Banach-Steinhaus theorem*

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March 17, 2025

Abstract

We formalize in Isabelle/HOL a result [2] due to S. Banach and H. Steinhaus [1] known as Banach-Steinhaus theorem or Uniform boundedness principle: a pointwise-bounded family of continuous linear operators from a Banach space to a normed space is uniformly bounded. Our approach is an adaptation to Isabelle/HOL of a proof due to A. Sokal [3].

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theory *Banach-Steinhaus-Missing*
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HOL−Analysis.Bounded-Linear-Function
HOL−Analysis.Line-Segment

begin

*Supported by the ERC consolidator grant CerQuS (819317), the PRG team grant Secure Quantum Technology (PRG946) from the Estonian Research Council, and the Estonian Centre of Exellence in IT (EXCITE) funded by ERDF.

1.1 Results missing for the proof of Banach-Steinhaus theorem

The results proved here are preliminaries for the proof of Banach-Steinhaus theorem using Sokal's approach, but they do not explicitly appear in Sokal's paper [3].

Notation for the norm

open-bundle *norm-syntax* **begin**
notation *norm* (‹‖-‖›)
end

Notation for apply bilinear function

open-bundle *blinfun-apply-syntax* **begin**
notation *blinfun-apply* (**infixr** ‹$*_v$› *70*)
end

lemma *bdd-above-plus*:
fixes *f*::‹*'a* ⇒ *real*›
assumes ‹*bdd-above* (*f* ‘ *S*)› **and** ‹*bdd-above* (*g* ‘ *S*)›
shows ‹*bdd-above* ((λ *x*. *f x* + *g x*) ‘ *S*)›

Explanation: If the images of two real-valued functions *f*,*g* are bounded above on a set *S*, then the image of their sum is bounded on *S*.

⟨*proof*⟩

The maximum of two functions

definition *pointwise-max*:: (*'a* ⇒ *'b*::*ord*) ⇒ (*'a* ⇒ *'b*) ⇒ (*'a* ⇒ *'b*) **where**
‹*pointwise-max f g* = (λ*x*. *max* (*f x*) (*g x*))›

lemma *max-Sup-absorb-left*:
fixes *f g*::‹*'a* ⇒ *real*›
assumes ‹*X* ≠ {}› **and** ‹*bdd-above* (*f* ‘ *X*)› **and** ‹*bdd-above* (*g* ‘ *X*)› **and** ‹*Sup* (*f* ‘ *X*) ≥ *Sup* (*g* ‘ *X*)›
shows ‹*Sup* ((*pointwise-max f g*) ‘ *X*) = *Sup* (*f* ‘ *X*)›

Explanation: For real-valued functions *f* and *g*, if the supremum of *f* is greater-equal the supremum of *g*, then the supremum of *max f g* equals the supremum of *f*. (Under some technical conditions.)

⟨*proof*⟩

lemma *max-Sup-absorb-right*:
fixes *f g*::‹*'a* ⇒ *real*›
assumes ‹*X* ≠ {}› **and** ‹*bdd-above* (*f* ‘ *X*)› **and** ‹*bdd-above* (*g* ‘ *X*)› **and** ‹*Sup* (*f* ‘ *X*) ≤ *Sup* (*g* ‘ *X*)›
shows ‹*Sup* ((*pointwise-max f g*) ‘ *X*) = *Sup* (*g* ‘ *X*)›

Explanation: For real-valued functions *f* and *g* and a nonempty set *X*, such that the *f* and *g* are bounded above on *X*, if the supremum of *f* on

X is lower-equal the supremum of g on X, then the supremum of *pointwise-max f g* on X equals the supremum of g. This is the right analog of *max-Sup-absorb-left*.

⟨*proof*⟩

lemma *max-Sup*:
fixes $f\ g$::‹$'a \Rightarrow real$›
assumes ‹$X \neq \{\}$› **and** ‹*bdd-above* (f ‘ X)› **and** ‹*bdd-above* (g ‘ X)›
shows ‹*Sup* ((*pointwise-max* $f\ g$) ‘ X) = *max* (*Sup* (f ‘ X)) (*Sup* (g ‘ X))›

Explanation: Let X be a nonempty set. Two supremum over X of the maximum of two real-value functions is equal to the maximum of their suprema over X, provided that the functions are bounded above on X.

⟨*proof*⟩

lemma *identity-telescopic*:
fixes x :: ‹- $\Rightarrow$ $'a$::*real-normed-vector*›
assumes ‹$x \longrightarrow l$›
shows ‹$(\lambda\ N.\ sum\ (\lambda\ k.\ x\ (Suc\ k) - x\ k)\ \{n..N\}) \longrightarrow l - x\ n$›

Expression of a limit as a telescopic series. Explanation: If x converges to l then the sum $\sum k = n..N.\ x\ (Suc\ k) - x\ k$ converges to $l - x\ n$ as N goes to infinity.

⟨*proof*⟩

lemma *bound-Cauchy-to-lim*:
assumes ‹$y \longrightarrow x$› **and** ‹$\bigwedge n.\ \|y\ (Suc\ n) - y\ n\| \leq c\hat{}n$› **and** ‹$y\ 0 = 0$› **and** ‹$c < 1$›
shows ‹$\|x - y\ (Suc\ n)\| \leq (c\ /\ (1 - c)) * c\ \hat{}\ n$›

Inequality about a sequence of approximations assuming that the sequence of differences is bounded by a geometric progression. Explanation: Let y be a sequence converging to x. If y satisfies the inequality $\|y\ (Suc\ n) - y\ n\| \leq c\ \hat{}\ n$ for some $c < 1$ and assuming $y\ 0 = 0$ then the inequality $\|x - y\ (Suc\ n)\| \leq (c\ /\ (1 - c)) * c\ \hat{}\ n$ holds.

⟨*proof*⟩

lemma *onorm-open-ball*:
includes *norm-syntax*
shows ‹$\|f\| = Sup\ \{\ \|f *_v x\|\ |\ x.\ \|x\| < 1\ \}$›

Explanation: Let f be a bounded linear operator. The operator norm of f is the supremum of $\|f *_v x\|$ for x such that $\|x\| < 1$.

⟨*proof*⟩

lemma *onorm-r*:

includes *norm-syntax*
assumes ‹$r > 0$›
shows ‹$\|f\| = Sup\ ((\lambda x.\ \|f *_v x\|)\ `\ (ball\ 0\ r))\ /\ r$›

Explanation: The norm of f is $1\ /\ r$ of the supremum of the norm of $f *_v x$ for x in the ball of radius r centered at the origin.

⟨*proof*⟩

Pointwise convergence

definition *pointwise-convergent-to* ::
‹($nat \Rightarrow ('a \Rightarrow 'b$::*topological-space*)) $\Rightarrow ('a \Rightarrow 'b) \Rightarrow bool$›
(‹((-)/ −*pointwise*→ (-))› [*60*, *60*] *60*) **where**
‹*pointwise-convergent-to* $x\ l = (\forall\ t::'a.\ (\lambda\ n.\ (x\ n)\ t) \longrightarrow l\ t)$›

lemma *linear-limit-linear*:
fixes f :: ‹- ⇒ ('a::*real-vector* ⇒ 'b::*real-normed-vector*)›
assumes ‹$\bigwedge n.\ linear\ (f\ n)$› **and** ‹$f$ −*pointwise*→ F›
shows ‹*linear* F›

Explanation: If a family of linear operators converges pointwise, then the limit is also a linear operator.

⟨*proof*⟩

lemma *non-Cauchy-unbounded*:
fixes a ::‹- ⇒ *real*›
assumes ‹$\bigwedge n.\ a\ n \geq 0$› **and** ‹$e > 0$›
and ‹$\forall M.\ \exists m.\ \exists n.\ m \geq M \wedge n \geq M \wedge m > n \wedge sum\ a\ \{Suc\ n..m\} \geq e$›
shows ‹$(\lambda n.\ (sum\ a\ \{0..n\})) \longrightarrow \infty$›

Explanation: If the sequence of partial sums of nonnegative terms is not Cauchy, then it converges to infinite.

⟨*proof*⟩

lemma *sum-Cauchy-positive*:
fixes a ::‹- ⇒ *real*›
assumes ‹$\bigwedge n.\ a\ n \geq 0$› **and** ‹$\exists K.\ \forall n.\ (sum\ a\ \{0..n\}) \leq K$›
shows ‹$Cauchy\ (\lambda n.\ sum\ a\ \{0..n\})$›

Explanation: If a series of nonnegative reals is bounded, then the series is Cauchy.

⟨*proof*⟩

lemma *convergent-series-Cauchy*:
fixes a::‹*nat* ⇒ *real*› **and** φ::‹*nat* ⇒ 'a::*metric-space*›
assumes ‹$\exists M.\ \forall n.\ sum\ a\ \{0..n\} \leq M$› **and** ‹$\bigwedge n.\ dist\ (\varphi\ (Suc\ n))\ (\varphi\ n) \leq a\ n$›
shows ‹$Cauchy\ \varphi$›

Explanation: Let a be a real-valued sequence and let φ be sequence in a metric space. If the partial sums of a are uniformly bounded and the distance between consecutive terms of φ are bounded by the sequence a, then φ is Cauchy.

⟨*proof*⟩

unbundle *blinfun-apply-syntax*

unbundle *no norm-syntax*

end

2 Banach-Steinhaus theorem

theory *Banach-Steinhaus*
  imports *Banach-Steinhaus-Missing*
begin

We formalize Banach-Steinhaus theorem as theorem *banach-steinhaus*. This theorem was originally proved in Banach-Steinhaus's paper [1]. For the proof, we follow Sokal's approach [3]. Furthermore, we prove as a corollary a result about pointwise convergent sequences of bounded operators whose domain is a Banach space.

2.1 Preliminaries for Sokal's proof of Banach-Steinhaus theorem

lemma *linear-plus-norm*:
  includes *norm-syntax*
  assumes ‹*linear f*›
  shows ‹$\|f\ \xi\| \leq max\ \|f\ (x + \xi)\|\ \|f\ (x - \xi)\|$›

Explanation: For arbitrary x and a linear operator f, $\|f\ \xi\|$ is upper bounded by the maximum of the norms of the shifts of f (i.e., $f\ (x + \xi)$ and $f\ (x - \xi)$).

⟨*proof*⟩

lemma *onorm-Sup-on-ball*:
  includes *norm-syntax*
  assumes ‹$r > 0$›
  shows $\|f\| \leq Sup\ (\ (\lambda x.\ \|f *_v x\|)\ `\ (ball\ x\ r)\ )\ /\ r$

Explanation: Let f be a bounded operator and let x be a point. For any $0 < r$, the operator norm of f is bounded above by the supremum of f applied to the open ball of radius r around x, divided by r.

⟨*proof*⟩

lemma *onorm-Sup-on-ball'*:
  includes *norm-syntax*
  assumes ‹$r > 0$› **and** ‹$\tau < 1$›
  shows ‹$\exists \xi \in ball\ x\ r.\ \ \tau * r * \|f\| \leq \|f *_v \xi\|$›

In the proof of Banach-Steinhaus theorem, we will use this variation of the lemma *onorm-Sup-on-ball*.

Explanation: Let f be a bounded operator, let x be a point and let r be a positive real number. For any real number $\tau < 1$, there is a point ξ in the open ball of radius r around x such that $\tau * r * \|f\| \leq \|f *_v \xi\|$.

⟨*proof*⟩

2.2 Banach-Steinhaus theorem

theorem *banach-steinhaus*:
  fixes f::‹$'c \Rightarrow ('a::banach \Rightarrow_L\ 'b::real\text{-}normed\text{-}vector)$›
  assumes ‹$\bigwedge x.\ bounded\ (range\ (\lambda n.\ (f\ n) *_v x))$›
  shows ‹$bounded\ (range\ f)$›

This is Banach-Steinhaus Theorem.

Explanation: If a family of bounded operators on a Banach space is pointwise bounded, then it is uniformly bounded.

⟨*proof*⟩

2.3 A consequence of Banach-Steinhaus theorem

corollary *bounded-linear-limit-bounded-linear*:
  fixes f::‹$nat \Rightarrow ('a::banach \Rightarrow_L\ 'b::real\text{-}normed\text{-}vector)$›
  assumes ‹$\bigwedge x.\ convergent\ (\lambda n.\ (f\ n) *_v x)$›
  shows ‹$\exists\, g.\ (\lambda n.\ (*_v)\ (f\ n))\ -pointwise\rightarrow\ (*_v)\ g$›

Explanation: If a sequence of bounded operators on a Banach space converges pointwise, then the limit is also a bounded operator.

⟨*proof*⟩

end

References

[1] S. Banach and H. Steinhaus. Sur le principe de la condensation de singularités. *Fundamenta Mathematicae*, 1(9):50–61, 1927.

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[3] A. D. Sokal. A really simple elementary proof of the uniform boundedness theorem. *The American Mathematical Monthly*, 118(5):450–452, 2011.