply (*clarsimp simp: zero-Integer-def one-Integer-def sdrop-all*
simp del: supto-simp
elim!: slengthE)
apply (*rule arg-cong, rule smap-cong[OF refl]*)
apply *clarsimp*
apply (*subst (3) sdrop-neg; fastforce simp: zero-Integer-def*)
done
qed *simp-all*

lemma *sdrop-elem-tails[iff]*:

assumes $xs \neq \perp$
shows $\text{sdrop} \cdot (\text{MkI} \cdot i) \cdot xs \in \text{sset} (\text{stails} \cdot xs)$
using *assms*
by (*clarsimp simp: tails-sdrop' zero-Integer-def one-Integer-def elim!: slengthE*)
(*metis add.left-neutral le-MkI-MkI le-cases not-less sdrop-all sdrop-neg zero-Integer-def zless-imp-add1-zle*)

fixrec *slast* :: [*'a*:] \rightarrow *'a* **where**

$\text{slast} \cdot [::] = \perp$
 $| [x \neq \perp; xs \neq \perp] \implies \text{slast} \cdot (x :# xs) = (\text{case } xs \text{ of } [::] \Rightarrow x \mid y :# ys \Rightarrow \text{slast} \cdot xs)$

lemma *slast-strict[simp]*:

$\text{slast} \cdot \perp = \perp$
by *fixrec-simp*

lemma *slast-singleton[simp]*: $\text{slast} \cdot [x:] = x$

by (*cases x = \perp ; simp*)

lemma *slast-sappend-ssnoc[simp]*:

assumes $xs \neq \perp$
shows $\text{slast} \cdot (xs :@ [x:]) = x$
using *assms*
proof(*induct xs*)
case (*scons y ys*) **then show** ?*case* **by** (*cases x = \perp ; simp; cases ys; simp*)
qed *simp-all*

fixrec *sbutlast* :: [*'a*:] \rightarrow [*'a*:] **where**

$\text{sbutlast} \cdot [::] = [::]$
 $| [x \neq \perp; xs \neq \perp] \implies \text{sbutlast} \cdot (x :# xs) = (\text{case } xs \text{ of } [::] \Rightarrow [::] \mid y :# ys \Rightarrow x :# \text{sbutlast} \cdot xs)$

lemma *sbutlast-strict*[*simp*]:

sbutlast· $\perp = \perp$

by *fixrec-simp*

lemma *sbutlast-sappend-ssnoc*[*simp*]:

assumes $x \neq \perp$

shows *sbutlast*·(*xs* :@ [:*x*:]) = *xs*

using *assms*

proof(*induct xs*)

case (*scons y ys*) **then show** ?*case* **by** (*cases ys*; *simp*)

qed *simp-all*

fixrec *prefix* :: [:'*a*::*Eq-def*:] \rightarrow [:'*a*:] \rightarrow *tr* **where**

prefix·*xs*· $\perp = \perp$

| $ys \neq \perp \implies$ *prefix*·[::]·*ys* = *TT*

| $\llbracket x \neq \perp; xs \neq \perp \rrbracket \implies$ *prefix*·(*x* :# *xs*)·[::] = *FF*

| $\llbracket x \neq \perp; xs \neq \perp; y \neq \perp; ys \neq \perp \rrbracket \implies$ *prefix*·(*x* :# *xs*)·(*y* :# *ys*) = (*eq*·*x*·*y* andalso *prefix*·*xs*·*ys*)

lemma *prefix-strict*[*simp*]: *prefix*· $\perp = \perp$

by (*clarsimp simp: cfun-eq-iff*) *fixrec-simp*

lemma *prefix-bottom-iff*[*simp*]: (*prefix*·*xs*·*ys* = \perp) \longleftrightarrow ($xs = \perp \vee ys = \perp$)

proof(*induct xs arbitrary: ys*)

case (*snil ys*) **then show** ?*case* **by** (*cases ys*) *simp-all*

next

case (*scons a xs*) **then show** ?*case* **by** (*cases ys*) *simp-all*

qed *simp*

lemma *prefix-definedD*:

assumes *prefix*·*xs*·*ys* = *TT*

shows $xs \neq \perp \wedge ys \neq \perp$

using *assms* **by** (*induct xs arbitrary: ys*) *auto*

lemma *prefix-refl*[*simp*]:

assumes $xs \neq \perp$

shows *prefix*·*xs*·*xs* = *TT*

using *assms* **by** (*induct xs*) *simp-all*

lemma *prefix-refl-conv*[*simp*]: (*prefix*·*xs*·*xs* = *TT*) \longleftrightarrow ($xs \neq \perp$)

by *auto*

lemma *prefix-of-snil*[*simp*]: *prefix*·*xs*·[::] = (*case xs of* [::] \Rightarrow *TT* | *x* :# *xs* \Rightarrow *FF*)

by (*cases xs*) *simp-all*

lemma *prefix-singleton-TT*:

shows *prefix*·[:*x*:]·*ys* = *TT* \longleftrightarrow ($x \neq \perp \wedge (\exists zs. zs \neq \perp \wedge ys = x :# zs)$)

by (*cases x* = \perp ; *clarsimp*; *cases ys*; *fastforce*)

lemma *prefix-singleton-FF*:

shows *prefix*·[:*x*:]·*ys* = *FF* \longleftrightarrow ($x \neq \perp \wedge (ys = [::] \vee (\exists z zs. z \neq \perp \wedge zs \neq \perp \wedge ys = z :# zs \wedge x \neq z))$)

by (*cases x* = \perp ; *clarsimp*; *cases ys*; *fastforce*)

lemma *prefix-FF-not-snilD*:

assumes *prefix*·*xs*·*ys* = *FF*

shows $xs \neq [::]$

using *assms* **by** (*cases xs*; *cases ys*; *simp*)

lemma *prefix-length*:

assumes $prefix.xs.ys = TT$

shows $le.(length.xs).(length.ys) = TT$

using *assms*

proof(*induct ys arbitrary: xs*)

case (*snil xs*) **then show** *?case* **by** (*cases xs*) *simp-all*

next

case (*scons a ys*) **then show** *?case* **by** (*cases xs*) (*simp-all add: le-plus-1*)

qed *simp*

lemma *prefix-length-strengthen*: $prefix.xs.ys = (le.(length.xs).(length.ys)$ *andalso* $prefix.xs.ys)$

by (*rule andalso-weaken-left*) (*auto dest: prefix-length*)

lemma *prefix-scons-snil*[*simp*]: $prefix.(x :# xs).[::] \neq TT$

by (*cases x :# xs \neq \perp*) *auto*

lemma *scons-prefix-scons*[*simp*]:

$(prefix.(x :# xs).(y :# ys) = TT) \longleftrightarrow (eq.x.y = TT \wedge prefix.xs.ys = TT)$

by (*cases x :# xs \neq \perp \wedge y :# ys \neq \perp*) *auto*

lemma *append-prefixD*:

assumes $prefix.(xs :@ ys).zs = TT$

shows $prefix.xs.zs = TT$

using *assms*

proof(*induct xs arbitrary: zs*)

case (*snil zs*) **then show** *?case* **using** *prefix.simps(2)* **by** *force*

next

case (*scons x xs zs*) **then show** *?case*

by (*metis prefix.simps(1) prefix-scons-snil sappend-scons scons-prefix-scons slist.exhaust*)

qed *simp*

lemma *same-prefix-prefix*[*simp*]:

assumes $xs \neq \perp$

shows $prefix.(xs :@ ys).(xs :@ zs) = prefix.ys.zs$

using *assms*

proof(*cases ys = \perp zs = \perp rule: bool.exhaust[case-product bool.exhaust]*)

case *False-False* **with** *assms* **show** *?thesis* **by** (*induct xs*) *simp-all*

qed *simp-all*

lemma *eq-prefix-TT*:

assumes $eq.xs.ys = TT$

shows $prefix.xs.ys = TT$

using *assms* **by** (*induct xs arbitrary: ys*) (*case-tac ys; simp*)+

lemma *prefix-eq-FF*:

assumes $prefix.xs.ys = FF$

shows $eq.xs.ys = FF$

using *assms* **by** (*induct xs arbitrary: ys*) (*case-tac ys; auto*)+

lemma *prefix-length-eq*:

shows $eq.xs.ys = (eq.(length.xs).(length.ys)$ *andalso* $prefix.xs.ys)$

proof(*induct xs arbitrary: ys*)

case (*snil ys*) **then show** *?case*

by (*cases ys; clarsimp simp: one-Integer-def elim!: lengthE*)

next

case (*scons x xs ys*) **then show** *?case*

by (*cases ys; clarsimp simp: zero-Integer-def one-Integer-def elim!: lengthE*)

qed *simp*

lemma *stake-slength-plus-1*:

shows $\text{stake} \cdot (\text{slength} \cdot xs + 1) \cdot (y \text{ :\# } ys) = y \text{ :\# } \text{stake} \cdot (\text{slength} \cdot xs) \cdot ys$

by (*cases* $xs = \perp \ y = \perp \ ys = \perp$ *rule*: *bool.exhaust*[*case-product bool.exhaust bool.exhaust*]; *clarsimp*)
(*auto simp*: *If2-def*[*symmetric*] *zero-Integer-def one-Integer-def split*: *If2-splits elim!*: *slengthE*)

lemma *sdrop-slength-plus-1*:

assumes $y \neq \perp$

shows $\text{sdrop} \cdot (\text{slength} \cdot xs + 1) \cdot (y \text{ :\# } ys) = \text{sdrop} \cdot (\text{slength} \cdot xs) \cdot ys$

using *assms*

by (*subst sdrop.simps*;

cases $xs = \perp$; *clarsimp*; *cases* $ys = \perp$;

clarsimp simp: *If2-def*[*symmetric*] *zero-Integer-def one-Integer-def split*: *If2-splits elim!*: *slengthE*)

lemma *eq-take-length-prefix*: $\text{prefix} \cdot xs \cdot ys = \text{eq} \cdot xs \cdot (\text{stake} \cdot (\text{slength} \cdot xs) \cdot ys)$

proof (*induct* xs *arbitrary*: ys)

case (*snil* ys) **show** *?case* **by** (*cases* ys ; *clarsimp*)

next

case (*scons* $x \ xs \ ys$)

note $IH = \text{this}$

show *?case*

proof (*cases* $\text{slength} \cdot xs = \perp$)

case *True* **then show** *?thesis* **by** *simp*

next

case *False*

show *?thesis*

proof (*cases* ys)

case *bottom*

then show *?thesis* **using** *False*

using *le-slength-plus*[*of* $xs \ 1$] **by** *simp*

next

case *snil* **then show** *?thesis* **using** *False* **and** $IH(1,2)$ **by** *simp*

next

case (*scons* $z \ zs$)

then show *?thesis*

using *False* **and** $IH(1,2) \ IH(3)$ [*of* zs]

by (*simp add*: *stake-slength-plus-1 monofun-cfun-arg*)

qed

qed

qed *simp*

lemma *prefix-sdrop-slength*:

assumes $\text{prefix} \cdot xs \cdot ys = TT$

shows $xs \text{ :@ } \text{sdrop} \cdot (\text{slength} \cdot xs) \cdot ys = ys$

using *assms* **by** (*induct* xs *arbitrary*: ys) (*case-tac* ys ; *simp add*: *sdrop-slength-plus-1*)**+**

lemma *prefix-sdrop-prefix-eq*:

assumes $\text{prefix} \cdot xs \cdot ys = TT$

shows $\text{eq} \cdot (\text{sdrop} \cdot (\text{slength} \cdot xs) \cdot ys) \cdot [::] = \text{eq} \cdot ys \cdot xs$

using *assms* **by** (*induct* xs *arbitrary*: ys) (*case-tac* ys ; *simp add*: *sdrop-slength-plus-1*)**+**

4 Knuth-Morris-Pratt matching according to Bird

4.1 Step 1: Specification

We begin with the specification of string matching given by Bird (2010, Chapter 16). (References to “Bird” in the following are to this text.) Note that we assume *eq* has some nice properties (see §2.2) and use strict lists.

fixrec *endswith* :: [*'a*::*Eq-def*:] → [*'a*:] → *tr* **where**
 [*simp del*]: *endswith.pat* = *selem.pat* oo *stails*

fixrec *matches* :: [*'a*::*Eq-def*:] → [*'a*:] → [*Integer*:] **where**
 [*simp del*]: *matches.pat* = *smap.length* oo *sfilter*·(*endswith.pat*) oo *sinits*

Bird describes *matches.pat.xs* as returning “a list of integers *p* such that *pat* is a suffix of *stake.p.xs*.”
 The following examples illustrate this behaviour:

lemma *matches*·[::]·[::] = [:0:]
unfolding *matches.unfold* *endswith.unfold* **by** *simp*

lemma *matches*·[::]·[:10::*Integer*, 20, 30:] = [:0, 1, 2, 3:]
unfolding *matches.unfold* *endswith.unfold* **by** *simp*

lemma *matches*·[:1::*Integer*, 2, 3, 1, 2:]·[:1, 2, 1, 2, 3, 1, 2, 3, 1, 2:] = [:7, 10:]
unfolding *matches.unfold* *endswith.unfold*
by (*simp add*: *sfilter-scons-let del*: *sfilter-strict-scons* *sfilter.simps*)

lemma *endswith-strict*[*simp*]:
endswith·⊥ = ⊥
endswith.pat·⊥ = ⊥
by (*fixrec-simp*; *simp add*: *cfun-eq-iff*)+

lemma *matches-strict*[*simp*]:
matches·⊥ = ⊥
matches.pat·⊥ = ⊥
by (*fixrec-simp*; *clarsimp simp*: *cfun-eq-iff*)+

Bird’s strategy for deriving KMP from this specification is encoded in the following lemmas: if we can rewrite *endswith* as a composition of a predicate with a *sfoldl*, then we can rewrite *matches* into a *sscanl*.

lemma *fork-sfoldl*:
shows *sfoldl*·*f1*·*z1* && *sfoldl*·*f2*·*z2* = *sfoldl*·(Λ (*a*, *b*) *z*. (*f1*·*a*·*z*, *f2*·*b*·*z*))·(*z1*, *z2*) (**is** ?*lhs* = ?*rhs*)
proof(*rule cfun-eqI*)
fix *xs* **show** ?*lhs*·*xs* = ?*rhs*·*xs*
by (*induct xs arbitrary*: *z1 z2*) *simp-all*
qed

lemma *smap-sfilter-split-cfcomp*: — Bird (16.4)
assumes *f*·⊥ = ⊥
assumes *p*·⊥ = ⊥
shows *smap*·*f* oo *sfilter*·(*p* oo *g*) = *smap*·*cfst* oo *sfilter*·(*p* oo *csnd*) oo *smap*·(*f* && *g*) (**is** ?*lhs* = ?*rhs*)
proof(*rule cfun-eqI*)
fix *xs* **show** ?*lhs*·*xs* = ?*rhs*·*xs*
using *assms* **by** (*induct xs*) (*simp-all add*: *If2-def[symmetric]* *split*: *If2-splits*)
qed

lemma *Bird-strategy*:
assumes *endswith*: *endswith.pat* = *p* oo *sfoldl*·*op*·*z*
assumes *step*: *step* = (Λ (*n*, *x*) *y*. (*n* + 1, *op*·*x*·*y*))
assumes *p*·⊥ = ⊥ — We can reasonably expect the predicate to be strict
shows *matches.pat* = *smap*·*cfst* oo *sfilter*·(*p* oo *csnd*) oo *sscanl*·*step*·(0, *z*)
apply (*simp add*: *matches.simps* *assoc-oo* *endswith*)
apply (*subst smap-sfilter-split-cfcomp*, *fastforce*, *fact*)
apply (*subst slength-sfoldl*)
apply (*subst fork-sfoldl*)
apply (*simp add*: *oo-assoc[symmetric]*)
apply (*subst sinits-sscanl*)
apply (*fold step*)

apply (*rule refl*)
done

Bird proceeds by reworking *endswith* into the form required by *Bird-strategy*. This is eased by an alternative definition of *endswith*.

lemma *sfilter-supto*:

assumes $0 \leq d$

shows $sfilter \cdot (\Lambda x. le \cdot (MkI \cdot n - x) \cdot (MkI \cdot d)) \cdot (supto \cdot (MkI \cdot m) \cdot (MkI \cdot n))$
 $= supto \cdot (MkI \cdot (if\ m \leq n - d\ then\ n - d\ else\ m)) \cdot (MkI \cdot n)$ (**is** $?sfilterp \cdot ?suptomn = -$)

proof(*cases* $m \leq n - d$)

case *True*

moreover

from *True* **assms** **have** $?sfilterp \cdot ?suptomn = ?sfilterp \cdot (supto \cdot (MkI \cdot m) \cdot (MkI \cdot (n - d - 1))) :@ supto \cdot (MkI \cdot (n - d)) \cdot (MkI \cdot n)$

using *supto-split1* **by** *auto*

moreover from *True* **assms** **have** $?sfilterp \cdot (supto \cdot (MkI \cdot m) \cdot (MkI \cdot (n - d - 1))) = [::]$ **by** *auto*

ultimately show *?thesis* **by** (*clarsimp intro!*: *trans[OF sfilter-cong[OF refl] sfilter-const-TT]*)

next

case *False* **then show** *?thesis*

by (*clarsimp intro!*: *trans[OF sfilter-cong[OF refl] sfilter-const-TT]*)

qed

lemma *endswith-eq-sdrop*: $endswith \cdot pat \cdot xs = eq \cdot pat \cdot (sdrop \cdot (slength \cdot xs - slength \cdot pat) \cdot xs)$

proof(*cases* $pat = \perp$ $xs = \perp$ *rule*: *bool.exhaust[case-product bool.exhaust]*)

case *False-False* **then show** *?thesis*

by (*cases* $endswith \cdot pat \cdot xs$;

simp only: *endswith.simps cfcomp2 stails-sdrop'*;

force simp: *zero-Integer-def one-Integer-def elim*: *slengthE*)

qed *simp-all*

lemma *endswith-def2*: — Bird p127

shows $endswith \cdot pat \cdot xs = eq \cdot pat \cdot (shead \cdot (sfilter \cdot (\Lambda x. prefix \cdot x \cdot pat) \cdot (stails \cdot xs)))$ (**is** $?lhs = ?rhs$)

proof(*cases* $pat = \perp$ $xs = \perp$ *rule*: *bool.exhaust[case-product bool.exhaust]*)

case *False-False*

from *False-False* **obtain** *patl xsl* **where** $*$: $slength \cdot xs = MkI \cdot xsl$ $0 \leq xsl$ $slength \cdot pat = MkI \cdot patl$ $0 \leq patl$

by (*meson Integer.exhaust slength-bottom-iff slength-ge-0*)

let $?patl \cdot xsl = if\ patl \leq xsl\ then\ xsl - patl\ else\ 0$

have $?rhs = eq \cdot pat \cdot (shead \cdot (sfilter \cdot (\Lambda x. le \cdot (slength \cdot x) \cdot (slength \cdot pat)\ andalso\ prefix \cdot x \cdot pat) \cdot (stails \cdot xs)))$

by (*subst prefix-slength-strengthen*) *simp*

also have $\dots = eq \cdot pat \cdot (shead \cdot (sfilter \cdot (\Lambda x. prefix \cdot x \cdot pat) \cdot (sfilter \cdot (\Lambda x. le \cdot (slength \cdot x) \cdot (slength \cdot pat)) \cdot (stails \cdot xs))))$

by (*simp add*: *sfilter-sfilter'*)

also have $\dots = eq \cdot pat \cdot (shead \cdot (smap \cdot (\Lambda k. sdrop \cdot k \cdot xs) \cdot (sfilter \cdot (\Lambda k. prefix \cdot (sdrop \cdot k \cdot xs) \cdot pat) \cdot (sfilter \cdot (\Lambda k. le \cdot (slength \cdot (sdrop \cdot k \cdot xs) \cdot pat) \cdot (MkI \cdot xsl))))))$

using $\langle slength \cdot xs = MkI \cdot xsl \rangle$ $\langle slength \cdot pat = MkI \cdot patl \rangle$

by (*simp add*: *stails-sdrop' sfilter-smap' cfcomp1 zero-Integer-def*)

also have $\dots = eq \cdot pat \cdot (shead \cdot (smap \cdot (\Lambda k. sdrop \cdot k \cdot xs) \cdot (sfilter \cdot (\Lambda k. prefix \cdot (sdrop \cdot k \cdot xs) \cdot pat) \cdot (sfilter \cdot (\Lambda k. le \cdot (MkI \cdot xsl - k) \cdot (MkI \cdot patl)) \cdot (supto \cdot (MkI \cdot 0) \cdot (MkI \cdot xsl))))))$

using $\langle slength \cdot xs = MkI \cdot xsl \rangle$

by (*subst* (2) *sfilter-cong[where* $p' = \Lambda x. le \cdot (MkI \cdot xsl - x) \cdot (MkI \cdot patl)$]; *fastforce simp*: *zero-Integer-def*)

also have $\dots = If\ prefix \cdot (sdrop \cdot (MkI \cdot ?patl \cdot xsl) \cdot xs) \cdot pat$

then $eq \cdot pat \cdot (sdrop \cdot (MkI \cdot ?patl \cdot xsl) \cdot xs)$

else $eq \cdot pat \cdot (shead \cdot (smap \cdot (\Lambda k. sdrop \cdot k \cdot xs) \cdot (sfilter \cdot (\Lambda x. prefix \cdot (sdrop \cdot x \cdot xs) \cdot pat) \cdot (supto \cdot (MkI \cdot (?patl \cdot xsl + 1)) \cdot (MkI \cdot xsl))))))$

using *False-False* $\langle 0 \leq xsl \rangle$ $\langle 0 \leq patl \rangle$

by (*subst sfilter-supto*) (*auto simp*: *If-distr one-Integer-def*)

also have $\dots = ?lhs$ (**is** *If ?c then - else ?else = -*)

proof(*cases* *?c*)

case *TT* **with** $\langle slength \cdot xs = MkI \cdot xsl \rangle$ $\langle slength \cdot pat = MkI \cdot patl \rangle$

show *?thesis* **by** (*simp add*: *endswith-eq-sdrop sdrop-neg zero-Integer-def*)

```

next
  case FF — Recursive case: the lists generated by supto are too short
  have ?else = shead·(smap·(Λ x. eq.pat·(sdrop·x·xs))·(sfilter·(Λ x. prefix·(sdrop·x·xs)·pat)·(supto·(MkI·(?patl·xsl + 1))·(MkI·xsl))))
  using FF by (subst shead-smap-distr[where f=eq.pat, symmetric]) (auto simp: cfcomp1)
  also have ... = shead·(smap·(Λ x. seq·x·FF)·(sfilter·(Λ x. prefix·(sdrop·x·xs)·pat)·(supto·(MkI·(?patl·xsl + 1))·(MkI·xsl))))
  using False-False * by (subst smap-cong[OF refl, where f'=Λ x. seq·x·FF]) (auto simp: zero-Integer-def split: if-splits)
  also from * FF have ... = ?lhs
  apply (auto 0 0 simp: shead-smap-distr seq-conv-if ends-with-eq-sdrop zero-Integer-def dest!: prefix-FF-not-snilD)
  apply (metis (full-types) le-MkI-MkI linorder-not-less order-refl prefix-FF-not-snilD sdrop-all zless-imp-add1-zle)
  using FF apply fastforce
  apply (metis add.left-neutral le-MkI-MkI linorder-not-less order-refl prefix-FF-not-snilD sdrop-0(1) sdrop-all zero-Integer-def zless-imp-add1-zle)
  done
  finally show ?thesis using FF by simp
qed (simp add: False-False)
finally show ?thesis ..
qed simp-all

```

Bird then generalizes $sfilter·(Λ x. prefix·x·pat)$ *oo stails* to *split*, where “ $split·pat·xs$ splits pat into two lists us and vs so that $us :@ vs = pat$ and us is the longest suffix of xs that is a prefix of pat .”

fixrec *split* :: [$'a::Eq-def$] \rightarrow [$'a$] \rightarrow [$'a$] \times [$'a$] **where** — Bird p128
 $[simp\ del]: split·pat·xs = If\ prefix·xs·pat\ then\ (xs,\ sdrop·(slength·xs)·pat)\ else\ split·pat·(stail·xs)$

lemma *split-strict*[*simp*]:
 shows $split·\perp = \perp$
 and $split·pat·\perp = \perp$
 by *fixrec-simp*+

lemma *split-bottom-iff*[*simp*]: $(split·pat·xs = \perp) \longleftrightarrow (pat = \perp \vee xs = \perp)$
 by (cases $pat = \perp$; *clarsimp*) (induct xs ; *subst split.unfold*; *simp*)

lemma *split-snil*[*simp*]:
 assumes $pat \neq \perp$
 shows $split·pat·[::] = ([::], pat)$
 using *assms* by *fixrec-simp*

lemma *split-pattern*: — Bird p128, observation

assumes $xs \neq \perp$
 assumes $split·pat·xs = (us, vs)$
 shows $us :@ vs = pat$
 using *assms*
proof(cases $pat = \perp$, *simp*, *induct xs arbitrary: us vs*)
 case *snil* then show ?case by (subst (*asm*) *split.unfold*) *simp*

next
 case (*scons x xs*) then show ?case
 by (subst (*asm*) (\exists) *split.unfold*)
 (*fastforce dest: prefix-sdrop-slength simp: If2-def[symmetric] split: If2-splits*)
qed *simp*

lemma *ends-with-split*: — Bird p128, after defining *split*

shows $ends-with·pat = snull\ oo\ csnd\ oo\ split·pat$
proof(*rule cfun-eqI*)
 fix xs show $ends-with·pat·xs = (snull\ oo\ csnd\ oo\ split·pat)·xs$
proof(cases $pat = \perp$, *simp*, *induct xs*)
 case (*scons x xs*) then show ?case

```

unfolding endswith-def2
by (subst split.unfold)
  (fastforce dest: prefix-sdrop-prefix-eq simp: If2-def[symmetric] If-distr snull-eq-snil split: If2-splits)
qed (simp-all add: snull-eq-snil endswith.simps)
qed

```

```

lemma split-length-lt:
  assumes pat  $\neq \perp$ 
  assumes xs  $\neq \perp$ 
  shows lt·(slength·(prod.fst (split.pat·xs)))·(slength·xs + 1) = TT
using assms
proof(induct xs)
  case (scons x xs) then show ?case
  by (subst split.unfold)
  (auto simp: If2-def[symmetric] one-Integer-def split: If2-splits elim!: slengthE elim: lt-trans)
qed simp-all

```

The predicate p required by *Bird-strategy* is therefore *snull oo csnd*. It remains to find op and z such that:

- $split.pat.[:] = z$
- $split.pat.(xs :@ [:x:]) = op.(split.pat.xs) \cdot x$

so that $split = sfoldl.op.z$.

We obtain $z = ([::], pat)$ directly from the definition of *split*.

Bird derives op on the basis of this crucial observation:

```

lemma split-snoc: — Bird p128
  shows split.pat·(xs :@ [:x:]) = split.pat·(cfst·(split.pat·xs) :@ [:x:])
proof(cases pat =  $\perp$ , simp, cases x =  $\perp$ , simp, induct xs)
  case (scons x xs) then show ?case
  apply –
  apply (subst (1 3) split.unfold)
  apply (clarsimp simp: If2-def[symmetric] split: If2-splits; intro conjI impI)
  apply (subst split.unfold; fastforce)
  apply (subst split.unfold; fastforce)
  apply (simp add: append-prefixD)
  done
qed simp-all

```

```

fixrec — Bird p129
  op :: ['a::Eq-def:]  $\rightarrow$  ['a:]  $\times$  ['a:]  $\rightarrow$  'a  $\rightarrow$  ['a:]  $\times$  ['a:]

```

where

```

[simp del]:
  op.pat·(us, vs)·x =
  ( If prefix·[:x:]·vs then (us :@ [:x:], stail·vs)
    else If snull·us then ([::], pat)
    else op.pat·(split.pat·(stail·us))·x )

```

```

lemma op-strict[simp]:

```

```

  op.pat· $\perp$  =  $\perp$ 
  op.pat·(us,  $\perp$ ) =  $\perp$ 
  op.pat·usvs· $\perp$  =  $\perp$ 
by fixrec-simp+

```

Bird demonstrates that op is partially correct wrt *split*, i.e., $op.pat.(split.pat.xs) \cdot x \sqsubseteq split.pat.(xs :@ [:x:])$. For total correctness we essentially prove that op terminates on well-defined arguments with an inductive argument.

```

lemma op-induct[case-names step]:

```

```

fixes usvs:: [:'a:] × 'b
assumes step:  $\bigwedge usvs. (\bigwedge usvs'. lt.(slength.(cfst.usvs')).(slength.(cfst.usvs)) = TT \implies P usvs') \implies P usvs$ 
shows  $P usvs$ 
proof(induct usvs rule: wf-induct)
  let ?r = { (usvs', usvs) | (usvs :: [:'a:] × 'b) (usvs' :: [:'a:] × 'b).  $lt.(slength.(cfst.usvs')).(slength.(cfst.usvs)) = TT$  }
  show wf ?r
  proof(rule wf-subset[OF wf-inv-image[where f= $\lambda(x, -). slength.x$ , OF wf-subset[OF wf-Integer-ge-less-than[where d=0]]]])
    let ?rslen = { ( $slength.us'$ ,  $slength.us$ ) | (us :: [:'a:]) (us' :: [:'a:]).  $lt.(slength.us').(slength.us) = TT$  }
    show ?rslen  $\subseteq$  Integer-ge-less-than 0
    apply (clarsimp simp: Integer-ge-less-than-def zero-Integer-def)
    apply (metis Integer.exhaust dist-eq-tr(4) dist-eq-tr(6) lt-Integer-bottom-iff lt-MkI-MkI slength-ge-0)
    done
    show ?r  $\subseteq$  inv-image ?rslen ( $\lambda(x, -). slength.x$ ) by (auto 0 3)
  qed
fix usvs
assume  $\forall usvs'. (usvs', usvs) \in ?r \longrightarrow P usvs'$ 
then show  $P usvs$ 
  by - (rule step; clarsimp; metis eq-fst-iff)
qed

```

```

lemma op-induct'[case-names step]:
assumes step:  $\bigwedge us. (\bigwedge us'. lt.(slength.us').(slength.us) = TT \implies P us') \implies P us$ 
shows  $P us$ 
by (rule op-induct[where P= $P \circ prod.fst$  and  $usvs=(us, vs)$  for  $vs::unit$ , simplified])
  (fastforce intro: step)

```

```

lemma split-snoc-op:
  split.pat.(xs :@ [:x:]) = op.pat.(split.pat.xs).x
proof(induct split.pat.xs arbitrary: xs rule: op-induct)
  case (step xs) show ?case
  proof(cases pat =  $\perp$  xs =  $\perp$  x =  $\perp$  rule: bool.exhaust[case-product bool.exhaust bool.exhaust])
    case False-False-False
    obtain us vs where  $*$ : split.pat.xs = (us, vs) by fastforce
    from False-False-False * have  $**$ : prefix.(us :@ [:x:]).pat = prefix[:x:].vs
      using split-pattern same-prefix-prefix sappend-bottom-iff by blast
    from False-False-False * **
    have  $***$ : sdrop.(slength.(us :@ [:x:])).pat = stail.vs if prefix.(us :@ [:x:]).pat = TT
      using sdrop-sappend-same[where xs= $us :@ [:x:]$  that
        by (cases vs; clarsimp) (fastforce dest!: split-pattern)
    from False-False-False * ** *** show ?thesis
    apply -
    apply (subst split-snoc)
    apply (subst split.unfold)
    apply (subst op.unfold)
    apply (fastforce simp: If2-def[symmetric] snull-FF-conv split: If2-splits intro: step split-length-lt)
    done
  qed simp-all
qed

```

```

lemma split-sfoldl-op:
assumes pat  $\neq \perp$ 
shows sfoldl.(op.pat).([:], pat) = split.pat (is ?lhs = ?rhs)
proof -
  have ?lhs.xs = ?rhs.xs for xs
    using assms by (induct xs rule: srev-induct) (simp-all add: split-snoc-op)
  then show ?thesis by (simp add: cfun-eq-iff)

```

qed

lemma *matches-op*:

shows $matches.pat = smap.cfst\ oo\ sfilter.(snull\ oo\ csnd\ oo\ csnd)$
 $oo\ sscanl.(\Lambda\ (n,\ usvs)\ x.\ (n + 1,\ op.pat.usvs.x)).(0,\ ([::],\ pat))$ **(is** $?lhs = ?rhs$)

proof(*cases pat = ⊥*)

case *True*

then have $?lhs.xs = ?rhs.xs$ **for** xs **by** (*cases xs; clarsimp*)

then show $?thesis$ **by** (*simp add: cfun-eq-iff*)

next

case *False* **then show** $?thesis$

apply (*subst (2) oo-assoc*)

apply (*rule Bird-strategy*)

apply (*simp-all add: endswith-split split-sfoldl-op oo-assoc*)

done

qed

Using *split-sfoldl-op* we can rewrite *op* into a more perspicuous form that exhibits how KMP handles the failure of the text to continue matching the pattern:

fixrec

$op' :: [:'a::Eq-def:] \rightarrow [:'a:] \times [:'a:] \rightarrow 'a \rightarrow [:'a:] \times [:'a:]$

where

[*simp del*]:

$op'.pat.(us, vs).x =$

(*If prefix[:x:].vs then (us :@ [:x:], stail.vs) — continue matching*

else If snull.us then ([::], pat) — fail at the start of the pattern: discard x

else sfoldl.(op'.pat).([::], pat).(stail.us :@ [:x:]) — fail later: discard shead.us and determine where to restart

)

Intuitively if x continues the pattern match then we extend the *split* of *pat* recorded in *us* and *vs*. Otherwise we need to find a prefix of *pat* to continue matching with. If we have yet to make any progress (i.e., $us = [::]$) we restart with the entire *pat* (aka z) and discard x . Otherwise, because a match cannot begin with $us :@ [:x:]$, we *split pat* (aka z) by iterating op' over $stail.us :@ [:x:]$. The remainder of the development is about memoising this last computation.

This is not yet the full KMP algorithm as it lacks what we call the ‘K’ optimisation, which we add in §4.2. Note that a termination proof for op' in HOL is tricky due to its use of higher-order nested recursion via *sfoldl*.

lemma *op'-strict[simp]*:

$op'.pat.\perp = \perp$

$op'.pat.(us, \perp) = \perp$

$op'.pat.usvs.\perp = \perp$

by *fixrec-simp+*

lemma *sfoldl-op'-strict[simp]*:

$op'.pat.(sfoldl.(op'.pat).(us, \perp).xs).x = \perp$

by (*induct xs arbitrary: x rule: srev-induct*) *simp-all*

lemma *op'-op*:

shows $op'.pat.usvs.x = op.pat.usvs.x$

proof(*cases pat = ⊥ x = ⊥ rule: bool.exhaust[case-product bool.exhaust]*)

case *True-False* **then show** $?thesis$

apply (*subst op'.unfold*)

apply (*subst op.unfold*)

apply *simp*

done

next

case *False-False* **then show** $?thesis$

proof(*induct usvs arbitrary: x rule: op-induct*)

case (*step usvs x*)

```

have *:  $sfoldl \cdot (op' \cdot pat) \cdot ([::], pat) \cdot xs = sfoldl \cdot (op \cdot pat) \cdot ([::], pat) \cdot xs$ 
  if  $lt \cdot (slength \cdot xs) \cdot (slength \cdot (cfst \cdot usvs)) = TT$  for  $xs$ 
using that
proof (induct xs rule: srev-induct)
  case (ssnoc x' xs')
  from  $ssnoc(1,2,4)$  have  $lt \cdot (slength \cdot xs') \cdot (slength \cdot (cfst \cdot usvs)) = TT$ 
    using lt-slength-0(2) lt-trans by auto
  moreover
  from step(2) ssnoc(1,2,4) have  $lt \cdot (slength \cdot (cfst \cdot (split \cdot pat \cdot xs'))) \cdot (slength \cdot (cfst \cdot usvs)) = TT$ 
    using lt-trans split-length-lt by (auto 10 0)
  ultimately show ?case by (simp add: ssnoc.hyps split-sfoldl-op split-snoc-op step)
qed simp-all
from step.prems show ?case
  apply (subst op'.unfold)
  apply (subst op.unfold)
  apply (clarsimp simp: If2-def[symmetric] snull-FF-conv split-sfoldl-op[symmetric] * split: If2-splits)
  apply (clarsimp simp: split-sfoldl-op step split-length-lt)
  done
qed
qed simp-all

```

4.2 Step 2: Data refinement and the ‘K’ optimisation

Bird memoises the restart computation in op' in two steps. The first reifies the control structure of op' into a non-wellfounded tree, which we discuss here. The second increases the sharing in this tree; see §4.6.

Briefly, we cache the $sfoldl \cdot (op' \cdot pat) \cdot ([::], pat) \cdot (stail \cdot us :@ [x:])$ computation in op' by finding a “representation” type t for the “abstract” type $[:'a:] \times [:'a:]$, a pair of functions rep, abs where $abs \circ rep = ID$, and then finding a derived form of op' that works on t rather than $[:'a:] \times [:'a:]$. We also take the opportunity to add the ‘K’ optimisation in the form of the *next* function.

As such steps are essentially *deus ex machina*, we try to provide some intuition after showing the new definitions.

domain $'a$ *tree* — Bird p130

= *Null*

| *Node* (*label* :: $'a$) (**lazy** *left* :: $'a$ *tree*) (**lazy** *right* :: $'a$ *tree*) — Strict in the label $'a$

fixrec *next* :: $[:'a::Eq-def:] \rightarrow ([:'a:] \times [:'a:])$ *tree* $\rightarrow ([:'a:] \times [:'a:])$ *tree* **where**

next · $[::] \cdot t = t$

| $[x \neq \perp; xs \neq \perp] \implies$

next · $(x :\# xs) \cdot \text{Null} = \text{Null}$

| $[x \neq \perp; xs \neq \perp] \implies$

next · $(x :\# xs) \cdot (\text{Node} \cdot (us, [::]) \cdot l \cdot r) = \text{Node} \cdot (us, [::]) \cdot l \cdot r$

| $[v \neq \perp; vs \neq \perp; x \neq \perp; xs \neq \perp] \implies$

next · $(x :\# xs) \cdot (\text{Node} \cdot (us, v :\# vs) \cdot l \cdot r) = \text{If } eq \cdot x \cdot v \text{ then } l \text{ else } \text{Node} \cdot (us, v :\# vs) \cdot l \cdot r$

fixrec — Bird p131 “an even simpler form”, with the ‘K’ optimisation

root2 :: $[:'a::Eq-def:] \rightarrow ([:'a:] \times [:'a:])$ *tree*

and *op2* :: $[:'a:] \rightarrow ([:'a:] \times [:'a:])$ *tree* $\rightarrow 'a \rightarrow ([:'a:] \times [:'a:])$ *tree*

and *rep2* :: $[:'a:] \rightarrow [:'a:] \times [:'a:] \rightarrow ([:'a:] \times [:'a:])$ *tree*

and *left2* :: $[:'a:] \rightarrow [:'a:] \times [:'a:] \rightarrow ([:'a:] \times [:'a:])$ *tree*

and *right2* :: $[:'a:] \rightarrow [:'a:] \times [:'a:] \rightarrow ([:'a:] \times [:'a:])$ *tree*

where

[*simp del*]:

root2 · *pat* = *rep2* · *pat* · $([::], \text{pat})$

| *op2* · *pat* · *Null* · $x = \text{root2} \cdot \text{pat}$

| $usvs \neq \perp \implies$

op2 · *pat* · $(\text{Node} \cdot usvs \cdot l \cdot r) \cdot x = \text{If } \text{prefix} \cdot [x] \cdot (\text{csnd} \cdot usvs) \text{ then } r \text{ else } \text{op2} \cdot \text{pat} \cdot l \cdot x$

| [*simp del*]:

rep2 · *pat* · *usvs* = $\text{Node} \cdot usvs \cdot (\text{left2} \cdot \text{pat} \cdot usvs) \cdot (\text{right2} \cdot \text{pat} \cdot usvs)$

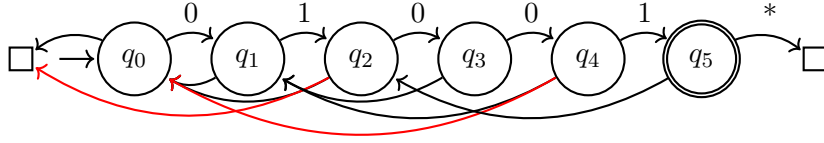


Figure 2: An example from Crochemore and Rytter (2002, §2.1). The MP tree for the pattern 01001 is drawn in black: right transitions are labelled with a symbol, whereas left transitions are unlabelled. The two ‘K’-optimised left transitions are shown in red. The boxes denote *Null*. The root node is q_0 .

```

| left2.pat·([:], vs) = next.vs·Null
| [[u ≠ ⊥; us ≠ ⊥]] ⇒
  left2.pat·(u :# us, vs) = next.vs·(sfoldl·(op2.pat)·(root2.pat)·us) — Note the use of op2 and next.
| right2.pat·(us, [:]) = Null — Unreachable
| [[v ≠ ⊥; vs ≠ ⊥]] ⇒
  right2.pat·(us, v :# vs) = rep2.pat·(us :@ [v:], vs)

```

```

fixrec abs2 :: ([:'a:] × [:'a:]) tree → [:'a:] × [:'a:] where
  usvs ≠ ⊥ ⇒ abs2·(Node.usvs.l.r) = usvs

```

```

fixrec matches2 :: [:'a:Eq-def:] → [:'a:] → [Integer:] where
[simp del]: matches2.pat = smap.cfst oo sfilter·(snull oo csnd oo abs2 oo csnd)
  oo sscanl·(Λ (n, x) y. (n + 1, op2.pat·x.y))·(0, root2.pat)

```

This tree can be interpreted as a sort of automaton¹, where *op2* goes *right* if the pattern continues with the next element of the text, and *left* otherwise, to determine how much of a prefix of the pattern could still be in play. Figure 2 visualises such an automaton for the pattern 01001, used by Crochemore and Rytter (2002, §2.1) to illustrate the difference between Morris-Pratt (MP) and Knuth-Morris-Pratt (KMP) preprocessing as we discuss below. Note that these are not the classical Mealy machines that correspond to regular expressions, where all outgoing transitions are labelled with symbols.

The following lemma shows how our sample automaton is encoded as a non-wellfounded tree.

lemma *concrete-tree-KMP*:

```

shows root2·[:0::Integer, 1, 0, 0, 1:]
  = (μ q0. Node·([:], [:0, 1, 0, 0, 1:])
    ·Null
    ·(μ q1. Node·([:0:], [:1, 0, 0, 1:])
      ·q0
      ·(μ q2. Node·([:0,1:], [:0, 0, 1:])
        ·Null — K optimisation: MP q0
        ·(Node·([:0,1,0:], [:0, 1:])
          ·q1
          ·(Node·([:0,1,0,0:], [:1:])
            ·q0 — K optimisation: MP q1
            ·(Node·([:0,1,0,0,1:], [:])·q2·Null))))))

```

(**is** ?lhs = fix·?F)

The sharing that we expect from a lazy (call-by-need) evaluator is here implied by the use of nested fixed points. The KMP preprocessor is expressed by the *left2* function, where *op2* is used to match the pattern against itself; the use of *op2* in *matches2* (“the driver”) is responsible for matching the (preprocessed) pattern against the text. This formally cashes in an observation by van der Woude (1989, §5), that these two algorithms are essentially the same, which has eluded other presentations².

Bird uses *Null* on a left path to signal to the driver that it should discard the current element of the text and restart matching from the beginning of the pattern (i.e. *root2*). This is a step towards the removal of *us* in §4.8. Note that the *Null* at the end of the rightmost path is unreachable: the rightmost *Node* has *vs* = [:] and therefore *op2* always takes the left branch.

¹Bird (2012, §3.1) suggests it can be thought of as a doubly-linked list, following Takeichi and Akama (1991).

²For instance, contrast our shared use of *op2* with the separated *match* and *rematch* functions of Ager et al. (2006, Figure 1).

The ‘K’ optimisation is perhaps best understood by example. Consider the automaton in Figure 2, and a text beginning with 011. Using the MP (black) transitions we take the path $\rightarrow q_0 \xrightarrow{0} q_1 \xrightarrow{1} \overbrace{q_2 \rightarrow q_0} \rightarrow \square$. Now, due to the failure of the comparison of the current element of the text (1) at q_2 , we can predict that the (identical) comparison at node q_0 will fail as well, and therefore have q_2 left-branch directly to \square . This saves a comparison in the driver at the cost of another in the preprocessor (in *next*). These optimisations are the red arrows in the diagram, and can in general save an arbitrary number of driver comparisons; consider the pattern 1^n for instance. More formally, *next* ensures that the heads of the suffixes of the pattern (*vs*) on consecutive labels on left paths are distinct; see below for a proof of this fact in our setting, and Gusfield (1997, §3.3.4) for a classical account. Unlike Bird’s suggestion (p134), our *next* function is not recursive.

We note in passing that while MP only allows *Null* on the left of the root node, *Null* can be on the left of any KMP node except for the rightmost (i.e., the one that signals a complete pattern match) where no optimisation is possible.

We proceed with the formalities of the data refinement.

schematic-goal *root2-op2-rep2-left2-right2-def*: — Obtain the definition of these functions as a single fixed point

```
( root2 :: ['a::Eq-def:] → (['a:] × ['a:]) tree
  , op2  :: ['a:] → (['a:] × ['a:]) tree → 'a → (['a:] × ['a:]) tree
  , rep2 :: ['a:] → ['a:] × ['a:] → (['a:] × ['a:]) tree
  , left2 :: ['a:] → ['a:] × ['a:] → (['a:] × ['a:]) tree
  , right2 :: ['a:] → ['a:] × ['a:] → (['a:] × ['a:]) tree )
= fix.?F
```

unfolding *op2-def root2-def rep2-def left2-def right2-def* by *simp*

lemma *abs2-strict[simp]*:

```
abs2.⊥ = ⊥
abs2.Null = ⊥
```

by *fixrec-simp+*

lemma *next-strict[simp]*:

```
next.⊥ = ⊥
next.xs.⊥ = ⊥
next.(x :# xs).(Node.(us, ⊥).l.r) = ⊥
```

apply *fixrec-simp*

apply (*cases xs; fixrec-simp; simp*)

apply (*cases x = ⊥; cases xs = ⊥; cases us = ⊥; fixrec-simp*)

done

lemma *next-Null[simp]*:

assumes $xs \neq \perp$

shows $next.xs.Null = Null$

using *assms* **by** (*cases xs*) *simp-all*

lemma *next-snil[simp]*:

assumes $xs \neq \perp$

shows $next.xs.(Node.(us, [::]).l.r) = Node.(us, [::]).l.r$

using *assms* **by** (*cases xs*) *simp-all*

lemma *op2-rep2-left2-right2-strict[simp]*:

```
op2.pat.⊥ = ⊥
op2.pat.(Node.(us, ⊥).l.r) = ⊥
op2.pat.(Node.usvs.l.r).⊥ = ⊥
rep2.pat.⊥ = ⊥
left2.pat.(⊥, vs) = ⊥
left2.pat.(us, ⊥) = ⊥
right2.pat.(us, ⊥) = ⊥
```

apply *fixrec-simp*

apply (*cases us = ⊥; fixrec-simp; simp*)

apply (*cases usvs = ⊥; fixrec-simp; simp*)

```

  apply fixrec-simp
  apply fixrec-simp
  apply (cases us; fixrec-simp)
  apply fixrec-simp
  done

```

lemma *snd-abs-root2-bottom*[simp]: $prod.snd (abs2.(root2.\perp)) = \perp$
by (simp add: root2.unfold rep2.unfold)

lemma *abs-rep2-ID'*[simp]: $abs2.(rep2.pat.usvs) = usvs$
by (cases usvs = \perp ; subst rep2.unfold; clarsimp)

lemma *abs-rep2-ID*: $abs2 \circ rep2.pat = ID$
by (clarsimp simp: cfun-eq-iff)

lemma *rep2-snoc-right2*: — Bird p131
assumes $prefix.[:x:].vs = TT$
shows $rep2.pat.(us :@ [:x:], stail.us) = right2.pat.(us, vs)$
using *assms* **by** (cases x = \perp ; cases vs; clarsimp)

lemma *not-prefix-op2-next*:
assumes $prefix.[:x:].xs = FF$
shows $op2.pat.(next.xs.(rep2.pat.usvs)).x = op2.pat.(rep2.pat.usvs).x$

proof —
obtain *us vs* **where** $usvs = (us, vs)$ **by** *force*
with *assms* **show** *?thesis*
by (cases xs; cases us; clarsimp; cases vs;
 clarsimp simp: rep2.simps prefix-singleton-FF If2-def[symmetric] split: If2-splits)

qed

Bird's appeal to *foldl-fusion* (p130) is too weak to justify this data refinement as his condition (iii) requires the worker functions to coincide on all representation values. Concretely he asks that:

$$rep2.pat.(op.pat.(abs2.t).x) = op2.pat.t.x \text{ — Bird (17.2)}$$

where t is an arbitrary tree. This does not hold for junk representations such as:

$$t = Node.(pat, [::]).Null.Null$$

Using worker/wrapper fusion (Gammie 2011; Gill and Hutton 2009) specialised to *scanl* (*scanl-ww-fusion*) we only need to establish this identity for valid representations, i.e., when t lies under the image of *rep2*. In pictures, we show that this diagram commutes:

$$\begin{array}{ccc}
 usvs & \xrightarrow{\Lambda usvs. op.pat.usvs.x} & usvs' \\
 \downarrow rep2.pat & & \downarrow rep2.pat \\
 t & \xrightarrow{\Lambda usvs. op2.pat.usvs.x} & t'
 \end{array}$$

Clearly this result self-composes: after an initial *rep2.pat* step, we can repeatedly simulate *op* steps with *op2* steps.

lemma *op-op2-refinement*:
assumes $pat \neq \perp$
shows $rep2.pat.(op.pat.usvs.x) = op2.pat.(rep2.pat.usvs).x$
proof(cases x = \perp usvs = \perp rule: bool.exhaust[case-product bool.exhaust])
case *False-False*
then have $x \neq \perp$ usvs $\neq \perp$ **by** *simp-all*
then show *?thesis*
proof(induct usvs arbitrary: x rule: op-induct)
case (*step usvs*)

```

obtain us vs where usvs: usvs = (us, vs) by fastforce
have *: sfoldl.(op2.pat).(root2.pat).xs = rep2.pat.(split.pat.xs) if lt.(length.xs).(length.us) = TT for xs
using that
proof(induct xs rule: srev-induct)
  case (ssnoc x xs)
  from ssnoc(1,2,4) have IH: sfoldl.(op2.pat).(root2.pat).xs = rep2.pat.(split.pat.xs)
    by — (rule ssnoc; auto intro: lt-trans dest: lt-length-0)
  obtain us' vs' where us'vs': split.pat.xs = (us', vs') by fastforce
  from ⟨pat ≠ ⊥⟩ ssnoc(1,2,4) usvs show ?case
    apply (clarsimp simp: split-sfoldl-op[symmetric] IH)
    apply (rule step(1)[simplified abs-rep2-ID', simplified, symmetric])
    using lt-trans split-length-lt split-sfoldl-op apply fastforce+
    done
qed (fastforce simp: ⟨pat ≠ ⊥⟩ root2.unfold) +
have **: If snul.us then rep2.pat.([:], pat) else rep2.pat.(op.pat.(split.pat.(stail.us)).x)
  = op2.pat.(left2.pat.(us, vs)).x if prefix[:x:].vs = FF
proof(cases us)
  case snul with that show ?thesis
    by simp (metis next-Null op2.simps(1) prefix.simps(1) prefix-FF-not-snulD root2.simps)
next
  case (scons u' us')
  from ⟨pat ≠ ⊥⟩ scons have lt.(length.(cfst.(split.pat.us^)).(length.us) = TT
    using split-length-lt by fastforce
  from ⟨pat ≠ ⊥⟩ ⟨x ≠ ⊥⟩ usvs that scons this show ?thesis
    by (clarsimp simp: * step(1)[simplified abs-rep2-ID'] not-prefix-op2-next)
qed simp
from ⟨usvs ≠ ⊥⟩ usvs show ?case
  apply (subst (2) rep2.unfold)
  apply (subst op2.unfold)
  apply (subst op.unfold)
  apply (clarsimp simp: If-distr rep2-snoc-right2 ** cong: If-cong)
  done
qed
qed (simp-all add: rep2.unfold)

```

Therefore the result of this data refinement is extensionally equal to the specification:

lemma *data-refinement:*

shows *matches = matches2*

proof(*intro cfun-eqI*)

fix *pat xs :: [:'a:] show matches.pat.xs = matches2.pat.xs*

proof(*cases pat = ⊥*)

case *True then show ?thesis by (cases xs; clarsimp simp: matches2.simps)*

next

case *False then show ?thesis*

unfolding *matches2.simps*

apply (*subst matches-op*) — Continue with previous derivation.

apply (*subst sscanl-wv-fusion[where wrap=ID ** abs2 and unwrap=ID ** rep2.pat and f'=Λ (n, x) y. (n + 1, op2.pat.x.y)]*)

apply (*simp add: abs-rep2-ID*)

apply (*simp add: op-op2-refinement*)

apply (*simp add: oo-assoc sfilter-smap root2.unfold*)

apply (*simp add: oo-assoc[symmetric]*)

done

qed

qed

This computation can be thought of as a pair coroutines with a producer (*root2/rep2*) / consumer (*op2*) structure. It turns out that laziness is not essential (see §6), though it does depend on being able to traverse incompletely

defined trees.

The key difficulty in defining this computation in HOL using present technology is that *op2* is neither terminating nor *friendly* in the terminology of Blanchette et al. (2017).

While this representation works for automata with this sort of structure, it is unclear how general it is; in particular it may not work so well if *left* branches can go forward as well as back. See also the commentary in Hinze and Jeuring (2001), who observe that sharing is easily lost, and so it is probably only useful in “closed” settings like the present one, unless the language is extended in unusual ways (Jeannin et al. 2017).

We conclude by proving that *rep2* produces trees that have the ‘K’ property, viz that labels on consecutive nodes on a left path do not start with the same symbol. This also establishes the productivity of *root2*. The pattern of proof used here – induction nested in coinduction – recurs in §4.6.

```

coinductive K :: ([:'a::Eq:] × [:'a:]) tree ⇒ bool where
  K Null
| [| usvs ≠ ⊥; K l; K r;
   ∧ v vs. csnd.usvs = v :# vs ⇒ l = Null ∨ (∃ v' vs'. csnd.(label.l) = v' :# vs' ∧ eq.v.v' = FF)
  |] ⇒ K (Node.usvs.l.r)

```

```

declare K.intros[intro!, simp]

```

```

lemma sfoldl-op2-root2-rep2-split:

```

```

  assumes pat ≠ ⊥
  shows sfoldl.(op2.pat).(root2.pat).xs = rep2.pat.(split.pat.xs)

```

```

proof(induct xs rule: srev-induct)

```

```

  case (ssnoc x xs) with ⟨pat ≠ ⊥⟩ ssnoc show ?case by (clarsimp simp: split-sfoldl-op[symmetric] op-op2-refinement)
qed (simp-all add: ⟨pat ≠ ⊥⟩ root2.unfold)

```

```

lemma K-rep2:

```

```

  assumes pat ≠ ⊥
  assumes us :@ vs = pat
  shows K (rep2.pat.(us, vs))

```

```

using assms

```

```

proof(coinduction arbitrary: us vs)

```

```

  case (K us vs) then show ?case
  proof(induct us arbitrary: vs rule: op-induct^)

```

```

    case (step us)
    from step.premss have us ≠ ⊥ vs ≠ ⊥ by auto
    show ?case
    proof(cases us)

```

```

      case bottom with ⟨us ≠ ⊥⟩ show ?thesis by simp

```

```

    next

```

```

      case snil with step.premss show ?thesis by (cases vs; force simp: rep2.simps)

```

```

    next

```

```

      case (scons u' us^)

```

```

      from ⟨pat ≠ ⊥⟩ sconss ⟨us ≠ ⊥⟩ ⟨vs ≠ ⊥⟩

```

```

      obtain usl vsl where splitl: split.pat.us' = (usl, vsl) usl ≠ ⊥ vsl ≠ ⊥ usl :@ vsl = pat

```

```

      by (metis (no-types, opaque-lifting) Rep-cfun-strict1 prod.collapse sappend-strict sappend-strict2 split-pattern)

```

```

      from sconss obtain l r where r: rep2.pat.(us, vs) = Node.(us, vs).l.r by (simp add: rep2.simps)

```

```

      moreover

```

```

      have (∃ us vs. l = rep2.pat.(us, vs) ∧ us :@ vs = pat) ∨ K l

```

```

      proof(cases vs)

```

```

        case snil with sconss splitl r show ?thesis

```

```

        by (clarsimp simp: rep2.simps sfoldl-op2-root2-rep2-split)

```

```

      next

```

```

        case sconss

```

```

        with ⟨pat ≠ ⊥⟩ ⟨us = u' :# us'⟩ ⟨u' ≠ ⊥⟩ ⟨us' ≠ ⊥⟩ ⟨vs ≠ ⊥⟩ r splitl show ?thesis

```

```

        apply (clarsimp simp: rep2.simps sfoldl-op2-root2-rep2-split)

```

```

        apply (cases vsl; cases usl; clarsimp simp: If2-def[symmetric] sfoldl-op2-root2-rep2-split split: If2-splits)

```

```

        apply (rename-tac ul' usl^)

```

```

apply (cut-tac us'=prod.fst (split.pat.usl') and vs=prod.snd (split.pat.usl') in step(1); clarsimp simp:
split-pattern)
apply (metis fst-conv lt-trans slength.simps(2) split-length-lt step.prem(1))
apply (erule disjE; clarsimp simp: sfoldl-op2-root2-rep2-split)
apply (rename-tac b l r)
apply (case-tac b; clarsimp simp: rep2.simps)
apply (auto simp: If2-def[symmetric] rep2.simps dest: split-pattern[rotated] split: If2-splits)
done
qed (simp add: ⟨vs ≠ ⊥⟩)
moreover
from ⟨us :@ vs = pat⟩ ⟨us ≠ ⊥⟩ ⟨vs ≠ ⊥⟩ r
have (∃ usr vsr. r = rep2.pat.(usr, vsr) ∧ usr :@ vsr = pat) ∨ K r
by (cases vs; clarsimp simp: rep2.simps)
moreover
have l = Null ∨ (∃ v' vs'. csnd.(label.l) = v' :# vs' ∧ eq.v.v' = FF) if vs = v :# vs' for v vs'
proof(cases vsl)
case snil with ⟨us :@ vs = pat⟩ ⟨us = u' :# us'⟩ splitl show ?thesis
using split-length-lt[where pat=pat and xs=us']
by (force elim: slengthE simp: one-Integer-def split: if-splits)
next
case scon
from splitl have lt.(slength.usl).(slength.us' + 1) = TT
by (metis fst-conv fst-strict split-bottom-iff split-length-lt)
with scon ⟨pat ≠ ⊥⟩ ⟨us = u' :# us'⟩ ⟨u' ≠ ⊥⟩ ⟨us' ≠ ⊥⟩ ⟨vs ≠ ⊥⟩ r splitl ⟨vs = v :# vs'⟩ show ?thesis
using step(1)[OF - ⟨pat ≠ ⊥⟩, where us'=prod.fst (split.pat.usl') and vs=prod.snd (split.pat.usl')]
by (clarsimp simp: rep2.simps sfoldl-op2-root2-rep2-split If2-def[symmetric] split: If2-splits)
qed (simp add: ⟨vsl ≠ ⊥⟩)
moreover note ⟨pat ≠ ⊥⟩ ⟨us ≠ ⊥⟩ ⟨vs ≠ ⊥⟩
ultimately show ?thesis by auto
qed
qed
qed

```

```

theorem K-root2:
assumes pat ≠ ⊥
shows K (root2.pat)
using assms unfolding root2.unfold by (simp add: K-rep2)

```

The remaining steps are as follows:

- 3. introduce an accumulating parameter (*grep*).
- 4. inline *rep* and simplify.
- 5. simplify to Bird's "simpler forms."
- 6. memoise *left*.
- 7. simplify, unfold *prefix*.
- 8. discard *us*.
- 9. factor out *pat*.

4.3 Step 3: Introduce an accumulating parameter (*grep*)

Next we prepare for the second memoization step (§4.6) by introducing an accumulating parameter to *rep2* that supplies the value of the left subtree.

We retain *rep2* as a wrapper for now, and inline *right2* to speed up simplification.

fixrec — Bird p131 / p132

```
root3 :: ['a::Eq-def:] → (['a:] × ['a:]) tree
and op3  :: ['a:] → (['a:] × ['a:]) tree → 'a → (['a:] × ['a:]) tree
and rep3 :: ['a:] → ['a:] × ['a:] → (['a:] × ['a:]) tree
and grep3 :: ['a:] → (['a:] × ['a:]) tree → ['a:] × ['a:] → (['a:] × ['a:]) tree
where
  [simp del]:
  root3.pat = rep3.pat.([::], pat)
| op3.pat.Null.x = root3.pat
| usvs ≠ ⊥ ⇒
  op3.pat.(Node.usvs.l.r).x = If prefix.[:x].(csnd.usvs) then r else op3.pat.l.x
| [simp del]: — Inline left2, factor out next.
  rep3.pat.usvs = grep3.pat.(case cfst.usvs of [::] ⇒ Null | u :# us ⇒ sfoldl.(op3.pat).(root3.pat).us).usvs
| [simp del]: — rep2 with left2 abstracted, right2 inlined.
  grep3.pat.l.usvs = Node.usvs.(next.(csnd.usvs).l).(case csnd.usvs of
    [::] ⇒ Null
  | v :# vs ⇒ rep3.pat.(cfst.usvs :@ [v:], vs))
```

schematic-goal root3-op3-rep3-grep3-def:

```
( root3 :: ['a::Eq-def:] → (['a:] × ['a:]) tree
, op3   :: ['a:] → (['a:] × ['a:]) tree → 'a → (['a:] × ['a:]) tree
, rep3  :: ['a:] → ['a:] × ['a:] → (['a:] × ['a:]) tree
, grep3 :: ['a:] → (['a:] × ['a:]) tree → ['a:] × ['a:] → (['a:] × ['a:]) tree )
= fix.?F
```

unfolding root3-def op3-def rep3-def grep3-def by simp

lemma r3-2:

```
(Λ (root, op, rep, grep). (root, op, rep)).
( root3 :: ['a::Eq-def:] → (['a:] × ['a:]) tree
, op3   :: ['a:] → (['a:] × ['a:]) tree → 'a → (['a:] × ['a:]) tree
, rep3  :: ['a:] → ['a:] × ['a:] → (['a:] × ['a:]) tree
, grep3 :: ['a:] → (['a:] × ['a:]) tree → ['a:] × ['a:] → (['a:] × ['a:]) tree )
= (Λ (root, op, rep, left, right). (root, op, rep)).
( root2 :: ['a::Eq-def:] → (['a:] × ['a:]) tree
, op2   :: ['a:] → (['a:] × ['a:]) tree → 'a → (['a:] × ['a:]) tree
, rep2  :: ['a::Eq-def:] → ['a:] × ['a:] → (['a:] × ['a:]) tree
, left2 :: ['a::Eq-def:] → ['a:] × ['a:] → (['a:] × ['a:]) tree
, right2 :: ['a::Eq-def:] → ['a:] × ['a:] → (['a:] × ['a:]) tree )
```

unfolding root2-op2-rep2-left2-right2-def root3-op3-rep3-grep3-def

apply (simp add: match-snil-match-scons-slist-case match-Null-match-Node-tree-case slist-case-distr tree-case-distr)

apply (simp add: fix-cprod fix-const) — Very slow. Sensitive to tuple order due to the asymmetry of fix-cprod.

apply (simp add: slist-case-distr)

done

4.4 Step 4: Inline rep

We further simplify by inlining rep3 into root3 and grep3.

fixrec

```
root4 :: ['a::Eq-def:] → (['a:] × ['a:]) tree
and op4  :: ['a:] → (['a:] × ['a:]) tree → 'a → (['a:] × ['a:]) tree
and grep4 :: ['a:] → (['a:] × ['a:]) tree → ['a:] × ['a:] → (['a:] × ['a:]) tree
where
  [simp del]:
  root4.pat = grep4.pat.Null.([::], pat)
| op4.pat.Null.x = root4.pat
| usvs ≠ ⊥ ⇒
  op4.pat.(Node.usvs.l.r).x = If prefix.[:x].(csnd.usvs) then r else op4.pat.l.x
| [simp del]:
```

$grep4.pat.l.usvs = Node.usvs.(next.(csnd.usvs).l).(case\ csnd.usvs\ of$
 $\quad [::] \Rightarrow Null$
 $\quad | v : \# us \Rightarrow grep4.pat.(case\ cfst.usvs\ :@ [::v] of$
 $\quad \quad [::] \Rightarrow Null \text{ — unreachable}$
 $\quad \quad | u : \# us \Rightarrow sfoldl.(op4.pat).(root4.pat).us).(cfst.usvs :@ [::v], us))$

schematic-goal $root4-op4-grep4-def$:

$(\ root4\ ::\ [::a::Eq-def:] \rightarrow ([::a:] \times [::a:])\ tree$
 $\ ,\ op4\ ::\ [::a:] \rightarrow ([::a:] \times [::a:])\ tree \rightarrow 'a \rightarrow ([::a:] \times [::a:])\ tree$
 $\ ,\ grep4\ ::\ [::a:] \rightarrow ([::a:] \times [::a:])\ tree \rightarrow [::a:] \times [::a:] \rightarrow ([::a:] \times [::a:])\ tree)$
 $=\ fix.\ ?F$

unfolding $root4-def\ op4-def\ grep4-def$ **by** $simp$

lemma $fix-syn4-permute$:

assumes $cont\ (\lambda(X1, X2, X3, X4). F1\ X1\ X2\ X3\ X4)$
assumes $cont\ (\lambda(X1, X2, X3, X4). F2\ X1\ X2\ X3\ X4)$
assumes $cont\ (\lambda(X1, X2, X3, X4). F3\ X1\ X2\ X3\ X4)$
assumes $cont\ (\lambda(X1, X2, X3, X4). F4\ X1\ X2\ X3\ X4)$
shows $fix-syn\ (\lambda(X1, X2, X3, X4). (F1\ X1\ X2\ X3\ X4, F2\ X1\ X2\ X3\ X4, F3\ X1\ X2\ X3\ X4, F4\ X1\ X2\ X3\ X4))$
 $=\ (\lambda(x1, x2, x4, x3). (x1, x2, x3, x4))$
 $(fix-syn\ (\lambda(X1, X2, X4, X3). (F1\ X1\ X2\ X3\ X4, F2\ X1\ X2\ X3\ X4, F4\ X1\ X2\ X3\ X4, F3\ X1\ X2\ X3\ X4)))$

by (*induct rule: parallel-fix-ind*) (*use assms in <auto simp: prod-cont-iff>*)

lemma $r4-3$:

$(\ root4\ ::\ [::a::Eq-def:] \rightarrow ([::a:] \times [::a:])\ tree$
 $\ ,\ op4\ ::\ [::a:] \rightarrow ([::a:] \times [::a:])\ tree \rightarrow 'a \rightarrow ([::a:] \times [::a:])\ tree$
 $\ ,\ grep4\ ::\ [::a:] \rightarrow ([::a:] \times [::a:])\ tree \rightarrow [::a:] \times [::a:] \rightarrow ([::a:] \times [::a:])\ tree)$
 $=\ (\Lambda\ (root, op, rep, grep). (root, op, grep)).$
 $(\ root3\ ::\ [::a::Eq-def:] \rightarrow ([::a:] \times [::a:])\ tree$
 $\ ,\ op3\ ::\ [::a:] \rightarrow ([::a:] \times [::a:])\ tree \rightarrow 'a \rightarrow ([::a:] \times [::a:])\ tree$
 $\ ,\ rep3\ ::\ [::a:] \rightarrow [::a:] \times [::a:] \rightarrow ([::a:] \times [::a:])\ tree$
 $\ ,\ grep3\ ::\ [::a:] \rightarrow ([::a:] \times [::a:])\ tree \rightarrow [::a:] \times [::a:] \rightarrow ([::a:] \times [::a:])\ tree)$

unfolding $root3-op3-rep3-grep3-def\ root4-op4-grep4-def$

apply (*clarsimp simp: slist-case-distr match-Null-match-Node-tree-case tree-case-distr eta-cfun*)

apply (*subst fix-syn4-permute; clarsimp simp: fix-cprod fix-const*) — Slow

done

4.5 Step 5: Simplify to Bird's “simpler forms”

The remainder of $left2$ in $grep4$ can be simplified by transforming the *case* scrutinee from $cfst.usvs :@ [::v]$ into $cfst.usvs$.

fixrec

$root5\ ::\ [::a::Eq-def:] \rightarrow ([::a:] \times [::a:])\ tree$
and $op5\ ::\ [::a::Eq-def:] \rightarrow ([::a:] \times [::a:])\ tree \rightarrow 'a \rightarrow ([::a:] \times [::a:])\ tree$
and $grep5\ ::\ [::a:] \rightarrow ([::a:] \times [::a:])\ tree \rightarrow [::a:] \times [::a:] \rightarrow ([::a:] \times [::a:])\ tree$
where

[*simp del*]:

$root5.pat = grep5.pat.Null.(::], pat)$

| $op5.pat.Null.x = root5.pat$

| $usvs \neq \perp \implies$

$op5.pat.(Node.usvs.l.r).x = If\ prefix.[::x].(csnd.usvs)\ then\ r\ else\ op5.pat.l.x$

[*simp del*]:

$grep5.pat.l.usvs = Node.usvs.(next.(csnd.usvs).l).(case\ csnd.usvs\ of$

$\quad [::] \Rightarrow Null$

$\quad | v : \# us \Rightarrow grep5.pat.(case\ cfst.usvs\ of \text{ — was } cfst.usvs :@ [::v]$

$\quad \quad [::] \Rightarrow root5.pat$

| $u : \# us \Rightarrow \text{sfoldl} \cdot (\text{op5} \cdot \text{pat}) \cdot (\text{root5} \cdot \text{pat}) \cdot (\text{us} : @ [:v]) \cdot (\text{cfst} \cdot \text{usvs} : @ [:v], \text{vs})$

schematic-goal *root5-op5-grep5-def*:

(*root5* :: [*a*::Eq-def:] \rightarrow ([*a*:] \times [*a*:]) *tree*
, *op5* :: [*a*:] \rightarrow ([*a*:] \times [*a*:]) *tree* \rightarrow '*a* \rightarrow ([*a*:] \times [*a*:]) *tree*
, *grep5* :: [*a*:] \rightarrow ([*a*:] \times [*a*:]) *tree* \rightarrow [*a*:] \times [*a*:] \rightarrow ([*a*:] \times [*a*:]) *tree*)
= *fix* . ?*F*

unfolding *root5-def op5-def grep5-def* **by** *simp*

lemma *op5-grep5-strict*[*simp*]:

op5 . *pat* . \perp = \perp
op5 . *pat* . (*Node* . (*us*, \perp) . *l* . *r*) = \perp
op5 . *pat* . (*Node* . *usvs* . *l* . *r*) . \perp = \perp
grep5 . *pat* . *l* . \perp = \perp
apply *fixrec-simp*
apply (*cases us* = \perp ; *fixrec-simp*; *simp*)
apply (*cases usvs* = \perp ; *fixrec-simp*; *simp*)
apply *fixrec-simp*

done

lemma *r5-4*:

(*root5* :: [*a*::Eq-def:] \rightarrow ([*a*:] \times [*a*:]) *tree*
, *op5* :: [*a*::Eq-def:] \rightarrow ([*a*:] \times [*a*:]) *tree* \rightarrow '*a* \rightarrow ([*a*:] \times [*a*:]) *tree*
, *grep5* :: [*a*:] \rightarrow ([*a*:] \times [*a*:]) *tree* \rightarrow [*a*:] \times [*a*:] \rightarrow ([*a*:] \times [*a*:]) *tree*)
= (*root4* :: [*a*::Eq-def:] \rightarrow ([*a*:] \times [*a*:]) *tree*
, *op4* :: [*a*::Eq-def:] \rightarrow ([*a*:] \times [*a*:]) *tree* \rightarrow '*a* \rightarrow ([*a*:] \times [*a*:]) *tree*
, *grep4* :: [*a*:] \rightarrow ([*a*:] \times [*a*:]) *tree* \rightarrow [*a*:] \times [*a*:] \rightarrow ([*a*:] \times [*a*:]) *tree*)

unfolding *root4-op4-grep4-def root5-op5-grep5-def*

by (*clarsimp simp*: *slist-case-distr slist-case-snoc stail-sappend cong*: *slist-case-cong*)

4.6 Step 6: Memoize left

The last substantial step is to memoise the computation of the left subtrees by tying the knot.

Note this makes the computation of *us* in the tree redundant; we remove it in §4.8.

fixrec — Bird p132

root6 :: [*a*::Eq-def:] \rightarrow ([*a*:] \times [*a*:]) *tree*

and *op6* :: [*a*::Eq-def:] \rightarrow ([*a*:] \times [*a*:]) *tree* \rightarrow '*a* \rightarrow ([*a*:] \times [*a*:]) *tree*

and *grep6* :: [*a*:] \rightarrow ([*a*:] \times [*a*:]) *tree* \rightarrow [*a*:] \times [*a*:] \rightarrow ([*a*:] \times [*a*:]) *tree*

where

[*simp del*]:

root6 . *pat* = *grep6* . *pat* . *Null* . ([:], *pat*)

| *op6* . *pat* . *Null* . *x* = *root6* . *pat*

| *usvs* \neq \perp \implies

op6 . *pat* . (*Node* . *usvs* . *l* . *r*) . *x* = *If prefix* . [*x*] . (*csnd* . *usvs*) *then r else op6* . *pat* . *l* . *x*

| [*simp del*]:

grep6 . *pat* . *l* . *usvs* = *Node* . *usvs* . (*next* . (*csnd* . *usvs*) . *l*) . (*case csnd* . *usvs of*

[:] \Rightarrow *Null*

| *v* : # *vs* \Rightarrow *grep6* . *pat* . (*op6* . *pat* . *l* . *v*) . (*cfst* . *usvs* : @ [:v], *vs*)

schematic-goal *root6-op6-grep6-def*:

(*root6* :: [*a*::Eq-def:] \rightarrow ([*a*:] \times [*a*:]) *tree*
, *op6* :: [*a*:] \rightarrow ([*a*:] \times [*a*:]) *tree* \rightarrow '*a* \rightarrow ([*a*:] \times [*a*:]) *tree*
, *grep6* :: [*a*:] \rightarrow ([*a*:] \times [*a*:]) *tree* \rightarrow [*a*:] \times [*a*:] \rightarrow ([*a*:] \times [*a*:]) *tree*)
= *fix* . ?*F*

unfolding *root6-def op6-def grep6-def* **by** *simp*

lemma *op6-grep6-strict*[*simp*]:

```

op6.pat.⊥ = ⊥
op6.pat.(Node.(us, ⊥).l.r) = ⊥
op6.pat.(Node.usvs.l.r).⊥ = ⊥
grep6.pat.l.⊥ = ⊥
apply fixrec-simp
apply (cases us = ⊥; fixrec-simp; simp)
apply (cases usvs = ⊥; fixrec-simp; simp)
apply fixrec-simp
done

```

Intuitively this step cashes in the fact that, in the context of $grep6.pat.l.usvs$, $sfoldl.(op6.pat).(root6.pat).us$ is equal to l .

Connecting this step with the previous one is not simply a matter of equational reasoning; we can see this by observing that the right subtree of $grep5.pat.l.usvs$ does not depend on l whereas that of $grep6.pat.l.usvs$ does, and therefore these cannot be extensionally equal. Furthermore the computations of the corresponding *roots* do not proceed in lockstep: consider the computation of the left subtree.

For our purposes it is enough to show that the trees $root5$ and $root6$ are equal, from which it follows that $op6 = op5$ by induction on its tree argument. The equality is established by exhibiting a *tree bisimulation* (*tree-bisim*) that relates the corresponding “producer” *grep* functions. Such a relation R must satisfy:

- $R \perp \perp$;
- $R \text{ Null Null}$; and
- if $R (\text{Node}.x.l.r) (\text{Node}.x'.l'.r')$ then $x = x'$, $R l l'$, and $R r r'$.

The following pair of *left* functions define suitable left paths from the corresponding *roots*.

```

fixrec left5 :: [:'a::Eq-def:] → [:'a:] → ([:'a:] × [:'a:]) tree where
  left5.pat.[:] = Null
|  $\llbracket u \neq \perp; us \neq \perp \rrbracket \implies$ 
  left5.pat.(u :# us) = sfoldl.(op5.pat).(root5.pat).us

```

```

fixrec left6 :: [:'a::Eq-def:] → [:'a:] → ([:'a:] × [:'a:]) tree where
  left6.pat.[:] = Null
|  $\llbracket u \neq \perp; us \neq \perp \rrbracket \implies$ 
  left6.pat.(u :# us) = sfoldl.(op6.pat).(root6.pat).us

```

inductive — This relation is not inductive.

```

root-bisim :: [:'a::Eq-def:] ⇒ ([:'a:] × [:'a:]) tree ⇒ ([:'a:] × [:'a:]) tree ⇒ bool
for pat :: [:'a:]

```

where

```

  bottom: root-bisim pat ⊥ ⊥
| Null: root-bisim pat Null Null
| gl:  $\llbracket pat \neq \perp; us \neq \perp; vs \neq \perp \rrbracket$ 
  ⇒ root-bisim pat (grep6.pat.(left6.pat.us).(us, vs)) (grep5.pat.(left5.pat.us).(us, vs))

```

declare $root-bisim.intros[intro!, simp]$

lemma $left6-left5-strict[simp]$:

```

  left6.pat.⊥ = ⊥
  left5.pat.⊥ = ⊥
by fixrec-simp+

```

lemma $op6-left6$: $\llbracket us \neq \perp; v \neq \perp \rrbracket \implies op6.pat.(left6.pat.us).v = left6.pat.(us :@ [:v:])$
by (cases us) simp-all

lemma $op5-left5$: $\llbracket us \neq \perp; v \neq \perp \rrbracket \implies op5.pat.(left5.pat.us).v = left5.pat.(us :@ [:v:])$
by (cases us) simp-all

lemma *root5-left5*: $v \neq \perp \implies \text{root5} \cdot \text{pat} = \text{left5} \cdot \text{pat} \cdot [:v]$
by *simp*

lemma *op5-sfoldl-left5*: $\llbracket us \neq \perp; u \neq \perp; v \neq \perp \rrbracket \implies$
 $\text{op5} \cdot \text{pat} \cdot (\text{sfoldl} \cdot (\text{op5} \cdot \text{pat}) \cdot (\text{root5} \cdot \text{pat}) \cdot \text{us}) \cdot v = \text{left5} \cdot \text{pat} \cdot (u \text{ :# } us \text{ :@ } [:v])$
by *simp*

lemma *root-bisim-root*:
assumes $\text{pat} \neq \perp$
shows *root-bisim pat (root6.pat) (root5.pat)*
unfolding *root6.unfold root5.unfold* **using** *assms*
by *simp (metis (no-types, lifting) left5.simps(1) left6.simps(1) root-bisim.simps slist.con-rews(3))*

lemma *next-grep6-cases*[*consumes 3, case-names gl nl*]:
assumes $vs \neq \perp$
assumes $xs \neq \perp$
assumes $P (\text{next} \cdot xs \cdot (\text{grep6} \cdot \text{pat} \cdot (\text{left6} \cdot \text{pat} \cdot \text{us}) \cdot (us, vs)))$
obtains $(gl) P (\text{grep6} \cdot \text{pat} \cdot (\text{left6} \cdot \text{pat} \cdot \text{us}) \cdot (us, vs)) \mid (nl) P (\text{next} \cdot vs \cdot (\text{left6} \cdot \text{pat} \cdot \text{us}))$
using *assms*
apply *atomize-elim*
apply (*subst grep6.unfold*)
apply (*subst (asm) grep6.unfold*)
apply (*cases xs; clarsimp*)
apply (*cases vs; clarsimp simp: If2-def[symmetric] split: If2-splits*)
done

lemma *root-bisim-op-next56*:
assumes *root-bisim pat t6 t5*
assumes $\text{prefix} \cdot [:x] \cdot xs = FF$
shows $\text{op6} \cdot \text{pat} \cdot (\text{next} \cdot xs \cdot t6) \cdot x = \text{op6} \cdot \text{pat} \cdot t6 \cdot x \wedge \text{op5} \cdot \text{pat} \cdot (\text{next} \cdot xs \cdot t5) \cdot x = \text{op5} \cdot \text{pat} \cdot t5 \cdot x$
using $\langle \text{root-bisim pat t6 t5} \rangle$
proof *cases*
case *Null* **with** *assms(2)* **show** *?thesis* **by** (*cases xs*) *simp-all*
next
case (*gl us vs*) **with** *assms(2)* **show** *?thesis*
apply (*cases x = \perp , simp*)
apply (*cases xs; clarsimp*)
apply (*subst (1 2) grep6.simps*)
apply (*subst (1 2) grep5.simps*)
apply (*cases vs; clarsimp simp: If2-def[symmetric] split: If2-splits*)
done
qed *simp*

The main part of establishing that *root-bisim* is a *tree-bisim* is in showing that the left paths constructed by the *greps* are *root-bisim*-related. We do this by inducting over the length of the pattern so far matched (*us*), as we did when proving that this tree has the ‘K’ property in §4.2.

lemma
assumes $\text{pat} \neq \perp$
shows *root-bisim-op: root-bisim pat t6 t5 \implies root-bisim pat (op6.pat.t6.x) (op5.pat.t5.x)* — *unused*
and *root-bisim-next-left: root-bisim pat (next-vs.(left6.pat.us)) (next-vs.(left5.pat.us))* (**is** *?rbnl us vs*)

proof –

let $?ogl5 = \lambda us \text{ vs. } \text{op5} \cdot \text{pat} \cdot (\text{grep5} \cdot \text{pat} \cdot (\text{left5} \cdot \text{pat} \cdot \text{us}) \cdot (us, vs)) \cdot x$
let $?ogl6 = \lambda us \text{ vs. } \text{op6} \cdot \text{pat} \cdot (\text{grep6} \cdot \text{pat} \cdot (\text{left6} \cdot \text{pat} \cdot \text{us}) \cdot (us, vs)) \cdot x$
let $?for5 = \lambda us. \text{sfoldl} \cdot (\text{op5} \cdot \text{pat}) \cdot (\text{root5} \cdot \text{pat}) \cdot \text{us}$
let $?for6 = \lambda us. \text{sfoldl} \cdot (\text{op6} \cdot \text{pat}) \cdot (\text{root6} \cdot \text{pat}) \cdot \text{us}$
have $\llbracket ?ogl6 \text{ us vs} = \text{Node} \cdot \text{usvs} \cdot \text{l.r}; \text{cfst} \cdot \text{usvs} \neq \perp; x \neq \perp \rrbracket \implies \text{le} \cdot (\text{slength} \cdot (\text{cfst} \cdot \text{usvs})) \cdot (\text{slength} \cdot \text{us} + 1) = TT$
and $\llbracket \text{op6} \cdot \text{pat} \cdot (\text{next} \cdot xs \cdot (\text{grep6} \cdot \text{pat} \cdot (\text{left6} \cdot \text{pat} \cdot \text{us}) \cdot (us, vs))) \cdot x = \text{Node} \cdot \text{usvs} \cdot \text{l.r}; \text{cfst} \cdot \text{usvs} \neq \perp; x \neq \perp; us \neq \perp; vs$

```

≠ ⊥] ⇒ le.(length.(cfst.usvs)).(length.us + 1) = TT
and [?for6 us = Node.usvs.l.r; cfst.usvs ≠ ⊥] ⇒ lt.(length.(cfst.usvs)).(length.us + 1) = TT
and [us ≠ ⊥; vs ≠ ⊥] ⇒ root-bisim pat (?ogl6 us vs) (?ogl5 us vs)
and root-bisim pat (?for6 us) (?for5 us)
and ?rbnl us vs
for usvs l r xs us vs
proof(induct us arbitrary: usvs l r vs x xs rule: op-induct)
case (step us)
have rbl: root-bisim pat (left6.pat.us) (left5.pat.us)
  by (cases us; fastforce intro: step(5) simp: left6.unfold left5.unfold)
{ case (1 usvs l r vs x)
  from rbl
  have L: le.(length.(prod.fst usvs')).(length.us + 1) = TT
    if op6.pat.(next.vs.(left6.pat.us)).x = Node.usvs'.l.r
    and cfst.usvs' ≠ ⊥
    and vs ≠ ⊥
    for usvs' l r
  proof cases
    case bottom with that show ?thesis by simp
  next
    case Null with that show ?thesis
      apply simp
      apply (subst (asm) root6.unfold)
      apply (subst (asm) grep6.unfold)
      apply (fastforce intro: le-plus-1)
      done
  next
    case (gl us'' vs'') show ?thesis
      proof(cases us)
        case bottom with that show ?thesis by simp
      next
        case snil with that show ?thesis
          apply simp
          apply (subst (asm) root6.unfold)
          apply (subst (asm) grep6.unfold)
          apply clarsimp
          done
      next
        case (scons ush ust)
          moreover from that gl scons ⟨x ≠ ⊥⟩ have le.(length.(cfst.usvs')).(length.us'' + 1) = TT
            apply simp
            apply (subst (asm) (2) grep6.unfold)
            apply (fastforce dest: step(2, 3)[rotated])
            done
          moreover from gl scons have lt.(length.us'').(length.(stail.us) + 1) = TT
            apply simp
            apply (subst (asm) grep6.unfold)
            apply (fastforce dest: step(3)[rotated])
            done
          ultimately show ?thesis
            apply clarsimp
            apply (metis Integer-le-both-plus-1 Ord-linear-class.le-trans le-iff-lt-or-eq)
            done
        qed
      qed
    from 1 show ?case
      apply (subst (asm) grep6.unfold)
      apply (cases vs; clarsimp simp: If2-def[symmetric] split: If2-splits)

```

```

    apply (rule L; fastforce)
    apply (metis (no-types, lifting) ab-semigroup-add-class.add-ac(1) fst-conv grep6.simps le-refl-Integer
sappend-snil-id-right slength.simps(2) slength-bottom-iff slength-sappend slist.con-rews(3) tree-injects')
    apply (rule L; fastforce)
  done }
note slength-ogl = this
{ case (2 usvs l r vs x xs)
  from 2 have xs ≠ ⊥ by clarsimp
  from ⟨vs ≠ ⊥⟩ ⟨xs ≠ ⊥⟩ 2(1) show ?case
  proof(cases rule: next-grep6-cases)
    case gl with ⟨cfst·usvs ≠ ⊥⟩ ⟨x ≠ ⊥⟩ show ?thesis using slength-ogl by blast
  next
    case nl
    from rbl show ?thesis
  proof cases
    case bottom with nl ⟨cfst·usvs ≠ ⊥⟩ show ?thesis by simp
  next
    case Null with nl ⟨us ≠ ⊥⟩ ⟨vs ≠ ⊥⟩ show ?thesis
    apply simp
    apply (subst (asm) root6.unfold)
    apply (subst (asm) grep6.unfold)
    apply (clarsimp simp: zero-Integer-def one-Integer-def elim!: slengthE)
  done
  next
    case (gl us'' vs'') show ?thesis
  proof(cases us)
    case bottom with ⟨us ≠ ⊥⟩ show ?thesis by simp
  next
    case snil with gl show ?thesis by (subst (asm) grep6.unfold) simp
  next
    case (scons u' us') with 2 nl gl show ?thesis
    apply clarsimp
    apply (subst (asm) (4) grep6.unfold)
    apply clarsimp
    apply (drule step(3)[rotated]; clarsimp)
    apply (drule step(2)[rotated]; clarsimp)
    apply (clarsimp simp: lt-le)
    apply (metis Integer-le-both-plus-1 Ord-linear-class.le-trans)
  done
  qed
  qed
  qed }
{ case (3 usvs l r) show ?case
  proof(cases us rule: srev-cases)
    case snil with 3 show ?thesis
    apply (subst (asm) root6.unfold)
    apply (subst (asm) grep6.unfold)
    apply fastforce
  done
  next
    case (ssnoc u' us')
    then have root-bisim pat (?for6 us') (?for5 us') by (fastforce intro: step(5))
    then show ?thesis
  proof cases
    case bottom with 3 ssnoc show ?thesis by simp
  next
    case Null with 3 ssnoc show ?thesis
    apply simp

```

```

    apply (subst (asm) root6.unfold)
    apply (subst (asm) grep6.unfold)
    apply (clarsimp simp: zero-Integer-def one-Integer-def elim!: slengthE)
  done
next
case (gl us'' vs'') with 3 ssnoc show ?thesis
  apply clarsimp
  apply (subst (asm) (2) grep6.unfold)
  apply (fastforce simp: zero-Integer-def one-Integer-def split: if-splits dest!: step(1)[rotated] step(3)[rotated]
elim!: slengthE)
  done
qed
qed (use 3 in simp) }
{ case (4 vs x) show ?case
proof (cases prefix[:x:].vs)
  case bottom then show ?thesis
    apply (subst grep6.unfold)
    apply (subst grep5.unfold)
    apply auto
  done
next
case TT with ⟨pat ≠ ⊥⟩ ⟨us ≠ ⊥⟩ show ?thesis
  apply (subst grep6.unfold)
  apply (subst grep5.unfold)
  apply (cases vs; clarsimp simp: op6-left6)
  apply (cases us; clarsimp simp del: left6.simps left5.simps simp add: root5-left5)
  apply (metis (no-types, lifting) op5-sfoldl-left5 root5-left5 root-bisim.simps sappend-bottom-iff slist.con-rews(3)
slist.con-rews(4))
  done
next
case FF with ⟨pat ≠ ⊥⟩ ⟨us ≠ ⊥⟩ show ?thesis
  apply (subst grep6.unfold)
  apply (subst grep5.unfold)
  using rbl
  apply (auto simp: root-bisim-op-next56 elim!: root-bisim.cases intro: root-bisim-root)
  apply (subst (asm) grep6.unfold)
  apply (cases us; fastforce dest: step(3)[rotated] intro: step(4))
  done
qed }
{ case 5 show ?case
proof (cases us rule: srev-cases)
  case (ssnoc u' us')
  then have root-bisim pat (?for6 us') (?for5 us') by (fastforce intro: step(5))
  then show ?thesis
proof cases
  case (gl us'' vs'') with ssnoc show ?thesis
    apply clarsimp
    apply (subst (asm) grep6.unfold)
    apply (fastforce dest: step(3)[rotated] intro: step(4))
  done
  qed (use ⟨pat ≠ ⊥⟩ ssnoc root-bisim-root in auto)
qed (use ⟨pat ≠ ⊥⟩ root-bisim-root in auto) }
{ case (6 vs)
from rbl ⟨pat ≠ ⊥⟩ show rbnl: ?rbnl us vs
proof cases
  case bottom then show ?thesis by fastforce
next
case Null then show ?thesis by (cases vs) auto

```

```

next
  case (gl us'' vs'') then show ?thesis
    apply clarsimp
    apply (subst grep5.unfold)
    apply (subst grep6.unfold)
    apply (subst (asm) grep5.unfold)
    apply (subst (asm) grep6.unfold)
    apply (cases us; clarsimp; cases vs''; clarsimp)
      apply (metis Rep-cfun-strict1 bottom left5.simps(2) left6.simps(2) next-snil next-strict(1) rbl)
      apply (cases vs; clarsimp)
    using rbl apply (fastforce dest: step(3)[rotated] intro: step(6) simp: If2-def[symmetric] simp del: eq-FF
split: If2-splits)+
  done
qed }
qed
from ⟨pat ≠ ⊥⟩ this(4) show root-bisim pat t6 t5 ⇒ root-bisim pat (op6.pat.t6.x) (op5.pat.t5.x)
  by (auto elim!: root-bisim.cases intro: root-bisim-root)
show ⟨root-bisim pat (next-vs.(left6.pat.us)) (next-vs.(left5.pat.us))⟩ by fact
qed

```

With this result in hand the remainder is technically fiddly but straightforward.

```

lemmas tree-bisimI = iffD2[OF fun-cong[OF tree-bisim-def[unfolded atomize-eq]], rule-format]

```

```

lemma tree-bisim-root-bisim:

```

```

  shows tree-bisim (root-bisim pat)
proof(rule tree-bisimI, erule root-bisim.cases, goal-cases bottom Null Node)
  case (Node x y us vs) then show ?case
    apply (subst (asm) grep5.unfold)
    apply (subst (asm) grep6.unfold)
    apply hypsubst-thin
    apply (clarsimp simp: root-bisim-next-left)
    apply (cases vs; clarsimp)
    apply (cases us; clarsimp simp del: left6.simps left5.simps simp add: op5-sfoldl-left5 op6-left6)
      apply (metis (no-types, lifting) root5-left5 root-bisim.simps slist.con-rews(3,4))
      apply (metis (no-types, lifting) op5-sfoldl-left5 root-bisim.simps sappend-bottom-iff slist.con-rews(3,4))
    done
qed simp-all

```

```

lemma r6-5:

```

```

  shows (root6.pat, op6.pat) = (root5.pat, op5.pat)
proof(cases pat = ⊥)
  case True
  from True have root6.pat = root5.pat
    apply (subst root6.unfold)
    apply (subst grep6.unfold)
    apply (subst root5.unfold)
    apply (subst grep5.unfold)
    apply simp
  done
  moreover
  from True ⟨root6.pat = root5.pat⟩ have op6.pat.t.x = op5.pat.t.x for t x
    by (induct t) simp-all
  ultimately show ?thesis by (simp add: cfun-eq-iff)

```

```

next

```

```

  case False
  then have root: root6.pat = root5.pat
    by (rule tree.coinduct[OF tree-bisim-root-bisim root-bisim-root])
  moreover

```

from *root* **have** $op6.pat.t.x = op5.pat.t.x$ **for** $t\ x$ **by** (*induct* t) *simp-all*
ultimately show *?thesis* **by** (*simp add: cfun-eq-iff*)
qed

We conclude this section by observing that accumulator-introduction is a well known technique (see, for instance, Hutton (2016, §13.6)), but the examples in the literature assume that the type involved is defined inductively. Bird adopts this strategy without considering what the mixed inductive/coinductive rule is that justifies the preservation of total correctness.

The difficulty of this step is why we wired in the ‘K’ opt earlier: it allows us to preserve the shape of the tree all the way from the data refinement to the final version.

4.7 Step 7: Simplify, unfold prefix

The next step (Bird, bottom of p132) is to move the case split in *grep6* on *vs* above the *Node* constructor, which makes *grep7* strict in that parameter and therefore not extensionally equal to *grep6*. We establish a weaker correspondence using fixed-point induction.

We also unfold *prefix* in *op6*.

fixrec

```

  root7 :: [:'a::Eq-def:] → ([:'a:] × [:'a:]) tree
and op7  :: [:'a::Eq-def:] → ([:'a:] × [:'a:]) tree → 'a → ([:'a:] × [:'a:]) tree
and grep7 :: [:'a:] → ([:'a:] × [:'a:]) tree → [:'a:] × [:'a:] → ([:'a:] × [:'a:]) tree
where
  [simp del]:
  root7.pat = grep7.pat.Null.([::], pat)
| op7.pat.Null.x = root7.pat
| op7.pat.(Node.(us, [::]).l.r).x = op7.pat.l.x — Unfold prefix
| [v ≠ ⊥; vs ≠ ⊥] ⇒
  op7.pat.(Node.(us, v :# vs).l.r).x = If eq.x.v then r else op7.pat.l.x
| [simp del]:
  grep7.pat.l.(us, [::]) = Node.(us, [::]).l.Null — Case split on vs hoisted above Node.
| [v ≠ ⊥; vs ≠ ⊥] ⇒
  grep7.pat.l.(us, v :# vs) = Node.(us, v :# vs).(next.(v :# vs).l).(grep7.pat.(op7.pat.l.v).(us :@ [:v:], vs))

```

schematic-goal *root7-op7-grep7-def*:

```

( root7 :: [:'a::Eq-def:] → ([:'a:] × [:'a:]) tree
, op7  :: [:'a:] → ([:'a:] × [:'a:]) tree → 'a → ([:'a:] × [:'a:]) tree
, grep7 :: [:'a:] → ([:'a:] × [:'a:]) tree → [:'a:] × [:'a:] → ([:'a:] × [:'a:]) tree )
= fix.?F

```

unfolding *root7-def op7-def grep7-def* **by** *simp*

lemma *r7-6-aux*:

assumes $pat \neq \perp$

shows

$(\Lambda (root, op, grep). (root.pat, seq.x.(op.pat.t.x), grep.pat.l.(us, vs))) \cdot$

```

( root7 :: [:'a::Eq-def:] → ([:'a:] × [:'a:]) tree
, op7  :: [:'a::Eq-def:] → ([:'a:] × [:'a:]) tree → 'a → ([:'a:] × [:'a:]) tree
, grep7 :: [:'a:] → ([:'a:] × [:'a:]) tree → [:'a:] × [:'a:] → ([:'a:] × [:'a:]) tree )
= (\Lambda (root, op, grep). (root.pat, seq.x.(op.pat.t.x), seq.vs.(grep.pat.l.(us, vs)))) \cdot
( root6 :: [:'a::Eq-def:] → ([:'a:] × [:'a:]) tree
, op6  :: [:'a::Eq-def:] → ([:'a:] × [:'a:]) tree → 'a → ([:'a:] × [:'a:]) tree
, grep6 :: [:'a:] → ([:'a:] × [:'a:]) tree → [:'a:] × [:'a:] → ([:'a:] × [:'a:]) tree )

```

unfolding *root6-op6-grep6-def root7-op7-grep7-def*

proof(*induct arbitrary: t x l us vs rule: parallel-fix-ind[case-names adm bottom step]*)

case (*step X Y t x l us vs*) **then show** *?case*

apply –

apply (*cases X, cases Y*)

apply (*rename-tac r10 o10 g10 r9 o9 g9*)

apply (*clarsimp simp: cfun-eq-iff assms match-Node-mplus-match-Node match-Null-match-Node-tree-case tree-case-distr match-snil-match-scons-slist-case slist-case-distr If-distr*)

apply (*intro allI conjI*)
apply (*case-tac t; clarsimp*)
apply (*rename-tac us vs l r*)
apply (*case-tac x = ⊥; clarsimp*)
apply (*case-tac vs; clarsimp; fail*)
apply (*case-tac vs; clarsimp*)
apply (*metis ID1 seq-simps(3)*)
done

qed *simp-all*

lemma *r7-6*:

assumes *pat ≠ ⊥*
shows *root7.pat = root6.pat*
and *x ≠ ⊥ ⇒ op7.pat.t.x = op6.pat.t.x*

using *r7-6-aux[OF assms]* **by** (*force simp: cfun-eq-iff dest: spec[where x=x]*)+

4.8 Step 8: Discard us

We now discard *us* from the tree as it is no longer used. This requires a new definition of *next*. This is essentially another data refinement.

fixrec *next'* :: *'a::Eq-def* → [*'a*] *tree* → [*'a*] *tree* **where**
next'.x.Null = Null
| *next'.x.(Node.[::].l.r) = Node.[::].l.r*
| $\llbracket v \neq \perp; vs \neq \perp; x \neq \perp \rrbracket \Longrightarrow$
next'.x.(Node.(v :# vs).l.r) = If eq.x.v then l else Node.(v :# vs).l.r

fixrec — Bird p133

root8 :: [*'a::Eq-def*] → [*'a*] *tree*
and *op8* :: [*'a*] → [*'a*] *tree* → *'a* → [*'a*] *tree*
and *grep8* :: [*'a*] → [*'a*] *tree* → [*'a*] → [*'a*] *tree*
where
[*simp del*]:
root8.pat = grep8.pat.Null.pat
| *op8.pat.Null.x = root8.pat*
| *op8.pat.(Node.[::].l.r).x = op8.pat.l.x*
| $\llbracket v \neq \perp; vs \neq \perp \rrbracket \Longrightarrow$
op8.pat.(Node.(v :# vs).l.r).x = If eq.x.v then r else op8.pat.l.x
| *grep8.pat.l.[::] = Node.[::].l.Null*
| $\llbracket v \neq \perp; vs \neq \perp \rrbracket \Longrightarrow$
grep8.pat.l.(v :# vs) = Node.(v :# vs).(next'.v.l).(grep8.pat.(op8.pat.l.v).vs)

fixrec *ok8* :: [*'a*] *tree* → *tr* **where**
vs ≠ ⊥ ⇒ ok8.(Node.vs.l.r) = snull.vs

schematic-goal *root8-op8-grep8-def*:

(*root8* :: [*'a::Eq-def*] → [*'a*] *tree*
, *op8* :: [*'a*] → [*'a*] *tree* → *'a* → [*'a*] *tree*
, *grep8* :: [*'a*] → [*'a*] *tree* → [*'a*] → [*'a*] *tree*)
= *fix.?F*

unfolding *op8-def root8-def grep8-def* **by** *simp*

lemma *next'-strict[simp]*:

next'.x.⊥ = ⊥
next'.⊥.(Node.(v :# vs).l.r) = ⊥

by (*cases v :# vs = ⊥; fixrec-simp; clarsimp*)+

lemma *root8-op8-grep8-strict*[*simp*]:

grep8.pat.l. \perp = \perp
op8.pat. \perp = \perp
root8. \perp = \perp

by *fixrec-simp+*

lemma *ok8-strict*[*simp*]:

ok8. \perp = \perp
ok8.Null = \perp

by *fixrec-simp+*

We cannot readily relate *next* and *next'* using worker/wrapper as the obvious abstraction is not invertible. Conversely the desired result is easily shown by fixed-point induction.

lemma *next'-next*:

assumes $v \neq \perp$
assumes $vs \neq \perp$
shows $next'.v \cdot (tree-map'.csnd.t) = tree-map'.csnd.(next \cdot (v :\# vs) \cdot t)$

proof(*induct t*)

case (*Node usvs' l r*) **with** *assms* **show** ?*case*
apply (*cases usvs'*; *clarsimp*)
apply (*rename-tac us'' vs''*)
apply (*case-tac vs''*; *clarsimp simp: If-distr*)
done

qed (*use assms in simp-all*)

lemma *r8-7*[*simplified*]:

shows $(\Lambda (root, op, grep). (root.pat$
 $, op.pat \cdot (tree-map'.csnd.t) \cdot x$
 $, grep.pat \cdot (tree-map'.csnd.l) \cdot (csnd.usvs))) \cdot (root8, op8, grep8)$
 $= (\Lambda (root, op, grep). (tree-map'.csnd \cdot (root.pat)$
 $, tree-map'.csnd \cdot (op.pat \cdot t \cdot x)$
 $, tree-map'.csnd \cdot (grep.pat \cdot l \cdot usvs))) \cdot (root7, op7, grep7)$

unfolding *root7-op7-grep7-def root8-op8-grep8-def*

proof(*induct arbitrary: l t usvs x rule: parallel-fix-ind[case-names adm bottom step]*)

case (*step X Y l t usvs x*) **then show** ?*case*
apply –
apply (*cases X*; *cases Y*)
apply (*clarsimp simp: cfun-eq-iff next'-next*
 $match-snil-match-scons-slist-case$ *slist-case-distr*
 $match-Node-mplus-match-Node$ *match-Null-match-Node-tree-case* *tree-case-distr*
 $cong: slist-case-cong$)
apply (*cases t*; *clarsimp simp: If-distr*)
apply (*rename-tac us vs l r*)
apply (*case-tac vs*; *fastforce*)
done

qed *simp-all*

Top-level equivalence follows from *lfp-fusion* specialized to *sscanl* (*sscanl-lfp-fusion*), which states that

$$smap.g \circ\circ sscanl.f.z = sscanl.f'.(g.z)$$

provided that *g* is strict and the following diagram commutes for $x \neq \perp$:

$$\begin{array}{ccc} a & \xrightarrow{\Lambda a. f \cdot a \cdot x} & a' \\ \downarrow g & & \downarrow g \\ b & \xrightarrow{\Lambda a. f' \cdot a \cdot x} & b' \end{array}$$

lemma *ok8-ok8*: *ok8 oo tree-map'.csnd = snull oo csnd oo abs2* (**is** *?lhs = ?rhs*)
proof(*rule cfun-eqI*)
 fix *t* **show** *?lhs·t = ?rhs·t*
 by (*cases t; clarsimp*) (*metis ok8.simps ok8-strict(1) snull-strict tree.con-rews(3)*)
qed

lemma *matches8*: — Bird p133

shows *matches.pat = smap·cfst oo sfilter·(ok8 oo csnd) oo sscanl·(Λ (n, x) y. (n + 1, op8·pat·x·y))·(0, root8·pat)*
(**is** *?lhs = ?rhs*)

proof(*cases pat = ⊥*)

case *True*

then have *?lhs·xs = ?rhs·xs for xs* **by** (*cases xs; clarsimp*)

then show *?thesis* **by** (*simp add: cfun-eq-iff*)

next

case *False*

then have ***: *matches.pat = smap·cfst oo sfilter·(snull oo csnd oo abs2 oo csnd) oo sscanl·(Λ (n, x) y. (n + 1, op7·pat·x·y))·(0, root7·pat)*

using *data-refinement*[**where** *'a='a*] *r3-2*[**where** *'a='a*] *r4-3*[**where** *'a='a*] *r5-4*[**where** *'a='a*] *r6-5(1)*[**where** *pat=pat*] *r7-6*[**where** *pat=pat*]

unfolding *matches2.unfold* **by** (*fastforce simp: oo-assoc cfun-eq-iff csplit-def intro!: cfun-arg-cong sscanl-cong*)

from *⟨pat ≠ ⊥⟩* **show** *?thesis*

apply —

apply (*subst conjunct1*[*OF r8-7*])

apply (*subst sscanl-lfp-fusion*[**where** *g=ID ** tree-map'.csnd* **and** *z=(0, root7·pat)*, *simplified*, *symmetric*])

prefer 2

apply (*subst oo-assoc, subst sfilter-smap, simp*)

apply (*simp add: oo-assoc*)

apply (*simp add: oo-assoc[symmetric]*)

apply (*subst oo-assoc, subst ok8-ok8*)

apply (*clarsimp simp: oo-assoc **)

apply (*rule refl*)

apply (*clarsimp simp: r8-7*)

done

qed

4.9 Step 9: Factor out pat (final version)

Finally we factor *pat* from these definitions and arrive at Bird's cyclic data structure, which when executed using lazy evaluation actually memoises the computation of *grep8*.

The *Letrec* syntax groups recursive bindings with *,* and separates these with *;*. Its lack of support for clausal definitions, and that *HOLCF case* does not support nested patterns, leads to some awkwardness.

fixrec *matchesf* :: *[:'a::Eq-def:] → [:'a:] → [:Integer:]* **where**

[simp del]: matchesf.pat =

 (*Letrec okf = (Λ (Node·vs·l·r). snull·vs);*

nextf = (Λ x t. case t of

Null ⇒ Null

 | *Node·vs·l·r ⇒ (case vs of*

[::] ⇒ t

 | *v :# vs ⇒ If eq·x·v then l else t));*

rootf = grepf·Null·pat,

opf = (Λ t x. case t of

Null ⇒ rootf

 | *Node·vs·l·r ⇒ (case vs of*

[::] ⇒ opf·l·x

 | *v :# vs ⇒ If eq·x·v then r else opf·l·x));*

grepf = (Λ l vs. case vs of

[::] ⇒ Node·[::]·l·Null

```

| v :# vs ⇒ Node.(v :# vs).(nextf.v.l).(grepf.(opf.l.v).vs));
stepf = (Λ (n, t) x. (n + 1, opf.t.x))
in smap.cfst oo sfilter.(okf oo csnd) oo sscanl.stepf.(0, rootf))

```

lemma *matchesf-ok8*: (Λ (Node.v.s.l.r). snull.v.s) = ok8
by (clarsimp simp: cfun-eq-iff; rename-tac x; case-tac x; clarsimp)

lemma *matchesf-next'*:

```

(Λ x t. case t of Null ⇒ Null | Node.v.s.l.r ⇒ (case vs of [::] ⇒ t | v :# vs ⇒ If eq.x.v then l else t)) = next'
apply (clarsimp simp: cfun-eq-iff next'.unfold
      match-snil-match-scons-slist-case slist-case-distr
      match-Node-mplus-match-Node match-Null-match-Node-tree-case tree-case-distr)
apply (simp cong: tree-case-cong)
apply (simp cong: slist-case-cong)
done

```

lemma *matchesf-8*:

```

fix.(Λ (Rootf, Opf, Grepf).
  ( Grepf.Null.pat
    , Λ t x. case t of Null ⇒ Rootf | Node.v.s.l.r ⇒
      (case vs of [::] ⇒ Opf.l.x | v :# vs ⇒ If eq.x.v then r else Opf.l.x)
    , Λ l vs. case vs of [::] ⇒ Node.[::].l.Null | v :# vs ⇒ Node.(v :# vs).(next'.v.l).(Grepf.(Opf.l.v).vs) )
= (Λ (root, op, grep). (root.pat, op.pat, grep.pat)).(root8, op8, grep8)
unfolding root8-op8-grep8-def
by (rule lfp-fusion[symmetric])
  (fastforce simp: cfun-eq-iff
    match-snil-match-scons-slist-case slist-case-distr
    match-Node-mplus-match-Node match-Null-match-Node-tree-case tree-case-distr)+

```

theorem *matches-final*:

```

shows matches = matchesf
by (clarsimp simp: cfun-eq-iff fix-const eta-cfun csplit-cfun3 CLetrec-def
      matches8 matchesf.unfold matchesf-next' matchesf-ok8 matchesf-8[simplified eta-cfun])

```

The final program above is easily syntactically translated into the Haskell shown in Figure 1, and one can expect GHC's list fusion machinery to compile the top-level driver into an efficient loop. Lochbihler and Maximova (2015) have mechanised this optimisation for Isabelle/HOL's code generator (and see also Huffman (2009)).

As we lack both pieces of infrastructure we show such a fusion is sound by hand.

lemma *fused-driver'*:

```

assumes g.⊥ = ⊥
assumes p.⊥ = ⊥
shows smap.g oo sfilter.p oo sscanl.f.z
  = (μ R. Λ z x.s. case x.s of
      [::] ⇒ If p.z then [g.z] else [::]
      | x :# xs ⇒ let z' = f.z.x in If p.z then g.z :# R.z'.xs else R.z'.xs).z
(is ?lhs = ?rhs)
proof(rule cfun-eqI)
  fix xs from assms show ?lhs.xs = ?rhs.xs
  by (induct xs arbitrary: z) (subst fix-eq; fastforce simp: If-distr Let-def)+
qed

```

5 Related work

Derivations of KMP matching are legion and we do not attempt to catalogue them here.

Bird and colleagues have presented versions of this story at least four times. All treat MP, not KMP (see §4.2), and use a style of equational reasoning with fold/unfold transformations (Burstall and Darlington 1977) that only

establishes partial correctness (see §1.1). Briefly:

- The second example of Bird (1977) is an imperative program that is similar to MP.
- Bird et al. (1989) devised the core of the derivation mechanized here, notably omitting a formal justification for the final data refinement step that introduces the circular data structure.
- Bird (2005) refines Bird et al. (1989) and derives Boyer-Moore matching (Gusfield 1997, §2.2) in a similar style.
- Bird (2010, Chapter 17) further refines Bird (2005) and is the basis of the work discussed here. Bird (2012, §3.1) contains some further relevant remarks.

Ager et al. (2006) show how KMP matchers (specialised to a given pattern) can be derived by the partial evaluation of an initial program in linear time. We observe that neither their approach, of incorporating the essence of KMP in their starting point, nor Bird’s of introducing it by data refinement (§4.2), provides a satisfying explanation of how KMP could be discovered; Pottier (2012) attempts to do this. In contrast to Bird, these and most other presentations make heavy use of arrays and array indexing which occludes the central insights.

6 Implementations

With varying amounts of effort we can translate our final program of §4.9 into a variety of languages. The most direct version, in Haskell, was shown in Figure 1. An ocaml version is similar due to that language’s support for laziness. In contrast Standard ML requires an encoding; we use backpatching as shown in Figure 4. In both cases the tree datatype can be made strict in the right branch as it is defined by primitive recursion on the pattern.

More interestingly, our derivation suggests that Bird’s KMP program can be computed using *rational* trees (also known as *regular* trees (Courcelle 1983)), which are traditionally supported by Prolog implementations. Our version is shown in Figure 3. This demonstrates that the program could instead be thought of as a computation over difference structures. Colmerauer (1982); Giannesini and Cohen (1984) provide more examples of this style of programming. We leave a proof of correctness to future work.

7 Concluding remarks

Our derivation leans heavily on domain theory’s ability to reason about partially-defined objects that are challenging to handle at present in a language of total functions. Conversely it is too abstract to capture the operational behaviour of the program as it does not model laziness. It would also be interesting to put the data refinement of §4.2 on a firmer foundation by deriving the memoizing datatype from the direct program of §4.1. Haskell fans may care to address the semantic discrepancies mentioned in §1.1.

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```

% -*- mode: prolog -*-
% Bird's Morris-Pratt string matcher
% Chapter 17, "Pearls of Functional Algorithm Design", 2010.
% - adapted to use rational trees.
% - with the 'K' (next) optimisation
% Tested with SWI Prolog, which has good support for rational trees.

% root/2 (+, -) det
root(Ws, T) :- grep(T, null, Ws, T).

% op/4 (?, +, +, -) det <-- Root may or may not be fully ground
op(Root, null, _X, Root).
op(Root, node([], L, _R), X, T) :- op(Root, L, X, T).
op(Root, node([V|_Vs], L, R), X, T) :-
    (X = V -> T = R ; op(Root, L, X, T)).

% next/3 (+, +, -) det
next(_X, null, null).
next(_X, node([], L, R), node([], L, R)).
next(X, node([V|Vs], L, R), T) :- ( X = V -> T = L ; T = node([V|Vs], L, R) ).

% grep/4 (+, +, +, -) det
grep(_Root, L, [], node([], L, null)).
grep(Root, L, [V|Vs], node([V|Vs], L1, R)) :-
    next(V, L, L1), op(Root, L, V, T), grep(Root, T, Vs, R).

% ok/1 (+) det
ok(node([], _L, _R)).

%% Driver

% matches_aux/5 (+, +, +, +, -) det
matches_aux(_Root, N, T, [], Ns) :- ( ok(T) -> Ns = [N] ; Ns = [] ).
matches_aux(Root, N, T, [X|Xs], Ns) :-
    N1 is N + 1, op(Root, T, X, T1),
    ( ok(T) -> ( Ns = [N|Ns1], matches_aux(Root, N1, T1, Xs, Ns1) )
      ; matches_aux(Root, N1, T1, Xs, Ns) ).

% matches/3 (+, +, -) det
matches(Ws, Txt, Ns) :- root(Ws, Root), matches_aux(Root, 0, Root, Txt, Ns).

% :- root([1,2,1], Root).
% :- root([1,2,1,1,2], Root).
% :- matches([1,2,3,1,2], [1,2,1,2,3,1,2,3,1,2], Ns).

```

Figure 3: The final KMP program transliterated into Prolog.

```

(* Bird's Morris-Pratt string matcher
   Chapter 17, "Pearls of Functional Algorithm Design", 2010.
   - with the 'K' (next) optimisation
   - using backpatching
*)

structure KMP :> sig val kmatches : ('a * 'a -> bool) -> 'a list -> 'a list -> int list end =
struct

datatype 'a thunk = Val of 'a | Thunk of unit -> 'a

type 'a lazy = 'a thunk ref

fun lazy (f: unit -> 'a) : 'a lazy =
  ref (Thunk f)

fun force (su : 'a lazy) : 'a =
  case !su of
    Val v => v
  | Thunk f => let val v = f () in su := Val v; v end

datatype 'a tree
  = Null
  | Node of 'a list * 'a tree lazy * 'a tree

type 'a ltree = 'a tree lazy

fun kmatches (eq: 'a * 'a -> bool) (ws: 'a list) : 'a list -> int list =
  let
    fun ok (t: 'a ltree) : bool = case force t of Node ([], l, r) => true | _ => false
    fun next (x: 'a) (t: 'a ltree) : 'a ltree =
      lazy (fn () => let val t = force t in case t of
        Null => Null
      | Node ([], _, _) => t
      | Node (v :: vs, l, _) => if eq (x, v) then force l else t end)
    (* Backpatching! *)
    val root : 'a ltree = lazy (fn () => raise Fail "blackhole")
    fun op' (t: 'a ltree) (x: 'a) : 'a ltree =
      lazy (fn () => case force t of
        Null => force root
      | Node (vvs, l, r) =>
        (case vvs of
          [] => force (op' l x)
        | v :: vs => if eq (x, v) then r else force (op' l x)))
    and grep (l: 'a ltree) (vvs: 'a list): 'a tree =
      ( (* print "grep: produce node\n"; *) case vvs of
        [] => Node ([], l, Null)
      | v :: vs => Node (vvs, next v l, grep (op' l v) vs) )
    val () = root := Thunk (fn () => grep (lazy (fn () => Null)) ws)
    fun step ((n, t): int * 'a ltree) (x: 'a) : int * 'a ltree = (n + 1, op' t x)
    fun rheight (t: 'a tree) =
      case t of Null => 0 | Node (_, _, r) => 1 + rheight r
    fun driver ((n, t): int * 'a ltree) (xxs: 'a list) : int list =
      case xxs of
        [] => if ok t then [n] else []
      | x :: xs => let val nt' = step (n, t) x
        in if ok t then n :: driver nt' xs else driver nt' xs end
  in
    driver (0, root)
  end
end;

```

Figure 4: The final KMP program transliterated into Standard ML.

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