The BKR Decision Procedure for Univariate Real Arithmetic

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Abstract

We formalize the univariate case of Ben-Or, Kozen, and Reif's decision procedure for first-order real arithmetic [1] (the BKR algorithm). We also formalize the univariate case of Renegar's variation [2] of the BKR algorithm. The two formalizations differ mathematically in minor ways (that have significant impact on the multivariate case), but are quite similar in proof structure. Both rely on sign-determination (finding the set of consistent sign assignments for a set of polynomials). The method used for sign-determination is similar to Tarski's original quantifier elimination algorithm (it stores key information in a matrix equation), but with a reduction step to keep complexity low.

Remark

Note that theories BKR_Decision and Renegar_Decision inherit oracles `holds_by_evaluation` and `cancel_type_definition` from Berlekamp_Zassenhaus.

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theory *More-Matrix*
imports *Jordan-Normal-Form.Matrix*
Jordan-Normal-Form.DL-Rank
Jordan-Normal-Form.VS-Connect
Jordan-Normal-Form.Gauss-Jordan-Elimination
begin

1 Kronecker Product

definition *kronecker-product* :: *'a* :: *ring mat* ⇒ *'a mat* ⇒ *'a mat* **where**
kronecker-product A B =
(*let ra* = *dim-row A*; *ca* = *dim-col A*;
rb = *dim-row B*; *cb* = *dim-col B*
in
mat (*ra*∗*rb*) (*ca*∗*cb*)
(λ(*i*,*j*).
A $$ (*i div rb*, *j div cb*) ∗
B $$ (*i mod rb*, *j mod cb*)
))

lemma *arith*:
assumes *d* < *a*
assumes *c* < *b*
shows *b*∗*d*+*c* < *a*∗(*b*::*nat*)
⟨*proof*⟩

lemma *dim-kronecker*[*simp*]:
dim-row (*kronecker-product A B*) = *dim-row A* ∗ *dim-row B*
dim-col (*kronecker-product A B*) = *dim-col A* ∗ *dim-col B*
⟨*proof*⟩

lemma *kronecker-inverse-index*:
assumes *r* < *dim-row A s* < *dim-col A*
assumes *v* < *dim-row B w* < *dim-col B*
shows *kronecker-product A B* $$ (*dim-row B*∗*r*+*v*, *dim-col B*∗*s*+*w*) = *A* $$ (*r*,*s*)
∗ *B* $$ (*v*,*w*)
⟨*proof*⟩

lemma *kronecker-distr-left*:
assumes *dim-row B* = *dim-row C dim-col B* = *dim-col C*
shows *kronecker-product A* (*B*+*C*) = *kronecker-product A B* + *kronecker-product*
A C
⟨*proof*⟩

lemma *kronecker-distr-right*:
assumes *dim-row B* = *dim-row C dim-col B* = *dim-col C*
shows *kronecker-product* (*B*+*C*) *A* = *kronecker-product B A* + *kronecker-product*

C A
⟨*proof*⟩

lemma *index-mat-mod*[*simp*]: $nr > 0$ & $nc > 0 \Longrightarrow$ *mat nr nc f* $$ (*i mod nr*,*j mod nc*) = *f* (*i mod nr*,*j mod nc*)
⟨*proof*⟩

lemma *kronecker-assoc*:
shows *kronecker-product A* (*kronecker-product B C*) = *kronecker-product* (*kronecker-product A B*) *C*
⟨*proof*⟩

lemma *sum-sum-mod-div*:
(∑ *ia* = *0*::*nat*..<*x*. ∑ *ja* = *0*..<*y*. *f ia ja*) =
(∑ *ia* = *0*..<*x*∗*y*. *f* (*ia div y*) (*ia mod y*))
⟨*proof*⟩

lemma *kronecker-of-mult*:
assumes *dim-col* (*A* :: *'a* :: *comm-ring mat*) = *dim-row C*
assumes *dim-col B* = *dim-row D*
shows *kronecker-product A B* ∗ *kronecker-product C D* = *kronecker-product* (*A* ∗ *C*) (*B* ∗ *D*)
⟨*proof*⟩

lemma *inverts-mat-length*:
assumes *square-mat A inverts-mat A B inverts-mat B A*
shows *dim-row B* = *dim-row A dim-col B* = *dim-col A*
⟨*proof*⟩

lemma *less-mult-imp-mod-less*:
m mod i < *i* **if** *m* < *n* ∗ *i* **for** *m n i* :: *nat*
⟨*proof*⟩

lemma *kronecker-one*:
shows *kronecker-product* ((1$_m$ *x*)::*'a* :: *ring-1 mat*) (1$_m$ *y*) = 1$_m$ (*x*∗*y*)
⟨*proof*⟩

lemma *kronecker-invertible*:
assumes *invertible-mat* (*A* :: *'a* :: *comm-ring-1 mat*) *invertible-mat B*
shows *invertible-mat* (*kronecker-product A B*)
⟨*proof*⟩

2 More DL Rank

instantiation *mat* :: (*conjugate*) *conjugate*
begin

definition *conjugate-mat* :: *'a* :: *conjugate mat* ⇒ *'a mat*

where *conjugate m = mat (dim-row m) (dim-col m) (λ(i,j). conjugate (m $$ (i,j)))*

lemma *dim-row-conjugate*[*simp*]: *dim-row (conjugate m) = dim-row m*
⟨*proof*⟩

lemma *dim-col-conjugate*[*simp*]: *dim-col (conjugate m) = dim-col m*
⟨*proof*⟩

lemma *carrier-vec-conjugate*[*simp*]: *m ∈ carrier-mat nr nc ⟹ conjugate m ∈ carrier-mat nr nc*
⟨*proof*⟩

lemma *mat-index-conjugate*[*simp*]:
shows *i < dim-row m ⟹ j < dim-col m ⟹ conjugate m $$ (i,j) = conjugate (m $$ (i,j))*
⟨*proof*⟩

lemma *row-conjugate*[*simp*]: *i < dim-row m ⟹ row (conjugate m) i = conjugate (row m i)*
⟨*proof*⟩

lemma *col-conjugate*[*simp*]: *i < dim-col m ⟹ col (conjugate m) i = conjugate (col m i)*
⟨*proof*⟩

lemma *rows-conjugate*: *rows (conjugate m) = map conjugate (rows m)*
⟨*proof*⟩

lemma *cols-conjugate*: *cols (conjugate m) = map conjugate (cols m)*
⟨*proof*⟩

instance
⟨*proof*⟩

end

abbreviation *conjugate-transpose* :: *'a*::*conjugate mat ⇒ 'a mat*
where *conjugate-transpose A ≡ conjugate* (A^T)

notation *conjugate-transpose* ((-^{H}) [*1000*])

lemma *transpose-conjugate*:
shows $(conjugate\ A)^T = A^H$
⟨*proof*⟩

lemma *vec-module-col-helper*:
fixes *A*:: (*'a* :: *field*) *mat*
shows (*0_v (dim-row A) ∈ LinearCombinations.module.span class-ring (⦇carrier =*

carrier-vec (*dim-row A*), *mult* = *undefined*, *one* = *undefined*, *zero* = 0_v (*dim-row A*), *add* = (+), *smult* = $(\cdot_v)$⦈ (*set* (*cols A*)))
⟨*proof*⟩

lemma *vec-module-col-helper2*:
fixes *A*:: ('*a* :: *field*) *mat*
shows ⋀*a x*. *x* ∈ *LinearCombinations.module.span class-ring*
⦇*carrier* = *carrier-vec* (*dim-row A*), *mult* = *undefined*, *one* = *undefined*,
zero = 0_v (*dim-row A*), *add* = (+), *smult* = $(\cdot_v)$⦈
(*set* (*cols A*)) ⟹
(⋀*a b v*. (*a* + *b*) $\cdot_v$ *v* = *a* $\cdot_v$ *v* + *b* $\cdot_v$ *v*) ⟹
a $\cdot_v$ *x*
∈ *LinearCombinations.module.span class-ring*
⦇*carrier* = *carrier-vec* (*dim-row A*), *mult* = *undefined*, *one* = *undefined*,
zero = 0_v (*dim-row A*), *add* = (+), *smult* = $(\cdot_v)$⦈
(*set* (*cols A*))
⟨*proof*⟩

lemma *vec-module-col*: *module* (*class-ring* :: '*a* :: *field ring*)
(*module-vec TYPE*('*a*)
(*dim-row A*)
⦇*carrier* :=
LinearCombinations.module.span
class-ring (*module-vec TYPE*('*a*) (*dim-row A*)) (*set* (*cols A*))⦈)
⟨*proof*⟩

lemma *vec-vs-col*: *vectorspace* (*class-ring* :: '*a* :: *field ring*)
(*module-vec TYPE*('*a*) (*dim-row A*)
⦇*carrier* :=
LinearCombinations.module.span
class-ring
(*module-vec TYPE*('*a*)
(*dim-row A*))
(*set* (*cols A*))⦈)
⟨*proof*⟩

lemma *cols-mat-mul-map*:
shows *cols* (*A* ∗ *B*) = *map* ((∗$_v$) *A*) (*cols B*)
⟨*proof*⟩

lemma *cols-mat-mul*:
shows *set* (*cols* (*A* ∗ *B*)) = (∗$_v$) *A* ‘ *set* (*cols B*)
⟨*proof*⟩

lemma *set-obtain-sublist*:
assumes *S* ⊆ *set ls*
obtains *ss* **where** *distinct ss S* = *set ss*
⟨*proof*⟩

lemma *mul-mat-of-cols*:
assumes $A \in$ *carrier-mat nr n*
assumes $\bigwedge j.\ j <$ *length cs* $\Longrightarrow$ *cs* ! $j \in$ *carrier-vec n*
shows $A *$ (*mat-of-cols n cs*) = *mat-of-cols nr* (*map* (($*_v$) A) *cs*)
⟨*proof*⟩

lemma *helper*:
fixes $x\ y\ z ::'a ::$ {*conjugatable-ring*, *comm-ring*}
shows $x * (y * z) = y * x * z$
⟨*proof*⟩

lemma *cscalar-prod-conjugate-transpose*:
fixes $x\ y ::'a ::$ {*conjugatable-ring*, *comm-ring*} *vec*
assumes $A \in$ *carrier-mat nr nc*
assumes $x \in$ *carrier-vec nr*
assumes $y \in$ *carrier-vec nc*
shows $x \cdot c\ (A *_v\ y) = (A^H *_v\ x) \cdot c\ y$
⟨*proof*⟩

lemma *mat-mul-conjugate-transpose-vec-eq-0*:
fixes $v ::'a ::$ {*conjugatable-ordered-ring*,*semiring-no-zero-divisors*,*comm-ring*}
vec
assumes $A \in$ *carrier-mat nr nc*
assumes $v \in$ *carrier-vec nr*
assumes $A *_v\ (A^H *_v\ v) = 0_v\ nr$
shows $A^H *_v\ v = 0_v\ nc$
⟨*proof*⟩

lemma *row-mat-of-cols*:
assumes $i < nr$
shows *row* (*mat-of-cols nr ls*) i = *vec* (*length ls*) ($\lambda j.$ (*ls* ! j) $\$i$)
⟨*proof*⟩

lemma *mat-of-cols-cons-mat-vec*:
fixes $v ::'a::$*comm-ring vec*
assumes $v \in$ *carrier-vec* (*length ls*)
assumes *dim-vec a* = *nr*
shows
mat-of-cols nr (*a* # *ls*) $*_v$ (*vCons m v*) =
$m \cdot_v\ a +$ *mat-of-cols nr ls* $*_v\ v$
⟨*proof*⟩

lemma *smult-vec-zero*:
fixes $v ::'a::$*ring vec*
shows $0 \cdot_v\ v = 0_v$ (*dim-vec v*)
⟨*proof*⟩

lemma *helper2*:

fixes A ::'a::*comm-ring mat*
fixes v ::'a *vec*
assumes $v \in$ *carrier-vec* (*length ss*)
assumes $\bigwedge x.\ x \in$ *set ls* $\Longrightarrow$ *dim-vec* $x =$ *nr*
shows
mat-of-cols nr ss $*_v$ $v =$
mat-of-cols nr (*ls* @ *ss*) $*_v$ (0_v (*length ls*) @$_v$ v)
⟨*proof*⟩

lemma *mat-of-cols-mult-mat-vec-permute-list*:
fixes v ::'a::*comm-ring list*
assumes *f permutes* {..<*length ss*}
assumes *length ss* = *length v*
shows
mat-of-cols nr (*permute-list f ss*) $*_v$ *vec-of-list* (*permute-list f v*) =
mat-of-cols nr ss $*_v$ *vec-of-list v*
⟨*proof*⟩

lemma *subindex-permutation*:
assumes *distinct ss set ss* ⊆ {..<*length ls*}
obtains *f* **where** *f permutes* {..<*length ls*}
permute-list f ls = *map* ((!) *ls*) (*filter* (λ*i*. *i* ∉ *set ss*) [*0*..<*length ls*]) @ *map* ((!) *ls*) *ss*
⟨*proof*⟩

lemma *subindex-permutation2*:
assumes *distinct ss set ss* ⊆ {..<*length ls*}
obtains *f* **where** *f permutes* {..<*length ls*}
ls = *permute-list f* (*map* ((!) *ls*) (*filter* (λ*i*. *i* ∉ *set ss*) [*0*..<*length ls*]) @ *map* ((!) *ls*) *ss*)
⟨*proof*⟩

lemma *distinct-list-subset-nths*:
assumes *distinct ss set ss* ⊆ *set ls*
obtains *ids* **where** *distinct ids set ids* ⊆ {..<*length ls*} *ss* = *map* ((!) *ls*) *ids*
⟨*proof*⟩

lemma *helper3*:
fixes A ::'a::*comm-ring mat*
assumes A: $A \in$ *carrier-mat nr nc*
assumes *ss*:*distinct ss set ss* ⊆ *set* (*cols A*)
assumes $v \in$ *carrier-vec* (*length ss*)
obtains c **where** *mat-of-cols nr ss* $*_v$ $v = A$ $*_v$ c *dim-vec c* = *nc*
⟨*proof*⟩

lemma *mat-mul-conjugate-transpose-sub-vec-eq-0*:
fixes A ::'a :: {*conjugatable-ordered-ring*,*semiring-no-zero-divisors*,*comm-ring*} *mat*

assumes $A \in$ *carrier-mat nr nc*
assumes *distinct ss set ss* $\subseteq$ *set* (*cols* (A^H))
assumes $v \in$ *carrier-vec* (*length ss*)
assumes $A *_v$ (*mat-of-cols nc ss* $*_v$ v) = 0_v *nr*
shows (*mat-of-cols nc ss* $*_v$ v) = 0_v *nc*
⟨*proof*⟩

lemma *Units-invertible*:
fixes *A*:: *'a*::*semiring-1 mat*
assumes $A \in$ *Units* (*ring-mat TYPE*('*a*) *n b*)
shows *invertible-mat A*
⟨*proof*⟩

lemma *invertible-Units*:
fixes *A*:: *'a*::*semiring-1 mat*
assumes *invertible-mat A*
shows $A \in$ *Units* (*ring-mat TYPE*('*a*) (*dim-row A*) *b*)
⟨*proof*⟩

lemma *invertible-det*:
fixes *A*:: *'a*::*field mat*
assumes $A \in$ *carrier-mat n n*
shows *invertible-mat A* $\longleftrightarrow$ *det A* $\neq$ *0*
⟨*proof*⟩

context *vec-space* **begin**

lemma *find-indices-distinct*:
assumes *distinct ss*
assumes $i <$ *length ss*
shows *find-indices* (*ss* ! *i*) *ss* = [*i*]
⟨*proof*⟩

lemma *lin-indpt-lin-comb-list*:
assumes *distinct ss*
assumes *lin-indpt* (*set ss*)
assumes *set ss* $\subseteq$ *carrier-vec n*
assumes *lincomb-list f ss* = 0_v *n*
assumes $i <$ *length ss*
shows *f i* = *0*
⟨*proof*⟩

lemma *span-mat-mul-subset*:
assumes $A \in$ *carrier-mat n d*
assumes $B \in$ *carrier-mat d nc*
shows *span* (*set* (*cols* (*A* ∗ *B*))) $\subseteq$ *span* (*set* (*cols A*))
⟨*proof*⟩

lemma *rank-mat-mul-right*:
assumes $A \in$ *carrier-mat n d*
assumes $B \in$ *carrier-mat d nc*
shows *rank* $(A * B) \leq$ *rank A*
⟨*proof*⟩

lemma *sumlist-drop*:
assumes $\bigwedge v.\ v \in$ *set ls* $\Longrightarrow$ *dim-vec v* $= n$
shows *sumlist ls* $=$ *sumlist* (*filter* $(\lambda v.\ v \neq 0_v\ n)$ *ls*)
⟨*proof*⟩

lemma *lincomb-list-alt*:
shows *lincomb-list c s* $=$
sumlist (*map2* $(\lambda i\ j.\ i \cdot_v s\ !\ j)$ (*map* $(\lambda i.\ c\ i)$ [*0..<length s*]) [*0..<length s*])
⟨*proof*⟩

lemma *lincomb-list-alt2*:
assumes $\bigwedge v.\ v \in$ *set s* $\Longrightarrow$ *dim-vec v* $= n$
assumes $\bigwedge i.\ i \in$ *set ls* $\Longrightarrow i <$ *length s*
shows
sumlist (*map2* $(\lambda i\ j.\ i \cdot_v s\ !\ j)$ (*map* $(\lambda i.\ c\ i)$ *ls*) *ls*) $=$
sumlist (*map2* $(\lambda i\ j.\ i \cdot_v s\ !\ j)$ (*map* $(\lambda i.\ c\ i)$ (*filter* $(\lambda i.\ c\ i \neq 0)$ *ls*)) (*filter* $(\lambda i.$ *c i* $\neq 0)$ *ls*))
⟨*proof*⟩

lemma *two-set*:
assumes *distinct ls*
assumes *set ls* $=$ *set* [*a*,*b*]
assumes $a \neq b$
shows *ls* $=$ [*a*,*b*] $\vee$ *ls* $=$ [*b*,*a*]
⟨*proof*⟩

lemma *filter-disj-inds*:
assumes $i <$ *length ls* $j <$ *length ls* $i \neq j$
shows *filter* $(\lambda ia.\ ia \neq j \longrightarrow ia = i)$ [*0..<length ls*] $=$ [*i*, *j*] $\vee$
filter $(\lambda ia.\ ia \neq j \longrightarrow ia = i)$ [*0..<length ls*] $=$ [*j*,*i*]
⟨*proof*⟩

lemma *lincomb-list-indpt-distinct*:
assumes $\bigwedge v.\ v \in$ *set ls* $\Longrightarrow$ *dim-vec v* $= n$
assumes
$\bigwedge c.$ *lincomb-list c ls* $= 0_v\ n \Longrightarrow (\forall i.\ i <$ (*length ls*) $\longrightarrow$ *c i* $= 0)$
shows *distinct ls*
⟨*proof*⟩

end

locale *conjugatable-vec-space* $=$ *vec-space f-ty n* **for**

f-ty::*'a*::*conjugatable-ordered-field itself*
and *n*
begin

lemma *transpose-rank-mul-conjugate-transpose*:
fixes *A* :: *'a mat*
assumes *A* ∈ *carrier-mat n nc*
shows *vec-space.rank nc* $A^H \leq$ *rank* $(A * A^H)$
⟨*proof*⟩

lemma *conjugate-transpose-rank-le*:
assumes *A* ∈ *carrier-mat n nc*
shows *vec-space.rank nc* $(A^H) \leq$ *rank A*
⟨*proof*⟩

lemma *conjugate-finsum*:
assumes *f*: *f* : *U* → *carrier-vec n*
shows *conjugate* (*finsum V f U*) = *finsum V* (*conjugate* ∘ *f*) *U*
⟨*proof*⟩

lemma *rank-conjugate-le*:
assumes *A*:*A* ∈ *carrier-mat n d*
shows *rank* (*conjugate* (*A*)) ≤ *rank A*
⟨*proof*⟩

lemma *rank-conjugate*:
assumes *A* ∈ *carrier-mat n d*
shows *rank* (*conjugate A*) = *rank A*
⟨*proof*⟩

end

lemma *conjugate-transpose-rank*:
fixes *A*::*'a*::{*conjugatable-ordered-field*} *mat*
shows *vec-space.rank* (*dim-row A*) *A* = *vec-space.rank* (*dim-col A*) (A^H)
⟨*proof*⟩

lemma *transpose-rank*:
fixes *A*::*'a*::{*conjugatable-ordered-field*} *mat*
shows *vec-space.rank* (*dim-row A*) *A* = *vec-space.rank* (*dim-col A*) (A^T)
⟨*proof*⟩

lemma *rank-mat-mul-left*:
fixes *A*::*'a*::{*conjugatable-ordered-field*} *mat*
assumes *A* ∈ *carrier-mat n d*
assumes *B* ∈ *carrier-mat d nc*
shows *vec-space.rank n* $(A * B) \leq$ *vec-space.rank d B*
⟨*proof*⟩

3 Results on Invertibility

definition *take-cols* :: *'a mat* ⇒ *nat list* ⇒ *'a mat*
where *take-cols A inds* = *mat-of-cols* (*dim-row A*) (*map* ((!) (*cols A*)) (*filter* ((>) (*dim-col A*)) *inds*))

definition *take-cols-var* :: *'a mat* ⇒ *nat list* ⇒ *'a mat*
where *take-cols-var A inds* = *mat-of-cols* (*dim-row A*) (*map* ((!) (*cols A*)) (*inds*))

definition *take-rows* :: *'a mat* ⇒ *nat list* ⇒ *'a mat*
where *take-rows A inds* = *mat-of-rows* (*dim-col A*) (*map* ((!) (*rows A*)) (*filter* ((>) (*dim-row A*)) *inds*))

lemma *cong1*:
x = *y* ⟹ *mat-of-cols n x* = *mat-of-cols n y*
⟨*proof*⟩

lemma *nth-filter*:
assumes *j* < *length* (*filter P ls*)
shows *P* ((*filter P ls*) ! *j*)
⟨*proof*⟩

lemma *take-cols-mat-mul*:
assumes *A* ∈ *carrier-mat nr n*
assumes *B* ∈ *carrier-mat n nc*
shows *A* ∗ *take-cols B inds* = *take-cols* (*A* ∗ *B*) *inds*
⟨*proof*⟩

lemma *take-cols-carrier-mat*:
assumes *A* ∈ *carrier-mat nr nc*
obtains *n* **where** *take-cols A inds* ∈ *carrier-mat nr n*
⟨*proof*⟩

lemma *take-cols-carrier-mat-strict*:
assumes *A* ∈ *carrier-mat nr nc*
assumes ⋀*i*. *i* ∈ *set inds* ⟹ *i* < *nc*
shows *take-cols A inds* ∈ *carrier-mat nr* (*length inds*)
⟨*proof*⟩

lemma *gauss-jordan-take-cols*:
assumes *gauss-jordan A* (*take-cols A inds*) = (*C*,*D*)
shows *D* = *take-cols C inds*
⟨*proof*⟩

lemma *dim-col-take-cols*:
assumes ⋀*j*. *j* ∈ *set inds* ⟹ *j* < *dim-col A*
shows *dim-col* (*take-cols A inds*) = *length inds*
⟨*proof*⟩

lemma *dim-col-take-rows*[*simp*]:
shows *dim-col* (*take-rows A inds*) = *dim-col A*
⟨*proof*⟩

lemma *cols-take-cols-subset*:
shows *set* (*cols* (*take-cols A inds*)) ⊆ *set* (*cols A*)
⟨*proof*⟩

lemma *dim-row-take-cols*[*simp*]:
shows *dim-row* (*take-cols A ls*) = *dim-row A*
⟨*proof*⟩

lemma *dim-row-append-rows*[*simp*]:
shows *dim-row* (*A* @$_r$ *B*) = *dim-row A* + *dim-row B*
⟨*proof*⟩

lemma *rows-inj*:
assumes *dim-col A* = *dim-col B*
assumes *rows A* = *rows B*
shows *A* = *B*
⟨*proof*⟩

lemma *append-rows-index*:
assumes *dim-col A* = *dim-col B*
assumes *i* < *dim-row A* + *dim-row B*
assumes *j* < *dim-col A*
shows (*A* @$_r$ *B*) $$ (*i*,*j*) = (*if i* < *dim-row A then A* $$ (*i*,*j*) *else B* $$ (*i*−*dim-row A*,*j*))
⟨*proof*⟩

lemma *row-append-rows*:
assumes *dim-col A* = *dim-col B*
assumes *i* < *dim-row A* + *dim-row B*
shows *row* (*A* @$_r$ *B*) *i* = (*if i* < *dim-row A then row A i else row B* (*i*−*dim-row A*))
⟨*proof*⟩

lemma *append-rows-mat-mul*:
assumes *dim-col A* = *dim-col B*
shows (*A* @$_r$ *B*) ∗ *C* = *A* ∗ *C* @$_r$ *B* ∗ *C*
⟨*proof*⟩

lemma *cardlt*:
shows *card* {*i*. *i* < (*n*::*nat*)} ≤ *n*
⟨*proof*⟩

lemma *row-echelon-form-zero-rows*:
assumes *row-ech*: *row-echelon-form A*
assumes *dim-asm*: *dim-col A* ≥ *dim-row A*

shows *take-rows A [0..<length (pivot-positions A)]* $@_r$ 0_m *(dim-row A − length (pivot-positions A)) (dim-col A) = A*
⟨*proof*⟩

lemma *length-pivot-positions-dim-row*:
assumes *row-echelon-form A*
shows *length (pivot-positions A) ≤ dim-row A*
⟨*proof*⟩

lemma *rref-pivot-positions*:
assumes *row-echelon-form R*
assumes *R*: *R ∈ carrier-mat nr nc*
shows ⋀*i j*. *(i,j) ∈ set (pivot-positions R) ⟹ i < nr ∧ j < nc*
⟨*proof*⟩

lemma *pivot-fun-monoton*:
assumes *pf*: *pivot-fun A f (dim-col A)*
assumes *dr*: *dim-row A = nr*
shows ⋀ *i*. *i < nr ⟹* (⋀ *k*. ((*k < nr ∧ i < k*) ⟶ *f i ≤ f k*))
⟨*proof*⟩

lemma *pivot-positions-contains*:
assumes *row-ech*: *row-echelon-form A*
assumes *dim-h*: *dim-col A ≥ dim-row A*
assumes *pivot-fun A f (dim-col A)*
shows ∀ *i < (length (pivot-positions A))*. *(i, f i) ∈ set (pivot-positions A)*
⟨*proof*⟩

lemma *pivot-positions-form-helper-1*:
shows *(a, b) ∈ set (pivot-positions-main-gen z A nr nc i j) ⟹ i ≤ a*
⟨*proof*⟩

lemma *pivot-positions-form-helper-2*:
shows *sorted-wrt (<) (map fst (pivot-positions-main-gen z A nr nc i j))*
⟨*proof*⟩

lemma *sorted-pivot-positions*:
shows *sorted-wrt (<) (map fst (pivot-positions A))*
⟨*proof*⟩

lemma *pivot-positions-form*:
assumes *row-ech*: *row-echelon-form A*
assumes *dim-h*: *dim-col A ≥ dim-row A*
shows ∀ *i < (length (pivot-positions A))*. *fst ((pivot-positions A) ! i) = i*
⟨*proof*⟩

lemma *take-cols-pivot-eq*:
assumes *row-ech*: *row-echelon-form A*
assumes *dim-h*: *dim-col A ≥ dim-row A*

shows *take-cols A* (*map snd* (*pivot-positions A*)) =
1$_m$ (*length* (*pivot-positions A*)) @$_r$
0$_m$ (*dim-row A* − *length* (*pivot-positions A*)) (*length* (*pivot-positions A*))
⟨*proof*⟩

lemma *rref-right-mul*:
assumes *row-echelon-form A*
assumes *dim-col A* ≥ *dim-row A*
shows
take-cols A (*map snd* (*pivot-positions A*)) ∗ *take-rows A* [*0*..<*length* (*pivot-positions A*)] = *A*
⟨*proof*⟩

context *conjugatable-vec-space* **begin**

lemma *lin-indpt-id*:
shows *lin-indpt* (*set* (*cols* (*1*$_m$ *n*)::*'a vec list*))
⟨*proof*⟩

lemma *lin-indpt-take-cols-id*:
shows *lin-indpt* (*set* (*cols* (*take-cols* (*1*$_m$ *n*) *inds*)))
⟨*proof*⟩

lemma *cols-id-unit-vecs*:
shows *cols* (*1*$_m$ *d*) = *unit-vecs d*
⟨*proof*⟩

lemma *distinct-cols-id*:
shows *distinct* (*cols* (*1*$_m$ *d*)::*'a vec list*)
⟨*proof*⟩

lemma *distinct-map-nth*:
assumes *distinct ls*
assumes *distinct inds*
assumes ⋀*j*. *j* ∈ *set inds* ⟹ *j* < *length ls*
shows *distinct* (*map* ((!) *ls*) *inds*)
⟨*proof*⟩

lemma *distinct-take-cols-id*:
assumes *distinct inds*
shows *distinct* (*cols* (*take-cols* (*1*$_m$ *n*) *inds*) :: *'a vec list*)
⟨*proof*⟩

lemma *rank-take-cols*:
assumes *distinct inds*
shows *rank* (*take-cols* (*1*$_m$ *n*) *inds*) = *length* (*filter* ((>) *n*) *inds*)
⟨*proof*⟩

lemma *rank-mul-left-invertible-mat*:

fixes A::'a *mat*
assumes *invertible-mat* A
assumes $A \in$ *carrier-mat* n n
assumes $B \in$ *carrier-mat* n *nc*
shows *rank* $(A * B) =$ *rank* B
⟨*proof*⟩

lemma *invertible-take-cols-rank*:
fixes A::'a *mat*
assumes *invertible-mat* A
assumes $A \in$ *carrier-mat* n n
assumes *distinct inds*
shows *rank* (*take-cols* A *inds*) = *length* (*filter* ((>) n) *inds*)
⟨*proof*⟩

lemma *rank-take-cols-leq*:
assumes R:$R \in$ *carrier-mat* n *nc*
shows *rank* (*take-cols* R *ls*) $\leq$ *rank* R
⟨*proof*⟩

lemma *rank-take-cols-geq*:
assumes R:$R \in$ *carrier-mat* n *nc*
assumes t:*take-cols* R *ls* $\in$ *carrier-mat* n r
assumes B:$B \in$ *carrier-mat* r *nc*
assumes $R =$ (*take-cols* R *ls*) $* B$
shows *rank* (*take-cols* R *ls*) $\geq$ *rank* R
⟨*proof*⟩

lemma *rref-drop-pivots*:
assumes *row-ech*: *row-echelon-form* R
assumes *dims*: $R \in$ *carrier-mat* n *nc*
assumes *order*: *nc* $\geq n$
shows *rank* (*take-cols* R (*map snd* (*pivot-positions* R))) = *rank* R
⟨*proof*⟩

lemma *gjs-and-take-cols-var*:
fixes A::'a *mat*
assumes A:$A \in$ *carrier-mat* n *nc*
assumes *order*: *nc* $\geq n$
shows (*take-cols* A (*map snd* (*pivot-positions* (*gauss-jordan-single* A)))) =
(*take-cols-var* A (*map snd* (*pivot-positions* (*gauss-jordan-single* A))))
⟨*proof*⟩

lemma *gauss-jordan-single-rank*:
fixes A::'a *mat*
assumes A:$A \in$ *carrier-mat* n *nc*
assumes *order*: *nc* $\geq n$
shows *rank* (*take-cols* A (*map snd* (*pivot-positions* (*gauss-jordan-single* A)))) =
rank A

⟨*proof*⟩

lemma *lin-indpt-subset-cols*:
  fixes *A*:: *'a mat*
  fixes *B*:: *'a vec set*
  assumes *A* ∈ *carrier-mat n n*
  assumes *inv*: *invertible-mat A*
  assumes *B* ⊆ *set* (*cols A*)
  shows *lin-indpt B*
⟨*proof*⟩

lemma *rank-invertible-subset-cols*:
  fixes *A*:: *'a mat*
  fixes *B*:: *'a vec list*
  assumes *inv*: *invertible-mat A*
  assumes *A-square*: *A* ∈ *carrier-mat n n*
  assumes *set-sub*: *set* (*B*) ⊆ *set* (*cols A*)
  assumes *dist-B*: *distinct B*
  shows *rank* (*mat-of-cols n B*) = *length B*
⟨*proof*⟩

end

end
theory *BKR-Algorithm*
  imports
    More-Matrix
    Sturm-Tarski.Sturm-Tarski
begin

4 Setup

definition *retrieve-polys*:: *real poly list* ⇒ *nat list* ⇒ *real poly list*
  where *retrieve-polys qss index-list* = (*map* (*nth qss*) *index-list*)

definition *construct-NofI*:: *real poly* ⇒ *real poly list* ⇒ *rat*
  where *construct-NofI p I* = *rat-of-int* (*changes-R-smods p* ((*pderiv p*)∗(*prod-list I*)))

definition *construct-rhs-vector*:: *real poly* ⇒ *real poly list* ⇒ *nat list list* ⇒ *rat vec*
   where *construct-rhs-vector p qs Is* = *vec-of-list* (*map* (λ *I*.(*construct-NofI p* (*retrieve-polys qs I*))) *Is*)

5 Base Case

definition *base-case-info*:: (*rat mat* × (*nat list list* × *rat list list*))
  where *base-case-info* =
    ((*mat-of-rows-list 2* [[*1*,*1*], [*1*,−*1*]]), ([[],[*0*]], [[*1*],[−*1*]]))

definition *base-case-solve-for-lhs*:: *real poly* ⇒ *real poly* ⇒ *rat vec*
where *base-case-solve-for-lhs p q* = (*mult-mat-vec* (*mat-of-rows-list 2* [[*1/2*, *1/2*], [*1/2*, −*1/2*]]) (*construct-rhs-vector p* [*q*] [[], [*0*]]))

thm *gauss-jordan-compute-inverse*

primrec *matr-option*:: *nat* ⇒ *'a*::{*one*, *zero*} *mat option* ⇒ *'a mat*
where *matr-option dimen None* = 1_m *dimen*
| *matr-option dimen* (*Some c*) = *c*

definition *mat-equal*:: *'a*:: *field mat* ⇒ *'a* :: *field mat* ⇒ *bool*
where *mat-equal A B* = (*dim-row A* = *dim-row B* ∧ *dim-col A* = *dim-col B* ∧ (*mat-to-list A*) = (*mat-to-list B*))

definition *mat-inverse-var* :: *'a* :: *field mat* ⇒ *'a mat option* **where**
mat-inverse-var A = (*if dim-row A* = *dim-col A then*
let one = 1_m (*dim-row A*) *in*
(*case gauss-jordan A one of*
(*B*, *C*) ⇒ *if* (*mat-equal B one*) *then Some C else None*) *else None*)

definition *solve-for-lhs*:: *real poly* ⇒ *real poly list* ⇒ *nat list list* ⇒ *rat mat* ⇒ *rat vec*
where *solve-for-lhs p qs subsets matr* =
mult-mat-vec (*matr-option* (*dim-row matr*) (*mat-inverse-var matr*)) (*construct-rhs-vector p qs subsets*)

6 Smashing

definition *subsets-smash*::*nat* ⇒ *nat list list* ⇒ *nat list list* ⇒ *nat list list*
where *subsets-smash n s1 s2* = *concat* (*map* (λ*l1*. *map* (λ *l2*. *l1* @ (*map* ((+) *n*) *l2*)) *s2*) *s1*)

definition *signs-smash*::*'a list list* ⇒ *'a list list* ⇒ *'a list list*
where *signs-smash s1 s2* = *concat* (*map* (λ*l1*. *map* (λ *l2*. *l1* @ *l2*) *s2*) *s1*)

definition *smash-systems*:: *real poly* ⇒ *real poly list* ⇒ *real poly list* ⇒ *nat list list* ⇒ *nat list list* ⇒
rat list list ⇒ *rat list list* ⇒ *rat mat* ⇒ *rat mat* ⇒
real poly list × (*rat mat* × (*nat list list* × *rat list list*))
where *smash-systems p qs1 qs2 subsets1 subsets2 signs1 signs2 mat1 mat2* =
(*qs1*@*qs2*, (*kronecker-product mat1 mat2*, (*subsets-smash* (*length qs1*) *subsets1 subsets2*, *signs-smash signs1 signs2*)))

fun *combine-systems*:: *real poly* ⇒ (*real poly list* × (*rat mat* × (*nat list list* × *rat list list*))) ⇒ (*real poly list* × (*rat mat* × (*nat list list* × *rat list list*)))

$\Rightarrow$ (*real poly list* $\times$ (*rat mat* $\times$ (*nat list list* $\times$ *rat list list*)))
where *combine-systems p* (*qs1*, *m1*, *sub1*, *sgn1*) (*qs2*, *m2*, *sub2*, *sgn2*) =
(*smash-systems p qs1 qs2 sub1 sub2 sgn1 sgn2 m1 m2*)

7 Reduction

definition *find-nonzeros-from-input-vec*:: *rat vec* $\Rightarrow$ *nat list*
where *find-nonzeros-from-input-vec lhs-vec* = *filter* (λi. *lhs-vec* \$ $i \neq 0$) [*0..<dim-vec lhs-vec*]

definition *take-indices*:: *'a list* $\Rightarrow$ *nat list* $\Rightarrow$ *'a list*
where *take-indices subsets indices* = *map* ((!) *subsets*) *indices*

definition *take-cols-from-matrix*:: *'a mat* $\Rightarrow$ *nat list* $\Rightarrow$ *'a mat*
where *take-cols-from-matrix matr indices-to-keep* =
mat-of-cols (*dim-row matr*) ((*take-indices* (*cols matr*) *indices-to-keep*):: *'a vec list*)

definition *take-rows-from-matrix*:: *'a mat* $\Rightarrow$ *nat list* $\Rightarrow$ *'a mat*
where *take-rows-from-matrix matr indices-to-keep* =
mat-of-rows (*dim-col matr*) ((*take-indices* (*rows matr*) *indices-to-keep*):: *'a vec list*)

fun *reduce-mat-cols*:: *'a mat* $\Rightarrow$ *rat vec* $\Rightarrow$ *'a mat*
where *reduce-mat-cols A lhs-vec* = *take-cols-from-matrix A* (*find-nonzeros-from-input-vec lhs-vec*)

definition *rows-to-keep*:: (*'a*::*field*) *mat* $\Rightarrow$ *nat list* **where**
rows-to-keep A = *map snd* (*pivot-positions* (*gauss-jordan-single* (A^T)))

fun *reduction-step*:: *rat mat* $\Rightarrow$ *rat list list* $\Rightarrow$ *nat list list* $\Rightarrow$ *rat vec* $\Rightarrow$ *rat mat* $\times$ (*nat list list* $\times$ *rat list list*)
where *reduction-step A signs subsets lhs-vec* =
(*let reduce-cols-A* = (*reduce-mat-cols A lhs-vec*);
rows-keep = *rows-to-keep reduce-cols-A in*
(*take-rows-from-matrix reduce-cols-A rows-keep*,
(*take-indices subsets rows-keep*,
take-indices signs (*find-nonzeros-from-input-vec lhs-vec*))))

fun *reduce-system*:: *real poly* $\Rightarrow$ (*real poly list* $\times$ (*rat mat* $\times$ (*nat list list* $\times$ *rat list list*))) $\Rightarrow$ (*rat mat* $\times$ (*nat list list* $\times$ *rat list list*))
where *reduce-system p* (*qs*,*m*,*subs*,*signs*) =
reduction-step m signs subs (*solve-for-lhs p qs subs m*)

8 Overall algorithm

fun *calculate-data*:: *real poly* ⇒ *real poly list* ⇒ (*rat mat* × (*nat list list* × *rat list list*))
where
calculate-data p qs =
(*let len* = *length qs in*
if len = *0 then*
(λ(*a*,*b*,*c*).(*a*,*b*,*map* (*drop 1*) *c*)) (*reduce-system p* ([*1*],*base-case-info*))
else if len ≤ *1 then reduce-system p* (*qs*,*base-case-info*)
else
(*let q1* = *take* (*len div 2*) *qs*; *left* = *calculate-data p q1*;
q2 = *drop* (*len div 2*) *qs*; *right* = *calculate-data p q2*;
comb = *combine-systems p* (*q1*,*left*) (*q2*,*right*) *in*
reduce-system p comb
)
)

definition *find-consistent-signs-at-roots*:: *real poly* ⇒ *real poly list* ⇒ *rat list list*
where [*code*]:
find-consistent-signs-at-roots p qs =
(*let* (*M*,*S*,Σ) = *calculate-data p qs in* Σ)

lemma *find-consistent-signs-at-roots-thm*:
shows *find-consistent-signs-at-roots p qs* = *snd* (*snd* (*calculate-data p qs*))
⟨*proof*⟩

end
theory *Matrix-Equation-Construction*

imports *BKR-Algorithm*
begin

9 Results with Sturm's Theorem

lemma *relprime*:
fixes *q*::*real poly*
assumes *coprime p q*
assumes *p* ≠ *0*
assumes *q* ≠ *0*
shows *changes-R-smods p* (*pderiv p*) = *card* {*x*. *poly p x* = *0* ∧ *poly q x* > *0*} + *card* {*x*. *poly p x* = *0* ∧ *poly q x* < *0*}
⟨*proof*⟩

lemma *card-eq-const-sum*:
fixes *k*:: *real*
assumes *finite A*

shows $k*card\ A = sum\ (\lambda x.\ k)\ A$
⟨*proof*⟩

lemma *restate-tarski*:
fixes *q*::*real poly*
assumes *coprime p q*
assumes $p \neq 0$
assumes $q \neq 0$
shows *changes-R-smods p ((pderiv p) * q) = card {x. poly p x = 0 ∧ poly q x > 0} − int(card {x. poly p x = 0 ∧ poly q x < 0})*
⟨*proof*⟩

lemma *restate-tarski2*:
fixes *q*::*real poly*
assumes $p \neq 0$
shows *changes-R-smods p ((pderiv p) * q) =*
int(card {x. poly p x = 0 ∧ poly q x > 0}) −
int(card {x. poly p x = 0 ∧ poly q x < 0})
⟨*proof*⟩

lemma *coprime-set-prod*:
fixes *I*:: *real poly set*
shows *finite I* $\Longrightarrow$ *((∀ q ∈ I. (coprime p q)) ⟶ (coprime p (∏ I)))*
⟨*proof*⟩

lemma *finite-nonzero-set-prod*:
fixes *I*:: *real poly set*
shows *nonzero-hyp*: *finite I* $\Longrightarrow$ *((∀ q ∈ I. q ≠ 0) ⟶ ∏ I ≠ 0)*
⟨*proof*⟩

10 Setting up the construction: Definitions

definition *characterize-root-list-p*:: *real poly ⇒ real list*
where *characterize-root-list-p p ≡ sorted-list-of-set({x. poly p x = 0}*::*real set)*

lemma *construct-NofI-prop*:
fixes *p*:: *real poly*
fixes *I*:: *real poly list*
assumes *nonzero*: *p≠0*
shows *construct-NofI p I =*
rat-of-int (int (card {x. poly p x = 0 ∧ poly (prod-list I) x > 0}) −
int (card {x. poly p x = 0 ∧ poly (prod-list I) x < 0}))
⟨*proof*⟩

definition *construct-s-vector*:: *real poly ⇒ real poly list list ⇒ rat vec*
where *construct-s-vector p Is = vec-of-list (map (λ I.(construct-NofI p I)) Is)*

definition *squash*::*'a*::*linordered-field* ⇒ *rat*
where *squash x* = (*if x* > *0 then 1*
else if x < *0 then* −*1*
else 0)

definition *signs-at*::*real poly list* ⇒ *real* ⇒ *rat list*
where *signs-at qs x* ≡
map (*squash* ∘ (λ*q*. *poly q x*)) *qs*

definition *characterize-consistent-signs-at-roots*:: *real poly* ⇒ *real poly list* ⇒ *rat list list*
where *characterize-consistent-signs-at-roots p qs* =
(*remdups* (*map* (*signs-at qs*) (*characterize-root-list-p p*)))

definition *consistent-sign-vec-copr*::*real poly list* ⇒ *real* ⇒ *rat list*
where *consistent-sign-vec-copr qs x* ≡
map (λ *q*. *if* (*poly q x* > *0*) *then* (*1*::*rat*) *else* (−*1*::*rat*)) *qs*

definition *characterize-consistent-signs-at-roots-copr*:: *real poly* ⇒ *real poly list* ⇒ *rat list list*
where *characterize-consistent-signs-at-roots-copr p qss* =
(*remdups* (*map* (*consistent-sign-vec-copr qss*) (*characterize-root-list-p p*)))

lemma *csa-list-copr-rel*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
assumes *nonzero*: *p*≠*0*
assumes *pairwise-rel-prime*: ∀ *q*. ((*List.member qs q*) ⟶ (*coprime p q*))
shows *characterize-consistent-signs-at-roots p qs* = *characterize-consistent-signs-at-roots-copr p qs*
⟨*proof*⟩

definition *list-constr*:: *nat list* ⇒ *nat* ⇒ *bool*
where *list-constr L n* ≡ *list-all* (λ*x*. *x* < *n*) *L*

definition *all-list-constr*:: *nat list list* ⇒ *nat* ⇒ *bool*
where *all-list-constr L n* ≡ (∀ *x*. *List.member L x* ⟶ *list-constr x n*)

definition *z*:: *nat list* ⇒ *rat list* ⇒ *rat*
where *z index-list sign-asg* ≡ (*prod-list* (*map* (*nth sign-asg*) *index-list*))

definition *mtx-row*:: *rat list list* ⇒ *nat list* ⇒ *rat list*
where *mtx-row sign-list index-list* ≡ (*map* ((*z index-list*)) *sign-list*)

definition *matrix-A*:: *rat list list* ⇒ *nat list list* ⇒ *rat mat*
where *matrix-A sign-list subset-list* =
(*mat-of-rows-list* (*length sign-list*) (*map* (λ*i* .(*mtx-row sign-list i*)) *subset-list*))

definition *alt-matrix-A*:: *rat list list* ⇒ *nat list list* ⇒ *rat mat*
where *alt-matrix-A signs subsets* = (*mat* (*length subsets*) (*length signs*)
(λ(*i*, *j*). *z* (*subsets* ! *i*) (*signs* ! *j*)))

lemma *alt-matrix-char*: *alt-matrix-A signs subsets* = *matrix-A signs subsets*
⟨*proof*⟩

lemma *subsets-are-rows*: ∀ *i* < (*length subsets*). *row* (*alt-matrix-A signs subsets*) *i*
= *vec* (*length signs*) (λ*j*. *z* (*subsets* ! *i*) (*signs* ! *j*))
⟨*proof*⟩

lemma *signs-are-cols*: ∀ *i* < (*length signs*). *col* (*alt-matrix-A signs subsets*) *i* =
vec (*length subsets*) (λ*j*. *z* (*subsets* ! *j*) (*signs* ! *i*))
⟨*proof*⟩

definition *construct-lhs-vector*:: *real poly* ⇒ *real poly list* ⇒ *rat list list* ⇒ *rat vec*
where *construct-lhs-vector p qs signs* ≡
vec-of-list (*map* (λ*w*. *rat-of-int* (*int* (*length* (*filter* (λ*v*. *v* = *w*) (*map* (*consistent-sign-vec-copr*
qs) (*characterize-root-list-p p*)))))) *signs*)

definition *satisfy-equation*:: *real poly* ⇒ *real poly list* ⇒ *nat list list* ⇒ *rat list list*
⇒ *bool*
where *satisfy-equation p qs subset-list sign-list* =
(*mult-mat-vec* (*matrix-A sign-list subset-list*) (*construct-lhs-vector p qs*
sign-list) = (*construct-rhs-vector p qs subset-list*))

11 Setting up the construction: Proofs

lemma *row-mat-of-rows-list*:
assumes *list-all* (λ*r*. *length r* = *nc*) *rs*
assumes *i* < *length rs*
shows *row* (*mat-of-rows-list nc rs*) *i* = *vec-of-list* (*nth rs i*)
⟨*proof*⟩

lemma *mult-mat-vec-of-list*:
assumes *length ls* = *nc*
assumes *list-all* (λ*r*. *length r* = *nc*) *rs*
shows *mat-of-rows-list nc rs* ∗$_v$ *vec-of-list ls* =
vec-of-list (*map* (λ*r*. *vec-of-list r* · *vec-of-list ls*) *rs*)
⟨*proof*⟩

lemma *mtx-row-length*:

list-all ($\lambda r.$ *length* r = *length signs*) (*map* (*mtx-row signs*) *ls*)
⟨*proof*⟩

thm *construct-lhs-vector-def*
thm *poly-roots-finite*

lemma *construct-lhs-vector-clean*:
assumes $p \neq 0$
assumes $i <$ *length signs*
shows (*construct-lhs-vector p qs signs*) $ i =
card {x. *poly p x* = 0 ∧ ((*consistent-sign-vec-copr qs x*) = (*nth signs i*))}
⟨*proof*⟩

lemma *construct-lhs-vector-cleaner*:
assumes $p \neq 0$
shows (*construct-lhs-vector p qs signs*) =
vec-of-list (*map* ($\lambda s.$ *rat-of-int* (*card* {x. *poly p x* = 0 ∧ ((*consistent-sign-vec-copr qs x*) = s)})) *signs*)
⟨*proof*⟩

lemma *z-signs*:
assumes *list-all* ($\lambda i.$ $i <$ *length signs*) I
assumes *list-all* ($\lambda s.$ $s = 1 \vee s = -1$) *signs*
shows (*z I signs* = 1) ∨ (*z I signs* = −1) ⟨*proof*⟩

lemma *z-lemma*:
fixes I:: *nat list*
fixes *sign*:: *rat list*
assumes *consistent*: *sign* ∈ *set* (*characterize-consistent-signs-at-roots-copr p qs*)
assumes *welldefined*: *list-constr I* (*length qs*)
shows (*z I sign* = 1) ∨ (*z I sign* = −1)
⟨*proof*⟩

lemma *in-set*:
fixes p:: *real poly*
assumes *nonzero*: $p \neq 0$
fixes *qs*:: *real poly list*
fixes I:: *nat list*
fixes *sign*:: *rat list*
fixes x:: *real*
assumes *root-p*: $x \in$ {x. *poly p x* = 0}
assumes *sign-fix*: *sign* = *consistent-sign-vec-copr qs x*
assumes *welldefined*: *list-constr I* (*length qs*)
shows *sign* ∈ *set* (*characterize-consistent-signs-at-roots-copr p qs*)
⟨*proof*⟩

lemma *nonzero-product*:
fixes *p*:: *real poly*
assumes *nonzero*: *p≠0*
fixes *qs*:: *real poly list*
assumes *pairwise-rel-prime-1*: ∀ *q*. ((*List.member qs q*) ⟶ (*coprime p q*))
fixes *I*:: *nat list*
fixes *x*:: *real*
assumes *root-p*: *x* ∈ {*x*. *poly p x* = *0*}
assumes *welldefined*: *list-constr I* (*length qs*)
shows (*poly* (*prod-list* (*retrieve-polys qs I*)) *x* > *0*) ∨ (*poly* (*prod-list* (*retrieve-polys qs I*)) *x* < *0*)
⟨*proof*⟩

lemma *horiz-vector-helper-pos-ind*:
fixes *p*:: *real poly*
assumes *nonzero*: *p≠0*
fixes *qs*:: *real poly list*
assumes *pairwise-rel-prime-1*: ∀ *q*. ((*List.member qs q*) ⟶ (*coprime p q*))
fixes *I*:: *nat list*
fixes *sign*:: *rat list*
fixes *x*:: *real*
assumes *root-p*: *x* ∈ {*x*. *poly p x* = *0*}
assumes *sign-fix*: *sign* = *consistent-sign-vec-copr qs x*
shows (*list-constr I* (*length qs*)) ⟶ (*poly* (*prod-list* (*retrieve-polys qs I*)) *x* > *0*) ⟷ (*z I sign* = *1*)
⟨*proof*⟩

lemma *horiz-vector-helper-pos*:
fixes *p*:: *real poly*
assumes *nonzero*: *p≠0*
fixes *qs*:: *real poly list*
assumes *pairwise-rel-prime-1*: ∀ *q*. ((*List.member qs q*) ⟶ (*coprime p q*))
fixes *I*:: *nat list*
fixes *sign*:: *rat list*
fixes *x*:: *real*
assumes *root-p*: *x* ∈ {*x*. *poly p x* = *0*}
assumes *sign-fix*: *sign* = *consistent-sign-vec-copr qs x*
assumes *welldefined*: *list-constr I* (*length qs*)
shows (*poly* (*prod-list* (*retrieve-polys qs I*)) *x* > *0*) ⟷ (*z I sign* = *1*)
⟨*proof*⟩

lemma *horiz-vector-helper-neg*:
fixes *p*:: *real poly*
assumes *nonzero*: *p≠0*
fixes *qs*:: *real poly list*
assumes *pairwise-rel-prime-1*: ∀ *q*. ((*List.member qs q*) ⟶ (*coprime p q*))
fixes *I*:: *nat list*

fixes *sign*:: *rat list*
fixes *x*:: *real*
assumes *root-p*: $x \in \{x.\ poly\ p\ x = 0\}$
assumes *sign-fix*: *sign* = *consistent-sign-vec-copr qs x*
assumes *welldefined*: *list-constr I* (*length qs*)
shows (*poly* (*prod-list* (*retrieve-polys qs I*)) *x* < *0*) $\longleftrightarrow$ (*z I sign* = -1)
⟨*proof*⟩

lemma *vec-of-list-dot-rewrite*:
assumes *length xs* = *length ys*
shows *vec-of-list xs* · *vec-of-list ys* =
sum-list (*map2* (∗) *xs ys*)
⟨*proof*⟩

lemma *lhs-dot-rewrite*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
fixes *I*:: *nat list*
fixes *signs*:: *rat list list*
assumes *nonzero*: *p*≠*0*
shows
(*vec-of-list* (*mtx-row signs I*) · (*construct-lhs-vector p qs signs*)) =
sum-list (*map* (λ*s*. (*z I s*) ∗ *rat-of-int* (*card* {*x*. *poly p x* = *0* ∧ *consistent-sign-vec-copr qs x* = *s*})) *signs*)
⟨*proof*⟩

lemma *sum-list-distinct-filter*:
fixes *f*:: *'a* ⇒ *int*
assumes *distinct xs distinct ys*
assumes *set ys* ⊆ *set xs*
assumes ⋀*x*. *x* ∈ *set xs* − *set ys* ⟹ *f x* = *0*
shows *sum-list* (*map f xs*) = *sum-list* (*map f ys*)
⟨*proof*⟩

lemma *construct-lhs-vector-drop-consistent*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
fixes *I*:: *nat list*
fixes *signs*:: *rat list list*
assumes *nonzero*: *p*≠*0*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
assumes *welldefined*: *list-constr I* (*length qs*)
shows
(*vec-of-list* (*mtx-row signs I*) · (*construct-lhs-vector p qs signs*)) =
(*vec-of-list* (*mtx-row* (*characterize-consistent-signs-at-roots-copr p qs*) *I*) ·
(*construct-lhs-vector p qs* (*characterize-consistent-signs-at-roots-copr p qs*)))

⟨*proof*⟩

lemma *matrix-equation-helper-step*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
fixes *I*:: *nat list*
fixes *signs*:: *rat list list*
assumes *nonzero*: *p≠0*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
assumes *welldefined*: *list-constr I* (*length qs*)
assumes *pairwise-rel-prime-1*: ∀ *q*. ((*List.member qs q*) ⟶ (*coprime p q*))
shows (*vec-of-list* (*mtx-row signs I*) ⋅ (*construct-lhs-vector p qs signs*)) =
rat-of-int (*card* {*x*. *poly p x* = *0* ∧ *poly* (*prod-list* (*retrieve-polys qs I*)) *x* > *0*})
−
rat-of-int (*card* {*x*. *poly p x* = *0* ∧ *poly* (*prod-list* (*retrieve-polys qs I*)) *x* < *0*})
⟨*proof*⟩

lemma *matrix-equation-main-step*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
fixes *I*:: *nat list*
fixes *signs*:: *rat list list*
assumes *nonzero*: *p≠0*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
assumes *welldefined*: *list-constr I* (*length qs*)
assumes *pairwise-rel-prime-1*: ∀ *q*. ((*List.member qs q*) ⟶ (*coprime p q*))
shows (*vec-of-list* (*mtx-row signs I*) ⋅ (*construct-lhs-vector p qs signs*)) =
construct-NofI p (*retrieve-polys qs I*)
⟨*proof*⟩

lemma *map-vec-vec-of-list-eq-intro*:
assumes *map f xs* = *map g ys*
shows *map-vec f* (*vec-of-list xs*) = *map-vec g* (*vec-of-list ys*)
⟨*proof*⟩

theorem *matrix-equation*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
fixes *subsets*:: *nat list list*
fixes *signs*:: *rat list list*
assumes *nonzero*: *p≠0*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
assumes *pairwise-rel-prime*: ∀ *q*. ((*List.member qs q*) ⟶ (*coprime p q*))

assumes *welldefined*: *all-list-constr* (*subsets*) (*length qs*)
shows *satisfy-equation p qs subsets signs*
⟨*proof*⟩

definition *roots*:: *real poly* ⇒ *real set*
where *roots p* = {*x*. *poly p x* = *0*}

definition *sgn*::*'a*::*linordered-field* ⇒ *rat*
where *sgn x* = (*if x* > *0 then 1*
else if x < *0 then* −*1*
else 0)

definition *sgn-vec*::*real poly list* ⇒ *real* ⇒ *rat list*
where *sgn-vec qs x* ≡ *map* (*sgn* ∘ (λ*q*. *poly q x*)) *qs*

definition *consistent-signs-at-roots*:: *real poly* ⇒ *real poly list* ⇒ *rat list set*
where *consistent-signs-at-roots p qs* =
(*sgn-vec qs*) ‘ (*roots p*)

lemma *consistent-signs-at-roots-eq*:
assumes *p* ≠ *0*
shows *consistent-signs-at-roots p qs* =
set (*characterize-consistent-signs-at-roots p qs*)
⟨*proof*⟩

abbreviation *w-vec*:: *real poly* ⇒ *real poly list* ⇒ *rat list list* ⇒ *rat vec*
where *w-vec* ≡ *construct-lhs-vector*

abbreviation *v-vec*:: *real poly* ⇒ *real poly list* ⇒ *nat list list* ⇒ *rat vec*
where *v-vec* ≡ *construct-rhs-vector*

abbreviation *M-mat*:: *rat list list* ⇒ *nat list list* ⇒ *rat mat*
where *M-mat* ≡ *matrix-A*

theorem *matrix-equation-pretty*:
assumes *p*≠*0*
assumes ⋀*q*. *q* ∈ *set qs* ⟹ *coprime p q*
assumes *distinct signs*
assumes *consistent-signs-at-roots p qs* ⊆ *set signs*
assumes ⋀*l i*. *l* ∈ *set subsets* ⟹ *i* ∈ *set l* ⟹ *i* < *length qs*
shows *M-mat signs subsets* ∗$_v$ *w-vec p qs signs* = *v-vec p qs subsets*
⟨*proof*⟩

end
theory *BKR-Proofs*
imports *Matrix-Equation-Construction*

begin

definition *well-def-signs*:: *nat* => *rat list list* ⇒ *bool*
where *well-def-signs num-polys sign-conds* ≡ ∀ *signs* ∈ *set*(*sign-conds*). (*length signs* = *num-polys*)

definition *satisfies-properties*:: *real poly* ⇒ *real poly list* ⇒ *nat list list* ⇒ *rat list list* ⇒ *rat mat* ⇒ *bool*
where *satisfies-properties p qs subsets signs matrix* =
(*all-list-constr subsets* (*length qs*) ∧ *well-def-signs* (*length qs*) *signs* ∧ *distinct signs*
∧ *satisfy-equation p qs subsets signs* ∧ *invertible-mat matrix* ∧ *matrix* = *matrix-A signs subsets*
∧ *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
)

12 Base Case

lemma *mat-base-case*:
shows *matrix-A* [[*1*],[−*1*]] [[],[*0*]] = (*mat-of-rows-list 2* [[*1*, *1*], [*1*, −*1*]])
⟨*proof*⟩

lemma *base-case-sgas*:
fixes *q p*:: *real poly*
assumes *nonzero*: *p* ≠ *0*
assumes *rel-prime*: *coprime p q*
shows *set* (*characterize-consistent-signs-at-roots-copr p* [*q*]) ⊆ {[*1*], [− *1*]}
⟨*proof*⟩

lemma *base-case-sgas-alt*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
assumes *len1*: *length qs* = *1*
assumes *nonzero*: *p* ≠ *0*
assumes *rel-prime*: ∀ *q*. (*List.member qs q*) ⟶ *coprime p q*
shows *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ {[*1*], [− *1*]}
⟨*proof*⟩

lemma *base-case-satisfy-equation*:
fixes *q p*:: *real poly*
assumes *nonzero*: *p* ≠ *0*
assumes *rel-prime*: *coprime p q*
shows *satisfy-equation p* [*q*] [[],[*0*]] [[*1*],[−*1*]]
⟨*proof*⟩

lemma *base-case-satisfy-equation-alt*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
assumes *len1*: *length qs* = *1*
assumes *nonzero*: *p* ≠ *0*

assumes *rel-prime*: $\forall\, q.\ (List.member\ qs\ q) \longrightarrow coprime\ p\ q$
shows *satisfy-equation p qs* [[],[0]] [[1],[−1]]
⟨*proof*⟩

lemma *base-case-matrix-eq*:
fixes *q p*:: *real poly*
assumes *nonzero*: $p \neq 0$
assumes *rel-prime*: *coprime p q*
shows (*mult-mat-vec* (*mat-of-rows-list 2* [[1, 1], [1, −1]]) (*construct-lhs-vector p* [*q*] [[1],[−1]]) =
(*construct-rhs-vector p* [*q*] [[],[0]]))
⟨*proof*⟩

lemma *less-two*:
shows $j < Suc\ (Suc\ 0) \longleftrightarrow j = 0 \vee j = 1$ ⟨*proof*⟩

lemma *inverse-mat-base-case*:
shows *inverts-mat* (*mat-of-rows-list 2* [[1/2, 1/2], [1/2, −(1/2)]]::*rat mat*) (*mat-of-rows-list 2* [[1, 1], [1, −1]]::*rat mat*)
⟨*proof*⟩

lemma *inverse-mat-base-case-2*:
shows *inverts-mat* (*mat-of-rows-list 2* [[1, 1], [1, −1]]::*rat mat*) (*mat-of-rows-list 2* [[1/2, 1/2], [1/2, −(1/2)]]::*rat mat*)
⟨*proof*⟩

lemma *base-case-invertible-mat*:
shows *invertible-mat* (*matrix-A* [[1], [− 1]] [[], [0]])
⟨*proof*⟩

13 Inductive Step

13.1 Lemmas on smashing subsets and signs

lemma *signs-smash-property*:
fixes *signs1 signs2* :: *'a list list*
fixes *a b*:: *nat*
shows $\forall$ (*elem* :: *'a list*). (*elem* ∈ (*set* (*signs-smash signs1 signs2*)) ⟶
($\exists$ *n m* :: *nat*.
elem = ((*nth signs1 n*)@(*nth signs2 m*))))
⟨*proof*⟩

lemma *signs-smash-property-set*:
fixes *signs1 signs2* :: *'a list list*
fixes *a b*:: *nat*
shows $\forall$ (*elem* :: *'a list*). (*elem* ∈ (*set* (*signs-smash signs1 signs2*)) ⟶
($\exists$ (*elem1*::*'a list*). $\exists$ (*elem2*::*'a list*).
(*elem1* ∈ *set*(*signs1*) ∧ *elem2* ∈ *set*(*signs2*) ∧ *elem* = (*elem1*@*elem2*))))
⟨*proof*⟩

lemma *subsets-smash-property*:
fixes *subsets1 subsets2* :: *nat list list*
fixes *n*:: *nat*
shows ∀ (*elem* :: *nat list*). (*List.member* (*subsets-smash n subsets1 subsets2*) *elem*) ⟶
(∃ (*elem1*::*nat list*). ∃ (*elem2*::*nat list*).
(*elem1* ∈ *set*(*subsets1*) ∧ *elem2* ∈ *set*(*subsets2*) ∧ *elem* = (*elem1* @ ((*map* ((+) *n*) *elem2*)))))
⟨*proof*⟩

13.2 Well-defined subsets preserved when smashing

lemma *list-constr-append*:
list-constr a n ∧ *list-constr b n* ⟶ *list-constr* (*a*@*b*) *n*
⟨*proof*⟩

lemma *well-def-step*:
fixes *qs1 qs2* :: *real poly list*
fixes *subsets1 subsets2* :: *nat list list*
assumes *well-def-subsets1*: *all-list-constr* (*subsets1*) (*length qs1*)
assumes *well-def-subsets2*: *all-list-constr* (*subsets2*) (*length qs2*)
shows *all-list-constr* ((*subsets-smash* (*length qs1*) *subsets1 subsets2*))
(*length* (*qs1* @ *qs2*))
⟨*proof*⟩

13.3 Well def signs preserved when smashing

lemma *well-def-signs-step*:
fixes *qs1 qs2* :: *real poly list*
fixes *signs1 signs2* :: *rat list list*
assumes *length qs1* > *0*
assumes *length qs2* > *0*
assumes *well-def1*: *well-def-signs* (*length qs1*) *signs1*
assumes *well-def2*: *well-def-signs* (*length qs2*) *signs2*
shows *well-def-signs* (*length* (*qs1*@*qs2*)) (*signs-smash signs1 signs2*)
⟨*proof*⟩

13.4 Distinct signs preserved when smashing

lemma *distinct-map-append*:
assumes *distinct ls*
shows *distinct* (*map* ((@) *xs*) *ls*)
⟨*proof*⟩

lemma *length-eq-append*:
assumes *length y* = *length b*
shows (*x* @ *y* = *a* @ *b*) ⟷ *x*=*a* ∧ *y* = *b*
⟨*proof*⟩

lemma *distinct-step*:
fixes *qs1 qs2* :: *real poly list*
fixes *signs1 signs2* :: *rat list list*
assumes *well-def1*: *well-def-signs n1 signs1*
assumes *well-def2*: *well-def-signs n2 signs2*
assumes *distinct1*: *distinct signs1*
assumes *distinct2*: *distinct signs2*
shows *distinct* (*signs-smash signs1 signs2*)
⟨*proof*⟩

13.5 Consistent sign assignments preserved when smashing

lemma *subset-smash-signs*:
fixes *a1 b1 a2 b2*:: *rat list list*
assumes *sub1*: *set a1* ⊆ *set a2*
assumes *sub2*: *set b1* ⊆ *set b2*
shows *set* (*signs-smash a1 b1*) ⊆ *set* (*signs-smash a2 b2*)
⟨*proof*⟩

lemma *subset-helper*:
fixes *p*:: *real poly*
fixes *qs1 qs2* :: *real poly list*
fixes *signs1 signs2* :: *rat list list*
shows ∀ *x* ∈ *set* (*characterize-consistent-signs-at-roots-copr p* (*qs1* @ *qs2*)).
∃ *x1* ∈ *set* (*characterize-consistent-signs-at-roots-copr p qs1*).
∃ *x2* ∈ *set* (*characterize-consistent-signs-at-roots-copr p qs2*).
x = *x1*@*x2*
⟨*proof*⟩

lemma *subset-step*:
fixes *p*:: *real poly*
fixes *qs1 qs2* :: *real poly list*
fixes *signs1 signs2* :: *rat list list*
assumes *csa1*: *set* (*characterize-consistent-signs-at-roots-copr p qs1*) ⊆ *set* (*signs1*)
assumes *csa2*: *set* (*characterize-consistent-signs-at-roots-copr p qs2*) ⊆ *set* (*signs2*)

shows *set* (*characterize-consistent-signs-at-roots-copr p*
(*qs1* @ *qs2*))
⊆ *set* (*signs-smash signs1 signs2*)
⟨*proof*⟩

13.6 Main Results

lemma *dim-row-mat-of-rows-list*[*simp*]:
shows *dim-row* (*mat-of-rows-list nr ls*) = *length ls*
⟨*proof*⟩

lemma *dim-col-mat-of-rows-list*[*simp*]:
shows *dim-col* (*mat-of-rows-list nr ls*) = *nr*
⟨*proof*⟩

lemma *dim-row-matrix-A*[*simp*]:
shows *dim-row* (*matrix-A signs subsets*) = *length subsets*
⟨*proof*⟩

lemma *dim-col-matrix-A*[*simp*]:
shows *dim-col* (*matrix-A signs subsets*) = *length signs*
⟨*proof*⟩

lemma *length-signs-smash*:
shows
length (*signs-smash signs1 signs2*) = *length signs1* ∗ *length signs2*
⟨*proof*⟩

lemma *length-subsets-smash*:
shows
length (*subsets-smash n subs1 subs2*) = *length subs1* ∗ *length subs2*
⟨*proof*⟩

lemma *length-eq-concat*:
assumes ⋀*x*. *x* ∈ *set ls* ⟹ *length x* = *n*
assumes *i* < *n* ∗ *length ls*
shows *concat ls* ! *i* = *ls* ! (*i div n*) ! (*i mod n*)
⟨*proof*⟩

lemma *z-append*:
assumes ⋀*i*. *i* ∈ *set xs* ⟹ *i* < *length as*
shows *z* (*xs* @ (*map* ((+) (*length as*)) *ys*)) (*as* @ *bs*) = *z xs as* ∗ *z ys bs*
⟨*proof*⟩

lemma *matrix-construction-is-kronecker-product*:
fixes *qs1* :: *real poly list*
fixes *subs1 subs2* :: *nat list list*
fixes *signs1 signs2* :: *rat list list*

assumes ⋀*l i*. *l* ∈ *set subs1* ⟹ *i* ∈ *set l* ⟹ *i* < *n1*
assumes ⋀*j*. *j* ∈ *set signs1* ⟹ *length j* = *n1*
shows
(*matrix-A* (*signs-smash signs1 signs2*) (*subsets-smash n1 subs1 subs2*)) =
kronecker-product (*matrix-A signs1 subs1*) (*matrix-A signs2 subs2*)
⟨*proof*⟩

lemma *inductive-step*:
fixes *p*:: *real poly*
fixes *qs1 qs2* :: *real poly list*
fixes *subsets1 subsets2* :: *nat list list*
fixes *signs1 signs2* :: *rat list list*

assumes *nonzero*: $p \neq 0$
assumes *nontriv1*: $length\ qs1 > 0$
assumes *nontriv2*: $length\ qs2 > 0$
assumes *pairwise-rel-prime1*: $\forall q.\ ((List.member\ qs1\ q) \longrightarrow (coprime\ p\ q))$
assumes *pairwise-rel-prime2*: $\forall q.\ ((List.member\ qs2\ q) \longrightarrow (coprime\ p\ q))$
assumes *welldefined-signs1*: $well\text{-}def\text{-}signs\ (length\ qs1)\ signs1$
assumes *welldefined-signs2*: $well\text{-}def\text{-}signs\ (length\ qs2)\ signs2$
assumes *distinct-signs1*: $distinct\ signs1$
assumes *distinct-signs2*: $distinct\ signs2$
assumes *all-info1*: $set\ (characterize\text{-}consistent\text{-}signs\text{-}at\text{-}roots\text{-}copr\ p\ qs1) \subseteq set(signs1)$
assumes *all-info2*: $set\ (characterize\text{-}consistent\text{-}signs\text{-}at\text{-}roots\text{-}copr\ p\ qs2) \subseteq set(signs2)$
assumes *welldefined-subsets1*: $all\text{-}list\text{-}constr\ (subsets1)\ (length\ qs1)$
assumes *welldefined-subsets2*: $all\text{-}list\text{-}constr\ (subsets2)\ (length\ qs2)$
assumes *invertibleMtx1*: $invertible\text{-}mat\ (matrix\text{-}A\ signs1\ subsets1)$
assumes *invertibleMtx2*: $invertible\text{-}mat\ (matrix\text{-}A\ signs2\ subsets2)$
shows $satisfy\text{-}equation\ p\ (qs1@qs2)\ (subsets\text{-}smash\ (length\ qs1)\ subsets1\ subsets2)$
$(signs\text{-}smash\ signs1\ signs2)$
$\wedge\ invertible\text{-}mat\ (matrix\text{-}A\ (signs\text{-}smash\ signs1\ signs2)\ (subsets\text{-}smash\ (length\ qs1)\ subsets1\ subsets2))$
⟨*proof*⟩

14 Reduction Step Proofs

definition *get-matrix*:: $(rat\ mat \times (nat\ list\ list \times rat\ list\ list)) \Rightarrow rat\ mat$
where $get\text{-}matrix\ data = fst(data)$

definition *get-subsets*:: $(rat\ mat \times (nat\ list\ list \times rat\ list\ list)) \Rightarrow nat\ list\ list$
where $get\text{-}subsets\ data = fst(snd(data))$

definition *get-signs*:: $(rat\ mat \times (nat\ list\ list \times rat\ list\ list)) \Rightarrow rat\ list\ list$
where $get\text{-}signs\ data = snd(snd(data))$

definition *reduction-signs*:: $real\ poly \Rightarrow real\ poly\ list \Rightarrow rat\ list\ list \Rightarrow nat\ list\ list$
$\Rightarrow rat\ mat \Rightarrow rat\ list\ list$
where $reduction\text{-}signs\ p\ qs\ signs\ subsets\ matr =$
$(take\text{-}indices\ signs\ (find\text{-}nonzeros\text{-}from\text{-}input\text{-}vec\ (solve\text{-}for\text{-}lhs\ p\ qs\ subsets\ matr)))$

definition *reduction-subsets*:: $real\ poly \Rightarrow real\ poly\ list \Rightarrow rat\ list\ list \Rightarrow nat\ list\ list \Rightarrow rat\ mat \Rightarrow nat\ list\ list$
where $reduction\text{-}subsets\ p\ qs\ signs\ subsets\ matr =$
$(take\text{-}indices\ subsets\ (rows\text{-}to\text{-}keep\ (reduce\text{-}mat\text{-}cols\ matr\ (solve\text{-}for\text{-}lhs\ p\ qs\ subsets\ matr))))$

lemma *reduction-signs-is-get-signs*: $reduction\text{-}signs\ p\ qs\ signs\ subsets\ m = get\text{-}signs\ (reduce\text{-}system\ p\ (qs, (m, (subsets, signs))))$
⟨*proof*⟩

lemma *reduction-subsets-is-get-subsets*: *reduction-subsets p qs signs subsets m* = *get-subsets* (*reduce-system p* (*qs*, (*m*, (*subsets*, *signs*))))
⟨*proof*⟩

lemma *find-zeros-from-vec-prop*:
fixes *input-vec* :: *rat vec*
shows ∀ *n* < (*dim-vec input-vec*). ((*input-vec* $ *n* ≠ *0*) ⟷
List.member (*find-nonzeros-from-input-vec input-vec*) *n*)
⟨*proof*⟩

14.1 Showing sign conditions preserved when reducing

lemma *take-indices-lem*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *arb-list* :: *'a list list*
fixes *index-list* :: *nat list*
fixes *n*:: *nat*
assumes *lest*: *n* < *length* (*take-indices arb-list index-list*)
assumes *well-def*: ∀ *q*.(*List.member index-list q* ⟶ *q* < *length arb-list*)
shows ∃ *k*<*length arb-list*.
(*take-indices arb-list index-list*) ! *n* = *arb-list* ! *k*
⟨*proof*⟩

lemma *invertible-means-mult-id*:
fixes *A*:: *'a*::*field mat*
assumes *assm*: *invertible-mat A*
shows *matr-option* (*dim-row A*)
(*mat-inverse* (*A*))∗*A* = *1*$_m$ (*dim-row A*)
⟨*proof*⟩

lemma *dim-invertible*:
fixes *A*:: *'a*::*field mat*
fixes *n*:: *nat*
assumes *assm*: *invertible-mat A*
assumes *dim*: *A* ∈ *carrier-mat n n*
shows *matr-option* (*dim-row A*)
(*mat-inverse* (*A*)) ∈ *carrier-mat n n*
⟨*proof*⟩

lemma *mult-assoc*:
fixes *A B*:: *rat mat*
fixes *v*:: *rat vec*
fixes *n*:: *nat*
assumes *A* ∈ *carrier-mat n n*
assumes *B* ∈ *carrier-mat n n*
assumes *dim-vec v* = *n*
shows *A* ∗$_v$ (*mult-mat-vec B v*) = (*A*∗*B*)∗$_v$ *v*
⟨*proof*⟩

lemma *size-of-mat*:
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
shows (*matrix-A signs subsets*) ∈ *carrier-mat* (*length subsets*) (*length signs*)
⟨*proof*⟩

lemma *size-of-lhs*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *signs* :: *rat list list*
shows *dim-vec* (*construct-lhs-vector p qs signs*) = *length signs*
⟨*proof*⟩

lemma *size-of-rhs*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
shows *dim-vec* (*construct-rhs-vector p qs subsets*) = *length subsets*
⟨*proof*⟩

lemma *same-size*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *invertible-mat*: *invertible-mat* (*matrix-A signs subsets*)
shows *length subsets* = *length signs*
⟨*proof*⟩

lemma *mat-equal-list-lem*:
fixes *A*:: *'a*::*field mat*
fixes *B*:: *'a*::*field mat*
shows (*dim-row A* = *dim-row B* ∧ *dim-col A* = *dim-col B* ∧ *mat-to-list A* = *mat-to-list B*)
⟹ *A* = *B*
⟨*proof*⟩

lemma *mat-inverse-same*: *mat-inverse-var A* = *mat-inverse A*
⟨*proof*⟩

lemma *construct-lhs-matches-solve-for-lhs*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *match*: *satisfy-equation p qs subsets signs*
assumes *invertible-mat*: *invertible-mat* (*matrix-A signs subsets*)
shows (*construct-lhs-vector p qs signs*) = *solve-for-lhs p qs subsets* (*matrix-A*

signs subsets)
⟨*proof*⟩

lemma *reduction-signs-set-helper-lemma*:
fixes *A*:: *rat list set*
fixes *C*:: *rat vec*
fixes *B*:: *rat list list*
assumes *dim-vec C* = *length B*
assumes *sub*: *A* ⊆ *set*(*B*)
assumes *not-in-hyp*: ∀ *n* < *dim-vec C*. *C* $ *n* = *0* ⟶ (*nth B n*) ∉ *A*
shows *A* ⊆ *set* (*take-indices B*
(*find-nonzeros-from-input-vec C*))
⟨*proof*⟩

lemma *reduction-signs-set-helper-lemma2*:
fixes *A*:: *rat list set*
fixes *C*:: *rat vec*
fixes *B*:: *rat list list*
assumes *dist*: *distinct B*
assumes *eq-len*: *dim-vec C* = *length B*
assumes *sub*: *A* ⊆ *set*(*B*)
assumes *not-in-hyp*: ∀ *n* < *dim-vec C*. *C* $ *n* ≠ *0* ⟶ (*nth B n*) ∈ *A*
shows *set* (*take-indices B*
(*find-nonzeros-from-input-vec C*)) ⊆ *A*
⟨*proof*⟩

lemma *reduction-doesnt-break-things-signs*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *nonzero*: *p* ≠ *0*
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation p qs subsets signs*
assumes *invertible-mat*: *invertible-mat* (*matrix-A signs subsets*)
shows *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*reduction-signs p qs signs subsets* (*matrix-A signs subsets*))
⟨*proof*⟩

lemma *reduction-deletes-bad-sign-conds*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *nonzero*: *p* ≠ *0*

assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation p qs subsets signs*
assumes *invertible-mat*: *invertible-mat* (*matrix-A signs subsets*)
shows *set* (*characterize-consistent-signs-at-roots-copr p qs*) = *set*(*reduction-signs p qs signs subsets* (*matrix-A signs subsets*))
⟨*proof*⟩

theorem *reduce-system-sign-conditions*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *nonzero*: *p* ≠ *0*
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation p qs subsets signs*
assumes *invertible-mat*: *invertible-mat* (*matrix-A signs subsets*)
shows *set* (*get-signs* (*reduce-system p* (*qs*, ((*matrix-A signs subsets*), (*subsets*, *signs*))))) = *set* (*characterize-consistent-signs-at-roots-copr p qs*)
⟨*proof*⟩

14.2 Showing matrix equation preserved when reducing

lemma *rows-to-keep-lem*:
fixes *A*:: ('*a*::*field*) *mat*
shows ⋀*ell*. *ell* ∈ *set* (*rows-to-keep A*) ⟹ *ell* < *dim-row A*
⟨*proof*⟩

lemma *reduce-system-matrix-equation-preserved*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *nonzero*: *p* ≠ *0*
assumes *welldefined-signs*: *well-def-signs* (*length qs*) *signs*
assumes *welldefined-subsets*: *all-list-constr* (*subsets*) (*length qs*)
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation p qs subsets signs*
assumes *invertible-mat*: *invertible-mat* (*matrix-A signs subsets*)
assumes *pairwise-rel-prime*: ∀ *q*. ((*List.member qs q*) ⟶ (*coprime p q*))
shows *satisfy-equation p qs* (*get-subsets* (*reduce-system p* (*qs*, ((*matrix-A signs subsets*), (*subsets*, *signs*)))))
(*get-signs* (*reduce-system p* (*qs*, ((*matrix-A signs subsets*), (*subsets*, *signs*)))))
⟨*proof*⟩

14.3 Showing matrix preserved

lemma *reduce-system-matrix-signs-helper-aux*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
fixes *S*:: *nat list*
assumes *well-def-h*: $\forall x.$ *List.member S x* $\longrightarrow$ $x <$ *length signs*
assumes *nonzero*: $p \neq 0$
shows *alt-matrix-A* (*take-indices signs S*) *subsets* = *take-cols-from-matrix* (*alt-matrix-A signs subsets*) *S*
⟨*proof*⟩

lemma *reduce-system-matrix-signs-helper*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
fixes *S*:: *nat list*
assumes *well-def-h*: $\forall x.$ *List.member S x* $\longrightarrow$ $x <$ *length signs*
assumes *nonzero*: $p \neq 0$
shows *matrix-A* (*take-indices signs S*) *subsets* = *take-cols-from-matrix* (*matrix-A signs subsets*) *S*
⟨*proof*⟩

lemma *reduce-system-matrix-subsets-helper-aux*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
fixes *S*:: *nat list*
assumes *inv*: *length subsets* $\geq$ *length signs*
assumes *well-def-h*: $\forall x.$ *List.member S x* $\longrightarrow$ $x <$ *length subsets*
assumes *nonzero*: $p \neq 0$
shows *alt-matrix-A signs* (*take-indices subsets S*) = *take-rows-from-matrix* (*alt-matrix-A signs subsets*) *S*
⟨*proof*⟩

lemma *reduce-system-matrix-subsets-helper*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
fixes *S*:: *nat list*
assumes *nonzero*: $p \neq 0$
assumes *inv*: *length subsets* $\geq$ *length signs*
assumes *well-def-h*: $\forall x.$ *List.member S x* $\longrightarrow$ $x <$ *length subsets*

shows *matrix-A signs* (*take-indices subsets S*) = *take-rows-from-matrix* (*matrix-A signs subsets*) *S*
⟨*proof*⟩

lemma *reduce-system-matrix-match*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *nonzero*: *p* ≠ *0*
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation p qs subsets signs*
assumes *inv*: *invertible-mat* (*matrix-A signs subsets*)
shows *matrix-A* (*get-signs* (*reduce-system p* (*qs*, ((*matrix-A signs subsets*), (*subsets*, *signs*))))) (*get-subsets* (*reduce-system p* (*qs*, ((*matrix-A signs subsets*), (*subsets*, *signs*))))) = (*get-matrix* (*reduce-system p* (*qs*, ((*matrix-A signs subsets*), (*subsets*, *signs*))))) ⟨*proof*⟩

14.4 Showing invertibility preserved when reducing

lemma *well-def-find-zeros-from-lhs-vec*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *len-eq*: *length subsets* = *length signs*
assumes *inv*: *invertible-mat* (*matrix-A signs subsets*)
assumes *nonzero*: *p* ≠ *0*
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation p qs subsets signs*
shows (⋀*j*. *j* ∈ *set* (*find-nonzeros-from-input-vec* (*solve-for-lhs p qs subsets* (*matrix-A signs subsets*))) ⟹ *j* < *length* (*cols* (*matrix-A signs subsets*)))
⟨*proof*⟩

lemma *take-cols-subsets-og-cols*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *len-eq*: *length subsets* = *length signs*
assumes *inv*: *invertible-mat* (*matrix-A signs subsets*)
assumes *nonzero*: *p* ≠ *0*

assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation p qs subsets signs*
shows *set* (*take-indices* (*cols* (*matrix-A signs subsets*))
(*find-nonzeros-from-input-vec* (*solve-for-lhs p qs subsets* (*matrix-A signs subsets*))))
⊆ *set* (*cols* (*matrix-A signs subsets*))
⟨*proof*⟩

lemma *reduction-doesnt-break-things-invertibility-step1*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *len-eq*: *length subsets* = *length signs*
assumes *inv*: *invertible-mat* (*matrix-A signs subsets*)
assumes *nonzero*: $p \neq 0$
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation p qs subsets signs*
shows *vec-space.rank* (*length signs*) (*reduce-mat-cols* (*matrix-A signs subsets*) (*solve-for-lhs p qs subsets* (*matrix-A signs subsets*))) =
(*length* (*find-nonzeros-from-input-vec* (*solve-for-lhs p qs subsets* (*matrix-A signs subsets*))))
⟨*proof*⟩

lemma *rechar-take-cols*: *take-cols-var A B* = *take-cols-from-matrix A B*
⟨*proof*⟩

lemma *rows-and-cols-transpose*: *rows M* = *cols* M^T
⟨*proof*⟩

lemma *take-rows-and-take-cols*: (*take-rows-from-matrix M r*) = (*take-cols-from-matrix* M^T *r*)^T
⟨*proof*⟩

lemma *reduction-doesnt-break-things-invertibility*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *len-eq*: *length subsets* = *length signs*
assumes *inv*: *invertible-mat* (*matrix-A signs subsets*)
assumes *nonzero*: $p \neq 0$
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots-copr p qs*) ⊆ *set*(*signs*)

assumes *match*: *satisfy-equation p qs subsets signs*
shows *invertible-mat* (*get-matrix* (*reduce-system p* (*qs*, ((*matrix-A signs subsets*), (*subsets*, *signs*))))))
⟨*proof*⟩

14.5 Well def signs preserved when reducing

lemma *reduction-doesnt-break-length-signs*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *length-init* : ∀ *x*∈ *set*(*signs*). *length x* = *length qs*
assumes *sat-eq*: *satisfy-equation p qs subsets signs*
assumes *inv-mat*: *invertible-mat* (*matrix-A signs subsets*)
shows ∀ *x* ∈ *set*(*reduction-signs p qs signs subsets* (*matrix-A signs subsets*)).
length x = *length qs*
⟨*proof*⟩

14.6 Distinct signs preserved when reducing

lemma *reduction-signs-are-distinct*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *sat-eq*: *satisfy-equation p qs subsets signs*
assumes *inv-mat*: *invertible-mat* (*matrix-A signs subsets*)
assumes *distinct-init*: *distinct signs*
shows *distinct* (*reduction-signs p qs signs subsets* (*matrix-A signs subsets*))
⟨*proof*⟩

14.7 Well def subsets preserved when reducing

lemma *reduction-doesnt-break-subsets*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *nonzero*: *p* ≠ *0*
assumes *length-init* : *all-list-constr subsets* (*length qs*)
assumes *sat-eq*: *satisfy-equation p qs subsets signs*
assumes *inv-mat*: *invertible-mat* (*matrix-A signs subsets*)
shows *all-list-constr* (*reduction-subsets p qs signs subsets* (*matrix-A signs subsets*)) (*length qs*)
⟨*proof*⟩

15 Overall Lemmas

lemma *combining-to-smash*: *combine-systems p* (*qs1*, *m1*, (*sub1*, *sgn1*)) (*qs2*, *m2*, (*sub2*, *sgn2*))
= *smash-systems p qs1 qs2 sub1 sub2 sgn1 sgn2 m1 m2*
⟨*proof*⟩

lemma *getter-functions*: *calculate-data p qs* = (*get-matrix* (*calculate-data p qs*), (*get-subsets* (*calculate-data p qs*), *get-signs* (*calculate-data p qs*)))
⟨*proof*⟩

15.1 Key properties preserved

15.1.1 Properties preserved when combining and reducing systems

lemma *combining-sys-satisfies-properties-helper*:
fixes *p*:: *real poly*
fixes *qs1* :: *real poly list*
fixes *qs2* :: *real poly list*
fixes *subsets1 subsets2* :: *nat list list*
fixes *signs1 signs2* :: *rat list list*
fixes *matrix1 matrix2*:: *rat mat*
assumes *nonzero*: *p* ≠ *0*
assumes *nontriv1*: *length qs1* > *0*
assumes *pairwise-rel-prime1*: ∀ *q*. ((*List.member qs1 q*) ⟶ (*coprime p q*))
assumes *nontriv2*: *length qs2* > *0*
assumes *pairwise-rel-prime2*: ∀ *q*. ((*List.member qs2 q*) ⟶ (*coprime p q*))
assumes *satisfies-properties-sys1*: *satisfies-properties p qs1 subsets1 signs1 matrix1*
assumes *satisfies-properties-sys2*: *satisfies-properties p qs2 subsets2 signs2 matrix2*
shows *satisfies-properties p* (*qs1*@*qs2*) (*get-subsets* (*snd* ((*combine-systems p* (*qs1*,(*matrix1*, (*subsets1*, *signs1*))) (*qs2*,(*matrix2*, (*subsets2*, *signs2*)))))))
(*get-signs* (*snd* ((*combine-systems p* (*qs1*,(*matrix1*, (*subsets1*, *signs1*))) (*qs2*,(*matrix2*, (*subsets2*, *signs2*)))))))
(*get-matrix* (*snd* ((*combine-systems p* (*qs1*,(*matrix1*, (*subsets1*, *signs1*))) (*qs2*,(*matrix2*, (*subsets2*, *signs2*)))))))
⟨*proof*⟩

lemma *combining-sys-satisfies-properties*:
fixes *p*:: *real poly*
fixes *qs1* :: *real poly list*
fixes *qs2* :: *real poly list*
assumes *nonzero*: *p* ≠ *0*
assumes *nontriv1*: *length qs1* > *0*
assumes *pairwise-rel-prime1*: ∀ *q*. ((*List.member qs1 q*) ⟶ (*coprime p q*))
assumes *nontriv2*: *length qs2* > *0*
assumes *pairwise-rel-prime2*: ∀ *q*. ((*List.member qs2 q*) ⟶ (*coprime p q*))
assumes *satisfies-properties-sys1*: *satisfies-properties p qs1* (*get-subsets* (*calculate-data*

p qs1)) (*get-signs* (*calculate-data p qs1*)) (*get-matrix* (*calculate-data p qs1*))
assumes *satisfies-properties-sys2*: *satisfies-properties p qs2* (*get-subsets* (*calculate-data p qs2*)) (*get-signs* (*calculate-data p qs2*)) (*get-matrix* (*calculate-data p qs2*))
shows *satisfies-properties p* (*qs1@qs2*) (*get-subsets* (*snd* ((*combine-systems p* (*qs1*,*calculate-data p qs1*) (*qs2*,*calculate-data p qs2*)))))
(*get-signs* (*snd* ((*combine-systems p* (*qs1*,*calculate-data p qs1*) (*qs2*,*calculate-data p qs2*)))))
(*get-matrix* (*snd* ((*combine-systems p* (*qs1*,*calculate-data p qs1*) (*qs2*,*calculate-data p qs2*)))))
⟨*proof*⟩

lemma *reducing-sys-satisfies-properties*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
fixes *matrix*:: *rat mat*
assumes *nonzero*: $p \neq 0$
assumes *nontriv*: *length qs* > 0
assumes *pairwise-rel-prime*: ∀ *q*. ((*List.member qs q*) ⟶ (*coprime p q*))
assumes *satisfies-properties-sys*: *satisfies-properties p qs subsets signs matrix*
shows *satisfies-properties p qs* (*get-subsets* (*reduce-system p* (*qs*,*matrix*,*subsets*,*signs*)))
(*get-signs* (*reduce-system p* (*qs*,*matrix*,*subsets*,*signs*)))
(*get-matrix* (*reduce-system p* (*qs*,*matrix*,*subsets*,*signs*)))
⟨*proof*⟩

15.1.2 For length 1 qs

lemma *length-1-calculate-data-satisfies-properties*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *nonzero*: $p \neq 0$
assumes *len1*: *length qs* = 1
assumes *pairwise-rel-prime*: ∀ *q*. ((*List.member qs q*) ⟶ (*coprime p q*))
shows *satisfies-properties p qs* (*get-subsets* (*calculate-data p qs*)) (*get-signs* (*calculate-data p qs*)) (*get-matrix* (*calculate-data p qs*))
⟨*proof*⟩

15.1.3 For arbitrary qs

lemma *append-not-distinct-helper*: (*List.member l1 m* ∧ *List.member l2 m*) ⟶ (*distinct* (*l1@l2*) = *False*)
⟨*proof*⟩

lemma *calculate-data-satisfies-properties*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*

fixes *signs* :: *rat list list*
shows $(p \neq 0 \wedge (length\ qs > 0) \wedge (\forall q.\ ((List.member\ qs\ q) \longrightarrow (coprime\ p\ q)))$
)
$\longrightarrow$ *satisfies-properties p qs* (*get-subsets* (*calculate-data p qs*)) (*get-signs* (*calculate-data p qs*)) (*get-matrix* (*calculate-data p qs*))
⟨*proof*⟩

15.2 Some key results on consistent sign assignments

lemma *find-consistent-signs-at-roots-len1*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *nonzero*: $p \neq 0$
assumes *len1*: $length\ qs = 1$
assumes *pairwise-rel-prime*: $\forall q.\ ((List.member\ qs\ q) \longrightarrow (coprime\ p\ q))$
shows *set* (*find-consistent-signs-at-roots p qs*) = *set* (*characterize-consistent-signs-at-roots-copr p qs*)
⟨*proof*⟩

lemma *smaller-sys-are-good*:
fixes *p*:: *real poly*
fixes *qs1* :: *real poly list*
fixes *qs2* :: *real poly list*
fixes *subsets* :: *nat list list*
fixes *signs* :: *rat list list*
assumes *nonzero*: $p \neq 0$
assumes *nontriv1*: $length\ qs1 > 0$
assumes *pairwise-rel-prime1*: $\forall q.\ ((List.member\ qs1\ q) \longrightarrow (coprime\ p\ q))$
assumes *nontriv2*: $length\ qs2 > 0$
assumes *pairwise-rel-prime2*: $\forall q.\ ((List.member\ qs2\ q) \longrightarrow (coprime\ p\ q))$
assumes *set*(*find-consistent-signs-at-roots p qs1*) = *set*(*characterize-consistent-signs-at-roots-copr p qs1*)
assumes *set*(*find-consistent-signs-at-roots p qs2*) = *set*(*characterize-consistent-signs-at-roots-copr p qs2*)
shows *set*(*snd*(*snd*(*reduce-system p* (*combine-systems p* (*qs1*,*calculate-data p qs1*) (*qs2*,*calculate-data p qs2*)))))
= *set*(*characterize-consistent-signs-at-roots-copr p* (*qs1*@*qs2*))
⟨*proof*⟩

lemma *find-consistent-signs-at-roots-1*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
shows $(p \neq 0 \wedge length\ qs > 0 \wedge$
$(\forall q.\ ((List.member\ qs\ q) \longrightarrow (coprime\ p\ q)))) \longrightarrow$
set(*find-consistent-signs-at-roots p qs*) = *set*(*characterize-consistent-signs-at-roots-copr p qs*)

⟨*proof*⟩

lemma *find-consistent-signs-at-roots-0*:
fixes *p*:: *real poly*
assumes $p \neq 0$
shows *set*(*find-consistent-signs-at-roots p* []) =
set(*characterize-consistent-signs-at-roots-copr p* [])
⟨*proof*⟩

lemma *find-consistent-signs-at-roots-copr*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
assumes $p \neq 0$
assumes $\bigwedge q.\ q \in set\ qs \Longrightarrow coprime\ p\ q$
shows *set*(*find-consistent-signs-at-roots p qs*) = *set*(*characterize-consistent-signs-at-roots-copr p qs*)
⟨*proof*⟩

lemma *find-consistent-signs-at-roots*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
assumes $p \neq 0$
assumes $\bigwedge q.\ q \in set\ qs \Longrightarrow coprime\ p\ q$
shows *set*(*find-consistent-signs-at-roots p qs*) = *set*(*characterize-consistent-signs-at-roots p qs*)
⟨*proof*⟩

end
theory *BKR-Decision*
imports *BKR-Algorithm*
Berlekamp-Zassenhaus.Factorize-Rat-Poly
Algebraic-Numbers.Real-Roots
BKR-Proofs
HOL.Deriv
begin

16 Algorithm

16.1 Parsing

datatype *'a fml* =
And 'a fml 'a fml
| *Or 'a fml 'a fml*
| *Gt 'a*
| *Geq 'a*
| *Lt 'a*
| *Leq 'a*
| *Eq 'a*
| *Neq 'a*

```
primrec lookup-sem :: nat fml ⇒ ('a::linordered-field list) ⇒ bool
  where
    lookup-sem (And l r) ls = (lookup-sem l ls ∧ lookup-sem r ls)
  | lookup-sem (Or l r) ls = (lookup-sem l ls ∨ lookup-sem r ls)
  | lookup-sem (Gt p) ls = (ls ! p > 0)
  | lookup-sem (Geq p) ls = (ls ! p ≥ 0)
  | lookup-sem (Lt p) ls = (ls ! p < 0)
  | lookup-sem (Leq p) ls = (ls ! p ≤ 0)
  | lookup-sem (Eq p) ls = (ls ! p = 0)
  | lookup-sem (Neq p) ls = (ls ! p ≠ 0)


primrec poly-list :: 'a fml ⇒ 'a list
  where
    poly-list (And l r) = poly-list l @ poly-list r
  | poly-list (Or l r) = poly-list l @ poly-list r
  | poly-list (Gt p) = [p]
  | poly-list (Geq p) = [p]
  | poly-list (Lt p) = [p]
  | poly-list (Leq p) = [p]
  | poly-list (Eq p) = [p]
  | poly-list (Neq p) = [p]

primrec index-of-aux :: 'a list ⇒ 'a ⇒ nat ⇒ nat where
  index-of-aux [] y n = n
| index-of-aux (x#xs) y n =
  (if x = y then n else index-of-aux xs y (n+1))

definition index-of :: 'a list ⇒ 'a ⇒ nat where
  index-of xs y = index-of-aux xs y 0

definition convert :: 'a fml ⇒ (nat fml × 'a list)
  where
    convert fml = (
    let ps = remdups (poly-list fml)
    in
    (map-fml (index-of ps) fml, ps)
  )
```

16.2 Factoring

```
definition factorize-rat-poly-monic :: rat poly ⇒ (rat × (rat poly × nat) list)
  where
    factorize-rat-poly-monic p = (
    let (c,fs) =  factorize-rat-poly p ;
        lcs = prod-list (map (λ(f,i). (lead-coeff f) ^ Suc i) fs) ;
        fs = map (λ(f,i). (normalize f, i)) fs
```

```
    in
    (c * lcs,fs)
  )
```

```
definition factorize-polys :: rat poly list ⇒ (rat poly list × (rat × (nat × nat) list)
list)
  where
    factorize-polys ps = (
    let fact-ps = map factorize-rat-poly-monic ps;
        factors = remdups (map fst (concat (map snd fact-ps))) ;
          data = map (λ(c,fs). (c, map (λ(f,pow). (index-of factors f, pow) ) fs))
fact-ps
    in
      (factors,data)
  )
```

```
definition undo-factorize :: rat × (nat × nat) list ⇒ rat list ⇒ rat
  where
    undo-factorize cfs signs =
   squash
    (case cfs of (c,fs) ⇒
    (c * prod-list (map (λ(f,pow). (signs ! f) ^ Suc pow) fs)))
```

```
definition undo-factorize-polys :: (rat × (nat × nat) list) list ⇒ rat list ⇒ rat list
  where
    undo-factorize-polys ls signs = map (λl. undo-factorize l signs) ls
```

16.3 Auxiliary Polynomial

```
definition crb:: real poly ⇒ int where
  crb p = ceiling (2 + max-list-non-empty (map (λ i. norm (coeff p i)) [0 ..< degree
p])
    / norm (lead-coeff p))
```

```
definition coprime-r :: real poly list ⇒ real poly
  where
      coprime-r ps = pderiv (prod-list ps) * ([:−(crb (prod-list ps)),1:]) * ([:(crb
(prod-list ps)),1:])
```

16.4 Setting Up the Procedure

```
definition insertAt :: nat ⇒ 'a  ⇒ 'a list ⇒ 'a list where
  insertAt n x ls = take n ls @ x # (drop n ls)
```

```
definition removeAt :: nat ⇒ 'a list ⇒ 'a list where
```

```
  removeAt n ls = take n ls @ (drop (n+1) ls)

definition find-sgas-aux:: real poly list ⇒ rat list list
  where find-sgas-aux in-list =
  concat (map (λi.
    map (λv. insertAt i 0 v) (find-consistent-signs-at-roots (in-list ! i) (removeAt i
in-list))
  ) [0..<length in-list])


definition find-sgas :: real poly list ⇒ rat list list
  where
    find-sgas ps = (
    let r = coprime-r ps in
    find-consistent-signs-at-roots r ps @ find-sgas-aux ps
  )


definition find-consistent-signs :: rat poly list ⇒ rat list list
  where
    find-consistent-signs ps = (
     let (fs,data) = factorize-polys ps;
         sgas = find-sgas (map (map-poly of-rat) fs);
         rsgas = map (undo-factorize-polys data) sgas
    in
    (if fs = [] then [(map (λx. if poly x 0 < 0 then −1 else if poly x 0 = 0 then 0
else 1) ps)] else rsgas)
  )
```

16.5 Deciding Univariate Problems

```
definition decide-universal :: rat poly fml ⇒ bool
  where [code]:
    decide-universal fml = (
    let (fml-struct,polys) = convert fml;
        conds = find-consistent-signs polys
    in
     list-all (lookup-sem fml-struct) conds
  )

definition decide-existential :: rat poly fml ⇒ bool
  where [code]:
    decide-existential fml = (
    let (fml-struct,polys) = convert fml;
        conds = find-consistent-signs polys
    in
      find (lookup-sem fml-struct) conds ≠ None
  )
```

17 Proofs

17.1 Parsing and Semantics

primrec *real-sem* :: *real poly fml* ⇒ *real* ⇒ *bool*
where
real-sem (*And l r*) *x* = (*real-sem l x* ∧ *real-sem r x*)
| *real-sem* (*Or l r*) *x* = (*real-sem l x* ∨ *real-sem r x*)
| *real-sem* (*Gt p*) *x* = (*poly p x* > *0*)
| *real-sem* (*Geq p*) *x* = (*poly p x* ≥ *0*)
| *real-sem* (*Lt p*) *x* = (*poly p x* < *0*)
| *real-sem* (*Leq p*) *x* = (*poly p x* ≤ *0*)
| *real-sem* (*Eq p*) *x* = (*poly p x* = *0*)
| *real-sem* (*Neq p*) *x* = (*poly p x* ≠ *0*)

primrec *fml-sem* :: *rat poly fml* ⇒ *real* ⇒ *bool*
where
fml-sem (*And l r*) *x* = (*fml-sem l x* ∧ *fml-sem r x*)
| *fml-sem* (*Or l r*) *x* = (*fml-sem l x* ∨ *fml-sem r x*)
| *fml-sem* (*Gt p*) *x* = (*rpoly p x* > *0*)
| *fml-sem* (*Geq p*) *x* = (*rpoly p x* ≥ *0*)
| *fml-sem* (*Lt p*) *x* = (*rpoly p x* < *0*)
| *fml-sem* (*Leq p*) *x* = (*rpoly p x* ≤ *0*)
| *fml-sem* (*Eq p*) *x* = (*rpoly p x* = *0*)
| *fml-sem* (*Neq p*) *x* = (*rpoly p x* ≠ *0*)

lemma *poly-list-set-fml*:
shows *set* (*poly-list fml*) = *set-fml fml*
⟨*proof*⟩

lemma *convert-semantics-lem*:
assumes ⋀*p*. *p* ∈ *set* (*poly-list fml*) ⟹
ls ! (*index-of ps p*) = *rpoly p x*
shows *fml-sem fml x* = *lookup-sem* (*map-fml* (*index-of ps*) *fml*) *ls*
⟨*proof*⟩

lemma *index-of-aux-more*:
shows *index-of-aux ls p n* ≥ *n*
⟨*proof*⟩

lemma *index-of-aux-lookup*:
assumes *p* ∈ *set ls*
shows (*index-of-aux ls p n*) − *n* < *length ls*
ls ! ((*index-of-aux ls p n*) − *n*) = *p*
⟨*proof*⟩

lemma *index-of-lookup*:
assumes *p* ∈ *set ls*
shows *index-of ls p* < *length ls*

ls ! (*index-of ls p*) = *p*
⟨*proof*⟩

lemma *convert-semantics*:
shows *fml-sem fml x* = *lookup-sem* (*fst* (*convert fml*)) (*map* (λ*p*. *rpoly p x*) (*snd* (*convert fml*)))
⟨*proof*⟩

lemma *convert-closed*:
shows ⋀*i*. *i* ∈ *set-fml* (*fst* (*convert fml*)) ⟹ *i* < *length* (*snd* (*convert fml*))
⟨*proof*⟩

definition *sign-vec*::*rat poly list* ⇒ *real* ⇒ *rat list*
where *sign-vec qs x* ≡
map (*squash* ∘ (λ*p*. *rpoly p x*)) *qs*

definition *consistent-sign-vectors*::*rat poly list* ⇒ *real set* ⇒ *rat list set*
where *consistent-sign-vectors qs S* = (*sign-vec qs*) ‘ *S*

lemma *sign-vec-semantics*:
assumes ⋀*i*. *i* ∈ *set-fml fml* ⟹ *i* < *length ls*
shows *lookup-sem fml* (*map* (λ*p*. *rpoly p x*) *ls*) = *lookup-sem fml* (*sign-vec ls x*)
⟨*proof*⟩

lemma *universal-lookup-sem*:
assumes ⋀*i*. *i* ∈ *set-fml fml* ⟹ *i* < *length qs*
assumes *set signs* = *consistent-sign-vectors qs UNIV*
shows (∀ *x*::*real*. *lookup-sem fml* (*map* (λ*p*. *rpoly p x*) *qs*)) ⟷
list-all (*lookup-sem fml*) *signs*
⟨*proof*⟩

lemma *existential-lookup-sem*:
assumes ⋀*i*. *i* ∈ *set-fml fml* ⟹ *i* < *length qs*
assumes *set signs* = *consistent-sign-vectors qs UNIV*
shows (∃ *x*::*real*. *lookup-sem fml* (*map* (λ*p*. *rpoly p x*) *qs*)) ⟷
find (*lookup-sem fml*) *signs* ≠ *None*
⟨*proof*⟩

17.2 Factoring Lemmas

interpretation *of-rat-poly-hom*: *map-poly-comm-semiring-hom of-rat*⟨*proof*⟩
interpretation *of-rat-poly-hom*: *map-poly-comm-ring-hom of-rat*⟨*proof*⟩
interpretation *of-rat-poly-hom*: *map-poly-idom-hom of-rat*⟨*proof*⟩

lemma *finite-prod-map-of-rat-poly-hom*:
shows *poly* (*real-of-rat-poly* (∏ (*a*,*b*)∈*s*. *f a b*)) *y* = (∏ (*a*,*b*)∈*s*. *poly* (*real-of-rat-poly*

$(f\ a\ b))\ y)$
⟨*proof*⟩

lemma *sign-vec-index-of*:
assumes $f \in$ *set ftrs*
shows *sign-vec ftrs x* ! (*index-of ftrs f*) = *squash* (*rpoly f x*)
⟨*proof*⟩

lemma *squash-idem*:
shows *squash* (*squash x*) = *squash x*
⟨*proof*⟩

lemma *squash-mult*:
shows *squash* ((*a*::*real*) ∗ *b*) = *squash a* ∗ *squash b*
⟨*proof*⟩

lemma *squash-prod-list*:
shows *squash* (*prod-list* (*ls*::*real list*)) = *prod-list* (*map squash ls*)
⟨*proof*⟩

lemma *squash-pow*:
shows *squash* ((*x*::*real*) ^ (*y*::*nat*)) = (*squash x*) ^ *y*
⟨*proof*⟩

lemma *squash-real-of-rat*[*simp*]:
shows *squash* (*real-of-rat x*) = *squash x*
⟨*proof*⟩

lemma *factorize-rat-poly-monic-irreducible-monic*:
assumes *factorize-rat-poly-monic f* = (*c*,*fs*)
assumes (*fi*,*i*) ∈ *set fs*
shows *irreducible fi* ∧ *monic fi*
⟨*proof*⟩

lemma *square-free-normalize*:
assumes *square-free p*
shows *square-free* (*normalize p*)
⟨*proof*⟩

lemma *coprime-normalize*:
assumes *coprime a b*
shows *coprime* (*normalize a*) *b*
⟨*proof*⟩

lemma *undo-normalize*:
shows *a* = *Polynomial.smult* (*unit-factor* (*lead-coeff a*)) (*normalize a*)
⟨*proof*⟩

lemma *finite-smult-distr*:

assumes *distinct fs*
shows $(\prod (x,y) \in set\ fs.\ Polynomial.smult\ ((f\ x\ y)::rat)\ (g\ x\ y)) =$
$Polynomial.smult\ (\prod (x,y) \in set\ fs.\ f\ x\ y)\ (\prod (x,y) \in set\ fs.\ g\ x\ y)$
⟨*proof*⟩

lemma *normalize-coprime-degree*:
assumes *normalize* $(f::rat\ poly) = normalize\ g$
assumes *coprime f g*
shows *degree f = 0*
⟨*proof*⟩

lemma *factorize-rat-poly-monic-square-free-factorization*:
assumes *res*: *factorize-rat-poly-monic f* = (*c*,*fs*)
shows *square-free-factorization f* (*c*,*fs*)
⟨*proof*⟩

lemma *undo-factorize-correct*:
assumes *factorize-rat-poly-monic p* = (*c*,*fs*)
assumes $\bigwedge f\ p.\ (f,p) \in set\ fs \Longrightarrow f \in set\ ftrs$
shows *undo-factorize* (*c*,*map* (λ(*f*,*pow*). (*index-of ftrs f*, *pow*)) *fs*) (*sign-vec ftrs x*) = *squash* (*rpoly p x*)
⟨*proof*⟩

lemma *length-sign-vec*[*simp*]:
shows *length* (*sign-vec ps x*) = *length ps* ⟨*proof*⟩

lemma *factorize-polys-has-factors*:
assumes *factorize-polys ps* = (*ftrs*,*data*)
assumes $p \in set\ ps$
assumes *factorize-rat-poly-monic p* = (*c*,*fs*)
shows *set* (*map fst fs*) ⊆ *set ftrs*
⟨*proof*⟩

lemma *factorize-polys-undo-factorize-polys*:
assumes *factorize-polys ps* = (*ftrs*,*data*)
shows *undo-factorize-polys data* (*sign-vec ftrs x*) = *sign-vec ps x*
⟨*proof*⟩

lemma *factorize-polys-irreducible-monic*:
assumes *factorize-polys ps* = (*fs*,*data*)
shows *distinct fs* $\bigwedge f.\ f \in set\ fs \Longrightarrow irreducible\ f \wedge monic\ f$
⟨*proof*⟩

lemma *factorize-polys-square-free*:
assumes *factorize-polys ps* = (*fs*,*data*)
shows $\bigwedge f.\ f \in set\ fs \Longrightarrow square\text{-}free\ f$
⟨*proof*⟩

lemma *irreducible-monic-coprime*:

assumes *f*: *monic f irreducible* (*f*::*rat poly*)
assumes *g*: *monic g irreducible* (*g*::*rat poly*)
assumes $f \neq g$
shows *coprime f g*
⟨*proof*⟩

lemma *factorize-polys-coprime*:
assumes *factorize-polys ps* = (*fs*,*data*)
shows $\bigwedge f\ g.\ f \in set\ fs \Longrightarrow g \in set\ fs \Longrightarrow f \neq g \Longrightarrow coprime\ f\ g$
⟨*proof*⟩

lemma *coprime-rat-poly-real-poly*:
assumes *coprime p* (*q*::*rat poly*)
shows *coprime* (*real-of-rat-poly p*) ((*real-of-rat-poly q*)::*real poly*)
⟨*proof*⟩

lemma *coprime-rat-poly-iff-coprimereal-poly*:
shows *coprime p* (*q*::*rat poly*) ⟷ *coprime* (*real-of-rat-poly p*) ((*real-of-rat-poly q*)::*real poly*)
⟨*proof*⟩

lemma *factorize-polys-map-distinct*:
assumes *factorize-polys ps* = (*fs*,*data*)
assumes *fss* = *map real-of-rat-poly fs*
shows *distinct fss*
⟨*proof*⟩

lemma *factorize-polys-map-square-free*:
assumes *factorize-polys ps* = (*fs*,*data*)
assumes *fss* = *map real-of-rat-poly fs*
shows $\bigwedge f.\ f \in set\ fss \Longrightarrow square\text{-}free\ f$
⟨*proof*⟩

lemma *factorize-polys-map-coprime*:
assumes *factorize-polys ps* = (*fs*,*data*)
assumes *fss* = *map real-of-rat-poly fs*
shows $\bigwedge f\ g.\ f \in set\ fss \Longrightarrow g \in set\ fss \Longrightarrow f \neq g \Longrightarrow coprime\ f\ g$
⟨*proof*⟩

lemma *coprime-prod-list*:
assumes $\bigwedge p.\ p \in set\ ps \Longrightarrow p \neq 0$
assumes *coprime* (*prod-list ps*) (*q*::*real poly*)
shows $\bigwedge p.\ p \in set\ ps \Longrightarrow coprime\ p\ q$
⟨*proof*⟩

lemma *factorize-polys-square-free-prod-list*:
assumes *factorize-polys ps* = (*fs*,*data*)
shows *square-free* (*prod-list fs*)

⟨*proof*⟩

lemma *factorize-polys-map-square-free-prod-list*:
assumes *factorize-polys ps* = (*fs*,*data*)
assumes *fss* = *map real-of-rat-poly fs*
shows *square-free* (*prod-list fss*)
⟨*proof*⟩

lemma *factorize-polys-map-coprime-pderiv*:
assumes *factorize-polys ps* = (*fs*,*data*)
assumes *fss* = *map real-of-rat-poly fs*
shows ⋀*f*. *f* ∈ *set fss* ⟹ *coprime f* (*pderiv* (*prod-list fss*))
⟨*proof*⟩

definition *pairwise-coprime-list*:: *rat poly list* ⇒ *bool*
where *pairwise-coprime-list qs* =
(∀ *m* < *length qs*. ∀ *n* < *length qs*.
m ≠ *n* ⟶ *coprime* (*qs* ! *n*) (*qs* ! *m*))

lemma *coprime-factorize*:
fixes *qs*:: *rat poly list*
shows *pairwise-coprime-list* (*fst*(*factorize-polys qs*))
⟨*proof*⟩

lemma *squarefree-factorization-degree*:
assumes *square-free-factorization p* (*c*,*fs*)
shows *degree p* = *sum-list* (*map* (λ(*f*,*c*). (*c*+*1*) ∗ *degree f*) *fs*)
⟨*proof*⟩

17.3 Auxiliary Polynomial Lemmas

definition *roots-of-coprime-r*:: *real poly list* ⇒ *real set*
where *roots-of-coprime-r qs* = {*x*. *poly* (*coprime-r qs*) *x* = *0*}

lemma *crb-lem-pos*:
fixes *x*:: *real*
fixes *p*:: *real poly*
assumes *x*: *poly p x* = *0*
assumes *p*: *p* ≠ *0*
shows *x* < *crb p*
⟨*proof*⟩

lemma *crb-lem-neg*:
fixes *x*:: *real*
fixes *p*:: *real poly*
assumes *x*: *poly p x* = *0*
assumes *p*: *p* ≠ *0*
shows *x* > −*crb p*

⟨*proof*⟩

lemma *prod-zero*:
shows $\forall x$. *poly* (*prod-list* (*qs*:: *rat poly list*)) $x = 0 \longleftrightarrow (\exists q \in set\ (qs).\ poly\ q\ x = 0)$
⟨*proof*⟩

lemma *coprime-r-zero1*: *poly* (*coprime-r qs*) (*crb* (*prod-list qs*)) = *0*
⟨*proof*⟩

lemma *coprime-r-zero2*: *poly* (*coprime-r qs*) (−*crb* (*prod-list qs*)) = *0*
⟨*proof*⟩

lemma *coprime-mult*:
fixes *a*:: *real poly*
fixes *b*:: *real poly*
fixes *c*:: *real poly*
assumes *algebraic-semidom-class.coprime a b*
assumes *algebraic-semidom-class.coprime a c*
shows *algebraic-semidom-class.coprime a* (*b*∗*c*)
⟨*proof*⟩

lemma *coprime-r-coprime-prop*:
fixes *ps*:: *rat poly list*
assumes *factorize-polys ps* = (*fs*,*data*)
assumes *fss* = *map real-of-rat-poly fs*
shows $\bigwedge f.\ f \in set\ fss \Longrightarrow coprime\ f\ (coprime\text{-}r\ fss)$
⟨*proof*⟩

lemma *coprime-r-nonzero*:
fixes *ps*:: *rat poly list*
assumes *factorize-polys ps* = (*fs*,*data*)
assumes *nonempty-fs*: *fs* ≠ []
assumes *fss-is*: *fss* = *map real-of-rat-poly fs*
shows (*coprime-r fss*) ≠ *0*
⟨*proof*⟩

lemma *Rolle-pderiv*:
fixes *q*:: *real poly*
fixes *x1 x2*:: *real*
shows $(x1 < x2 \wedge poly\ q\ x1 = 0 \wedge poly\ q\ x2 = 0) \longrightarrow (\exists w.\ x1 < w \wedge w < x2 \wedge poly\ (pderiv\ q)\ w = 0)$
⟨*proof*⟩

lemma *coprime-r-roots-prop*:
fixes *qs*:: *real poly list*
assumes *pairwise-rel-prime*: $\forall$ *q1 q2*. (*q1* ≠ *q2* ∧ (*List.member qs q1*) ∧ (*List.member*

qs q2))⟶ *coprime q1 q2*
shows ∀ *x1*. ∀ *x2*. ((*x1* < *x2* ∧ (∃ *q1* ∈ *set* (*qs*). (*poly q1 x1*) = *0*) ∧ (∃ *q2*∈ *set*(*qs*). (*poly q2 x2*) = *0*)) ⟶ (∃ *q*. *x1* < *q* ∧ *q* < *x2* ∧ *poly* (*coprime-r qs*) *q* = *0*))
⟨*proof*⟩

17.4 Setting Up the Procedure: Lemmas

definition *has-no-zeros*::*rat list* ⇒ *bool*
where *has-no-zeros l* = (*0* ∉ *set l*)

lemma *hnz-prop*: *has-no-zeros l* ⟷ ¬(∃ *k* < *length l*. *l* ! *k* = *0*)
⟨*proof*⟩

definition *cast-rat-list*:: *rat poly list* ⇒ *real poly list*
where *cast-rat-list qs* = *map real-of-rat-poly qs*

definition *consistent-sign-vectors-r*::*real poly list* ⇒ *real set* ⇒ *rat list set*
where *consistent-sign-vectors-r qs S* = (*signs-at qs*) ‘ *S*

lemma *consistent-sign-vectors-consistent-sign-vectors-r*:
shows*consistent-sign-vectors-r* (*cast-rat-list qs*) *S* = *consistent-sign-vectors qs S*
⟨*proof*⟩

lemma *coprime-over-reals-coprime-over-rats*:
fixes *qs*:: *rat poly list*
assumes *csa-in*: *csa* ∈ (*consistent-sign-vectors qs UNIV*)
assumes *p1p2*: *p1*≠*p2* ∧ *p1* < *length csa* ∧ *p2* < *length csa* ∧ *csa* ! *p1* = *0* ∧ *csa* ! *p2* = *0*
shows ¬ *algebraic-semidom-class.coprime* (*nth qs p1*) (*nth qs p2*)
⟨*proof*⟩

lemma *zero-above*:
fixes *qs*:: *rat poly list*
fixes *x1*:: *real*
assumes *hnz*: *has-no-zeros* (*sign-vec qs x1*)
shows (∀ *x2* > *x1*. ((*sign-vec qs x1*) ≠ (*sign-vec qs x2*)) ⟶
(∃ (*r*::*real*) > *x1*. (*r* ≤ *x2* ∧ (∃ *q* ∈ *set*(*qs*). *rpoly q r* = *0*))))
⟨*proof*⟩

lemma *zero-below*:
fixes *qs*:: *rat poly list*
fixes *x1*:: *real*
assumes *hnz*: *has-no-zeros* (*sign-vec qs x1*)
shows ∀ *x2* < *x1*. ((*sign-vec qs x1*) ≠ (*sign-vec qs x2*)) ⟶
(∃ (*r*::*real*) < *x1*. (*r* ≥ *x2* ∧ (∃ *q* ∈ *set*(*qs*). *rpoly q r* = *0*)))
⟨*proof*⟩

lemma *sorted-list-lemma*:
fixes *l*:: *real list*
fixes *a b*:: *real*
fixes *n*:: *nat*
assumes $a < b$
assumes $(n + 1) < length\ l$
assumes *strict-sort*: *sorted-wrt* $(<)\ l$
assumes *lt-a*: $(l\ !\ n) < a$
assumes *b-lt*: $b < (l\ !\ (n+1))$
shows $\neg(\exists\,(x::real).\ (List.member\ l\ x \wedge a \leq x \wedge x \leq b))$
⟨*proof*⟩

lemma *roots-of-coprime-r-location-property*:
fixes *qs*:: *rat poly list*
fixes *sga*:: *rat list*
fixes *zer-list*
assumes *pairwise-rel-prime*: *pairwise-coprime-list qs*
assumes *all-squarefree*: $\bigwedge q.\ q \in set\ qs \Longrightarrow rsquarefree\ q$
assumes *x1-prop*: *sga* = *sign-vec qs x1*
assumes *hnz*: *has-no-zeros sga*
assumes *zer-list-prop*: *zer-list* = *sorted-list-of-set* $\{(x::real).\ (\exists\, q \in set(qs).\ (rpoly\ q\ x = 0))\}$
shows *zer-list* $\neq []$ $\longrightarrow$ $((x1 < (zer\text{-}list\ !\ 0)) \vee (x1 > (zer\text{-}list\ !\ (length\ zer\text{-}list - 1)) \vee$
$(\exists\ n < (length\ zer\text{-}list - 1).\ x1 > (zer\text{-}list\ !\ n) \wedge x1 < (zer\text{-}list\ !\ (n+1)))))$
⟨*proof*⟩

lemma *roots-of-coprime-r-capture-sgas-without-zeros*:
fixes *qs*:: *rat poly list*
fixes *sga*:: *rat list*
assumes *pairwise-rel-prime*: *pairwise-coprime-list qs*
assumes *all-squarefree*: $\bigwedge q.\ q \in set\ qs \Longrightarrow rsquarefree\ q$
assumes *ex-x1*: *sga* = *sign-vec qs x1*
assumes *hnz*: *has-no-zeros sga*
shows $(\exists\, w \in (roots\text{-}of\text{-}coprime\text{-}r\ (cast\text{-}rat\text{-}list\ qs)).\ sga = (sign\text{-}vec\ qs\ w))$
⟨*proof*⟩

lemma *find-csas-lemma-nozeros*:
fixes *qs*:: *rat poly list*
assumes *fs*: *factorize-polys qs* = (*fs*,*data*)
assumes *fs* $\neq []$
shows $(csa \in (consistent\text{-}sign\text{-}vectors\ fs\ UNIV) \wedge has\text{-}no\text{-}zeros\ csa) \longleftrightarrow$
csa ∈ *set* (*find-consistent-signs-at-roots* (*coprime-r* (*cast-rat-list fs*)) (*cast-rat-list fs*))
⟨*proof*⟩

lemma *range-eq*:
fixes *a*
fixes *b*
shows $a = b \longrightarrow range\ a = range\ b$
⟨*proof*⟩

lemma *removeAt-distinct*:
shows $distinct\ fss \Longrightarrow distinct\ (removeAt\ i\ fss)$
⟨*proof*⟩

lemma *consistent-signs-atw*:
assumes $\bigwedge p.\ p \in set\ fs \Longrightarrow poly\ p\ x \neq 0$
shows $consistent\text{-}sign\text{-}vec\text{-}copr\ fs\ x = signs\text{-}at\ fs\ x$
⟨*proof*⟩

lemma *characterize-consistent-signs-at-roots-rw*:
assumes $p \neq 0$
assumes *copr*: $\bigwedge q.\ q \in set\ fs \Longrightarrow coprime\ p\ q$
shows $set\ (characterize\text{-}consistent\text{-}signs\text{-}at\text{-}roots\ p\ fs) =$
$consistent\text{-}sign\text{-}vectors\text{-}r\ fs\ (\{x.\ poly\ p\ x = 0\})$
⟨*proof*⟩

lemma *signs-at-insert*:
assumes $i \leq length\ qs$
shows $insertAt\ i\ (squash\ (poly\ p\ x))\ (signs\text{-}at\ qs\ x) =$
$signs\text{-}at\ (insertAt\ i\ p\ qs)\ x$
⟨*proof*⟩

lemma *length-removeAt*:
assumes $i < length\ ls$
shows $length\ (removeAt\ i\ ls) = length\ ls - 1$
⟨*proof*⟩

lemma *insertAt-removeAt*:
assumes $i < length\ ls$
shows $insertAt\ i\ (ls!i)\ (removeAt\ i\ ls) = ls$
⟨*proof*⟩

lemma *insertAt-nth*:
assumes $i \leq length\ ls$
shows $insertAt\ i\ n\ ls\ !\ i = n$
⟨*proof*⟩

lemma *length-signs-at*[*simp*]:
shows $length\ (signs\text{-}at\ qs\ x) = length\ qs$
⟨*proof*⟩

lemma *find-csa-at-index*:

assumes $i < length\ fss$
assumes $\bigwedge p\ q.\ p \in set\ fss \Longrightarrow q \in set\ fss \Longrightarrow p \neq q \Longrightarrow coprime\ p\ q$
assumes $\bigwedge p.\ p \in set\ fss \Longrightarrow p \neq 0$
assumes *distinct fss*
shows
$set\ (map\ (\lambda v.\ insertAt\ i\ 0\ v)\ (find\text{-}consistent\text{-}signs\text{-}at\text{-}roots\ (fss\ !\ i)\ (removeAt\ i\ fss))) =$
$\{v \in consistent\text{-}sign\text{-}vectors\text{-}r\ fss\ UNIV.\ v\ !\ i = 0\}$
⟨*proof*⟩

lemma *in-set-concat-map*:
assumes $i < length\ ls$
assumes $x \in set\ (f\ (ls\ !\ i))$
shows $x \in set\ (concat\ (map\ f\ ls))$
⟨*proof*⟩

lemma *find-csas-lemma-onezero-gen*:
fixes *qs*:: *rat poly list*
assumes *fs*: *factorize-polys qs* = (*fs*,*data*)
assumes *fss*: *fss* = *cast-rat-list fs*
shows $(csa \in (consistent\text{-}sign\text{-}vectors\text{-}r\ fss\ UNIV) \wedge \neg(has\text{-}no\text{-}zeros\ csa))$
$\longleftrightarrow\ csa \in set\ (find\text{-}sgas\text{-}aux\ fss)$
⟨*proof*⟩

lemma *mem-append*: $List.member\ (l1@l2)\ m \longleftrightarrow (List.member\ l1\ m \vee List.member\ l2\ m)$
⟨*proof*⟩

lemma *same-sign-cond-rationals-reals*:
fixes *qs*:: *rat poly list*
assumes *lenh*: $length\ (fst(factorize\text{-}polys\ qs)) > 0$
shows $set(find\text{-}sgas\ (map\ (map\text{-}poly\ of\text{-}rat)\ (fst(factorize\text{-}polys\ qs)))) = consistent\text{-}sign\text{-}vectors\ (fst(factorize\text{-}polys\ qs))\ UNIV$
⟨*proof*⟩

lemma *factorize-polys-undo-factorize-polys-set*:
assumes *factorize-polys ps* = (*ftrs*,*data*)
assumes *sgas* = *find-sgas* (*map* (*map-poly of-rat*) *ftrs*)
assumes *sgas-set*: *set* (*sgas*) = *consistent-sign-vectors ftrs UNIV*
shows $set\ (map\ (undo\text{-}factorize\text{-}polys\ data)\ sgas) = consistent\text{-}sign\text{-}vectors\ ps\ UNIV$
⟨*proof*⟩

lemma *main-step-aux1*:
fixes *qs*:: *rat poly list*
assumes *empty*: $(fst(factorize\text{-}polys\ qs)) = []$
shows $set\ (find\text{-}consistent\text{-}signs\ qs) = consistent\text{-}sign\text{-}vectors\ qs\ UNIV$
⟨*proof*⟩

lemma *main-step-aux2*:
fixes *qs*:: *rat poly list*
assumes *lenh*: *length* (*fst*(*factorize-polys qs*)) > *0*
shows *set* (*find-consistent-signs qs*) = *consistent-sign-vectors qs UNIV*
⟨*proof*⟩

lemma *main-step*:
fixes *qs*:: *rat poly list*
shows *set* (*find-consistent-signs qs*) = *consistent-sign-vectors qs UNIV*
⟨*proof*⟩

17.5 Decision Procedure: Main Lemmas

lemma *decide-univ-lem-helper*:
assumes (*fml-struct*,*polys*) = *convert fml*
shows (∀ *x*::*real*. *lookup-sem fml-struct* (*map* (λ*p*. *rpoly p x*) *polys*)) ⟷
(*decide-universal fml*)
⟨*proof*⟩

lemma *decide-exis-lem-helper*:
assumes (*fml-struct*,*polys*) = *convert fml*
shows (∃ *x*::*real*. *lookup-sem fml-struct* (*map* (λ*p*. *rpoly p x*) *polys*)) ⟷
(*decide-existential fml*)
⟨*proof*⟩

theorem *decision-procedure*:
shows (∀ *x*::*real*. *fml-sem fml x*) ⟷ (*decide-universal fml*)
∃ *x*::*real*. *fml-sem fml x* ⟷ (*decide-existential fml*)
⟨*proof*⟩

end
theory *Renegar-Algorithm*
imports *BKR-Algorithm*
begin

definition *construct-NofI-R*:: *real poly* ⇒ *real poly list* ⇒ *real poly list* ⇒ *rat*
where *construct-NofI-R p I1 I2* = (
let new-p = *sum-list* (*map* (λ*x*. *x*^*2*) (*p* # *I1*)) *in*
rat-of-int (*changes-R-smods new-p* ((*pderiv new-p*)∗(*prod-list I2*))))

definition *construct-rhs-vector-R*:: *real poly* ⇒ *real poly list* ⇒ (*nat list* ∗ *nat list*) *list* ⇒ *rat vec*
where *construct-rhs-vector-R p qs Is* =
vec-of-list (*map* (λ(*I1*,*I2*).
(*construct-NofI-R p* (*retrieve-polys qs I1*) (*retrieve-polys qs I2*))) *Is*)

18 Base Case

definition *base-case-info-R*:: (*rat mat* × ((*nat list* ∗ *nat list*) *list* × *rat list list*))
where *base-case-info-R* =
((*mat-of-rows-list 3* [[*1*,*1*,*1*], [*0*,*1*,*0*], [*1*,*0*,−*1*]]),([([], []),([*0*], []),([], [*0*])], [[*1*],[*0*],[−*1*]]))

definition *base-case-solve-for-lhs*:: *real poly* ⇒ *real poly* ⇒ *rat vec*
where *base-case-solve-for-lhs p q* = (*mult-mat-vec* (*mat-of-rows-list 3* [[*1/2*, −*1/2*, *1/2*], [*0*, *1*, *0*], [*1/2*, −*1/2*, −*1/2*]]) (*construct-rhs-vector-R p* [*q*] [([], []),([*0*], []),([], [*0*])]))

definition *solve-for-lhs-R*:: *real poly* ⇒ *real poly list* ⇒ (*nat list* ∗ *nat list*) *list* ⇒ *rat mat* ⇒ *rat vec*
where *solve-for-lhs-R p qs subsets matr* =
mult-mat-vec (*matr-option* (*dim-row matr*) (*mat-inverse-var matr*)) (*construct-rhs-vector-R p qs subsets*)

19 Smashing

definition *subsets-smash-R*::*nat* ⇒ (*nat list*∗*nat list*) *list* ⇒ (*nat list*∗*nat list*) *list* ⇒ (*nat list*∗*nat list*) *list*
where *subsets-smash-R n s1 s2* = *concat* (*map* (λ*l1*. *map* (λ *l2*. (((*fst l1*) @ (*map* ((+) *n*) (*fst l2*))), (*snd l1*) @ (*map* ((+) *n*) (*snd l2*)))) *s2*) *s1*)

definition *smash-systems-R*:: *real poly* ⇒ *real poly list* ⇒ *real poly list* ⇒ (*nat list* ∗ *nat list*) *list* ⇒ (*nat list* ∗ *nat list*) *list* ⇒
rat list list ⇒ *rat list list* ⇒ *rat mat* ⇒ *rat mat* ⇒
real poly list × (*rat mat* × ((*nat list* ∗ *nat list*) *list* × *rat list list*))
where *smash-systems-R p qs1 qs2 subsets1 subsets2 signs1 signs2 mat1 mat2* =
(*qs1*@*qs2*, (*kronecker-product mat1 mat2*, (*subsets-smash-R* (*length qs1*) *subsets1 subsets2*, *signs-smash signs1 signs2*)))

fun *combine-systems-R*:: *real poly* ⇒ (*real poly list* × (*rat mat* × ((*nat list* ∗ *nat list*) *list* × *rat list list*))) ⇒ (*real poly list* × (*rat mat* × ((*nat list* ∗ *nat list*) *list* × *rat list list*)))
⇒ (*real poly list* × (*rat mat* × ((*nat list* ∗ *nat list*) *list* × *rat list list*)))
where *combine-systems-R p* (*qs1*, *m1*, *sub1*, *sgn1*) (*qs2*, *m2*, *sub2*, *sgn2*) =
(*smash-systems-R p qs1 qs2 sub1 sub2 sgn1 sgn2 m1 m2*)

20 Reduction

fun *reduction-step-R*:: *rat mat* ⇒ *rat list list* ⇒ (*nat list*∗*nat list*) *list* ⇒ *rat vec* ⇒ *rat mat* × ((*nat list*∗*nat list*) *list* × *rat list list*)
where *reduction-step-R A signs subsets lhs-vec* =
(*let reduce-cols-A* = (*reduce-mat-cols A lhs-vec*);
rows-keep = *rows-to-keep reduce-cols-A in*

(*take-rows-from-matrix reduce-cols-A rows-keep*,
(*take-indices subsets rows-keep*,
take-indices signs (*find-nonzeros-from-input-vec lhs-vec*))))

fun *reduce-system-R*:: *real poly* ⇒ (*real poly list* × (*rat mat* × ((*nat list*∗*nat list*) *list* × *rat list list*))) ⇒ (*rat mat* × ((*nat list*∗*nat list*) *list* × *rat list list*))
where *reduce-system-R p* (*qs*,*m*,*subs*,*signs*) =
reduction-step-R m signs subs (*solve-for-lhs-R p qs subs m*)

21 Overall algorithm

fun *calculate-data-R*:: *real poly* ⇒ *real poly list* ⇒ (*rat mat* × ((*nat list*∗*nat list*) *list* × *rat list list*))
where
calculate-data-R p qs =
(*let len* = *length qs in*
if len = *0 then*
(λ(*a*,*b*,*c*).(*a*,*b*,*map* (*drop 1*) *c*)) (*reduce-system-R p* ([*1*],*base-case-info-R*))
else if len ≤ *1 then reduce-system-R p* (*qs*,*base-case-info-R*)
else
(*let q1* = *take* (*len div 2*) *qs*; *left* = *calculate-data-R p q1*;
q2 = *drop* (*len div 2*) *qs*; *right* = *calculate-data-R p q2*;
comb = *combine-systems-R p* (*q1*,*left*) (*q2*,*right*) *in*
reduce-system-R p comb
)
)

definition *find-consistent-signs-at-roots-R*:: *real poly* ⇒ *real poly list* ⇒ *rat list list*
where [*code*]:
find-consistent-signs-at-roots-R p qs =
(*let* (*M*,*S*,Σ) = *calculate-data-R p qs in* Σ)

lemma *find-consistent-signs-at-roots-thm-R*:
shows *find-consistent-signs-at-roots-R p qs* = *snd* (*snd* (*calculate-data-R p qs*))
⟨*proof*⟩

end
theory *Renegar-Proofs*
imports *Renegar-Algorithm*
BKR-Proofs
begin

22 Tarski Queries Changed

lemma *construct-NofI-R-relation*:
fixes *p*:: *real poly*
fixes *I1*:: *real poly list*

fixes *I2*:: *real poly list*
shows *construct-NofI-R p I1 I2* =
construct-NofI (*sum-list* (*map power2* (*p* # *I1*))) *I2*
⟨*proof*⟩

lemma *sum-list-map-power2*:
shows *sum-list* (*map power2 ls*) ≥ (*0*::*real poly*)
⟨*proof*⟩

lemma *sum-list-map-power2-poly*:
shows *poly* (*sum-list* (*map power2* (*ls*::*real poly list*))) *x* ≥ (*0*::*real*)
⟨*proof*⟩

lemma *construct-NofI-R-prop-helper*:
fixes *p*:: *real poly*
fixes *I1*:: *real poly list*
fixes *I2*:: *real poly list*
assumes *nonzero*: *p*≠*0*
shows *construct-NofI-R p I1 I2* =
rat-of-int (*int* (*card* {*x*. *poly* (*sum-list* (*map* (λ*x*. *x*^*2*) (*p* # *I1*))) *x* = *0* ∧ *poly* (*prod-list I2*) *x* > *0*}) −
int (*card* {*x*. *poly* (*sum-list* (*map* (λ*x*. *x*^*2*) (*p* # *I1*))) *x* = *0* ∧ *poly* (*prod-list I2*) *x* < *0*}))
⟨*proof*⟩

lemma *zer-iff*:
fixes *p*:: *real poly*
fixes *ls*:: *real poly list*
shows *poly* (*sum-list* (*map* (λ*x*. *x*^*2*) *ls*)) *x* = *0* ⟷ (∀ *i* ∈ *set ls*. *poly i x* = *0*)
⟨*proof*⟩

lemma *construct-NofI-prop-R*:
fixes *p*:: *real poly*
fixes *I1*:: *real poly list*
fixes *I2*:: *real poly list*
assumes *nonzero*: *p*≠*0*
shows *construct-NofI-R p I1 I2* =
rat-of-int (*int* (*card* {*x*. *poly p x* = *0* ∧ (∀ *q* ∈ *set I1*. *poly q x* = *0*) ∧ *poly* (*prod-list I2*) *x* > *0*}) −
int (*card* {*x*. *poly p x* = *0* ∧ (∀ *q* ∈ *set I1*. *poly q x* = *0*) ∧ *poly* (*prod-list I2*) *x* < *0*}))
⟨*proof*⟩

23 Matrix Equation

definition *map-sgas*:: *rat list* ⇒ *rat list*
where *map-sgas l* = *map* (λ*r*. (*1* − *r*^*2*)) *l*

definition *z-R*:: (*nat list*∗*nat list*) ⇒ *rat list* ⇒ *rat*

where *z-R index-list sign-asg* ≡ (*prod-list* (*map* (*nth* (*map-sgas sign-asg*)) (*fst*(*index-list*))))∗(*prod-list* (*map* (*nth sign-asg*) (*snd*(*index-list*))))

definition *mtx-row-R*:: *rat list list* ⇒ (*nat list* ∗ *nat list*) ⇒ *rat list*
where *mtx-row-R sign-list index-list* ≡ (*map* ((*z-R index-list*)) *sign-list*)

definition *matrix-A-R*:: *rat list list* ⇒ (*nat list* ∗ *nat list*) *list* ⇒ *rat mat*
where *matrix-A-R sign-list subset-list* =
(*mat-of-rows-list* (*length sign-list*) (*map* (λ*i* .(*mtx-row-R sign-list i*)) *subset-list*))

definition *all-list-constr-R*:: (*nat list*∗*nat list*) *list* ⇒ *nat* ⇒ *bool*
where *all-list-constr-R L n* ≡ (∀ *x*. *List.member L x* ⟶ (*list-constr* (*fst x*) *n* ∧ *list-constr* (*snd x*) *n*))

definition *alt-matrix-A-R*:: *rat list list* ⇒ (*nat list*∗*nat list*) *list* ⇒ *rat mat*
where *alt-matrix-A-R signs subsets* = (*mat* (*length subsets*) (*length signs*)
(λ(*i*, *j*). *z-R* (*subsets* ! *i*) (*signs* ! *j*)))

lemma *alt-matrix-char-R*: *alt-matrix-A-R signs subsets* = *matrix-A-R signs subsets*
⟨*proof*⟩

lemma *subsets-are-rows-R*: ∀ *i* < (*length subsets*). *row* (*alt-matrix-A-R signs subsets*) *i* = *vec* (*length signs*) (λ*j*. *z-R* (*subsets* ! *i*) (*signs* ! *j*))
⟨*proof*⟩

lemma *signs-are-cols-R*: ∀ *i* < (*length signs*). *col* (*alt-matrix-A-R signs subsets*) *i* = *vec* (*length subsets*) (λ*j*. *z-R* (*subsets* ! *j*) (*signs* ! *i*))
⟨*proof*⟩

definition *consistent-sign-vec*::*real poly list* ⇒ *real* ⇒ *rat list*
where *consistent-sign-vec qs x* ≡
map (λ *q*. *if* (*poly q x* > *0*) *then* (*1*::*rat*) *else* (*if* (*poly q x* = *0*) *then* (*0*::*rat*) *else* (−*1*::*rat*))) *qs*

definition *construct-lhs-vector-R*:: *real poly* ⇒ *real poly list* ⇒ *rat list list* ⇒ *rat vec*
where *construct-lhs-vector-R p qs signs* ≡
vec-of-list (*map* (λ*w*. *rat-of-int* (*int* (*length* (*filter* (λ*v*. *v* = *w*) (*map* (*consistent-sign-vec qs*) (*characterize-root-list-p p*)))))) *signs*)

definition *satisfy-equation-R*:: *real poly* ⇒ *real poly list* ⇒ (*nat list*∗*nat list*) *list* ⇒ *rat list list* ⇒ *bool*
where *satisfy-equation-R p qs subset-list sign-list* =
(*mult-mat-vec* (*matrix-A-R sign-list subset-list*) (*construct-lhs-vector-R p qs sign-list*) = (*construct-rhs-vector-R p qs subset-list*))

lemma *construct-lhs-vector-clean-R*:
assumes $p \neq 0$
assumes $i < length\ signs$
shows $(construct\text{-}lhs\text{-}vector\text{-}R\ p\ qs\ signs)\ \$\ i =$
$card\ \{x.\ poly\ p\ x = 0 \wedge ((consistent\text{-}sign\text{-}vec\ qs\ x) = (nth\ signs\ i))\}$
⟨*proof*⟩

lemma *construct-lhs-vector-cleaner-R*:
assumes $p \neq 0$
shows $(construct\text{-}lhs\text{-}vector\text{-}R\ p\ qs\ signs) =$
$vec\text{-}of\text{-}list\ (map\ (\lambda s.\ rat\text{-}of\text{-}int\ (card\ \{x.\ poly\ p\ x = 0 \wedge ((consistent\text{-}sign\text{-}vec\ qs\ x) = s)\}))\ signs)$
⟨*proof*⟩

lemma *z-signs-R2*:
fixes *I*:: *nat list*
fixes *signs*:: *rat list*
assumes *lf*: $list\text{-}all\ (\lambda i.\ i < length\ signs)\ I$
assumes *la*: $list\text{-}all\ (\lambda s.\ s = 1 \vee s = 0 \vee s = -1)\ signs$
shows $(prod\text{-}list\ (map\ (nth\ signs)\ I)) = 1 \vee$
$(prod\text{-}list\ (map\ (nth\ signs)\ I)) = 0 \vee$
$(prod\text{-}list\ (map\ (nth\ signs)\ I)) = -1$ ⟨*proof*⟩

lemma *z-signs-R1*:
fixes *I*:: *nat list*
fixes *signs*:: *rat list*
assumes *lf*: $list\text{-}all\ (\lambda i.\ i < length\ signs)\ I$
assumes *la*: $list\text{-}all\ (\lambda s.\ s = 1 \vee s = 0 \vee s = -1)\ signs$
shows $(prod\text{-}list\ (map\ (nth\ (map\text{-}sgas\ signs))\ I)) = 1 \vee$
$(prod\text{-}list\ (map\ (nth\ (map\text{-}sgas\ signs))\ I)) = 0$ ⟨*proof*⟩

lemma *z-signs-R*:
fixes *I*:: $(nat\ list * nat\ list)$
fixes *signs*:: *rat list*
assumes *lf*: $list\text{-}all\ (\lambda i.\ i < length\ signs)\ (fst(I))$
assumes *ls*: $list\text{-}all\ (\lambda i.\ i < length\ signs)\ (snd(I))$
assumes *la*: $list\text{-}all\ (\lambda s.\ s = 1 \vee s = 0 \vee s = -1)\ signs$
shows $(z\text{-}R\ I\ signs = 1) \vee (z\text{-}R\ I\ signs = 0) \vee (z\text{-}R\ I\ signs = -1)$
⟨*proof*⟩

lemma *z-lemma-R*:
fixes *I*:: $nat\ list * nat\ list$
fixes *sign*:: *rat list*
assumes *consistent*: $sign \in set\ (characterize\text{-}consistent\text{-}signs\text{-}at\text{-}roots\ p\ qs)$
assumes *welldefined1*: $list\text{-}constr\ (fst\ I)\ (length\ qs)$
assumes *welldefined2*: $list\text{-}constr\ (snd\ I)\ (length\ qs)$
shows $(z\text{-}R\ I\ sign = 1) \vee (z\text{-}R\ I\ sign = 0) \vee (z\text{-}R\ I\ sign = -1)$

⟨*proof*⟩

lemma *in-set-R*:
fixes *p*:: *real poly*
assumes *nonzero*: *p≠0*
fixes *qs*:: *real poly list*
fixes *sign*:: *rat list*
fixes *x*:: *real*
assumes *root-p*: *x* ∈ {*x*. *poly p x* = *0*}
assumes *sign-fix*: *sign* = *consistent-sign-vec qs x*
shows *sign* ∈ *set* (*characterize-consistent-signs-at-roots p qs*)
⟨*proof*⟩

lemma *consistent-signs-prop-R*:
fixes *p*:: *real poly*
assumes *nonzero*: *p≠0*
fixes *qs*:: *real poly list*
fixes *sign*:: *rat list*
fixes *x*:: *real*
assumes *root-p*: *x* ∈ {*x*. *poly p x* = *0*}
assumes *sign-fix*: *sign* = *consistent-sign-vec qs x*
shows *list-all* (λ*s*. *s* = *1* ∨ *s* = *0* ∨ *s* = −*1*) *sign*
⟨*proof*⟩

lemma *horiz-vector-helper-pos-ind-R1*:
fixes *p*:: *real poly*
assumes *nonzero*: *p≠0*
fixes *qs*:: *real poly list*
fixes *I*:: *nat list*
fixes *sign*:: *rat list*
fixes *x*:: *real*
assumes *root-p*: *x* ∈ {*x*. *poly p x* = *0*}
assumes *sign-fix*: *sign* = *consistent-sign-vec qs x*
assumes *asm*: *list-constr I* (*length qs*)
shows (*prod-list* (*map* (*nth* (*map-sgas sign*)) *I*)) = *1* ⟷
(∀ *p* ∈ *set* (*retrieve-polys qs I*). *poly p x* = *0*)
⟨*proof*⟩

lemma *csv-length-same-as-qlist*:
fixes *p*:: *real poly*
assumes *nonzero*: *p≠0*
fixes *qs*:: *real poly list*
fixes *sign*:: *rat list*
fixes *x*:: *real*
assumes *root-p*: *x* ∈ {*x*. *poly p x* = *0*}
assumes *sign-fix*: *sign* = *consistent-sign-vec qs x*
shows *length sign* = *length qs*

⟨*proof*⟩

lemma *horiz-vector-helper-zer-ind-R2*:
fixes *p*:: *real poly*
assumes *nonzero*: *p≠0*
fixes *qs*:: *real poly list*
fixes *I*:: *nat list*
fixes *sign*:: *rat list*
fixes *x*:: *real*
assumes *root-p*: *x* ∈ {*x*. *poly p x* = *0*}
assumes *sign-fix*: *sign* = *consistent-sign-vec qs x*
assumes *asm*: *list-constr I* (*length qs*)
shows (*prod-list* (*map* (*nth sign*) *I*)) = *0* ⟷
(*poly* (*prod-list* (*retrieve-polys qs I*)) *x* = *0*)
⟨*proof*⟩

lemma *horiz-vector-helper-pos-ind-R2*:
fixes *p*:: *real poly*
assumes *nonzero*: *p≠0*
fixes *qs*:: *real poly list*
fixes *I*:: *nat list*
fixes *sign*:: *rat list*
fixes *x*:: *real*
assumes *root-p*: *x* ∈ {*x*. *poly p x* = *0*}
assumes *sign-fix*: *sign* = *consistent-sign-vec qs x*
assumes *asm*: *list-constr I* (*length qs*)
shows (*prod-list* (*map* (*nth sign*) *I*)) = *1* ⟷
(*poly* (*prod-list* (*retrieve-polys qs I*)) *x* > *0*)
⟨*proof*⟩

lemma *horiz-vector-helper-pos-ind-R*:
fixes *p*:: *real poly*
assumes *nonzero*: *p≠0*
fixes *qs*:: *real poly list*
fixes *I*:: *nat list* ∗ *nat list*
fixes *sign*:: *rat list*
fixes *x*:: *real*
assumes *root-p*: *x* ∈ {*x*. *poly p x* = *0*}
assumes *sign-fix*: *sign* = *consistent-sign-vec qs x*
assumes *asm1*: *list-constr* (*fst I*) (*length qs*)
assumes *asm2*: *list-constr* (*snd I*) (*length qs*)
shows ((∀ *p* ∈ *set* (*retrieve-polys qs* (*fst I*)). *poly p x* = *0*) ∧ (*poly* (*prod-list* (*retrieve-polys qs* (*snd I*))) *x* > *0*)) ⟷ (*z-R I sign* = *1*)
⟨*proof*⟩

lemma *horiz-vector-helper-pos-R*:
fixes *p*:: *real poly*
assumes *nonzero*: *p≠0*
fixes *qs*:: *real poly list*

fixes I:: *nat list*∗*nat list*
fixes *sign*:: *rat list*
fixes x:: *real*
assumes *root-p*: $x \in \{x.\ poly\ p\ x = 0\}$
assumes *sign-fix*: $sign = consistent\text{-}sign\text{-}vec\ qs\ x$
assumes *welldefined1*: $list\text{-}constr\ (fst\ I)\ (length\ qs)$
assumes *welldefined2*: $list\text{-}constr\ (snd\ I)\ (length\ qs)$
shows $((\forall p \in set\ (retrieve\text{-}polys\ qs\ (fst\ I)).\ poly\ p\ x = 0) \wedge (poly\ (prod\text{-}list\ (retrieve\text{-}polys\ qs\ (snd\ I)))\ x > 0)) \longleftrightarrow (z\text{-}R\ I\ sign = 1)$
⟨*proof*⟩

lemma *horiz-vector-helper-neg-R*:
fixes p:: *real poly*
assumes *nonzero*: $p \neq 0$
fixes *qs*:: *real poly list*
fixes I:: *nat list*∗*nat list*
fixes *sign*:: *rat list*
fixes x:: *real*
assumes *root-p*: $x \in \{x.\ poly\ p\ x = 0\}$
assumes *sign-fix*: $sign = consistent\text{-}sign\text{-}vec\ qs\ x$
assumes *welldefined1*: $list\text{-}constr\ (fst\ I)\ (length\ qs)$
assumes *welldefined2*: $list\text{-}constr\ (snd\ I)\ (length\ qs)$
shows $((\forall p \in set\ (retrieve\text{-}polys\ qs\ (fst\ I)).\ poly\ p\ x = 0) \wedge (poly\ (prod\text{-}list\ (retrieve\text{-}polys\ qs\ (snd\ I)))\ x < 0)) \longleftrightarrow (z\text{-}R\ I\ sign = -1)$
⟨*proof*⟩

lemma *lhs-dot-rewrite*:
fixes p:: *real poly*
fixes *qs*:: *real poly list*
fixes I:: *nat list*∗*nat list*
fixes *signs*:: *rat list list*
assumes *nonzero*: $p \neq 0$
shows
$(vec\text{-}of\text{-}list\ (mtx\text{-}row\text{-}R\ signs\ I) \cdot (construct\text{-}lhs\text{-}vector\text{-}R\ p\ qs\ signs)) =$
$sum\text{-}list\ (map\ (\lambda s.\ (z\text{-}R\ I\ s)\ *\ rat\text{-}of\text{-}int\ (card\ \{x.\ poly\ p\ x = 0 \wedge consistent\text{-}sign\text{-}vec\ qs\ x = s\}))\ signs)$
⟨*proof*⟩

lemma *construct-lhs-vector-drop-consistent-R*:
fixes p:: *real poly*
fixes *qs*:: *real poly list*
fixes I:: *nat list*∗*nat list*
fixes *signs*:: *rat list list*
assumes *nonzero*: $p \neq 0$
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: $set\ (characterize\text{-}consistent\text{-}signs\text{-}at\text{-}roots\ p\ qs) \subseteq set(signs)$
assumes *welldefined1*: $list\text{-}constr\ (fst\ I)\ (length\ qs)$
assumes *welldefined2*: $list\text{-}constr\ (snd\ I)\ (length\ qs)$

shows
(*vec-of-list* (*mtx-row-R signs I*) · (*construct-lhs-vector-R p qs signs*)) =
(*vec-of-list* (*mtx-row-R* (*characterize-consistent-signs-at-roots p qs*) *I*) ·
(*construct-lhs-vector-R p qs* (*characterize-consistent-signs-at-roots p qs*)))
⟨*proof*⟩

lemma *matrix-equation-helper-step-R*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
fixes *I*:: *nat list*∗*nat list*
fixes *signs*:: *rat list list*
assumes *nonzero*: *p*≠*0*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
assumes *welldefined1*: *list-constr* (*fst I*) (*length qs*)
assumes *welldefined2*: *list-constr* (*snd I*) (*length qs*)
shows (*vec-of-list* (*mtx-row-R signs I*) · (*construct-lhs-vector-R p qs signs*)) =
rat-of-int (*card* {*x*. *poly p x* = *0* ∧ (∀ *p* ∈ *set* (*retrieve-polys qs* (*fst I*)). *poly p x* = *0*) ∧ *poly* (*prod-list* (*retrieve-polys qs* (*snd I*))) *x* > *0*}) −
rat-of-int (*card* {*x*. *poly p x* = *0* ∧ (∀ *p* ∈ *set* (*retrieve-polys qs* (*fst I*)). *poly p x* = *0*) ∧ *poly* (*prod-list* (*retrieve-polys qs* (*snd I*))) *x* < *0*})
⟨*proof*⟩

lemma *matrix-equation-main-step-R*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
fixes *I*:: *nat list*∗*nat list*
fixes *signs*:: *rat list list*
assumes *nonzero*: *p*≠*0*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
assumes *welldefined1*: *list-constr* (*fst I*) (*length qs*)
assumes *welldefined2*: *list-constr* (*snd I*) (*length qs*)
shows (*vec-of-list* (*mtx-row-R signs I*) ·
(*construct-lhs-vector-R p qs signs*)) =
construct-NofI-R p (*retrieve-polys qs* (*fst I*)) (*retrieve-polys qs* (*snd I*))
⟨*proof*⟩

lemma *mtx-row-length-R*:
list-all (λ*r*. *length r* = *length signs*) (*map* (*mtx-row-R signs*) *ls*)
⟨*proof*⟩

theorem *matrix-equation-R*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
fixes *subsets*:: (*nat list*∗*nat list*) *list*
fixes *signs*:: *rat list list*
assumes *nonzero*: *p*≠*0*

assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
assumes *welldefined*: *all-list-constr-R* (*subsets*) (*length qs*)
shows *satisfy-equation-R p qs subsets signs*
⟨*proof*⟩

lemma *consistent-signs-at-roots-eq*:
assumes *p* ≠ *0*
shows *consistent-signs-at-roots p qs* =
set (*characterize-consistent-signs-at-roots p qs*)
⟨*proof*⟩

abbreviation *w-vec-R*:: *real poly* ⇒ *real poly list* ⇒ *rat list list* ⇒ *rat vec*
where *w-vec-R* ≡ *construct-lhs-vector-R*

abbreviation *v-vec-R*:: *real poly* ⇒ *real poly list* ⇒ (*nat list*∗*nat list*) *list* ⇒ *rat vec*
where *v-vec-R* ≡ *construct-rhs-vector-R*

abbreviation *M-mat-R*:: *rat list list* ⇒ (*nat list*∗*nat list*) *list* ⇒ *rat mat*
where *M-mat-R* ≡ *matrix-A-R*

theorem *matrix-equation-pretty*:
fixes *subsets*:: (*nat list*∗*nat list*) *list*
assumes *p*≠*0*
assumes *distinct signs*
assumes *consistent-signs-at-roots p qs* ⊆ *set signs*
assumes ⋀*a b i*. (*a*, *b*) ∈ *set* (*subsets*) ⟹ (*i* ∈ *set a* ∨ *i* ∈ *set b*) ⟹ *i* < *length qs*
shows *M-mat-R signs subsets* ∗$_v$ *w-vec-R p qs signs* = *v-vec-R p qs subsets*
⟨*proof*⟩

24 Base Case

definition *satisfies-properties-R*:: *real poly* ⇒ *real poly list* ⇒ (*nat list*∗*nat list*) *list* ⇒ *rat list list* ⇒ *rat mat* ⇒ *bool*
where *satisfies-properties-R p qs subsets signs matrix* =
(*all-list-constr-R subsets* (*length qs*) ∧ *well-def-signs* (*length qs*) *signs* ∧ *distinct signs*
∧ *satisfy-equation-R p qs subsets signs* ∧ *invertible-mat matrix* ∧ *matrix* = *matrix-A-R signs subsets*
∧ *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
)

lemma *mat-base-case-R*:
shows *matrix-A-R* [[*1*],[*0*],[−*1*]] [([], []),([*0*], []),([], [*0*])] = (*mat-of-rows-list 3* [[*1*,*1*,*1*], [*0*,*1*,*0*], [*1*,*0*,−*1*]])

⟨*proof*⟩

lemma *base-case-sgas-R*:
fixes *q p*:: *real poly*
assumes *nonzero*: $p \neq 0$
shows *set* (*characterize-consistent-signs-at-roots p* [*q*]) ⊆ {[*1*],[*0*], [− *1*]}
⟨*proof*⟩

lemma *base-case-sgas-alt-R*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
assumes *len1*: *length qs* = *1*
assumes *nonzero*: $p \neq 0$
shows *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ {[*1*], [*0*], [− *1*]}
⟨*proof*⟩

lemma *base-case-satisfy-equation-R*:
fixes *q p*:: *real poly*
assumes *nonzero*: $p \neq 0$
shows *satisfy-equation-R p* [*q*] [([], []),([*0*], []),([], [*0*])] [[*1*],[*0*],[−*1*]]
⟨*proof*⟩

lemma *base-case-satisfy-equation-alt-R*:
fixes *p*:: *real poly*
fixes *qs*:: *real poly list*
assumes *len1*: *length qs* = *1*
assumes *nonzero*: $p \neq 0$
shows *satisfy-equation-R p qs* [([], []),([*0*], []),([], [*0*])] [[*1*],[*0*],[−*1*]]
⟨*proof*⟩

lemma *base-case-matrix-eq*:
fixes *q p*:: *real poly*
assumes *nonzero*: $p \neq 0$
shows (*mult-mat-vec* (*mat-of-rows-list 3* [[*1*,*1*,*1*], [*0*,*1*,*0*], [*1*,*0*,−*1*]]) (*construct-lhs-vector-R p* [*q*] [[*1*],[*0*],[−*1*]]) =
(*construct-rhs-vector-R p* [*q*] [([], []),([*0*], []),([], [*0*])]))
⟨*proof*⟩

lemma *less-three*: (*n*::*nat*) < *Suc* (*Suc* (*Suc 0*)) ⟷ *n* = *0* ∨ *n* = *1* ∨ *n* = *2*
⟨*proof*⟩

lemma *inverse-mat-base-case-R*:
shows *inverts-mat* (*mat-of-rows-list 3* [[*1/2*, −*1/2*, *1/2*], [*0*, *1*, *0*], [*1/2*, −*1/2*, −*1/2*]]::*rat mat*) (*mat-of-rows-list 3* [[*1*,*1*,*1*], [*0*,*1*,*0*], [*1*,*0*,−*1*]]::*rat mat*)
⟨*proof*⟩

lemma *inverse-mat-base-case-2-R*:
shows *inverts-mat* (*mat-of-rows-list 3* [[*1*,*1*,*1*], [*0*,*1*,*0*], [*1*,*0*,−*1*]]::*rat mat*) (*mat-of-rows-list 3* [[*1/2*, −*1/2*, *1/2*], [*0*, *1*, *0*], [*1/2*, −*1/2*, −*1/2*]]:: *rat mat*)

⟨*proof*⟩

lemma *base-case-invertible-mat-R*:
shows *invertible-mat* (*matrix-A-R* [[*1*],[*0*], [− *1*]] [([], []),([*0*], []),([], [*0*])])
⟨*proof*⟩

25 Inductive Step

25.1 Lemmas on smashing subsets

definition *subsets-first-component-list*::(*nat list*∗*nat list*) *list* ⇒ *nat list list*
where *subsets-first-component-list I* = *map* (λ*I*. (*fst I*)) *I*

definition *subsets-second-component-list*::(*nat list*∗*nat list*) *list* ⇒ *nat list list*
where *subsets-second-component-list I* = *map* (λ*I*. (*snd I*)) *I*

definition *smash-list-list*::(*'a list*∗*'a list*) *list* ⇒ (*'a list*∗*'a list*) *list* ⇒ (*'a list*∗*'a list*) *list*
where *smash-list-list s1 s2* = *concat* (*map* (λ*l1*. *map* (λ*l2*. (*fst l1* @ *fst l2*, *snd l1* @ *snd l2*)) *s2*) *s1*)

lemma *smash-list-list-property-set*:
fixes *l1 l2* :: (*'a list*∗*'a list*) *list*
fixes *a b*:: *nat*
shows ∀ (*elem* :: (*'a list*∗*'a list*)). (*elem* ∈ (*set* (*smash-list-list l1 l2*)) ⟶
(∃ (*elem1*:: (*'a list*∗*'a list*)). ∃ (*elem2*:: (*'a list*∗*'a list*)).
(*elem1* ∈ *set*(*l1*) ∧ *elem2* ∈ *set*(*l2*) ∧ *elem* = (*fst elem1*@ *fst elem2*, *snd elem1* @ *snd elem2*))))
⟨*proof*⟩

lemma *subsets-smash-property-R*:
fixes *subsets1 subsets2* :: (*nat list*∗*nat list*) *list*
fixes *n*:: *nat*
shows ∀ (*elem* :: *nat list*∗*nat list*). (*List.member* (*subsets-smash-R n subsets1 subsets2*) *elem*) ⟶
(∃ (*elem1*::*nat list*∗*nat list*). ∃ (*elem2*::*nat list*∗*nat list*).
(*elem1* ∈ *set*(*subsets1*) ∧ *elem2* ∈ *set*(*subsets2*) ∧ *elem* = ((*fst elem1*) @ (*map* ((+) *n*) (*fst elem2*)), (*snd elem1*) @ (*map* ((+) *n*) (*snd elem2*)))))
⟨*proof*⟩

25.2 Well-defined subsets preserved when smashing

lemma *well-def-step-R*:
fixes *qs1 qs2* :: *real poly list*
fixes *subsets1 subsets2* :: (*nat list*∗*nat list*) *list*
assumes *well-def-subsets1*: *all-list-constr-R* (*subsets1*) (*length qs1*)
assumes *well-def-subsets2*: *all-list-constr-R* (*subsets2*) (*length qs2*)
shows *all-list-constr-R* ((*subsets-smash-R* (*length qs1*) *subsets1 subsets2*))

(*length* (*qs1* @ *qs2*))
⟨*proof*⟩

25.3 Consistent Sign Assignments Preserved When Smashing

lemma *subset-helper-R*:
fixes *p*:: *real poly*
fixes *qs1 qs2* :: *real poly list*
fixes *signs1 signs2* :: *rat list list*
shows ∀ *x* ∈ *set* (*characterize-consistent-signs-at-roots p* (*qs1* @ *qs2*)).
∃ *x1* ∈ *set* (*characterize-consistent-signs-at-roots p qs1*).
∃ *x2* ∈ *set* (*characterize-consistent-signs-at-roots p qs2*).
x = *x1*@*x2*
⟨*proof*⟩

lemma *subset-step-R*:
fixes *p*:: *real poly*
fixes *qs1 qs2* :: *real poly list*
fixes *signs1 signs2* :: *rat list list*
assumes *csa1*: *set* (*characterize-consistent-signs-at-roots p qs1*) ⊆ *set* (*signs1*)
assumes *csa2*: *set* (*characterize-consistent-signs-at-roots p qs2*) ⊆ *set* (*signs2*)
shows *set* (*characterize-consistent-signs-at-roots p*
(*qs1* @ *qs2*))
⊆ *set* (*signs-smash signs1 signs2*)
⟨*proof*⟩

25.4 Main Results

lemma *dim-row-matrix-A-R*[*simp*]:
shows *dim-row* (*matrix-A-R signs subsets*) = *length subsets*
⟨*proof*⟩

lemma *dim-col-matrix-A-R*[*simp*]:
shows *dim-col* (*matrix-A-R signs subsets*) = *length signs*
⟨*proof*⟩

lemma *length-subsets-smash-R*:
shows
length (*subsets-smash-R n subs1 subs2*) = *length subs1* ∗ *length subs2*
⟨*proof*⟩

lemma *z-append-R*:
fixes *xs*:: (*nat list* ∗ *nat list*)
assumes ⋀*i*. *i* ∈ *set* (*fst xs*) ⟹ *i* < *length as*
assumes ⋀*i*. *i* ∈ *set* (*snd xs*) ⟹ *i* < *length as*
shows *z-R* ((*fst xs*) @ (*map* ((+) (*length as*)) (*fst ys*)), (*snd xs*) @ (*map* ((+) (*length as*)) (*snd ys*))) (*as* @ *bs*) = *z-R xs as* ∗ *z-R ys bs*
⟨*proof*⟩

lemma *matrix-construction-is-kronecker-product-R*:
fixes *qs1* :: *real poly list*
fixes *subs1 subs2* :: (*nat list*∗*nat list*) *list*
fixes *signs1 signs2* :: *rat list list*

assumes $\bigwedge l\ i.\ l \in set\ subs1 \Longrightarrow (i \in set\ (fst\ l) \vee i \in set\ (snd\ l)) \Longrightarrow i < n1$
assumes $\bigwedge j.\ j \in set\ signs1 \Longrightarrow length\ j = n1$
shows (*matrix-A-R* (*signs-smash signs1 signs2*) (*subsets-smash-R n1 subs1 subs2*))
=
kronecker-product (*matrix-A-R signs1 subs1*) (*matrix-A-R signs2 subs2*)
⟨*proof*⟩

lemma *inductive-step-R*:
fixes *p*:: *real poly*
fixes *qs1 qs2* :: *real poly list*
fixes *subsets1 subsets2* :: (*nat list*∗*nat list*) *list*
fixes *signs1 signs2* :: *rat list list*
assumes *nonzero*: $p \neq 0$
assumes *nontriv1*: *length qs1* > *0*
assumes *nontriv2*: *length qs2* > *0*
assumes *welldefined-signs1*: *well-def-signs* (*length qs1*) *signs1*
assumes *welldefined-signs2*: *well-def-signs* (*length qs2*) *signs2*
assumes *distinct-signs1*: *distinct signs1*
assumes *distinct-signs2*: *distinct signs2*
assumes *all-info1*: *set* (*characterize-consistent-signs-at-roots p qs1*) ⊆ *set*(*signs1*)
assumes *all-info2*: *set* (*characterize-consistent-signs-at-roots p qs2*) ⊆ *set*(*signs2*)
assumes *welldefined-subsets1*: *all-list-constr-R* (*subsets1*) (*length qs1*)
assumes *welldefined-subsets2*: *all-list-constr-R* (*subsets2*) (*length qs2*)
assumes *invertibleMtx1*: *invertible-mat* (*matrix-A-R signs1 subsets1*)
assumes *invertibleMtx2*: *invertible-mat* (*matrix-A-R signs2 subsets2*)
shows *satisfy-equation-R p* (*qs1*@*qs2*) (*subsets-smash-R* (*length qs1*) *subsets1 subsets2*) (*signs-smash signs1 signs2*)
∧ *invertible-mat* (*matrix-A-R* (*signs-smash signs1 signs2*) (*subsets-smash-R* (*length qs1*) *subsets1 subsets2*))
⟨*proof*⟩

26 Reduction Step Proofs

definition *get-matrix-R*:: (*rat mat* × ((*nat list*∗*nat list*) *list* × *rat list list*)) ⇒ *rat mat*
where *get-matrix-R data* = *fst*(*data*)

definition *get-subsets-R*:: (*rat mat* × ((*nat list*∗*nat list*) *list* × *rat list list*)) ⇒ (*nat list*∗*nat list*) *list*
where *get-subsets-R data* = *fst*(*snd*(*data*))

definition *get-signs-R*:: (*rat mat* × ((*nat list*∗*nat list*) *list* × *rat list list*)) ⇒ *rat list list*
where *get-signs-R data* = *snd*(*snd*(*data*))

definition *reduction-signs-R*:: *real poly* ⇒ *real poly list* ⇒ *rat list list* ⇒ (*nat list*∗*nat list*) *list* ⇒ *rat mat* ⇒ *rat list list*
where *reduction-signs-R p qs signs subsets matr* =
(*take-indices signs* (*find-nonzeros-from-input-vec* (*solve-for-lhs-R p qs subsets matr*)))

definition *reduction-subsets-R*:: *real poly* ⇒ *real poly list* ⇒ *rat list list* ⇒ (*nat list*∗*nat list*) *list* ⇒ *rat mat* ⇒ (*nat list*∗*nat list*) *list*
where *reduction-subsets-R p qs signs subsets matr* =
(*take-indices subsets* (*rows-to-keep* (*reduce-mat-cols matr* (*solve-for-lhs-R p qs subsets matr*))))

lemma *reduction-signs-is-get-signs-R*: *reduction-signs-R p qs signs subsets m* = *get-signs-R* (*reduce-system-R p* (*qs*, (*m*, (*subsets*, *signs*))))
⟨*proof*⟩

lemma *reduction-subsets-is-get-subsets-R*: *reduction-subsets-R p qs signs subsets m* = *get-subsets-R* (*reduce-system-R p* (*qs*, (*m*, (*subsets*, *signs*))))
⟨*proof*⟩

26.1 Showing sign conditions preserved when reducing

lemma *take-indices-lem-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *arb-list* :: (*'a list*∗*'a list*) *list*
fixes *index-list* :: *nat list*
fixes *n*:: *nat*
assumes *lest*: *n* < *length* (*take-indices arb-list index-list*)
assumes *well-def*: ∀ *q*.(*List.member index-list q* ⟶ *q* < *length arb-list*)
shows ∃ *k*<*length arb-list*.
(*take-indices arb-list index-list*) ! *n* = *arb-list* ! *k*
⟨*proof*⟩

lemma *size-of-mat-R*:
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
shows (*matrix-A-R signs subsets*) ∈ *carrier-mat* (*length subsets*) (*length signs*)
⟨*proof*⟩

lemma *size-of-lhs-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *signs* :: *rat list list*

shows *dim-vec* (*construct-lhs-vector-R p qs signs*) = *length signs*
⟨*proof*⟩

lemma *size-of-rhs-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
shows *dim-vec* (*construct-rhs-vector-R p qs subsets*) = *length subsets*
⟨*proof*⟩

lemma *same-size-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *invertible-mat*: *invertible-mat* (*matrix-A-R signs subsets*)
shows *length subsets* = *length signs*
⟨*proof*⟩

lemma *construct-lhs-matches-solve-for-lhs-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *match*: *satisfy-equation-R p qs subsets signs*
assumes *invertible-mat*: *invertible-mat* (*matrix-A-R signs subsets*)
shows (*construct-lhs-vector-R p qs signs*) = *solve-for-lhs-R p qs subsets* (*matrix-A-R signs subsets*)
⟨*proof*⟩

lemma *reduction-doesnt-break-things-signs-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *nonzero*: *p* ≠ *0*
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation-R p qs subsets signs*
assumes *invertible-mat*: *invertible-mat* (*matrix-A-R signs subsets*)
shows *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*reduction-signs-R p qs signs subsets* (*matrix-A-R signs subsets*))
⟨*proof*⟩

lemma *reduction-deletes-bad-sign-conds-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*

fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *nonzero*: $p \neq 0$
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation-R p qs subsets signs*
assumes *invertible-mat*: *invertible-mat* (*matrix-A-R signs subsets*)
shows *set* (*characterize-consistent-signs-at-roots p qs*) = *set*(*reduction-signs-R p qs signs subsets* (*matrix-A-R signs subsets*))
⟨*proof*⟩

theorem *reduce-system-sign-conditions-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *nonzero*: $p \neq 0$
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation-R p qs subsets signs*
assumes *invertible-mat*: *invertible-mat* (*matrix-A-R signs subsets*)
shows *set* (*get-signs-R* (*reduce-system-R p* (*qs*, ((*matrix-A-R signs subsets*), (*subsets*, *signs*))))) = *set* (*characterize-consistent-signs-at-roots p qs*)
⟨*proof*⟩

26.2 Showing matrix equation preserved when reducing

lemma *reduce-system-matrix-equation-preserved-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *nonzero*: $p \neq 0$
assumes *welldefined-signs*: *well-def-signs* (*length qs*) *signs*
assumes *welldefined-subsets*: *all-list-constr-R* (*subsets*) (*length qs*)
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation-R p qs subsets signs*
assumes *invertible-mat*: *invertible-mat* (*matrix-A-R signs subsets*)
shows *satisfy-equation-R p qs* (*get-subsets-R* (*reduce-system-R p* (*qs*, ((*matrix-A-R signs subsets*), (*subsets*, *signs*))))) (*get-signs-R* (*reduce-system-R p* (*qs*, ((*matrix-A-R signs subsets*), (*subsets*, *signs*)))))
⟨*proof*⟩

26.3 Showing matrix preserved

lemma *reduce-system-matrix-signs-helper-aux-R*:

fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
fixes *S*:: *nat list*
assumes *well-def-h*: ∀ *x*. *List.member S x* ⟶ *x* < *length signs*
assumes *nonzero*: *p* ≠ *0*
shows *alt-matrix-A-R* (*take-indices signs S*) *subsets* = *take-cols-from-matrix* (*alt-matrix-A-R signs subsets*) *S*
⟨*proof*⟩

lemma *reduce-system-matrix-signs-helper-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
fixes *S*:: *nat list*
assumes *well-def-h*: ∀ *x*. *List.member S x* ⟶ *x* < *length signs*
assumes *nonzero*: *p* ≠ *0*
shows *matrix-A-R* (*take-indices signs S*) *subsets* = *take-cols-from-matrix* (*matrix-A-R signs subsets*) *S*
⟨*proof*⟩

lemma *reduce-system-matrix-subsets-helper-aux-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗ *nat list*) *list*
fixes *signs* :: *rat list list*
fixes *S*:: *nat list*
assumes *inv*: *length subsets* ≥ *length signs*
assumes *well-def-h*: ∀ *x*. *List.member S x* ⟶ *x* < *length subsets*
assumes *nonzero*: *p* ≠ *0*
shows *alt-matrix-A-R signs* (*take-indices subsets S*) = *take-rows-from-matrix* (*alt-matrix-A-R signs subsets*) *S*
⟨*proof*⟩

lemma *reduce-system-matrix-subsets-helper-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
fixes *S*:: *nat list*
assumes *nonzero*: *p* ≠ *0*
assumes *inv*: *length subsets* ≥ *length signs*
assumes *well-def-h*: ∀ *x*. *List.member S x* ⟶ *x* < *length subsets*
shows *matrix-A-R signs* (*take-indices subsets S*) = *take-rows-from-matrix* (*matrix-A-R signs subsets*) *S*

⟨*proof*⟩

lemma *reduce-system-matrix-match-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *nonzero*: $p \neq 0$
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation-R p qs subsets signs*
assumes *inv*: *invertible-mat* (*matrix-A-R signs subsets*)
shows *matrix-A-R* (*get-signs-R* (*reduce-system-R p* (*qs*, ((*matrix-A-R signs subsets*), (*subsets*, *signs*))))) (*get-subsets-R* (*reduce-system-R p* (*qs*, ((*matrix-A-R signs subsets*), (*subsets*, *signs*))))) = (*get-matrix-R* (*reduce-system-R p* (*qs*, ((*matrix-A-R signs subsets*), (*subsets*, *signs*)))))
⟨*proof*⟩

26.4 Showing invertibility preserved when reducing

thm *conjugatable-vec-space.gauss-jordan-single-rank*

thm *vec-space.full-rank-lin-indpt*

lemma *well-def-find-zeros-from-lhs-vec-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *len-eq*: *length subsets* = *length signs*
assumes *inv*: *invertible-mat* (*matrix-A-R signs subsets*)
assumes *nonzero*: $p \neq 0$
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation-R p qs subsets signs*
shows (⋀*j*. *j* ∈ *set* (*find-nonzeros-from-input-vec* (*solve-for-lhs-R p qs subsets* (*matrix-A-R signs subsets*))) ⟹ *j* < *length* (*cols* (*matrix-A-R signs subsets*)))
⟨*proof*⟩

lemma *take-cols-subsets-og-cols-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*

fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *len-eq*: *length subsets* = *length signs*
assumes *inv*: *invertible-mat* (*matrix-A-R signs subsets*)
assumes *nonzero*: $p \neq 0$
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation-R p qs subsets signs*
shows *set* (*take-indices* (*cols* (*matrix-A-R signs subsets*))
(*find-nonzeros-from-input-vec* (*solve-for-lhs-R p qs subsets* (*matrix-A-R signs subsets*))))
⊆ *set* (*cols* (*matrix-A-R signs subsets*))
⟨*proof*⟩

lemma *reduction-doesnt-break-things-invertibility-step1-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *len-eq*: *length subsets* = *length signs*
assumes *inv*: *invertible-mat* (*matrix-A-R signs subsets*)
assumes *nonzero*: $p \neq 0$
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation-R p qs subsets signs*
shows *vec-space.rank* (*length signs*) (*reduce-mat-cols* (*matrix-A-R signs subsets*) (*solve-for-lhs-R p qs subsets* (*matrix-A-R signs subsets*))) =
(*length* (*find-nonzeros-from-input-vec* (*solve-for-lhs-R p qs subsets* (*matrix-A-R signs subsets*))))
⟨*proof*⟩

lemma *reduction-doesnt-break-things-invertibility-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *len-eq*: *length subsets* = *length signs*
assumes *inv*: *invertible-mat* (*matrix-A-R signs subsets*)
assumes *nonzero*: $p \neq 0$
assumes *welldefined-signs1*: *well-def-signs* (*length qs*) *signs*
assumes *distinct-signs*: *distinct signs*
assumes *all-info*: *set* (*characterize-consistent-signs-at-roots p qs*) ⊆ *set*(*signs*)
assumes *match*: *satisfy-equation-R p qs subsets signs*
shows *invertible-mat* (*get-matrix-R* (*reduce-system-R p* (*qs*, ((*matrix-A-R signs subsets*), (*subsets*, *signs*)))))

⟨*proof*⟩

26.5 Well def signs preserved when reducing

lemma *reduction-doesnt-break-length-signs-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *length-init* : ∀ *x*∈ *set*(*signs*). *length x* = *length qs*
assumes *sat-eq*: *satisfy-equation-R p qs subsets signs*
assumes *inv-mat*: *invertible-mat* (*matrix-A-R signs subsets*)
shows ∀ *x* ∈ *set*(*reduction-signs-R p qs signs subsets* (*matrix-A-R signs subsets*)).

length x = *length qs*
⟨*proof*⟩

26.6 Distinct signs preserved when reducing

lemma *reduction-signs-are-distinct-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *sat-eq*: *satisfy-equation-R p qs subsets signs*
assumes *inv-mat*: *invertible-mat* (*matrix-A-R signs subsets*)
assumes *distinct-init*: *distinct signs*
shows *distinct* (*reduction-signs-R p qs signs subsets* (*matrix-A-R signs subsets*))
⟨*proof*⟩

26.7 Well def subsets preserved when reducing

lemma *reduction-doesnt-break-subsets-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗ *nat list*) *list*
fixes *signs* :: *rat list list*
assumes *nonzero*: *p* ≠ *0*
assumes *length-init* : *all-list-constr-R subsets* (*length qs*)
assumes *sat-eq*: *satisfy-equation-R p qs subsets signs*
assumes *inv-mat*: *invertible-mat* (*matrix-A-R signs subsets*)
shows *all-list-constr-R* (*reduction-subsets-R p qs signs subsets* (*matrix-A-R signs subsets*)) (*length qs*)
⟨*proof*⟩

27 Overall Lemmas

lemma *combining-to-smash-R*: *combine-systems-R p* (*qs1*, *m1*, (*sub1*, *sgn1*)) (*qs2*, *m2*, (*sub2*, *sgn2*))

= *smash-systems-R p qs1 qs2 sub1 sub2 sgn1 sgn2 m1 m2*
⟨*proof*⟩

lemma *getter-functions-R*: *calculate-data-R p qs* = (*get-matrix-R* (*calculate-data-R p qs*), (*get-subsets-R* (*calculate-data-R p qs*), *get-signs-R* (*calculate-data-R p qs*)))
⟨*proof*⟩

27.1 Key properties preserved

27.1.1 Properties preserved when combining and reducing systems

lemma *combining-sys-satisfies-properties-helper-R*:
fixes *p*:: *real poly*
fixes *qs1* :: *real poly list*
fixes *qs2* :: *real poly list*
fixes *subsets1 subsets2* :: (*nat list* ∗ *nat list*) *list*
fixes *signs1 signs2* :: *rat list list*
fixes *matrix1 matrix2*:: *rat mat*
assumes *nonzero*: *p* ≠ *0*
assumes *nontriv1*: *length qs1* > *0*
assumes *nontriv2*: *length qs2* > *0*
assumes *satisfies-properties-sys1*: *satisfies-properties-R p qs1 subsets1 signs1 matrix1*
assumes *satisfies-properties-sys2*: *satisfies-properties-R p qs2 subsets2 signs2 matrix2*
shows *satisfies-properties-R p* (*qs1*@*qs2*) (*get-subsets-R* (*snd* ((*combine-systems-R p* (*qs1*,(*matrix1*, (*subsets1*, *signs1*))) (*qs2*,(*matrix2*, (*subsets2*, *signs2*)))))))
(*get-signs-R* (*snd* ((*combine-systems-R p* (*qs1*,(*matrix1*, (*subsets1*, *signs1*))) (*qs2*,(*matrix2*, (*subsets2*, *signs2*)))))))
(*get-matrix-R* (*snd* ((*combine-systems-R p* (*qs1*,(*matrix1*, (*subsets1*, *signs1*))) (*qs2*,(*matrix2*, (*subsets2*, *signs2*)))))))
⟨*proof*⟩

lemma *combining-sys-satisfies-properties-R*:
fixes *p*:: *real poly*
fixes *qs1* :: *real poly list*
fixes *qs2* :: *real poly list*
assumes *nonzero*: *p* ≠ *0*
assumes *nontriv1*: *length qs1* > *0*
assumes *nontriv2*: *length qs2* > *0*
assumes *satisfies-properties-sys1*: *satisfies-properties-R p qs1* (*get-subsets-R* (*calculate-data-R p qs1*)) (*get-signs-R* (*calculate-data-R p qs1*)) (*get-matrix-R* (*calculate-data-R p qs1*))
assumes *satisfies-properties-sys2*: *satisfies-properties-R p qs2* (*get-subsets-R* (*calculate-data-R p qs2*)) (*get-signs-R* (*calculate-data-R p qs2*)) (*get-matrix-R* (*calculate-data-R p qs2*))
shows *satisfies-properties-R p* (*qs1*@*qs2*) (*get-subsets-R* (*snd* ((*combine-systems-R p* (*qs1*,*calculate-data-R p qs1*) (*qs2*,*calculate-data-R p qs2*)))))
(*get-signs-R* (*snd* ((*combine-systems-R p* (*qs1*,*calculate-data-R p qs1*) (*qs2*,*calculate-data-R*

p qs2)))))
(*get-matrix-R* (*snd* ((*combine-systems-R p* (*qs1*,*calculate-data-R p qs1*) (*qs2*,*calculate-data-R p qs2*)))))
⟨*proof*⟩

lemma *reducing-sys-satisfies-properties-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
fixes *matrix*:: *rat mat*
assumes *nonzero*: $p \neq 0$
assumes *nontriv*: *length qs* > 0
assumes *satisfies-properties-sys*: *satisfies-properties-R p qs subsets signs matrix*
shows *satisfies-properties-R p qs* (*get-subsets-R* (*reduce-system-R p* (*qs*,*matrix*,*subsets*,*signs*)))
(*get-signs-R* (*reduce-system-R p* (*qs*,*matrix*,*subsets*,*signs*)))
(*get-matrix-R* (*reduce-system-R p* (*qs*,*matrix*,*subsets*,*signs*)))
⟨*proof*⟩

27.1.2 For length 1 qs

lemma *length-1-calculate-data-satisfies-properties-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *nonzero*: $p \neq 0$
assumes *len1*: *length qs* = 1
shows *satisfies-properties-R p qs* (*get-subsets-R* (*calculate-data-R p qs*)) (*get-signs-R* (*calculate-data-R p qs*)) (*get-matrix-R* (*calculate-data-R p qs*))
⟨*proof*⟩

27.1.3 For arbitrary qs

lemma *calculate-data-satisfies-properties-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
shows ($p \neq 0 \wedge$ (*length qs* > 0))
⟶ *satisfies-properties-R p qs* (*get-subsets-R* (*calculate-data-R p qs*)) (*get-signs-R* (*calculate-data-R p qs*)) (*get-matrix-R* (*calculate-data-R p qs*))
⟨*proof*⟩

27.2 Some key results on consistent sign assignments

lemma *find-consistent-signs-at-roots-len1-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*

fixes *signs* :: *rat list list*
assumes *nonzero*: $p \neq 0$
assumes *len1*: *length qs* = *1*
shows *set* (*find-consistent-signs-at-roots-R p qs*) = *set* (*characterize-consistent-signs-at-roots p qs*)
⟨*proof*⟩

lemma *smaller-sys-are-good-R*:
fixes *p*:: *real poly*
fixes *qs1* :: *real poly list*
fixes *qs2* :: *real poly list*
fixes *subsets* :: (*nat list*∗*nat list*) *list*
fixes *signs* :: *rat list list*
assumes *nonzero*: $p \neq 0$
assumes *nontriv1*: *length qs1* > *0*
assumes *nontriv2*: *length qs2* > *0*
assumes *set*(*find-consistent-signs-at-roots-R p qs1*) = *set*(*characterize-consistent-signs-at-roots p qs1*)
assumes *set*(*find-consistent-signs-at-roots-R p qs2*) = *set*(*characterize-consistent-signs-at-roots p qs2*)
shows *set*(*snd*(*snd*(*reduce-system-R p* (*combine-systems-R p* (*qs1*,*calculate-data-R p qs1*) (*qs2*,*calculate-data-R p qs2*)))))
= *set*(*characterize-consistent-signs-at-roots p* (*qs1*@*qs2*))
⟨*proof*⟩

lemma *find-consistent-signs-at-roots-1-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
shows ($p \neq 0 \wedge$ *length qs* > *0*) ⟶
set(*find-consistent-signs-at-roots-R p qs*) = *set*(*characterize-consistent-signs-at-roots p qs*)
⟨*proof*⟩

lemma *find-consistent-signs-at-roots-0-R*:
fixes *p*:: *real poly*
assumes $p \neq 0$
shows *set*(*find-consistent-signs-at-roots-R p* []) =
set(*characterize-consistent-signs-at-roots p* [])
⟨*proof*⟩

lemma *find-consistent-signs-at-roots-R*:
fixes *p*:: *real poly*
fixes *qs* :: *real poly list*
assumes $p \neq 0$
shows *set*(*find-consistent-signs-at-roots-R p qs*) = *set*(*characterize-consistent-signs-at-roots p qs*)
⟨*proof*⟩

end

theory *Renegar-Decision*
imports *Renegar-Proofs*
BKR-Decision
begin

28 Algorithm

definition *consistent-sign-vectors-R*::*real poly list* ⇒ *real set* ⇒ *rat list set*
where *consistent-sign-vectors-R qs S* = (*consistent-sign-vec qs*) ‘ *S*

primrec *prod-list-var*:: *real poly list* ⇒ *real poly*
where *prod-list-var* [] = *1*
| *prod-list-var* (*h*# *T*) = (*if h* = *0 then* (*prod-list-var T*) *else* (*h*∗ *prod-list-var T*))

primrec *check-all-const-deg*:: *real poly list* ⇒ *bool*
where *check-all-const-deg* [] = *True*
| *check-all-const-deg* (*h*# *T*) = (*if degree h* = *0 then* (*check-all-const-deg T*) *else False*)

definition *poly-f* :: *real poly list* ⇒ *real poly*
where
poly-f ps =
(*if* (*check-all-const-deg ps* = *True*) *then* [:*0*, *1*:] *else*
(*pderiv* (*prod-list-var ps*)) ∗ (*prod-list-var ps*)∗ ([:−(*crb* (*prod-list-var ps*)),*1*:]) ∗ ([:(*crb* (*prod-list-var ps*)),*1*:]))

definition *find-consistent-signs-R* :: *real poly list* ⇒ *rat list list*
where
find-consistent-signs-R ps = *find-consistent-signs-at-roots-R* (*poly-f ps*) *ps*

definition *decide-universal-R* :: *real poly fml* ⇒ *bool*
where [*code*]:
decide-universal-R fml = (
let (*fml-struct*,*polys*) = *convert fml*;
conds = *find-consistent-signs-R polys*
in
list-all (*lookup-sem fml-struct*) *conds*
)

definition *decide-existential-R* :: *real poly fml* ⇒ *bool*
where [*code*]:
decide-existential-R fml = (
let (*fml-struct*,*polys*) = *convert fml*;
conds = *find-consistent-signs-R polys*
in
find (*lookup-sem fml-struct*) *conds* ≠ *None*
)

28.1 Proofs

definition *roots-of-poly-f*:: *real poly list* ⇒ *real set*
where *roots-of-poly-f qs* = {*x*. *poly* (*poly-f qs*) *x* = *0*}

lemma *prod-list-var-nonzero*:
shows *prod-list-var qs* ≠ *0*
⟨*proof*⟩

lemma *q-dvd-prod-list-var-prop*:
assumes *q* ∈ *set qs*
assumes *q* ≠ *0*
shows *q dvd prod-list-var qs* ⟨*proof*⟩

lemma *check-all-const-deg-prop*:
shows *check-all-const-deg l* = *True* ⟷ (∀ *p* ∈ *set*(*l*). *degree p* = *0*)
⟨*proof*⟩

lemma *poly-f-nonzero*:
fixes *qs* :: *real poly list*
shows (*poly-f qs*) ≠ *0*
⟨*proof*⟩

lemma *poly-f-roots-prop-1*:
fixes *qs*:: *real poly list*
assumes *non-const*: *check-all-const-deg qs* = *False*
shows ∀ *x1*. ∀ *x2*. ((*x1* < *x2* ∧ (∃ *q1* ∈ *set* (*qs*). *q1* ≠ *0* ∧ (*poly q1 x1*) = *0*) ∧ (∃ *q2*∈ *set*(*qs*). *q2* ≠ *0* ∧ (*poly q2 x2*) = *0*)) ⟶ (∃ *q*. *x1* < *q* ∧ *q* < *x2* ∧ *poly* (*poly-f qs*) *q* = *0*))
⟨*proof*⟩

lemma *main-step-aux1-R*:
fixes *qs*:: *real poly list*
assumes *non-const*: *check-all-const-deg qs* = *True*
shows *set* (*find-consistent-signs-R qs*) = *consistent-sign-vectors-R qs UNIV*
⟨*proof*⟩

lemma *sorted-list-lemma-var*:
fixes *l*:: *real list*
fixes *x*:: *real*
assumes *length l* > *1*
assumes *strict-sort*: *sorted-wrt* (<) *l*
assumes *x-not-in*: ¬ (*List.member l x*)
assumes *lt-a*: *x* > (*l* ! *0*)
assumes *b-lt*: *x* < (*l* ! (*length l* − *1*))
shows (∃ *n*. *n* < *length l* − *1* ∧ *x* > *l* ! *n* ∧ *x* < *l* !(*n*+*1*)) ⟨*proof*⟩

lemma *all-sample-points-prop*:
assumes *is-not-const*: *check-all-const-deg qs* = *False*
assumes *s-is*: *S* = (*characterize-root-list-p* (*pderiv* (*prod-list-var qs*) ∗ (*prod-list-var qs*) ∗ ([:−(*crb* (*prod-list-var qs*)),1:]) ∗ ([:(*crb* (*prod-list-var qs*)),1:])))
shows *consistent-sign-vectors-R qs UNIV* = *consistent-sign-vectors-R qs* (*set S*)
⟨*proof*⟩

lemma *main-step-aux2-R*:
fixes *qs*:: *real poly list*
assumes *is-not-const*: *check-all-const-deg qs* = *False*
shows *set* (*find-consistent-signs-R qs*) = *consistent-sign-vectors-R qs UNIV*
⟨*proof*⟩

lemma *main-step-R*:
fixes *qs*:: *real poly list*
shows *set* (*find-consistent-signs-R qs*) = *consistent-sign-vectors-R qs UNIV*
⟨*proof*⟩

lemma *consistent-sign-vec-semantics-R*:
assumes ⋀*i*. *i* ∈ *set-fml fml* ⟹ *i* < *length ls*
shows *lookup-sem fml* (*map* (λ*p*. *poly p x*) *ls*) = *lookup-sem fml* (*consistent-sign-vec ls x*)
⟨*proof*⟩

lemma *universal-lookup-sem-R*:
assumes ⋀*i*. *i* ∈ *set-fml fml* ⟹ *i* < *length qs*
assumes *set signs* = *consistent-sign-vectors-R qs UNIV*
shows (∀ *x*::*real*. *lookup-sem fml* (*map* (λ*p*. *poly p x*) *qs*)) ⟷
list-all (*lookup-sem fml*) *signs*
⟨*proof*⟩

lemma *existential-lookup-sem-R*:
assumes ⋀*i*. *i* ∈ *set-fml fml* ⟹ *i* < *length qs*
assumes *set signs* = *consistent-sign-vectors-R qs UNIV*
shows (∃ *x*::*real*. *lookup-sem fml* (*map* (λ*p*. *poly p x*) *qs*)) ⟷
find (*lookup-sem fml*) *signs* ≠ *None*
⟨*proof*⟩

lemma *decide-univ-lem-helper-R*:
fixes *fml*:: *real poly fml*
assumes (*fml-struct*,*polys*) = *convert fml*
shows (∀ *x*::*real*. *lookup-sem fml-struct* (*map* (λ*p*. *poly p x*) *polys*)) ⟷ (*decide-universal-R fml*)
⟨*proof*⟩

lemma *decide-exis-lem-helper-R*:
fixes *fml*:: *real poly fml*
assumes (*fml-struct*,*polys*) = *convert fml*

shows ($\exists$ *x*::*real*. *lookup-sem fml-struct* (*map* (λ*p*. *poly p x*) *polys*)) $\longleftrightarrow$ (*decide-existential-R fml*)
⟨*proof*⟩

lemma *convert-semantics-lem-R*:
assumes $\bigwedge$*p*. *p* $\in$ *set* (*poly-list fml*) $\Longrightarrow$
ls ! (*index-of ps p*) = *poly p x*
shows *real-sem fml x* = *lookup-sem* (*map-fml* (*index-of ps*) *fml*) *ls*
⟨*proof*⟩

lemma *convert-semantics-R*:
shows *real-sem fml x* = *lookup-sem* (*fst* (*convert fml*)) (*map* (λ*p*. *poly p x*) (*snd* (*convert fml*)))
⟨*proof*⟩

theorem *decision-procedure-R*:
shows ($\forall$ *x*::*real*. *real-sem fml x*) $\longleftrightarrow$ (*decide-universal-R fml*)
$\exists$ *x*::*real*. *real-sem fml x* $\longleftrightarrow$ (*decide-existential-R fml*)
⟨*proof*⟩

end

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