Banach-Steinhaus theorem*

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Abstract

We formalize in Isabelle/HOL a result [2] due to S. Banach and H. Steinhaus [1] known as Banach-Steinhaus theorem or Uniform boundedness principle: a pointwise-bounded family of continuous linear operators from a Banach space to a normed space is uniformly bounded. Our approach is an adaptation to Isabelle/HOL of a proof due to A. Sokal [3].

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theory *Banach-Steinhaus-Missing*
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begin

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1.1 Results missing for the proof of Banach-Steinhaus theorem

The results proved here are preliminaries for the proof of Banach-Steinhaus theorem using Sokal's approach, but they do not explicitly appear in Sokal's paper [3].

Notation for the norm

bundle *notation-norm* **begin**
notation *norm* (‖-‖)
end

bundle *no-notation-norm* **begin**
no-notation *norm* (‖-‖)
end

unbundle *notation-norm*

Notation for apply bilinear function

bundle *notation-blinfun-apply* **begin**
notation *blinfun-apply* (**infixr** $*_v$ *70*)
end

bundle *no-notation-blinfun-apply* **begin**
no-notation *blinfun-apply* (**infixr** $*_v$ *70*)
end

unbundle *notation-blinfun-apply*

lemma *bdd-above-plus*:
  fixes *f*::‹'*a* ⇒ *real*›
  assumes ‹*bdd-above* (*f* ‘ *S*)› **and** ‹*bdd-above* (*g* ‘ *S*)›
  shows ‹*bdd-above* ((λ *x*. *f x* + *g x*) ‘ *S*)›

Explanation: If the images of two real-valued functions f,g are bounded above on a set S, then the image of their sum is bounded on S.

⟨*proof*⟩

The maximum of two functions

definition *pointwise-max*:: ('*a* ⇒ '*b*::*ord*) ⇒ ('*a* ⇒ '*b*) ⇒ ('*a* ⇒ '*b*) **where**
  ‹*pointwise-max f g* = (λ*x*. *max* (*f x*) (*g x*))›

lemma *max-Sup-absorb-left*:
  fixes *f g*::‹'*a* ⇒ *real*›
  assumes ‹*X* ≠ {}› **and** ‹*bdd-above* (*f* ‘ *X*)› **and** ‹*bdd-above* (*g* ‘ *X*)› **and** ‹*Sup* (*f* ‘ *X*) ≥ *Sup* (*g* ‘ *X*)›
  shows ‹*Sup* ((*pointwise-max f g*) ‘ *X*) = *Sup* (*f* ‘ *X*)›

Explanation: For real-valued functions f and g, if the supremum of f is greater-equal the supremum of g, then the supremum of *max f g* equals the supremum of *f*. (Under some technical conditions.)

⟨*proof*⟩

lemma *max-Sup-absorb-right*:
fixes *f g*::‹′*a* ⇒ *real*›
assumes ‹$X \neq \{\}$› **and** ‹*bdd-above* (*f* ‘ *X*)› **and** ‹*bdd-above* (*g* ‘ *X*)› **and** ‹*Sup* (*f* ‘ *X*) ≤ *Sup* (*g* ‘ *X*)›
shows ‹*Sup* ((*pointwise-max f g*) ‘ *X*) = *Sup* (*g* ‘ *X*)›

Explanation: For real-valued functions f and g and a nonempty set X, such that the f and g are bounded above on X, if the supremum of f on X is lower-equal the supremum of g on X, then the supremum of *pointwise-max f g* on X equals the supremum of *g*. This is the right analog of *max-Sup-absorb-left*.

⟨*proof*⟩

lemma *max-Sup*:
fixes *f g*::‹′*a* ⇒ *real*›
assumes ‹$X \neq \{\}$› **and** ‹*bdd-above* (*f* ‘ *X*)› **and** ‹*bdd-above* (*g* ‘ *X*)›
shows ‹*Sup* ((*pointwise-max f g*) ‘ *X*) = *max* (*Sup* (*f* ‘ *X*)) (*Sup* (*g* ‘ *X*))›

Explanation: Let X be a nonempty set. Two supremum over X of the maximum of two real-value functions is equal to the maximum of their suprema over X, provided that the functions are bounded above on X.

⟨*proof*⟩

lemma *identity-telescopic*:
fixes *x* :: ‹- ⇒ ′*a*::*real-normed-vector*›
assumes ‹$x \longrightarrow l$›
shows ‹(λ *N*. *sum* (λ *k*. *x* (*Suc k*) − *x k*) {*n*..*N*}) $\longrightarrow$ *l* − *x n*›

Expression of a limit as a telescopic series. Explanation: If x converges to l then the sum $\sum k = n..N.\ x\ (Suc\ k) - x\ k$ converges to $l - x\ n$ as N goes to infinity.

⟨*proof*⟩

lemma *bound-Cauchy-to-lim*:
assumes ‹$y \longrightarrow x$› **and** ‹$\bigwedge n.\ \|y\ (Suc\ n) - y\ n\| \leq c\hat{}n$› **and** ‹*y 0* = *0*› **and** ‹*c* < *1*›
shows ‹$\|x - y\ (Suc\ n)\| \leq (c\ /\ (1 - c)) * c\ \hat{}\ n$›

Inequality about a sequence of approximations assuming that the sequence of differences is bounded by a geometric progression. Explanation: Let y be a sequence converging to x. If y satisfies the inequality $\|y\ (Suc\ n) - y\ n\| \leq c\ \hat{}\ n$ for some $c < 1$ and assuming $y\ 0 = (0::'a)$ then the inequality $\|x - y\ (Suc\ n)\| \leq (c\ /\ (1 - c)) * c\ \hat{}\ n$ holds.

⟨*proof*⟩

lemma *onorm-open-ball*:
  includes *notation-norm*
  shows ‹$\|f\| = Sup\ \{\ \|f *_v x\|\ |\ x.\ \|x\| < 1\ \}$›

Explanation: Let f be a bounded linear operator. The operator norm of f is the supremum of $\|f *_v x\|$ for x such that $\|x\| < 1$.

⟨*proof*⟩

lemma *onorm-r*:
  includes *notation-norm*
  assumes ‹$r > 0$›
  shows ‹$\|f\| = Sup\ ((\lambda x.\ \|f *_v x\|)$ ‘ $(ball\ 0\ r))\ /\ r$›

Explanation: The norm of f is $1\ /\ r$ of the supremum of the norm of $f *_v x$ for x in the ball of radius r centered at the origin.

⟨*proof*⟩

Pointwise convergence

definition *pointwise-convergent-to* ::
  ‹(*nat* ⇒ ('*a* ⇒ '*b*::*topological-space*)) ⇒ ('*a* ⇒ '*b*) ⇒ *bool*›
  (‹((-)/ −*pointwise*→ (-))› [*60*, *60*] *60*) **where**
  ‹*pointwise-convergent-to* $x\ l = (\forall\ t{::}'a.\ (\lambda\ n.\ (x\ n)\ t) \longrightarrow l\ t)$›

lemma *linear-limit-linear*:
  fixes f :: ‹- ⇒ ('*a*::*real-vector* ⇒ '*b*::*real-normed-vector*)›
  assumes ‹$\bigwedge n.$ *linear* $(f\ n)$› **and** ‹f −*pointwise*→ F›
  shows ‹*linear* F›

Explanation: If a family of linear operators converges pointwise, then the limit is also a linear operator.

⟨*proof*⟩

lemma *non-Cauchy-unbounded*:
  fixes a ::‹- ⇒ *real*›
  assumes ‹$\bigwedge n.\ a\ n \geq 0$› **and** ‹$e > 0$›
    and ‹$\forall M.\ \exists m.\ \exists n.\ m \geq M \land n \geq M \land m > n \land sum\ a\ \{Suc\ n..m\} \geq e$›
  shows ‹$(\lambda n.\ (sum\ a\ \{0..n\})) \longrightarrow \infty$›

Explanation: If the sequence of partial sums of nonnegative terms is not Cauchy, then it converges to infinite.

⟨*proof*⟩

lemma *sum-Cauchy-positive*:
  fixes a ::‹- ⇒ *real*›
  assumes ‹$\bigwedge n.\ a\ n \geq 0$› **and** ‹$\exists K.\ \forall n.\ (sum\ a\ \{0..n\}) \leq K$›
  shows ‹*Cauchy* $(\lambda n.\ sum\ a\ \{0..n\})$›

Explanation: If a series of nonnegative reals is bounded, then the series is Cauchy.

⟨*proof*⟩

lemma *convergent-series-Cauchy*:
fixes a::‹*nat* ⇒ *real*› **and** φ::‹*nat* ⇒ *'a*::*metric-space*›
assumes ‹$\exists M.\ \forall n.\ sum\ a\ \{0..n\} \leq M$› **and** ‹$\bigwedge n.\ dist\ (\varphi\ (Suc\ n))\ (\varphi\ n) \leq a\ n$›
shows ‹*Cauchy* φ›

Explanation: Let a be a real-valued sequence and let φ be sequence in a metric space. If the partial sums of a are uniformly bounded and the distance between consecutive terms of φ are bounded by the sequence a, then φ is Cauchy.

⟨*proof*⟩

unbundle *notation-blinfun-apply*

unbundle *no-notation-norm*

end

2 Banach-Steinhaus theorem

theory *Banach-Steinhaus*
imports *Banach-Steinhaus-Missing*
begin

We formalize Banach-Steinhaus theorem as theorem *banach-steinhaus*. This theorem was originally proved in Banach-Steinhaus's paper [1]. For the proof, we follow Sokal's approach [3]. Furthermore, we prove as a corollary a result about pointwise convergent sequences of bounded operators whose domain is a Banach space.

2.1 Preliminaries for Sokal's proof of Banach-Steinhaus theorem

lemma *linear-plus-norm*:
includes *notation-norm*
assumes ‹*linear* f›
shows ‹$\|f\ \xi\| \leq max\ \|f\ (x + \xi)\|\ \|f\ (x - \xi)\|$›

Explanation: For arbitrary x and a linear operator f, $\|f\ \xi\|$ is upper bounded by the maximum of the norms of the shifts of f (i.e., $f\ (x + \xi)$ and $f\ (x - \xi)$).

⟨*proof*⟩

lemma *onorm-Sup-on-ball*:
includes *notation-norm*
assumes ‹$r > 0$›
shows $\|f\| \leq Sup\ (\ (\lambda x.\ \|f *_v x\|)\ `\ (ball\ x\ r)\ )\ /\ r$

Explanation: Let f be a bounded operator and let x be a point. For any $0 < r$, the operator norm of f is bounded above by the supremum of f applied to the open ball of radius r around x, divided by r.

⟨*proof*⟩

lemma *onorm-Sup-on-ball′*:
includes *notation-norm*
assumes ‹$r > 0$› **and** ‹$\tau < 1$›
shows ‹$\exists \xi \in ball\ x\ r.\ \ \tau * r * \|f\| \leq \|f *_v \xi\|$›

In the proof of Banach-Steinhaus theorem, we will use this variation of the lemma *onorm-Sup-on-ball*.

Explanation: Let f be a bounded operator, let x be a point and let r be a positive real number. For any real number $\tau < 1$, there is a point ξ in the open ball of radius r around x such that $\tau * r * \|f\| \leq \|f *_v \xi\|$.

⟨*proof*⟩

2.2 Banach-Steinhaus theorem

theorem *banach-steinhaus*:
fixes f::‹$'c \Rightarrow ('a::banach \Rightarrow_L\ 'b::real\text{-}normed\text{-}vector)$›
assumes ‹$\bigwedge x.\ bounded\ (range\ (\lambda n.\ (f\ n) *_v x))$›
shows ‹$bounded\ (range\ f)$›

This is Banach-Steinhaus Theorem.

Explanation: If a family of bounded operators on a Banach space is pointwise bounded, then it is uniformly bounded.

⟨*proof*⟩

2.3 A consequence of Banach-Steinhaus theorem

corollary *bounded-linear-limit-bounded-linear*:
fixes f::‹$nat \Rightarrow ('a::banach \Rightarrow_L\ 'b::real\text{-}normed\text{-}vector)$›
assumes ‹$\bigwedge x.\ convergent\ (\lambda n.\ (f\ n) *_v x)$›
shows ‹$\exists\, g.\ (\lambda n.\ (*_v)\ (f\ n))\ -pointwise\rightarrow\ (*_v)\ g$›

Explanation: If a sequence of bounded operators on a Banach space converges pointwise, then the limit is also a bounded operator.

⟨*proof*⟩

end

References

[1] S. Banach and H. Steinhaus. Sur le principe de la condensation de singularités. *Fundamenta Mathematicae*, 1(9):50–61, 1927.

[2] M. S. Moslehian and E. W. Weisstein. Uniform boundedness principle. *From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/UniformBoundednessPrinciple.html.*

[3] A. D. Sokal. A really simple elementary proof of the uniform boundedness theorem. *The American Mathematical Monthly*, 118(5):450–452, 2011.