Spivey's Generalized Recurrence for Bell Numbers

Lukas Bulwahn

April 19, 2020

Abstract

This entry defines the Bell numbers [1] as the cardinality of set partitions for a carrier set of given size, and derives Spivey's generalized recurrence relation for Bell numbers [2] following his elegant and intuitive combinatorial proof.

As the set construction for the combinatorial proof requires construction of three intermediate structures, the main difficulty of the formalization is handling the overall combinatorial argument in a structured way. The introduced proof structure allows us to compose the combinatorial argument from its subparts, and supports to keep track how the detailed proof steps are related to the overall argument. To obtain this structure, this entry uses set monad notation for the set construction's definition, introduces suitable predicates and rules, and follows a repeating structure in its Isar proof.

Contents

1 Bell Numbers and Spivey's Generalized Recurrence

theory *Bell-Numbers*
imports
HOL−Library.FuncSet
HOL−Library.Monad-Syntax

HOL−Library.Stirling
Card-Partitions.Injectivity-Solver
Card-Partitions.Card-Partitions
begin

1.1 Preliminaries

1.1.1 Additions to FuncSet

lemma *extensional-funcset-ext*:
assumes $f \in A \to_E B$ $g \in A \to_E B$
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x$
shows $f = g$
⟨*proof*⟩

1.1.2 Additions for Injectivity Proofs

lemma *inj-on-impl-inj-on-image*:
assumes *inj-on f A*
assumes $\bigwedge x.\ x \in X \Longrightarrow x \subseteq A$
shows *inj-on* $((`)\ f)\ X$
⟨*proof*⟩

lemma *injectivity-union*:
assumes $A \cup B = C \cup D$
assumes $P\ A\ P\ C$
assumes $Q\ B\ Q\ D$
$\bigwedge S\ T.\ P\ S \Longrightarrow Q\ T \Longrightarrow S \cap T = \{\}$
shows $A = C \wedge B = D$
⟨*proof*⟩

lemma *injectivity-image*:
assumes $f\ `\ A = g\ `\ A$
assumes $\forall x{\in}A.\ invert\ (f\ x) = x \wedge invert\ (g\ x) = x$
shows $\forall x{\in}A.\ f\ x = g\ x$
⟨*proof*⟩

lemma *injectivity-image-union*:
assumes $(\lambda X.\ X \cup F\ X)\ `\ P = (\lambda X.\ X \cup G\ X)\ `\ P'$
assumes $\forall X \in P.\ X \subseteq A\ \forall X \in P'.\ X \subseteq A$
assumes $\forall X \in P.\ \forall y{\in}F\ X.\ y \notin A\ \forall X \in P'.\ \forall y{\in}G\ X.\ y \notin A$
shows $P = P'$
⟨*proof*⟩

1.2 Definition of Bell Numbers

definition *Bell* :: *nat* ⇒ *nat*
where
Bell n = *card* {*P*. *partition-on* {*0..<n*} *P*}

lemma *Bell-altdef*:
  assumes *finite A*
  shows $Bell\ (card\ A) = card\ \{P.\ partition\text{-}on\ A\ P\}$
$\langle proof \rangle$

lemma *Bell-0*:
  $Bell\ 0 = 1$
$\langle proof \rangle$

1.3 Construction of the Partitions

definition $construct\text{-}partition\text{-}on :: 'a\ set \Rightarrow 'a\ set \Rightarrow 'a\ set\ set\ set$
where
  $construct\text{-}partition\text{-}on\ B\ C =$
    $do\ \{$
      $k\ \leftarrow \{0..card\ B\};$
      $j\ \leftarrow \{0..card\ C\};$
      $P\ \leftarrow \{P.\ partition\text{-}on\ C\ P \wedge card\ P = j\};$
      $B' \leftarrow \{B'.\ B' \subseteq B \wedge card\ B' = k\};$
      $Q\ \leftarrow \{Q.\ partition\text{-}on\ B'\ Q\};$
      $f\ \leftarrow (B - B') \rightarrow_E P;$
      $P'\ \leftarrow \{(\lambda X.\ X \cup \{x \in B - B'.\ f\,x = X\})\ `\ P\};$
      $\{P' \cup Q\}$
    $\}$

lemma *construct-partition-on*:
  assumes *finite B finite C*
  assumes $B \cap C = \{\}$
  shows $construct\text{-}partition\text{-}on\ B\ C = \{P.\ partition\text{-}on\ (B \cup C)\ P\}$
$\langle proof \rangle$

1.4 Injectivity of the Set Construction

lemma *injectivity*:
  assumes $B \cap C = \{\}$
  assumes P: $(partition\text{-}on\ C\ P \wedge card\ P = j) \wedge (partition\text{-}on\ C\ P' \wedge card\ P' = j')$
  assumes B': $(B' \subseteq B \wedge card\ B' = k) \wedge (B'' \subseteq B \wedge card\ B'' = k')$
  assumes Q: $partition\text{-}on\ B'\ Q \wedge partition\text{-}on\ B''\ Q'$
  assumes f: $f \in B - B' \rightarrow_E P \wedge g \in B - B'' \rightarrow_E P'$
  assumes P': $P'' = (\lambda X.\ X \cup \{x \in B - B'.\ f\,x = X\})\ `\ P \wedge$
    $P''' = (\lambda X.\ X \cup \{x \in B - B''.\ g\ x = X\})\ `\ P'$
  assumes *eq-result*: $P'' \cup Q = P''' \cup Q'$
  shows $f = g$ **and** $Q = Q'$ **and** $B' = B''$
    and $P = P'$ **and** $j = j'$ **and** $k = k'$
$\langle proof \rangle$

1.5 The Generalized Bell Recurrence Relation

theorem *Bell-eq*:

Bell $(n + m) = (\sum k \leq n. \sum j \leq m. j$ ˆ $(n - k) *$ *Stirling* $m\ j * (n$ *choose* $k) *$ *Bell* $k)$
⟨*proof*⟩

1.6 Corollaries of the Generalized Bell Recurrence

corollary *Bell-Stirling-eq*:
Bell $m = (\sum j \leq m.$ *Stirling* $m\ j)$
⟨*proof*⟩

corollary *Bell-recursive-eq*:
Bell $(n + 1) = (\sum k \leq n. (n$ *choose* $k) *$ *Bell* $k)$
⟨*proof*⟩

end

References

[1] N. J. A. Sloane. A000110: Bell or exponential numbers: number of ways to partition a set of n labeled elements. In *The On-Line Encyclopedia of Integer Sequences*. https://oeis.org/A000110.

[2] M. Z. Spivey. A generalized recurrence for Bell numbers. *Journal of Integer Sequences*, 11, 2008. Electronic copy available at https://cs.uwaterloo.ca/journals/JIS/VOL11/Spivey/spivey25.pdf.