

# Operations on Bounded Natural Functors

Jasmin Christian Blanchette      Andrei Popescu      Dmitriy Traytel

May 26, 2024

## Abstract

This entry formalizes the closure property of bounded natural functors (BNFs) under seven operations. These operations and the corresponding proofs constitute the core of Isabelle’s (co)datatype package. To be close to the implemented tactics, the proofs are deliberately formulated as detailed apply scripts. The (co)datatypes together with (co)induction principles and (co)recursors are byproducts of the fixpoint operations LFP and GFP. Composition of BNFs is subdivided into four simpler operations: Compose, Kill, Lift, and Permute. The N2M operation provides mutual (co)induction principles and (co)recursors for nested (co)datatypes.

## Contents

<b>1</b>	<b>Least Fixpoint (a.k.a. Datatype)</b>	<b>1</b>
1.1	Algebra . . . . .	2
1.2	Morphism . . . . .	3
1.3	Bounds . . . . .	6
1.4	Minimal Algebras . . . . .	7
1.5	Initiality . . . . .	18
1.6	Initial Algebras . . . . .	18
1.7	The datatype . . . . .	26
1.8	The Result as an BNF . . . . .	35
<b>2</b>	<b>Greatest Fixpoint (a.k.a. Codatatype)</b>	<b>51</b>
2.1	Coalgebra . . . . .	53
2.2	Type-coalgebra . . . . .	54
2.3	Morphism . . . . .	54
2.4	Bisimulations . . . . .	57
2.5	The Tree Coalgebra . . . . .	66
2.6	Quotient Coalgebra . . . . .	106
2.7	Coinduction . . . . .	115
2.8	The Result as an BNF . . . . .	116
<b>3</b>	<b>Normalized Composition of BNFs</b>	<b>145</b>
<b>4</b>	<b>Removing Live Variables</b>	<b>147</b>
<b>5</b>	<b>Adding New Live Variables</b>	<b>150</b>
<b>6</b>	<b>Changing the Order of Live Variables</b>	<b>151</b>
<b>7</b>	<b>Mutual View on Nested Datatypes</b>	<b>153</b>
7.1	Nested Definition . . . . .	153
7.2	Isomorphic Mutual Definition . . . . .	154
7.3	Mutualization . . . . .	154
7.3.1	Iterators . . . . .	154
7.3.2	Recursors . . . . .	154
7.3.3	Induction . . . . .	155

<b>8</b>	<b>Mutual View on Nested Coatypes</b>	<b>156</b>
8.1	Nested definition . . . . .	156
8.2	Isomorphic Mutual Definition . . . . .	156
8.3	Mutualization . . . . .	156
8.3.1	Coiterators . . . . .	156
8.3.2	Corecursors . . . . .	157
8.3.3	Coinduction . . . . .	158

## 1 Least Fixpoint (a.k.a. Datatype)

**unbundle** *cardinal\_syntax*

**ML**  $\langle open\ Ctr\_Sugar\_Util \rangle$   
**notation** *BNF\_Def.convolve* ( $\langle\_ , \_ \rangle$ )

'b1 = ('a, 'b1, 'b2) F1  
'b2 = ('a, 'b1, 'b2) F2

To build a witness scenario, let us assume

('a, 'b1, 'b2) F1 = 'a \* 'b1 + 'a \* 'b2  
('a, 'b1, 'b2) F2 = unit + 'b1 \* 'b2

**declare**  $[[bnf\_internals]]$

**bnf-axiomatization** (*F1set1*: 'a, *F1set2*: 'b1, *F1set3*: 'b2) *F1*  
[wits: 'a  $\Rightarrow$  'b1  $\Rightarrow$  ('a, 'b1, 'b2) *F1* 'a  $\Rightarrow$  'b2  $\Rightarrow$  ('a, 'b1, 'b2) *F1*]  
**for map**: *F1map* rel: *F1rel*

**bnf-axiomatization** (*F2set1*: 'a, *F2set2*: 'b1, *F2set3*: 'b2) *F2*  
[wits: ('a, 'b1, 'b2) *F2*]  
**for map**: *F2map* rel: *F2rel*

**abbreviation** *F1in* :: 'a1 set  $\Rightarrow$  'a2 set  $\Rightarrow$  'a3 set  $\Rightarrow$  (('a1, 'a2, 'a3) *F1*) set **where**  
*F1in* A1 A2 A3  $\equiv \{x. F1set1\ x \subseteq A1 \wedge F1set2\ x \subseteq A2 \wedge F1set3\ x \subseteq A3\}$

**abbreviation** *F2in* :: 'a1 set  $\Rightarrow$  'a2 set  $\Rightarrow$  'a3 set  $\Rightarrow$  (('a1, 'a2, 'a3) *F2*) set **where**  
*F2in* A1 A2 A3  $\equiv \{x. F2set1\ x \subseteq A1 \wedge F2set2\ x \subseteq A2 \wedge F2set3\ x \subseteq A3\}$

**lemma** *F1map\_comp\_id*: *F1map* g1 g2 g3 (*F1map* id f2 f3 x) = *F1map* g1 (g2 o f2) (g3 o f3) x

**apply** (*rule trans*)  
**apply** (*rule F1.map\_comp*)  
**unfolding** *o\_id*  
**apply** (*rule refl*)  
**done**

**lemmas** *F1in\_mono23* = *F1.in\_mono*[*OF subset\_refl*]

**lemma** *F1map\_congL*:  $[\forall a \in F1set2\ x. f\ a = a; \forall a \in F1set3\ x. g\ a = a] \Longrightarrow$

*F1map* id f g x = x  
**apply** (*rule trans*)  
**apply** (*rule F1.map\_cong0*)  
**apply** (*rule refl*)  
**apply** (*rule trans*)  
**apply** (*erule bspec*)  
**apply** *assumption*  
**apply** (*rule sym*)  
**apply** (*rule id\_apply*)  
**apply** (*rule trans*)  
**apply** (*erule bspec*)  
**apply** *assumption*  
**apply** (*rule sym*)  
**apply** (*rule id\_apply*)  
**apply** (*rule F1.map\_id*)  
**done**

**lemma** *F2map\_comp\_id*:  $F2map\ g1\ g2\ g3\ (F2map\ id\ f2\ f3\ x) = F2map\ g1\ (g2\ o\ f2)\ (g3\ o\ f3)\ x$   
**apply** (*rule trans*)  
**apply** (*rule F2.map\_comp*)  
**unfolding** *o\_id*  
**apply** (*rule refl*)  
**done**

**lemmas** *F2in\_mono23* = *F2.in\_mono*[*OF subset\_refl*]

**lemma** *F2map\_congL*:  $\llbracket \forall a \in F2set2\ x.\ f\ a = a; \forall a \in F2set3\ x.\ g\ a = a \rrbracket \implies$   
 $F2map\ id\ f\ g\ x = x$   
**apply** (*rule trans*)  
**apply** (*rule F2.map\_cong0*)  
**apply** (*rule refl*)  
**apply** (*rule trans*)  
**apply** (*erule bspec*)  
**apply** *assumption*  
**apply** (*rule sym*)  
**apply** (*rule id\_apply*)  
**apply** (*rule trans*)  
**apply** (*erule bspec*)  
**apply** *assumption*  
**apply** (*rule sym*)  
**apply** (*rule id\_apply*)  
**apply** (*rule F2.map\_id*)  
**done**

## 1.1 Algebra

**definition** *alg where*

*alg B1 B2 s1 s2* =  
 $((\forall x \in F1in\ (UNIV :: 'a\ set)\ B1\ B2.\ s1\ x \in B1) \wedge (\forall y \in F2in\ (UNIV :: 'a\ set)\ B1\ B2.\ s2\ y \in B2))$

**lemma** *alg\_F1set*:  $\llbracket alg\ B1\ B2\ s1\ s2; F1set2\ x \subseteq B1; F1set3\ x \subseteq B2 \rrbracket \implies s1\ x \in B1$   
**apply** (*tactic <dtac @{\context} @{\thm iffD1[OF alg\_def]} 1>*)  
**apply** (*erule conjE*)  
**apply** (*erule bspec*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI[OF subset\_UNIV]*)  
**apply** (*erule conjI*)  
**apply** *assumption*  
**done**

**lemma** *alg\_F2set*:  $\llbracket alg\ B1\ B2\ s1\ s2; F2set2\ x \subseteq B1; F2set3\ x \subseteq B2 \rrbracket \implies s2\ x \in B2$   
**apply** (*tactic <dtac @{\context} @{\thm iffD1[OF alg\_def]} 1>*)  
**apply** (*erule conjE*)  
**apply** (*erule bspec*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI[OF subset\_UNIV]*)  
**apply** (*erule conjI*)  
**apply** *assumption*  
**done**

**lemma** *alg\_not\_empty*:

*alg B1 B2 s1 s2*  $\implies B1 \neq \{\} \wedge B2 \neq \{\}$   
**apply** (*rule conjI*)  
**apply** (*rule notI*)  
**apply** (*tactic <hyp\_subst\_tac @{\context} 1>*)  
**apply** (*frule alg\_F1set*)

**apply** (*rule subset\_emptyI*)

```

apply (erule F1.wit1 F1.wit2 F2.wit)

apply (rule subsetI)
apply (erule F1.wit1 F1.wit2 F2.wit)

apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (tactic ⟨FIRST' (map (fn thm => rtac @ {context} thm THEN' assume_tac @ {context}) @ {thms alg_F1set
alg_F2set}) 1⟩)

apply (rule subset_emptyI)
apply (erule F1.wit1 F1.wit2 F2.wit)

apply (rule subsetI)
apply (erule F1.wit1 F1.wit2 F2.wit)
apply (erule FalseE)

apply (erule emptyE)

apply (rule notI)
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (erule alg_F2set)

apply (rule subsetI)
apply (rule FalseE)
apply (erule F1.wit1 F1.wit2 F2.wit)

apply (rule subset_emptyI)
apply (erule F1.wit1 F1.wit2 F2.wit)

apply (erule emptyE)
done

```

## 1.2 Morphism

**definition mor where**

$$\begin{aligned}
\text{mor } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ f \ g = & \\
& (((\forall a \in B1. f \ a \in B1') \wedge (\forall a \in B2. g \ a \in B2')) \wedge \\
& ((\forall z \in F1in \ (UNIV :: 'a \ set) \ B1 \ B2. f \ (s1 \ z) = s1' \ (F1map \ id \ f \ g \ z)) \wedge \\
& (\forall z \in F2in \ (UNIV :: 'a \ set) \ B1 \ B2. g \ (s2 \ z) = s2' \ (F2map \ id \ f \ g \ z))))
\end{aligned}$$

**lemma morE1:**  $\llbracket \text{mor } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ f \ g; z \in F1in \ UNIV \ B1 \ B2 \rrbracket$

```

 $\implies f \ (s1 \ z) = s1' \ (F1map \ id \ f \ g \ z)$ 
apply (tactic ⟨dtac @ {context} @ {thm iffD1[OF mor_def]} 1⟩)
apply (erule conjE)+
apply (erule bspec)
apply assumption
done

```

**lemma morE2:**  $\llbracket \text{mor } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ f \ g; z \in F2in \ UNIV \ B1 \ B2 \rrbracket$

```

 $\implies g \ (s2 \ z) = s2' \ (F2map \ id \ f \ g \ z)$ 
apply (tactic ⟨dtac @ {context} @ {thm iffD1[OF mor_def]} 1⟩)
apply (erule conjE)+
apply (erule bspec)
apply assumption
done

```

**lemma mor\_incl:**  $\llbracket B1 \subseteq B1'; B2 \subseteq B2' \rrbracket \implies \text{mor } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1 \ s2 \ id \ id$

```

apply (tactic ⟨rtac @ {context} (@ {thm mor_def}) RS iffD2) 1⟩)
apply (rule conjI)

apply (rule conjI)
apply (rule ballI)

```

**apply** (*erule subsetD*)  
**apply** (*erule ssubst\_mem*[*OF id\_apply*])

**apply** (*rule ballI*)  
**apply** (*erule subsetD*)  
**apply** (*erule ssubst\_mem*[*OF id\_apply*])

**apply** (*rule conjI*)  
**apply** (*rule ballI*)  
**apply** (*rule trans*)  
**apply** (*rule id\_apply*)  
**apply** (*tactic <stac @*{*context*} *@*{*thm F1.map\_id*} *1*>)  
**apply** (*rule refl*)

**apply** (*rule ballI*)  
**apply** (*rule trans*)  
**apply** (*rule id\_apply*)  
**apply** (*tactic <stac @*{*context*} *@*{*thm F2.map\_id*} *1*>)  
**apply** (*rule refl*)  
**done**

**lemma** *mor\_comp*:

$\llbracket \text{mor } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ f \ g; \\ \text{mor } B1' \ B2' \ s1' \ s2' \ B1'' \ B2'' \ s1'' \ s2'' \ f' \ g' \rrbracket \implies \\ \text{mor } B1 \ B2 \ s1 \ s2 \ B1'' \ B2'' \ s1'' \ s2'' \ (f' \circ f) \ (g' \circ g)$   
**apply** (*tactic <dtac @*{*context*} (*@*{*thm mor\_def*} *RS iffD1*) *1*>)  
**apply** (*tactic <dtac @*{*context*} (*@*{*thm mor\_def*} *RS iffD1*) *1*>)  
**apply** (*tactic <rtac @*{*context*} (*@*{*thm mor\_def*} *RS iffD2*) *1*>)  
**apply** (*erule conjE*)  
**apply** (*rule conjI*)

**apply** (*rule conjI*)  
**apply** (*rule ballI*)  
**apply** (*erule ssubst\_mem*[*OF o\_apply*])  
**apply** (*erule bspec*)  
**apply** (*erule bspec*)  
**apply** *assumption*

**apply** (*rule ballI*)  
**apply** (*erule ssubst\_mem*[*OF o\_apply*])  
**apply** (*erule bspec*)  
**apply** (*erule bspec*)  
**apply** *assumption*

**apply** (*rule conjI*)  
**apply** (*rule ballI*)  
**apply** (*erule trans*[*OF o\_apply*])  
**apply** (*rule trans*)  
**apply** (*rule trans*)  
**apply** (*erule bspec*[*rotated*])  
**apply** *assumption*  
**apply** (*erule arg\_cong*)  
**apply** (*erule CollectE conjE*)  
**apply** (*erule bspec*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI*)  
**apply** (*rule subset\_UNIV*)  
**apply** (*rule conjI*)  
**apply** (*rule ord\_eq\_le\_trans*)  
**apply** (*rule F1.set\_map*(*2*))  
**apply** (*rule image\_subsetI*)  
**apply** (*erule bspec*)  
**apply** (*erule subsetD*)

```

apply assumption
apply (rule ord_eq_le_trans)
apply (rule F1.set_map(3))
apply (rule image_subsetI)
apply (erule bspec)
apply (erule subsetD)
apply assumption
apply (rule arg_cong[OF F1map_comp_id])

```

```

apply (rule ballI)
apply (rule trans[OF o_apply])
apply (rule trans)
apply (rule trans)
apply (drule bspec[rotated])
apply assumption
apply (erule arg_cong)
apply (erule CollectE conjE)+
apply (erule bspec)
apply (rule CollectI)
apply (rule conjI)
apply (rule subset_UNIV)
apply (rule conjI)
apply (rule ord_eq_le_trans)
apply (rule F2.set_map(2))
apply (rule image_subsetI)
apply (erule bspec)
apply (erule subsetD)
apply assumption
apply (rule ord_eq_le_trans)
apply (rule F2.set_map(3))
apply (rule image_subsetI)
apply (erule bspec)
apply (erule subsetD)
apply assumption
apply (rule arg_cong[OF F2map_comp_id])
done

```

```

lemma mor_cong:  $\llbracket f' = f; g' = g; \text{mor } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ f \ g \rrbracket \implies$ 
 $\text{mor } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ f' \ g'$ 
apply (tactic <hyp_subst_tac @ {context} 1>)
apply assumption
done

```

```

lemma mor_str:
 $\text{mor } UNIV \ UNIV \ (F1map \ id \ s1 \ s2) \ (F2map \ id \ s1 \ s2) \ UNIV \ UNIV \ s1 \ s2 \ s1 \ s2$ 
apply (rule iffD2)
apply (rule mor_def)
apply (rule conjI)
apply (rule conjI)
apply (rule ballI)
apply (rule UNIV_I)
apply (rule ballI)
apply (rule UNIV_I)

```

```

apply (rule conjI)
apply (rule ballI)
apply (rule refl)
apply (rule ballI)
apply (rule refl)
done

```

### 1.3 Bounds

```

type-synonym  $bd\_type\_F1' = bd\_type\_F1 + (bd\_type\_F1, bd\_type\_F1, bd\_type\_F1) \ F1$ 

```

**type-synonym**  $bd\_type\_F2' = bd\_type\_F2 + (bd\_type\_F2, bd\_type\_F2, bd\_type\_F2) F2$

**type-synonym**  $SucFbd\_type = ((bd\_type\_F1' + bd\_type\_F2') set)$

**type-synonym**  $'a1 ASucFbd\_type = (SucFbd\_type \Rightarrow ('a1 + bool))$

**abbreviation**  $F1bd' \equiv bd\_F1 + c \mid UNIV :: (bd\_type\_F1, bd\_type\_F1, bd\_type\_F1) F1 set$

**lemma**  $F1set1\_bd\_incr: \bigwedge x. |F1set1 x| < o F1bd'$

by (rule ordLess\_ordLeq\_trans[OF F1.set\_bd(1) ordLeq\_csum1[OF F1.bd\_Card\_order]])

**lemma**  $F1set2\_bd\_incr: \bigwedge x. |F1set2 x| < o F1bd'$

by (rule ordLess\_ordLeq\_trans[OF F1.set\_bd(2) ordLeq\_csum1[OF F1.bd\_Card\_order]])

**lemma**  $F1set3\_bd\_incr: \bigwedge x. |F1set3 x| < o F1bd'$

by (rule ordLess\_ordLeq\_trans[OF F1.set\_bd(3) ordLeq\_csum1[OF F1.bd\_Card\_order]])

**lemmas**  $F1bd'_Card\_order = Card\_order\_csum$

**lemmas**  $F1bd'_Cinfinite = Cinfinite\_csum1[OF F1.bd\_Cinfinite]$

**lemmas**  $F1bd'_Cnotzero = Cinfinite\_Cnotzero[OF F1bd'_Cinfinite]$

**lemmas**  $F1bd'_card\_order = card\_order\_csum[OF F1.bd\_card\_order card\_of\_card\_order\_on]$

**abbreviation**  $F2bd' \equiv bd\_F2 + c \mid UNIV :: (bd\_type\_F2, bd\_type\_F2, bd\_type\_F2) F2 set$

**lemma**  $F2set1\_bd\_incr: \bigwedge x. |F2set1 x| < o F2bd'$

by (rule ordLess\_ordLeq\_trans[OF F2.set\_bd(1) ordLeq\_csum1[OF F2.bd\_Card\_order]])

**lemma**  $F2set2\_bd\_incr: \bigwedge x. |F2set2 x| < o F2bd'$

by (rule ordLess\_ordLeq\_trans[OF F2.set\_bd(2) ordLeq\_csum1[OF F2.bd\_Card\_order]])

**lemma**  $F2set3\_bd\_incr: \bigwedge x. |F2set3 x| < o F2bd'$

by (rule ordLess\_ordLeq\_trans[OF F2.set\_bd(3) ordLeq\_csum1[OF F2.bd\_Card\_order]])

**lemmas**  $F2bd'_Card\_order = Card\_order\_csum$

**lemmas**  $F2bd'_Cinfinite = Cinfinite\_csum1[OF F2.bd\_Cinfinite]$

**lemmas**  $F2bd'_Cnotzero = Cinfinite\_Cnotzero[OF F2bd'_Cinfinite]$

**lemmas**  $F2bd'_card\_order = card\_order\_csum[OF F2.bd\_card\_order card\_of\_card\_order\_on]$

**abbreviation**  $SucFbd$  where  $SucFbd \equiv cardSuc (F1bd' + c F2bd')$

**abbreviation**  $ASucFbd$  where  $ASucFbd \equiv ( \mid UNIV \mid + c ctwo ) \hat{c} SucFbd$

**lemma**  $F1set1\_bd: |F1set1 x| < o bd\_F1 + c bd\_F2$

apply (rule ordLess\_ordLeq\_trans)

apply (rule F1.set\_bd(1))

apply (rule ordLeq\_csum1)

apply (rule F1.bd\_Card\_order)

done

**lemma**  $F1set2\_bd: |F1set2 x| < o bd\_F1 + c bd\_F2$

apply (rule ordLess\_ordLeq\_trans)

apply (rule F1.set\_bd(2))

apply (rule ordLeq\_csum1)

apply (rule F1.bd\_Card\_order)

done

**lemma**  $F1set3\_bd: |F1set3 x| < o bd\_F1 + c bd\_F2$

apply (rule ordLess\_ordLeq\_trans)

apply (rule F1.set\_bd(3))

apply (rule ordLeq\_csum1)

apply (rule F1.bd\_Card\_order)

done

**lemma**  $F2set1\_bd: |F2set1 x| < o bd\_F1 + c bd\_F2$

apply (rule ordLess\_ordLeq\_trans)

apply (rule F2.set\_bd(1))

apply (rule ordLeq\_csum2)

apply (rule F2.bd\_Card\_order)

done

**lemma**  $F2set2\_bd: |F2set2 x| < o bd\_F1 + c bd\_F2$

apply (rule ordLess\_ordLeq\_trans)

**apply** (rule *F2.set\_bd(2)*)  
**apply** (rule *ordLeq\_csum2*)  
**apply** (rule *F2.bd\_Card\_order*)  
**done**

**lemma** *F2set3\_bd: |F2set3 x| <o bd\_F1 +c bd\_F2*  
**apply** (rule *ordLess\_ordLeq\_trans*)  
**apply** (rule *F2.set\_bd(3)*)  
**apply** (rule *ordLeq\_csum2*)  
**apply** (rule *F2.bd\_Card\_order*)  
**done**

**lemmas** *SucFbd\_Card\_order = cardSuc\_Card\_order[OF Card\_order\_csum]*  
**lemmas** *SucFbd\_Cinfinite = Cinfinite\_cardSuc[OF Cinfinite\_csum1 [OF F1bd'\_Cinfinite]]*  
**lemmas** *SucFbd\_Cnotzero = Cinfinite\_Cnotzero[OF SucFbd\_Cinfinite]*  
**lemmas** *worel\_SucFbd = Card\_order\_wo\_rel[OF SucFbd\_Card\_order]*  
**lemmas** *ASucFbd\_Cinfinite = Cinfinite\_cexp[OF ordLeq\_csum2 [OF Card\_order\_ctwo] SucFbd\_Cinfinite]*

## 1.4 Minimal Algebras

**abbreviation** *min\_G1* **where**  
*min\_G1 As1\_As2 i*  $\equiv (\bigcup j \in \text{underS } \text{SucFbd } i. \text{fst } (As1\_As2\ j))$

**abbreviation** *min\_G2* **where**  
*min\_G2 As1\_As2 i*  $\equiv (\bigcup j \in \text{underS } \text{SucFbd } i. \text{snd } (As1\_As2\ j))$

**abbreviation** *min\_H* **where**  
*min\_H s1 s2 As1\_As2 i*  $\equiv$   
 $(\text{min\_G1 } As1\_As2\ i \cup s1 \text{ ' (F1in (UNIV :: 'a set) (min\_G1 } As1\_As2\ i) (\text{min\_G2 } As1\_As2\ i)),$   
 $\text{min\_G2 } As1\_As2\ i \cup s2 \text{ ' (F2in (UNIV :: 'a set) (min\_G1 } As1\_As2\ i) (\text{min\_G2 } As1\_As2\ i)))$

**abbreviation** *min\_algs* **where**  
*min\_algs s1 s2*  $\equiv \text{wo\_rel.worec } \text{SucFbd } (\text{min\_H } s1\ s2)$

**definition** *min\_alg1* **where**  
*min\_alg1 s1 s2*  $= (\bigcup i \in \text{Field } \text{SucFbd}. \text{fst } (\text{min\_algs } s1\ s2\ i))$

**definition** *min\_alg2* **where**  
*min\_alg2 s1 s2*  $= (\bigcup i \in \text{Field } \text{SucFbd}. \text{snd } (\text{min\_algs } s1\ s2\ i))$

**lemma** *min\_algs:*  
*i*  $\in \text{Field } \text{SucFbd} \implies \text{min\_algs } s1\ s2\ i = \text{min\_H } s1\ s2\ (\text{min\_algs } s1\ s2)\ i$   
**apply** (rule *fun\_cong[OF wo\_rel.worec\_fixpoint[OF worel\_SucFbd]]*)  
**apply** (rule *iffD2*)  
**apply** (rule *meta\_eq\_to\_obj\_eq*)  
**apply** (rule *wo\_rel.adm\_wo\_def[OF worel\_SucFbd]*)  
**apply** (rule *allI*)+  
**apply** (rule *impI*)

**apply** (rule *iffD2*)  
**apply** (rule *prod.inject*)  
**apply** (rule *conjI*)

**apply** (rule *arg\_cong2[of \_ \_ \_ \_ (U)]*)  
**apply** (rule *SUP\_cong*)  
**apply** (rule *refl*)  
**apply** (rule *bspec*)  
**apply** *assumption*  
**apply** (rule *arg\_cong*)

**apply** (rule *image\_cong*)  
**apply** (rule *arg\_cong2[of \_ \_ \_ \_ F1in UNIV]*)  
**apply** (rule *SUP\_cong*)  
**apply** (rule *refl*)



```

apply (drule bspec)
apply assumption
apply (erule arg_cong)
apply (rule SUP_cong)
apply (rule refl)
apply (drule bspec)
apply assumption
apply (erule arg_cong)
apply (rule refl)

apply (rule arg_cong2[of _ _ _ _ (U)])
apply (rule SUP_cong)
apply (rule refl)
apply (drule bspec)
apply assumption
apply (erule arg_cong)

apply (rule image_cong)
apply (rule arg_cong2[of _ _ _ _ F2in UNIV])
apply (rule SUP_cong)
apply (rule refl)
apply (drule bspec)
apply assumption
apply (erule arg_cong)
apply (rule SUP_cong)
apply (rule refl)
apply (drule bspec)
apply assumption
apply (erule arg_cong)
apply (rule refl)
done

corollary min_algs1:  $i \in \text{Field SucFbd} \implies \text{fst} (\text{min\_algs } s1 \ s2 \ i) =$ 
  min_G1 (min_algs s1 s2) i  $\cup$ 
  s1 ' (F1in UNIV (min_G1 (min_algs s1 s2) i) (min_G2 (min_algs s1 s2) i))
apply (rule trans)
apply (erule arg_cong[OF min_algs])
apply (rule fst_conv)
done

corollary min_algs2:  $i \in \text{Field SucFbd} \implies \text{snd} (\text{min\_algs } s1 \ s2 \ i) =$ 
  min_G2 (min_algs s1 s2) i  $\cup$ 
  s2 ' (F2in UNIV (min_G1 (min_algs s1 s2) i) (min_G2 (min_algs s1 s2) i))
apply (rule trans)
apply (erule arg_cong[OF min_algs])
apply (rule snd_conv)
done

lemma min_algs_mono1: relChain SucFbd (%i. fst (min_algs s1 s2 i))
apply (tactic <rtac @ {context} @ {thm iffD2[OF meta_eq_to_obj_eq[OF relChain_def]]} 1>)
apply (rule allI)+
apply (rule impI)
apply (rule case_split)
apply (rule xt1(3))
apply (rule min_algs1)
apply (erule FieldI2)
apply (rule subsetI)
apply (rule UnI1)
apply (rule UN_I)
apply (erule underS_I)
apply assumption
apply assumption
apply (rule equalityD1)

```

```

apply (drule notnotD)
apply (erule arg_cong)
done

```

```

lemma min_algs_mono2: relChain SucFbd (%i. snd (min_algs s1 s2 i))
apply (tactic <rtac @{context} @<{thm iffD2[OF meta_eq_to_obj_eq[OF relChain_def]]} 1>)
apply (rule allI)+
apply (rule impI)
apply (rule case_split)
apply (rule xt1(3))
apply (rule min_algs2)
apply (erule FieldI2)
apply (rule subsetI)
apply (rule UnI1)
apply (rule UN_I)
apply (erule underS_I)
apply assumption
apply assumption
apply (rule equalityD1)
apply (drule notnotD)
apply (erule arg_cong)
done

```

```

lemma SucFbd_limit: [[x1 ∈ Field SucFbd & x2 ∈ Field SucFbd]]
⇒ ∃ y ∈ Field SucFbd. (x1 ≠ y ∧ (x1, y) ∈ SucFbd) ∧ (x2 ≠ y ∧ (x2, y) ∈ SucFbd)
apply (erule conjE)+
apply (rule rev_mp)
apply (rule Cinfinitelimit_finite)
apply (rule finite.insertI)
apply (rule finite.insertI)
apply (rule finite.emptyI)
apply (erule insert_subsetI)
apply (erule insert_subsetI)
apply (rule empty_subsetI)
apply (rule SucFbd_Cinfinitelimit)
apply (rule impI)
apply (erule bexE)
apply (rule beXI)

apply (rule conjI)

apply (erule bspec)
apply (rule insertI1)

apply (erule bspec)
apply (rule insertI2)
apply (rule insertI1)
apply assumption
done

```

```

lemma alg_min_alg: alg (min_alg1 s1 s2) (min_alg2 s1 s2) s1 s2
apply (tactic <rtac @{context} @<{thm alg_def} RS iffD2) 1>)
apply (rule conjI)
apply (rule ballI)
apply (erule CollectE conjE)+

apply (rule bexE)
apply (rule cardSuc_UNION_Cinfinitelimit)
apply (rule Cinfinitelimit_csum1)
apply (rule F1bd'_Cinfinitelimit)
apply (rule min_algs_mono1)
apply (erule subset_trans[OF _ equalityD1[OF min_alg1_def]])
apply (rule ordLeq_transitive)

```

**apply** (rule ordLess\_imp\_ordLeq[OF F1set2\_bd\_incr])  
**apply** (rule ordLeq\_csum1)  
**apply** (rule F1bd'\_Card\_order)

**apply** (rule bexE)  
**apply** (rule cardSuc\_UNION\_Cinfinite)  
**apply** (rule Cinfinite\_csum1)  
**apply** (rule F1bd'\_Cinfinite)  
**apply** (rule min\_algs\_mono2)  
**apply** (erule subset\_trans[OF equalityD1[OF min\_alg2\_def]])  
**apply** (rule ordLeq\_transitive)  
**apply** (rule ordLess\_imp\_ordLeq[OF F1set3\_bd\_incr])  
**apply** (rule ordLeq\_csum1)  
**apply** (rule F1bd'\_Card\_order)

**apply** (rule bexE)  
**apply** (rule SucFbd\_limit)  
**apply** (erule conjI)  
**apply** assumption  
**apply** (rule subsetD[OF equalityD2[OF min\_alg1\_def]])  
**apply** (rule UN\_I)  
**apply** (erule thin\_rl)  
**apply** (erule thin\_rl)  
**apply** (erule thin\_rl)  
**apply** (erule thin\_rl)  
**apply** (erule thin\_rl)  
**apply** (erule thin\_rl)  
**apply** (erule thin\_rl)  
**apply** assumption  
**apply** (rule subsetD)  
**apply** (rule equalityD2)  
**apply** (rule min\_algs1)  
**apply** assumption  
**apply** (rule UnI2)  
**apply** (rule image\_eqI)  
**apply** (rule refl)  
**apply** (rule CollectI)  
**apply** (erule asm\_rl)  
**apply** (erule thin\_rl)  
**apply** (erule thin\_rl)  
**apply** (erule conjE)+

**apply** (rule conjI)  
**apply** assumption

**apply** (rule conjI)  
**apply** (erule subset\_trans)  
**apply** (rule subsetI)  
**apply** (rule UN\_I)  
**apply** (erule underS\_I)  
**apply** assumption  
**apply** assumption

**apply** (erule subset\_trans)  
**apply** (erule UN\_upper[OF underS\_I])  
**apply** assumption

**apply** (rule ballI)  
**apply** (erule CollectE conjE)+

**apply** (rule bexE)

```

apply (rule cardSuc_UNION_Cinfinite)
  apply (rule Cinfinite_csum1)
  apply (rule F1bd'_Cinfinite)
  apply (rule min_algs_mono1)

apply (erule subset_trans[OF equalityD1[OF min_alg1_def]])
apply (rule ordLeq_transitive)
apply (rule ordLess_imp_ordLeq[OF F2set2_bd_incr])
apply (rule ordLeq_csum2)
apply (rule F2bd'_Card_order)

apply (rule bezE)
apply (rule cardSuc_UNION_Cinfinite)
  apply (rule Cinfinite_csum1)
  apply (rule F1bd'_Cinfinite)
  apply (rule min_algs_mono2)

apply (erule subset_trans[OF equalityD1[OF min_alg2_def]])
apply (rule ordLeq_transitive)
apply (rule ordLess_imp_ordLeq[OF F2set3_bd_incr])
apply (rule ordLeq_csum2)
apply (rule F2bd'_Card_order)

apply (rule bezE)
apply (rule SucFbd_limit)
apply (erule conjI)
apply assumption
apply (rule subsetD[OF equalityD2[OF min_alg2_def]])
apply (rule UN_I)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply assumption
apply (rule subsetD)
apply (rule equalityD2)
apply (rule min_algs2)
apply assumption
apply (rule UnI2)
apply (rule image_eqI)
apply (rule refl)
apply (rule CollectI)
apply (rule conjI)
apply assumption

apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule conjE)+
apply (rule conjI)
apply (erule subset_trans)
apply (rule UN_upper)
apply (erule underS_I)
apply assumption

apply (erule subset_trans)
apply (rule UN_upper)
apply (erule underS_I)
apply assumption
done

```

**lemmas** *SucFbd\_ASucFbd* = *ordLess\_ordLeq\_trans*[*OF*  
*ordLess\_ctwo\_cexp*  
*cexp\_mono1*[*OF* *ordLeq\_csum2*[*OF* *Card\_order\_ctwo*]],  
*OF* *SucFbd\_Card\_order* *SucFbd\_Card\_order*]

**lemma** *card\_of\_min\_algs*:  
**fixes** *s1* :: ('a, 'b, 'c) *F1*  $\Rightarrow$  'b **and** *s2* :: ('a, 'b, 'c) *F2*  $\Rightarrow$  'c  
**shows** *i*  $\in$  *Field SucFbd*  $\longrightarrow$   
( |fst (*min\_algs* *s1* *s2* *i*)|  $\leq_o$  (*ASucFbd* :: 'a *ASucFbd\_type* *rel*)  $\wedge$  |snd (*min\_algs* *s1* *s2* *i*)|  $\leq_o$  (*ASucFbd* :: 'a  
*ASucFbd\_type* *rel*) )  
**apply** (*rule well\_order\_induct\_imp*[*of* \_ %*i*. ( |fst (*min\_algs* *s1* *s2* *i*)|  $\leq_o$  *ASucFbd*  $\wedge$  |snd (*min\_algs* *s1* *s2* *i*)|  
 $\leq_o$  *ASucFbd* ), *OF* *worel\_SucFbd*])  
**apply** (*rule impI*)  
**apply** (*rule conjI*)  
**apply** (*rule ordIso\_ordLeq\_trans*)  
**apply** (*rule card\_of\_ordIso\_subst*)  
**apply** (*erule min\_algs1*)  
**apply** (*rule Un\_Cinfinite\_bound*)  
  
**apply** (*rule UNION\_Cinfinite\_bound*)  
  
**apply** (*rule ordLess\_imp\_ordLeq*)  
**apply** (*rule ordLess\_transitive*)  
**apply** (*rule card\_of\_underS*)  
**apply** (*rule SucFbd\_Card\_order*)  
**apply** *assumption*  
**apply** (*rule SucFbd\_ASucFbd*)  
  
**apply** (*rule ballI*)  
**apply** (*erule allE*)  
**apply** (*drule mp*)  
**apply** (*erule underS\_E*)  
**apply** (*drule mp*)  
**apply** (*erule underS\_Field*)  
**apply** (*erule conjE*)  
**apply** *assumption*  
  
**apply** (*rule ASucFbd\_Cinfinite*)  
  
**apply** (*rule ordLeq\_transitive*)  
**apply** (*rule card\_of\_image*)  
**apply** (*rule ordLeq\_transitive*)  
**apply** (*rule F1.in\_bd*)  
**apply** (*rule ordLeq\_transitive*)  
**apply** (*rule cexp\_mono1*)  
**apply** (*rule csum\_mono1*)  
**apply** (*rule csum\_mono2*)  
**apply** (*rule csum\_cinfinite\_bound*)  
**apply** (*rule UNION\_Cinfinite\_bound*)  
  
**apply** (*rule ordLess\_imp\_ordLeq*)  
**apply** (*rule ordLess\_transitive*)  
**apply** (*rule card\_of\_underS*)  
**apply** (*rule SucFbd\_Card\_order*)  
**apply** *assumption*  
**apply** (*rule SucFbd\_ASucFbd*)  
  
**apply** (*rule ballI*)  
**apply** (*erule allE*)  
**apply** (*drule mp*)  
**apply** (*erule underS\_E*)  
**apply** (*drule mp*)

```

    apply (erule underS_Field)
    apply (erule conjE)+
    apply assumption

    apply (rule ASucFbd_Cinfinite)

    apply (rule UNION_Cinfinite_bound)

    apply (rule ordLess_imp_ordLeq)
    apply (rule ordLess_transitive)
    apply (rule card_of_underS)
    apply (rule SucFbd_Card_order)
    apply assumption
    apply (rule SucFbd_ASucFbd)

    apply (rule ballI)
    apply (erule allE)
    apply (drule mp)
    apply (erule underS_E)
    apply (drule mp)
    apply (erule underS_Field)
    apply (erule conjE)+
    apply assumption

    apply (rule ASucFbd_Cinfinite)

    apply (rule card_of_Card_order)
    apply (rule card_of_Card_order)
    apply (rule ASucFbd_Cinfinite)

    apply (rule F1bd'_Card_order)
    apply (rule ordIso_ordLeq_trans)
    apply (rule cexp_cong1)
    apply (rule ordIso_transitive)
    apply (rule csum_cong1)
    apply (rule ordIso_transitive)
    apply (tactic ⟨BNF_Tactics.mk_rotate_eq_tac @ {context}
      (rtac @ {context} @ {thm ordIso_refl} THEN'
        FIRST' [rtac @ {context} @ {thm card_of_Card_order},
          rtac @ {context} @ {thm Card_order_csum},
          rtac @ {context} @ {thm Card_order_cexp}]
        @ {thm ordIso_transitive} @ {thm csum_assoc} @ {thm csum_com} @ {thm csum_cong}
        [1,2] [2,1] 1)⟩)
    apply (rule csum_absorb1)
    apply (rule ASucFbd_Cinfinite)

    apply (rule ordLeq_transitive)
    apply (rule ordLeq_csum1)
    apply (tactic ⟨FIRST' [rtac @ {context} @ {thm Card_order_csum}, rtac @ {context} @ {thm card_of_Card_order}]
1)⟩)
    apply (rule ordLeq_cexp1)
    apply (rule SucFbd_Cnotzero)
    apply (rule Card_order_csum)
    apply (rule csum_absorb1)
    apply (rule ASucFbd_Cinfinite)
    apply (rule ctwo_ordLeq_Cinfinite)
    apply (rule ASucFbd_Cinfinite)
    apply (rule F1bd'_Card_order)
    apply (rule ordIso_imp_ordLeq)
    apply (rule cexp_cprod_ordLeq)

    apply (rule Card_order_csum)
    apply (rule SucFbd_Cinfinite)

```

```

apply (rule F1bd'_Cnotzero)
apply (rule ordLeq_transitive)
apply (rule ordLeq_csum1)
apply (rule F1bd'_Card_order)
apply (rule cardSuc_ordLeq)
apply (rule Card_order_csum)

apply (rule ASucFbd_Cinfinite)

apply (rule ordIso_ordLeq_trans)
apply (rule card_of_ordIso_subst)
apply (erule min_algs2)
apply (rule Un_Cinfinite_bound)

apply (rule UNION_Cinfinite_bound)

apply (rule ordLess_imp_ordLeq)
apply (rule ordLess_transitive)
apply (rule card_of_underS)
apply (rule SucFbd_Card_order)
apply assumption
apply (rule SucFbd_ASucFbd)

apply (rule ballI)
apply (erule allE)
apply (drule mp)
apply (erule underS_E)
apply (drule mp)
apply (erule underS_Field)
apply (erule conjE)+
apply assumption

apply (rule ASucFbd_Cinfinite)

apply (rule ordLeq_transitive)
apply (rule card_of_image)
apply (rule ordLeq_transitive)
apply (rule F2.in_bd)
apply (rule ordLeq_transitive)
apply (rule cexp_mono1)
apply (rule csum_mono1)
apply (rule csum_mono2)
apply (rule csum_cinfinite_bound)
apply (rule UNION_Cinfinite_bound)

apply (rule ordLess_imp_ordLeq)
apply (rule ordLess_transitive)
apply (rule card_of_underS)
apply (rule SucFbd_Card_order)
apply assumption
apply (rule SucFbd_ASucFbd)

apply (rule ballI)
apply (erule allE)
apply (drule mp)
apply (erule underS_E)
apply (drule mp)
apply (erule underS_Field)
apply (erule conjE)+
apply assumption

apply (rule ASucFbd_Cinfinite)

```

```

apply (rule UNION_Cinfinite_bound)

apply (rule ordLess_imp_ordLeq)
apply (rule ordLess_transitive)
apply (rule card_of_underS)
  apply (rule SucFbd_Card_order)
apply assumption
apply (rule SucFbd_ASucFbd)

apply (rule ballI)
apply (erule allE)
apply (drule mp)
  apply (erule underS_E)
apply (drule mp)
  apply (erule underS_Field)
apply (erule conjE)+
apply assumption

apply (rule ASucFbd_Cinfinite)

apply (rule card_of_Card_order)
apply (rule card_of_Card_order)
apply (rule ASucFbd_Cinfinite)

apply (rule F2bd'_Card_order)
apply (rule ordIso_ordLeq_trans)
apply (rule cexp_cong1)

apply (rule ordIso_transitive)
apply (rule csum_cong1)
apply (rule ordIso_transitive)
apply (tactic ⟨BNF_Tactics.mk_rotate_eq_tac @ {context}
  (rtac @ {context} @ {thm ordIso_refl} THEN'
    FIRST' [rtac @ {context} @ {thm card_of_Card_order},
    rtac @ {context} @ {thm Card_order_csum},
    rtac @ {context} @ {thm Card_order_cexp}]
    @ {thm ordIso_transitive} @ {thm csum_assoc} @ {thm csum_com} @ {thm csum_cong}
    [1,2] [2,1] 1)⟩)
apply (rule csum_absorb1)
apply (rule ASucFbd_Cinfinite)

apply (rule ordLeq_transitive)
apply (rule ordLeq_csum1)
apply (tactic ⟨FIRST' [rtac @ {context} @ {thm Card_order_csum}, rtac @ {context} @ {thm card_of_Card_order}]
1)⟩)
apply (rule ordLeq_cexp1)
apply (rule SucFbd_Cnotzero)
apply (rule Card_order_csum)

apply (rule csum_absorb1)
apply (rule ASucFbd_Cinfinite)
apply (rule ctwo_ordLeq_Cinfinite)
apply (rule ASucFbd_Cinfinite)
apply (rule F2bd'_Card_order)
apply (rule ordIso_imp_ordLeq)
apply (rule cexp_cprod_ordLeq)
  apply (rule Card_order_csum)
apply (rule SucFbd_Cinfinite)
apply (rule F2bd'_Cnotzero)
apply (rule ordLeq_transitive)
apply (rule ordLeq_csum2)
apply (rule F2bd'_Card_order)
apply (rule cardSuc_ordLeq)

```



```

apply (rule Card_order_csum)

apply (rule ASucFbd_Cinfinite)
done

lemma card_of_min_alg1:
  fixes s1 :: ('a, 'b, 'c) F1  $\Rightarrow$  'b and s2 :: ('a, 'b, 'c) F2  $\Rightarrow$  'c
  shows |min_alg1 s1 s2|  $\leq o$  (ASucFbd :: 'a ASucFbd_type rel)
  apply (rule ordIso_ordLeq_trans)
  apply (rule card_of_ordIso_subst[OF min_alg1_def])
  apply (rule UNION_Cinfinite_bound)

  apply (rule ordIso_ordLeq_trans)
  apply (rule card_of_Field_ordIso)
  apply (rule SucFbd_Card_order)
  apply (rule ordLess_imp_ordLeq)
  apply (rule SucFbd_ASucFbd)

  apply (rule ballI)
  apply (drule rev_mp)
  apply (rule card_of_min_algs)
  apply (erule conjE)+
  apply assumption
  apply (rule ASucFbd_Cinfinite)
done

lemma card_of_min_alg2:
  fixes s1 :: ('a, 'b, 'c) F1  $\Rightarrow$  'b and s2 :: ('a, 'b, 'c) F2  $\Rightarrow$  'c
  shows |min_alg2 s1 s2|  $\leq o$  (ASucFbd :: 'a ASucFbd_type rel)
  apply (rule ordIso_ordLeq_trans)
  apply (rule card_of_ordIso_subst[OF min_alg2_def])
  apply (rule UNION_Cinfinite_bound)

  apply (rule ordIso_ordLeq_trans)
  apply (rule card_of_Field_ordIso)
  apply (rule SucFbd_Card_order)
  apply (rule ordLess_imp_ordLeq)
  apply (rule SucFbd_ASucFbd)

  apply (rule ballI)
  apply (drule rev_mp)
  apply (rule card_of_min_algs)
  apply (erule conjE)+
  apply assumption
  apply (rule ASucFbd_Cinfinite)
done

lemma least_min_algs: alg B1 B2 s1 s2  $\implies$ 
  i  $\in$  Field SucFbd  $\longrightarrow$ 
  fst (min_algs s1 s2 i)  $\subseteq$  B1  $\wedge$  snd (min_algs s1 s2 i)  $\subseteq$  B2
  apply (rule well_order_induct_imp[of _ %i. (fst (min_algs s1 s2 i)  $\subseteq$  B1  $\wedge$  snd (min_algs s1 s2 i)  $\subseteq$  B2), OF
  worel_SucFbd])
  apply (rule impI)
  apply (rule conjI)
  apply (rule ord_eq_le_trans)
  apply (erule min_algs1)
  apply (rule Un_least)
  apply (rule UN_least)
  apply (erule allE)
  apply (drule mp)
  apply (erule underS_E)
  apply (drule mp)
  apply (erule underS_Field)

```

```

apply (erule conjE)+
apply assumption
apply (rule image_subsetI)
apply (erule CollectE conjE)+
apply (erule alg_F1set)

```

```

apply (erule subset_trans)
apply (rule UN_least)
apply (erule allE)
apply (drule mp)
  apply (erule underS_E)
apply (drule mp)
  apply (erule underS_Field)
apply (erule conjE)+
apply assumption

```

```

apply (erule subset_trans)
apply (rule UN_least)
apply (erule allE)
apply (drule mp)
  apply (erule underS_E)
apply (drule mp)
  apply (erule underS_Field)
apply (erule conjE)+
apply assumption

```

```

apply (rule ord_eq_le_trans)
apply (erule min_algs2)
apply (rule Un_least)
apply (rule UN_least)
apply (erule allE)
apply (drule mp)
  apply (erule underS_E)
apply (drule mp)
  apply (erule underS_Field)
apply (erule conjE)+
apply assumption
apply (rule image_subsetI)
apply (erule CollectE conjE)+
apply (erule alg_F2set)

```

```

apply (erule subset_trans)
apply (rule UN_least)
apply (erule allE)
apply (drule mp)
  apply (erule underS_E)
apply (drule mp)
  apply (erule underS_Field)
apply (erule conjE)+
apply assumption

```

```

apply (erule subset_trans)
apply (rule UN_least)
apply (erule allE)
apply (drule mp)
  apply (erule underS_E)
apply (drule mp)
  apply (erule underS_Field)
apply (erule conjE)+
apply assumption
done

```

**lemma** least\_min\_alg1:  $alg\ B1\ B2\ s1\ s2 \implies min\_alg1\ s1\ s2 \subseteq B1$

```

apply (rule ord_eq_le_trans[OF min_alg1_def])
apply (rule UN_least)
apply (drule least_min_algs)
apply (drule mp)
  apply assumption
apply (erule conjE)+
apply assumption
done

```

```

lemma least_min_alg2: alg B1 B2 s1 s2  $\implies$  min_alg2 s1 s2  $\subseteq$  B2
apply (rule ord_eq_le_trans[OF min_alg2_def])
apply (rule UN_least)
apply (drule least_min_algs)
apply (drule mp)
  apply assumption
apply (erule conjE)+
apply assumption
done

```

```

lemma mor_incl_min_alg:
  alg B1 B2 s1 s2  $\implies$ 
  mor (min_alg1 s1 s2) (min_alg2 s1 s2) s1 s2 B1 B2 s1 s2 id id
apply (rule mor_incl)
apply (erule least_min_alg1)
apply (erule least_min_alg2)
done

```

## 1.5 Initiality

The following "happens" to be the type (for our particular construction) of the initial algebra carrier:

```

type-synonym 'a1 F1init_type = ('a1, 'a1 ASucFbd_type, 'a1 ASucFbd_type) F1  $\Rightarrow$  'a1 ASucFbd_type
type-synonym 'a1 F2init_type = ('a1, 'a1 ASucFbd_type, 'a1 ASucFbd_type) F2  $\Rightarrow$  'a1 ASucFbd_type

```

```

typedef 'a1 IIT =
  UNIV ::
  (('a1 ASucFbd_type set  $\times$  'a1 ASucFbd_type set)  $\times$  ('a1 F1init_type  $\times$  'a1 F2init_type)) set
by (rule exI) (rule UNIV_I)

```

## 1.6 Initial Algebras

```

abbreviation II :: 'a1 IIT set where
  II  $\equiv$  {Abs_IIT ((B1, B2), (s1, s2)) | B1 B2 s1 s2. alg B1 B2 s1 s2}

```

```

definition str_init1 where
  str_init1 (dummy :: 'a1)
  (y::('a1, 'a1 IIT  $\Rightarrow$  'a1 ASucFbd_type, 'a1 IIT  $\Rightarrow$  'a1 ASucFbd_type) F1)
  (i :: 'a1 IIT) =
  fst (snd (Rep_IIT i))
  (F1map id ( $\lambda$ f :: 'a1 IIT  $\Rightarrow$  'a1 ASucFbd_type. f i) ( $\lambda$ f. f i) y)

```

```

definition str_init2 where
  str_init2 (dummy :: 'a1) y (i :: 'a1 IIT) =
  snd (snd (Rep_IIT i)) (F2map id ( $\lambda$ f. f i) ( $\lambda$ f. f i) y)

```

```

abbreviation car_init1 where
  car_init1 dummy  $\equiv$  min_alg1 (str_init1 dummy) (str_init2 dummy)

```

```

abbreviation car_init2 where
  car_init2 dummy  $\equiv$  min_alg2 (str_init1 dummy) (str_init2 dummy)

```

```

lemma alg_select:
   $\forall i \in II.$  alg (fst (fst (Rep_IIT i))) (snd (fst (Rep_IIT i)))
  (fst (snd (Rep_IIT i))) (snd (snd (Rep_IIT i)))
apply (rule ballI)
apply (erule CollectE exE conjE)+
apply (tactic <hyp_subst_tac @ {context} 1 >)
unfolding fst_conv snd_conv Abs_IIT_inverse[OF UNIV_I]

```

**apply** *assumption*  
**done**

**lemma** *mor\_select*:

```

[[i ∈ II;
  mor (fst (fst (Rep_IIT i))) (snd (fst (Rep_IIT i)))
    (fst (snd (Rep_IIT i))) (snd (snd (Rep_IIT i))) UNIV UNIV s1' s2' f g]] ==>
mor (car_init1 dummy) (car_init2 dummy) (str_init1 dummy) (str_init2 dummy) UNIV UNIV s1' s2' (f ∘ (λh.
h i)) (g ∘ (λh. h i))
apply (rule mor_cong)
apply (rule sym)
apply (rule o_id)
apply (rule sym)
apply (rule o_id)
apply (tactic <rtac @ {context} (Thm.permute_prem 0 1 @ {thm mor_comp}) 1 >)
apply (tactic <etac @ {context} (Thm.permute_prem 0 1 @ {thm mor_comp}) 1 >)
apply (tactic <rtac @ {context} (@ {thm mor_def} RS iffD2) 1 >)
apply (rule conjI)

apply (rule conjI)
apply (rule ballI)
apply (erule bspec[rotated])
apply (erule CollectE)
apply assumption

apply (rule ballI)
apply (erule bspec[rotated])
apply (erule CollectE)
apply assumption

apply (rule conjI)
apply (rule ballI)
apply (rule str_init1_def)

apply (rule ballI)
apply (rule str_init2_def)

apply (rule mor_incl_min_alg)

apply (erule thin_rl)+
apply (tactic <rtac @ {context} (@ {thm alg_def} RS iffD2) 1 >)
apply (rule conjI)
apply (rule ballI)
apply (erule CollectE conjE)+
apply (rule CollectI)
apply (rule ballI)
apply (erule bspec[OF alg_select])
apply (rule ssubst_mem[OF str_init1_def])
apply (erule alg_F1set)

apply (rule ord_eq_le_trans)
apply (rule F1.set_map(2))
apply (rule subset_trans)
apply (erule image_mono)
apply (rule image_Collect_subsetI)
apply (erule bspec)
apply assumption

apply (rule ord_eq_le_trans)
apply (rule F1.set_map(3))
apply (rule subset_trans)
apply (erule image_mono)
apply (rule image_Collect_subsetI)

```

**apply** (*erule bspec*)  
**apply** *assumption*

**apply** (*rule ballI*)  
**apply** (*erule CollectE conjE*)  
**apply** (*rule CollectI*)  
**apply** (*rule ballI*)  
**apply** (*frule bspec*[*OF alg\_select*])  
**apply** (*rule ssubst\_mem*[*OF str\_init2\_def*])  
**apply** (*erule alg\_F2set*)

**apply** (*rule ord\_eq\_le\_trans*)  
**apply** (*rule F2.set\_map*(2))  
**apply** (*rule subset\_trans*)  
**apply** (*erule image\_mono*)  
**apply** (*rule image\_Collect\_subsetI*)  
**apply** (*erule bspec*)  
**apply** *assumption*

**apply** (*rule ord\_eq\_le\_trans*)  
**apply** (*rule F2.set\_map*(3))  
**apply** (*rule subset\_trans*)  
**apply** (*erule image\_mono*)  
**apply** (*rule image\_Collect\_subsetI*)  
**apply** (*erule bspec*)  
**apply** *assumption*  
**done**

**lemma** *init\_unique\_mor*:

$\llbracket a1 \in \text{car\_init1 dummy}; a2 \in \text{car\_init2 dummy};$   
*mor* (*car\_init1 dummy*) (*car\_init2 dummy*) (*str\_init1 dummy*) (*str\_init2 dummy*) *B1 B2 s1 s2 f1 f2*;  
*mor* (*car\_init1 dummy*) (*car\_init2 dummy*) (*str\_init1 dummy*) (*str\_init2 dummy*) *B1 B2 s1 s2 g1 g2*  $\rrbracket \implies$   
*f1 a1 = g1 a1*  $\wedge$  *f2 a2 = g2 a2*

**apply** (*rule conjI*)  
**apply** (*erule prop\_restrict*)  
**apply** (*erule thin\_rl*)  
**apply** (*rule least\_min\_alg1*)  
**apply** (*tactic*  $\langle \text{rtac } @\{\text{context}\} (@\{\text{thm alg\_def}\} \text{RS iffD2}) 1 \rangle$ )  
**apply** (*rule conjI*)  
**apply** (*rule ballI*)  
**apply** (*rule CollectI*)  
**apply** (*erule CollectE conjE*)  
**apply** (*rule conjI*)

**apply** (*rule alg\_F1set*[*OF alg\_min\_alg*])  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)

**apply** (*rule trans*)  
**apply** (*erule morE1*)  
**apply** (*rule subsetD*)  
**apply** (*rule F1in\_mono23*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** *assumption*

**apply** (*rule trans*)  
**apply** (*rule arg\_cong*[*OF F1.map\_cong0*])  
    **apply** (*rule refl*)  
    **apply** (*erule prop\_restrict*)  
    **apply** *assumption*  
**apply** (*erule prop\_restrict*)  
**apply** *assumption*

**apply** (*rule sym*)  
**apply** (*erule morE1*)  
**apply** (*rule subsetD*)  
    **apply** (*rule F1in\_mono23*)  
    **apply** (*rule Collect\_restrict*)  
    **apply** (*rule Collect\_restrict*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI*)  
    **apply** *assumption*  
**apply** (*rule conjI*)  
    **apply** *assumption*  
**apply** *assumption*

**apply** (*rule ballI*)  
**apply** (*rule CollectI*)  
**apply** (*erule CollectE conjE*)+  
**apply** (*rule conjI*)

**apply** (*rule alg\_F2set*[*OF alg\_min\_alg*])  
    **apply** (*erule subset\_trans*)  
    **apply** (*rule Collect\_restrict*)  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)

**apply** (*rule trans*)  
**apply** (*erule morE2*)  
**apply** (*rule subsetD*)  
    **apply** (*rule F2in\_mono23*)  
    **apply** (*rule Collect\_restrict*)  
    **apply** (*rule Collect\_restrict*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI*)  
    **apply** *assumption*  
**apply** (*rule conjI*)  
    **apply** *assumption*  
**apply** *assumption*

**apply** (*rule trans*)  
**apply** (*rule arg\_cong*[*OF F2.map\_cong0*])  
    **apply** (*rule refl*)  
    **apply** (*erule prop\_restrict*)  
    **apply** *assumption*  
**apply** (*erule prop\_restrict*)  
**apply** *assumption*

**apply** (*rule sym*)  
**apply** (*erule morE2*)  
**apply** (*rule subsetD*)  
    **apply** (*rule F2in\_mono23*)  
    **apply** (*rule Collect\_restrict*)  
    **apply** (*rule Collect\_restrict*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI*)  
**apply** *assumption*

**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** *assumption*

**apply** (*erule thin\_rl*)  
**apply** (*erule prop\_restrict*)  
**apply** (*rule least\_min\_alg2*)  
**apply** (*tactic* <*rtac* @{*context*} (@{*thm alg\_def*} *RS iffD2*) 1 >)  
**apply** (*rule conjI*)  
**apply** (*rule ballI*)  
**apply** (*rule CollectI*)  
**apply** (*erule CollectE conjE*)+  
**apply** (*rule conjI*)

**apply** (*rule alg\_F1set*[*OF alg\_min\_alg*])  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)

**apply** (*rule trans*)  
**apply** (*erule morE1*)  
**apply** (*rule subsetD*)  
**apply** (*rule F1in\_mono23*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** *assumption*

**apply** (*rule trans*)  
**apply** (*rule arg\_cong*[*OF F1.map\_cong0*])  
**apply** (*rule refl*)  
**apply** (*erule prop\_restrict*)  
**apply** *assumption*  
**apply** (*erule prop\_restrict*)  
**apply** *assumption*

**apply** (*rule sym*)  
**apply** (*erule morE1*)  
**apply** (*rule subsetD*)  
**apply** (*rule F1in\_mono23*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** *assumption*

**apply** (*rule ballI*)  
**apply** (*rule CollectI*)  
**apply** (*erule CollectE conjE*)+  
**apply** (*rule conjI*)

**apply** (*rule alg\_F2set*[*OF alg\_min\_alg*])  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)

**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)

**apply** (*rule trans*)  
**apply** (*erule morE2*)  
**apply** (*rule subsetD*)  
**apply** (*rule F2in\_mono23*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** *assumption*

**apply** (*rule trans*)  
**apply** (*rule arg\_cong[OF F2.map\_cong0]*)  
**apply** (*rule refl*)  
**apply** (*erule prop\_restrict*)  
**apply** *assumption*  
**apply** (*erule prop\_restrict*)  
**apply** *assumption*

**apply** (*rule sym*)  
**apply** (*erule morE2*)  
**apply** (*rule subsetD*)  
**apply** (*rule F2in\_mono23*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** *assumption*  
**done**

**abbreviation** *closed where*

*closed dummy phi1 phi2*  $\equiv ((\forall x \in F1in UNIV (car\_init1 dummy) (car\_init2 dummy)).$   
 $(\forall z \in F1set2 x. phi1 z) \wedge (\forall z \in F1set3 x. phi2 z) \longrightarrow phi1 (str\_init1 dummy x)) \wedge$   
 $(\forall x \in F2in UNIV (car\_init1 dummy) (car\_init2 dummy)).$   
 $(\forall z \in F2set2 x. phi1 z) \wedge (\forall z \in F2set3 x. phi2 z) \longrightarrow phi2 (str\_init2 dummy x))$ )

**lemma** *init\_induct: closed dummy phi1 phi2*  $\implies$

$(\forall x \in car\_init1 dummy. phi1 x) \wedge (\forall x \in car\_init2 dummy. phi2 x)$

**apply** (*rule conjI*)  
**apply** (*rule ballI*)  
**apply** (*erule prop\_restrict*)  
**apply** (*rule least\_min\_alg1*)  
**apply** (*tactic <rtac @ {context} (@ {thm alg\_def} RS iffD2) 1 >*)

**apply** (*rule conjI*)  
**apply** (*rule ballI*)  
**apply** (*rule CollectI*)  
**apply** (*erule CollectE conjE*)+  
**apply** (*rule conjI*)

**apply** (*rule alg\_F1set[OF alg\_min\_alg]*)  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)



**apply** (*rule mp*)  
**apply** (*erule bspec*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** (*rule conjI*)  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)

**apply** (*rule conjI*)  
**apply** (*rule ballI*)  
**apply** (*erule prop\_restrict*)  
**apply** *assumption*  
**apply** (*rule ballI*)  
**apply** (*erule prop\_restrict*)  
**apply** *assumption*

**apply** (*rule ballI*)  
**apply** (*rule CollectI*)  
**apply** (*erule CollectE conjE*)+  
**apply** (*rule conjI*)

**apply** (*rule alg\_F2set*[*OF alg\_min\_alg*])  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)

**apply** (*rule mp*)  
**apply** (*erule bspec*)  
**apply** (*rule CollectI*)  
**apply** (*rule conjI*)  
**apply** *assumption*  
**apply** (*rule conjI*)  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)  
**apply** (*erule subset\_trans*)  
**apply** (*rule Collect\_restrict*)

**apply** (*rule conjI*)  
**apply** (*rule ballI*)  
**apply** (*erule prop\_restrict*)  
**apply** *assumption*  
**apply** (*rule ballI*)  
**apply** (*erule prop\_restrict*)  
**apply** *assumption*

**apply** (*rule ballI*)  
**apply** (*erule prop\_restrict*)  
**apply** (*rule least\_min\_alg2*)  
**apply** (*tactic*  $\langle \text{rtac } @\{\text{context}\} (@\{\text{thm alg\_def}\} \text{RS iffD2}) 1 \rangle$ )

**apply** (*rule conjI*)  
**apply** (*rule ballI*)  
**apply** (*rule CollectI*)  
**apply** (*erule CollectE conjE*)+  
**apply** (*rule conjI*)

**apply** (*rule alg\_F1set*[*OF alg\_min\_alg*])

```

apply (erule subset_trans)
apply (rule Collect_restrict)
apply (erule subset_trans)
apply (rule Collect_restrict)

```

```

apply (rule mp)
apply (erule bspec)
apply (rule CollectI)
apply (rule conjI)
apply assumption
apply (rule conjI)
apply (erule subset_trans)
apply (rule Collect_restrict)
apply (erule subset_trans)
apply (rule Collect_restrict)

```

```

apply (rule conjI)
apply (rule ballI)
apply (erule prop_restrict)
apply assumption
apply (rule ballI)
apply (erule prop_restrict)
apply assumption

```

```

apply (rule ballI)
apply (rule CollectI)
apply (erule CollectE conjE)+
apply (rule conjI)

```

```

apply (rule alg_F2set[OF alg_min_alg])
apply (erule subset_trans)
apply (rule Collect_restrict)
apply (erule subset_trans)
apply (rule Collect_restrict)

```

```

apply (rule mp)
apply (erule bspec)
apply (rule CollectI)
apply (rule conjI)
apply assumption
apply (rule conjI)
apply (erule subset_trans)
apply (rule Collect_restrict)
apply (erule subset_trans)
apply (rule Collect_restrict)

```

```

apply (rule conjI)
apply (rule ballI)
apply (erule prop_restrict)
apply assumption
apply (rule ballI)
apply (erule prop_restrict)
apply assumption
done

```

## 1.7 The datatype

```

typedef (overloaded) 'a1 IF1 = car_init1 (undefined :: 'a1)
apply (rule iffD2)
apply (rule ex_in_conv)
apply (rule conjunct1)
apply (rule alg_not_empty)
apply (rule alg_min_alg)

```

```

done

typedef (overloaded) 'a1 IF2 = car_init2 (undefined :: 'a1)
  apply (rule iffD2)
  apply (rule ex_in_conv)
  apply (rule conjunct2)
  apply (rule alg_not_empty)
  apply (rule alg_min_alg)
done

definition ctor1 where ctor1 = Abs_IF1 o str_init1 undefined o F1map id Rep_IF1 Rep_IF2
definition ctor2 where ctor2 = Abs_IF2 o str_init2 undefined o F2map id Rep_IF1 Rep_IF2

lemma mor_Rep_IF:
  mor (UNIV :: 'a IF1 set) (UNIV :: 'a IF2 set) ctor1 ctor2
  (car_init1 undefined) (car_init2 undefined) (str_init1 undefined) (str_init2 undefined) Rep_IF1 Rep_IF2
  unfolding mor_def ctor1_def ctor2_def o_apply
  apply (rule conjI)
  apply (rule conjI)
  apply (rule ballI)
  apply (rule Rep_IF1)
  apply (rule ballI)
  apply (rule Rep_IF2)

  apply (rule conjI)
  apply (rule ballI)
  apply (rule Abs_IF1_inverse)
  apply (rule alg_F1set[OF alg_min_alg])
  apply (rule ord_eq_le_trans[OF F1.set_map(2)])
  apply (rule image_subsetI)
  apply (rule Rep_IF1)
  apply (rule ord_eq_le_trans[OF F1.set_map(3)])
  apply (rule image_subsetI)
  apply (rule Rep_IF2)

  apply (rule ballI)
  apply (rule Abs_IF2_inverse)
  apply (rule alg_F2set[OF alg_min_alg])
  apply (rule ord_eq_le_trans[OF F2.set_map(2)])
  apply (rule image_subsetI)
  apply (rule Rep_IF1)
  apply (rule ord_eq_le_trans[OF F2.set_map(3)])
  apply (rule image_subsetI)
  apply (rule Rep_IF2)
done

lemma mor_Abs_IF:
  mor (car_init1 undefined) (car_init2 undefined)
  (str_init1 undefined) (str_init2 undefined) UNIV UNIV ctor1 ctor2 Abs_IF1 Abs_IF2
  unfolding mor_def ctor1_def ctor2_def o_apply
  apply (rule conjI)
  apply (rule conjI)
  apply (rule ballI)
  apply (rule UNIV_I)
  apply (rule ballI)
  apply (rule UNIV_I)

  apply (rule conjI)
  apply (rule ballI)
  apply (erule CollectE conjE)+
  apply (rule sym[OF arg_cong[OF trans[OF F1map_comp_id F1map_congL]]])
  apply (rule ballI[OF trans[OF o_apply]])
  apply (erule Abs_IF1_inverse[OF subsetD])

```

```

apply assumption
apply (rule ballI[OF trans[OF o_apply]])
apply (erule Abs_IF2_inverse[OF subsetD])
apply assumption

apply (rule ballI)
apply (erule CollectE conjE)+
apply (rule sym[OF arg_cong[OF trans[OF F2map_comp_id F2map_congL]]])
apply (rule ballI[OF trans[OF o_apply]])
apply (erule Abs_IF1_inverse[OF subsetD])
apply assumption
apply (rule ballI[OF trans[OF o_apply]])
apply (erule Abs_IF2_inverse[OF subsetD])
apply assumption
done

```

**lemma** *copy*:

```

[[alg B1 B2 s1 s2; bij_betw f B1' B1; bij_betw g B2' B2]] ==>
  ∃ f' g'. alg B1' B2' f' g' ∧ mor B1' B2' f' g' B1 B2 s1 s2 f g
apply (rule exI)+
apply (rule conjI)
apply (tactic <rtac @{context} (@{thm alg_def} RS iffD2) 1>)
apply (rule conjI)
apply (rule ballI)
apply (erule CollectE conjE)+
apply (rule subsetD)
apply (rule equalityD1)
apply (erule bij_betw_imp_surj_on[OF bij_betw_the_inv_into])
apply (rule imageI)
apply (erule alg_F1set)
apply (rule ord_eq_le_trans)
apply (rule F1.set_map(2))
apply (rule subset_trans)
apply (erule image_mono)
apply (rule equalityD1)
apply (erule bij_betw_imp_surj_on)
apply (rule ord_eq_le_trans)
apply (rule F1.set_map(3))
apply (rule subset_trans)
apply (erule image_mono)
apply (rule equalityD1)
apply (erule bij_betw_imp_surj_on)

apply (rule ballI)
apply (erule CollectE conjE)+
apply (rule subsetD)
apply (rule equalityD1)
apply (erule bij_betw_imp_surj_on[OF bij_betw_the_inv_into])
apply (rule imageI)
apply (erule alg_F2set)
apply (rule ord_eq_le_trans)
apply (rule F2.set_map(2))
apply (rule subset_trans)
apply (erule image_mono)
apply (rule equalityD1)
apply (erule bij_betw_imp_surj_on)
apply (rule ord_eq_le_trans)
apply (rule F2.set_map(3))
apply (rule subset_trans)
apply (erule image_mono)
apply (rule equalityD1)
apply (erule bij_betw_imp_surj_on)

```

```

apply (tactic <rtac @{{context}} (@{thm mor_def} RS iffD2) 1>)
apply (rule conjI)
apply (rule conjI)
  apply (erule bij_betwE)
apply (erule bij_betwE)

```

```

apply (rule conjI)
apply (rule ballI)
apply (erule CollectE conjE)+
apply (erule f_the_inv_into_f_bij_betw)
apply (erule alg_F1set)
  apply (rule ord_eq_le_trans)
    apply (rule F1.set_map(2))
apply (rule subset_trans)
  apply (erule image_mono)
apply (rule equalityD1)
apply (erule bij_betw_imp_surj_on)
apply (rule ord_eq_le_trans)
apply (rule F1.set_map(3))
apply (rule subset_trans)
  apply (erule image_mono)
apply (rule equalityD1)
apply (erule bij_betw_imp_surj_on)

```

```

apply (rule ballI)
apply (erule CollectE conjE)+
apply (erule f_the_inv_into_f_bij_betw)
apply (erule alg_F2set)
  apply (rule ord_eq_le_trans)
    apply (rule F2.set_map(2))
apply (rule subset_trans)
  apply (erule image_mono)
apply (rule equalityD1)
apply (erule bij_betw_imp_surj_on)
apply (rule ord_eq_le_trans)
apply (rule F2.set_map(3))
apply (rule subset_trans)
  apply (erule image_mono)
apply (rule equalityD1)
apply (erule bij_betw_imp_surj_on)
done

```

**lemma** *init\_ex\_mor*:

```

 $\exists f g. mor\ UNIV\ UNIV\ ctor1\ ctor2\ UNIV\ UNIV\ s1\ s2\ f\ g$ 
apply (insert ex_bij_betw[OF card_of_min_alg1, of s1 s2]
  ex_bij_betw[OF card_of_min_alg2, of s1 s2])
apply (erule exE)+
apply (rule rev_mp)
  apply (rule copy[OF alg_min_alg])
    apply assumption
  apply assumption
apply (rule impI)
apply (erule exE conjE)+

```

```

apply (rule exI)+
apply (rule mor_comp)
  apply (rule mor_Rep_IF)
apply (rule mor_select)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply assumption

```

```

unfolding fst_conv snd_conv Abs_IIT_inverse[OF UNIV_I]
apply (erule mor_comp)
apply (rule mor_incl)
  apply (rule subset_UNIV)
apply (rule subset_UNIV)
done

```

Iteration

**abbreviation** fold where

```

fold s1 s2  $\equiv$  (SOME f. mor UNIV UNIV ctor1 ctor2 UNIV UNIV s1 s2 (fst f) (snd f))

```

**definition** fold1 where fold1 s1 s2 = fst (fold s1 s2)

**definition** fold2 where fold2 s1 s2 = snd (fold s1 s2)

**lemma** mor\_fold:

```

mor UNIV UNIV ctor1 ctor2 UNIV UNIV s1 s2 (fold1 s1 s2) (fold2 s1 s2)

```

```

unfolding fold1_def fold2_def

```

```

apply (rule rev_mp)

```

```

  apply (rule init_ex_mor)

```

```

apply (rule impI)

```

```

apply (erule exE)

```

```

apply (erule exE)

```

```

apply (rule someI[of % $(f :: ('a \text{ IF1} \Rightarrow 'b) \times ('a \text{ IF2} \Rightarrow 'c))$ ].

```

```

mor UNIV UNIV ctor1 ctor2 UNIV UNIV s1 s2 (fst f) (snd f)])

```

```

apply (erule mor_cong[OF fst_conv snd_conv])

```

```

done

```

**ML**  $\langle$

```

val fold1 = rule_by_tactic @{context}

```

```

  (rtac @{context} CollectI 1 THEN BNF_Util.CONJ_WRAP (K (rtac @{context} @{thm subset_UNIV} 1)) (1
upto 3))

```

```

  @{thm morE1[OF mor_fold]}

```

```

val fold2 = rule_by_tactic @{context}

```

```

  (rtac @{context} CollectI 1 THEN BNF_Util.CONJ_WRAP (K (rtac @{context} @{thm subset_UNIV} 1)) (1
upto 3))

```

```

  @{thm morE2[OF mor_fold]}

```

$\rangle$

**theorem** fold1:

```

(fold1 s1 s2) (ctor1 x) = s1 (F1map id (fold1 s1 s2) (fold2 s1 s2) x)

```

```

apply (rule morE1)

```

```

  apply (rule mor_fold)

```

```

apply (rule CollectI)

```

```

apply (rule conjI)

```

```

  apply (rule subset_UNIV)

```

```

apply (rule conjI)

```

```

  apply (rule subset_UNIV)

```

```

apply (rule subset_UNIV)

```

```

done

```

**theorem** fold2:

```

(fold2 s1 s2) (ctor2 x) = s2 (F2map id (fold1 s1 s2) (fold2 s1 s2) x)

```

```

apply (rule morE2)

```

```

  apply (rule mor_fold)

```

```

apply (rule CollectI)

```

```

apply (rule conjI)

```

```

  apply (rule subset_UNIV)

```

```

apply (rule conjI)

```

```

  apply (rule subset_UNIV)

```

```

apply (rule subset_UNIV)

```

```

done

```

**lemma** *mor\_UNIV*: *mor UNIV UNIV s1 s2 UNIV UNIV s1' s2' f g*  $\longleftrightarrow$

*f o s1 = s1' o F1map id f g*  $\wedge$  *g o s2 = s2' o F2map id f g*

```

apply (rule iffI)
apply (rule conjI)
apply (rule ext)
apply (rule trans)
apply (rule o_apply)
apply (rule trans)
apply (erule morE1)
apply (rule CollectI)
apply (rule conjI)
apply (rule subset_UNIV)
apply (rule conjI)
apply (rule subset_UNIV)
apply (rule subset_UNIV)
apply (rule sym[OF o_apply])

```

```

apply (rule ext)
apply (rule trans)
apply (rule o_apply)
apply (rule trans)
apply (erule morE2)
apply (rule CollectI)
apply (rule conjI)
apply (rule subset_UNIV)
apply (rule conjI)
apply (rule subset_UNIV)
apply (rule subset_UNIV)
apply (rule sym[OF o_apply])

```

```

apply (tactic <rtac @{context}> (@{thm mor_def} RS iffD2) 1)

```

```

apply (rule conjI)
apply (rule conjI)
apply (rule ballI)
apply (rule UNIV_I)
apply (rule ballI)
apply (rule UNIV_I)
apply (erule conjE)
apply (drule iffD1[OF fun_eq_iff])
apply (drule iffD1[OF fun_eq_iff])
apply (rule conjI)
apply (rule ballI)
apply (erule allE)+
apply (rule trans)
apply (erule trans[OF sym[OF o_apply]])
apply (rule o_apply)
apply (rule ballI)
apply (erule allE)+
apply (rule trans)
apply (erule trans[OF sym[OF o_apply]])
apply (rule o_apply)
done

```

**lemma** *fold\_unique\_mor*: *mor UNIV UNIV ctor1 ctor2 UNIV UNIV s1 s2 f g*  $\implies$

*f = fold1 s1 s2*  $\wedge$  *g = fold2 s1 s2*

```

apply (rule conjI)
apply (rule surj_fun_eq)
apply (rule type_definition.Abs_image[OF type_definition_IF1])
apply (rule ballI)
apply (rule conjunct1)
apply (rule init_unique_mor)
apply assumption
apply (rule Rep_IF2)

```

```

apply (rule mor_comp)
  apply (rule mor_Abs_IF)
apply assumption
apply (rule mor_comp)
  apply (rule mor_Abs_IF)
apply (rule mor_fold)

apply (rule surj_fun_eq)
  apply (rule type_definition.Abs_image[OF type_definition_IF2])
apply (rule ballI)
apply (rule conjunct2)
apply (rule init_unique_mor)
  apply (rule Rep_IF1)
  apply assumption
apply (rule mor_comp)
  apply (rule mor_Abs_IF)
apply assumption
apply (rule mor_comp)
  apply (rule mor_Abs_IF)
apply (rule mor_fold)
done

lemmas fold_unique = fold_unique_mor[OF iffD2[OF mor_UNIV], OF conjI]

lemmas fold1_ctor = sym[OF conjunct1[OF fold_unique_mor[OF mor_incl[OF subset_UNIV subset_UNIV]]]]
lemmas fold2_ctor = sym[OF conjunct2[OF fold_unique_mor[OF mor_incl[OF subset_UNIV subset_UNIV]]]]

Case distinction

lemmas ctor1_o_fold1 =
  trans[OF conjunct1[OF fold_unique_mor[OF mor_comp[OF mor_fold mor_str]]] fold1_ctor]
lemmas ctor2_o_fold2 =
  trans[OF conjunct2[OF fold_unique_mor[OF mor_comp[OF mor_fold mor_str]]] fold2_ctor]

definition dtor1 = fold1 (F1map id ctor1 ctor2) (F2map id ctor1 ctor2)
definition dtor2 = fold2 (F1map id ctor1 ctor2) (F2map id ctor1 ctor2)

ML <Local_Defs.fold @ {context} @ {thms dtor1_def} @ {thm ctor1_o_fold1}>
ML <Local_Defs.fold @ {context} @ {thms dtor2_def} @ {thm ctor2_o_fold2}>

lemma ctor1_o_dtor1: ctor1 o dtor1 = id
  unfolding dtor1_def
  apply (rule ctor1_o_fold1)
done

lemma ctor2_o_dtor2: ctor2 o dtor2 = id
  unfolding dtor2_def
  apply (rule ctor2_o_fold2)
done

lemma dtor1_o_ctor1: dtor1 o ctor1 = id
  apply (rule ext)
  apply (rule trans[OF o_apply])
  apply (rule trans[OF fun_cong[OF dtor1_def]])
  apply (rule trans[OF fold1])
  apply (rule trans[OF F1map_comp_id])
  apply (rule trans[OF F1map_congL])
  apply (rule ballI)
  apply (rule trans[OF fun_cong[OF ctor1_o_fold1] id_apply])
  apply (rule ballI)
  apply (rule trans[OF fun_cong[OF ctor2_o_fold2] id_apply])
  apply (rule sym[OF id_apply])
done

```



```

lemma dtor2_o_ctor2: dtor2 o ctor2 = id
  apply (rule ext)
  apply (rule trans[OF o_apply])
  apply (rule trans[OF fun_cong[OF dtor2_def]])
  apply (rule trans[OF fold2])
  apply (rule trans[OF F2map_comp_id])
  apply (rule trans[OF F2map_congL])
  apply (rule ballI)
  apply (rule trans[OF fun_cong[OF ctor1_o_fold1] id_apply])
  apply (rule ballI)
  apply (rule trans[OF fun_cong[OF ctor2_o_fold2] id_apply])
  apply (rule sym[OF id_apply])
done

```

```

lemmas dtor1_ctor1 = pointfree_idE[OF dtor1_o_ctor1]
lemmas dtor2_ctor2 = pointfree_idE[OF dtor2_o_ctor2]
lemmas ctor1_dtor1 = pointfree_idE[OF ctor1_o_dtor1]
lemmas ctor2_dtor2 = pointfree_idE[OF ctor2_o_dtor2]

```

```

lemmas bij_dtor1 = o_bij[OF ctor1_o_dtor1 dtor1_o_ctor1]
lemmas inj_dtor1 = bij_is_inj[OF bij_dtor1]
lemmas surj_dtor1 = bij_is_surj[OF bij_dtor1]
lemmas dtor1_nchotomy = surjD[OF surj_dtor1]
lemmas dtor1_diff = inj_eq[OF inj_dtor1]
lemmas dtor1_cases = exE[OF dtor1_nchotomy]
lemmas bij_dtor2 = o_bij[OF ctor2_o_dtor2 dtor2_o_ctor2]
lemmas inj_dtor2 = bij_is_inj[OF bij_dtor2]
lemmas surj_dtor2 = bij_is_surj[OF bij_dtor2]
lemmas dtor2_nchotomy = surjD[OF surj_dtor2]
lemmas dtor2_diff = inj_eq[OF inj_dtor2]
lemmas dtor2_cases = exE[OF dtor2_nchotomy]

```

```

lemmas bij_ctor1 = o_bij[OF dtor1_o_ctor1 ctor1_o_dtor1]
lemmas inj_ctor1 = bij_is_inj[OF bij_ctor1]
lemmas surj_ctor1 = bij_is_surj[OF bij_ctor1]
lemmas ctor1_nchotomy = surjD[OF surj_ctor1]
lemmas ctor1_diff = inj_eq[OF inj_ctor1]
lemmas ctor1_cases = exE[OF ctor1_nchotomy]
lemmas bij_ctor2 = o_bij[OF dtor2_o_ctor2 ctor2_o_dtor2]
lemmas inj_ctor2 = bij_is_inj[OF bij_ctor2]
lemmas surj_ctor2 = bij_is_surj[OF bij_ctor2]
lemmas ctor2_nchotomy = surjD[OF surj_ctor2]
lemmas ctor2_diff = inj_eq[OF inj_ctor2]
lemmas ctor2_cases = exE[OF ctor2_nchotomy]

```

Primitive recursion

**definition** *rec1* **where**

*rec1* *s1 s2* = *snd* o *fold1* (<*ctor1* o *F1map id fst fst, s1*>) (<*ctor2* o *F2map id fst fst, s2*>)

**definition** *rec2* **where**

*rec2* *s1 s2* = *snd* o *fold2* (<*ctor1* o *F1map id fst fst, s1*>) (<*ctor2* o *F2map id fst fst, s2*>)

```

lemma fold1_o_ctor1: fold1 s1 s2 o ctor1 = s1 o F1map id (fold1 s1 s2) (fold2 s1 s2)
  by (tactic <rtac @ {context} (BNF_Tactics.mk_pointfree2 @ {context} @ {thm fold1}) 1>)
lemma fold2_o_ctor2: fold2 s1 s2 o ctor2 = s2 o F2map id (fold1 s1 s2) (fold2 s1 s2)
  by (tactic <rtac @ {context} (BNF_Tactics.mk_pointfree2 @ {context} @ {thm fold2}) 1>)

```

**lemmas** *fst\_rec1\_pair* =

```

trans[OF conjunct1[OF fold_unique[OF
  trans[OF o_assoc[symmetric] trans[OF arg_cong2[of _ _ _ _ (o), OF refl
    trans[OF fold1_o_ctor1 convol_o]]], OF trans[OF fst_convol]]]
  trans[OF o_assoc[symmetric] trans[OF arg_cong2[of _ _ _ _ (o), OF refl
    trans[OF fold2_o_ctor2 convol_o]]], OF trans[OF fst_convol]]]]]
fold1_ctor, unfolded F1.map_comp0[of id, unfolded id_o] F2.map_comp0[of id, unfolded id_o] o_assoc,

```

*OF refl refl*  
**lemmas** *fst\_rec2\_pair* =  
*trans*[*OF conjunct2*[*OF fold\_unique*[*OF*  
*trans*[*OF o\_assoc*[*symmetric*] *trans*[*OF arg\_cong2*[*of \_ \_ \_ \_ (o)*, *OF refl*  
*trans*[*OF fold1\_o\_ctor1 convol\_o*]]], *OF trans*[*OF fst\_convol*]]  
*trans*[*OF o\_assoc*[*symmetric*] *trans*[*OF arg\_cong2*[*of \_ \_ \_ \_ (o)*, *OF refl*  
*trans*[*OF fold2\_o\_ctor2 convol\_o*]]], *OF trans*[*OF fst\_convol*]]]]  
*fold2\_ctor*, *unfolded F1.map\_comp0*[*of id*, *unfolded id\_o*] *F2.map\_comp0*[*of id*, *unfolded id\_o*] *o\_assoc*,  
*OF refl refl*]

**theorem** *rec1*: *rec1 s1 s2 (ctor1 x) = s1 (F1map id (<id, rec1 s1 s2>) (<id, rec2 s1 s2>) x)*

**unfolding** *rec1\_def rec2\_def o\_apply fold1 snd\_convol'*  
*convol\_expand\_snd*[*OF fst\_rec1\_pair*] *convol\_expand\_snd*[*OF fst\_rec2\_pair*] ..

**theorem** *rec2*: *rec2 s1 s2 (ctor2 x) = s2 (F2map id (<id, rec1 s1 s2>) (<id, rec2 s1 s2>) x)*

**unfolding** *rec1\_def rec2\_def o\_apply fold2 snd\_convol'*  
*convol\_expand\_snd*[*OF fst\_rec1\_pair*] *convol\_expand\_snd*[*OF fst\_rec2\_pair*] ..

**lemma** *rec\_unique*:

*f o ctor1 = s1 o F1map id <id, f> <id, g>  $\implies$*

*g o ctor2 = s2 o F2map id <id, f> <id, g>  $\implies f = rec1 s1 s2 \wedge g = rec2 s1 s2$*

**unfolding** *rec1\_def rec2\_def convol\_expand\_snd'*[*OF fst\_rec1\_pair*] *convol\_expand\_snd'*[*OF fst\_rec2\_pair*]

**apply** (*rule fold\_unique*)

**apply** (*unfold convol\_o id\_o o\_id F1.map\_comp0*[*symmetric*] *F2.map\_comp0*[*symmetric*]

*F1.map\_id0 F2.map\_id0 o\_assoc*[*symmetric*] *fst\_convol*)

**apply** (*erule arg\_cong2*[*of \_ \_ \_ \_ BNF\_Def.convol, OF refl*])

**apply** (*erule arg\_cong2*[*of \_ \_ \_ \_ BNF\_Def.convol, OF refl*])

**done**

Induction

**theorem** *ctor\_induct*:

$\llbracket \bigwedge x. (\bigwedge a. a \in F1set2\ x \implies \phi1\ a) \implies (\bigwedge a. a \in F1set3\ x \implies \phi2\ a) \implies \phi1\ (ctor1\ x);$   
 $\bigwedge x. (\bigwedge a. a \in F2set2\ x \implies \phi1\ a) \implies (\bigwedge a. a \in F2set3\ x \implies \phi2\ a) \implies \phi2\ (ctor2\ x) \rrbracket \implies$   
 $\phi1\ a \wedge \phi2\ b$

**apply** (*rule mp*)

**apply** (*rule impI*)

**apply** (*erule conjE*)

**apply** (*rule conjI*)

**apply** (*rule iffD1*[*OF arg\_cong*[*OF Rep\_IF1\_inverse*]])

**apply** (*erule bspec*[*OF \_ Rep\_IF1*])

**apply** (*rule iffD1*[*OF arg\_cong*[*OF Rep\_IF2\_inverse*]])

**apply** (*erule bspec*[*OF \_ Rep\_IF2*])

**apply** (*rule init\_induct*)

**apply** (*rule conjI*)

**apply** (*drule asm\_rl*)

**apply** (*erule thin\_rl*)

**apply** (*rule ballI*)

**apply** (*rule impI*)

**apply** (*rule iffD2*[*OF arg\_cong*[*OF morE1*[*OF mor\_Abs\_IF*]]])

**apply** *assumption*

**apply** (*erule CollectE conjE*)**+**

**apply** (*drule meta\_spec*)

**apply** (*drule meta\_mp*)

**apply** (*rule iffD1*[*OF arg\_cong*[*OF Rep\_IF1\_inverse*]])

**apply** (*erule bspec*)

**apply** (*drule rev\_subsetD*)

**apply** (*rule equalityD1*)

**apply** (*rule F1.set\_map*(2))

**apply** (*erule imageE*)

**apply** (*tactic <hyp\_subst\_tac @*{*context*} *1*>)

```

apply (rule ssubst_mem[OF Abs_IF1_inverse])
apply (erule subsetD)
apply assumption
apply assumption

```

```

apply (drule meta_mp)
apply (rule iffD1[OF arg_cong[OF Rep_IF2_inverse]])
apply (erule bspec)
apply (drule rev_subsetD)
apply (rule equalityD1)
apply (rule F1.set_map(3))
apply (erule imageE)
apply (tactic ⟨hyp_subst_tac @{context} 1⟩)
apply (rule ssubst_mem[OF Abs_IF2_inverse])
apply (erule subsetD)
apply assumption
apply assumption

```

```

apply assumption

```

```

apply (erule thin_rl)
apply (drule asm_rl)
apply (rule ballI)
apply (rule impI)
apply (rule iffD2[OF arg_cong[OF morE2[OF mor_Abs_IF]])]
apply assumption
apply (erule CollectE conjE)+
apply (drule meta_spec)
apply (drule meta_mp)
apply (rule iffD1[OF arg_cong[OF Rep_IF1_inverse]])
apply (erule bspec)
apply (drule rev_subsetD)
apply (rule equalityD1)
apply (rule F2.set_map(2))
apply (erule imageE)
apply (tactic ⟨hyp_subst_tac @{context} 1⟩)
apply (rule ssubst_mem[OF Abs_IF1_inverse])
apply (erule subsetD)
apply assumption
apply assumption

```

```

apply (drule meta_mp)
apply (rule iffD1[OF arg_cong[OF Rep_IF2_inverse]])
apply (erule bspec)
apply (drule rev_subsetD)
apply (rule equalityD1)
apply (rule F2.set_map(3))
apply (erule imageE)
apply (tactic ⟨hyp_subst_tac @{context} 1⟩)
apply (rule ssubst_mem[OF Abs_IF2_inverse])
apply (erule subsetD)
apply assumption
apply assumption

```

```

apply assumption
done

```

**theorem** *ctor\_induct2*:

$$\begin{aligned}
& \llbracket \bigwedge x y. (\bigwedge a b. a \in F1set2\ x \implies b \in F1set2\ y \implies \mathit{phi1}\ a\ b) \implies \\
& \quad (\bigwedge a b. a \in F1set3\ x \implies b \in F1set3\ y \implies \mathit{phi2}\ a\ b) \implies \mathit{phi1}\ (\mathit{ctor1}\ x)\ (\mathit{ctor1}\ y); \\
& \bigwedge x y. (\bigwedge a b. a \in F2set2\ x \implies b \in F2set2\ y \implies \mathit{phi1}\ a\ b) \implies \\
& \quad (\bigwedge a b. a \in F2set3\ x \implies b \in F2set3\ y \implies \mathit{phi2}\ a\ b) \implies \mathit{phi2}\ (\mathit{ctor2}\ x)\ (\mathit{ctor2}\ y) \rrbracket \implies \\
& \mathit{phi1}\ a1\ b1 \wedge \mathit{phi2}\ a2\ b2
\end{aligned}$$

```

apply (rule rev_mp)
apply (rule ctor_induct[of %a1. (∀ x. phi1 a1 x) %a2. (∀ y. phi2 a2 y) a1 a2])
apply (rule allI[OF conjunct1[OF ctor_induct[OF asm_rl TrueI]])
apply (drule meta_spec2)
apply (erule thin_rl)
apply (tactic ⟨(dtac @ {context} @ {thm meta_mp} THEN_ALL_NEW Goal.norm_hhf_tac @ {context}) 1⟩)
apply (drule meta_spec)+
apply (erule meta_mp[OF spec])
apply assumption
apply (drule meta_mp)
apply (drule meta_spec)+
apply (erule meta_mp[OF spec])
apply assumption
apply assumption

apply (rule allI[OF conjunct2[OF ctor_induct[OF TrueI asm_rl]])
apply (erule thin_rl)
apply (drule meta_spec2)
apply (drule meta_mp)
apply (drule meta_spec)+
apply (erule meta_mp[OF spec])
apply assumption
apply (erule meta_mp)
apply (drule meta_spec)+
apply (erule meta_mp[OF spec])
apply assumption

apply (rule impI)
apply (erule conjE allE)+
apply (rule conjI)
apply assumption
apply assumption
done

```

## 1.8 The Result as an BNF

The map operator

**abbreviation** *IF1map* **where**  $IF1map\ f \equiv fold1\ (ctor1\ o\ (F1map\ f\ id\ id))\ (ctor2\ o\ (F2map\ f\ id\ id))$

**abbreviation** *IF2map* **where**  $IF2map\ f \equiv fold2\ (ctor1\ o\ (F1map\ f\ id\ id))\ (ctor2\ o\ (F2map\ f\ id\ id))$

**theorem** *IF1map*:

$(IF1map\ f)\ o\ ctor1 = ctor1\ o\ (F1map\ f\ (IF1map\ f)\ (IF2map\ f))$

```

apply (rule ext)
apply (rule trans[OF o_apply])
apply (rule trans[OF fold1])
apply (rule trans[OF o_apply])
apply (rule trans[OF arg_cong[OF F1map_comp_id]])
apply (rule trans[OF arg_cong[OF F1.map_cong0]])
apply (rule refl)
apply (rule trans[OF o_apply])
apply (rule id_apply)
apply (rule trans[OF o_apply])
apply (rule id_apply)
apply (rule sym[OF o_apply])
done

```

**theorem** *IF2map*:

$(IF2map\ f)\ o\ ctor2 = ctor2\ o\ (F2map\ f\ (IF1map\ f)\ (IF2map\ f))$

```

apply (rule ext)
apply (rule trans[OF o_apply])
apply (rule trans[OF fold2])
apply (rule trans[OF o_apply])
apply (rule trans[OF arg_cong[OF F2map_comp_id]])

```

```

apply (rule trans[OF arg_cong[OF F2.map_cong0]])
  apply (rule refl)
  apply (rule trans[OF o_apply])
  apply (rule id_apply)
  apply (rule trans[OF o_apply])
  apply (rule id_apply)
apply (rule sym[OF o_apply])
done

```

```

lemmas IF1map_simps = o_eq_dest[OF IF1map]
lemmas IF2map_simps = o_eq_dest[OF IF2map]

```

**lemma** IFmap\_unique:

```

[[u o ctor1 = ctor1 o F1map f u v; v o ctor2 = ctor2 o F2map f u v]] ==>
  u = IF1map f ^& v = IF2map f
apply (rule fold_unique)
unfolding o_assoc[symmetric] F1.map_comp0[symmetric] F2.map_comp0[symmetric] id_o o_id
apply assumption
apply assumption
done

```

**theorem** IF1map\_id: IF1map id = id

```

apply (rule sym)
apply (rule conjunct1[OF IFmap_unique])
apply (rule trans[OF id_o])
apply (rule trans[OF sym[OF o_id]])
apply (rule arg_cong[OF sym[OF F1.map_id0]])
apply (rule trans[OF id_o])
apply (rule trans[OF sym[OF o_id]])
apply (rule arg_cong[OF sym[OF F2.map_id0]])
done

```

**theorem** IF2map\_id: IF2map id = id

```

apply (rule sym)
apply (rule conjunct2[OF IFmap_unique])
apply (rule trans[OF id_o])
apply (rule trans[OF sym[OF o_id]])
apply (rule arg_cong[OF sym[OF F1.map_id0]])
apply (rule trans[OF id_o])
apply (rule trans[OF sym[OF o_id]])
apply (rule arg_cong[OF sym[OF F2.map_id0]])
done

```

**theorem** IF1map\_comp: IF1map (g o f) = IF1map g o IF1map f

```

apply (rule sym)
apply (rule conjunct1[OF IFmap_unique])
apply (rule ext)
apply (rule trans[OF o_apply])
apply (rule trans[OF o_apply])
apply (rule trans[OF arg_cong[OF IF1map_simps]])
apply (rule trans[OF IF1map_simps])
apply (rule trans[OF arg_cong[OF F1.map_comp]])
apply (rule sym[OF o_apply])
apply (rule ext)
apply (rule trans[OF o_apply])
apply (rule trans[OF o_apply])
apply (rule trans[OF arg_cong[OF IF2map_simps]])
apply (rule trans[OF IF2map_simps])
apply (rule trans[OF arg_cong[OF F2.map_comp]])
apply (rule sym[OF o_apply])
done

```

**theorem** IF2map\_comp: IF2map (g o f) = IF2map g o IF2map f

```

apply (rule sym)
apply (tactic ‹rtac @_{context} (Thm.permute_premis 0 1 @_{thm conjunct2[OF IFmap_unique]}) 1›)
apply (rule ext)
apply (rule trans[OF o_apply])
apply (rule trans[OF o_apply])
apply (rule trans[OF arg_cong[OF IF2map_simps]])
apply (rule trans[OF IF2map_simps])
apply (rule trans[OF arg_cong[OF F2.map_comp]])
apply (rule sym[OF o_apply])
apply (rule ext)
apply (rule trans[OF o_apply])
apply (rule trans[OF o_apply])
apply (rule trans[OF arg_cong[OF IF1map_simps]])
apply (rule trans[OF IF1map_simps])
apply (rule trans[OF arg_cong[OF F1.map_comp]])
apply (rule sym[OF o_apply])
done

```

The bound

**abbreviation** *IFbd* **where**  $IFbd \equiv bd\_F1 + c \ bd\_F2$

**theorem** *IFbd\_card\_order*: *card\_order IFbd*

```

apply (rule card_order_csum)
apply (rule F1.bd_card_order)
apply (rule F2.bd_card_order)
done

```

**lemma** *IFbd\_Cinfinite*: *Cinfinite IFbd*

```

apply (rule Cinfinite_csum1)
apply (rule F1.bd_Cinfinite)
done

```

**lemma** *IFbd\_regularCard*: *regularCard IFbd*

```

apply (rule regularCard_csum)
apply (rule F1.bd_Cinfinite)
apply (rule F2.bd_Cinfinite)
apply (rule F1.bd_regularCard)
apply (rule F2.bd_regularCard)
done

```

**lemmas** *IFbd\_cinfinite = conjunct1 [OF IFbd\_Cinfinite]*

The set operator

**abbreviation** *IF1col* **where**  $IF1col \equiv (\lambda X. F1set1\ X \cup (\bigcup (F1set2\ X) \cup \bigcup (F1set3\ X)))$

**abbreviation** *IF2col* **where**  $IF2col \equiv (\lambda X. F2set1\ X \cup (\bigcup (F2set2\ X) \cup \bigcup (F2set3\ X)))$

**abbreviation** *IF1set* **where**  $IF1set \equiv fold1\ IF1col\ IF2col$

**abbreviation** *IF2set* **where**  $IF2set \equiv fold2\ IF1col\ IF2col$

**abbreviation** *IF1in* **where**  $IF1in\ A \equiv \{x. IF1set\ x \subseteq A\}$

**abbreviation** *IF2in* **where**  $IF2in\ A \equiv \{x. IF2set\ x \subseteq A\}$

**lemma** *IF1set*:  $IF1set\ o\ ctor1 = IF1col\ o\ (F1map\ id\ IF1set\ IF2set)$

```

apply (rule ext)
apply (rule trans[OF o_apply])
apply (rule trans[OF fold1])
apply (rule sym[OF o_apply])
done

```

**lemma** *IF2set*:  $IF2set\ o\ ctor2 = IF2col\ o\ (F2map\ id\ IF1set\ IF2set)$

```

apply (rule ext)
apply (rule trans[OF o_apply])
apply (rule trans[OF fold2])

```

```

apply (rule sym[OF o_apply])
done

```

**theorem** *IF1set\_simps*:

```

 $IF1set (ctor1 x) = F1set1 x \cup ((\bigcup a \in F1set2 x. IF1set a) \cup (\bigcup a \in F1set3 x. IF2set a))$ 
apply (rule trans[OF o_eq_dest[OF IF1set]])
apply (rule arg_cong2[of _ _ _ _ (U)])
apply (rule trans[OF F1.set_map(1) trans[OF fun_cong[OF image_id] id_apply]])
apply (rule arg_cong2[of _ _ _ _ (U)])
apply (rule arg_cong[OF F1.set_map(2)])
apply (rule arg_cong[OF F1.set_map(3)])
done

```

**theorem** *IF2set\_simps*:

```

 $IF2set (ctor2 x) = F2set1 x \cup ((\bigcup a \in F2set2 x. IF1set a) \cup (\bigcup a \in F2set3 x. IF2set a))$ 
apply (rule trans[OF o_eq_dest[OF IF2set]])
apply (rule arg_cong2[of _ _ _ _ (U)])
apply (rule trans[OF F2.set_map(1) trans[OF fun_cong[OF image_id] id_apply]])
apply (rule arg_cong2[of _ _ _ _ (U)])
apply (rule arg_cong[OF F2.set_map(2)])
apply (rule arg_cong[OF F2.set_map(3)])
done

```

**lemmas** *F1set1\_IF1set = xt1(3)[OF IF1set\_simps Un\_upper1]*

**lemmas** *F1set2\_IF1set = subset\_trans[OF UN\_upper subset\_trans[OF Un\_upper1 xt1(3)[OF IF1set\_simps Un\_upper2]]]*

**lemmas** *F1set3\_IF1set = subset\_trans[OF UN\_upper subset\_trans[OF Un\_upper2 xt1(3)[OF IF1set\_simps Un\_upper2]]]*

**lemmas** *F2set1\_IF2set = xt1(3)[OF IF2set\_simps Un\_upper1]*

**lemmas** *F2set2\_IF2set = subset\_trans[OF UN\_upper subset\_trans[OF Un\_upper1 xt1(3)[OF IF2set\_simps Un\_upper2]]]*

**lemmas** *F2set3\_IF2set = subset\_trans[OF UN\_upper subset\_trans[OF Un\_upper2 xt1(3)[OF IF2set\_simps Un\_upper2]]]*

The BNF conditions for IF

**lemma** *IFset\_natural*:

$f ' (IF1set x) = IF1set (IF1map f x) \wedge f ' (IF2set y) = IF2set (IF2map f y)$

```

apply (rule ctor_induct[of _ _ x y])

```

```

apply (rule trans)
apply (rule image_cong)
apply (rule IF1set_simps)
apply (rule refl)
apply (rule sym)
apply (rule trans[OF arg_cong[of _ _ IF1set, OF IF1map_simps] trans[OF IF1set_simps]])

```

```

apply (rule sym)
apply (rule trans)
apply (rule image_Un)
apply (rule arg_cong2[of _ _ _ _ (U)])
apply (rule sym)
apply (rule F1.set_map(1))

```

```

apply (rule trans)
apply (rule image_Un)
apply (rule arg_cong2[of _ _ _ _ (U)])
apply (rule trans)
apply (rule image_UN)
apply (rule trans)
apply (rule SUP_cong)
apply (rule refl)
apply (tactic <Goal.assume_rule_tac @{context} 1>)
apply (rule sym)
apply (rule trans)
apply (rule SUP_cong)
apply (rule F1.set_map(2))

```

```

  apply (rule refl)
  apply (rule UN_simps(10))

```

```

apply (rule trans)
  apply (rule image_UN)
  apply (rule trans)
  apply (rule SUP_cong)
  apply (rule refl)
  apply (tactic ‹Goal.assume_rule_tac @{context} 1›)
  apply (rule sym)
  apply (rule trans)
  apply (rule SUP_cong)
  apply (rule F1.set_map(3))
  apply (rule refl)
  apply (rule UN_simps(10))

```

```

apply (rule trans)
  apply (rule image_cong)
  apply (rule IF2set_simps)
  apply (rule refl)
  apply (rule sym)
  apply (rule trans[OF arg_cong[of _ _ IF2set, OF IF2map_simps] trans[OF IF2set_simps]])

```

```

apply (rule sym)
  apply (rule trans)
  apply (rule image_Un)
  apply (rule arg_cong2[of _ _ _ _ (U)])
  apply (rule sym)
  apply (rule F2.set_map(1))

```

```

apply (rule trans)
  apply (rule image_Un)
  apply (rule arg_cong2[of _ _ _ _ (U)])

```

```

apply (rule trans)
  apply (rule image_UN)
  apply (rule trans)
  apply (rule SUP_cong)
  apply (rule refl)
  apply (tactic ‹Goal.assume_rule_tac @{context} 1›)
  apply (rule sym)
  apply (rule trans)
  apply (rule SUP_cong)
  apply (rule F2.set_map(2))
  apply (rule refl)
  apply (rule UN_simps(10))

```

```

apply (rule trans)
  apply (rule image_UN)
  apply (rule trans)
  apply (rule SUP_cong)
  apply (rule refl)
  apply (tactic ‹Goal.assume_rule_tac @{context} 1›)
  apply (rule sym)
  apply (rule trans)
  apply (rule SUP_cong)
  apply (rule F2.set_map(3))
  apply (rule refl)
  apply (rule UN_simps(10))
done

```

**theorem** *IF1set\_natural*:  $IF1set \circ (IF1map \ f) = image \ f \circ IF1set$



```

apply (rule ext)
apply (rule trans)
  apply (rule o_apply)
apply (rule sym)
apply (rule trans)
  apply (rule o_apply)
apply (rule conjunct1)
apply (rule IFset_natural)
done

```

**theorem** *IF2set\_natural*:  $IF2set\ o\ (IF2map\ f) = image\ f\ o\ IF2set$

```

apply (rule ext)
apply (rule trans)
  apply (rule o_apply)
apply (rule sym)
apply (rule trans)
  apply (rule o_apply)
apply (rule conjunct2)
apply (rule IFset_natural)
done

```

**lemma** *IFmap\_cong*:

```

 $((\forall a \in IF1set\ x.\ f\ a = g\ a) \longrightarrow IF1map\ f\ x = IF1map\ g\ x) \wedge$ 
 $((\forall a \in IF2set\ y.\ f\ a = g\ a) \longrightarrow IF2map\ f\ y = IF2map\ g\ y)$ 
apply (rule ctor_induct[of _ _ x y])

```

```

apply (rule impI)
apply (rule trans)
  apply (rule IF1map_simps)
apply (rule trans)
  apply (rule arg_cong[OF F1.map_cong0])
    apply (erule bspec)
    apply (erule rev_subsetD)
    apply (rule F1set1_IF1set)
  apply (rule mp)
  apply (tactic <Goal.assume_rule_tac @ {context} 1>)
apply (rule ballI)
apply (erule bspec)
apply (erule rev_subsetD)
apply (erule F1set2_IF1set)
apply (rule mp)
  apply (tactic <Goal.assume_rule_tac @ {context} 1>)
apply (rule ballI)
apply (erule bspec)
apply (erule rev_subsetD)
apply (erule F1set3_IF1set)
apply (rule sym)
apply (rule IF1map_simps)

```

```

apply (rule impI)
apply (rule trans)
  apply (rule IF2map_simps)
apply (rule trans)
  apply (rule arg_cong[OF F2.map_cong0])
    apply (erule bspec)
    apply (erule rev_subsetD)
    apply (rule F2set1_IF2set)
  apply (rule mp)
  apply (tactic <Goal.assume_rule_tac @ {context} 1>)
apply (rule ballI)
apply (erule bspec)
apply (erule rev_subsetD)
apply (erule F2set2_IF2set)

```

```

apply (rule mp)
apply (tactic ⟨Goal.assume_rule_tac @ {context} 1⟩)
apply (rule ballI)
apply (erule bspec)
apply (erule rev_subsetD)
apply (erule F2set3_IF2set)
apply (rule sym)
apply (rule IF2map_simps)
done

```

**theorem IF1map\_cong:**  
 $(\bigwedge a. a \in IF1set\ x \implies f\ a = g\ a) \implies IF1map\ f\ x = IF1map\ g\ x$   
**apply** (rule mp)  
**apply** (rule conjunct1)  
**apply** (rule IFmap\_cong)  
**apply** (rule ballI)  
**apply** (tactic ⟨Goal.assume\_rule\_tac @ {context} 1⟩)  
**done**

**theorem IF2map\_cong:**  
 $(\bigwedge a. a \in IF2set\ x \implies f\ a = g\ a) \implies IF2map\ f\ x = IF2map\ g\ x$   
**apply** (rule mp)  
**apply** (rule conjunct2)  
**apply** (rule IFmap\_cong)  
**apply** (rule ballI)  
**apply** (tactic ⟨Goal.assume\_rule\_tac @ {context} 1⟩)  
**done**

**lemma IFset\_bd:**  
 $|IF1set\ (x :: 'a\ IF1)| < o\ IFbd \wedge |IF2set\ (y :: 'a\ IF2)| < o\ IFbd$   
**apply** (rule ctor\_induct[of \_ \_ x y])

```

apply (rule ordIso_ordLess_trans)
apply (rule card_of_ordIso_subst)
apply (rule IF1set_simps)
apply (rule Un_Cinfinite_bound_strict)
apply (rule F1set1_bd)
apply (rule Un_Cinfinite_bound_strict)
apply (rule regularCard_UNION_bound)
apply (rule IFbd_Cinfinite)
apply (rule IFbd_regularCard)
apply (rule F1set2_bd)
apply (tactic ⟨Goal.assume_rule_tac @ {context} 1⟩)
apply (rule regularCard_UNION_bound)
apply (rule IFbd_Cinfinite)
apply (rule IFbd_regularCard)
apply (rule F1set3_bd)
apply (tactic ⟨Goal.assume_rule_tac @ {context} 1⟩)
apply (rule IFbd_Cinfinite)
apply (rule IFbd_Cinfinite)

```

```

apply (rule ordIso_ordLess_trans)
apply (rule card_of_ordIso_subst)
apply (rule IF2set_simps)
apply (rule Un_Cinfinite_bound_strict)
apply (rule F2set1_bd)
apply (rule Un_Cinfinite_bound_strict)
apply (rule regularCard_UNION_bound)
apply (rule IFbd_Cinfinite)
apply (rule IFbd_regularCard)
apply (rule F2set2_bd)
apply (tactic ⟨Goal.assume_rule_tac @ {context} 1⟩)
apply (rule regularCard_UNION_bound)

```

```

    apply (rule IFbd_Cinfinite)
    apply (rule IFbd_regularCard)
    apply (rule F2set3_bd)
    apply (tactic ‹Goal.assume_rule_tac @{{context}} 1›)
    apply (rule IFbd_Cinfinite)
    apply (rule IFbd_Cinfinite)
done

```

**lemmas**  $IF1set\_bd = conjunct1[OF IFset\_bd]$

**lemmas**  $IF2set\_bd = conjunct2[OF IFset\_bd]$

**definition**  $IF1rel$  **where**

```

IF1rel R =
  (BNF_Def.Grp (IF1in (Collect (case_prod R))) (IF1map fst)) ^--1 OO
  (BNF_Def.Grp (IF1in (Collect (case_prod R))) (IF1map snd))

```

**definition**  $IF2rel$  **where**

```

IF2rel R =
  (BNF_Def.Grp (IF2in (Collect (case_prod R))) (IF2map fst)) ^--1 OO
  (BNF_Def.Grp (IF2in (Collect (case_prod R))) (IF2map snd))

```

**lemma**  $in\_IF1rel$ :

```

IF1rel R x y ‹ $\longleftrightarrow$ › (‹ $\exists z. z \in IF1in (Collect (case\_prod R)) \wedge IF1map\ fst\ z = x \wedge IF1map\ snd\ z = y$ ›)
unfolding IF1rel_def by (rule predicate2_eqD[OF OO_Grp_alt])

```

**lemma**  $in\_IF2rel$ :

```

IF2rel R x y ‹ $\longleftrightarrow$ › (‹ $\exists z. z \in IF2in (Collect (case\_prod R)) \wedge IF2map\ fst\ z = x \wedge IF2map\ snd\ z = y$ ›)
unfolding IF2rel_def by (rule predicate2_eqD[OF OO_Grp_alt])

```

**lemma**  $IF1rel\_F1rel$ :  $IF1rel\ R\ (ctor1\ a)\ (ctor1\ b) \longleftrightarrow F1rel\ R\ (IF1rel\ R)\ (IF2rel\ R)\ a\ b$

```

apply (rule iffI)
apply (tactic ‹dtac @{{context}} (@{{thm in_IF1rel[THEN iffD1]}) 1›)+
apply (erule exE conjE CollectE)+
apply (rule iffD2)
apply (rule F1.in_rel)
apply (rule exI)
apply (rule conjI)
apply (rule CollectI)
apply (rule conjI)
apply (rule ord_eq_le_trans)
apply (rule F1.set_map(1))
apply (rule ord_eq_le_trans)
apply (rule trans[OF fun_cong[OF image_id] id_apply])
apply (rule subset_trans)
apply (rule F1set1_IF1set)
apply (erule ord_eq_le_trans[OF arg_cong[OF ctor1_dtor1]])

```

```

apply (rule conjI)
apply (rule ord_eq_le_trans)
apply (rule F1.set_map(2))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule iffD2)
apply (rule in_IF1rel)
apply (rule exI)
apply (rule conjI)
apply (rule CollectI)
apply (erule subset_trans[OF F1set2_IF1set])
apply (erule ord_eq_le_trans[OF arg_cong[OF ctor1_dtor1]])
apply (rule conjI)
apply (rule refl)
apply (rule refl)

```

```

apply (rule ord_eq_le_trans)
  apply (rule F1.set_map(3))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule iffD2)
  apply (rule in_IF2rel)
apply (rule exI)
apply (rule conjI)
  apply (rule CollectI)
apply (rule subset_trans)
  apply (rule F1set3_IF1set)
  apply assumption
apply (erule ord_eq_le_trans[OF arg_cong[OF ctor1_dtor1]])
apply (rule conjI)
  apply (rule refl)
apply (rule refl)
apply (rule conjI)

```

```

apply (rule trans)
  apply (rule F1.map_comp)
apply (rule trans)
apply (rule F1.map_cong0)
  apply (rule fun_cong[OF o_id])
  apply (rule trans)
    apply (rule o_apply)
    apply (rule fst_conv)
apply (rule trans)
  apply (rule o_apply)
apply (rule fst_conv)
apply (rule iffD1[OF ctor1_diff])
apply (rule trans)
  apply (rule sym)
apply (rule IF1map_simps)
apply (erule trans[OF arg_cong[OF ctor1_dtor1]])

```

```

apply (rule trans)
apply (rule F1.map_comp)
apply (rule trans)
apply (rule F1.map_cong0)
  apply (rule fun_cong[OF o_id])
  apply (rule trans)
    apply (rule o_apply)
    apply (rule snd_conv)
apply (rule trans)
  apply (rule o_apply)
apply (rule snd_conv)
apply (rule iffD1[OF ctor1_diff])
apply (rule trans)
  apply (rule sym)
apply (rule IF1map_simps)
apply (erule trans[OF arg_cong[OF ctor1_dtor1]])

```

```

apply (tactic <dtac @{context} (@{thm F1.in_rel[THEN iffD1]}) 1>)
apply (erule exE conjE CollectE)+
apply (rule iffD2)
  apply (rule in_IF1rel)
apply (rule exI)
apply (rule conjI)
apply (rule CollectI)
apply (rule ord_eq_le_trans)

```

```

apply (rule IF1set_simps)
apply (rule Un_least)
apply (rule ord_eq_le_trans)
  apply (rule box_equals[OF _ refl])
    apply (rule F1.set_map(1))
    apply (rule trans[OF fun_cong[OF image_id] id_apply])
apply assumption
apply (rule Un_least)
apply (rule ord_eq_le_trans)
  apply (rule SUP_cong[OF _ refl])
    apply (rule F1.set_map(2))
apply (rule UN_least)
apply (drule rev_subsetD)
  apply (erule image_mono)
apply (erule imageE)
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_proxE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (tactic <dtac @{context} (@{thm in_IF1rel[THEN iffD1]}) 1>)
apply (drule someI_ex)
apply (erule conjE)+
apply (erule CollectD)

apply (rule ord_eq_le_trans)
apply (rule SUP_cong[OF _ refl])
apply (rule F1.set_map(3))
apply (rule UN_least)
apply (drule rev_subsetD)
  apply (erule image_mono)
apply (erule imageE)
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_proxE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (tactic <dtac @{context} (@{thm in_IF2rel[THEN iffD1]}) 1>)
apply (drule someI_ex)
apply (erule conjE)+
apply (erule CollectD)

apply (rule conjI)
apply (rule trans)
  apply (rule IF1map_simps)
apply (rule iffD2[OF ctor1_diff])
apply (rule trans)
  apply (rule F1.map_comp)
apply (rule trans)
apply (rule F1.map_cong0)
  apply (rule fun_cong[OF o_id])
  apply (rule trans[OF o_apply])
apply (drule rev_subsetD)
  apply assumption
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_proxE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (tactic <dtac @{context} (@{thm in_IF1rel[THEN iffD1]}) 1>)
apply (drule someI_ex)
apply (erule conjE)+
apply assumption
apply (rule trans[OF o_apply])
apply (drule rev_subsetD)
  apply assumption
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_proxE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst

```

```

apply (tactic <dtac @{context} (@{thm in_IF2rel[THEN iffD1]}) 1>)
apply (drule someI_ex)
apply (erule conjE)+
apply assumption
apply assumption

apply (rule trans)
apply (rule IF1map_simps)
apply (rule iffD2[OF ctor1_diff])
apply (rule trans)
apply (rule F1.map_comp)
apply (rule trans)
apply (rule F1.map_cong0)
apply (rule fun_cong[OF o_id])
apply (rule trans[OF o_apply])
apply (drule rev_subsetD)
apply assumption
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (tactic <dtac @{context} (@{thm in_IF1rel[THEN iffD1]}) 1>)
apply (drule someI_ex)
apply (erule conjE)+
apply assumption
apply (rule trans[OF o_apply])
apply (drule rev_subsetD)
apply assumption
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (tactic <dtac @{context} (@{thm in_IF2rel[THEN iffD1]}) 1>)
apply (drule someI_ex)
apply (erule conjE)+
apply assumption
apply assumption
done

```

**lemma** *IF2rel\_F2rel*:  $IF2rel\ R\ (ctor2\ a)\ (ctor2\ b) \longleftrightarrow F2rel\ R\ (IF1rel\ R)\ (IF2rel\ R)\ a\ b$

```

apply (rule iffI)
apply (tactic <dtac @{context} (@{thm in_IF2rel[THEN iffD1]}) 1>)+
apply (erule exE conjE CollectE)+
apply (rule iffD2)
apply (rule F2.in_rel)
apply (rule exI)
apply (rule conjI)
apply (rule CollectI)
apply (rule conjI)
apply (rule ord_eq_le_trans)
apply (rule F2.set_map(1))
apply (rule ord_eq_le_trans)
apply (rule trans[OF fun_cong[OF image_id] id_apply])
apply (rule subset_trans)
apply (rule F2set1_IF2set)
apply (erule ord_eq_le_trans[OF arg_cong[OF ctor2_dtor2]])

apply (rule conjI)
apply (rule ord_eq_le_trans)
apply (rule F2.set_map(2))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule iffD2)
apply (rule in_IF1rel)

```

```

apply (rule exI)
apply (rule conjI)
apply (rule CollectI)
apply (rule subset_trans)
  apply (rule F2set2_IF2set)
  apply assumption
apply (erule ord_eq_le_trans[OF arg_cong[OF ctor2_dtor2]])
apply (rule conjI)
apply (rule refl)
apply (rule refl)

```

```

apply (rule ord_eq_le_trans)
apply (rule F2.set_map(3))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule iffD2)
  apply (rule in_IF2rel)
apply (rule exI)
apply (rule conjI)
apply (rule CollectI)
apply (rule subset_trans)
  apply (rule F2set3_IF2set)
  apply assumption
apply (erule ord_eq_le_trans[OF arg_cong[OF ctor2_dtor2]])
apply (rule conjI)
  apply (rule refl)
apply (rule refl)
apply (rule conjI)

```

```

apply (rule trans)
apply (rule F2.map_comp)
apply (rule trans)
apply (rule F2.map_cong0)
  apply (rule fun_cong[OF o_id])
apply (rule trans)
  apply (rule o_apply)
apply (rule fst_conv)
apply (rule trans)
  apply (rule o_apply)
apply (rule fst_conv)
apply (rule iffD1[OF ctor2_diff])
apply (rule trans)
  apply (rule sym)
apply (rule IF2map_simps)
apply (erule trans[OF arg_cong[OF ctor2_dtor2]])

```

```

apply (rule trans)
apply (rule F2.map_comp)
apply (rule trans)
apply (rule F2.map_cong0)
  apply (rule fun_cong[OF o_id])
apply (rule trans)
  apply (rule o_apply)
apply (rule snd_conv)
apply (rule trans)
  apply (rule o_apply)
apply (rule snd_conv)
apply (rule iffD1[OF ctor2_diff])
apply (rule trans)
apply (rule sym)
apply (rule IF2map_simps)

```

```

apply (erule trans[OF arg_cong[OF ctor2_dtor2]])

apply (tactic <dtac @ {context} (@ {thm F2.in_rel[THEN iffD1]}) 1>)
apply (erule exE conjE CollectE)+
apply (rule iffD2)
  apply (rule in_IF2rel)
apply (rule exI)
apply (rule conjI)
  apply (rule CollectI)
  apply (rule ord_eq_le_trans)
  apply (rule IF2set_simps)
apply (rule Un_least)
  apply (rule ord_eq_le_trans)
  apply (rule trans)
  apply (rule trans)
  apply (rule arg_cong[OF dtor2_ctor2])
  apply (rule F2.set_map(1))
  apply (rule trans[OF fun_cong[OF image_id] id_apply])
apply assumption
apply (rule Un_least)
apply (rule ord_eq_le_trans)
  apply (rule trans[OF arg_cong[OF dtor2_ctor2]])
  apply (rule arg_cong[OF F2.set_map(2)])
apply (rule UN_least)
apply (drule rev_subsetD)
  apply (erule image_mono)
apply (erule imageE)
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_proxE iffD1[OF prod.inject, elim_format] conjE)+
apply (tactic <hyp_subst_tac @ {context} 1>)
apply (tactic <dtac @ {context} (@ {thm in_IF1rel[THEN iffD1]}) 1>)
apply (drule someI_ex)
apply (erule conjE)+
apply (erule CollectD)

apply (rule ord_eq_le_trans)
apply (rule trans[OF arg_cong[OF dtor2_ctor2]])
apply (rule arg_cong[OF F2.set_map(3)])
apply (rule UN_least)
apply (drule rev_subsetD)
apply (erule image_mono)
apply (erule imageE)
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_proxE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (tactic <dtac @ {context} (@ {thm in_IF2rel[THEN iffD1]}) 1>)
apply (drule someI_ex)
apply (erule exE conjE)+
apply (erule CollectD)

apply (rule conjI)
apply (rule trans)
  apply (rule arg_cong[OF dtor2_ctor2])
apply (rule trans)
  apply (rule IF2map_simps)
apply (rule iffD2)
apply (rule ctor2_diff)
apply (rule trans)
apply (rule F2.map_comp)
apply (rule trans)
apply (rule F2.map_cong0)
  apply (rule fun_cong[OF o_id])
  apply (rule trans[OF o_apply])

```



```

apply (drule rev_subsetD)
apply assumption
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (tactic <dtac @{context} (@{thm in_IF1rel[THEN iffD1]}) 1>)
apply (drule someI_ex)
apply (erule conjE)+
apply assumption
apply (rule trans[OF o_apply])
apply (drule rev_subsetD)
apply assumption
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (tactic <dtac @{context} (@{thm in_IF2rel[THEN iffD1]}) 1>)
apply (drule someI_ex)
apply (erule conjE)+
apply assumption
apply assumption

```

```

apply (rule trans)
apply (rule arg_cong[OF dtor2_ctor2])
apply (rule trans)
apply (rule IF2map_simps)
apply (rule iffD2)
apply (rule ctor2_diff)
apply (rule trans)
apply (rule F2.map_comp)
apply (rule trans)
apply (rule F2.map_cong0)
apply (rule fun_cong[OF o_id])
apply (rule trans[OF o_apply])
apply (drule rev_subsetD)
apply assumption
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (tactic <dtac @{context} (@{thm in_IF1rel[THEN iffD1]}) 1>)
apply (drule someI_ex)
apply (erule conjE)+
apply assumption
apply (rule trans[OF o_apply])
apply (drule rev_subsetD)
apply assumption
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (tactic <dtac @{context} (@{thm in_IF2rel[THEN iffD1]}) 1>)
apply (drule someI_ex)
apply (erule conjE)+
apply assumption
apply assumption
done

```

lemma Irel\_induct:

```

assumes IH1:  $\forall x y. F1rel P1 P2 P3 x y \longrightarrow P2 (ctor1 x) (ctor1 y)$ 
and IH2:  $\forall x y. F2rel P1 P2 P3 x y \longrightarrow P3 (ctor2 x) (ctor2 y)$ 
shows  $IF1rel P1 \leq P2 \wedge IF2rel P1 \leq P3$ 
unfolding le_fun_def le_bool_def all_simps(1,2)[symmetric]
apply (rule allI)+
apply (rule ctor_induct2)
apply (rule impI)

```

```

apply (drule iffD1[OF IF1rel_F1rel])
apply (rule mp[OF spec2[OF IH1]])
apply (erule F1.rel_mono_strong0)
  apply (rule ballI[OF ballI[OF imp_refl]])
  apply (drule asm_rl)
  apply (erule thin_rl)
apply (rule ballI[OF ballI])
apply assumption
apply (erule thin_rl)
apply (drule asm_rl)
apply (rule ballI[OF ballI])
apply assumption

```

```

apply (rule impI)
apply (drule iffD1[OF IF2rel_F2rel])
apply (rule mp[OF spec2[OF IH2]])
apply (erule F2.rel_mono_strong0)
  apply (rule ballI[OF ballI[OF imp_refl]])
  apply (drule asm_rl)
  apply (erule thin_rl)
apply (rule ballI[OF ballI])
apply assumption
apply (erule thin_rl)
apply (drule asm_rl)
apply (rule ballI[OF ballI])
apply assumption
done

```

**lemma** le\_IFrel\_Comp:

```

((IF1rel R OO IF1rel S) x1 y1  $\longrightarrow$  IF1rel (R OO S) x1 y1)  $\wedge$ 
((IF2rel R OO IF2rel S) x2 y2  $\longrightarrow$  IF2rel (R OO S) x2 y2)
apply (rule ctor_induct2[of _ _ x1 y1 x2 y2])
apply (rule impI)
apply (erule nchotomy_relcomppE[OF ctor1_nchotomy])
apply (drule iffD1[OF IF1rel_F1rel])
apply (drule iffD1[OF IF1rel_F1rel])
apply (rule iffD2[OF IF1rel_F1rel])
apply (rule F1.rel_mono_strong0)
  apply (rule iffD2[OF predicate2_eqD[OF F1.rel_compp]])
  apply (rule relcomppI)
  apply assumption
  apply assumption
apply (rule ballI impI)+
apply assumption
apply (rule ballI)+
apply assumption
apply (rule ballI)+
apply assumption

```

```

apply (rule impI)
apply (erule nchotomy_relcomppE[OF ctor2_nchotomy])
apply (drule iffD1[OF IF2rel_F2rel])
apply (drule iffD1[OF IF2rel_F2rel])
apply (rule iffD2[OF IF2rel_F2rel])
apply (rule F2.rel_mono_strong0)
  apply (rule iffD2[OF predicate2_eqD[OF F2.rel_compp]])
  apply (rule relcomppI)
  apply assumption
  apply assumption
apply (rule ballI impI)+
apply assumption
apply (rule ballI)+
apply assumption

```

```

apply (rule ballI)+
apply assumption
done

lemma le_IF1rel_Comp: IF1rel R1 OO IF1rel R2 ≤ IF1rel (R1 OO R2)
  by (rule predicate2I) (erule mp[OF conjunct1[OF le_IFrel_Comp]])

lemma le_IF2rel_Comp: IF2rel R1 OO IF2rel R2 ≤ IF2rel (R1 OO R2)
  by (rule predicate2I) (erule mp[OF conjunct2[OF le_IFrel_Comp]])

context includes lifting_syntax
begin

lemma fold_transfer:
  ((F1rel R S T ==>> S) ==>> (F2rel R S T ==>> T) ==>> IF1rel R ==>> S) fold1 fold1 ^
  ((F1rel R S T ==>> S) ==>> (F2rel R S T ==>> T) ==>> IF2rel R ==>> T) fold2 fold2
  unfolding rel_fun_def_butlast_all_conj_distrib[symmetric] imp_conjR[symmetric]
  unfolding rel_fun_iff_leq_vimage2p
  apply (rule allI impI)+
  apply (rule Irel_induct)
  apply (rule allI impI vimage2pI)+
  apply (unfold fold1 fold2) [1]
  apply (erule predicate2D_vimage2p)
  apply (rule rel_funD[OF rel_funD[OF rel_funD[OF rel_funD[OF F1.map_transfer]]]])
  apply (rule id_transfer)
  apply (rule vimage2p_rel_fun)
  apply (rule vimage2p_rel_fun)
  apply assumption

  apply (rule allI impI vimage2pI)+
  apply (unfold fold1 fold2) [1]
  apply (erule predicate2D_vimage2p)
  apply (rule rel_funD[OF rel_funD[OF rel_funD[OF rel_funD[OF F2.map_transfer]]]])
  apply (rule id_transfer)
  apply (rule vimage2p_rel_fun)
  apply (rule vimage2p_rel_fun)
  apply assumption
done

end

definition IF1wit x = ctor1 (wit2_F1 x (ctor2 wit_F2))
definition IF2wit = ctor2 wit_F2

lemma IF1wit: x ∈ IF1set (IF1wit y) ⇒ x = y
  unfolding IF1wit_def
  by (elim UnE F1.wit2[elim_format] F2.wit[elim_format] UN_E FalseE |
    rule refl | hypsubst | assumption | unfold IF1set_simps IF2set_simps)+

lemma IF2wit: x ∈ IF2set IF2wit ⇒ False
  unfolding IF2wit_def
  by (elim UnE F2.wit[elim_format] UN_E FalseE |
    rule refl | hypsubst | assumption | unfold IF2set_simps)+

ML <
  BNF_FP_Util.mk_xtor_co_iter_o_map_thms BNF_Util.Least_FP false 1 @ {thm fold_unique}
  @ {thms IF1map IF2map} (map (BNF_Tactics.mk_pointfree2 @ {context}) @ {thms fold1 fold2})
  @ {thms F1.map_comp0[symmetric] F2.map_comp0[symmetric]} @ {thms F1.map_cong0 F2.map_cong0}
  >

ML <
  BNF_FP_Util.mk_xtor_co_iter_o_map_thms BNF_Util.Least_FP true 1 @ {thm rec_unique}

```

```

@{thms IF1map IF2map} (map (BNF_Tactics.mk_pointfree2 @{context}) @{thms rec1 rec2})
@{thms F1.map_comp0[symmetric] F2.map_comp0[symmetric]} @{thms F1.map_cong0 F2.map_cong0}

```

```

bnf 'a IF1
  map: IF1map
  sets: IF1set
  bd: IFbd
  wits: IF1wit
  rel: IF1rel
  apply –
    apply (rule IF1map_id)
    apply (rule IF1map_comp)
    apply (erule IF1map_cong)
    apply (rule IF1set_natural)
    apply (rule IFbd_card_order)
    apply (rule IFbd_cinfinite)
    apply (rule IFbd_regularCard)
    apply (rule IF1set_bd)
    apply (rule le_IF1rel_Comp)
    apply (rule IF1rel_def[unfolded OO_Grp_alt mem_Collect_eq])
apply (erule IF1wit)
done

```

```

bnf 'a IF2
  map: IF2map
  sets: IF2set
  bd: IFbd
  wits: IF2wit
  rel: IF2rel
  apply –
    apply (rule IF2map_id)
    apply (rule IF2map_comp)
    apply (erule IF2map_cong)
    apply (rule IF2set_natural)
    apply (rule IFbd_card_order)
    apply (rule IFbd_cinfinite)
    apply (rule IFbd_regularCard)
    apply (rule IF2set_bd)
    apply (rule le_IF2rel_Comp)
    apply (rule IF2rel_def[unfolded OO_Grp_alt mem_Collect_eq])
apply (erule IF2wit)
done

```

## 2 Greatest Fixpoint (a.k.a. Codatatype)

**unbundle** *cardinal\_syntax*

```

'b1 = ('a, 'b1, 'b2) F1
'b2 = ('a, 'b1, 'b2) F2

```

To build a witness scenario, let us assume

```

('a, 'b1, 'b2) F1 = 'a * 'b1 + 'a * 'b2
('a, 'b1, 'b2) F2 = unit + 'b1 * 'b2

```

**ML** *<open Ctr\_Sugar\_Util>*

**declare** *[[bnf\_internals]]*

```

bnf-axiomatization (F1set1: 'a, F1set2: 'b1, F1set3: 'b2) F1
  [wits: 'a ⇒ 'b1 ⇒ ('a, 'b1, 'b2) F1 'a ⇒ 'b2 ⇒ ('a, 'b1, 'b2) F1]
  for map: F1map rel: F1rel

```

```

bnf-axiomatization (F2set1: 'a, F2set2: 'b1, F2set3: 'b2) F2

```

[wits: ('a, 'b1, 'b2) F2]  
 for map: F2map rel: F2rel

**lemma** *F1rel\_cong*:  $\llbracket R1 = S1; R2 = S2; R3 = S3 \rrbracket \implies F1rel R1 R2 R3 = F1rel S1 S2 S3$   
 by *hypsubst rule*

**lemma** *F2rel\_cong*:  $\llbracket R1 = S1; R2 = S2; R3 = S3 \rrbracket \implies F2rel R1 R2 R3 = F2rel S1 S2 S3$   
 by *hypsubst rule*

**abbreviation** *F1in* :: 'a1 set  $\Rightarrow$  'a2 set  $\Rightarrow$  'a3 set  $\Rightarrow$  (('a1, 'a2, 'a3) F1) set **where**  
*F1in* A1 A2 A3  $\equiv \{x. F1set1 x \subseteq A1 \wedge F1set2 x \subseteq A2 \wedge F1set3 x \subseteq A3\}$

**abbreviation** *F2in* :: 'a1 set  $\Rightarrow$  'a2 set  $\Rightarrow$  'a3 set  $\Rightarrow$  (('a1, 'a2, 'a3) F2) set **where**  
*F2in* A1 A2 A3  $\equiv \{x. F2set1 x \subseteq A1 \wedge F2set2 x \subseteq A2 \wedge F2set3 x \subseteq A3\}$

**lemma** *F1map\_comp\_id*:  $F1map g1 g2 g3 (F1map id f2 f3 x) = F1map g1 (g2 o f2) (g3 o f3) x$   
 apply (rule *trans*)  
 apply (rule *F1.map\_comp*)  
 unfolding *o\_id*  
 apply (rule *refl*)  
 done

**lemmas** *F1in\_mono23* = *F1.in\_mono*[*OF subset\_refl*]

**lemmas** *F1in\_mono23'* = *subsetD*[*OF F1in\_mono23*]

**lemma** *F1map\_congL*:  $\llbracket \forall a \in F1set2 x. f a = a; \forall a \in F1set3 x. g a = a \rrbracket \implies$   
 $F1map id f g x = x$   
 apply (rule *trans*)  
 apply (rule *F1.map\_cong0*)  
 apply (rule *refl*)  
 apply (rule *trans*)  
 apply (erule *bspec*)  
 apply *assumption*  
 apply (rule *sym*)  
 apply (rule *id\_apply*)  
 apply (rule *trans*)  
 apply (erule *bspec*)  
 apply *assumption*  
 apply (rule *sym*)  
 apply (rule *id\_apply*)  
 apply (rule *F1.map\_id*)  
 done

**lemma** *F2map\_comp\_id*:  $F2map g1 g2 g3 (F2map id f2 f3 x) = F2map g1 (g2 o f2) (g3 o f3) x$   
 apply (rule *trans*)  
 apply (rule *F2.map\_comp*)  
 unfolding *o\_id*  
 apply (rule *refl*)  
 done

**lemmas** *F2in\_mono23* = *F2.in\_mono*[*OF subset\_refl*]

**lemmas** *F2in\_mono23'* = *subsetD*[*OF F2in\_mono23*]

**lemma** *F2map\_congL*:  $\llbracket \forall a \in F2set2 x. f a = a; \forall a \in F2set3 x. g a = a \rrbracket \implies$   
 $F2map id f g x = x$   
 apply (rule *trans*)  
 apply (rule *F2.map\_cong0*)  
 apply (rule *refl*)  
 apply (rule *trans*)  
 apply (erule *bspec*)  
 apply *assumption*  
 apply (rule *sym*)  
 apply (rule *id\_apply*)  
 apply (rule *trans*)

```

  apply (erule bspec)
  apply assumption
  apply (rule sym)
  apply (rule id_apply)
  apply (rule F2.map_id)
done

```

## 2.1 Coalgebra

**definition** *coalg where*

```

coalg B1 B2 s1 s2 =
  (( $\forall a \in B1. s1 a \in F1in (UNIV :: 'a set) B1 B2$ )  $\wedge$  ( $\forall a \in B2. s2 a \in F2in (UNIV :: 'a set) B1 B2$ ))

```

**lemmas** *coalg\_F1in* = *bspec*[*OF conjunct1*[*OF iffD1*[*OF coalg\_def*]]]

**lemmas** *coalg\_F2in* = *bspec*[*OF conjunct2*[*OF iffD1*[*OF coalg\_def*]]]

**lemma** *coalg\_F1set2*:

```

[[coalg B1 B2 s1 s2; a  $\in$  B1]]  $\implies$  F1set2 (s1 a)  $\subseteq$  B1
  apply (tactic <dtac @{context} @{thm iffD1[OF coalg_def]} 1>)
  apply (erule conjE)
  apply (drule bspec[rotated])
  apply assumption
  apply (erule CollectE conjE)+
  apply assumption
done

```

**lemma** *coalg\_F1set3*:

```

[[coalg B1 B2 s1 s2; a  $\in$  B1]]  $\implies$  F1set3 (s1 a)  $\subseteq$  B2
  apply (tactic <dtac @{context} @{thm iffD1[OF coalg_def]} 1>)
  apply (erule conjE)
  apply (drule bspec[rotated])
  apply assumption
  apply (erule CollectE conjE)+
  apply assumption
done

```

**lemma** *coalg\_F2set2*:

```

[[coalg B1 B2 s1 s2; a  $\in$  B2]]  $\implies$  F2set2 (s2 a)  $\subseteq$  B1
  apply (tactic <dtac @{context} @{thm iffD1[OF coalg_def]} 1>)
  apply (erule conjE)
  apply (drule bspec[rotated])
  apply assumption
  apply (erule CollectE conjE)+
  apply assumption
done

```

**lemma** *coalg\_F2set3*:

```

[[coalg B1 B2 s1 s2; a  $\in$  B2]]  $\implies$  F2set3 (s2 a)  $\subseteq$  B2
  apply (tactic <dtac @{context} @{thm iffD1[OF coalg_def]} 1>)
  apply (erule conjE)
  apply (drule bspec[rotated])
  apply assumption
  apply (erule CollectE conjE)+
  apply assumption
done

```

## 2.2 Type-coalgebra

**abbreviation** *tcoalg s1 s2*  $\equiv$  *coalg UNIV UNIV s1 s2*

**lemma** *tcoalg*: *tcoalg s1 s2*

```

  apply (tactic <rtac @{context} (@{thm coalg_def} RS iffD2) 1>)
  apply (rule conjI)
  apply (rule ballI)

```

```

apply (rule CollectI)
apply (rule conjI)
apply (rule subset_UNIV)
apply (rule conjI)
apply (rule subset_UNIV)
apply (rule subset_UNIV)
apply (rule ballI)
apply (rule CollectI)
apply (rule conjI)
apply (rule subset_UNIV)
apply (rule conjI)
apply (rule subset_UNIV)
apply (rule subset_UNIV)
done

```

## 2.3 Morphism

**definition** *mor where*

```

mor B1 B2 s1 s2 B1' B2' s1' s2' f g =
  ((( $\forall a \in B1. f a \in B1'$ )  $\wedge$  ( $\forall a \in B2. g a \in B2'$ ))  $\wedge$ 
  (( $\forall z \in B1. F1map (id :: 'a \Rightarrow 'a) f g (s1 z) = s1' (f z)$ )  $\wedge$ 
  ( $\forall z \in B2. F2map (id :: 'a \Rightarrow 'a) f g (s2 z) = s2' (g z)$ )))

```

**lemma** *mor\_image1*: *mor* B1 B2 s1 s2 B1' B2' s1' s2' f g  $\implies$   $f' B1 \subseteq B1'$

```

apply (tactic <dtac @{\context} @{\thm iffD1[OF mor_def]} 1>)
apply (erule conjE)+
apply (rule image_subsetI)
apply (erule bspec)
apply assumption
done

```

**lemma** *mor\_image2*: *mor* B1 B2 s1 s2 B1' B2' s1' s2' f g  $\implies$   $g' B2 \subseteq B2'$

```

apply (tactic <dtac @{\context} @{\thm iffD1[OF mor_def]} 1>)
apply (erule conjE)+
apply (rule image_subsetI)
apply (erule bspec)
apply assumption
done

```

**lemmas** *mor\_image1'* = subsetD[OF *mor\_image1 imageI*]

**lemmas** *mor\_image2'* = subsetD[OF *mor\_image2 imageI*]

**lemma** *morE1*:  $\llbracket$ *mor* B1 B2 s1 s2 B1' B2' s1' s2' f g;  $z \in B1$  $\rrbracket$

```

 $\implies$  F1map id f g (s1 z) = s1' (f z)
apply (tactic <dtac @{\context} @{\thm iffD1[OF mor_def]} 1>)
apply (erule conjE)+
apply (erule bspec)
apply assumption
done

```

**lemma** *morE2*:  $\llbracket$ *mor* B1 B2 s1 s2 B1' B2' s1' s2' f g;  $z \in B2$  $\rrbracket$

```

 $\implies$  F2map id f g (s2 z) = s2' (g z)
apply (tactic <dtac @{\context} @{\thm iffD1[OF mor_def]} 1>)
apply (erule conjE)+
apply (erule bspec)
apply assumption
done

```

**lemma** *mor\_incl*:  $\llbracket B1 \subseteq B1'; B2 \subseteq B2' \rrbracket \implies$  *mor* B1 B2 s1 s2 B1' B2' s1 s2 id id

```

apply (tactic <rtac @{\context} (@{\thm mor_def} RS iffD2) 1>)
apply (rule conjI)
apply (rule conjI)
apply (rule ballI)
apply (rule ssubst_mem[OF id_apply])

```

**apply** (erule subsetD)  
**apply** assumption

**apply** (rule ballI)  
**apply** (rule ssubst\_mem[OF id\_apply])  
**apply** (erule subsetD)  
**apply** assumption

**apply** (rule conjI)  
**apply** (rule ballI)  
**apply** (rule trans[OF F1.map\_id])  
**apply** (rule sym)  
**apply** (rule arg\_cong[OF id\_apply])  
**apply** (rule ballI)  
**apply** (rule trans[OF F2.map\_id])  
**apply** (rule sym)  
**apply** (rule arg\_cong[OF id\_apply])  
**done**

**lemmas** mor\_id = mor\_incl[OF subset\_refl subset\_refl]

**lemma** mor\_comp:

$\llbracket \text{mor } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ f \ g; \\ \text{mor } B1' \ B2' \ s1' \ s2' \ B1'' \ B2'' \ s1'' \ s2'' \ f' \ g' \rrbracket \implies \\ \text{mor } B1 \ B2 \ s1 \ s2 \ B1'' \ B2'' \ s1'' \ s2'' \ (f' \circ f) \ (g' \circ g)$   
**apply** (tactic <rtac @ {context} (@ {thm mor\_def} RS iffD2) 1 >)  
**apply** (rule conjI)

**apply** (rule conjI)  
**apply** (rule ballI)  
**apply** (rule ssubst\_mem[OF o\_apply])  
**apply** (erule mor\_image1')  
**apply** (erule mor\_image1')  
**apply** assumption

**apply** (rule ballI)  
**apply** (rule ssubst\_mem[OF o\_apply])  
**apply** (erule mor\_image2')  
**apply** (erule mor\_image2')  
**apply** assumption

**apply** (rule conjI)  
**apply** (rule ballI)  
**apply** (tactic <stac @ {context} @ {thm o\_apply} 1 >)  
**apply** (rule trans)  
**apply** (rule sym[OF F1map\_comp\_id])  
**apply** (rule trans)  
**apply** (erule arg\_cong[OF morE1])  
**apply** assumption  
**apply** (erule morE1)  
**apply** (erule mor\_image1')  
**apply** assumption

**apply** (rule ballI)  
**apply** (tactic <stac @ {context} @ {thm o\_apply} 1 >)  
**apply** (rule trans)  
**apply** (rule sym[OF F2map\_comp\_id])  
**apply** (rule trans)  
**apply** (erule arg\_cong[OF morE2])  
**apply** assumption  
**apply** (erule morE2)  
**apply** (erule mor\_image2')  
**apply** assumption



done

**lemma** *mor\_cong*:  $\llbracket f' = f; g' = g; \text{mor } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ f \ g \rrbracket \implies$   
 $\text{mor } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ f' \ g'$   
**apply** (tactic <hyp\_subst\_tac @ {context} 1 >)  
**apply** assumption  
**done**

**lemma** *mor\_UNIV*:  $\text{mor } UNIV \ UNIV \ s1 \ s2 \ UNIV \ UNIV \ s1' \ s2' \ f1 \ f2 \longleftrightarrow$   
 $F1map \ id \ f1 \ f2 \ o \ s1 = s1' \ o \ f1 \wedge F2map \ id \ f1 \ f2 \ o \ s2 = s2' \ o \ f2$   
**apply** (rule iffI)  
**apply** (rule conjI)  
**apply** (rule ext)  
**apply** (rule trans)  
**apply** (rule trans)  
**apply** (rule o\_apply)  
**apply** (erule morE1)  
**apply** (rule UNIV\_I)  
**apply** (rule sym[OF o\_apply])  
**apply** (rule ext)  
**apply** (rule trans)  
**apply** (rule trans)  
**apply** (rule o\_apply)  
**apply** (erule morE2)  
**apply** (rule UNIV\_I)  
**apply** (rule sym[OF o\_apply])

**apply** (tactic <rtac @ {context} (@ {thm mor\_def} RS iffD2) 1 >)  
**apply** (rule conjI)  
**apply** (rule conjI)  
**apply** (rule ballI)  
**apply** (rule UNIV\_I)  
**apply** (rule ballI)  
**apply** (rule UNIV\_I)  
**apply** (rule conjI)  
**apply** (tactic <dtac @ {context} (BNF\_Util.mk\_conjunctN 2 1) 1 >)  
**apply** (rule ballI)  
**apply** (erule o\_eq\_dest)  
**apply** (tactic <dtac @ {context} (BNF\_Util.mk\_conjunctN 2 2) 1 >)  
**apply** (rule ballI)  
**apply** (erule o\_eq\_dest)  
**done**

**lemma** *mor\_str*:  
 $\text{mor } UNIV \ UNIV \ s1 \ s2 \ UNIV \ UNIV \ (F1map \ id \ s1 \ s2) \ (F2map \ id \ s1 \ s2) \ s1 \ s2$   
**apply** (rule iffD2)  
**apply** (rule mor\_UNIV)  
**apply** (rule conjI)  
**apply** (rule refl)  
**apply** (rule refl)  
**done**

**lemma** *mor\_case\_sum*:  
 $\text{mor } UNIV \ UNIV \ s1 \ s2 \ UNIV \ UNIV \ (\text{case\_sum } (F1map \ id \ Inl \ Inl \ o \ s1) \ s1') \ (\text{case\_sum } (F2map \ id \ Inl \ Inl \ o \ s2) \ s2') \ Inl \ Inl$   
**apply** (tactic <rtac @ {context} (@ {thm mor\_UNIV} RS iffD2) 1 >)  
**apply** (rule conjI)  
**apply** (rule sym)  
**apply** (rule case\_sum\_o\_inj(1))  
**apply** (rule sym)  
**apply** (rule case\_sum\_o\_inj(1))  
**done**

## 2.4 Bisimulations

**definition** *bis* where

$$\begin{aligned}
 & \text{bis } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ R1 \ R2 = \\
 & ((R1 \subseteq B1 \times B1' \wedge R2 \subseteq B2 \times B2') \wedge \\
 & ((\forall b1 \ b1'. (b1, b1') \in R1 \longrightarrow \\
 & (\exists z \in F1 \text{in UNIV } R1 \ R2. \\
 & \quad F1\text{map id fst fst } z = s1 \ b1 \wedge F1\text{map id snd snd } z = s1' \ b1')) \wedge \\
 & (\forall b2 \ b2'. (b2, b2') \in R2 \longrightarrow \\
 & (\exists z \in F2 \text{in UNIV } R1 \ R2. \\
 & \quad F2\text{map id fst fst } z = s2 \ b2 \wedge F2\text{map id snd snd } z = s2' \ b2'))))
 \end{aligned}$$

**lemma** *bis\_cong*:  $\llbracket \text{bis } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ R1 \ R2; R1' = R1; R2' = R2 \rrbracket \implies$

*bis*  $B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ R1' \ R2'$

**apply** (*tactic*  $\langle \text{hyp\_subst\_tac } @\{\text{context}\} \ 1 \rangle$ )

**apply** *assumption*

**done**

**lemma** *bis\_Frel*:

$$\begin{aligned}
 & \text{bis } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ R1 \ R2 \longleftrightarrow \\
 & (R1 \subseteq B1 \times B1' \wedge R2 \subseteq B2 \times B2') \wedge \\
 & ((\forall b1 \ b1'. (b1, b1') \in R1 \longrightarrow F1\text{rel } (=) \ (in\_rel \ R1) \ (in\_rel \ R2) \ (s1 \ b1) \ (s1' \ b1')) \wedge \\
 & (\forall b2 \ b2'. (b2, b2') \in R2 \longrightarrow F2\text{rel } (=) \ (in\_rel \ R1) \ (in\_rel \ R2) \ (s2 \ b2) \ (s2' \ b2')))
 \end{aligned}$$

**apply** (*rule trans*[*OF bis\_def*])

**apply** (*rule iffI*)

**apply** (*erule conjE*)

**apply** (*erule conjI*)

**apply** (*rule conjI*)

**apply** (*rule allI*)

**apply** (*rule allI*)

**apply** (*rule impI*)

**apply** (*tactic*  $\langle \text{dtac } @\{\text{context}\} \ (BNF\_Util.mk\_conjunctN \ 2 \ 1) \ 1 \rangle$ )

**apply** (*erule allE*)<sub>+</sub>

**apply** (*erule impE*)

**apply** *assumption*

**apply** (*erule bexE*)

**apply** (*erule conjE CollectE*)<sub>+</sub>

**apply** (*rule iffD2*[*OF F1.in\_rel*])

**apply** (*rule exI*)

**apply** (*rule conjI*[*rotated*])

**apply** (*rule conjI*)

**apply** (*rule trans*)

**apply** (*rule trans*)

**apply** (*rule F1.map\_comp*)

**apply** (*rule F1.map\_cong0*)

**apply** (*rule fst\_diag\_id*)

**apply** (*rule fun\_cong*[*OF o\_id*])

**apply** (*rule fun\_cong*[*OF o\_id*])

**apply** *assumption*

**apply** (*rule trans*)

**apply** (*rule trans*)

**apply** (*rule F1.map\_comp*)

**apply** (*rule F1.map\_cong0*)

**apply** (*rule snd\_diag\_id*)

**apply** (*rule fun\_cong*[*OF o\_id*])

**apply** (*rule fun\_cong*[*OF o\_id*])

**apply** *assumption*

**apply** (*rule CollectI*)

**apply** (*rule conjI*)

**apply** (*rule ord\_eq\_le\_trans*)

**apply** (*rule F1.set\_map*(1))

```

apply (rule subset_trans)
apply (erule image_mono)
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule refl)
apply (rule conjI)
apply (rule ord_eq_le_trans)
  apply (rule trans)
    apply (rule F1.set_map(2))
    apply (rule trans[OF fun_cong[OF image_id] id_apply])
  apply (erule Collect_case_prod_in_rel_leI)
apply (rule ord_eq_le_trans)
apply (rule trans)
  apply (rule F1.set_map(3))
apply (rule trans[OF fun_cong[OF image_id] id_apply])
apply (erule Collect_case_prod_in_rel_leI)

apply (rule allI)
apply (rule allI)
apply (rule impI)
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule allE)+
apply (erule impE)
apply assumption
apply (erule bexE)
apply (erule conjE CollectE)+
apply (rule iffD2[OF F2.in_rel])
apply (rule exI)
apply (rule conjI[rotated])
apply (rule conjI)
  apply (rule trans)
  apply (rule trans)
    apply (rule F2.map_comp)
  apply (rule F2.map_cong0)
    apply (rule fst_diag_id)
    apply (rule fun_cong[OF o_id])
  apply (rule fun_cong[OF o_id])
apply assumption

apply (rule trans)
apply (rule trans)
  apply (rule F2.map_comp)
apply (rule F2.map_cong0)
  apply (rule snd_diag_id)
apply (rule fun_cong[OF o_id])
apply (rule fun_cong[OF o_id])
apply assumption

apply (rule CollectI)
apply (rule conjI)
apply (rule ord_eq_le_trans)
  apply (rule F2.set_map(1))
apply (rule subset_trans)
  apply (erule image_mono)
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule refl)
apply (rule conjI)
apply (rule ord_eq_le_trans)
  apply (rule trans)
  apply (rule F2.set_map(2))

```

```

    apply (rule trans[OF fun_cong[OF image_id] id_apply])
    apply (erule Collect_case_prod_in_rel_leI)
  apply (rule ord_eq_le_trans)
  apply (rule trans)
    apply (rule F2.set_map(3))
    apply (rule trans[OF fun_cong[OF image_id] id_apply])
    apply (erule Collect_case_prod_in_rel_leI)

  apply (erule conjE)
  apply (erule conjI)

  apply (rule conjI)
  apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
  apply (rule allI)
  apply (rule allI)
  apply (rule impI)
  apply (erule allE)
  apply (erule allE)
  apply (erule impE)
  apply assumption
  apply (drule iffD1[OF F1.in_rel])
  apply (erule exE conjE CollectE Collect_case_prod_in_rel_leE)+

  apply (rule bexI)
  apply (rule conjI)
  apply (rule trans)
    apply (rule F1.map_comp)
    apply (tactic <stac @{context} @{thm id_o} 1>)
    apply (tactic <stac @{context} @{thm o_id} 1>)
    apply (tactic <stac @{context} @{thm o_id} 1>)
    apply assumption

  apply (rule trans)
  apply (rule F1.map_comp)
  apply (tactic <stac @{context} @{thm id_o} 1>)
  apply (tactic <stac @{context} @{thm o_id} 1>)
  apply (tactic <stac @{context} @{thm o_id} 1>)
  apply (rule trans)
  apply (rule F1.map_cong0)
  apply (rule Collect_case_prodD)
  apply (erule subsetD)
  apply assumption
  apply (rule refl)
  apply (rule refl)
  apply assumption

  apply (rule CollectI)
  apply (rule conjI)
  apply (rule subset_UNIV)

  apply (rule conjI)
  apply (rule ord_eq_le_trans)
  apply (rule trans)
    apply (rule F1.set_map(2))
    apply (rule trans[OF fun_cong[OF image_id] id_apply])
    apply assumption

  apply (rule ord_eq_le_trans)
  apply (rule trans)
    apply (rule F1.set_map(3))
    apply (rule trans[OF fun_cong[OF image_id] id_apply])
    apply assumption

```

```

apply (tactic ‹dtac @{{context}} (BNF_Util.mk_conjunctN 2 2) 1›)
apply (rule allI)
apply (rule allI)
apply (rule impI)
apply (erule allE)
apply (erule allE)
apply (erule impE)
apply assumption
apply (drule iffD1[OF F2.in_rel])
apply (erule exE conjE CollectE Collect_case_prod_in_rel_leE)+

```

```

apply (rule bezI)
apply (rule conjI)
apply (rule trans)
apply (rule F2.map_comp)
apply (tactic ‹stac @{{context}} @{{thm id_o}} 1›)
apply (tactic ‹stac @{{context}} @{{thm o_id}} 1›)
apply (tactic ‹stac @{{context}} @{{thm o_id}} 1›)
apply assumption

```

```

apply (rule trans)
apply (rule F2.map_comp)
apply (tactic ‹stac @{{context}} @{{thm id_o}} 1›)
apply (tactic ‹stac @{{context}} @{{thm o_id}} 1›)
apply (tactic ‹stac @{{context}} @{{thm o_id}} 1›)
apply (rule trans)
apply (rule F2.map_cong0)
apply (rule Collect_case_prodD)
apply (erule subsetD)
apply assumption
apply (rule refl)
apply (rule refl)
apply assumption

```

```

apply (rule CollectI)
apply (rule conjI)
apply (rule subset_UNIV)

```

```

apply (rule conjI)
apply (rule ord_eq_le_trans)
apply (rule trans)
apply (rule F2.set_map(2))
apply (rule trans[OF fun_cong[OF image_id] id_apply])
apply assumption

```

```

apply (rule ord_eq_le_trans)
apply (rule trans)
apply (rule F2.set_map(3))
apply (rule trans[OF fun_cong[OF image_id] id_apply])
apply assumption
done

```

**lemma** *bis\_converse*:

```

bis B1 B2 s1 s2 B1' B2' s1' s2' R1 R2  $\implies$ 
bis B1' B2' s1' s2' B1 B2 s1 s2 (R1-1) (R2-1)
apply (tactic ‹rtac @{{context}} (@{{thm bis_Frel}} RS iffD2) 1›)
apply (tactic ‹dtac @{{context}} (@{{thm bis_Frel}} RS iffD1) 1›)
apply (erule conjE)+
apply (rule conjI)

```

```

apply (rule conjI)

```

```

apply (rule iffD1[OF converse_subset_swap])
apply (erule subset_trans)
apply (rule equalityD2)
apply (rule converse_Times)

```

```

apply (rule iffD1[OF converse_subset_swap])
apply (erule subset_trans)
apply (rule equalityD2)
apply (rule converse_Times)

```

```

apply (rule conjI)
apply (rule allI)
apply (rule allI)
apply (rule impI)
apply (rule predicate2D[OF eq_refl[OF F1rel_cong]])
  apply (rule conversesep_eq)
  apply (rule conversesep_in_rel)
  apply (rule conversesep_in_rel)
apply (rule predicate2D[OF eq_refl[OF sym[OF F1.rel_conversep]]])
apply (erule allE)+
apply (rule conversesepI)
apply (erule mp)
apply (erule converseD)

```

```

apply (rule allI)
apply (rule allI)
apply (rule impI)
apply (rule predicate2D[OF eq_refl[OF F2rel_cong]])
  apply (rule conversesep_eq)
  apply (rule conversesep_in_rel)
  apply (rule conversesep_in_rel)
apply (rule predicate2D[OF eq_refl[OF sym[OF F2.rel_conversep]]])
apply (erule allE)+
apply (rule conversesepI)
apply (erule mp)
apply (erule converseD)
done

```

**lemma** bis\_Comp:

```

[[bis B1 B2 s1 s2 B1' B2' s1' s2' P1 P2;
 bis B1' B2' s1' s2' B1'' B2'' s1'' s2'' Q1 Q2]] ==>
 bis B1 B2 s1 s2 B1'' B2'' s1'' s2'' (P1 O Q1) (P2 O Q2)
apply (tactic <rtac @ {context} (@ {thm bis_Frel[THEN iffD2]}) 1>)
apply (tactic <dtac @ {context} (@ {thm bis_Frel[THEN iffD1]}) 1>)+
apply (erule conjE)+
apply (rule conjI)
apply (rule conjI)
apply (erule relcomp_subset_Sigma)
apply assumption
apply (erule relcomp_subset_Sigma)
apply assumption

```

```

apply (rule conjI)
apply (rule allI)+
apply (rule impI)
apply (rule predicate2D[OF eq_refl[OF F1rel_cong]])
  apply (rule eq_OO)
  apply (rule relcompp_in_rel)
  apply (rule relcompp_in_rel)
apply (rule predicate2D[OF eq_refl[OF sym[OF F1.rel_compp]]])
apply (erule relcompE)
apply (tactic <dtac @ {context} (@ {thm prod.inject[THEN iffD1]}) 1>)
apply (erule conjE)

```

```

apply (tactic <hyp_subst_tac @ {context} 1 >)
apply (erule allE)+
apply (rule relcomppI)
  apply (erule mp)
  apply assumption
apply (erule mp)
apply assumption

```

```

apply (rule allI)+
apply (rule impI)
apply (rule predicate2D[OF eq_refl[OF F2rel_cong]])
  apply (rule eq_OO)
  apply (rule relcompp_in_rel)
  apply (rule relcompp_in_rel)
apply (rule predicate2D[OF eq_refl[OF sym[OF F2.rel_compp]])]
apply (erule relcompE)
apply (tactic <dtac @ {context} (@ {thm prod.inject[THEN iffD1]}) 1 >)
apply (erule conjE)
apply (tactic <hyp_subst_tac @ {context} 1 >)
apply (erule allE)+
apply (rule relcomppI)
  apply (erule mp)
  apply assumption
apply (erule mp)
apply assumption
done

```

```

lemma bis_Gr:  $\llbracket \text{coalg } B1 \ B2 \ s1 \ s2; \text{mor } B1 \ B2 \ s1 \ s2 \ B1' \ B2' \ s1' \ s2' \ f1 \ f2 \rrbracket \implies$ 
  bis B1 B2 s1 s2 B1' B2' s1' s2' (BNF_Def.Gr B1 f1) (BNF_Def.Gr B2 f2)
unfolding bis_Frel eq_alt in_rel_Gr F1.rel_Grp F2.rel_Grp
apply (rule conjI)
apply (rule conjI)
  apply (rule iffD2[OF Gr_incl])
  apply (erule mor_image1)
apply (rule iffD2[OF Gr_incl])
apply (erule mor_image2)

```

```

apply (rule conjI)
apply (rule allI)
apply (rule allI)
apply (rule impI)
apply (rule GrpI)
  apply (erule trans[OF morE1])
  apply (erule GrD1)
  apply (erule arg_cong[OF GrD2])
apply (erule coalg_F1in)
apply (erule GrD1)

```

```

apply (rule allI)
apply (rule allI)
apply (rule impI)
apply (rule GrpI)
  apply (erule trans[OF morE2])
  apply (erule GrD1)
  apply (erule arg_cong[OF GrD2])
apply (erule coalg_F2in)
apply (erule GrD1)
done

```

```

lemmas bis_image2 = bis_cong[OF bis_Comp[OF bis_converse[OF bis_Gr] bis_Gr] image2_Gr image2_Gr]
lemmas bis_diag = bis_cong[OF bis_Gr[OF _ mor_id] Id_on_Gr Id_on_Gr]

```

```

lemma bis_Union:  $\forall i \in I. \text{bis } B1 \ B2 \ s1 \ s2 \ B1 \ B2 \ s1 \ s2 \ (R1i \ i) \ (R2i \ i) \implies$ 

```

```

bis B1 B2 s1 s2 B1 B2 s1 s2 ( $\bigcup_{i \in I}. R1i\ i$ ) ( $\bigcup_{i \in I}. R2i\ i$ )
unfolding bis_def
apply (rule conjI)
apply (rule conjI)
  apply (rule UN_least)
  apply (drule bspec)
  apply assumption
  apply (drule conjunct1)
  apply (tactic <etac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
  apply (rule UN_least)
  apply (drule bspec)
  apply assumption
  apply (drule conjunct1)
  apply (tactic <etac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)

```

```

apply (rule conjI)
apply (rule allI)+
apply (rule impI)
apply (erule UN_E)
apply (drule bspec)
  apply assumption
  apply (drule conjunct2)
  apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
  apply (erule allE)+
  apply (drule mp)
  apply assumption
  apply (erule bexE)
  apply (rule bexE)
  apply assumption
  apply (rule F1in_mono23')
  apply (erule SUP_upper2[OF _ subset_refl])
  apply (erule SUP_upper2[OF _ subset_refl])
  apply assumption

```

```

apply (rule allI)+
apply (rule impI)
apply (erule UN_E)
apply (drule bspec)
  apply assumption
  apply (drule conjunct2)
  apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
  apply (erule allE)+
  apply (drule mp)
  apply assumption
  apply (erule bexE)
  apply (rule bexE)
  apply assumption
  apply (rule F2in_mono23')
  apply (erule SUP_upper2[OF _ subset_refl])
  apply (erule SUP_upper2[OF _ subset_refl])
  apply assumption
done

```

**abbreviation**  $sbis\ B1\ B2\ s1\ s2\ R1\ R2 \equiv bis\ B1\ B2\ s1\ s2\ B1\ B2\ s1\ s2\ R1\ R2$

**definition**  $lsbis1$  where  $lsbis1\ B1\ B2\ s1\ s2 =$   
 $(\bigcup R \in \{(R1, R2) \mid R1\ R2 . sbis\ B1\ B2\ s1\ s2\ R1\ R2\}. fst\ R)$

**definition**  $lsbis2$  where  $lsbis2\ B1\ B2\ s1\ s2 =$   
 $(\bigcup R \in \{(R1, R2) \mid R1\ R2 . sbis\ B1\ B2\ s1\ s2\ R1\ R2\}. snd\ R)$



```

lemma sbis_lsbis:
  sbis B1 B2 s1 s2 (lsbis1 B1 B2 s1 s2) (lsbis2 B1 B2 s1 s2)
  apply (tactic <rtac @{context}> (Thm.permute_premis 0 1 @{thm bis_cong}) 1 >)
    apply (rule lsbis1_def)
    apply (rule lsbis2_def)
    apply (rule bis_Union)
    apply (rule ballI)
    apply (erule CollectE exE conjE)+
    apply (tactic <hyp_subst_tac @{context}> 1 >)
    apply (erule bis_cong)
    apply (rule fst_conv)
    apply (rule snd_conv)
  done

lemmas lsbis1_incl = conjunct1[OF conjunct1[OF iffD1[OF bis_def]], OF sbis_lsbis]
lemmas lsbis2_incl = conjunct2[OF conjunct1[OF iffD1[OF bis_def]], OF sbis_lsbis]
lemmas lsbisE1 =
  mp[OF spec[OF spec[OF conjunct1[OF conjunct2[OF iffD1[OF bis_def]], OF sbis_lsbis]]]]
lemmas lsbisE2 =
  mp[OF spec[OF spec[OF conjunct2[OF conjunct2[OF iffD1[OF bis_def]], OF sbis_lsbis]]]]

lemma incl_lsbis1: sbis B1 B2 s1 s2 R1 R2  $\implies$  R1  $\subseteq$  lsbis1 B1 B2 s1 s2
  apply (rule xt1(3))
  apply (rule lsbis1_def)
  apply (rule SUP_upper2)
  apply (rule CollectI)
  apply (rule exI)+
  apply (rule conjI)
  apply (rule refl)
  apply assumption
  apply (rule equalityD2)
  apply (rule fst_conv)
  done

lemma incl_lsbis2: sbis B1 B2 s1 s2 R1 R2  $\implies$  R2  $\subseteq$  lsbis2 B1 B2 s1 s2
  apply (rule xt1(3))
  apply (rule lsbis2_def)
  apply (rule SUP_upper2)
  apply (rule CollectI)
  apply (rule exI)+
  apply (rule conjI)
  apply (rule refl)
  apply assumption
  apply (rule equalityD2)
  apply (rule snd_conv)
  done

lemma equiv_lsbis1: coalg B1 B2 s1 s2  $\implies$  equiv B1 (lsbis1 B1 B2 s1 s2)
  apply (rule iffD2[OF equiv_def])

  apply (rule conjI)
  apply (rule iffD2[OF refl_on_def])
  apply (rule conjI)
  apply (rule lsbis1_incl)
  apply (rule ballI)
  apply (rule subsetD)
  apply (rule incl_lsbis1)
  apply (rule bis_diag)
  apply assumption
  apply (erule Id_onI)

  apply (rule conjI)
  apply (rule iffD2[OF sym_def])

```

```

apply (rule allI)+
apply (rule impI)
apply (rule subsetD)
  apply (rule incl_lsbis1)
  apply (rule bis_converse)
  apply (rule sbis_lsbis)
apply (erule converseI)

```

```

apply (rule iffD2[OF trans_def])
apply (rule allI)+
apply (rule impI)+
apply (rule subsetD)
  apply (rule incl_lsbis1)
  apply (rule bis_Comp)
  apply (rule sbis_lsbis)
  apply (rule sbis_lsbis)
apply (erule relcompI)
apply assumption
done

```

**lemma** *equiv\_lsbis2*:  $\text{coalg } B1 \ B2 \ s1 \ s2 \implies \text{equiv } B2 \ (\text{lsbis2 } B1 \ B2 \ s1 \ s2)$

```

unfolding equiv_def refl_on_def sym_def trans_def
apply (rule conjI)

```

```

  apply (rule conjI)
    apply (rule lsbis2_incl)
  apply (rule ballI)
  apply (rule subsetD)
    apply (rule incl_lsbis2)
    apply (rule bis_diag)
  apply assumption
apply (erule Id_onI)

```

```

apply (rule conjI)

```

```

  apply (rule allI)+
  apply (rule impI)
  apply (rule subsetD)
    apply (rule incl_lsbis2)
    apply (rule bis_converse)
    apply (rule sbis_lsbis)
  apply (erule converseI)

```

```

apply (rule allI)+
apply (rule impI)+
apply (rule subsetD)
  apply (rule incl_lsbis2)
  apply (rule bis_Comp)
  apply (rule sbis_lsbis)
  apply (rule sbis_lsbis)
apply (erule relcompI)
apply assumption
done

```

## 2.5 The Tree Coalgebra

```

typedef bd_type_F = UNIV :: (bd_type_F1 + bd_type_F2) suc set
apply (rule exI) apply (rule UNIV_I)
done

```

```

type-synonym 'a carrier = ((bd_type_F + bd_type_F) list set ×
  ((bd_type_F + bd_type_F) list ⇒ (('a, bd_type_F, bd_type_F) F1 + ('a, bd_type_F, bd_type_F) F2)))

```

```

abbreviation bd_F ≡ dir_image (card_suc (bd_F1 + c bd_F2)) Abs_bd_type_F

```

**lemmas**  $sum\_card\_order = card\_order\_csum[OF F1.bd\_card\_order F2.bd\_card\_order]$   
**lemmas**  $sum\_Cinfinite = Cinfinite\_csum1[OF F1.bd\_Cinfinite]$   
**lemmas**  $bd\_F = dir\_image[OF Abs\_bd\_type\_F\_inject[OF UNIV\_I UNIV\_I] Card\_order\_card\_suc[OF sum\_card\_order]]$   
**lemmas**  $bd\_F\_Cinfinite = Cinfinite\_cong[OF bd\_F\_Cinfinite\_card\_suc[OF sum\_Cinfinite sum\_card\_order]]$   
**lemmas**  $bd\_F\_Card\_order = Card\_order\_ordIso[OF Card\_order\_card\_suc[OF sum\_card\_order] ordIso\_symmetric[OF bd\_F]]$   
**lemma**  $bd\_F\_card\_order: card\_order\ bd\_F$   
**apply** (rule  $card\_order\_dir\_image$ )  
**apply** (rule  $bijI'$ )  
**apply** (rule  $Abs\_bd\_type\_F\_inject[OF UNIV\_I UNIV\_I]$ )  
**apply** (rule  $Abs\_bd\_type\_F\_cases$ )  
**apply** (erule  $exI$ )  
**apply** (rule  $card\_order\_card\_suc$ )  
**apply** (rule  $sum\_card\_order$ )  
**done**  
**lemmas**  $bd\_F\_regularCard = regularCard\_ordIso[OF bd\_F\_Cinfinite\_card\_suc[OF sum\_Cinfinite sum\_card\_order] regularCard\_card\_suc[OF sum\_card\_order sum\_Cinfinite]]$   
]

**lemmas**  $F1set1\_bd' = ordLess\_transitive[OF F1.set\_bd(1) ordLess\_ordIso\_trans[OF ordLeq\_ordLess\_trans[OF ordLeq\_csum1[OF F1.bd\_Card\_order] card\_suc\_greater[OF sum\_card\_order]] bd\_F]]$   
**lemmas**  $F1set2\_bd' = ordLess\_transitive[OF F1.set\_bd(2) ordLess\_ordIso\_trans[OF ordLeq\_ordLess\_trans[OF ordLeq\_csum1[OF F1.bd\_Card\_order] card\_suc\_greater[OF sum\_card\_order]] bd\_F]]$   
**lemmas**  $F1set3\_bd' = ordLess\_transitive[OF F1.set\_bd(3) ordLess\_ordIso\_trans[OF ordLeq\_ordLess\_trans[OF ordLeq\_csum1[OF F1.bd\_Card\_order] card\_suc\_greater[OF sum\_card\_order]] bd\_F]]$

**lemmas**  $F2set1\_bd' = ordLess\_transitive[OF F2.set\_bd(1) ordLess\_ordIso\_trans[OF ordLeq\_ordLess\_trans[OF ordLeq\_csum2[OF F2.bd\_Card\_order] card\_suc\_greater[OF sum\_card\_order]] bd\_F]]$   
**lemmas**  $F2set2\_bd' = ordLess\_transitive[OF F2.set\_bd(2) ordLess\_ordIso\_trans[OF ordLeq\_ordLess\_trans[OF ordLeq\_csum2[OF F2.bd\_Card\_order] card\_suc\_greater[OF sum\_card\_order]] bd\_F]]$   
**lemmas**  $F2set3\_bd' = ordLess\_transitive[OF F2.set\_bd(3) ordLess\_ordIso\_trans[OF ordLeq\_ordLess\_trans[OF ordLeq\_csum2[OF F2.bd\_Card\_order] card\_suc\_greater[OF sum\_card\_order]] bd\_F]]$

**abbreviation**  $Succ1\ Kl\ kl \equiv \{k1. Inl\ k1 \in BNF\_Greatest\_Fixpoint.Succ\ Kl\ kl\}$   
**abbreviation**  $Succ2\ Kl\ kl \equiv \{k2. Inr\ k2 \in BNF\_Greatest\_Fixpoint.Succ\ Kl\ kl\}$

**definition**  $isNode1$  **where**  
 $isNode1\ Kl\ lab\ kl = (\exists x1. lab\ kl = Inl\ x1 \wedge F1set2\ x1 = Succ1\ Kl\ kl \wedge F1set3\ x1 = Succ2\ Kl\ kl)$

**definition**  $isNode2$  **where**  
 $isNode2\ Kl\ lab\ kl = (\exists x2. lab\ kl = Inr\ x2 \wedge F2set2\ x2 = Succ1\ Kl\ kl \wedge F2set3\ x2 = Succ2\ Kl\ kl)$

**abbreviation**  $isTree$  **where**  
 $isTree\ Kl\ lab \equiv ([\ ] \in Kl \wedge (\forall kl \in Kl. (\forall k1 \in Succ1\ Kl\ kl. isNode1\ Kl\ lab\ (kl \text{ @ } [Inl\ k1])) \wedge (\forall k2 \in Succ2\ Kl\ kl. isNode2\ Kl\ lab\ (kl \text{ @ } [Inr\ k2])))$

**definition**  $carT1$  **where**  
 $carT1 = \{(Kl :: (bd\_type\_F + bd\_type\_F)\ list\ set,\ lab) \mid Kl\ lab. isTree\ Kl\ lab \wedge isNode1\ Kl\ lab\ [\ ]\}$

**definition**  $carT2$  **where**  
 $carT2 = \{(Kl :: (bd\_type\_F + bd\_type\_F)\ list\ set,\ lab) \mid Kl\ lab. isTree\ Kl\ lab \wedge isNode2\ Kl\ lab\ [\ ]\}$

**definition**  $strT1$  **where**  
 $strT1 = (case\_prod\ (\%Kl\ lab. case\_sum\ (F1map\ id\ (\lambda k1. (BNF\_Greatest\_Fixpoint.Shift\ Kl\ (Inl\ k1), BNF\_Greatest\_Fixpoint.shift\ lab\ (Inl\ k1))))$

( $\lambda k2. (BNF\_Greatest\_Fixpoint.Shift\ Kl\ (Inr\ k2),\ BNF\_Greatest\_Fixpoint.shift\ lab\ (Inr\ k2))$ )) undefined (lab []))

**definition** *strT2* **where**

*strT2* = (case\_prod (%Kl lab. case\_sum undefined (F2map id  
( $\lambda k1. (BNF\_Greatest\_Fixpoint.Shift\ Kl\ (Inl\ k1),\ BNF\_Greatest\_Fixpoint.shift\ lab\ (Inl\ k1)$ ))  
( $\lambda k2. (BNF\_Greatest\_Fixpoint.Shift\ Kl\ (Inr\ k2),\ BNF\_Greatest\_Fixpoint.shift\ lab\ (Inr\ k2)$ )) (lab [])))

**lemma** *coalg\_T*: *coalg carT1 carT2 strT1 strT2*

**unfolding** *coalg\_def carT1\_def carT2\_def isNode1\_def isNode2\_def*

**apply** (rule *conjI*)

**apply** (rule *ballI*)

**apply** (erule *CollectE exE conjE*)+

**apply** (tactic <hyp\_subst\_tac @ {context} 1 >)

**apply** (rule *ssubst\_mem*[*OF trans*[*OF trans*[*OF fun\_cong*[*OF strT1\_def*] *prod.case*]])

**apply** (erule *trans*[*OF arg\_cong*])

**apply** (rule *sum.case*(1))

**apply** (rule *CollectI*)

**apply** (rule *conjI*)

**apply** (rule *subset\_UNIV*)

**apply** (rule *conjI*)

**apply** (rule *ord\_eq\_le\_trans*[*OF F1.set\_map*(2)])

**apply** (rule *image\_subsetI*)

**apply** (rule *CollectI*)

**apply** (rule *exI*)+

**apply** (rule *conjI*)

**apply** (rule *refl*)

**apply** (rule *conjI*)

**apply** (rule *conjI*)

**apply** (erule *empty\_Shift*)

**apply** (erule *rev\_subsetD*)

**apply** (erule *equalityD1*)

**apply** (erule *CollectD*)

**apply** (rule *ballI*)

**apply** (rule *conjI*)

**apply** (rule *ballI*)

**apply** (erule *CollectE*)

**apply** (erule *ShiftD*)

**apply** (erule *bspec*)

**apply** (erule *thin\_rl*)

**apply** *assumption*

**apply** (tactic <dtac @ {context} (BNF\_Util.mk\_conjunctN 2 1) 1 >)

**apply** (erule *bspec*)

**apply** (rule *CollectI*)

**apply** (erule *subsetD*[*OF equalityD1*[*OF Succ\_Shift*]])

**apply** (erule *exE conjE*)+

**apply** (rule *exI*)

**apply** (rule *conjI*)

**apply** (rule *trans*[*OF fun\_cong*[*OF shift\_def*]])

**apply** (rule *trans*[*OF arg\_cong*[*OF sym*[*OF append\_Cons*]])

**apply** *assumption*

**apply** (rule *conjI*)

**apply** (erule *trans*)

**apply** (rule *Collect\_cong*)

**apply** (rule *eqset\_imp\_iff*)

**apply** (rule *sym*)

**apply** (rule *trans*)

**apply** (rule *Succ\_Shift*)

**apply** (rule *arg\_cong*[*OF sym*[*OF append\_Cons*]])

**apply** (erule *trans*)

**apply** (rule *Collect\_cong*)

**apply** (rule *eqset\_imp\_iff*)

**apply** (rule *sym*)

```

apply (rule trans)
apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])

```

```

apply (rule ballI)
apply (erule CollectE)
apply (drule ShiftD)
apply (drule bspec)
apply (erule thin_rl)
apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule bspec)
apply (rule CollectI)
apply (erule subsetD[OF equalityD1[OF Succ_Shift]])
apply (erule exE conjE)+
apply (rule exI)
apply (rule conjI)
apply (rule trans[OF fun_cong[OF shift_def]])
apply (rule trans[OF arg_cong[OF sym[OF append_Cons]])]
apply assumption
apply (rule conjI)
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])

```

```

apply (drule bspec)
apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule bspec)
apply (erule subsetD[OF equalityD1])
apply assumption
apply (erule exE conjE)+
apply (rule exI)
apply (rule conjI)
apply (rule trans[OF fun_cong[OF shift_def]])
apply (erule trans[OF arg_cong[OF sym[OF append_Nil]])]
apply (rule conjI)
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Nil]])
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Nil]])

```

```

apply (rule ord_eq_le_trans[OF F1.set_map(3)])
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
apply (rule refl)
apply (rule conjI)
apply (rule conjI)
  apply (erule empty_Shift)
  apply (drule rev_subsetD)
  apply (erule equalityD1)
apply (erule CollectD)
apply (rule ballI)
apply (rule conjI)
apply (rule ballI)
apply (erule CollectE)
apply (drule ShiftD)
apply (drule bspec)
  apply (erule thin_rl)
  apply assumption
apply (tactic ‹dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1›)
apply (drule bspec)
apply (rule CollectI)
apply (erule subsetD[OF equalityD1[OF Succ_Shift]])
apply (erule exE conjE)+
apply (rule exI)
apply (rule conjI)
apply (rule trans[OF fun_cong[OF shift_def]])
apply (rule trans[OF arg_cong[OF sym[OF append_Cons]]])
apply assumption
apply (rule conjI)
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
  apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
  apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])

```

```

apply (rule ballI)
apply (erule CollectE)
apply (drule ShiftD)
apply (drule bspec)
apply (erule thin_rl)
apply assumption
apply (tactic ‹dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1›)
apply (drule bspec)
apply (rule CollectI)
apply (erule subsetD[OF equalityD1[OF Succ_Shift]])
apply (erule exE conjE)+
apply (rule exI)
apply (rule conjI)
apply (rule trans[OF fun_cong[OF shift_def]])
apply (rule trans[OF arg_cong[OF sym[OF append_Cons]]])

```

```

apply assumption
apply (rule conjI)
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])

```

```

apply (drule bspec)
apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1 >)
apply (drule bspec)
apply (erule subsetD[OF equalityD1])
apply assumption
apply (erule exE conjE) +
apply (rule exI)
apply (rule conjI)
apply (rule trans[OF fun_cong[OF shift_def]])
apply (erule trans[OF arg_cong[OF sym[OF append_Nil]]])
apply (rule conjI)
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Nil]])
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Nil]])

```

```

apply (rule ballI)
apply (erule CollectE exE conjE) +
apply (tactic <hyp_subst_tac @{context} 1 >)
apply (rule ssubst_mem[OF trans[OF fun_cong[OF strT2_def] prod.case]])
apply (rule ssubst_mem)
apply (rule trans)
apply (erule arg_cong)
apply (rule sum.case(2))
apply (rule CollectI)
apply (rule conjI)
apply (rule subset_UNIV)
apply (rule conjI)
apply (rule ord_eq_le_trans[OF F2.set_map(2)])
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule exI) +
apply (rule conjI)

```

```

apply (rule refl)
apply (rule conjI)
apply (rule conjI)
  apply (erule empty_Shift)
  apply (drule rev_subsetD)
  apply (erule equalityD1)
  apply (erule CollectD)
apply (rule ballI)
apply (rule conjI)
  apply (rule ballI)
  apply (erule CollectE)
  apply (drule ShiftD)
apply (drule bspec)
  apply (erule thin_rl)
  apply assumption
apply (tactic ‹dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1›)
apply (drule bspec)
  apply (rule CollectI)
  apply (erule subsetD[OF equalityD1[OF Succ_Shift]])
apply (erule exE conjE)+
apply (rule exI)
apply (rule conjI)
  apply (rule trans[OF fun_cong[OF shift_def]])
  apply (rule trans[OF arg_cong[OF sym[OF append_Cons]]])
  apply assumption
apply (rule conjI)
  apply (erule trans)
  apply (rule Collect_cong)
  apply (rule eqset_imp_iff)
  apply (rule sym)
  apply (rule trans)
  apply (rule Succ_Shift)
  apply (rule arg_cong[OF sym[OF append_Cons]])
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
  apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])

```

```

apply (rule ballI)
apply (erule CollectE)
apply (drule ShiftD)
apply (drule bspec)
  apply (erule thin_rl)
  apply assumption
apply (tactic ‹dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1›)
apply (drule bspec)
  apply (rule CollectI)
  apply (erule subsetD[OF equalityD1[OF Succ_Shift]])
apply (erule exE conjE)+
apply (rule exI)
apply (rule conjI)
  apply (rule trans[OF fun_cong[OF shift_def]])
  apply (rule trans[OF arg_cong[OF sym[OF append_Cons]]])
  apply assumption
apply (rule conjI)
  apply (erule trans)
  apply (rule Collect_cong)
  apply (rule eqset_imp_iff)
  apply (rule sym)

```



```

apply (rule trans)
  apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
  apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])

apply (drule bspec)
apply assumption
apply (tactic ⟨dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1⟩)
apply (drule bspec)
apply (erule subsetD[OF equalityD1])
apply assumption
apply (erule exE conjE)+
apply (rule exI)
apply (rule conjI)
apply (rule trans[OF fun_cong[OF shift_def]])
apply (erule trans[OF arg_cong[OF sym[OF append_Nil]]])
apply (rule conjI)
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
  apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Nil]])
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
  apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Nil]])

apply (rule ord_eq_le_trans[OF F2.set_map(3)])
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (rule conjI)
apply (rule conjI)
  apply (erule empty_Shift)
apply (drule rev_subsetD)
  apply (erule equalityD1)
apply (erule CollectD)
apply (rule ballI)
apply (rule conjI)
apply (rule ballI)
apply (erule CollectE)
apply (drule ShiftD)
apply (drule bspec)
  apply (erule thin_rl)
apply assumption
apply (tactic ⟨dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1⟩)
apply (drule bspec)
  apply (rule CollectI)
  apply (erule subsetD[OF equalityD1[OF Succ_Shift]])
apply (erule exE conjE)+

```

```

apply (rule exI)
apply (rule conjI)
  apply (rule trans[OF fun_cong[OF shift_def]])
  apply (rule trans[OF arg_cong[OF sym[OF append_Cons]]])
  apply assumption
apply (rule conjI)
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
  apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
  apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])

apply (rule ballI)
apply (erule CollectE)
apply (drule ShiftD)
apply (drule bspec)
apply (erule thin_rl)
apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule bspec)
apply (rule CollectI)
apply (erule subsetD[OF equalityD1[OF Succ_Shift]])
apply (erule exE conjE)+
apply (rule exI)
apply (rule conjI)
apply (rule trans[OF fun_cong[OF shift_def]])
apply (rule trans[OF arg_cong[OF sym[OF append_Cons]]])
apply assumption
apply (rule conjI)
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
  apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
  apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Cons]])

apply (drule bspec)
apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule bspec)
apply (erule subsetD[OF equalityD1])
apply assumption
apply (erule exE conjE)+
apply (rule exI)
apply (rule conjI)

```

```

apply (rule trans[OF fun_cong[OF shift_def]])
apply (erule trans[OF arg_cong[OF sym[OF append_Nil]]])
apply (rule conjI)
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Nil]])
apply (erule trans)
apply (rule Collect_cong)
apply (rule eqset_imp_iff)
apply (rule sym)
apply (rule trans)
apply (rule Succ_Shift)
apply (rule arg_cong[OF sym[OF append_Nil]])
done

```

```

abbreviation tobd_F12 where tobd_F12 s1 x  $\equiv$  toCard (F1set2 (s1 x)) bd_F
abbreviation tobd_F13 where tobd_F13 s1 x  $\equiv$  toCard (F1set3 (s1 x)) bd_F
abbreviation tobd_F22 where tobd_F22 s2 x  $\equiv$  toCard (F2set2 (s2 x)) bd_F
abbreviation tobd_F23 where tobd_F23 s2 x  $\equiv$  toCard (F2set3 (s2 x)) bd_F
abbreviation frombd_F12 where frombd_F12 s1 x  $\equiv$  fromCard (F1set2 (s1 x)) bd_F
abbreviation frombd_F13 where frombd_F13 s1 x  $\equiv$  fromCard (F1set3 (s1 x)) bd_F
abbreviation frombd_F22 where frombd_F22 s2 x  $\equiv$  fromCard (F2set2 (s2 x)) bd_F
abbreviation frombd_F23 where frombd_F23 s2 x  $\equiv$  fromCard (F2set3 (s2 x)) bd_F

```

```

lemmas tobd_F12_inj = toCard_inj[OF ordLess_imp_ordLeq[OF F1set2_bd'] bd_F_Card_order]
lemmas tobd_F13_inj = toCard_inj[OF ordLess_imp_ordLeq[OF F1set3_bd'] bd_F_Card_order]
lemmas tobd_F22_inj = toCard_inj[OF ordLess_imp_ordLeq[OF F2set2_bd'] bd_F_Card_order]
lemmas tobd_F23_inj = toCard_inj[OF ordLess_imp_ordLeq[OF F2set3_bd'] bd_F_Card_order]
lemmas frombd_F12_tobd_F12 = fromCard_toCard[OF ordLess_imp_ordLeq[OF F1set2_bd'] bd_F_Card_order]
lemmas frombd_F13_tobd_F13 = fromCard_toCard[OF ordLess_imp_ordLeq[OF F1set3_bd'] bd_F_Card_order]
lemmas frombd_F22_tobd_F22 = fromCard_toCard[OF ordLess_imp_ordLeq[OF F2set2_bd'] bd_F_Card_order]
lemmas frombd_F23_tobd_F23 = fromCard_toCard[OF ordLess_imp_ordLeq[OF F2set3_bd'] bd_F_Card_order]

```

**definition** Lev **where**

```

Lev s1 s2 = rec_nat (%a. {[]}, %b. {[]})
  (%n rec.
    (%a1.
      {Inl (tobd_F12 s1 a1 b1) # kl | b1 kl. b1  $\in$  F1set2 (s1 a1)  $\wedge$  kl  $\in$  fst rec b1}  $\cup$ 
      {Inr (tobd_F13 s1 a1 b2) # kl | b2 kl. b2  $\in$  F1set3 (s1 a1)  $\wedge$  kl  $\in$  snd rec b2},
      %a2.
      {Inl (tobd_F22 s2 a2 b1) # kl | b1 kl. b1  $\in$  F2set2 (s2 a2)  $\wedge$  kl  $\in$  fst rec b1}  $\cup$ 
      {Inr (tobd_F23 s2 a2 b2) # kl | b2 kl. b2  $\in$  F2set3 (s2 a2)  $\wedge$  kl  $\in$  snd rec b2}))

```

**abbreviation** Lev1 **where** Lev1 s1 s2 n  $\equiv$  fst (Lev s1 s2 n)

**abbreviation** Lev2 **where** Lev2 s1 s2 n  $\equiv$  snd (Lev s1 s2 n)

```

lemmas Lev1_0 = fun_cong[OF fstI[OF rec_nat_0_imp[OF Lev_def]]]
lemmas Lev2_0 = fun_cong[OF sndI[OF rec_nat_0_imp[OF Lev_def]]]
lemmas Lev1_Suc = fun_cong[OF fstI[OF rec_nat_Suc_imp[OF Lev_def]]]
lemmas Lev2_Suc = fun_cong[OF sndI[OF rec_nat_Suc_imp[OF Lev_def]]]

```

**definition** rv **where**

```

rv s1 s2 = rec_list (%b1. Inl b1, %b2. Inr b2)
  (%k kl rec.
    (%b1. case_sum (%k1. fst rec (frombd_F12 s1 b1 k1)) (%k2. snd rec (frombd_F13 s1 b1 k2)) k,
      %b2. case_sum (%k1. fst rec (frombd_F22 s2 b2 k1)) (%k2. snd rec (frombd_F23 s2 b2 k2)) k))

```

**abbreviation** *rv1* **where** *rv1 s1 s2 kl*  $\equiv$  *fst (rv s1 s2 kl)*  
**abbreviation** *rv2* **where** *rv2 s1 s2 kl*  $\equiv$  *snd (rv s1 s2 kl)*

**lemmas** *rv1\_Nil* = *fun\_cong[OF fstI[OF rec\_list\_Nil\_imp[OF rv\_def]]]*  
**lemmas** *rv2\_Nil* = *fun\_cong[OF sndI[OF rec\_list\_Nil\_imp[OF rv\_def]]]*  
**lemmas** *rv1\_Cons* = *fun\_cong[OF fstI[OF rec\_list\_Cons\_imp[OF rv\_def]]]*  
**lemmas** *rv2\_Cons* = *fun\_cong[OF sndI[OF rec\_list\_Cons\_imp[OF rv\_def]]]*

**abbreviation** *Lab1 s1 s2 b1 kl*  $\equiv$   
*(case\_sum (%k. Inl (F1map id (tobd\_F12 s1 k) (tobd\_F13 s1 k) (s1 k)))*  
*(%k. Inr (F2map id (tobd\_F22 s2 k) (tobd\_F23 s2 k) (s2 k))) (rv1 s1 s2 kl b1))*

**abbreviation** *Lab2 s1 s2 b2 kl*  $\equiv$   
*(case\_sum (%k. Inl (F1map id (tobd\_F12 s1 k) (tobd\_F13 s1 k) (s1 k)))*  
*(%k. Inr (F2map id (tobd\_F22 s2 k) (tobd\_F23 s2 k) (s2 k))) (rv2 s1 s2 kl b2))*

**definition** *beh1 s1 s2 a* =  $(\bigcup n. Lev1 s1 s2 n a, Lab1 s1 s2 a)$   
**definition** *beh2 s1 s2 a* =  $(\bigcup n. Lev2 s1 s2 n a, Lab2 s1 s2 a)$

**lemma** *length\_Lev*:  
 $\forall kl b1 b2. (kl \in Lev1 s1 s2 n b1 \longrightarrow length\ kl = n) \wedge$   
 $(kl \in Lev2 s1 s2 n b2 \longrightarrow length\ kl = n)$

**apply** (*rule nat\_induct*)  
**apply** (*rule allI*)  
**apply** (*rule conjI*)  
**apply** (*rule impI*)  
**apply** (*drule subsetD[OF equalityD1[OF Lev1\_0]]*)  
**apply** (*erule singletonE*)  
**apply** (*erule ssubst*)  
**apply** (*rule list.size(3)*)

**apply** (*rule impI*)  
**apply** (*drule subsetD[OF equalityD1[OF Lev2\_0]]*)  
**apply** (*erule singletonE*)  
**apply** (*erule ssubst*)  
**apply** (*rule list.size(3)*)

**apply** (*rule allI*)  
**apply** (*rule conjI*)  
**apply** (*rule impI*)  
**apply** (*drule subsetD[OF equalityD1[OF Lev1\_Suc]]*)  
**apply** (*erule UnE*)  
**apply** (*erule CollectE exE conjE*)  
**apply** (*tactic <hyp\_subst\_tac @ {context} 1>*)  
**apply** (*rule trans*)  
**apply** (*rule length\_Cons*)  
**apply** (*rule arg\_cong[of \_ \_ Suc]*)  
**apply** (*erule allE*)  
**apply** (*tactic <dtac @ {context} (BNF\_Util.mk\_conjunctN 2 1) 1>*)  
**apply** (*erule mp*)  
**apply** (*assumption*)

**apply** (*erule CollectE exE conjE*)  
**apply** (*tactic <hyp\_subst\_tac @ {context} 1>*)  
**apply** (*rule trans*)  
**apply** (*rule length\_Cons*)  
**apply** (*rule arg\_cong[of \_ \_ Suc]*)  
**apply** (*erule allE*)  
**apply** (*tactic <dtac @ {context} (BNF\_Util.mk\_conjunctN 2 2) 1>*)  
**apply** (*erule mp*)  
**apply** (*assumption*)

```

apply (rule impI)
apply (drule subsetD[OF equalityD1[OF Lev2_Suc]])
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (tactic <hyp_subst_tac @ {context} 1>)
apply (rule trans)
apply (rule length_Cons)
apply (rule arg_cong[of _ _ Suc])
apply (erule allE)+
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply assumption

```

```

apply (erule CollectE exE conjE)+
apply (tactic <hyp_subst_tac @ {context} 1>)
apply (rule trans)
apply (rule length_Cons)
apply (rule arg_cong[of _ _ Suc])
apply (erule allE)+
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply assumption
done

```

```

lemmas length_Lev1 = mp[OF conjunct1[OF spec[OF spec [OF spec[OF length_Lev]]]]]
lemmas length_Lev2 = mp[OF conjunct2[OF spec[OF spec [OF spec[OF length_Lev]]]]]

```

```

lemma length_Lev1': kl ∈ Lev1 s1 s2 n a ⇒ kl ∈ Lev1 s1 s2 (length kl) a
apply (frule length_Lev1)
apply (erule ssubst)
apply assumption
done

```

```

lemma length_Lev2': kl ∈ Lev2 s1 s2 n a ⇒ kl ∈ Lev2 s1 s2 (length kl) a
apply (frule length_Lev2)
apply (erule ssubst)
apply assumption
done

```

**lemma** rv\_last:

```

∀ k b1 b2.
  ((∃ b1'. rv1 s1 s2 (kl @ [Inl k]) b1 = Inl b1') ∧
   (∃ b1'. rv1 s1 s2 (kl @ [Inr k]) b1 = Inr b1')) ∧
  ((∃ b2'. rv2 s1 s2 (kl @ [Inl k]) b2 = Inl b2') ∧
   (∃ b2'. rv2 s1 s2 (kl @ [Inr k]) b2 = Inr b2'))
apply (rule list.induct[of _ kl])
apply (rule allI)+
apply (rule conjI)
apply (rule conjI)
apply (rule exI)
apply (rule trans[OF arg_cong[OF append_Nil]])
apply (rule trans[OF rv1_Cons])
apply (rule trans[OF arg_cong[OF sum.case(1)]])
apply (rule rv1_Nil)
apply (rule exI)
apply (rule trans[OF arg_cong[OF append_Nil]])
apply (rule trans[OF rv1_Cons])
apply (rule trans[OF arg_cong[OF sum.case(2)]])
apply (rule rv2_Nil)
apply (rule conjI)
apply (rule exI)

```

```

apply (rule trans[OF arg_cong[OF append_Nil]])
apply (rule trans[OF rv2_Cons])
apply (rule trans[OF arg_cong[OF sum.case(1)]])
apply (rule rv1_Nil)
apply (rule exI)
apply (rule trans[OF arg_cong[OF append_Nil]])
apply (rule trans[OF rv2_Cons])
apply (rule trans[OF arg_cong[OF sum.case(2)]])
apply (rule rv2_Nil)

```

```

apply (rule allI)+
apply (rule sum.exhaust)
apply (rule conjI)
apply (erule allE)+
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (rule conjI)
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule exE)
apply (rule exI)
apply (rule trans[OF arg_cong[OF append_Cons]])
apply (rule trans[OF rv1_Cons])
apply (erule trans[OF arg_cong[OF sum.case_cong_weak]])
apply (rule trans[OF arg_cong[OF sum.case(1)]])
apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule exE)
apply (rule exI)
apply (rule trans[OF arg_cong[OF append_Cons]])
apply (rule trans[OF rv1_Cons])
apply (erule trans[OF arg_cong[OF sum.case_cong_weak]])
apply (rule trans[OF arg_cong[OF sum.case(1)]])
apply assumption

```

```

apply (erule allE)+
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (rule conjI)
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule exE)
apply (rule exI)
apply (rule trans[OF arg_cong[OF append_Cons]])
apply (rule trans[OF rv2_Cons])
apply (erule trans[OF arg_cong[OF sum.case_cong_weak]])
apply (rule trans[OF arg_cong[OF sum.case(1)]])
apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule exE)
apply (rule exI)
apply (rule trans[OF arg_cong[OF append_Cons]])
apply (rule trans[OF rv2_Cons])
apply (erule trans[OF arg_cong[OF sum.case_cong_weak]])
apply (rule trans[OF arg_cong[OF sum.case(1)]])
apply assumption

```

```

apply (rule conjI)
apply (erule allE)+
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (rule conjI)
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule exE)
apply (rule exI)
apply (rule trans[OF arg_cong[OF append_Cons]])
apply (rule trans[OF rv1_Cons])
apply (erule trans[OF arg_cong[OF sum.case_cong_weak]])

```

```

apply (rule trans[OF arg_cong[OF sum.case(2)]])
apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule exE)
apply (rule exI)
apply (rule trans[OF arg_cong[OF append_Cons]])
apply (rule trans[OF rv1_Cons])
apply (erule trans[OF arg_cong[OF sum.case_cong_weak]])
apply (rule trans[OF arg_cong[OF sum.case(2)]])
apply assumption

apply (erule allE)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (rule conjI)
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule exE)
apply (rule exI)
apply (rule trans[OF arg_cong[OF append_Cons]])
apply (rule trans[OF rv2_Cons])
apply (erule trans[OF arg_cong[OF sum.case_cong_weak]])
apply (rule trans[OF arg_cong[OF sum.case(2)]])
apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule exE)
apply (rule exI)
apply (rule trans[OF arg_cong[OF append_Cons]])
apply (rule trans[OF rv2_Cons])
apply (erule trans[OF arg_cong[OF sum.case_cong_weak]])
apply (rule trans[OF arg_cong[OF sum.case(2)]])
apply assumption
done

```

```

lemmas rv_last' = spec[OF spec[OF spec[OF rv_last]]]
lemmas rv1_Inl_last = conjunct1[OF conjunct1[OF rv_last']]
lemmas rv1_Inr_last = conjunct2[OF conjunct1[OF rv_last']]
lemmas rv2_Inl_last = conjunct1[OF conjunct2[OF rv_last']]
lemmas rv2_Inr_last = conjunct2[OF conjunct2[OF rv_last']]

```

**lemma** Fset\_Lev:

```

 $\forall kl\ b1\ b2\ b1'\ b2'\ b1''\ b2''.$ 
( $kl \in Lev1\ s1\ s2\ n\ b1 \longrightarrow$ 
  ( $(rv1\ s1\ s2\ kl\ b1 = Inl\ b1' \longrightarrow$ 
    ( $b1'' \in F1set2\ (s1\ b1') \longrightarrow$ 
       $kl @ [Inl\ (tobd\_F12\ s1\ b1'\ b1'')] \in Lev1\ s1\ s2\ (Suc\ n)\ b1) \wedge$ 
      ( $b2'' \in F1set3\ (s1\ b1') \longrightarrow$ 
         $kl @ [Inr\ (tobd\_F13\ s1\ b1'\ b2'')] \in Lev1\ s1\ s2\ (Suc\ n)\ b1)) \wedge$ 
      ( $rv1\ s1\ s2\ kl\ b1 = Inr\ b2' \longrightarrow$ 
        ( $b1'' \in F2set2\ (s2\ b2') \longrightarrow$ 
           $kl @ [Inl\ (tobd\_F22\ s2\ b2'\ b1'')] \in Lev1\ s1\ s2\ (Suc\ n)\ b1) \wedge$ 
          ( $b2'' \in F2set3\ (s2\ b2') \longrightarrow$ 
             $kl @ [Inr\ (tobd\_F23\ s2\ b2'\ b2'')] \in Lev1\ s1\ s2\ (Suc\ n)\ b1)))) \wedge$ 
      ( $kl \in Lev2\ s1\ s2\ n\ b2 \longrightarrow$ 
        ( $(rv2\ s1\ s2\ kl\ b2 = Inl\ b1' \longrightarrow$ 
          ( $b1'' \in F1set2\ (s1\ b1') \longrightarrow$ 
             $kl @ [Inl\ (tobd\_F12\ s1\ b1'\ b1'')] \in Lev2\ s1\ s2\ (Suc\ n)\ b2) \wedge$ 
            ( $b2'' \in F1set3\ (s1\ b1') \longrightarrow$ 
               $kl @ [Inr\ (tobd\_F13\ s1\ b1'\ b2'')] \in Lev2\ s1\ s2\ (Suc\ n)\ b2)) \wedge$ 
              ( $rv2\ s1\ s2\ kl\ b2 = Inr\ b2' \longrightarrow$ 
                ( $b1'' \in F2set2\ (s2\ b2') \longrightarrow$ 
                   $kl @ [Inl\ (tobd\_F22\ s2\ b2'\ b1'')] \in Lev2\ s1\ s2\ (Suc\ n)\ b2) \wedge$ 
                  ( $b2'' \in F2set3\ (s2\ b2') \longrightarrow$ 
                     $kl @ [Inr\ (tobd\_F23\ s2\ b2'\ b2'')] \in Lev2\ s1\ s2\ (Suc\ n)\ b2))))))$ 
apply (rule nat_induct[of _ n])

```

```

apply (rule allI)+
apply (rule conjI)
apply (rule impI)
apply (drule subsetD[OF equalityD1[OF Lev1_0]])
apply (erule singletonE)
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (rule conjI)
apply (rule impI)
apply (drule trans[OF sym])
apply (rule rv1_Nil)
apply (drule Inl_inject)
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (rule conjI)
apply (rule impI)
apply (rule ssubst_mem[OF append_Nil])
apply (rule subsetD[OF equalityD2])
apply (rule Lev1_Suc)
apply (rule UnI1)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
apply (rule refl)
apply (erule conjI)
apply (rule subsetD[OF equalityD2])
apply (rule Lev1_0)
apply (rule singletonI)
apply (rule impI)
apply (rule ssubst_mem[OF append_Nil])
apply (rule subsetD[OF equalityD2])
apply (rule Lev1_Suc)
apply (rule UnI2)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
apply (rule refl)
apply (erule conjI)
apply (rule subsetD[OF equalityD2])
apply (rule Lev2_0)
apply (rule singletonI)

apply (rule impI)
apply (drule trans[OF sym])
apply (rule rv1_Nil)
apply (erule notE[OF Inr_not_Inl])

apply (rule impI)
apply (drule rev_subsetD[OF _ equalityD1])
apply (rule Lev2_0)
apply (erule singletonE)
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (rule conjI)
apply (rule impI)
apply (drule trans[OF sym])
apply (rule rv2_Nil)
apply (erule notE[OF Inl_not_Inr])

apply (rule impI)
apply (drule trans[OF sym])
apply (rule rv2_Nil)
apply (drule Inr_inject)
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (tactic ⟨stac @ {context} @ {thm append_Nil} 1⟩)+

```



```

apply (rule conjI)
apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev2_Suc)
apply (rule UnI1)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev1_0)
apply (rule singletonI)
apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev2_Suc)
apply (rule UnI2)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev2_0)
apply (rule singletonI)

```

```

apply (rule allI)+
apply (rule conjI)
apply (rule impI)
apply (drule subsetD[OF equalityD1[OF Lev1_Suc]])
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (rule conjI)
apply (rule impI)
apply (rule conjI)
  apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev1_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI1)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (rule conjI)
  apply assumption
apply (drule sym[OF trans[OF sym]])
  apply (rule trans[OF rv1_Cons])
  apply (rule trans[OF sum.case(1)])
  apply (erule arg_cong[OF frombd_F12_tobd_F12])
apply (erule allE)+
apply (tactic ⟨dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1⟩)
apply (drule mp)
  apply assumption
apply (tactic ⟨dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1⟩)
apply (drule mp)
  apply assumption
apply (tactic ⟨dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1⟩)
apply (erule mp)
apply assumption

```

```

apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev1_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI1)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (rule conjI)
apply assumption
apply (drule sym[OF trans[OF sym]])
apply (rule trans[OF rv1_Cons])
apply (rule trans[OF sum.case(1)])
apply (erule arg_cong[OF frombd_F12_tobd_F12])
apply (erule alle)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1›)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1›)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1›)
apply (erule mp)
apply assumption

```

```

apply (rule impI)
apply (rule conjI)
apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev1_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI1)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (rule conjI)
apply assumption
apply (drule sym[OF trans[OF sym]])
apply (rule trans[OF rv1_Cons])
apply (rule trans[OF sum.case(1)])
apply (erule arg_cong[OF frombd_F12_tobd_F12])
apply (erule alle)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1›)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1›)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1›)
apply (erule mp)
apply assumption

```

```

apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev1_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI1)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)

```

```

apply (rule conjI)
  apply assumption
apply (drule sym[OF trans[OF sym]])
  apply (rule trans[OF rv1_Cons])
  apply (rule trans[OF sum.case(1)])
  apply (erule arg_cong[OF frombd_F12_tobd_F12])
apply (erule allE)+
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply assumption

```

```

apply (erule CollectE exE conjE)+
apply (tactic <hyp_subst_tac @{\context} 1>)
apply (tactic <stac @{\context} @{\thm rv1_Cons} 1>)
apply (tactic <stac @{\context} @{\thm sum.case(2)} 1>)
apply (tactic <stac @{\context} @{\thm frombd_F13_tobd_F13} 1>)
apply assumption
apply (rule conjI)
apply (rule impI)
apply (rule conjI)
  apply (rule impI)
  apply (rule subsetD[OF equalityD2])
  apply (rule Lev1_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI2)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply (erule allE)+
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply assumption

```

```

apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev1_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI2)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply (erule allE)+
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)

```

```

apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply assumption

```

```

apply (rule impI)
apply (rule conjI)
apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev1_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI2)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply (erule allE)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply assumption

```

```

apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev1_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI2)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply (erule allE)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply assumption

```

```

apply (rule impI)
apply (drule rev_subsetD[OF equalityD1])
  apply (rule Lev2_Suc)
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (tactic <hyp_subst_tac @{context} 1>)
apply (tactic <stac @{context} @{thm rv2_Cons 1>)
apply (tactic <stac @{context} @{thm sum.case(1)> 1>)
apply (tactic <stac @{context} @{thm frombd_F22_tobd_F22 1>)
  apply assumption
apply (rule conjI)
apply (rule impI)

```

```

apply (rule conjI)
apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev2_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI1)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply (erule allE)+
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply assumption

```

```

apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev2_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI1)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply (erule allE)+
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply assumption

```

```

apply (rule impI)
apply (rule conjI)
apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev2_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI1)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply (erule allE)+
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)

```

**apply** (*erule mp*)  
**apply** *assumption*

**apply** (*rule impI*)  
**apply** (*rule subsetD*[*OF equalityD2*])  
  **apply** (*rule Lev2\_Suc*)  
**apply** (*rule ssubst\_mem*[*OF append\_Cons*])  
**apply** (*rule UnI1*)  
**apply** (*rule CollectI*)  
**apply** (*rule exI*)+  
**apply** (*rule conjI*)  
  **apply** (*rule refl*)  
**apply** (*erule conjI*)  
**apply** (*erule allE*)+  
**apply** (*tactic <dtac* @*{context}* (*BNF\_Util.mk\_conjunctN 2 1*) 1 $\rangle$ )  
**apply** (*drule mp*)  
  **apply** *assumption*  
**apply** (*tactic <dtac* @*{context}* (*BNF\_Util.mk\_conjunctN 2 2*) 1 $\rangle$ )  
**apply** (*drule mp*)  
  **apply** *assumption*  
**apply** (*tactic <dtac* @*{context}* (*BNF\_Util.mk\_conjunctN 2 2*) 1 $\rangle$ )  
**apply** (*erule mp*)  
**apply** *assumption*

**apply** (*erule CollectE exE conjE*)+  
**apply** (*tactic <hyp\_subst\_tac* @*{context}* 1 $\rangle$ )  
**apply** (*tactic <stac* @*{context}* @*{thm rv2\_Cons}* 1 $\rangle$ )  
**apply** (*tactic <stac* @*{context}* @*{thm sum.case(2)}* 1 $\rangle$ )  
**apply** (*tactic <stac* @*{context}* @*{thm frombd\_F23\_tobd\_F23}* 1 $\rangle$ )  
  **apply** *assumption*  
**apply** (*rule conjI*)  
**apply** (*rule impI*)  
**apply** (*rule conjI*)  
**apply** (*rule impI*)  
**apply** (*rule subsetD*[*OF equalityD2*])  
  **apply** (*rule Lev2\_Suc*)  
**apply** (*rule ssubst\_mem*[*OF append\_Cons*])  
**apply** (*rule UnI2*)  
**apply** (*rule CollectI*)  
**apply** (*rule exI*)+  
**apply** (*rule conjI*)  
  **apply** (*rule refl*)  
**apply** (*erule conjI*)  
**apply** (*erule allE*)+  
**apply** (*tactic <dtac* @*{context}* (*BNF\_Util.mk\_conjunctN 2 2*) 1 $\rangle$ )  
**apply** (*drule mp*)  
  **apply** *assumption*  
**apply** (*tactic <dtac* @*{context}* (*BNF\_Util.mk\_conjunctN 2 1*) 1 $\rangle$ )  
**apply** (*drule mp*)  
  **apply** *assumption*  
**apply** (*tactic <dtac* @*{context}* (*BNF\_Util.mk\_conjunctN 2 1*) 1 $\rangle$ )  
**apply** (*erule mp*)  
**apply** *assumption*

**apply** (*rule impI*)  
**apply** (*rule subsetD*[*OF equalityD2*])  
  **apply** (*rule Lev2\_Suc*)  
**apply** (*rule ssubst\_mem*[*OF append\_Cons*])  
**apply** (*rule UnI2*)  
**apply** (*rule CollectI*)  
**apply** (*rule exI*)+  
**apply** (*rule conjI*)  
**apply** (*rule refl*)

```

apply (erule conjI)
apply (erule allE)+
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply assumption

```

```

apply (rule impI)
apply (rule conjI)
apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev2_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI2)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply (erule allE)+
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply assumption

```

```

apply (rule impI)
apply (rule subsetD[OF equalityD2])
  apply (rule Lev2_Suc)
apply (rule ssubst_mem[OF append_Cons])
apply (rule UnI2)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply (erule allE)+
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{\context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply assumption
done

```

```

lemmas Fset_Lev' = spec[OF spec[OF spec[OF spec[OF spec[OF spec[OF spec[OF Fset_Lev]]]]]]]]]]]]]]
lemmas F1set2_Lev1 = mp[OF conjunct1[OF mp[OF conjunct1[OF mp[OF conjunct1[OF Fset_Lev]]]]]]]]]]]]]]
lemmas F1set2_Lev2 = mp[OF conjunct1[OF mp[OF conjunct1[OF mp[OF conjunct2[OF Fset_Lev]]]]]]]]]]]]]]
lemmas F2set2_Lev1 = mp[OF conjunct1[OF mp[OF conjunct2[OF mp[OF conjunct1[OF Fset_Lev]]]]]]]]]]]]]]
lemmas F2set2_Lev2 = mp[OF conjunct1[OF mp[OF conjunct2[OF mp[OF conjunct2[OF Fset_Lev]]]]]]]]]]]]]]
lemmas F1set3_Lev1 = mp[OF conjunct2[OF mp[OF conjunct1[OF mp[OF conjunct1[OF Fset_Lev]]]]]]]]]]]]]]

```

**lemmas**  $F1set3\_Lev2 = mp[OF conjunct2[OF mp[OF conjunct1[OF mp[OF conjunct2[OF Fset\_Lev]]]]]]$   
**lemmas**  $F2set3\_Lev1 = mp[OF conjunct2[OF mp[OF conjunct2[OF mp[OF conjunct1[OF Fset\_Lev]]]]]]$   
**lemmas**  $F2set3\_Lev2 = mp[OF conjunct2[OF mp[OF conjunct2[OF mp[OF conjunct2[OF Fset\_Lev]]]]]]$

**lemma**  $Fset\_image\_Lev:$

$\forall kl\ k\ b1\ b2\ b1'\ b2'.$

$(kl \in Lev1\ s1\ s2\ n\ b1 \longrightarrow$   
 $(kl \ @ \ [Inl\ k] \in Lev1\ s1\ s2\ (Suc\ n)\ b1 \longrightarrow$   
 $(rv1\ s1\ s2\ kl\ b1 = Inl\ b1' \longrightarrow k \in tobd\_F12\ s1\ b1' \ ' \ F1set2\ (s1\ b1')) \wedge$   
 $(rv1\ s1\ s2\ kl\ b1 = Inr\ b2' \longrightarrow k \in tobd\_F22\ s2\ b2' \ ' \ F2set2\ (s2\ b2')))) \wedge$   
 $(kl \ @ \ [Inr\ k] \in Lev1\ s1\ s2\ (Suc\ n)\ b1 \longrightarrow$   
 $(rv1\ s1\ s2\ kl\ b1 = Inl\ b1' \longrightarrow k \in tobd\_F13\ s1\ b1' \ ' \ F1set3\ (s1\ b1')) \wedge$   
 $(rv1\ s1\ s2\ kl\ b1 = Inr\ b2' \longrightarrow k \in tobd\_F23\ s2\ b2' \ ' \ F2set3\ (s2\ b2')))) \wedge$   
 $(kl \in Lev2\ s1\ s2\ n\ b2 \longrightarrow$   
 $(kl \ @ \ [Inl\ k] \in Lev2\ s1\ s2\ (Suc\ n)\ b2 \longrightarrow$   
 $(rv2\ s1\ s2\ kl\ b2 = Inl\ b1' \longrightarrow k \in tobd\_F12\ s1\ b1' \ ' \ F1set2\ (s1\ b1')) \wedge$   
 $(rv2\ s1\ s2\ kl\ b2 = Inr\ b2' \longrightarrow k \in tobd\_F22\ s2\ b2' \ ' \ F2set2\ (s2\ b2')))) \wedge$   
 $(kl \ @ \ [Inr\ k] \in Lev2\ s1\ s2\ (Suc\ n)\ b2 \longrightarrow$   
 $(rv2\ s1\ s2\ kl\ b2 = Inl\ b1' \longrightarrow k \in tobd\_F13\ s1\ b1' \ ' \ F1set3\ (s1\ b1')) \wedge$   
 $(rv2\ s1\ s2\ kl\ b2 = Inr\ b2' \longrightarrow k \in tobd\_F23\ s2\ b2' \ ' \ F2set3\ (s2\ b2'))))$   
**apply** (rule nat\_induct[of \_ n])

**apply** (rule allI)+  
**apply** (rule conjI)  
**apply** (rule impI)  
**apply** (drule subsetD[OF equalityD1[OF Lev1\_0]])  
**apply** (erule singletonE)  
**apply** (tactic <hyp\_subst\_tac @ {context} 1>)  
**apply** (rule conjI)  
**apply** (rule impI)  
**apply** (rule conjI)  
**apply** (rule impI)  
**apply** (drule trans[OF sym])  
**apply** (rule rv1\_Nil)  
**apply** (drule ssubst\_mem[OF sym[OF append\_Nil]])  
**apply** (drule subsetD[OF equalityD1[OF Lev1\_Suc]])  
**apply** (drule Inl\_inject)  
**apply** (tactic <hyp\_subst\_tac @ {context} 1>)  
**apply** (erule UnE)  
**apply** (erule CollectE exE conjE)+  
**apply** (drule list.inject[THEN iffD1])  
**apply** (erule conjE)  
**apply** (drule Inl\_inject)  
**apply** (tactic <hyp\_subst\_tac @ {context} 1>)  
**apply** (erule imageI)  
**apply** (erule CollectE exE conjE)+  
**apply** (drule list.inject[THEN iffD1])  
**apply** (erule conjE)  
**apply** (erule notE[OF Inl\_not\_Inr])  
**apply** (rule impI)  
**apply** (drule trans[OF sym])  
**apply** (rule rv1\_Nil)  
**apply** (erule notE[OF Inr\_not\_Inl])

**apply** (rule impI)  
**apply** (rule conjI)  
**apply** (rule impI)  
**apply** (drule ssubst\_mem[OF sym[OF append\_Nil]])  
**apply** (drule subsetD[OF equalityD1[OF Lev1\_Suc]])  
**apply** (drule trans[OF sym])  
**apply** (rule rv1\_Nil)  
**apply** (drule Inl\_inject)  
**apply** (tactic <hyp\_subst\_tac @ {context} 1>)



```

apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (erule notE[OF Inr_not_Inl])
apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (drule Inr_inject)
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (erule imageI)
apply (rule impI)
apply (drule trans[OF sym])
apply (rule rv1_Nil)
apply (erule notE[OF Inr_not_Inl])

apply (rule impI)
apply (drule subsetD[OF equalityD1[OF Lev2_0]])
apply (erule singletonE)
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (rule conjI)
apply (rule impI)
apply (rule conjI)
apply (rule impI)
apply (drule trans[OF sym])
apply (rule rv2_Nil)
apply (erule notE[OF Inl_not_Inr])
apply (rule impI)
apply (drule ssubst_mem[OF sym[OF append_Nil]])
apply (drule subsetD[OF equalityD1[OF Lev2_Suc]])
apply (drule trans[OF sym])
apply (rule rv2_Nil)
apply (drule Inr_inject)
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (drule Inl_inject)
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (erule imageI)
apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (erule notE[OF Inl_not_Inr])

apply (rule impI)
apply (rule conjI)
apply (rule impI)
apply (drule trans[OF sym])
apply (rule rv2_Nil)
apply (erule notE[OF Inl_not_Inr])
apply (rule impI)
apply (drule ssubst_mem[OF sym[OF append_Nil]])
apply (drule subsetD[OF equalityD1[OF Lev2_Suc]])
apply (drule trans[OF sym])
apply (rule rv2_Nil)
apply (drule Inr_inject)
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)

```

```

apply (erule notE[OF Inr_not_Inl])
apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (drule Inr_inject)
apply (tactic <hyp_subst_tac @ {context} 1>)
apply (erule imageI)

```

```

apply (rule allI)+
apply (rule conjI)
apply (rule impI)
apply (drule subsetD[OF equalityD1[OF Lev1_Suc]])
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (tactic <hyp_subst_tac @ {context} 1>)
apply (rule conjI)
apply (rule impI)
apply (drule ssubst_mem[OF sym[OF append_Cons]])
apply (drule subsetD[OF equalityD1[OF Lev1_Suc]])
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (drule Inl_inject)
apply (tactic <dtac @ {context}
(Thm.permute_premis 0 2 (@ {thm tobd_F12_inj} RS iffD1)) 1>)
apply assumption
apply assumption
apply (tactic <hyp_subst_tac @ {context} 1>)
apply (rule conjI)
apply (rule impI)
apply (drule trans[OF sym])
apply (rule trans[OF rv1_Cons])
apply (rule trans[OF arg_cong[OF sum.case(1)]])
apply (erule arg_cong[OF frombd_F12_tobd_F12])
apply (erule allE)+
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
apply assumption
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
apply assumption
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply (erule sym)

apply (rule impI)
apply (drule trans[OF sym])
apply (rule trans[OF rv1_Cons])
apply (rule trans[OF arg_cong[OF sum.case(1)]])
apply (erule arg_cong[OF frombd_F12_tobd_F12])
apply (erule allE)+
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
apply assumption
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
apply assumption
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply (erule sym)

```

```

apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (erule notE[OF Inl_not_Inr])

apply (rule impI)
apply (drule ssubst_mem[OF sym[OF append_Cons]])
apply (drule subsetD[OF equalityD1[OF Lev1_Suc]])
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (drule Inl_inject)
apply (tactic <dtac @ {context}
(Thm.permute_premis 0 2 @ {thm tobd_F12_inj[THEN iffD1]}) 1>)
  apply assumption
  apply assumption
apply (tactic <hyp_subst_tac @ {context} 1>)
apply (rule conjI)
apply (rule impI)
apply (drule trans[OF sym])
  apply (rule trans[OF rv1_Cons])
  apply (rule trans[OF arg_cong[OF sum.case(1)]])
  apply (erule arg_cong[OF frombd_F12_tobd_F12])
apply (erule allE)+
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply (erule sym)

apply (rule impI)
apply (drule trans[OF sym])
apply (rule trans[OF rv1_Cons])
apply (rule trans[OF arg_cong[OF sum.case(1)]])
apply (erule arg_cong[OF frombd_F12_tobd_F12])
apply (erule allE)+
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply (erule sym)

apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (erule notE[OF Inl_not_Inr])

apply (erule CollectE exE conjE)+
apply (tactic <hyp_subst_tac @ {context} 1>)
apply (rule conjI)
apply (rule impI)
apply (drule ssubst_mem[OF sym[OF append_Cons]])
apply (drule subsetD[OF equalityD1[OF Lev1_Suc]])

```

```

apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (erule notE[OF Inr_not_Inl])

apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (drule Inr_inject)
apply (tactic <dtac @ {context}
(Thm.permute_premis 0 2 @ {thm tobd_F13_inj[THEN iffD1]}) 1>)
apply assumption
apply assumption
apply (tactic <hyp_subst_tac @ {context} 1>)
apply (rule conjI)
apply (rule impI)
apply (drule trans[OF sym])
apply (rule trans[OF rv1_Cons])
apply (rule trans[OF arg_cong[OF sum.case(2)]])
apply (erule arg_cong[OF frombd_F13_tobd_F13])
apply (erule allE)+
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
apply assumption
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
apply assumption
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply (erule sym)

apply (rule impI)
apply (drule trans[OF sym])
apply (rule trans[OF rv1_Cons])
apply (rule trans[OF arg_cong[OF sum.case(2)]])
apply (erule arg_cong[OF frombd_F13_tobd_F13])
apply (erule allE)+
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
apply assumption
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
apply assumption
apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply (erule sym)

apply (rule impI)
apply (drule ssubst_mem[OF sym[OF append_Cons]])
apply (drule subsetD[OF equalityD1[OF Lev1_Suc]])
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (erule notE[OF Inr_not_Inl])

apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (drule Inr_inject)
apply (tactic <dtac @ {context}
(Thm.permute_premis 0 2 @ {thm tobd_F13_inj[THEN iffD1]}) 1>)

```

```

  apply assumption
  apply assumption
  apply (tactic <hyp_subst_tac @ {context} 1 >)
  apply (rule conjI)
  apply (rule impI)
  apply (drule trans[OF sym])
  apply (rule trans[OF rv1_Cons])
  apply (rule trans[OF arg_cong[OF sum.case(2)]])
  apply (erule arg_cong[OF frombd_F13_tobd_F13])
  apply (erule allE)+
  apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1 >)
  apply (drule mp)
  apply assumption
  apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1 >)
  apply (drule mp)
  apply assumption
  apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 1) 1 >)
  apply (erule mp)
  apply (erule sym)

```

```

  apply (rule impI)
  apply (drule trans[OF sym])
  apply (rule trans[OF rv1_Cons])
  apply (rule trans[OF arg_cong[OF sum.case(2)]])
  apply (erule arg_cong[OF frombd_F13_tobd_F13])
  apply (erule allE)+
  apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1 >)
  apply (drule mp)
  apply assumption
  apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1 >)
  apply (drule mp)
  apply assumption
  apply (tactic <dtac @ {context} (BNF_Util.mk_conjunctN 2 2) 1 >)
  apply (erule mp)
  apply (erule sym)

```

```

  apply (rule impI)
  apply (drule subsetD[OF equalityD1[OF Lev2_Suc]])
  apply (erule UnE)
  apply (erule CollectE exE conjE)+
  apply (tactic <hyp_subst_tac @ {context} 1 >)
  apply (rule conjI)
  apply (rule impI)
  apply (drule ssubst_mem[OF sym[OF append_Cons]])
  apply (drule subsetD[OF equalityD1[OF Lev2_Suc]])
  apply (erule UnE)
  apply (erule CollectE exE conjE)+
  apply (drule list.inject[THEN iffD1])
  apply (erule conjE)
  apply (drule Inl_inject)
  apply (tactic <dtac @ {context}
  (Thm.permute_premis 0 2 @ {thm tobd_F22_inj[THEN iffD1]}) 1 >)
  apply assumption
  apply assumption
  apply (tactic <hyp_subst_tac @ {context} 1 >)
  apply (rule conjI)
  apply (rule impI)
  apply (drule trans[OF sym])
  apply (rule trans[OF rv2_Cons])
  apply (rule trans[OF arg_cong[OF sum.case(1)]])
  apply (erule arg_cong[OF frombd_F22_tobd_F22])

```

```

apply (erule allE)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply (erule sym)

apply (rule impI)
apply (drule trans[OF sym])
apply (rule trans[OF rv2_Cons])
apply (rule trans[OF arg_cong[OF sum.case(1)]])
apply (erule arg_cong[OF frombd_F22_tobd_F22])
apply (erule allE)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply (erule sym)

apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (erule notE[OF Inl_not_Inr])

apply (rule impI)
apply (drule subst_mem[OF sym[OF append_Cons]])
apply (drule subsetD[OF equalityD1[OF Lev2_Suc]])
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (drule Inl_inject)
apply (tactic <dtac @{context}
(Thm.permute_premis 0 2 @{thm tobd_F22_inj[THEN iffD1]}) 1>)
  apply assumption
  apply assumption
apply (tactic <hyp_subst_tac @{context} 1>)
apply (rule conjI)
apply (rule impI)
apply (drule trans[OF sym])
  apply (rule trans[OF rv2_Cons])
  apply (rule trans[OF arg_cong[OF sum.case(1)]])
  apply (erule arg_cong[OF frombd_F22_tobd_F22])
apply (erule allE)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply (erule sym)

apply (rule impI)

```

```

apply (drule trans[OF sym])
apply (rule trans[OF rv2_Cons])
apply (rule trans[OF arg_cong[OF sum.case(1)]])
apply (erule arg_cong[OF frombd_F22_tobd_F22])
apply (erule allE)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply (erule sym)

```

```

apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (erule notE[OF Inl_not_Inr])

```

```

apply (erule CollectE exE conjE)+
apply (tactic <hyp_subst_tac @{context} 1>)
apply (rule conjI)
apply (rule impI)
apply (drule ssubst_mem[OF sym[OF append_Cons]])
apply (drule subsetD[OF equalityD1[OF Lev2_Suc]])
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (erule notE[OF Inr_not_Inl])

```

```

apply (erule CollectE exE conjE)+
apply (drule list.inject[THEN iffD1])
apply (erule conjE)
apply (drule Inr_inject)
apply (tactic <dtac @{context}
(Thm.permute_premis 0 2 @{thm tobd_F23_inj[THEN iffD1]}) 1>)
apply assumption
apply assumption
apply (tactic <hyp_subst_tac @{context} 1>)
apply (rule conjI)
apply (rule impI)
apply (drule trans[OF sym])
apply (rule trans[OF rv2_Cons])
apply (rule trans[OF arg_cong[OF sum.case(2)]])
apply (erule arg_cong[OF frombd_F23_tobd_F23])
apply (erule allE)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (drule mp)
apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (drule mp)
apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply (erule sym)

```

```

apply (rule impI)
apply (drule trans[OF sym])
apply (rule trans[OF rv2_Cons])
apply (rule trans[OF arg_cong[OF sum.case(2)]])
apply (erule arg_cong[OF frombd_F23_tobd_F23])

```

```

apply (erule allE)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply (erule sym)

```

```

apply (rule impI)
apply (erule ssubst_mem[OF sym[OF append_Cons]])
apply (erule subsetD[OF equalityD1[OF Lev2_Suc]])
apply (erule UnE)
  apply (erule CollectE exE conjE)+
  apply (erule list.inject[THEN iffD1])
  apply (erule conjE)
apply (erule notE[OF Inr_not_Inl])

```

```

apply (erule CollectE exE conjE)+
apply (erule list.inject[THEN iffD1])
apply (erule conjE)
apply (erule Inr_inject)
apply (tactic <dtac @{context}
  (Thm.permute_premis 0 2 @{thm tobd_F23_inj[THEN iffD1]}) 1>)
  apply assumption
  apply assumption
apply (tactic <hyp_subst_tac @{context} 1>)
apply (rule conjI)
apply (rule impI)
apply (erule trans[OF sym])
apply (erule trans[OF rv2_Cons])
apply (erule trans[OF arg_cong[OF sum.case(2)]])
apply (erule arg_cong[OF frombd_F23_tobd_F23])
apply (erule allE)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply (erule sym)

```

```

apply (rule impI)
apply (erule trans[OF sym])
  apply (erule trans[OF rv2_Cons])
  apply (erule trans[OF arg_cong[OF sum.case(2)]])
  apply (erule arg_cong[OF frombd_F23_tobd_F23])
apply (erule allE)+
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
  apply assumption
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply (erule sym)
done

```



```

lemmas Fset_image_Lev' =
  spec[OF spec[OF spec[OF spec[OF spec[OF spec[OF Fset_image_Lev]]]]]]]
lemmas F1set2_image_Lev1 =
  mp[OF conjunct1[OF mp[OF conjunct1[OF mp[OF conjunct1[OF Fset_image_Lev']]]]]]
lemmas F1set2_image_Lev2 =
  mp[OF conjunct1[OF mp[OF conjunct1[OF mp[OF conjunct2[OF Fset_image_Lev']]]]]]
lemmas F1set3_image_Lev1 =
  mp[OF conjunct1[OF mp[OF conjunct2[OF mp[OF conjunct1[OF Fset_image_Lev']]]]]]
lemmas F1set3_image_Lev2 =
  mp[OF conjunct1[OF mp[OF conjunct2[OF mp[OF conjunct2[OF Fset_image_Lev']]]]]]
lemmas F2set2_image_Lev1 =
  mp[OF conjunct2[OF mp[OF conjunct1[OF mp[OF conjunct1[OF Fset_image_Lev']]]]]]
lemmas F2set2_image_Lev2 =
  mp[OF conjunct2[OF mp[OF conjunct1[OF mp[OF conjunct2[OF Fset_image_Lev']]]]]]
lemmas F2set3_image_Lev1 =
  mp[OF conjunct2[OF mp[OF conjunct2[OF mp[OF conjunct1[OF Fset_image_Lev']]]]]]
lemmas F2set3_image_Lev2 =
  mp[OF conjunct2[OF mp[OF conjunct2[OF mp[OF conjunct2[OF Fset_image_Lev']]]]]]

```

**lemma** *mor\_beh*:

*mor UNIV UNIV s1 s2 carT1 carT2 strT1 strT2 (beh1 s1 s2) (beh2 s1 s2)*

```

apply (rule mor_cong)
  apply (rule ext[OF beh1_def])
  apply (rule ext[OF beh2_def])
apply (tactic <rtac @<context> (<@<thm mor_def> RS iffD2) 1>)
apply (rule conjI)
apply (rule conjI)
apply (rule ballI)
apply (rule subsetD[OF equalityD2[OF carT1_def]])
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
apply (rule refl)
apply (rule conjI)
apply (rule conjI)
apply (rule UN_I)
  apply (rule UNIV_I)
apply (rule subsetD)
  apply (rule equalityD2)
  apply (rule Lev1_0)
apply (rule singletonI)

```

```

apply (rule ballI)
apply (erule UN_E)
apply (rule conjI)
apply (rule ballI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (rule rev_mp[OF rv1_Inl_last_impI])
apply (erule exE)
apply (rule iffD2[OF isNode1_def])
apply (rule exI)
apply (rule conjI)
apply (erule trans[OF sum.case_cong_weak])
apply (rule sum.case(1))

```

```

apply (rule conjI)
apply (rule trans[OF F1.set_map(2)])
apply (rule equalityI)
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule SuccI)
apply (rule UN_I[OF UNIV_I])
apply (erule F1set2_Lev1)

```

```

apply assumption
apply assumption
apply (rule subsetI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (rule F1set2_image_Lev1)
  apply assumption
  apply (drule length_Lev1)
  apply (tactic <hyp_subst_tac @ {context} 1>)
  apply (drule length_Lev1')
  apply (erule subsetD[OF equalityD1[OF arg_cong[OF length_append_singleton]]])
apply assumption

```

```

apply (rule trans[OF F1.set_map(3)])
apply (rule equalityI)
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule SuccI)
apply (rule UN_I[OF UNIV_I])
apply (erule F1set3_Lev1)
  apply assumption
apply assumption
apply (rule subsetI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (rule F1set3_image_Lev1)
  apply assumption
  apply (drule length_Lev1)
  apply (tactic <hyp_subst_tac @ {context} 1>)
  apply (drule length_Lev1')
  apply (erule subsetD[OF equalityD1[OF arg_cong[OF length_append_singleton]]])
apply assumption

```

```

apply (rule ballI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (rule rev_mp[OF rv1_Inr_last_impI])
apply (erule exE)
apply (rule iffD2[OF isNode2_def])
apply (rule exI)
apply (rule conjI)
apply (erule trans[OF sum.case_cong_weak])
apply (rule sum.case(2))

```

```

apply (rule conjI)
apply (rule trans[OF F2.set_map(2)])
apply (rule equalityI)
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule SuccI)
apply (rule UN_I[OF UNIV_I])
apply (erule F2set2_Lev1)
  apply assumption
apply assumption
apply (rule subsetI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (erule thin_rl)
apply (erule thin_rl)

```

```

apply (erule thin_rl)
apply (erule thin_rl)
apply (rule F2set2_image_Lev1)
  apply assumption
  apply (drule length_Lev1)
  apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
  apply (drule length_Lev1')
  apply (erule subsetD[OF equalityD1[OF arg_cong[OF length_append_singleton]]])
apply assumption

apply (rule trans[OF F2.set_map(3)])
apply (rule equalityI)
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule SuccI)
apply (rule UN_I[OF UNIV_I])
apply (erule F2set3_Lev1)
  apply assumption
apply assumption
apply (rule subsetI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (rule F2set3_image_Lev1)
  apply assumption
  apply (drule length_Lev1)
  apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
  apply (drule length_Lev1')
  apply (erule subsetD[OF equalityD1[OF arg_cong[OF length_append_singleton]]])
apply assumption

apply (rule iffD2[OF isNode1_def])
apply (rule exI)
apply (rule conjI)
apply (rule trans[OF sum.case_cong_weak])
  apply (rule rv1_Nil)
apply (rule sum.case(1))

apply (rule conjI)
apply (rule trans[OF F1.set_map(2)])
apply (rule equalityI)
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule SuccI)
apply (rule UN_I[OF UNIV_I])
apply (rule F1set2_Lev1)
  apply (rule subsetD[OF equalityD2])
  apply (rule Lev1_0)
  apply (rule singletonI)
apply (rule rv1_Nil)
apply assumption
apply (rule subsetI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (rule F1set2_image_Lev1)
  apply (rule subsetD[OF equalityD2[OF Lev1_0]])
  apply (rule singletonI)
apply (drule length_Lev1')
apply (erule subsetD[OF equalityD1[OF arg_cong[OF
  trans[OF length_append_singleton arg_cong[of __ Suc, OF list.size(3)]]]])
apply (rule rv1_Nil)

```

```

apply (rule trans[OF F1.set_map(3)])
apply (rule equalityI)
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule SuccI)
apply (rule UN_I[OF UNIV_I])
apply (rule F1set3_Lev1)
  apply (rule subsetD[OF equalityD2])
    apply (rule Lev1_0)
      apply (rule singletonI)
        apply (rule rv1_Nil)
          apply assumption
apply (rule subsetI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (rule F1set3_image_Lev1)
  apply (rule subsetD[OF equalityD2[OF Lev1_0]])
    apply (rule singletonI)
      apply (drule length_Lev1')
        apply (erule subsetD[OF equalityD1[OF arg_cong[OF
          trans[OF length_append_singleton arg_cong[of __ Suc, OF list.size(3)]]]])
          apply (rule rv1_Nil)

```

```

apply (rule ballI)
apply (rule subsetD[OF equalityD2[OF carT2_def]])
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
apply (rule refl)
apply (rule conjI)
apply (rule conjI)
apply (rule UN_I)
  apply (rule UNIV_I)
apply (rule subsetD)
  apply (rule equalityD2)
  apply (rule Lev2_0)
apply (rule singletonI)

```

```

apply (rule ballI)
apply (erule UN_E)
apply (rule conjI)
apply (rule ballI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (rule rev_mp[OF rv2_Inl_last_impI])
apply (erule exE)
apply (rule iffD2[OF isNode1_def])
apply (rule exI)
apply (rule conjI)
apply (erule trans[OF sum.case_cong_weak])
apply (rule sum.case(1))

```

```

apply (rule conjI)
apply (rule trans[OF F1.set_map(2)])
apply (rule equalityI)
  apply (rule image_subsetI)
    apply (rule CollectI)
      apply (rule SuccI)
        apply (rule UN_I[OF UNIV_I])
          apply (erule F1set2_Lev2)
            apply assumption
              apply assumption
                apply (rule subsetI)

```

```

apply (erule CollectE SuccD[elim_format] UN_E)+
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (rule F1set2_image_Lev2)
  apply assumption
  apply (drule length_Lev2)
  apply (tactic <hyp_subst_tac @ {context} 1>)
  apply (drule length_Lev2')
  apply (erule subsetD[OF equalityD1[OF arg_cong[OF length_append_singleton]]])
apply assumption

```

```

apply (rule trans[OF F1.set_map(3)])
apply (rule equalityI)
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule SuccI)
apply (rule UN_I[OF UNIV_I])
apply (erule F1set3_Lev2)
  apply assumption
apply assumption
apply (rule subsetI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (rule F1set3_image_Lev2)
  apply assumption
  apply (drule length_Lev2)
  apply (tactic <hyp_subst_tac @ {context} 1>)
  apply (drule length_Lev2')
  apply (erule subsetD[OF equalityD1[OF arg_cong[OF length_append_singleton]]])
apply assumption

```

```

apply (rule ballI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (rule rev_mp[OF rv2_Inr_last_impI])
apply (erule exE)
apply (rule iffD2[OF isNode2_def])
apply (rule exI)
apply (rule conjI)
  apply (erule trans[OF sum.case_cong_weak])
  apply (rule sum.case(2))

```

```

apply (rule conjI)
apply (rule trans[OF F2.set_map(2)])
apply (rule equalityI)
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule SuccI)
apply (rule UN_I[OF UNIV_I])
apply (erule F2set2_Lev2)
  apply assumption
apply assumption
apply (rule subsetI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (rule F2set2_image_Lev2)

```

```

apply assumption
apply (drule length_Lev2)
apply (tactic <hyp_subst_tac @ {context} 1 >)
apply (drule length_Lev2')
apply (erule subsetD[OF equalityD1[OF arg_cong[OF length_append_singleton]]])
apply assumption

```

```

apply (rule trans[OF F2.set_map(3)])
apply (rule equalityI)
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule SuccI)
apply (rule UN_I[OF UNIV_I])
apply (erule F2set3_Lev2)
apply assumption
apply assumption
apply (rule subsetI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (erule thin_rl)
apply (rule F2set3_image_Lev2)
apply assumption
apply (drule length_Lev2)
apply (tactic <hyp_subst_tac @ {context} 1 >)
apply (drule length_Lev2^)
apply (erule subsetD[OF equalityD1[OF arg_cong[OF length_append_singleton]]])
apply assumption

```

```

apply (rule iffD2[OF isNode2_def])
apply (rule exI)
apply (rule conjI)
apply (rule trans[OF sum.case_cong_weak])
apply (rule rv2_Nil)
apply (rule sum.case(2))

```

```

apply (rule conjI)
apply (rule trans[OF F2.set_map(2)])
apply (rule equalityI)
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule SuccI)
apply (rule UN_I[OF UNIV_I])
apply (rule F2set2_Lev2)
apply (rule subsetD[OF equalityD2[OF Lev2_0]])
apply (rule singletonI)
apply (rule rv2_Nil)
apply assumption
apply (rule subsetI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (rule F2set2_image_Lev2)
apply (rule subsetD[OF equalityD2[OF Lev2_0]])
apply (rule singletonI)
apply (drule length_Lev2^)
apply (erule subsetD[OF equalityD1[OF arg_cong[OF
trans[OF length_append_singleton arg_cong[of _ _ Suc, OF list.size(3)]]]]])
apply (rule rv2_Nil)

```

```

apply (rule trans[OF F2.set_map(3)])
apply (rule equalityI)
apply (rule image_subsetI)
apply (rule CollectI)

```

```

apply (rule SuccI)
apply (rule UN_I[OF UNIV_I])
apply (rule F2set3_Lev2)
  apply (rule subsetD[OF equalityD2])
    apply (rule Lev2_0)
    apply (rule singletonI)
  apply (rule rv2_Nil)
apply assumption
apply (rule subsetI)
apply (erule CollectE SuccD[elim_format] UN_E)+
apply (rule F2set3_image_Lev2)
  apply (rule subsetD[OF equalityD2[OF Lev2_0]])
  apply (rule singletonI)
apply (drule length_Lev2')
apply (erule subsetD[OF equalityD1[OF arg_cong[OF
  trans[OF length_append_singleton arg_cong[of _ _ Suc, OF list.size(3)]]]])
apply (rule rv2_Nil)

apply (rule conjI)
apply (rule ballI)
apply (rule sym)
apply (rule trans)
  apply (rule trans[OF fun_cong[OF strT1_def] prod.case])
apply (tactic <CONVERSION (Conv.top_conv
  (K (Conv.try_conv (Conv.rewr_conv (@{thm rv1_Nil} RS eq_reflection)))) @){context} 1>)
apply (rule trans[OF sum.case_cong_weak])
  apply (rule sum.case(1))
apply (rule trans[OF sum.case(1)])
apply (rule trans[OF F1map_comp_id])
apply (rule F1.map_cong0[OF refl])
apply (rule trans)
  apply (rule o_apply)
apply (rule iffD2)
  apply (rule prod.inject)
apply (rule conjI)
  apply (rule trans)
  apply (rule Shift_def)

apply (rule equalityI)
apply (rule subsetI)
apply (erule thin_rl)
apply (erule CollectE UN_E)+
apply (drule length_Lev1')
apply (drule asm_rl)
apply (erule thin_rl)
apply (drule rev_subsetD[OF _ equalityD1])
  apply (rule trans[OF arg_cong[OF length_Cons]])
  apply (rule Lev1_Suc)
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (tactic <dtac @){context} @){thm list.inject[THEN iffD1]} 1>)
apply (erule conjE)
apply (drule Inl_inject)
apply (tactic <dtac @){context}
(Thm.permute_premis 0 2 @){thm tobd_F12_inj[THEN iffD1]} 1>)
  apply assumption
  apply assumption
  apply (tactic <hyp_subst_tac @){context} 1>)
  apply (erule UN_I[OF UNIV_I])
apply (erule CollectE exE conjE)+

```

```

apply (tactic <dtac @{\context} @{\thm list.inject[THEN iffD1]} 1>)
apply (erule conjE)
apply (erule notE[OF Inl_not_Inr])

apply (rule UN_least)
apply (rule subsetI)
apply (rule CollectI)
apply (rule UN_I[OF UNIV_I])
apply (rule subsetD[OF equalityD2])
  apply (rule Lev1_Suc)
apply (rule UnI1)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply assumption

apply (rule trans)
  apply (rule shift_def)
apply (rule iffD2)
  apply (rule fun_eq_iff)
apply (rule allI)
apply (tactic <CONVERSION (Conv.top_conv
  (K (Conv.try_conv (Conv.rewr_conv (@{\thm rv1_Cons} RS eq_reflection)))) @{\context}) 1>)
apply (rule sum.case_cong_weak)
apply (rule trans[OF sum.case(1)])
apply (drule frombd_F12_tobd_F12)
apply (erule arg_cong)

apply (rule trans)
apply (rule o_apply)
apply (rule iffD2)
apply (rule prod.inject)
apply (rule conjI)
apply (rule trans)
  apply (rule Shift_def)

apply (rule equalityI)
apply (rule subsetI)
apply (erule thin_rl)
apply (erule CollectE UN_E)+
apply (drule length_Lev1^)
apply (drule asm_rl)
apply (erule thin_rl)
apply (drule rev_subsetD[OF _ equalityD1])
  apply (rule trans[OF arg_cong[OF length_Cons]])
  apply (rule Lev1_Suc)
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (tactic <dtac @{\context} @{\thm list.inject[THEN iffD1]} 1>)
apply (erule conjE)
apply (erule notE[OF Inr_not_Inl])
apply (erule CollectE exE conjE)+
apply (tactic <dtac @{\context} @{\thm list.inject[THEN iffD1]} 1>)
apply (erule conjE)
apply (drule Inr_inject)
apply (tactic <dtac @{\context}
(Thm.permute_premis 0 2 @{\thm tobd_F13_inj[THEN iffD1]}) 1>)
  apply assumption
  apply assumption

```



```

apply (tactic <hyp_subst_tac @ {context} 1 >)
apply (erule UN_I[OF UNIV_I])

```

```

apply (rule UN_least)
apply (rule subsetI)
apply (rule CollectI)
apply (rule UN_I[OF UNIV_I])
apply (rule subsetD[OF equalityD2])
  apply (rule Lev1_Suc)
apply (rule UnI2)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply assumption

```

```

apply (rule trans)
apply (rule shift_def)
apply (rule iffD2)
  apply (rule fun_eq_iff)
apply (rule allI)
apply (rule sum.case_cong_weak)
apply (rule trans[OF rv1_Cons])
apply (rule trans[OF sum.case(2)])
apply (erule arg_cong[OF frombd_F13_tobd_F13])

```

```

apply (rule ballI)
apply (rule sym)
apply (rule trans)
  apply (rule trans[OF fun_cong[OF strT2_def] prod.case])
apply (rule trans[OF sum.case_cong_weak[OF trans[OF sum.case_cong_weak]]])
  apply (rule rv2_Nil)
  apply (rule sum.case(2))
apply (rule trans[OF sum.case(2)])
apply (rule trans[OF F2map_comp_id])
apply (rule F2.map_cong0[OF refl])
apply (rule trans)
  apply (rule o_apply)
apply (rule iffD2)
apply (rule prod.inject)
apply (rule conjI)
apply (rule trans)
  apply (rule Shift_def)

```

```

apply (rule equalityI)
apply (rule subsetI)
apply (erule thin_rl)
apply (erule CollectE UN_E)+
apply (drule length_Lev2^)
apply (drule asm_rl)
apply (erule thin_rl)
apply (drule rev_subsetD[OF _ equalityD1])
apply (rule trans[OF arg_cong[OF length_Cons]])
apply (rule Lev2_Suc)
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (tactic <dtac @ {context} @ {thm list.inject[THEN iffD1]} 1 >)
apply (erule conjE)

```

```

apply (drule Inl_inject)
apply (tactic <dtac @{context}
(Thm.permute_premis 0 2 @{thm tobd_F22_inj[THEN iffD1]}) 1>)
  apply assumption
  apply assumption
  apply (tactic <hyp_subst_tac @{context} 1>)
  apply (erule UN_I[OF UNIV_I])
  apply (erule CollectE exE conjE)+
  apply (tactic <dtac @{context} @{thm list.inject[THEN iffD1]}) 1>)
  apply (erule conjE)
  apply (erule notE[OF Inl_not_Inr])

```

```

apply (rule UN_least)
apply (rule subsetI)
apply (rule CollectI)
apply (rule UN_I[OF UNIV_I])
apply (rule subsetD[OF equalityD2])
  apply (rule Lev2_Suc)
apply (rule UnI1)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply assumption

```

```

apply (rule trans)
apply (rule shift_def)
apply (rule iffD2)
apply (rule fun_eq_iff)
apply (rule allI)
apply (rule sum.case_cong_weak)
apply (rule trans[OF rv2_Cons])
apply (rule trans[OF arg_cong[OF sum.case(1)]])
apply (erule arg_cong[OF frombd_F22_tobd_F22])

```

```

apply (rule trans)
apply (rule o_apply)
apply (rule iffD2)
apply (rule prod.inject)
apply (rule conjI)
apply (rule trans)
apply (rule Shift_def)

```

```

apply (rule equalityI)
apply (rule subsetI)
apply (erule thin_rl)
apply (erule CollectE UN_E)+
apply (drule length_Lev2')
apply (drule asm_rl)
apply (erule thin_rl)
apply (drule rev_subsetD[OF equalityD1])
  apply (rule trans[OF arg_cong[OF length_Cons]])
  apply (rule Lev2_Suc)
apply (erule UnE)
apply (erule CollectE exE conjE)+
apply (tactic <dtac @{context} @{thm list.inject[THEN iffD1]}) 1>)
apply (erule conjE)
apply (erule notE[OF Inr_not_Inl])
apply (erule CollectE exE conjE)+
apply (tactic <dtac @{context} @{thm list.inject[THEN iffD1]}) 1>)

```

```

apply (erule conjE)
apply (drule Inr_inject)
apply (tactic `dtac @ {context}`)
(Thm.permute_premis 0 2 @ {thm tobd_F23_inj [THEN iffD1]}) 1 >
  apply assumption
  apply assumption
apply (tactic `hyp_subst_tac @ {context}`) 1 >
apply (erule UN_I [OF UNIV_I])

```

```

apply (rule UN_least)
apply (rule subsetI)
apply (rule CollectI)
apply (rule UN_I [OF UNIV_I])
apply (rule subsetD [OF equalityD2])
  apply (rule Lev2_Suc)
apply (rule UnI2)
apply (rule CollectI)
apply (rule exI)+
apply (rule conjI)
  apply (rule refl)
apply (erule conjI)
apply assumption

```

```

apply (rule trans)
  apply (rule shift_def)
apply (rule iffD2)
  apply (rule fun_eq_iff)
apply (rule allI)

```

```

apply (rule sum.case_cong_weak)
apply (rule trans [OF rv2_Cons])
apply (rule trans [OF arg_cong [OF sum.case(2)]])
apply (erule arg_cong [OF frombd_F23_tobd_F23])
done

```

## 2.6 Quotient Coalgebra

**abbreviation** *car\_final1* **where**  
*car\_final1*  $\equiv$  *carT1* // (*lsbis1 carT1 carT2 strT1 strT2*)

**abbreviation** *car\_final2* **where**  
*car\_final2*  $\equiv$  *carT2* // (*lsbis2 carT1 carT2 strT1 strT2*)

**abbreviation** *str\_final1* **where**  
*str\_final1*  $\equiv$  *univ* (F1map *id*  
 (*Equiv\_Relations.proj* (*lsbis1 carT1 carT2 strT1 strT2*))  
 (*Equiv\_Relations.proj* (*lsbis2 carT1 carT2 strT1 strT2*)) *o strT1*)

**abbreviation** *str\_final2* **where**  
*str\_final2*  $\equiv$  *univ* (F2map *id*  
 (*Equiv\_Relations.proj* (*lsbis1 carT1 carT2 strT1 strT2*))  
 (*Equiv\_Relations.proj* (*lsbis2 carT1 carT2 strT1 strT2*)) *o strT2*)

**lemma** *congruent\_strQ1*: *congruent* (*lsbis1 carT1 carT2 strT1 strT2* :: 'a carrier rel)  
 (F1map *id* (*Equiv\_Relations.proj* (*lsbis1 carT1 carT2 strT1 strT2* :: 'a carrier rel))  
 (*Equiv\_Relations.proj* (*lsbis2 carT1 carT2 strT1 strT2* :: 'a carrier rel)) *o strT1*)

```

apply (rule congruentI)
apply (drule lsbisE1)
apply (erule bexE conjE CollectE)+
apply (rule trans [OF o_apply])
apply (erule trans [OF arg_cong [OF sym]])
apply (rule trans [OF F1map_comp_id])
apply (rule trans [OF F1.map_cong0])
  apply (rule refl)
apply (rule equiv_proj)
apply (rule equiv_lsbis1)

```

```

    apply (rule coalg_T)
    apply (erule subsetD)
    apply assumption
    apply (rule equiv_proj)
    apply (rule equiv_lsbis2)
    apply (rule coalg_T)
    apply (erule subsetD)
    apply assumption
    apply (rule sym)
    apply (rule trans[OF o_apply])
    apply (erule trans[OF arg_cong[OF sym]])
    apply (rule F1map_comp_id)
    done

```

```

lemma congruent_strQ2: congruent (lsbis2 carT1 carT2 strT1 strT2 :: 'a carrier rel)
(F2map id (Equiv_Relations.proj (lsbis1 carT1 carT2 strT1 strT2 :: 'a carrier rel))
(Equiv_Relations.proj (lsbis2 carT1 carT2 strT1 strT2 :: 'a carrier rel)) o strT2)
apply (rule congruentI)
apply (drule lsbisE2)
apply (erule bexE conjE CollectE)+
apply (rule trans[OF o_apply])
apply (erule trans[OF arg_cong[OF sym]])
apply (rule trans[OF F2map_comp_id])
apply (rule trans[OF F2.map_cong0])
  apply (rule refl)
  apply (rule equiv_proj)
  apply (rule equiv_lsbis1)
  apply (rule coalg_T)
  apply (erule subsetD)
  apply assumption
apply (rule equiv_proj)
apply (rule equiv_lsbis2)
apply (rule coalg_T)
apply (erule subsetD)
apply assumption
apply (rule sym)
apply (rule trans[OF o_apply])
apply (erule trans[OF arg_cong[OF sym]])
apply (rule F2map_comp_id)
done

```

```

lemma coalg_final:
coalg car_final1 car_final2 str_final1 str_final2
apply (tactic ‹rtac @_{context} (@_{thm coalg_def} RS iffD2) 1›)
apply (rule conjI)
apply (rule univ_preserves)
  apply (rule equiv_lsbis1)
  apply (rule coalg_T)
apply (rule congruent_strQ1)
apply (rule ballI)
apply (rule ssubst_mem)
apply (rule o_apply)
apply (rule CollectI)
apply (rule conjI)
apply (rule subset_UNIV)
apply (rule conjI)
apply (rule ord_eq_le_trans[OF F1.set_map(2)])
apply (rule image_subsetI)
apply (rule iffD2)
  apply (rule proj_in_iff)
  apply (rule equiv_lsbis1[OF coalg_T])
apply (erule rev_subsetD)
apply (erule coalg_F1set2[OF coalg_T])

```

```

apply (rule ord_eq_le_trans[OF F1.set_map(3)])
apply (rule image_subsetI)
apply (rule iffD2)
  apply (rule proj_in_iff)
  apply (rule equiv_lsbis2[OF coalg_T])
apply (erule rev_subsetD)
apply (erule coalg_F1set3[OF coalg_T])

```

```

apply (rule univ_preserves)
  apply (rule equiv_lsbis2)
  apply (rule coalg_T)
apply (rule congruent_strQ2)
apply (rule ballI)
apply (tactic ⟨stac @ {context} @ {thm o_apply} 1⟩)
apply (rule CollectI)
apply (rule conjI)
  apply (rule subset_UNIV)
apply (rule conjI)
apply (rule ord_eq_le_trans[OF F2.set_map(2)])
apply (rule image_subsetI)
apply (rule iffD2)
  apply (rule proj_in_iff)
  apply (rule equiv_lsbis1[OF coalg_T])
apply (erule rev_subsetD)
apply (erule coalg_F2set2[OF coalg_T])
apply (rule ord_eq_le_trans[OF F2.set_map(3)])
apply (rule image_subsetI)
apply (rule iffD2)
  apply (rule proj_in_iff)
  apply (rule equiv_lsbis2[OF coalg_T])
apply (erule rev_subsetD)
apply (erule coalg_F2set3[OF coalg_T])
done

```

**lemma** mor\_T\_final:

```

mor carT1 carT2 strT1 strT2 car_final1 car_final2 str_final1 str_final2
(Equiv_Relations.proj (lsbis1 carT1 carT2 strT1 strT2))
(Equiv_Relations.proj (lsbis2 carT1 carT2 strT1 strT2))
apply (tactic ⟨rtac @ {context} (@ {thm mor_def} RS iffD2) 1⟩)
apply (rule conjI)
apply (rule conjI)
  apply (rule ballI)
  apply (rule iffD2)
  apply (rule proj_in_iff)
  apply (rule equiv_lsbis1[OF coalg_T])
apply assumption
apply (rule ballI)
apply (rule iffD2)
  apply (rule proj_in_iff)
  apply (rule equiv_lsbis2[OF coalg_T])
apply assumption

apply (rule conjI)
apply (rule ballI)
apply (rule sym)
apply (rule trans)
apply (rule univ_commute)
  apply (rule equiv_lsbis1[OF coalg_T])
  apply (rule congruent_strQ1)
apply assumption
apply (rule o_apply)

apply (rule ballI)

```

```

apply (rule sym)
apply (rule trans)
apply (rule univ_commute)
  apply (rule equiv_lsbis2[OF coalg_T])
  apply (rule congruent_strQ2)
apply assumption
apply (rule o_apply)
done

```

```

lemmas mor_final = mor_comp[OF mor_beh mor_T_final]
lemmas in_car_final1 = mor_image1'[OF mor_final UNIV_I]
lemmas in_car_final2 = mor_image2'[OF mor_final UNIV_I]

```

```

typedef (overloaded) 'a JF1 = car_final1 :: 'a carrier set set
by (rule exI) (rule in_car_final1)

```

```

typedef (overloaded) 'a JF2 = car_final2 :: 'a carrier set set
by (rule exI) (rule in_car_final2)

```

**definition** dtor1 **where**

```

  dtor1 x = F1map id Abs_JF1 Abs_JF2 (str_final1 (Rep_JF1 x))

```

**definition** dtor2 **where**

```

  dtor2 x = F2map id Abs_JF1 Abs_JF2 (str_final2 (Rep_JF2 x))

```

**lemma** mor\_Rep\_JF: mor UNIV UNIV dtor1 dtor2

```

  car_final1 car_final2 str_final1 str_final2
  Rep_JF1 Rep_JF2

```

**unfolding** mor\_def dtor1\_def dtor2\_def

```

apply (rule conjI)
apply (rule conjI)
  apply (rule ballI)
  apply (rule Rep_JF1)
apply (rule ballI)
apply (rule Rep_JF2)

```

```

apply (rule conjI)
apply (rule ballI)
apply (rule trans[OF F1map_comp_id])
apply (rule F1map_congL)
apply (rule ballI)
apply (rule trans[OF o_apply])
apply (rule Abs_JF1_inverse)
apply (erule rev_subsetD)
apply (rule coalg_F1set2)
  apply (rule coalg_final)
apply (rule Rep_JF1)
apply (rule ballI)
apply (rule trans[OF o_apply])
apply (rule Abs_JF2_inverse)
apply (erule rev_subsetD)
apply (rule coalg_F1set3)
  apply (rule coalg_final)
apply (rule Rep_JF1)

```

```

apply (rule ballI)
apply (rule trans[OF F2map_comp_id])
apply (rule F2map_congL)
apply (rule ballI)
apply (rule trans[OF o_apply])

```

```

apply (rule Abs_JF1_inverse)
apply (erule rev_subsetD)
apply (rule coalg_F2set2)
  apply (rule coalg_final)
apply (rule Rep_JF2)
apply (rule ballI)
apply (rule trans[OF o_apply])
apply (rule Abs_JF2_inverse)
apply (erule rev_subsetD)
apply (rule coalg_F2set3)
  apply (rule coalg_final)
apply (rule Rep_JF2)
done

```

**lemma** *mor\_Abs\_JF*: *mor car\_final1 car\_final2 str\_final1 str\_final2*

*UNIV UNIV dtor1 dtor2 Abs\_JF1 Abs\_JF2*

**unfolding** *mor\_def dtor1\_def dtor2\_def*

```

apply (rule conjI)
apply (rule conjI)
  apply (rule ballI)
  apply (rule UNIV_I)
apply (rule ballI)
apply (rule UNIV_I)

```

```

apply (rule conjI)
apply (rule ballI)
apply (erule sym[OF arg_cong[OF Abs_JF1_inverse]])
apply (rule ballI)
apply (erule sym[OF arg_cong[OF Abs_JF2_inverse]])
done

```

**definition** *unfold1* **where**

*unfold1 s1 s2 x =*

*Abs\_JF1 ((Equiv\_Relations.proj (lsbis1 carT1 carT2 strT1 strT2) o beh1 s1 s2) x)*

**definition** *unfold2* **where**

*unfold2 s1 s2 x =*

*Abs\_JF2 ((Equiv\_Relations.proj (lsbis2 carT1 carT2 strT1 strT2) o beh2 s1 s2) x)*

**lemma** *mor\_unfold*:

*mor UNIV UNIV s1 s2 UNIV UNIV dtor1 dtor2 (unfold1 s1 s2) (unfold2 s1 s2)*

```

apply (rule iffD2)
apply (rule mor_UNIV)
apply (rule conjI)
apply (rule ext)
apply (rule sym[OF trans[OF o_apply]])
apply (rule trans[OF dtor1_def])
apply (rule trans[OF arg_cong[OF unfold1_def]])
apply (rule trans[OF arg_cong[OF Abs_JF1_inverse[OF in_car_final1]])
apply (rule trans[OF arg_cong[OF sym[OF morE1[OF mor_final UNIV_I]])])
apply (rule trans[OF F1map_comp_id])
apply (rule sym[OF trans[OF o_apply]])
apply (rule F1.map_cong0)
  apply (rule refl)
apply (rule trans[OF unfold1_def])
apply (rule sym[OF o_apply])
apply (rule trans[OF unfold2_def])
apply (rule sym[OF o_apply])

```

```

apply (rule ext)
apply (rule sym[OF trans[OF o_apply]])
apply (rule trans[OF dtor2_def])
apply (rule trans[OF arg_cong[OF unfold2_def]])
apply (rule trans[OF arg_cong[OF Abs_JF2_inverse[OF in_car_final2]])

```

```

apply (rule trans[OF arg_cong[OF sym[OF morE2[OF mor_final UNIV_I]]]])
apply (rule trans[OF F2map_comp_id])
apply (rule sym[OF trans[OF o_apply]])
apply (rule F2.map_cong0)
  apply (rule refl)
  apply (rule trans[OF unfold1_def])
  apply (rule sym[OF o_apply])
apply (rule trans[OF unfold2_def])
apply (rule sym[OF o_apply])
done

```

**lemmas** *unfold1* = sym[OF morE1[OF mor\_unfold UNIV\_I]]

**lemmas** *unfold2* = sym[OF morE2[OF mor\_unfold UNIV\_I]]

**lemma** *JF\_cind*: sbis UNIV UNIV dtor1 dtor2 R1 R2  $\implies R1 \subseteq Id \wedge R2 \subseteq Id$

```

apply (rule rev_mp)
apply (tactic ⟨forward_tac @ {context} @ {thms bis_def[THEN iffD1]} 1⟩)
apply (erule conjE)+
apply (rule bis_cong)
  apply (rule bis_Comp)
  apply (rule bis_converse)
  apply (rule bis_Gr)
  apply (rule tcoalg)
apply (rule mor_Rep_JF)
apply (rule bis_Comp)
apply assumption
apply (rule bis_Gr)
apply (rule tcoalg)
apply (rule mor_Rep_JF)
apply (erule relImage_Gr)
apply (erule relImage_Gr)

```

```

apply (rule impI)
apply (rule rev_mp)
apply (rule bis_cong)
  apply (rule bis_Comp)
  apply (rule bis_Gr)
  apply (rule coalg_T)
  apply (rule mor_T_final)
apply (rule bis_Comp)
apply (rule sbis_lsbis)
apply (rule bis_converse)
apply (rule bis_Gr)
apply (rule coalg_T)
apply (rule mor_T_final)
apply (rule relInvImage_Gr[OF lsbis1_incl])
apply (rule relInvImage_Gr[OF lsbis2_incl])

```

```

apply (rule impI)
apply (rule conjI)
apply (rule subset_trans)
apply (rule relInvImage_UNIV_relImage)
apply (rule subset_trans)
apply (rule relInvImage_mono)
apply (rule subset_trans)
apply (erule incl_lsbis1)
apply (rule ord_eq_le_trans)
apply (rule sym[OF relImage_relInvImage])
apply (rule xt1(3))
apply (rule Sigma_cong)
  apply (rule proj_image)
  apply (rule proj_image)
apply (rule lsbis1_incl)

```



```

apply (rule subset_trans)
apply (rule relImage_mono)
apply (rule incl_lsbis1)
apply assumption
apply (rule relImage_proj)
apply (rule equiv_lsbis1[OF coalg_T])
apply (rule relInvImage_Id_on)
apply (rule Rep_JF1_inject)

apply (rule subset_trans)
apply (rule relInvImage_UNIV_relImage)
apply (rule subset_trans)
apply (rule relInvImage_mono)
apply (rule subset_trans)
apply (erule incl_lsbis2)
apply (rule ord_eq_le_trans)
apply (rule sym[OF relImage_relInvImage])
apply (rule xt1(3))
apply (rule Sigma_cong)
apply (rule proj_image)
apply (rule proj_image)
apply (rule lsbis2_incl)
apply (rule subset_trans)
apply (rule relImage_mono)
apply (rule incl_lsbis2)
apply assumption
apply (rule relImage_proj)
apply (rule equiv_lsbis2[OF coalg_T])
apply (rule relInvImage_Id_on)
apply (rule Rep_JF2_inject)
done

lemmas JF_cind1 = conjunct1[OF JF_cind]
lemmas JF_cind2 = conjunct2[OF JF_cind]

lemma unfold_unique_mor:
  mor UNIV UNIV s1 s2 UNIV UNIV dtor1 dtor2 f1 f2  $\implies$ 
  f1 = unfold1 s1 s2  $\wedge$  f2 = unfold2 s1 s2
apply (rule conjI)
apply (rule ext)
apply (erule IdD[OF subsetD[OF JF_cind1[OF bis_image2[OF tcoalg _ tcoalg]]]])
apply (rule mor_comp[OF mor_final mor_Abs_JF])
apply (rule image2_eqI)
apply (rule refl)
apply (rule trans[OF arg_cong[OF unfold1_def]])
apply (rule sym[OF o_apply])
apply (rule UNIV_I)

apply (rule ext)
apply (erule IdD[OF subsetD[OF JF_cind2[OF bis_image2[OF tcoalg _ tcoalg]]]])
apply (rule mor_comp[OF mor_final mor_Abs_JF])
apply (rule image2_eqI)
apply (rule refl)
apply (rule trans[OF arg_cong[OF unfold2_def]])
apply (rule sym[OF o_apply])
apply (rule UNIV_I)
done

lemmas unfold_unique = unfold_unique_mor[OF iffD2[OF mor_UNIV], OF conjI]
lemmas unfold1_dtor = sym[OF conjunct1[OF unfold_unique_mor[OF mor_id]]]
lemmas unfold2_dtor = sym[OF conjunct2[OF unfold_unique_mor[OF mor_id]]]

lemmas unfold1_o_dtor1 =

```

$trans[OF\ conjunct1[OF\ unfold\_unique\_mor[OF\ mor\_comp[OF\ mor\_str\ mor\_unfold]]] unfold1\_dctor]$   
**lemmas**  $unfold2\_o\_dctor2 =$   
 $trans[OF\ conjunct2[OF\ unfold\_unique\_mor[OF\ mor\_comp[OF\ mor\_str\ mor\_unfold]]] unfold2\_dctor]$

**definition**  $ctor1$  **where**  $ctor1 = unfold1 (F1map\ id\ dctor1\ dctor2) (F2map\ id\ dctor1\ dctor2)$

**definition**  $ctor2$  **where**  $ctor2 = unfold2 (F1map\ id\ dctor1\ dctor2) (F2map\ id\ dctor1\ dctor2)$

**lemma**  $ctor1\_o\_dctor1$ :  
 $ctor1\ o\ dctor1 = id$   
**unfolding**  $ctor1\_def$   
**apply**  $(rule\ unfold1\_o\_dctor1)$   
**done**

**lemma**  $ctor2\_o\_dctor2$ :  
 $ctor2\ o\ dctor2 = id$   
**unfolding**  $ctor2\_def$   
**apply**  $(rule\ unfold2\_o\_dctor2)$   
**done**

**lemma**  $dctor1\_o\_ctor1$ :  
 $dctor1\ o\ ctor1 = id$   
**unfolding**  $ctor1\_def$   
**apply**  $(rule\ ext)$   
**apply**  $(rule\ trans[OF\ o\_apply])$   
**apply**  $(rule\ trans[OF\ unfold1])$   
**apply**  $(rule\ trans[OF\ F1map\_comp\_id])$   
**apply**  $(rule\ trans[OF\ F1map\_congL])$   
**apply**  $(rule\ ballI)$   
**apply**  $(rule\ trans[OF\ fun\_cong[OF\ unfold1\_o\_dctor1\ id\_apply])$   
**apply**  $(rule\ ballI)$   
**apply**  $(rule\ trans[OF\ fun\_cong[OF\ unfold2\_o\_dctor2\ id\_apply])$   
**apply**  $(rule\ sym[OF\ id\_apply])$   
**done**

**lemma**  $dctor2\_o\_ctor2$ :  
 $dctor2\ o\ ctor2 = id$   
**unfolding**  $ctor2\_def$   
**apply**  $(rule\ ext)$   
**apply**  $(rule\ trans[OF\ o\_apply])$   
**apply**  $(rule\ trans[OF\ unfold2])$   
**apply**  $(rule\ trans[OF\ F2map\_comp\_id])$   
**apply**  $(rule\ trans[OF\ F2map\_congL])$   
**apply**  $(rule\ ballI)$   
**apply**  $(rule\ trans[OF\ fun\_cong[OF\ unfold1\_o\_dctor1\ id\_apply])$   
**apply**  $(rule\ ballI)$   
**apply**  $(rule\ trans[OF\ fun\_cong[OF\ unfold2\_o\_dctor2\ id\_apply])$   
**apply**  $(rule\ sym[OF\ id\_apply])$   
**done**

**lemmas**  $dctor1\_ctor1 = pointfree\_idE[OF\ dctor1\_o\_ctor1]$

**lemmas**  $dctor2\_ctor2 = pointfree\_idE[OF\ dctor2\_o\_ctor2]$

**lemmas**  $ctor1\_dctor1 = pointfree\_idE[OF\ ctor1\_o\_dctor1]$

**lemmas**  $ctor2\_dctor2 = pointfree\_idE[OF\ ctor2\_o\_dctor2]$

**lemmas**  $bij\_dctor1 = o\_bij[OF\ ctor1\_o\_dctor1\ dctor1\_o\_ctor1]$

**lemmas**  $inj\_dctor1 = bij\_is\_inj[OF\ bij\_dctor1]$

**lemmas**  $surj\_dctor1 = bij\_is\_surj[OF\ bij\_dctor1]$

**lemmas**  $dctor1\_nchotomy = surjD[OF\ surj\_dctor1]$

**lemmas**  $dctor1\_diff = inj\_eq[OF\ inj\_dctor1]$

**lemmas**  $dctor1\_cases = exE[OF\ dctor1\_nchotomy]$

**lemmas**  $bij\_dctor2 = o\_bij[OF\ ctor2\_o\_dctor2\ dctor2\_o\_ctor2]$

**lemmas**  $inj\_dctor2 = bij\_is\_inj[OF\ bij\_dctor2]$

**lemmas** *surj\_dtor2* = *bij\_is\_surj*[*OF* *bij\_dtor2*]  
**lemmas** *dtor2\_nchotomy* = *surjD*[*OF* *surj\_dtor2*]  
**lemmas** *dtor2\_diff* = *inj\_eq*[*OF* *inj\_dtor2*]  
**lemmas** *dtor2\_cases* = *exE*[*OF* *dtor2\_nchotomy*]

**lemmas** *bij\_ctor1* = *o\_bij*[*OF* *dtor1\_o\_ctor1* *ctor1\_o\_dtor1*]  
**lemmas** *inj\_ctor1* = *bij\_is\_inj*[*OF* *bij\_ctor1*]  
**lemmas** *surj\_ctor1* = *bij\_is\_surj*[*OF* *bij\_ctor1*]  
**lemmas** *ctor1\_nchotomy* = *surjD*[*OF* *surj\_ctor1*]  
**lemmas** *ctor1\_diff* = *inj\_eq*[*OF* *inj\_ctor1*]  
**lemmas** *ctor1\_cases* = *exE*[*OF* *ctor1\_nchotomy*]  
**lemmas** *bij\_ctor2* = *o\_bij*[*OF* *dtor2\_o\_ctor2* *ctor2\_o\_dtor2*]  
**lemmas** *inj\_ctor2* = *bij\_is\_inj*[*OF* *bij\_ctor2*]  
**lemmas** *surj\_ctor2* = *bij\_is\_surj*[*OF* *bij\_ctor2*]  
**lemmas** *ctor2\_nchotomy* = *surjD*[*OF* *surj\_ctor2*]  
**lemmas** *ctor2\_diff* = *inj\_eq*[*OF* *inj\_ctor2*]  
**lemmas** *ctor2\_cases* = *exE*[*OF* *ctor2\_nchotomy*]

**lemmas** *ctor1\_unfold1* = *iffD1*[*OF* *dtor1\_diff* *trans*[*OF* *unfold1* *sym*[*OF* *dtor1\_ctor1*]]]  
**lemmas** *ctor2\_unfold2* = *iffD1*[*OF* *dtor2\_diff* *trans*[*OF* *unfold2* *sym*[*OF* *dtor2\_ctor2*]]]

**definition** *corec1* **where** *corec1* *s1* *s2* =  
*unfold1* (*case\_sum* (*F1map* *id* *Inl* *Inl* *o* *dtor1*) *s1*)  
(*case\_sum* (*F2map* *id* *Inl* *Inl* *o* *dtor2*) *s2*) *o* *Inr*

**definition** *corec2* **where** *corec2* *s1* *s2* =  
*unfold2* (*case\_sum* (*F1map* *id* *Inl* *Inl* *o* *dtor1*) *s1*)  
(*case\_sum* (*F2map* *id* *Inl* *Inl* *o* *dtor2*) *s2*) *o* *Inr*

**lemma** *dtor1\_o\_unfold1*: *dtor1* *o* *unfold1* *s1* *s2* = *F1map* *id* (*unfold1* *s1* *s2*) (*unfold2* *s1* *s2*) *o* *s1*  
**by** (*tactic*  $\langle$ *rtac*  $\@$ {*context*} (*BNF\_Tactics.mk\_pointfree2*  $\@$ {*context*}  $\@$ {*thm* *unfold1*})  $1$  $\rangle$ )

**lemma** *dtor2\_o\_unfold2*: *dtor2* *o* *unfold2* *s1* *s2* = *F2map* *id* (*unfold1* *s1* *s2*) (*unfold2* *s1* *s2*) *o* *s2*  
**by** (*tactic*  $\langle$ *rtac*  $\@$ {*context*} (*BNF\_Tactics.mk\_pointfree2*  $\@$ {*context*}  $\@$ {*thm* *unfold2*})  $1$  $\rangle$ )

**lemma** *corec1\_Inl\_sum*:  
*unfold1* (*case\_sum* (*F1map* *id* *Inl* *Inl* *o* *dtor1*) *s1*) (*case\_sum* (*F2map* *id* *Inl* *Inl* *o* *dtor2*) *s2*) *o* *Inl* = *id*  
**apply** (*rule* *trans*[*OF* *conjunct1*[*OF* *unfold\_unique*] *unfold1\_dtor*])  
**apply** (*rule* *trans*[*OF* *arg\_cong2*[*of* \_ \_ \_ \_ (*o*), *OF* *F1.map\_comp0*[*of* *id*, *unfolded\_id\_o*] *refl*]])  
**apply** (*rule* *sym*[*OF* *trans*[*OF* *o\_assoc*]])  
**apply** (*rule* *trans*[*OF* *arg\_cong2*[*of* \_ \_ \_ \_ (*o*), *OF* *dtor1\_o\_unfold1* *refl*]])  
**apply** (*rule* *box\_equals*[*OF* \_ *o\_assoc* *o\_assoc*])  
**apply** (*rule* *arg\_cong2*[*of* \_ \_ \_ \_ (*o*), *OF* *refl* *case\_sum\_o\_inj*( $1$ )])  
**apply** (*rule* *trans*[*OF* *arg\_cong2*[*of* \_ \_ \_ \_ (*o*), *OF* *F2.map\_comp0*[*of* *id*, *unfolded\_id\_o*] *refl*]])  
**apply** (*rule* *sym*[*OF* *trans*[*OF* *o\_assoc*]])  
**apply** (*rule* *trans*[*OF* *arg\_cong2*[*of* \_ \_ \_ \_ (*o*), *OF* *dtor2\_o\_unfold2* *refl*]])  
**apply** (*rule* *box\_equals*[*OF* \_ *o\_assoc* *o\_assoc*])  
**apply** (*rule* *arg\_cong2*[*of* \_ \_ \_ \_ (*o*), *OF* *refl* *case\_sum\_o\_inj*( $1$ )])  
**done**

**lemma** *corec2\_Inl\_sum*:  
*unfold2* (*case\_sum* (*F1map* *id* *Inl* *Inl* *o* *dtor1*) *s1*) (*case\_sum* (*F2map* *id* *Inl* *Inl* *o* *dtor2*) *s2*) *o* *Inl* = *id*  
**apply** (*rule* *trans*[*OF* *conjunct2*[*OF* *unfold\_unique*] *unfold2\_dtor*])  
**apply** (*rule* *trans*[*OF* *arg\_cong2*[*of* \_ \_ \_ \_ (*o*), *OF* *F1.map\_comp0*[*of* *id*, *unfolded\_id\_o*] *refl*]])  
**apply** (*rule* *sym*[*OF* *trans*[*OF* *o\_assoc*]])  
**apply** (*rule* *trans*[*OF* *arg\_cong2*[*of* \_ \_ \_ \_ (*o*), *OF* *dtor1\_o\_unfold1* *refl*]])  
**apply** (*rule* *box\_equals*[*OF* \_ *o\_assoc* *o\_assoc*])  
**apply** (*rule* *arg\_cong2*[*of* \_ \_ \_ \_ (*o*), *OF* *refl* *case\_sum\_o\_inj*( $1$ )])  
**apply** (*rule* *trans*[*OF* *arg\_cong2*[*of* \_ \_ \_ \_ (*o*), *OF* *F2.map\_comp0*[*of* *id*, *unfolded\_id\_o*] *refl*]])  
**apply** (*rule* *sym*[*OF* *trans*[*OF* *o\_assoc*]])  
**apply** (*rule* *trans*[*OF* *arg\_cong2*[*of* \_ \_ \_ \_ (*o*), *OF* *dtor2\_o\_unfold2* *refl*]])  
**apply** (*rule* *box\_equals*[*OF* \_ *o\_assoc* *o\_assoc*])  
**apply** (*rule* *arg\_cong2*[*of* \_ \_ \_ \_ (*o*), *OF* *refl* *case\_sum\_o\_inj*( $1$ )])

done

**lemma** *case\_sum\_expand\_Inr*:  $f \circ \text{Inl} = g \implies \text{case\_sum } g (f \circ \text{Inr}) = f$   
by (auto split: sum.splits)

**theorem** *corec1*:

*dtor1* (corec1 s1 s2 a) =  
F1map id (case\_sum id (corec1 s1 s2)) (case\_sum id (corec2 s1 s2)) (s1 a)  
**unfolding** corec1\_def corec2\_def o\_apply unfold1 sum.case  
case\_sum\_expand\_Inr[OF corec1\_Inl\_sum] case\_sum\_expand\_Inr[OF corec2\_Inl\_sum] ..

**theorem** *corec2*:

*dtor2* (corec2 s1 s2 a) =  
F2map id (case\_sum id (corec1 s1 s2)) (case\_sum id (corec2 s1 s2)) (s2 a)  
**unfolding** corec1\_def corec2\_def o\_apply unfold2 sum.case  
case\_sum\_expand\_Inr[OF corec1\_Inl\_sum] case\_sum\_expand\_Inr[OF corec2\_Inl\_sum] ..

**lemma** *corec\_unique*:

F1map id (case\_sum id f1) (case\_sum id f2)  $\circ$  s1 = dtor1  $\circ$  f1  $\implies$   
F2map id (case\_sum id f1) (case\_sum id f2)  $\circ$  s2 = dtor2  $\circ$  f2  $\implies$   
f1 = corec1 s1 s2  $\wedge$  f2 = corec2 s1 s2  
**unfolding** corec1\_def corec2\_def case\_sum\_expand\_Inr'[OF corec1\_Inl\_sum] case\_sum\_expand\_Inr'[OF corec2\_Inl\_sum]  
**apply** (rule unfold\_unique)  
**apply** (unfold o\_case\_sum\_id\_o\_o\_id F1.map\_comp0[symmetric] F2.map\_comp0[symmetric])  
F1.map\_id0 F2.map\_id0 o\_assoc case\_sum\_o\_inj(1)  
**apply** (erule arg\_cong2[of \_ \_ \_ \_ case\_sum, OF refl])  
**apply** (erule arg\_cong2[of \_ \_ \_ \_ case\_sum, OF refl])  
done

## 2.7 Coinduction

**lemma** *Frel\_coind*:

$\llbracket \forall a b. \text{phi1 } a b \longrightarrow \text{F1rel } (=) \text{ phi1 phi2 (dtor1 } a) (\text{dtor1 } b);$   
 $\forall a b. \text{phi2 } a b \longrightarrow \text{F2rel } (=) \text{ phi1 phi2 (dtor2 } a) (\text{dtor2 } b) \rrbracket \implies$   
 $(\text{phi1 } a1 b1 \longrightarrow a1 = b1) \wedge (\text{phi2 } a2 b2 \longrightarrow a2 = b2)$   
**apply** (rule rev\_mp)  
**apply** (rule JF\_cind)  
**apply** (rule iffD2)  
**apply** (rule bis\_Frel)  
**apply** (rule conjI)  
  
**apply** (rule conjI)  
**apply** (rule ord\_le\_eq\_trans[OF subset\_UNIV UNIV\_Times\_UNIV[THEN sym]])  
**apply** (rule ord\_le\_eq\_trans[OF subset\_UNIV UNIV\_Times\_UNIV[THEN sym]])  
  
**apply** (rule conjI)  
**apply** (rule allI)+  
**apply** (rule impI)  
**apply** (erule allE)+  
**apply** (rule predicate2D[OF eq\_refl[OF F1rel\_cong]])  
**apply** (rule refl)  
**apply** (rule in\_rel\_Collect\_case\_prod\_eq[symmetric])  
**apply** (rule in\_rel\_Collect\_case\_prod\_eq[symmetric])  
**apply** (erule mp)  
**apply** (erule CollectE)  
**apply** (erule case\_prodD)  
  
**apply** (rule allI)+  
**apply** (rule impI)  
**apply** (erule allE)+  
**apply** (rule predicate2D[OF eq\_refl[OF F2rel\_cong]])  
**apply** (rule refl)  
**apply** (rule in\_rel\_Collect\_case\_prod\_eq[symmetric])  
**apply** (rule in\_rel\_Collect\_case\_prod\_eq[symmetric])

```

apply (erule mp)
apply (erule CollectE)
apply (erule case_prodD)

```

```

apply (rule impI)
apply (erule conjE)+

```

```

apply (rule conjI)
apply (rule impI)
apply (rule IdD)
apply (erule subsetD)
apply (rule CollectI)
apply (erule case_prodI)

```

```

apply (rule impI)
apply (rule IdD)
apply (erule subsetD)
apply (rule CollectI)
apply (erule case_prodI)
done

```

## 2.8 The Result as an BNF

**abbreviation** *JF1map* **where**

```

JF1map u  $\equiv$  unfold1 (F1map u id id o dtor1) (F2map u id id o dtor2)

```

**abbreviation** *JF2map* **where**

```

JF2map u  $\equiv$  unfold2 (F1map u id id o dtor1) (F2map u id id o dtor2)

```

**lemma** *JF1map*: dtor1 o *JF1map* u = F1map u (*JF1map* u) (*JF2map* u) o dtor1

```

apply (rule ext)
apply (rule sym[OF trans[OF o_apply]])
apply (rule sym[OF trans[OF o_apply]])
apply (rule trans[OF unfold1])
apply (rule box_equals[OF F1.map_comp _ F1.map_cong0, rotated])
  apply (rule fun_cong[OF id_o])
  apply (rule fun_cong[OF o_id])
  apply (rule fun_cong[OF o_id])
apply (rule sym[OF arg_cong[OF o_apply]])
done

```

**lemma** *JF2map*: dtor2 o *JF2map* u = F2map u (*JF1map* u) (*JF2map* u) o dtor2

```

apply (rule ext)
apply (rule sym[OF trans[OF o_apply]])
apply (rule sym[OF trans[OF o_apply]])
apply (rule trans[OF unfold2])
apply (rule box_equals[OF F2.map_comp _ F2.map_cong0, rotated])
  apply (rule fun_cong[OF id_o])
  apply (rule fun_cong[OF o_id])
  apply (rule fun_cong[OF o_id])
apply (rule sym[OF arg_cong[OF o_apply]])
done

```

**lemmas** *JF1map\_simps* = o\_eq\_dest[OF *JF1map*]

**lemmas** *JF2map\_simps* = o\_eq\_dest[OF *JF2map*]

**theorem** *JF1map\_id*: *JF1map* id = id

```

apply (rule trans)
apply (rule conjunct1)
apply (rule unfold_unique)
  apply (rule sym[OF JF1map])
  apply (rule sym[OF JF2map])
apply (rule unfold1_dtor)
done

```

**theorem** *JF2map\_id*:  $JF2map\ id = id$

```
apply (rule trans)
apply (rule conjunct2)
apply (rule unfold_unique)
apply (rule sym[OF JF1map])
apply (rule sym[OF JF2map])
apply (rule unfold2_dtor)
done
```

**lemma** *JFmap\_unique*:

```
[[dtor1 o u = F1map f u v o dtor1; dtor2 o v = F2map f u v o dtor2]] ==>
u = JF1map f ^& v = JF2map f
apply (rule unfold_unique)
unfolding o_assoc F1.map_comp0[symmetric] F2.map_comp0[symmetric] id_o o_id
apply (erule sym)
apply (erule sym)
done
```

**theorem** *JF1map\_comp*:  $JF1map\ (g\ o\ f) = JF1map\ g\ o\ JF1map\ f$

```
apply (rule sym)
apply (rule conjunct1)
apply (rule JFmap_unique)
apply (rule trans[OF o_assoc])
apply (rule trans[OF arg_cong2[of _ _ _ _ (o), OF JF1map refl]])
apply (rule trans[OF sym[OF o_assoc]])
apply (rule trans[OF arg_cong[OF JF1map]])
apply (rule trans[OF o_assoc])
apply (rule arg_cong2[of _ _ _ _ (o), OF sym[OF F1.map_comp0] refl])
```

```
apply (rule trans[OF o_assoc])
apply (rule trans[OF arg_cong2[of _ _ _ _ (o), OF JF2map refl]])
apply (rule trans[OF sym[OF o_assoc]])
apply (rule trans[OF arg_cong[OF JF2map]])
apply (rule trans[OF o_assoc])
apply (rule arg_cong2[of _ _ _ _ (o), OF sym[OF F2.map_comp0] refl])
done
```

**theorem** *JF2map\_comp*:  $JF2map\ (g\ o\ f) = JF2map\ g\ o\ JF2map\ f$

```
apply (rule sym)
apply (rule conjunct2)
apply (tactic <rtac @&{context}> (Thm.permute_prem 0 1 @&{thm unfold_unique}> 1)>
```

```
apply (rule trans[OF o_assoc])
apply (rule trans[OF arg_cong[OF sym[OF F2.map_comp0]])
apply (rule sym[OF trans[OF o_assoc]])
apply (rule trans[OF arg_cong2[OF JF2map refl]])
apply (rule trans[OF sym[OF o_assoc]])
apply (rule trans[OF arg_cong[OF JF2map]])
apply (rule trans[OF o_assoc])
apply (rule trans[OF arg_cong2[OF sym[OF F2.map_comp0] refl]])
apply (rule ext)
apply (rule trans[OF o_apply])
apply (rule sym)
apply (rule trans[OF o_apply])
apply (rule F2.map_comp0)
apply (rule trans[OF o_apply])
apply (rule id_apply)
apply (rule trans[OF o_apply])
apply (rule arg_cong[OF id_apply])
apply (rule trans[OF o_apply])
apply (rule arg_cong[OF id_apply])
```

```

apply (rule trans[OF o_assoc])
apply (rule trans[OF arg_cong[OF sym[OF F1.map_comp0]]])
apply (rule sym[OF trans[OF o_assoc]])
apply (rule trans[OF arg_cong2[OF JF1map refl]])
apply (rule trans[OF sym[OF o_assoc]])
apply (rule trans[OF arg_cong[OF JF1map]])
apply (rule trans[OF o_assoc])
apply (rule trans[OF arg_cong2[OF sym[OF F1.map_comp0] refl]])
apply (rule ext)
apply (rule trans[OF o_apply])
apply (rule sym)
apply (rule trans[OF o_apply])
apply (rule F1.map_cong0)
  apply (rule trans[OF o_apply])
  apply (rule id_apply)
  apply (rule trans[OF o_apply])
  apply (rule arg_cong[OF id_apply])
apply (rule trans[OF o_apply])
apply (rule arg_cong[OF id_apply])
done

```

**definition** *JFcol* where

```

JFcol = rec_nat (%a. {}, %b. {})
  (%n rec.
    (%a. F1set1 (dtor1 a)  $\cup$ 
      (( $\bigcup a' \in F1set2$  (dtor1 a). fst rec a')  $\cup$ 
       ( $\bigcup a' \in F1set3$  (dtor1 a). snd rec a')),
    %b. F2set1 (dtor2 b)  $\cup$ 
      (( $\bigcup b' \in F2set2$  (dtor2 b). fst rec b')  $\cup$ 
       ( $\bigcup b' \in F2set3$  (dtor2 b). snd rec b'))))

```

**abbreviation** *JF1col* where *JF1col* *n*  $\equiv$  fst (*JFcol* *n*)

**abbreviation** *JF2col* where *JF2col* *n*  $\equiv$  snd (*JFcol* *n*)

**lemmas** *JF1col\_0* = fun\_cong[OF fstI[OF rec\_nat\_0\_imp[OF *JFcol\_def*]]]

**lemmas** *JF2col\_0* = fun\_cong[OF sndI[OF rec\_nat\_0\_imp[OF *JFcol\_def*]]]

**lemmas** *JF1col\_Suc* = fun\_cong[OF fstI[OF rec\_nat\_Suc\_imp[OF *JFcol\_def*]]]

**lemmas** *JF2col\_Suc* = fun\_cong[OF sndI[OF rec\_nat\_Suc\_imp[OF *JFcol\_def*]]]

**lemma** *JFcol\_minimal*:

```

[[ $\bigwedge a$ . F1set1 (dtor1 a)  $\subseteq$  K1 a;
   $\bigwedge b$ . F2set1 (dtor2 b)  $\subseteq$  K2 b;
   $\bigwedge a a'$ .  $a' \in F1set2$  (dtor1 a)  $\implies$  K1 a'  $\subseteq$  K1 a;
   $\bigwedge a b'$ .  $b' \in F1set3$  (dtor1 a)  $\implies$  K2 b'  $\subseteq$  K1 a;
   $\bigwedge b a'$ .  $a' \in F2set2$  (dtor2 b)  $\implies$  K1 a'  $\subseteq$  K2 b;
   $\bigwedge b b'$ .  $b' \in F2set3$  (dtor2 b)  $\implies$  K2 b'  $\subseteq$  K2 b]]  $\implies$ 
 $\forall a b$ . JF1col n a  $\subseteq$  K1 a  $\wedge$  JF2col n b  $\subseteq$  K2 b

```

```

apply (rule nat_induct)
apply (rule allI)+
apply (rule conjI)
apply (rule ord_eq_le_trans)
  apply (rule JF1col_0)
  apply (rule empty_subsetI)

```

```

apply (rule ord_eq_le_trans)
apply (rule JF2col_0)
apply (rule empty_subsetI)

```

```

apply (rule allI)+
apply (rule conjI)
apply (rule ord_eq_le_trans)
apply (rule JF1col_Suc)

```

**apply** (rule *Un\_least*)  
**apply** *assumption*  
**apply** (rule *Un\_least*)  
**apply** (rule *UN\_least*)  
**apply** (erule *allE conjE*)+  
**apply** (rule *subset\_trans*)  
**apply** *assumption*  
**apply** *assumption*

**apply** (rule *UN\_least*)  
**apply** (erule *allE conjE*)+  
**apply** (rule *subset\_trans*)  
**apply** *assumption*  
**apply** *assumption*

**apply** (rule *ord\_eq\_le\_trans*)  
**apply** (rule *JF2col\_Suc*)  
**apply** (rule *Un\_least*)  
**apply** *assumption*  
**apply** (rule *Un\_least*)  
**apply** (rule *UN\_least*)  
**apply** (erule *allE conjE*)+  
**apply** (rule *subset\_trans*)  
**apply** *assumption*  
**apply** *assumption*

**apply** (rule *UN\_least*)  
**apply** (erule *allE conjE*)+  
**apply** (rule *subset\_trans*)  
**apply** *assumption*  
**apply** *assumption*  
**done**

**lemma** *JFset\_minimal*:

$$\begin{aligned} & \llbracket \bigwedge a. F1set1 (dtor1 a) \subseteq K1 a; \\ & \bigwedge b. F2set1 (dtor2 b) \subseteq K2 b; \\ & \bigwedge a a'. a' \in F1set2 (dtor1 a) \implies K1 a' \subseteq K1 a; \\ & \bigwedge a b'. b' \in F1set3 (dtor1 a) \implies K2 b' \subseteq K1 a; \\ & \bigwedge b a'. a' \in F2set2 (dtor2 b) \implies K1 a' \subseteq K2 b; \\ & \bigwedge b b'. b' \in F2set3 (dtor2 b) \implies K2 b' \subseteq K2 b \rrbracket \implies \\ & (\bigcup n. JF1col n a) \subseteq K1 a \wedge (\bigcup n. JF2col n b) \subseteq K2 b \end{aligned}$$

**apply** (rule *conjI*)  
**apply** (rule *UN\_least*)  
**apply** (rule *rev\_mp*)  
**apply** (rule *JFcol\_minimal*)  
**apply** *assumption*  
**apply** *assumption*  
**apply** *assumption*  
**apply** *assumption*  
**apply** *assumption*  
**apply** *assumption*  
**apply** (rule *impI*)  
**apply** (erule *allE conjE*)+  
**apply** *assumption*

**apply** (rule *UN\_least*)  
**apply** (rule *rev\_mp*)  
**apply** (rule *JFcol\_minimal*)  
**apply** *assumption*  
**apply** *assumption*  
**apply** *assumption*



```

  apply assumption
  apply assumption
  apply assumption
  apply (rule impI)
  apply (erule allE conjE)+
  apply assumption
done

```

**abbreviation**  $JF1set$  **where**  $JF1set\ a \equiv (\bigcup n. JF1col\ n\ a)$   
**abbreviation**  $JF2set$  **where**  $JF2set\ a \equiv (\bigcup n. JF2col\ n\ a)$

**lemma**  $F1set1\_incl\_JF1set$ :  
 $F1set1\ (dtor1\ a) \subseteq JF1set\ a$   
 apply (rule SUP\_upper2)  
 apply (rule UNIV\_I)  
 apply (rule ord\_le\_eq\_trans)  
 apply (rule Un\_upper1)  
 apply (rule sym)  
 apply (rule JF1col\_Suc)  
done

**lemma**  $F2set1\_incl\_JF2set$ :  
 $F2set1\ (dtor2\ a) \subseteq JF2set\ a$   
 apply (rule SUP\_upper2)  
 apply (rule UNIV\_I)  
 apply (rule ord\_le\_eq\_trans)  
 apply (rule Un\_upper1)  
 apply (rule sym)  
 apply (rule JF2col\_Suc)  
done

**lemma**  $F1set2\_JF1set\_incl\_JF1set$ :  
 $a' \in F1set2\ (dtor1\ a) \implies JF1set\ a' \subseteq JF1set\ a$   
 apply (rule UN\_least)  
 apply (rule subsetI)  
 apply (rule UN\_I)  
 apply (rule UNIV\_I)  
 apply (rule subsetD)  
 apply (rule equalityD2)  
 apply (rule JF1col\_Suc)  
 apply (rule UnI2)  
 apply (tactic <rtac @ {context} (BNF\_Util.mk\_UnIN 2 1) 1>)  
 apply (erule UN\_I)  
 apply assumption  
done

**lemma**  $F1set3\_JF2set\_incl\_JF1set$ :  
 $a' \in F1set3\ (dtor1\ a) \implies JF2set\ a' \subseteq JF1set\ a$   
 apply (rule UN\_least)  
 apply (rule subsetI)  
 apply (rule UN\_I)  
 apply (rule UNIV\_I)  
 apply (rule subsetD)  
 apply (rule equalityD2)  
 apply (rule JF1col\_Suc)  
 apply (rule UnI2)  
 apply (tactic <rtac @ {context} (BNF\_Util.mk\_UnIN 2 2) 1>)  
 apply (erule UN\_I)  
 apply assumption  
done

**lemma**  $F2set2\_JF1set\_incl\_JF2set$ :  
 $a' \in F2set2\ (dtor2\ a) \implies JF1set\ a' \subseteq JF2set\ a$

```

apply (rule UN_least)
apply (rule subsetI)
apply (rule UN_I)
  apply (rule UNIV_I)
apply (rule subsetD)
  apply (rule equalityD2)
  apply (rule JF2col_Suc)
apply (rule UnI2)
apply (tactic <rtac @_{context} (BNF_Util.mk_UnIN 2 1) 1>)
apply (erule UN_I)
apply assumption
done

```

```

lemma F2set3_JF2set_incl_JF2set:
   $a' \in F2set3 \text{ (dtor2 } a) \implies JF2set \ a' \subseteq JF2set \ a$ 
  apply (rule UN_least)
  apply (rule subsetI)
  apply (rule UN_I)
    apply (rule UNIV_I)
  apply (rule subsetD)
    apply (rule equalityD2)
    apply (rule JF2col_Suc)
  apply (rule UnI2)
  apply (tactic <rtac @_{context} (BNF_Util.mk_UnIN 2 2) 1>)
  apply (erule UN_I)
  apply assumption
done

```

```

lemmas F1set1_JF1set = subsetD[OF F1set1_incl_JF1set]
lemmas F2set1_JF2set = subsetD[OF F2set1_incl_JF2set]
lemmas F1set2_JF1set_JF1set = subsetD[OF F1set2_JF1set_incl_JF1set]
lemmas F1set3_JF2set_JF1set = subsetD[OF F1set3_JF2set_incl_JF1set]
lemmas F2set2_JF1set_JF2set = subsetD[OF F2set2_JF1set_incl_JF2set]
lemmas F2set3_JF2set_JF2set = subsetD[OF F2set3_JF2set_incl_JF2set]

```

```

lemma JFset_le:
  fixes a :: 'a JF1 and b :: 'a JF2
  shows
     $JF1set \ a \subseteq F1set1 \text{ (dtor1 } a) \cup (\bigcup (JF1set \ ' \ F1set2 \text{ (dtor1 } a)) \cup \bigcup (JF2set \ ' \ F1set3 \text{ (dtor1 } a))) \wedge$ 
     $JF2set \ b \subseteq F2set1 \text{ (dtor2 } b) \cup (\bigcup (JF1set \ ' \ F2set2 \text{ (dtor2 } b)) \cup \bigcup (JF2set \ ' \ F2set3 \text{ (dtor2 } b)))$ 
  apply (rule JFset_minimal)
    apply (rule Un_upper1)
    apply (rule Un_upper1)
  apply (rule subsetI)
  apply (rule UnI2)
  apply (tactic <rtac @_{context} (BNF_Util.mk_UnIN 2 1) 1>)
  apply (erule UN_I)
  apply (erule UnE)
    apply (erule F1set1_JF1set)
  apply (erule UnE)+
  apply (erule UN_E)
  apply (erule F1set2_JF1set_JF1set)
  apply assumption
  apply (erule UN_E)
  apply (erule F1set3_JF2set_JF1set)
  apply assumption
  apply (rule subsetI)
  apply (rule UnI2)
  apply (tactic <rtac @_{context} (BNF_Util.mk_UnIN 2 2) 1>)
  apply (erule UN_I)
  apply (erule UnE)
    apply (erule F2set1_JF2set)
  apply (erule UnE)+

```

```

apply (erule UN_E)
apply (erule F2set2_JF1set_JF2set)
apply assumption
apply (erule UN_E)
apply (erule F2set3_JF2set_JF2set)
apply assumption
apply (rule subsetI)
apply (rule UnI2)
apply (tactic <rtac @{\context} (BNF_Util.mk_UnIN 2 1) 1>)
apply (erule UN_I)
apply (erule UnE)+
apply (erule F1set1_JF1set)
apply (erule UnE)+
apply (erule UN_E)
apply (erule F1set2_JF1set_JF1set)
apply assumption
apply (erule UN_E)
apply (erule F1set3_JF2set_JF1set)
apply assumption
apply (rule subsetI)
apply (rule UnI2)
apply (tactic <rtac @{\context} (BNF_Util.mk_UnIN 2 2) 1>)
apply (erule UN_I)
apply (erule UnE)+
apply (erule F2set1_JF2set)
apply (erule UnE)+
apply (erule UN_E)
apply (erule F2set2_JF1set_JF2set)
apply assumption
apply (erule UN_E)
apply (erule F2set3_JF2set_JF2set)
apply assumption
done

```

**theorem** *JF1set\_simps*:

```

JF1set a = F1set1 (dtor1 a) ∪
  ((∪ b ∈ F1set2 (dtor1 a). JF1set b) ∪
   (∪ b ∈ F1set3 (dtor1 a). JF2set b))
apply (rule equalityI)
apply (rule conjunct1[OF JFset_le])
apply (rule Un_least)
apply (rule F1set1_incl_JF1set)
apply (rule Un_least)
apply (rule UN_least)
apply (erule F1set2_JF1set_incl_JF1set)
apply (rule UN_least)
apply (erule F1set3_JF2set_incl_JF1set)
done

```

**theorem** *JF2set\_simps*:

```

JF2set a = F2set1 (dtor2 a) ∪
  ((∪ b ∈ F2set2 (dtor2 a). JF1set b) ∪
   (∪ b ∈ F2set3 (dtor2 a). JF2set b))
apply (rule equalityI)
apply (rule conjunct2[OF JFset_le])
apply (rule Un_least)
apply (rule F2set1_incl_JF2set)
apply (rule Un_least)
apply (rule UN_least)
apply (erule F2set2_JF1set_incl_JF2set)
apply (rule UN_least)
apply (erule F2set3_JF2set_incl_JF2set)
done

```

**lemma** *JFcol\_natural*:

```

 $\forall b1\ b2.\ u \ ' (JF1col\ n\ b1) = JF1col\ n\ (JF1map\ u\ b1) \wedge$ 
 $u \ ' (JF2col\ n\ b2) = JF2col\ n\ (JF2map\ u\ b2)$ 
apply (rule nat_induct)
apply (rule allI)+
unfolding JF1col_0 JF2col_0
apply (rule conjI)
apply (rule image_empty)
apply (rule image_empty)

apply (rule allI)+
apply (rule conjI)
apply (unfold JF1col_Suc JF1map_simps image_Un image_UN UN_simps(10)
  F1.set_map(1) F1.set_map(2) F1.set_map(3)) [1]
apply (rule arg_cong2[of _ _ _ _ (U)])
apply (rule refl)
apply (rule arg_cong2[of _ _ _ _ (U)])
apply (rule SUP_cong[OF refl])
apply (erule allE)+
apply (tactic <etac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (rule SUP_cong[OF refl])
apply (erule allE)+
apply (tactic <etac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)

apply (unfold JF2col_Suc JF2map_simps image_Un image_UN UN_simps(10)
  F2.set_map(1) F2.set_map(2) F2.set_map(3)) [1]
apply (rule arg_cong2[of _ _ _ _ (U)])
apply (rule refl)
apply (rule arg_cong2[of _ _ _ _ (U)])
apply (rule SUP_cong[OF refl])
apply (erule allE)+
apply (tactic <etac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (rule SUP_cong[OF refl])
apply (erule allE)+
apply (tactic <etac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
done

```

**theorem** *JF1set\_natural*:  $JF1set\ o\ (JF1map\ u) = image\ u\ o\ JF1set$

```

apply (rule ext)
apply (rule trans[OF o_apply])
apply (rule sym)
apply (rule trans[OF o_apply])
apply (rule trans[OF image_UN])
apply (rule SUP_cong[OF refl])
apply (rule conjunct1)
apply (rule spec[OF spec[OF JFcol_natural]])
done

```

**theorem** *JF2set\_natural*:  $JF2set\ o\ (JF2map\ u) = image\ u\ o\ JF2set$

```

apply (rule ext)
apply (rule trans[OF o_apply])
apply (rule sym)
apply (rule trans[OF o_apply])
apply (rule trans[OF image_UN])
apply (rule SUP_cong[OF refl])
apply (rule conjunct2)
apply (rule spec[OF spec[OF JFcol_natural]])
done

```

**theorem** *JFmap\_cong0*:

```

 $((\forall p \in JF1set\ a.\ u\ p = v\ p) \longrightarrow JF1map\ u\ a = JF1map\ v\ a) \wedge$ 
 $((\forall p \in JF2set\ b.\ u\ p = v\ p) \longrightarrow JF2map\ u\ b = JF2map\ v\ b)$ 

```

```

apply (rule rev_mp)
apply (rule Frel_coind[of
  %b c.  $\exists a. a \in \{a. \forall p \in JF1set\ a. u\ p = v\ p\} \wedge b = JF1map\ u\ a \wedge c = JF1map\ v\ a$ 
  %b c.  $\exists a. a \in \{a. \forall p \in JF2set\ a. u\ p = v\ p\} \wedge b = JF2map\ u\ a \wedge c = JF2map\ v\ a$ ])
apply (intro allI impI iffD2[OF F1.in_rel])

apply (erule exE conjE)+
apply (tactic <hyp_subst_tac @ {context} 1>)
apply (rule exI)

apply (rule conjI[rotated])
apply (rule conjI)
apply (rule trans[OF F1.map_comp])
apply (rule sym)
apply (rule trans[OF JF1map_simps])
apply (rule F1.map_cong0)
apply (rule sym[OF trans[OF o_apply]])
apply (rule fst_conv)
apply (rule sym[OF fun_cong[OF fst_convolfunfolding convolfunfolding]])
apply (rule sym[OF fun_cong[OF fst_convolfunfolding convolfunfolding]])

apply (rule trans[OF F1.map_comp])
apply (rule sym)
apply (rule trans[OF JF1map_simps])
apply (rule F1.map_cong0)
apply (rule sym[OF trans[OF o_apply]])
apply (rule trans[OF snd_conv])
apply (erule CollectE)
apply (erule bspec)
apply (erule F1set1_JF1set)
apply (rule sym[OF fun_cong[OF snd_convolfunfolding convolfunfolding]])
apply (rule sym[OF fun_cong[OF snd_convolfunfolding convolfunfolding]])

apply (rule CollectI)
apply (rule conjI)
apply (rule ord_eq_le_trans)
apply (rule F1.set_map(1))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule refl)

apply (rule conjI)
apply (rule ord_eq_le_trans)
apply (rule F1.set_map(2))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule exI)
apply (rule conjI)
apply (rule CollectI)
apply (rule ballI)
apply (erule CollectE)
apply (erule bspec)
apply (erule F1set2_JF1set_JF1set)
apply assumption
apply (rule conjI[OF refl refl])

apply (rule ord_eq_le_trans)
apply (rule F1.set_map(3))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)

```

```

apply (rule exI)
apply (rule conjI)
apply (rule CollectI)
apply (rule ballI)
apply (erule CollectE)
apply (erule bspec)
apply (erule F1set3_JF2set_JF1set)
apply assumption
apply (rule conjI[OF refl refl])

```

```

apply (intro allI impI iffD2[OF F2.in_rel])

```

```

apply (erule exE conjE)+
apply (tactic <hyp_subst_tac @{context} 1>)
apply (rule exI)

```

```

apply (rule conjI[rotated])
apply (rule conjI)
apply (rule trans[OF F2.map_comp])
apply (rule sym)
apply (rule trans[OF JF2map_simps])
apply (rule F2.map_cong0)
apply (rule sym[OF trans[OF o_apply]])
apply (rule fst_conv)
apply (rule sym[OF fun_cong[OF fst_conv[unfolded convol_def]])])
apply (rule sym[OF fun_cong[OF fst_conv[unfolded convol_def]])])

```

```

apply (rule trans[OF F2.map_comp])
apply (rule sym)
apply (rule trans[OF JF2map_simps])
apply (rule F2.map_cong0)
apply (rule sym[OF trans[OF o_apply]])
apply (rule trans[OF snd_conv])
apply (erule CollectE)
apply (erule bspec)
apply (erule F2set1_JF2set)
apply (rule sym[OF fun_cong[OF snd_conv[unfolded convol_def]])])
apply (rule sym[OF fun_cong[OF snd_conv[unfolded convol_def]])])

```

```

apply (rule CollectI)
apply (rule conjI)
apply (rule ord_eq_le_trans)
apply (rule F2.set_map(1))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule refl)

```

```

apply (rule conjI)
apply (rule ord_eq_le_trans)
apply (rule F2.set_map(2))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule exI)
apply (rule conjI)
apply (rule CollectI)
apply (rule ballI)
apply (erule CollectE)
apply (erule bspec)

```

```

apply (erule F2set2_JF1set_JF2set)
apply assumption
apply (rule conjI[OF refl refl])

```

```

apply (rule ord_eq_le_trans)
apply (rule F2.set_map(3))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule exI)
apply (rule conjI)
apply (rule CollectI)
apply (rule ballI)
apply (erule CollectE)
apply (erule bspec)
apply (erule F2set3_JF2set_JF2set)
apply assumption
apply (rule conjI[OF refl refl])

```

```

apply (rule impI)

```

```

apply (rule conjI)
apply (rule impI)
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (erule mp)
apply (rule exI)
apply (rule conjI)
apply (erule CollectI)
apply (rule conjI)
apply (rule refl)
apply (rule refl)

```

```

apply (rule impI)
apply (tactic <dtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (erule mp)
apply (rule exI)
apply (rule conjI)
apply (erule CollectI)
apply (rule conjI)
apply (rule refl)
apply (rule refl)
done

```

```

lemmas JF1map_cong0 = mp[OF conjunct1[OF JFmap_cong0]]

```

```

lemmas JF2map_cong0 = mp[OF conjunct2[OF JFmap_cong0]]

```

```

lemma JFcol_bd:  $\forall (j1 :: 'a \text{ JF1}) (j2 :: 'a \text{ JF2}). |JF1col\ n\ j1| <_o\ bd\_F \wedge |JF2col\ n\ j2| <_o\ bd\_F$ 

```

```

apply (rule nat_induct)
apply (rule allI)+
apply (rule conjI)
apply (rule ordIso_ordLess_trans)
apply (rule card_of_ordIso_subst)
apply (rule JF1col_0)
apply (rule Cinfinite_gt_empty)
apply (rule bd_F_Cinfinite)
apply (rule ordIso_ordLess_trans)
apply (rule card_of_ordIso_subst)
apply (rule JF2col_0)
apply (rule Cinfinite_gt_empty)
apply (rule bd_F_Cinfinite)

```

```

apply (rule allI)+
apply (rule conjI)
apply (rule ordIso_ordLess_trans)
  apply (rule card_of_ordIso_subst)
  apply (rule JF1col_Suc)
apply (rule Un_Cinfinite_bound_strict)
  apply (rule F1set1_bd')
apply (rule Un_Cinfinite_bound_strict)
  apply (rule regularCard_UNION_bound)
    apply (rule bd_F_Cinfinite)
    apply (rule bd_F_regularCard)
  apply (rule F1set2_bd')
apply (erule allE)+
apply (tactic <etac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (rule regularCard_UNION_bound)
  apply (rule bd_F_Cinfinite)
  apply (rule bd_F_regularCard)
apply (rule F1set3_bd')
apply (erule allE)+
apply (tactic <etac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (rule bd_F_Cinfinite)
apply (rule bd_F_Cinfinite)

```

```

apply (rule ordIso_ordLess_trans)
apply (rule card_of_ordIso_subst)
apply (rule JF2col_Suc)
apply (rule Un_Cinfinite_bound_strict)
  apply (rule F2set1_bd')
apply (rule Un_Cinfinite_bound_strict)
  apply (rule regularCard_UNION_bound)
    apply (rule bd_F_Cinfinite)
    apply (rule bd_F_regularCard)
  apply (rule F2set2_bd')
apply (erule allE)+
apply (tactic <etac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (rule regularCard_UNION_bound)
  apply (rule bd_F_Cinfinite)
  apply (rule bd_F_regularCard)
apply (rule F2set3_bd')
apply (erule allE)+
apply (tactic <etac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (rule bd_F_Cinfinite)
apply (rule bd_F_Cinfinite)
done

```

```

theorem JF1set_bd: |JF1set j| <o bd_F
apply (rule regularCard_UNION_bound)
  apply (rule bd_F_Cinfinite)
  apply (rule bd_F_regularCard)
apply (rule ordIso_ordLess_trans)
apply (rule card_of_nat)
apply (rule ordLess_ordIso_trans)
apply (rule natLeq_ordLess_cinfinite)
  apply (rule sum_Cinfinite)
  apply (rule sum_card_order)
apply (rule bd_F)
apply (tactic <rtac @{context} (BNF_Util.mk_conjunctN 2 1) 1>)
apply (rule spec[OF spec[OF JFcol_bd]])
done

```

```

theorem JF2set_bd: |JF2set j| <o bd_F
apply (rule regularCard_UNION_bound)
  apply (rule bd_F_Cinfinite)

```



```

apply (rule bd_F_regularCard)
apply (rule ordIso_ordLess_trans)
apply (rule card_of_nat)
apply (rule ordLess_ordIso_trans)
apply (rule natLeq_ordLess_cinfinite)
  apply (rule sum_Cinfinite)
apply (rule sum_card_order)
apply (rule bd_F)
apply (tactic <rtac @{context} (BNF_Util.mk_conjunctN 2 2) 1>)
apply (rule spec[OF spec[OF JFcol_bd]])
done

```

**abbreviation**  $JF2wit \equiv \text{ctor2 wit\_F2}$

**theorem**  $JF2wit: \bigwedge x. x \in JF2set \ JF2wit \implies \text{False}$

```

apply (drule rev_subsetD)
  apply (rule equalityD1)
  apply (rule JF2set_simps)
unfolding dtor2_ctor2
apply (erule UnE)
  apply (erule F2.wit)
apply (erule UnE)
  apply (erule UN_E)
  apply (erule F2.wit)
apply (erule UN_E)
apply (erule F2.wit)
done

```

**abbreviation**  $JF1wit \equiv (\%a. \text{ctor1 (wit2\_F1 a JF2wit)})$

**theorem**  $JF1wit: \bigwedge x. x \in JF1set (JF1wit a) \implies x = a$

```

apply (drule rev_subsetD)
  apply (rule equalityD1)
  apply (rule JF1set_simps)
unfolding dtor1_ctor1
apply (erule UnE)+
  apply (erule F1.wit2)
apply (erule UnE)
  apply (erule UN_E)
  apply (drule F1.wit2)
  apply (erule FalseE)
apply (erule UN_E)
apply (drule F1.wit2)
apply (tactic <hyp_subst_tac @{context} 1>)
apply (drule rev_subsetD)
  apply (rule equalityD1)
  apply (rule JF2set_simps)
unfolding dtor2_ctor2
apply (erule UnE)+
  apply (drule F2.wit)
  apply (erule FalseE)
apply (erule UnE)
  apply (erule UN_E)
  apply (drule F2.wit)
  apply (erule FalseE)
apply (erule UN_E)
apply (drule F2.wit)
apply (erule FalseE)
done

```

**context**

fixes phi1 :: 'a ⇒ 'a JF1 ⇒ bool and phi2 :: 'a ⇒ 'a JF2 ⇒ bool  
begin

lemmas JFset\_induct =

JFset\_minimal[of %b1. {a ∈ JF1set b1 . phi1 a b1} %b2. {a ∈ JF2set b2 . phi2 a b2},  
unfolding subset\_Collect\_iff[OF F1set1\_incl\_JF1set] subset\_Collect\_iff[OF F2set1\_incl\_JF2set]  
subset\_Collect\_iff[OF subset\_refl],  
OF ballI ballI  
subset\_CollectI[OF F1set2\_JF1set\_incl\_JF1set]  
subset\_CollectI[OF F1set3\_JF2set\_incl\_JF1set]  
subset\_CollectI[OF F2set2\_JF1set\_incl\_JF2set]  
subset\_CollectI[OF F2set3\_JF2set\_incl\_JF2set]]

end

ML ⟨rule\_by\_tactic @ {context} (ALLGOALS (TRY o etac @ {context} asm\_rl)) @ {thm JFset\_induct}⟩

abbreviation JF1in where JF1in B ≡ {a. JF1set a ⊆ B}

abbreviation JF2in where JF2in B ≡ {a. JF2set a ⊆ B}

definition JF1rel where

JF1rel R = (BNF\_Def.Grp (JF1in (Collect (case\_prod R))) (JF1map fst))<sup>^--1</sup> OO  
(BNF\_Def.Grp (JF1in (Collect (case\_prod R))) (JF1map snd))

definition JF2rel where

JF2rel R = (BNF\_Def.Grp (JF2in (Collect (case\_prod R))) (JF2map fst))<sup>^--1</sup> OO  
(BNF\_Def.Grp (JF2in (Collect (case\_prod R))) (JF2map snd))

lemma in\_JF1rel:

JF1rel R x y ⟷ (∃ z. z ∈ JF1in (Collect (case\_prod R)) ∧ JF1map fst z = x ∧ JF1map snd z = y)  
by (rule predicate2\_eqD[OF trans[OF JF1rel\_def OO\_Grp\_alt]])

lemma in\_JF2rel:

JF2rel R x y ⟷ (∃ z. z ∈ JF2in (Collect (case\_prod R)) ∧ JF2map fst z = x ∧ JF2map snd z = y)  
by (rule predicate2\_eqD[OF trans[OF JF2rel\_def OO\_Grp\_alt]])

lemma J\_rel\_coind\_ind:

[∀ x y. R2 x y ⟶ (f x y ∈ F1in (Collect (case\_prod R1)) (Collect (case\_prod R2)) (Collect (case\_prod R3))) ∧  
F1map fst fst fst (f x y) = dtor1 x ∧  
F1map snd snd snd (f x y) = dtor1 y];  
∀ x y. R3 x y ⟶ (g x y ∈ F2in (Collect (case\_prod R1)) (Collect (case\_prod R2)) (Collect (case\_prod R3))) ∧  
F2map fst fst fst (g x y) = dtor2 x ∧  
F2map snd snd snd (g x y) = dtor2 y]] ⟹  
(∀ a ∈ JF1set z1. ∀ x y. R2 x y ∧ z1 = unfold1 (case\_prod f) (case\_prod g) (x, y) ⟶ R1 (fst a) (snd a)) ∧  
(∀ a ∈ JF2set z2. ∀ x y. R3 x y ∧ z2 = unfold2 (case\_prod f) (case\_prod g) (x, y) ⟶ R1 (fst a) (snd a))  
apply (tactic ⟨rtac @ {context} (rule\_by\_tactic @ {context} (ALLGOALS (TRY o etac @ {context} asm\_rl))  
@ {thm JFset\_induct} of  
λ a z1. ∀ x y. R2 x y ∧ z1 = unfold1 (case\_prod f) (case\_prod g) (x, y) ⟶ R1 (fst a) (snd a)  
λ a z2. ∀ x y. R3 x y ∧ z2 = unfold2 (case\_prod f) (case\_prod g) (x, y) ⟶ R1 (fst a) (snd a)  
z1 z2}} 1) )  
apply (rule allI impI)+  
apply (erule conjE)  
apply (drule spec2)  
apply (erule thin\_rl)  
apply (drule mp)  
apply assumption  
apply (erule CollectE conjE Collect\_case\_prodD[OF subsetD] rev\_subsetD)+  
apply hypsubst

```

unfolding unfold1 F1.set_map(1) prod.case image_id id_apply
  apply (rule subset_refl)

  apply (rule allI impI)+
  apply (erule conjE)
  apply (erule thin_rl)
  apply (drule spec2)
  apply (drule mp)
  apply assumption
  apply (erule CollectE conjE Collect_case_prodD[OF subsetD] rev_subsetD)+
  apply hypsubst
unfolding unfold2 F2.set_map(1) prod.case image_id id_apply
  apply (rule subset_refl)

```

```

  apply (rule impI allI)+
  apply (erule conjE)
  apply (drule spec2)
  apply (erule thin_rl)
  apply (drule mp)
  apply assumption
  apply (erule CollectE conjE)+
  apply (tactic <hyp_subst_tac @{context} 1>)
unfolding unfold1 F1.set_map(2) prod.case image_id id_apply
  apply (erule imageE)
  apply (tactic <hyp_subst_tac @{context} 1>)
  apply (erule allE mp)+
  apply (rule conjI)
  apply (erule Collect_case_prodD[OF subsetD])
  apply assumption
  apply (rule arg_cong[OF surjective_pairing])

```

```

  apply (rule impI allI)+
  apply (erule conjE)
  apply (drule spec2)
  apply (erule thin_rl)
  apply (drule mp)
  apply assumption
  apply (erule CollectE conjE)+
  apply (tactic <hyp_subst_tac @{context} 1>)
unfolding unfold1 F1.set_map(3) prod.case image_id id_apply
  apply (erule imageE)
  apply (tactic <hyp_subst_tac @{context} 1>)
  apply (erule allE mp)+
  apply (rule conjI)
  apply (erule Collect_case_prodD[OF subsetD])
  apply assumption
  apply (rule arg_cong[OF surjective_pairing])

```

```

  apply (rule impI allI)+
  apply (erule conjE)
  apply (erule thin_rl)
  apply (drule spec2)
  apply (drule mp)
  apply assumption
  apply (erule CollectE conjE)+
  apply (tactic <hyp_subst_tac @{context} 1>)
unfolding unfold2 F2.set_map(2) prod.case image_id id_apply
  apply (erule imageE)

```

```

apply (tactic <hyp_subst_tac @ {context} 1 >)
apply (erule allE mp)+
apply (rule conjI)
  apply (erule Collect_case_prodD[OF subsetD])
  apply assumption
apply (rule arg_cong[OF surjective_pairing])

```

```

apply (rule impI allI)+
apply (erule conjE)
apply (erule thin_rl)
apply (drule spec2)
apply (drule mp)
  apply assumption
apply (erule CollectE conjE)+
apply (tactic <hyp_subst_tac @ {context} 1 >)
unfolding unfold2 F2.set_map(3) prod.case_image_id id_apply
apply (erule imageE)
apply (tactic <hyp_subst_tac @ {context} 1 >)
apply (erule allE mp)+
apply (rule conjI)
  apply (erule Collect_case_prodD[OF subsetD])
  apply assumption
apply (rule arg_cong[OF surjective_pairing])
done

```

**lemma** *J\_rel\_coind\_coind1*:

```


$$\llbracket \forall x y. R2\ x\ y \longrightarrow (f\ x\ y \in F1in\ (Collect\ (case\_prod\ R1))\ (Collect\ (case\_prod\ R2))\ (Collect\ (case\_prod\ R3))) \wedge$$


$$F1map\ fst\ fst\ fst\ (f\ x\ y) = dtor1\ x \wedge$$


$$F1map\ snd\ snd\ snd\ (f\ x\ y) = dtor1\ y;$$


$$\forall x y. R3\ x\ y \longrightarrow (g\ x\ y \in F2in\ (Collect\ (case\_prod\ R1))\ (Collect\ (case\_prod\ R2))\ (Collect\ (case\_prod\ R3))) \wedge$$


$$F2map\ fst\ fst\ fst\ (g\ x\ y) = dtor2\ x \wedge$$


$$F2map\ snd\ snd\ snd\ (g\ x\ y) = dtor2\ y \rrbracket \Longrightarrow$$


$$((\exists y. R2\ x1\ y \wedge x1' = JF1map\ fst\ (unfold1\ (case\_prod\ f)\ (case\_prod\ g)\ (x1,\ y))) \longrightarrow x1' = x1) \wedge$$


$$((\exists y. R3\ x2\ y \wedge x2' = JF2map\ fst\ (unfold2\ (case\_prod\ f)\ (case\_prod\ g)\ (x2,\ y))) \longrightarrow x2' = x2)$$

apply (rule Frel_coind[of
  
$$\lambda x1' x1. \exists y. R2\ x1\ y \wedge x1' = JF1map\ fst\ (unfold1\ (case\_prod\ f)\ (case\_prod\ g)\ (x1,\ y))$$

  
$$\lambda x2' x2. \exists y. R3\ x2\ y \wedge x2' = JF2map\ fst\ (unfold2\ (case\_prod\ f)\ (case\_prod\ g)\ (x2,\ y))$$

  
$$x1'\ x1$$

  
$$x2'\ x2$$

])
apply (intro allI impI iffD2[OF F1.in_rel])

```

```

apply (erule exE conjE)+
apply (drule spec2)
apply (erule thin_rl)
apply (drule mp)
  apply assumption
apply (erule CollectE conjE)+
apply (tactic <hyp_subst_tac @ {context} 1 >)
apply (rule exI)
apply (rule conjI[rotated])
apply (rule conjI)
  apply (rule trans[OF F1.map_comp])
  apply (rule sym[OF trans[OF JF1map_simps]])
  apply (rule trans[OF arg_cong[OF unfold1]])
  apply (rule trans[OF F1.map_comp F1.map_cong0])
  apply (rule trans[OF fun_cong[OF o_id]])
  apply (rule sym[OF fun_cong[OF fst_diag_fst]])
  apply (rule sym[OF trans[OF o_apply]])
  apply (rule fst_conv)
  apply (rule sym[OF trans[OF o_apply]])
  apply (rule fst_conv)

```

```

apply (rule trans[OF F1.map_comp])
apply (rule trans[OF F1.map_cong0])
  apply (rule fun_cong[OF snd_diag fst])
  apply (rule trans[OF o_apply])
  apply (rule snd_conv)
apply (rule trans[OF o_apply])
apply (rule snd_conv)
apply (erule trans[OF arg_cong[OF prod.case]])

```

```

apply (unfold prod.case o_def fst_conv snd_conv) []
apply (rule CollectI)
apply (rule conjI)
apply (rule ord_eq_le_trans[OF F1.set_map(1)])
apply (rule image_subsetI CollectI case_prodI)+
apply (rule refl)

```

```

apply (rule conjI)
apply (rule ord_eq_le_trans[OF F1.set_map(2)])
apply (rule image_subsetI CollectI case_prodI exI)+
apply (rule conjI)
  apply (erule Collect_case_prodD[OF subsetD])
  apply assumption
apply (rule arg_cong[OF surjective_pairing])

```

```

apply (rule ord_eq_le_trans[OF F1.set_map(3)])
apply (rule image_subsetI CollectI case_prodI exI)+
apply (rule conjI)
  apply (erule Collect_case_prodD[OF subsetD])
  apply assumption
apply (rule arg_cong[OF surjective_pairing])

```

```

apply (intro allI impI iffD2[OF F2.in_rel])

```

```

apply (erule exE conjE)+
apply (erule thin_rl)
apply (drule spec2)
apply (drule mp)
  apply assumption
apply (erule CollectE conjE)+
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (rule exI)
apply (rule conjI[rotated])
apply (rule conjI)
  apply (rule trans[OF F2.map_comp])
  apply (rule sym[OF trans[OF JF2map_simps]])
  apply (rule trans[OF arg_cong[OF unfold2]])
  apply (rule trans[OF F2.map_comp F2.map_cong0])
    apply (rule fun_cong[OF trans[OF o_id fst_diag fst[symmetric]]])
    apply (rule sym[OF trans[OF o_apply]])
    apply (rule fst_conv)
  apply (rule sym[OF trans[OF o_apply]])
  apply (rule fst_conv)
apply (rule trans[OF F2.map_comp])
apply (rule trans[OF F2.map_cong0])
  apply (rule fun_cong[OF snd_diag fst])
  apply (rule trans[OF o_apply])
  apply (rule snd_conv)
  apply (rule trans[OF o_apply])
  apply (rule snd_conv)
apply (erule trans[OF arg_cong[OF prod.case]])

```

```

apply (unfold prod.case o_def fst_conv snd_conv) []
apply (rule CollectI)
apply (rule conjI)
  apply (rule ord_eq_le_trans[OF F2.set_map(1)])
  apply (rule image_subsetI)
  apply (rule CollectI)
  apply (rule case_prodI)
  apply (rule refl)

```

```

apply (rule conjI)
apply (rule ord_eq_le_trans[OF F2.set_map(2)])
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule exI)
apply (rule conjI)
  apply (erule Collect_case_prodD[OF subsetD])
  apply assumption
apply (rule arg_cong[OF surjective_pairing])

```

```

apply (rule ord_eq_le_trans[OF F2.set_map(3)])
apply (rule image_subsetI CollectI case_prodI exI)+
apply (rule conjI)
apply (erule Collect_case_prodD[OF subsetD])
apply assumption
apply (rule arg_cong[OF surjective_pairing])
done

```

**lemma** *J\_rel\_coind\_coind2*:

```

[[ $\forall x y. R2 x y \longrightarrow (f x y \in F1in (Collect (case\_prod R1)) (Collect (case\_prod R2)) (Collect (case\_prod R3))) \wedge$ 
 $F1map\ fst\ fst\ fst\ (f\ x\ y) = dtor1\ x \wedge$ 
 $F1map\ snd\ snd\ snd\ (f\ x\ y) = dtor1\ y$ ];
 $\forall x y. R3 x y \longrightarrow (g x y \in F2in (Collect (case\_prod R1)) (Collect (case\_prod R2)) (Collect (case\_prod R3))) \wedge$ 
 $F2map\ fst\ fst\ fst\ (g\ x\ y) = dtor2\ x \wedge$ 
 $F2map\ snd\ snd\ snd\ (g\ x\ y) = dtor2\ y$ ]]  $\implies$ 
(( $\exists x. R2\ x\ y1 \wedge y1' = JF1map\ snd\ (unfold1\ (case\_prod\ f)\ (case\_prod\ g)\ (x,\ y1))$ )  $\longrightarrow y1' = y1$ )  $\wedge$ 
(( $\exists x. R3\ x\ y2 \wedge y2' = JF2map\ snd\ (unfold2\ (case\_prod\ f)\ (case\_prod\ g)\ (x,\ y2))$ )  $\longrightarrow y2' = y2$ )
apply (rule Frel_coind[of
   $\lambda y1' y1. \exists x. R2\ x\ y1 \wedge y1' = JF1map\ snd\ (unfold1\ (case\_prod\ f)\ (case\_prod\ g)\ (x,\ y1))$ 
   $\lambda y2' y2. \exists x. R3\ x\ y2 \wedge y2' = JF2map\ snd\ (unfold2\ (case\_prod\ f)\ (case\_prod\ g)\ (x,\ y2))$ 
   $y1'\ y1$ 
   $y2'\ y2$ 
])
apply (intro allI impI iffD2[OF F1.in_rel])

```

```

apply (erule exE conjE)+
apply (erule spec2)
apply (erule thin_rl)
apply (erule mp)
apply assumption
apply (erule CollectE conjE)+
apply (tactic <hyp_subst_tac @{context} 1>)
apply (rule exI)
apply (rule conjI[rotated])
apply (rule conjI)
  apply (rule trans[OF F1.map_comp])
  apply (rule sym[OF trans[OF JF1map_simps]])
  apply (rule trans[OF arg_cong[OF unfold1]])
  apply (rule trans[OF F1.map_comp F1.map_cong0])
    apply (rule trans[OF fun_cong[OF o_id]])
    apply (rule sym[OF fun_cong[OF fst_diag_snd]])
  apply (rule sym[OF trans[OF o_apply]])
  apply (rule fst_conv)

```

```

apply (rule sym[OF trans[OF o_apply]])
apply (rule fst_conv)
apply (rule trans[OF F1.map_comp])
apply (rule trans[OF F1.map_cong0])
  apply (rule fun_cong[OF snd_diag_snd])
  apply (rule trans[OF o_apply])
  apply (rule snd_conv)
apply (rule trans[OF o_apply])
apply (rule snd_conv)
apply (erule trans[OF arg_cong[OF prod.case]])

apply (unfold prod.case o_def fst_conv snd_conv) []
apply (rule CollectI)
apply (rule conjI)
apply (rule ord_eq_le_trans[OF F1.set_map(1)])
apply (rule image_subsetI CollectI case_prodI)+
apply (rule refl)

apply (rule conjI)
apply (rule ord_eq_le_trans[OF F1.set_map(2)])
apply (rule image_subsetI CollectI case_prodI exI)+
apply (rule conjI)
  apply (erule Collect_case_prodD[OF subsetD])
  apply assumption
apply (rule arg_cong[OF surjective_pairing])

apply (rule ord_eq_le_trans[OF F1.set_map(3)])
apply (rule image_subsetI CollectI case_prodI exI)+
apply (rule conjI)
  apply (erule Collect_case_prodD[OF subsetD])
  apply assumption
apply (rule arg_cong[OF surjective_pairing])

apply (intro allI impI iffD2[OF F2.in_rel])

apply (erule exE conjE)+
apply (erule thin_rl)
apply (drule spec2)
apply (drule mp)
  apply assumption
apply (erule CollectE conjE)+
apply (tactic ⟨hyp_subst_tac @ {context} 1⟩)
apply (rule exI)
apply (rule conjI[rotated])
apply (rule conjI)
  apply (rule trans[OF F2.map_comp])
  apply (rule sym[OF trans[OF JF2map_simps]])
  apply (rule trans[OF arg_cong[OF unfold2]])
  apply (rule trans[OF F2.map_comp F2.map_cong0])
    apply (rule trans[OF fun_cong[OF o_id]])
    apply (rule sym[OF fun_cong[OF fst_diag_snd]])
  apply (rule sym[OF trans[OF o_apply]])
  apply (rule fst_conv)
apply (rule sym[OF trans[OF o_apply]])
apply (rule fst_conv)
apply (rule trans[OF F2.map_comp])
apply (rule trans[OF F2.map_cong0])
  apply (rule fun_cong[OF snd_diag_snd])
  apply (rule trans[OF o_apply])
  apply (rule snd_conv)
apply (rule trans[OF o_apply])

```

```

apply (rule snd_conv)
apply (erule trans[OF arg_cong[OF prod.case]])

```

```

apply (unfold prod.case o_def fst_conv snd_conv) []
apply (rule CollectI)
apply (rule conjI)
apply (rule ord_eq_le_trans[OF F2.set_map(1)])
apply (rule image_subsetI CollectI case_prodI)+
apply (rule refl)

```

```

apply (rule conjI)
apply (rule ord_eq_le_trans[OF F2.set_map(2)])
apply (rule image_subsetI CollectI case_prodI exI)+
apply (rule conjI)
apply (erule Collect_case_prodD[OF subsetD])
apply assumption
apply (rule arg_cong[OF surjective_pairing])

```

```

apply (rule ord_eq_le_trans[OF F2.set_map(3)])
apply (rule image_subsetI CollectI case_prodI exI)+
apply (rule conjI)
apply (erule Collect_case_prodD[OF subsetD])
apply assumption
apply (rule arg_cong[OF surjective_pairing])
done

```

lemma *J\_rel\_coind*:

```

assumes CIH1:  $\forall x2 y2. R2 x2 y2 \longrightarrow F1rel R1 R2 R3 (dtor1 x2) (dtor1 y2)$ 
and CIH2:  $\forall x3 y3. R3 x3 y3 \longrightarrow F2rel R1 R2 R3 (dtor2 x3) (dtor2 y3)$ 
shows  $R2 \leq JF1rel R1 \wedge R3 \leq JF2rel R1$ 
apply (insert CIH1 CIH2)
unfolding F1.in_rel F2.in_rel ex_simps(6)[symmetric] choice_iff
apply (erule exE)+
apply (rule conjI)

```

```

apply (rule predicate2I)
apply (rule iffD2[OF in_JF1rel])
apply (rule exI conjI)+
apply (rule CollectI)
apply (rule rev_mp[OF conjunct1[OF J_rel_coind_ind]])
apply assumption
apply assumption
apply (erule thin_rl)
apply (erule thin_rl)
apply (rule impI)
apply (rule subsetI CollectI iffD2[OF case_prod_beta])+
apply (drule bspec)
apply assumption
apply (erule allE mp conjE)+
apply (erule conjI[OF _ refl])

```

```

apply (rule conjI)
apply (rule rev_mp[OF conjunct1[OF J_rel_coind_coind1]])
apply assumption
apply assumption
apply (erule thin_rl)
apply (erule thin_rl)
apply (rule impI)
apply (erule mp)
apply (rule exI)
apply (erule conjI[OF _ refl])

```

```

apply (rule rev_mp[OF conjunct1[OF J_rel_coind_coind2]])

```



```

  apply assumption
  apply assumption
  apply (erule thin_rl)
  apply (erule thin_rl)
  apply (rule impI)
  apply (erule mp)
  apply (rule exI)
  apply (erule conjI[OF _ refl])

```

```

apply (rule predicate2I)
apply (rule iffD2[OF in_JF2rel])
apply (rule exI conjI)+
  apply (rule rev_mp[OF conjunct2[OF J_rel_coind_ind]])
    apply assumption
    apply assumption
  apply (erule thin_rl)
  apply (erule thin_rl)
  apply (rule impI)
  apply (rule CollectI)
  apply (rule subsetI CollectI iffD2[OF case_prod_beta])+
  apply (erule bspec)
  apply assumption
  apply (erule allE mp conjE)+
  apply (erule conjI[OF _ refl])

```

```

apply (rule conjI)
apply (rule rev_mp[OF conjunct2[OF J_rel_coind_coind1]])
  apply assumption
  apply assumption
  apply (erule thin_rl)
  apply (erule thin_rl)
  apply (rule impI)
  apply (erule mp)
  apply (rule exI)
  apply (erule conjI[OF _ refl])

```

```

apply (rule rev_mp[OF conjunct2[OF J_rel_coind_coind2]])
  apply assumption
  apply assumption
  apply (erule thin_rl)
  apply (erule thin_rl)
  apply (rule impI)
  apply (erule mp)
  apply (rule exI)
  apply (erule conjI[OF _ refl])
done

```

lemma *JF1rel\_F1rel*:  $JF1rel\ R\ a\ b \longleftrightarrow F1rel\ R\ (JF1rel\ R)\ (JF2rel\ R)\ (dtor1\ a)\ (dtor1\ b)$

```

apply (rule iffI)
  apply (erule iffD1[OF in_JF1rel])
  apply (erule exE conjE CollectE)+
  apply (rule iffD2[OF F1.in_rel])
  apply (rule exI)
  apply (rule conjI)
  apply (rule CollectI)
  apply (rule conjI)
  apply (rule ord_eq_le_trans)
  apply (rule F1.set_map(1))
  apply (rule ord_eq_le_trans)
  apply (rule trans[OF fun_cong[OF image_id] id_apply])
  apply (erule subset_trans[OF F1set1_incl_JF1set])

```

```

apply (rule conjI)
apply (rule ord_eq_le_trans)
  apply (rule F1.set_map(2))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule iffD2[OF in_JF1rel])
apply (rule exI)
apply (rule conjI)
  apply (rule CollectI)
apply (erule subset_trans[OF F1set2_JF1set_incl_JF1set])
apply assumption
apply (rule conjI)
  apply (rule refl)
apply (rule refl)

```

```

apply (rule ord_eq_le_trans)
apply (rule F1.set_map(3))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule iffD2[OF in_JF2rel])
apply (rule exI)
apply (rule conjI)
  apply (rule CollectI)
apply (erule subset_trans[OF F1set3_JF2set_incl_JF1set])
apply assumption
apply (rule conjI)
  apply (rule refl)
apply (rule refl)
apply (rule conjI)

```

```

apply (rule trans)
apply (rule F1.map_comp)
apply (rule trans)
apply (rule F1.map_cong0)
  apply (rule fun_cong[OF o_id])
apply (rule trans)
  apply (rule o_apply)
apply (rule fst_conv)
apply (rule trans)
  apply (rule o_apply)
apply (rule fst_conv)
apply (rule trans)
apply (rule sym)
apply (rule JF1map_simps)
apply (rule iffD2[OF dtor1_diff])
apply assumption

```

```

apply (rule trans)
apply (rule F1.map_comp)
apply (rule trans)
apply (rule F1.map_cong0)
  apply (rule fun_cong[OF o_id])
apply (rule trans)
  apply (rule o_apply)
apply (rule snd_conv)
apply (rule trans)
  apply (rule o_apply)
apply (rule snd_conv)
apply (rule trans)
apply (rule sym)

```

```

apply (rule JF1map_simps)
apply (rule iffD2[OF dtor1_diff])
apply assumption

apply (drule iffD1[OF F1.in_rel])
apply (erule exE conjE CollectE) +
apply (rule iffD2[OF in_JF1rel])
apply (rule exI)
apply (rule conjI)
apply (rule CollectI)
apply (rule ord_eq_le_trans)
apply (rule JF1set_simps)
apply (rule Un_least)
apply (rule ord_eq_le_trans)
  apply (rule box_equals)
    apply (rule F1.set_map(1))
      apply (rule arg_cong[OF sym[OF dtor1_ctor1]])
      apply (rule trans[OF fun_cong[OF image_id] id_apply])
apply assumption
apply (rule Un_least)
apply (rule ord_eq_le_trans)
  apply (rule SUP_cong[OF _ refl])
  apply (rule box_equals[OF _ _ refl])
  apply (rule F1.set_map(2))
  apply (rule arg_cong[OF sym[OF dtor1_ctor1]])
apply (rule UN_least)
apply (drule rev_subsetD)
  apply (erule image_mono)
apply (erule imageE)
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE) +
apply hypsubst
apply (drule iffD1[OF in_JF1rel])
apply (drule someI_ex)
apply (erule conjE) +
apply (erule CollectE conjE) +
apply assumption

apply (rule ord_eq_le_trans)
apply (rule trans[OF arg_cong[OF dtor1_ctor1]])
apply (rule arg_cong[OF F1.set_map(3)])
apply (rule UN_least)
apply (drule rev_subsetD)
apply (erule image_mono)
apply (erule imageE)
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE) +
apply hypsubst
apply (drule iffD1[OF in_JF2rel])
apply (drule someI_ex)
apply (erule exE conjE) +
apply (erule CollectD)

apply (rule conjI)

apply (rule iffD1[OF dtor1_diff])
apply (rule trans)
apply (rule JF1map_simps)
apply (rule box_equals)
  apply (rule F1.map_comp)
  apply (rule arg_cong[OF sym[OF dtor1_ctor1]])
apply (rule trans)
apply (rule F1.map_cong0)

```

```

    apply (rule fun_cong[OF o_id])
    apply (rule trans[OF o_apply])
    apply (drule rev_subsetD)
    apply assumption
    apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
    apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
    apply hypsubst
    apply (drule iffD1[OF in_JF1rel])
    apply (drule someI_ex)
    apply (erule conjE)+
    apply assumption
    apply (rule trans[OF o_apply])
    apply (drule rev_subsetD)
    apply assumption
    apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
    apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
    apply hypsubst
    apply (drule iffD1[OF in_JF2rel])
    apply (drule someI_ex)
    apply (erule conjE)+
    apply assumption
    apply assumption

apply (rule iffD1[OF dtor1_diff])
apply (rule trans)
  apply (rule JF1map_simps)
apply (rule trans)
  apply (rule arg_cong[OF dtor1_ctor1])
apply (rule trans)
  apply (rule F1.map_comp)
apply (rule trans)
  apply (rule F1.map_cong0)
  apply (rule fun_cong[OF o_id])
  apply (rule trans[OF o_apply])
  apply (drule rev_subsetD)
  apply assumption
  apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
  apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
  apply hypsubst
  apply (drule iffD1[OF in_JF1rel])
  apply (drule someI_ex)
  apply (erule conjE)+
  apply assumption
  apply (rule trans[OF o_apply])
  apply (drule rev_subsetD)
  apply assumption
  apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
  apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
  apply hypsubst
  apply (drule iffD1[OF in_JF2rel])
  apply (drule someI_ex)
  apply (erule conjE)+
  apply assumption
  apply assumption
done

```

```

lemma JF2rel_F2rel: JF2rel R a b  $\longleftrightarrow$  F2rel R (JF1rel R) (JF2rel R) (dtor2 a) (dtor2 b)
  apply (rule iffI)
  apply (drule iffD1[OF in_JF2rel])
  apply (erule exE conjE CollectE)+
  apply (rule iffD2[OF F2.in_rel])
  apply (rule exI)
  apply (rule conjI)

```

```

apply (rule CollectI)
apply (rule conjI)
  apply (rule ord_eq_le_trans)
    apply (rule F2.set_map(1))
  apply (rule ord_eq_le_trans)
    apply (rule trans[OF fun_cong[OF image_id] id_apply])
apply (rule subset_trans)
  apply (rule F2set1_incl_JF2set)
apply assumption

```

```

apply (rule conjI)
apply (rule ord_eq_le_trans)
  apply (rule F2.set_map(2))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule iffD2[OF in_JF1rel])
apply (rule exI)
apply (rule conjI)
  apply (rule CollectI)
  apply (rule subset_trans)
    apply (rule F2set2_JF1set_incl_JF2set)
  apply assumption
apply assumption
apply (rule conjI)
  apply (rule refl)
apply (rule refl)

```

```

apply (rule ord_eq_le_trans)
apply (rule F2.set_map(3))
apply (rule image_subsetI)
apply (rule CollectI)
apply (rule case_prodI)
apply (rule iffD2[OF in_JF2rel])
apply (rule exI)
apply (rule conjI)
  apply (rule CollectI)
  apply (rule subset_trans)
    apply (rule F2set3_JF2set_incl_JF2set)
  apply assumption
apply assumption
apply (rule conjI)
  apply (rule refl)
apply (rule refl)
apply (rule conjI)

```

```

apply (rule trans)
apply (rule F2.map_comp)
apply (rule trans)
apply (rule F2.map_cong0)
  apply (rule fun_cong[OF o_id])
  apply (rule trans)
  apply (rule o_apply)
  apply (rule fst_conv)
apply (rule trans)
  apply (rule o_apply)
apply (rule fst_conv)
apply (rule trans)
apply (rule sym)
apply (rule JF2map_simps)
apply (rule iffD2)
  apply (rule dtor2_diff)
apply assumption

```

```

apply (rule trans)
apply (rule F2.map_comp)
apply (rule trans)
apply (rule F2.map_cong0)
  apply (rule fun_cong[OF o_id])
  apply (rule trans)
  apply (rule o_apply)
  apply (rule snd_conv)
apply (rule trans)
  apply (rule o_apply)
  apply (rule snd_conv)
apply (rule trans)
apply (rule sym)
apply (rule JF2map_simps)
apply (rule iffD2)
  apply (rule dtor2_diff)
apply assumption

apply (drule iffD1[OF F2.in_rel])
apply (erule exE conjE CollectE)+
apply (rule iffD2[OF in_JF2rel])
apply (rule exI)
apply (rule conjI)
apply (rule CollectI)
apply (rule ord_eq_le_trans)
  apply (rule JF2set_simps)
apply (rule Un_least)
apply (rule ord_eq_le_trans)
  apply (rule trans)
  apply (rule trans)
  apply (rule arg_cong[OF dtor2_ctor2])
  apply (rule F2.set_map(1))
  apply (rule trans[OF fun_cong[OF image_id] id_apply])
apply assumption
apply (rule Un_least)
apply (rule ord_eq_le_trans)
  apply (rule trans[OF arg_cong[OF dtor2_ctor2]])
  apply (rule arg_cong[OF F2.set_map(2)])
apply (rule UN_least)
apply (drule rev_subsetD)
  apply (erule image_mono)
apply (erule imageE)
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (drule iffD1[OF in_JF1rel])
apply (drule someI_ex)
apply (erule conjE)+
apply (erule CollectD)

apply (rule ord_eq_le_trans)
apply (rule trans[OF arg_cong[OF dtor2_ctor2]])
apply (rule arg_cong[OF F2.set_map(3)])
apply (rule UN_least)
apply (drule rev_subsetD)
  apply (erule image_mono)
apply (erule imageE)
apply (drule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (drule iffD1[OF in_JF2rel])
apply (drule someI_ex)

```

```

apply (erule exE conjE)+
apply (erule CollectD)

apply (rule conjI)

apply (rule iffD1)
  apply (rule dtor2_diff)
apply (rule trans)
  apply (rule JF2map_simps)
apply (rule trans)
  apply (rule arg_cong[OF dtor2_ctor2])
apply (rule trans)
  apply (rule F2.map_comp)
apply (rule trans)
  apply (rule F2.map_cong0)
    apply (rule fun_cong[OF o_id])
    apply (rule trans[OF o_apply])
    apply (erule rev_subsetD)
    apply assumption
  apply (erule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (erule iffD1[OF in_JF1rel])
apply (erule someI_ex)
apply (erule conjE)+
apply assumption
apply (rule trans[OF o_apply])
apply (erule rev_subsetD)
  apply assumption
apply (erule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (erule iffD1[OF in_JF2rel])
apply (erule someI_ex)
apply (erule conjE)+
apply assumption
apply assumption

apply (rule iffD1)
  apply (rule dtor2_diff)
apply (rule trans)
  apply (rule JF2map_simps)
apply (rule trans)
  apply (rule arg_cong[OF dtor2_ctor2])
apply (rule trans)
  apply (rule F2.map_comp)
apply (rule trans)
  apply (rule F2.map_cong0)
    apply (rule fun_cong[OF o_id])
    apply (rule trans[OF o_apply])
    apply (erule rev_subsetD)
    apply assumption
  apply (erule ssubst_mem[OF surjective_pairing[symmetric]])
apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (erule iffD1[OF in_JF1rel])
apply (erule someI_ex)
apply (erule conjE)+
apply assumption
apply (rule trans[OF o_apply])
apply (erule rev_subsetD)
  apply assumption
apply (erule ssubst_mem[OF surjective_pairing[symmetric]])

```

```

apply (erule CollectE case_prodE iffD1[OF prod.inject, elim_format] conjE)+
apply hypsubst
apply (drule iffD1[OF in_JF2rel])
apply (drule someI_ex)
apply (erule conjE)+
apply assumption
apply assumption
done

```

**lemma** JFrel\_Comp\_le:

```

JF1rel R1 OO JF1rel R2 ≤ JF1rel (R1 OO R2) ∧ JF2rel R1 OO JF2rel R2 ≤ JF2rel (R1 OO R2)
apply (rule J_rel_coind[OF allI[OF allI[OF impI]] allI[OF allI[OF impI]]])
apply (rule predicate2D[OF eq_refl[OF sym[OF F1.rel_compp]]])
apply (erule relcomppE)
apply (rule relcomppI)
apply (erule iffD1[OF JF1rel_F1rel])
apply (erule iffD1[OF JF1rel_F1rel])
apply (rule predicate2D[OF eq_refl[OF sym[OF F2.rel_compp]]])
apply (erule relcomppE)
apply (rule relcomppI)
apply (erule iffD1[OF JF2rel_F2rel])
apply (erule iffD1[OF JF2rel_F2rel])
done

```

**context includes** lifting\_syntax

**begin**

**lemma** unfold\_transfer:

```

((S ==> F1rel R S T) ==> (T ==> F2rel R S T) ==> S ==> JF1rel R) unfold1 unfold1 ∧
((S ==> F1rel R S T) ==> (T ==> F2rel R S T) ==> T ==> JF2rel R) unfold2 unfold2
unfolding rel_fun_def_butlast_all_conj_distrib[symmetric] imp_conjR[symmetric]
unfolding rel_fun_iff_geq_image2p
apply (rule allI impI)+
apply (rule J_rel_coind)
apply (rule allI impI)+
apply (erule image2pE)
apply hypsubst
apply (unfold unfold1 unfold2) [1]
apply (rule rel_funD[OF rel_funD[OF rel_funD[OF rel_funD[OF F1.map_transfer]]]])
apply (rule id_transfer)
apply (rule rel_fun_image2p)
apply (rule rel_fun_image2p)
apply (erule predicate2D)
apply (erule image2pI)

apply (rule allI impI)+
apply (erule image2pE)
apply hypsubst
apply (unfold unfold1 unfold2) [1]
apply (rule rel_funD[OF rel_funD[OF rel_funD[OF rel_funD[OF F2.map_transfer]]]])
apply (rule id_transfer)
apply (rule rel_fun_image2p)
apply (rule rel_fun_image2p)
apply (erule predicate2D)
apply (erule image2pI)
done

```

**end**

**ML** <

```

BNF_FP_Util.mk_xtor_co_iter_o_map_thms BNF_Util.Greatest_FP false 1 @ {thm unfold_unique}
@ {thms JF1map JF2map} (map (BNF_Tactics.mk_pointfree2 @ {context}) @ {thms unfold1 unfold2})
@ {thms F1.map_comp0[symmetric] F2.map_comp0[symmetric]} @ {thms F1.map_cong0 F2.map_cong0}

```



```

>
ML <
  BNF_FP_Util.mk_xtor_co_iter_o_map_thms BNF_Util.Greatest_FP true 1 @ {thm corec_unique}
  @ {thms JF1map JF2map} (map (BNF_Tactics.mk_pointfree2 @ {context}) @ {thms corec1 corec2})
  @ {thms F1.map_comp0[symmetric] F2.map_comp0[symmetric]} @ {thms F1.map_cong0 F2.map_cong0}
>

```

```

bnf 'a JF1
  map: JF1map
  sets: JF1set
  bd: bd_F
  wits: JF1wit
  rel: JF1rel
    apply -
      apply (rule JF1map_id)
      apply (rule JF1map_comp)
      apply (erule JF1map_cong0[OF ballI])
      apply (rule JF1set_natural)
      apply (rule bd_F_card_order)
      apply (rule conjunct1[OF bd_F_Cinfinite])
      apply (rule bd_F_regularCard)
      apply (rule JF1set_bd)
      apply (rule conjunct1[OF JFrel_Comp_le])
      apply (rule JF1rel_def[unfolded OO_Grp_alt mem_Collect_eq])
apply (erule JF1wit)
done

```

```

bnf 'a JF2
  map: JF2map
  sets: JF2set
  bd: bd_F
  wits: JF2wit
  rel: JF2rel
    apply -
      apply (rule JF2map_id)
      apply (rule JF2map_comp)
      apply (erule JF2map_cong0[OF ballI])
      apply (rule JF2set_natural)
      apply (rule bd_F_card_order)
      apply (rule conjunct1[OF bd_F_Cinfinite])
      apply (rule bd_F_regularCard)
      apply (rule JF2set_bd)
      apply (rule conjunct2[OF JFrel_Comp_le])
      apply (rule JF2rel_def[unfolded OO_Grp_alt mem_Collect_eq])
apply (erule JF2wit)
done

```

### 3 Normalized Composition of BNFs

Expected normal form: outer m-ary BNF is composed with m inner n-ary BNFs.

```

unbundle cardinal_syntax

```

```

declare [[bnf_internals]]
bnf-axiomatization (dead 'p1, F1set1: 'a1, F1set2: 'a2) F1
  [wits: ('p1, 'a1, 'a2) F1]
  for map: F1map rel: F1rel
bnf-axiomatization (dead 'p2, F2set1: 'a1, F2set2: 'a2) F2
  [wits: 'a1  $\Rightarrow$  ('p2, 'a1, 'a2) F2 'a2  $\Rightarrow$  ('p2, 'a1, 'a2) F2]
  for map: F2map rel: F2rel

```

**bnf-axiomatization** (*dead* 'p3, F3set1: 'a1, F3set2: 'a2) F3

[*wits*: 'a1  $\Rightarrow$  'a2  $\Rightarrow$  ('p3, 'a1, 'a2) F3]

for *map*: F3map *rel*: F3rel

**bnf-axiomatization** (*dead* 'p, Gset1: 'b1, Gset2: 'b2, Gset3: 'b3) G

[*wits*: 'b1  $\Rightarrow$  'b3  $\Rightarrow$  ('p, 'b1, 'b2, 'b3) G 'b2  $\Rightarrow$  'b3  $\Rightarrow$  ('p, 'b1, 'b2, 'b3) G]

for *map*: Gmap *rel*: Grel

**type-synonym** ('p1, 'p2, 'p3, 'p, 'a1, 'a2) H =

('p, ('p1, 'a1, 'a2) F1, ('p2, 'a1, 'a2) F2, ('p3, 'a1, 'a2) F3) G

**type-synonym** ('p1, 'p2, 'p3, 'p) Hbd\_type =

('p1 bd\_type\_F1 + 'p2 bd\_type\_F2 + 'p3 bd\_type\_F3)  $\times$  'p bd\_type\_G

**abbreviation** F1in where F1in A1 A2  $\equiv$  {x. F1set1 x  $\subseteq$  A1  $\wedge$  F1set2 x  $\subseteq$  A2}

**abbreviation** F2in where F2in A1 A2  $\equiv$  {x. F2set1 x  $\subseteq$  A1  $\wedge$  F2set2 x  $\subseteq$  A2}

**abbreviation** F3in where F3in A1 A2  $\equiv$  {x. F3set1 x  $\subseteq$  A1  $\wedge$  F3set2 x  $\subseteq$  A2}

**abbreviation** Gin where Gin A1 A2 A3  $\equiv$  {x. Gset1 x  $\subseteq$  A1  $\wedge$  Gset2 x  $\subseteq$  A2  $\wedge$  Gset3 x  $\subseteq$  A3}

**abbreviation** Gset where

Gset  $\equiv$  BNF\_Def.collect {Gset1, Gset2, Gset3}

**abbreviation** Hmap :: ('a1  $\Rightarrow$  'b1)  $\Rightarrow$  ('a2  $\Rightarrow$  'b2)  $\Rightarrow$

('p1, 'p2, 'p3, 'p, 'a1, 'a2) H  $\Rightarrow$  ('p1, 'p2, 'p3, 'p, 'b1, 'b2) H where

Hmap f g  $\equiv$  Gmap (F1map f g) (F2map f g) (F3map f g)

**abbreviation** Hset1 :: ('p1, 'p2, 'p3, 'p, 'a1, 'a2) H  $\Rightarrow$  'a1 set where

Hset1  $\equiv$  Union o Gset o Gmap F1set1 F2set1 F3set1

**abbreviation** Hset2 :: ('p1, 'p2, 'p3, 'p, 'a1, 'a2) H  $\Rightarrow$  'a2 set where

Hset2  $\equiv$  Union o Gset o Gmap F1set2 F2set2 F3set2

**lemma** Hset1\_alt:

Hset1 = Union o BNF\_Def.collect {image F1set1 o Gset1, image F2set1 o Gset2, image F3set1 o Gset3}

by (tactic  $\langle$ BNF\_Comp\_Tactics.mk\_comp\_set\_alt\_tac @ {context} @ {thm G.collect\_set\_map}  $\rangle$ )

**lemma** Hset2\_alt:

Hset2 = Union o BNF\_Def.collect {image F1set2 o Gset1, image F2set2 o Gset2, image F3set2 o Gset3}

by (tactic  $\langle$ BNF\_Comp\_Tactics.mk\_comp\_set\_alt\_tac @ {context} @ {thm G.collect\_set\_map}  $\rangle$ )

**abbreviation** Hbd where

Hbd  $\equiv$  (bd\_F1 +c bd\_F2 +c bd\_F3) \*c bd\_G

**theorem** Hmap\_id: Hmap id id = id

unfolding G.map\_id0 F1.map\_id0 F2.map\_id0 F3.map\_id0 ..

**theorem** Hmap\_comp: Hmap (f1 o g1) (f2 o g2) = Hmap f1 f2 o Hmap g1 g2

unfolding G.map\_comp0 F1.map\_comp0 F2.map\_comp0 F3.map\_comp0 ..

**theorem** Hmap\_cong:  $\llbracket \bigwedge z. z \in \text{Hset1 } x \implies f1 \ z = g1 \ z; \bigwedge z. z \in \text{Hset2 } x \implies f2 \ z = g2 \ z \rrbracket \implies$

Hmap f1 f2 x = Hmap g1 g2 x

by (tactic  $\langle$ BNF\_Comp\_Tactics.mk\_comp\_map\_cong0\_tac @ {context}

$\rrbracket$  @ {thms Hset1\_alt Hset2\_alt} @ {thm G.map\_cong0} @ {thms F1.map\_cong0 F2.map\_cong0 F3.map\_cong0}  $\rangle$ )

**theorem** Hset1\_natural: Hset1 o Hmap f1 f2 = image f1 o Hset1

by (tactic  $\langle$ BNF\_Comp\_Tactics.mk\_comp\_set\_map0\_tac @ {context} @ {thm refl} @ {thm G.map\_comp0} @ {thm G.map\_cong0}

@ {thm G.collect\_set\_map} @ {thms F1.set\_map0(1) F2.set\_map0(1) F3.set\_map0(1)}  $\rangle$ )

**theorem** Hset2\_natural: Hset2 o Hmap f1 f2 = image f2 o Hset2

by (tactic  $\langle$ BNF\_Comp\_Tactics.mk\_comp\_set\_map0\_tac @ {context} @ {thm refl} @ {thm G.map\_comp0} @ {thm G.map\_cong0}

@ {thm G.collect\_set\_map} @ {thms F1.set\_map0(2) F2.set\_map0(2) F3.set\_map0(2)}  $\rangle$ )

**theorem** Hbd\_card\_order: card\_order Hbd

by (tactic  $\langle$ BNF\_Comp\_Tactics.mk\_comp\_bd\_card\_order\_tac @ {context}

@{thms F1.bd\_card\_order F2.bd\_card\_order F3.bd\_card\_order} @ {thm G.bd\_card\_order}>>)

**theorem** *Hbd\_cinfinite*: *cinfinite Hbd*

by (tactic <BNF\_Comp\_Tactics.mk\_comp\_bd\_cinfinite\_tac @ {context}  
@ {thm F1.bd\_cinfinite} @ {thm G.bd\_cinfinite}>>)

**theorem** *Hbd\_regularCard*: *regularCard Hbd*

by (tactic <BNF\_Comp\_Tactics.mk\_comp\_bd\_regularCard\_tac @ {context}  
@ {thms F1.bd\_regularCard F2.bd\_regularCard F3.bd\_regularCard} @ {thm G.bd\_regularCard}  
@ {thms F1.bd\_Cinfinite F2.bd\_Cinfinite F3.bd\_Cinfinite} @ {thm G.bd\_Cinfinite}>>)

**theorem** *Hset1\_bd*:  $|Hset1 (x :: ('p1, 'p2, 'p3, 'p, 'a1, 'a2) H) | < o$

(*Hbd* :: ('p1, 'p2, 'p3, 'p) *Hbd\_type* rel)

by (tactic <BNF\_Comp\_Tactics.mk\_comp\_set\_bd\_tac @ {context} @ {thm refl} NONE @ {thm Hset1\_alt}  
@ {thms comp\_single\_set\_bd\_strict[OF F1.bd\_Cinfinite F1.bd\_regularCard G.bd\_Cinfinite  
G.bd\_regularCard F1.set\_bd(1) G.set\_bd(1)]  
comp\_single\_set\_bd\_strict[OF F2.bd\_Cinfinite F2.bd\_regularCard G.bd\_Cinfinite  
G.bd\_regularCard F2.set\_bd(1) G.set\_bd(2)]  
comp\_single\_set\_bd\_strict[OF F3.bd\_Cinfinite F3.bd\_regularCard G.bd\_Cinfinite  
G.bd\_regularCard F3.set\_bd(1) G.set\_bd(3)]}  
@ {thms Cinfinite\_cprod2[OF Cinfinite\_Cnotzero[OF G.bd\_Cinfinite] F1.bd\_Cinfinite]  
Cinfinite\_cprod2[OF Cinfinite\_Cnotzero[OF G.bd\_Cinfinite] F2.bd\_Cinfinite]  
Cinfinite\_cprod2[OF Cinfinite\_Cnotzero[OF G.bd\_Cinfinite] F3.bd\_Cinfinite]}>>)

**theorem** *Hset2\_bd*:  $|Hset2 (x :: ('p1, 'p2, 'p3, 'p, 'a1, 'a2) H) | < o$

(*Hbd* :: ('p1, 'p2, 'p3, 'p) *Hbd\_type* rel)

by (tactic <BNF\_Comp\_Tactics.mk\_comp\_set\_bd\_tac @ {context} @ {thm refl} NONE @ {thm Hset2\_alt}  
@ {thms comp\_single\_set\_bd\_strict[OF F1.bd\_Cinfinite F1.bd\_regularCard G.bd\_Cinfinite  
G.bd\_regularCard F1.set\_bd(2) G.set\_bd(1)]  
comp\_single\_set\_bd\_strict[OF F2.bd\_Cinfinite F2.bd\_regularCard G.bd\_Cinfinite  
G.bd\_regularCard F2.set\_bd(2) G.set\_bd(2)]  
comp\_single\_set\_bd\_strict[OF F3.bd\_Cinfinite F3.bd\_regularCard G.bd\_Cinfinite  
G.bd\_regularCard F3.set\_bd(2) G.set\_bd(3)]}  
@ {thms Cinfinite\_cprod2[OF Cinfinite\_Cnotzero[OF G.bd\_Cinfinite] F1.bd\_Cinfinite]  
Cinfinite\_cprod2[OF Cinfinite\_Cnotzero[OF G.bd\_Cinfinite] F2.bd\_Cinfinite]  
Cinfinite\_cprod2[OF Cinfinite\_Cnotzero[OF G.bd\_Cinfinite] F3.bd\_Cinfinite]}>>)

**abbreviation** *Hin* where  $Hin A1 A2 \equiv \{x. Hset1 x \subseteq A1 \wedge Hset2 x \subseteq A2\}$

**lemma** *Hin\_alt*:  $Hin A1 A2 = Gin (F1in A1 A2) (F2in A1 A2) (F3in A1 A2)$

by (tactic <BNF\_Comp\_Tactics.mk\_comp\_in\_alt\_tac @ {context} @ {thms Hset1\_alt Hset2\_alt}>>)

**definition** *Hwit1* where  $Hwit1 b c = wit1\_G wit\_F1 (wit\_F3 b c)$

**definition** *Hwit21* where  $Hwit21 b c = wit2\_G (wit1\_F2 b) (wit\_F3 b c)$

**definition** *Hwit22* where  $Hwit22 b c = wit2\_G (wit2\_F2 c) (wit\_F3 b c)$

**lemma** *Hwit1*:

$\bigwedge x. x \in Hset1 (Hwit1 b c) \implies x = b$

$\bigwedge x. x \in Hset2 (Hwit1 b c) \implies x = c$

**unfolding** *Hwit1\_def*

by (tactic <BNF\_Comp\_Tactics.mk\_comp\_wit\_tac @ {context} [] @ {thms G.wit1 G.wit2}  
@ {thm G.collect\_set\_map} @ {thms F1.wit F2.wit1 F2.wit2 F3.wit}>>)

**lemma** *Hwit21*:

$\bigwedge x. x \in Hset1 (Hwit21 b c) \implies x = b$

$\bigwedge x. x \in Hset2 (Hwit21 b c) \implies x = c$

**unfolding** *Hwit21\_def*

by (tactic <BNF\_Comp\_Tactics.mk\_comp\_wit\_tac @ {context} [] @ {thms G.wit1 G.wit2}  
@ {thm G.collect\_set\_map} @ {thms F1.wit F2.wit1 F2.wit2 F3.wit}>>)

**lemma** *Hwit22*:

$\bigwedge x. x \in Hset1 (Hwit22 b c) \implies x = b$

$\bigwedge x. x \in \text{Hset2} (\text{Hwit22 } b \ c) \implies x = c$   
**unfolding** *Hwit22\_def*  
**by** (*tactic*  $\langle \text{BNF\_Comp\_Tactics.mk\_comp\_wit\_tac} \ @\{\text{context}\} \ [] \ @\{\text{thms } G.\text{wit1 } G.\text{wit2}\} \ @\{\text{thm } G.\text{collect\_set\_map}\} \ @\{\text{thms } F1.\text{wit } F2.\text{wit1 } F2.\text{wit2 } F3.\text{wit}\} \rangle$ )

**lemma** *Grel\_cong*:  $\llbracket R1 = S1; R2 = S2; R3 = S3 \rrbracket \implies \text{Grel } R1 \ R2 \ R3 = \text{Grel } S1 \ S2 \ S3$   
**by** *hypsubst* (*rule refl*)

**definition** *Hrel* **where**

$\text{Hrel } R1 \ R2 = (\text{BNF\_Def.Grp } (\text{Hin } (\text{Collect } (\text{case\_prod } R1))) (\text{Collect } (\text{case\_prod } R2)))) (\text{Hmap } \text{fst } \text{fst}) \ ^{-1} \ \text{OO}$   
 $(\text{BNF\_Def.Grp } (\text{Hin } (\text{Collect } (\text{case\_prod } R1))) (\text{Collect } (\text{case\_prod } R2)))) (\text{Hmap } \text{snd } \text{snd})$

**lemmas** *Hrel\_unfold* =  $\text{trans}[\text{OF } \text{Hrel\_def } \text{trans}[\text{OF } \text{OO\_Grp\_cong}[\text{OF } \text{Hin\_alt}]$   
 $\text{trans}[\text{OF } \text{arg\_cong2}[\text{of } \_ \_ \_ \_ \text{relcompp}, \text{OF } \text{trans}[\text{OF } \text{arg\_cong}[\text{of } \_ \_ \_ \_ \text{conversep}, \text{OF } \text{sym}[\text{OF } G.\text{rel\_Grp}]]$   
 $G.\text{rel\_conversep}[\text{symmetric}]] \text{sym}[\text{OF } G.\text{rel\_Grp}]]$   
 $\text{trans}[\text{OF } G.\text{rel\_compp}[\text{symmetric}] \ \text{Grel\_cong}[\text{OF } \text{sym}[\text{OF } F1.\text{rel\_compp\_Grp}] \ \text{sym}[\text{OF } F2.\text{rel\_compp\_Grp}]$   
 $\text{sym}[\text{OF } F3.\text{rel\_compp\_Grp}]]]]]$

**bnf** *H*: (*'p1, 'p2, 'p3, 'p, 'a1, 'a2*) *H*

*map*: *Hmap*

*sets*: *Hset1 Hset2*

*bd*: *Hbd* :: (*'p1, 'p2, 'p3, 'p*) *Hbd\_type* *rel*

*rel*: *Hrel*

**apply** –

**apply** (*rule Hmap\_id*)

**apply** (*rule Hmap\_comp*)

**apply** (*erule Hmap\_cong*) **apply** *assumption*

**apply** (*rule Hset1\_natural*)

**apply** (*rule Hset2\_natural*)

**apply** (*rule Hbd\_card\_order*)

**apply** (*rule Hbd\_cinfinite*)

**apply** (*rule Hbd\_regularCard*)

**apply** (*rule Hset1\_bd*)

**apply** (*rule Hset2\_bd*)

**apply** (*unfold Hrel\_unfold G.rel\_compp[symmetric] F1.rel\_compp[symmetric] F2.rel\_compp[symmetric] F3.rel\_compp[symmetric]*  
*eq\_OO*) [1] **apply** (*rule order\_refl*)

**apply** (*rule Hrel\_def[unfolded OO\_Grp\_alt mem\_Collect\_eq]*)

**done**

## 4 Removing Live Variables

**unbundle** *cardinal\_syntax*

**declare**  $\llbracket \text{bnf\_internals} \rrbracket$

**bnf-axiomatization** (*dead 'p, Fset1: 'a1, Fset2: 'a2, Fset3: 'a3*) *F* **for** *map: Fmap* *rel: Frel*

**abbreviation** *F1map* :: (*'a2*  $\implies$  *'b2*)  $\implies$  (*'a3*  $\implies$  *'b3*)  $\implies$  (*'p, 'a1, 'a2, 'a3*) *F*  $\implies$  (*'p, 'a1, 'b2, 'b3*) *F* **where**  
*F1map*  $\equiv$  *Fmap id*

**abbreviation** *F2map* :: (*'a3*  $\implies$  *'b3*)  $\implies$  (*'p, 'a1, 'a2, 'a3*) *F*  $\implies$  (*'p, 'a1, 'a2, 'b3*) *F* **where**  
*F2map*  $\equiv$  *Fmap id id*

**abbreviation** *F1set1*  $\equiv$  *Fset2*

**abbreviation** *F1set2*  $\equiv$  *Fset3*

**abbreviation** *F2set*  $\equiv$  *Fset3*

**theorem** *F1map\_id*: *F1map id id* = *id*

**by** (*rule F.map\_id0*)

**theorem** *F2map\_id*:  $F2map\ id = id$   
**by** (rule *F.map\_id0*)

**theorem** *F1map\_comp*:  $F1map\ (f1\ o\ g1)\ (f2\ o\ g2) = F1map\ f1\ f2\ o\ F1map\ g1\ g2$   
**by** (unfold *F.map\_comp0*[*symmetric*] *o\_id*) (rule *refl*)

**theorem** *F2map\_comp*:  $F2map\ (f\ o\ g) = F2map\ f\ o\ F2map\ g$   
**by** (unfold *F.map\_comp0*[*symmetric*] *o\_id*) (rule *refl*)

**theorem** *F1map\_cong*:  $\llbracket \bigwedge z. z \in F1set1\ x \implies f1\ z = g1\ z; \bigwedge z. z \in F1set2\ x \implies f2\ z = g2\ z \rrbracket$   
 $\implies F1map\ f1\ f2\ x = F1map\ g1\ g2\ x$   
**apply** (rule *F.map\_cong0*)  
**apply** (rule *refl*)  
**apply** *assumption*  
**apply** *assumption*  
**done**

**theorem** *F2map\_cong*:  $\llbracket \bigwedge z. z \in F2set\ x \implies f\ z = g\ z \rrbracket \implies F2map\ f\ x = F2map\ g\ x$   
**apply** (rule *F.map\_cong0*)  
**apply** (rule *refl*)  
**apply** (rule *refl*)  
**apply** *assumption*  
**done**

**theorem** *F1set1\_natural*:  $F1set1\ o\ F1map\ f1\ f2 = image\ f1\ o\ F1set1$   
**by** (rule *F.set\_map0*(2))

**theorem** *F1set2\_natural*:  $F1set2\ o\ F1map\ f1\ f2 = image\ f2\ o\ F1set2$   
**by** (rule *F.set\_map0*(3))

**theorem** *F2set\_natural*:  $F2set\ o\ F2map\ f = image\ f\ o\ F2set$   
**by** (rule *F.set\_map0*(3))

**abbreviation** *Fin* ::  $'a1\ set \Rightarrow 'a2\ set \Rightarrow 'a3\ set \Rightarrow ((p, 'a1, 'a2, 'a3)\ F)\ set$  **where**  
 $Fin\ A1\ A2\ A3 \equiv \{x. Fset1\ x \subseteq A1 \wedge Fset2\ x \subseteq A2 \wedge Fset3\ x \subseteq A3\}$

**abbreviation** *F1in* ::  $'a2\ set \Rightarrow 'a3\ set \Rightarrow ((p, 'a1, 'a2, 'a3)\ F)\ set$  **where**  
 $F1in\ A1\ A2 \equiv \{x. F1set1\ x \subseteq A1 \wedge F1set2\ x \subseteq A2\}$

**lemma** *F1in\_alt*:  $F1in\ A2\ A3 = Fin\ UNIV\ A2\ A3$   
**by** (tactic  $\langle BNF\_Comp\_Tactics.kill\_in\_alt\_tac\ @\{context\} \rangle$ )

**abbreviation** *F2in* ::  $'a3\ set \Rightarrow ((p, 'a1, 'a2, 'a3)\ F)\ set$  **where**  
 $F2in\ A \equiv \{x. F2set\ x \subseteq A\}$

**lemma** *F2in\_alt*:  $F2in\ A3 = Fin\ UNIV\ UNIV\ A3$   
**by** (tactic  $\langle BNF\_Comp\_Tactics.kill\_in\_alt\_tac\ @\{context\} \rangle$ )

**lemma** *Frel\_cong*:  $\llbracket R1 = S1; R2 = S2; R3 = S3 \rrbracket \implies Frel\ R1\ R2\ R3 = Frel\ S1\ S2\ S3$   
**by** *hypsubst* (rule *refl*)

**definition** *F1rel* **where**  
 $F1rel\ R1\ R2 = (BNF\_Def.Grp\ (F1in\ (Collect\ (case\_prod\ R1)))\ (Collect\ (case\_prod\ R2)))\ (F1map\ fst\ fst) \hat{\ } - - 1$   
 $OO$   
 $(BNF\_Def.Grp\ (F1in\ (Collect\ (case\_prod\ R1)))\ (Collect\ (case\_prod\ R2)))\ (F1map\ snd\ snd))$

**lemmas** *F1rel\_unfold* =  $trans[OF\ F1rel\_def\ trans[OF\ OO\_Grp\_cong[OF\ F1in\_alt]$   
 $trans[OF\ arg\_cong2[of\ \_ \_ \_ \_ relcomp, OF\ trans[OF\ arg\_cong[of\ \_ \_ \_ \_ conversep, OF\ sym[OF\ F.rel\_Grp]]$   
 $F.rel\_conversep[symmetric]]\ sym[OF\ F.rel\_Grp]]$   
 $trans[OF\ F.rel\_comp[symmetric]]\ Frel\_cong[OF\ trans[OF\ Grp\_UNIV\_id[OF\ refl]\ eq\_alt[symmetric]]\ Grp\_fst\_snd$   
 $Grp\_fst\_snd]]]]]$

**definition** *F2rel* **where**

$F2rel\ R1 = (BNF\_Def.Grp\ (F2in\ (Collect\ (case\_prod\ R1)))\ (F2map\ fst))\ \hat{-}\ -1\ OO$   
 $(BNF\_Def.Grp\ (F2in\ (Collect\ (case\_prod\ R1)))\ (F2map\ snd))$

**lemmas**  $F2rel\_unfold = trans[OF\ F2rel\_def\ trans[OF\ OO\_Grp\_cong[OF\ F2in\_alt]$   
 $trans[OF\ arg\_cong2[of\ \_ \_ \_ \_ relcomp, OF\ trans[OF\ arg\_cong[of\ \_ \_ conversep, OF\ sym[OF\ F.rel\_Grp]]$   
 $F.rel\_conversep[symmetric]]\ sym[OF\ F.rel\_Grp]]$   
 $trans[OF\ F.rel\_compp[symmetric]\ Frel\_cong[OF\ trans[OF\ Grp\_UNIV\_id[OF\ refl]\ eq\_alt[symmetric]]\ trans[OF$   
 $Grp\_UNIV\_id[OF\ refl]\ eq\_alt[symmetric]]\ Grp\_fst\_snd]]]]]$

**bnf**  $F1: ('p, 'a1, 'a2, 'a3)\ F$   
 $map: F1map$   
 $sets: F1set1\ F1set2$   
 $bd: bd\_F :: ('p\ bd\_type\_F)\ rel$   
 $rel: F1rel$   
**apply** -  
**apply** (rule  $F1map\_id$ )  
**apply** (rule  $F1map\_comp$ )  
**apply** (erule  $F1map\_cong$ ) **apply** assumption  
**apply** (rule  $F1set1\_natural$ )  
**apply** (rule  $F1set2\_natural$ )  
**apply** (rule  $F.bd\_card\_order$ )  
**apply** (rule  $F.bd\_cinfinit$ )  
**apply** (rule  $F.bd\_regularCard$ )  
**apply** (rule  $F.set\_bd(2)$ )  
**apply** (rule  $F.set\_bd(3)$ )  
**apply** (unfold  $F1rel\_unfold\ F.rel\_compp[symmetric]\ eq\_OO$ ) [1] **apply** (rule  $order\_refl$ )  
**apply** (rule  $F1rel\_def[unfolded\ OO\_Grp\_alt\ mem\_Collect\_eq]$ )  
**done**

**bnf**  $F2: ('p, 'a1, 'a2, 'a3)\ F$   
 $map: F2map$   
 $sets: F2set$   
 $bd: bd\_F :: ('p\ bd\_type\_F)\ rel$   
 $rel: F2rel$   
**apply** -  
**apply** (rule  $F2map\_id$ )  
**apply** (rule  $F2map\_comp$ )  
**apply** (erule  $F2map\_cong$ )  
**apply** (rule  $F2set\_natural$ )  
**apply** (rule  $F.bd\_card\_order$ )  
**apply** (rule  $F.bd\_cinfinit$ )  
**apply** (rule  $F.bd\_regularCard$ )  
**apply** (rule  $F.set\_bd(3)$ )  
**apply** (unfold  $F2rel\_unfold\ F.rel\_compp[symmetric]\ eq\_OO$ ) [1] **apply** (rule  $order\_refl$ )  
**apply** (rule  $F2rel\_def[unfolded\ OO\_Grp\_alt\ mem\_Collect\_eq]$ )  
**done**

## 5 Adding New Live Variables

**unbundle**  $cardinal\_syntax$

**declare**  $[[bnf\_internals]]$

**bnf-axiomatization** (dead  $'p, Fset1: 'a1, Fset2: 'a2$ )  $F$

$[wits: 'a1 \Rightarrow 'a2 \Rightarrow ('p, 'a1, 'a2)\ F]$

**for**  $map: Fmap\ rel: Frel$

**type-synonym**  $('p, 'a1, 'a2, 'a3, 'a4)\ F' = ('p, 'a3, 'a4)\ F$

**abbreviation**  $F'map :: ('a1 \Rightarrow 'b1) \Rightarrow ('a2 \Rightarrow 'b2) \Rightarrow ('a3 \Rightarrow 'b3) \Rightarrow ('a4 \Rightarrow 'b4) \Rightarrow ('p, 'a1, 'a2, 'a3, 'a4)\ F'$

$\Rightarrow ('p, 'b1, 'b2, 'b3, 'b4)\ F'$  **where**

$F'map\ f1\ f2\ f3\ f4 \equiv Fmap\ f3\ f4$

**abbreviation**  $F'set1 :: ('p, 'a1, 'a2, 'a3, 'a4) F' \Rightarrow 'a1 \text{ set}$  **where**  
 $F'set1 \equiv \lambda\_ . \{\}$

**abbreviation**  $F'set2 :: ('p, 'a1, 'a2, 'a3, 'a4) F' \Rightarrow 'a2 \text{ set}$  **where**  
 $F'set2 \equiv \lambda\_ . \{\}$

**abbreviation**  $F'set3 :: ('p, 'a1, 'a2, 'a3, 'a4) F' \Rightarrow 'a3 \text{ set}$  **where**  
 $F'set3 \equiv F'set1$

**abbreviation**  $F'set4 :: ('p, 'a1, 'a2, 'a3, 'a4) F' \Rightarrow 'a4 \text{ set}$  **where**  
 $F'set4 \equiv F'set2$

**abbreviation**  $F'bd$  **where**  
 $F'bd \equiv bd\_F$

**theorem**  $F'map\_id$ :  $F'map \ id \ id \ id \ id \ id = id$   
**by** (rule  $F.map\_id0$ )

**theorem**  $F'map\_comp$ :  
 $F'map \ (f1 \ o \ g1) \ (f2 \ o \ g2) \ (f3 \ o \ g3) \ (f4 \ o \ g4) = F'map \ f1 \ f2 \ f3 \ f4 \ o \ F'map \ g1 \ g2 \ g3 \ g4$   
**by** (rule  $F.map\_comp0$ )

**theorem**  $F'map\_cong$ :  
 $\llbracket \bigwedge z. z \in F'set1 \ x \Longrightarrow f1 \ z = g1 \ z; \bigwedge z. z \in F'set2 \ x \Longrightarrow f2 \ z = g2 \ z;$   
 $\bigwedge z. z \in F'set3 \ x \Longrightarrow f3 \ z = g3 \ z; \bigwedge z. z \in F'set4 \ x \Longrightarrow f4 \ z = g4 \ z \rrbracket$   
 $\Longrightarrow F'map \ f1 \ f2 \ f3 \ f4 \ x = F'map \ g1 \ g2 \ g3 \ g4 \ x$   
**apply** (tactic  $\langle BNF\_Util.rtac \ @\{context\} \ @\{thm \ F.map\_cong0\} \ 1 \ THEN \ REPEAT\_DETERM\_N \ 2 \ (assume\_tac \ @\{context\} \ 1) \rangle$ )  
**apply**  $assumption+$   
**done**

**theorem**  $F'set1\_natural$ :  $F'set1 \ o \ F'map \ f1 \ f2 \ f3 \ f4 = image \ f1 \ o \ F'set1$   
**by** (tactic  $\langle BNF\_Comp\_Tactics.empty\_natural\_tac \ @\{context\} \rangle$ )

**theorem**  $F'set2\_natural$ :  $F'set2 \ o \ F'map \ f1 \ f2 \ f3 \ f4 = image \ f2 \ o \ F'set2$   
**by** (tactic  $\langle BNF\_Comp\_Tactics.empty\_natural\_tac \ @\{context\} \rangle$ )

**theorem**  $F'set3\_natural$ :  $F'set3 \ o \ F'map \ f1 \ f2 \ f3 \ f4 = image \ f3 \ o \ F'set3$   
**by** (rule  $F.set\_map0(1)$ )

**theorem**  $F'set4\_natural$ :  $F'set4 \ o \ F'map \ f1 \ f2 \ f3 \ f4 = image \ f4 \ o \ F'set4$   
**by** (rule  $F.set\_map0(2)$ )

**theorem**  $F'bd\_card\_order$ :  $card\_order \ bd\_F$   
**by** (rule  $F.bd\_card\_order$ )

**theorem**  $F'bd\_cinfinte$ :  $cinfinte \ bd\_F$   
**by** (rule  $F.bd\_cinfinte$ )

**theorem**  $F'bd\_regularCard$ :  $regularCard \ bd\_F$   
**by** (rule  $F.bd\_regularCard$ )

**theorem**  $F'set1\_bd$ :  $|F'set1 \ x| < o \ F'bd$   
**by** (tactic  $\langle BNF\_Comp\_Tactics.mk\_lift\_set\_bd\_tac \ @\{context\} \ @\{thm \ F.bd\_Cinfinte\} \rangle$ )

**theorem**  $F'set2\_bd$ :  $|F'set2 \ x| < o \ F'bd$   
**by** (tactic  $\langle BNF\_Comp\_Tactics.mk\_lift\_set\_bd\_tac \ @\{context\} \ @\{thm \ F.bd\_Cinfinte\} \rangle$ )

**theorem**  $F'set3\_bd$ :  $|F'set3 \ (x :: ('c, 'a, 'd) \ F)| < o \ (F'bd :: 'c \ bd\_type\_F \ rel)$   
**by** (rule  $F.set\_bd(1)$ )

**theorem**  $F'set4\_bd$ :  $|F'set4 \ (x :: ('c, 'a, 'd) \ F)| < o \ (F'bd :: 'c \ bd\_type\_F \ rel)$   
**by** (rule  $F.set\_bd(2)$ )

**abbreviation**  $F'in :: 'a1\ set \Rightarrow 'a2\ set \Rightarrow 'a3\ set \Rightarrow 'a4\ set \Rightarrow ((p, 'a1, 'a2, 'a3, 'a4)\ F)\ set$  **where**  
 $F'in\ A1\ A2\ A3\ A4 \equiv \{x.\ F'set1\ x \subseteq A1 \wedge F'set2\ x \subseteq A2 \wedge F'set3\ x \subseteq A3 \wedge F'set4\ x \subseteq A4\}$

**definition**  $F'rel$  **where**

$F'rel\ R1\ R2\ R3\ R4 = (BNF\_Def.Grp\ (F'in\ (Collect\ (case\_prod\ R1)))\ (Collect\ (case\_prod\ R2)))\ (Collect\ (case\_prod\ R3))\ (Collect\ (case\_prod\ R4))\ (F'map\ fst\ fst\ fst\ fst)^{-1}\ OO$   
 $(BNF\_Def.Grp\ (F'in\ (Collect\ (case\_prod\ R1)))\ (Collect\ (case\_prod\ R2)))\ (Collect\ (case\_prod\ R3))\ (Collect\ (case\_prod\ R4))\ (F'map\ snd\ snd\ snd\ snd)$

**lemmas**  $F'rel\_unfold = trans[OF\ F'rel\_def[unfolded\ eqTrueI[OF\ empty\_subsetI]\ simp\_thms(22)]\ trans[OF\ OO\_Grp\_cong[OF\ refl]\ sym[OF\ F.rel\_compp\_Grp]]]$

**bnf**  $F'$ :  $(p, 'a1, 'a2, 'a3, 'a4)\ F'$   
 $map: F'map$   
 $sets: F'set1\ F'set2\ F'set3\ F'set4$   
 $bd: F'bd :: 'p\ bd\_type\_F\ rel$   
 $wits: wit\_F$   
 $rel: F'rel$   
 $plugins\ del: lifting\ transfer$   
**apply** –  
**apply** (rule  $F'map\_id$ )  
**apply** (rule  $F'map\_comp$ )  
**apply** (erule  $F'map\_cong$ ) **apply** *assumption*+  
**apply** (rule  $F'set1\_natural$ )  
**apply** (rule  $F'set2\_natural$ )  
**apply** (rule  $F'set3\_natural$ )  
**apply** (rule  $F'set4\_natural$ )  
**apply** (rule  $F'bd\_card\_order$ )  
**apply** (rule  $F'bd\_cinfinte$ )  
**apply** (rule  $F'bd\_regularCard$ )  
**apply** (rule  $F'set1\_bd$ )  
**apply** (rule  $F'set2\_bd$ )  
**apply** (rule  $F'set3\_bd$ )  
**apply** (rule  $F'set4\_bd$ )  
**apply** (unfold  $F'rel\_unfold\ F.rel\_compp[symmetric]\ eq\_OO$ ) [1] **apply** (rule  $order\_refl$ )  
**apply** (rule  $F'rel\_def[unfolded\ OO\_Grp\_alt\ mem\_Collect\_eq]$ )  
**apply** (erule  $F.wit\ emptyE$ )  
**done**

## 6 Changing the Order of Live Variables

**unbundle**  $cardinal\_syntax$

**declare**  $[[bnf\_internals]]$

**bnf-axiomatization**  $(dead\ 'p,\ Fset1: 'a1,\ Fset2: 'a2,\ Fset3: 'a3)\ F$  **for**  $map: Fmap\ rel: Frel$

**type-synonym**  $(p, 'a1, 'a2, 'a3)\ F' = (p, 'a3, 'a1, 'a2)\ F$

**abbreviation**  $Fin :: 'a1\ set \Rightarrow 'a2\ set \Rightarrow 'a3\ set \Rightarrow ((p, 'a1, 'a2, 'a3)\ F)\ set$  **where**

$Fin\ A1\ A2\ A3 \equiv \{x.\ Fset1\ x \subseteq A1 \wedge Fset2\ x \subseteq A2 \wedge Fset3\ x \subseteq A3\}$

**abbreviation**  $F'map :: ('a1 \Rightarrow 'b1) \Rightarrow ('a2 \Rightarrow 'b2) \Rightarrow ('a3 \Rightarrow 'b3) \Rightarrow (p, 'a1, 'a2, 'a3)\ F' \Rightarrow (p, 'b1, 'b2, 'b3)\ F'$  **where**

$F'map\ f\ g\ h \equiv Fmap\ h\ f\ g$

**abbreviation**  $F'set1 :: (p, 'a1, 'a2, 'a3)\ F' \Rightarrow 'a1\ set$  **where**

$F'set1 \equiv Fset2$

**abbreviation**  $F'set2 :: (p, 'a1, 'a2, 'a3)\ F' \Rightarrow 'a2\ set$  **where**

$F'set2 \equiv Fset3$



**abbreviation**  $F'set3 :: ('p, 'a1, 'a2, 'a3) F' \Rightarrow 'a3 \text{ set}$  **where**  
 $F'set3 \equiv Fset1$

**abbreviation**  $F'bd$  **where**  
 $F'bd \equiv bd\_F$

**theorem**  $F'map\_id: F'map \text{ id id id} = \text{id}$   
**by** (rule  $F.map\_id0$ )

**theorem**  $F'map\_comp: F'map (f1 \circ g1) (f2 \circ g2) (f3 \circ g3) = F'map f1 f2 f3 \circ F'map g1 g2 g3$   
**by** (rule  $F.map\_comp0$ )

**theorem**  $F'map\_cong: [\bigwedge z. z \in F'set1 \ x \Longrightarrow f1 \ z = g1 \ z; \bigwedge z. z \in F'set2 \ x \Longrightarrow f2 \ z = g2 \ z; \bigwedge z. z \in F'set3 \ x \Longrightarrow f3 \ z = g3 \ z]$   
 $\Longrightarrow F'map f1 f2 f3 \ x = F'map g1 g2 g3 \ x$   
**apply** (rule  $F.map\_cong0$ )  
**apply**  $assumption+$   
**done**

**theorem**  $F'set1\_natural: F'set1 \circ F'map f1 f2 f3 = \text{image } f1 \circ F'set1$   
**by** (rule  $F.set\_map0(2)$ )

**theorem**  $F'set2\_natural: F'set2 \circ F'map f1 f2 f3 = \text{image } f2 \circ F'set2$   
**by** (rule  $F.set\_map0(3)$ )

**theorem**  $F'set3\_natural: F'set3 \circ F'map f1 f2 f3 = \text{image } f3 \circ F'set3$   
**by** (rule  $F.set\_map0(1)$ )

**theorem**  $F'bd\_card\_order: \text{card\_order } F'bd$   
**by** (rule  $F.bd\_card\_order$ )

**theorem**  $F'bd\_cinfinte: \text{cinfinte } F'bd$   
**by** (rule  $F.bd\_cinfinte$ )

**theorem**  $F'bd\_regularCard: \text{regularCard } F'bd$   
**by** (rule  $F.bd\_regularCard$ )

**theorem**  $F'set1\_bd: |F'set1 \ (x :: ('c, 'a, 'b, 'd) F)| < o \ (F'bd :: 'c \text{ bd\_type\_} F \text{ rel})$   
**by** (rule  $F.set\_bd(2)$ )

**theorem**  $F'set2\_bd: |F'set2 \ (x :: ('c, 'a, 'b, 'd) F)| < o \ (F'bd :: 'c \text{ bd\_type\_} F \text{ rel})$   
**by** (rule  $F.set\_bd(3)$ )

**theorem**  $F'set3\_bd: |F'set3 \ (x :: ('c, 'a, 'b, 'd) F)| < o \ (F'bd :: 'c \text{ bd\_type\_} F \text{ rel})$   
**by** (rule  $F.set\_bd(1)$ )

**abbreviation**  $F'in :: 'a1 \text{ set} \Rightarrow 'a2 \text{ set} \Rightarrow 'a3 \text{ set} \Rightarrow ((p, 'a1, 'a2, 'a3) F') \text{ set}$  **where**  
 $F'in \ A1 \ A2 \ A3 \equiv \{x. F'set1 \ x \subseteq A1 \ \wedge \ F'set2 \ x \subseteq A2 \ \wedge \ F'set3 \ x \subseteq A3\}$

**lemma**  $F'in\_alt: F'in \ A1 \ A2 \ A3 = \text{Fin } A3 \ A1 \ A2$   
**apply** (rule  $\text{Collect\_cong}$ )  
**by** (tactic  $\langle \text{BNF\_Tactics.mk\_rotate\_eq\_tac } @\{\text{context}\}$   
 $(\text{BNF\_Util.rtac } @\{\text{context}\} \ @\{\text{thm refl}\}) \ @\{\text{thm trans}\} \ @\{\text{thm conj\_assoc}\} \ @\{\text{thm conj\_commute}\} \ @\{\text{thm conj\_cong}\}$   
 $[1, 2, 3] [3, 1, 2] 1 \rangle$ )

**definition**  $F'rel$  **where**  
 $F'rel \ R1 \ R2 \ R3 = (\text{BNF\_Def.Grp } (F'in \ (\text{Collect } (\text{case\_prod } R1)) \ (\text{Collect } (\text{case\_prod } R2)) \ (\text{Collect } (\text{case\_prod } R3)))) \ (F'map \ \text{fst } \text{fst } \text{fst}) \ ^{-1} \ \text{OO}$   
 $(\text{BNF\_Def.Grp } (F'in \ (\text{Collect } (\text{case\_prod } R1)) \ (\text{Collect } (\text{case\_prod } R2)) \ (\text{Collect } (\text{case\_prod } R3)))) \ (F'map \ \text{snd } \text{snd } \text{snd}))$

**lemmas**  $F'rel\_unfold = \text{trans}[OF \ F'rel\_def \ \text{trans}[OF \ \text{OO\_Grp\_cong}[OF \ F'in\_alt] \ \text{sym}[OF \ F.rel\_compp\_Grp]]]$

```

bnf F': ('p, 'a1, 'a2, 'a3) F'
  map: F'map
  sets: F'set1 F'set2 F'set3
  bd: F'bd :: 'p bd_type_F rel
  rel: F'rel
    apply -
    apply (rule F'map_id)
    apply (rule F'map_comp)
    apply (erule F'map_cong) apply assumption+
    apply (rule F'set1_natural)
    apply (rule F'set2_natural)
    apply (rule F'set3_natural)
    apply (rule F'bd_card_order)
    apply (rule F'bd_cinfinite)
    apply (rule F'bd_regularCard)
    apply (rule F'set1_bd)
    apply (rule F'set2_bd)
    apply (rule F'set3_bd)
    apply (unfold F'rel_unfold F.rel_compp[symmetric] eq_OO) [1] apply (rule order_refl)
    apply (rule F'rel_def[unfolded OO_Grp_alt mem_Collect_eq])
  done

```

## 7 Mutual View on Nested Datatypes

**notation** *BNF\_Def.convolve* (<\_, >\_)

**declare** [[*bnf\_internals*]]

**declare** [[*typedef\_overloaded*]]

**bnf-axiomatization** ('a, 'b) F0 [*wits*: 'a  $\Rightarrow$  ('a, 'b) F0]

**bnf-axiomatization** ('a, 'b) G0 [*wits*: 'a  $\Rightarrow$  ('a, 'b) G0]

### 7.1 Nested Definition

**datatype** 'a F = CF ('a, 'a F) F0

**datatype** 'a G = CG ('a, ('a G) F) G0

**type-synonym** ('b, 'c) F\_pre\_F = ('c, 'b) F0

**type-synonym** ('c, 'a) G\_pre\_G = ('a, 'c F) G0

**term** *ctor\_fold\_F* :: (('b, 'c) F\_pre\_F  $\Rightarrow$  'b)  $\Rightarrow$  'c F  $\Rightarrow$  'b

**term** *ctor\_fold\_G* :: (('c, 'a) G\_pre\_G  $\Rightarrow$  'c)  $\Rightarrow$  'a G  $\Rightarrow$  'c

**term** *ctor\_rec\_F* :: (('c F  $\times$  'b, 'c) F\_pre\_F  $\Rightarrow$  'b)  $\Rightarrow$  'c F  $\Rightarrow$  'b

**term** *ctor\_rec\_G* :: (('a G  $\times$  'c, 'a) G\_pre\_G  $\Rightarrow$  'c)  $\Rightarrow$  'a G  $\Rightarrow$  'c

**thm** *F.ctor\_rel\_induct*

**thm** *G.ctor\_rel\_induct*[*unfolded rel\_pre\_G\_def id\_apply*]

### 7.2 Isomorphic Mutual Definition

**datatype** 'a G<sub>M</sub> = CG ('a, 'a GF<sub>M</sub>) G0

**and** 'a GF<sub>M</sub> = CF ('a G<sub>M</sub>, 'a GF<sub>M</sub>) F0

**type-synonym** ('b, 'c) GF<sub>M</sub>\_pre\_GF<sub>M</sub> = ('c, 'b) F0

**type-synonym** ('c, 'a) G<sub>M</sub>\_pre\_G<sub>M</sub> = ('a, 'c) G0

**term** *ctor\_fold\_G<sub>M</sub>* :: (('c, 'a) G<sub>M</sub>\_pre\_G<sub>M</sub>  $\Rightarrow$  'b)  $\Rightarrow$  (('c, 'b) GF<sub>M</sub>\_pre\_GF<sub>M</sub>  $\Rightarrow$  'c)  $\Rightarrow$  'a G<sub>M</sub>  $\Rightarrow$  'b

**term** *ctor\_fold\_GF<sub>M</sub>* :: (('c, 'a) G<sub>M</sub>\_pre\_G<sub>M</sub>  $\Rightarrow$  'b)  $\Rightarrow$  (('c, 'b) GF<sub>M</sub>\_pre\_GF<sub>M</sub>  $\Rightarrow$  'c)  $\Rightarrow$  'a GF<sub>M</sub>  $\Rightarrow$  'c

**term** *ctor\_rec\_G<sub>M</sub>* :: (('a GF<sub>M</sub>  $\times$  'c, 'a) G<sub>M</sub>\_pre\_G<sub>M</sub>  $\Rightarrow$  'b)  $\Rightarrow$  (('a GF<sub>M</sub>  $\times$  'c, 'a G<sub>M</sub>  $\times$  'b) GF<sub>M</sub>\_pre\_GF<sub>M</sub>  $\Rightarrow$  'c)  $\Rightarrow$  'a G<sub>M</sub>  $\Rightarrow$  'b

**term**  $ctor\_rec\_GF_M :: (('a\ GF_M \times 'c, 'a)\ GF_M\_pre\_G_M \Rightarrow 'b) \Rightarrow (('a\ GF_M \times 'c, 'a\ GF_M \times 'b)\ GF_M\_pre\_GF_M \Rightarrow 'c) \Rightarrow 'a\ GF_M \Rightarrow 'c$

**thm**  $G_M\_GF_M.ctor\_rel\_induct[unfolded\ rel\_pre\_G_M\_def\ rel\_pre\_GF_M\_def]$

## 7.3 Mutualization

### 7.3.1 Iterators

**definition**  $n2m\_ctor\_fold\_G :: (('c, 'a)\ GF_M\_pre\_G_M \Rightarrow 'b) \Rightarrow (('c, 'b)\ GF_M\_pre\_GF_M \Rightarrow 'c) \Rightarrow 'a\ G \Rightarrow 'b$   
**where**  $n2m\_ctor\_fold\_G\ s1\ s2 = ctor\_fold\_G\ (s1\ o\ map\_pre\_G_M\ id\ (id :: unit \Rightarrow unit)\ (ctor\_fold\_F\ (s2\ o\ BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf))\ o\ BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf)$

**definition**  $n2m\_ctor\_fold\_G\_F :: (('c, 'a)\ GF_M\_pre\_G_M \Rightarrow 'b) \Rightarrow (('c, 'b)\ GF_M\_pre\_GF_M \Rightarrow 'c) \Rightarrow 'a\ GF \Rightarrow 'c$

**where**  $n2m\_ctor\_fold\_G\_F\ s1\ s2 = ctor\_fold\_F\ (s2\ o\ map\_pre\_GF_M\ (id :: unit \Rightarrow unit)\ (n2m\_ctor\_fold\_G\ s1\ s2)\ id\ o\ BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf)$

**lemma**  $G\_ctor\_o\_fold: ctor\_fold\_G\ s\ o\ ctor\_G = s\ o\ map\_pre\_G\ id\ (ctor\_fold\_G\ s)$

**unfolding**  $fun\_eq\_iff\ o\_apply\ G.ctor\_fold\ \mathbf{by}\ simp$

**lemma**  $F\_ctor\_o\_fold: ctor\_fold\_F\ s\ o\ ctor\_F = s\ o\ map\_pre\_F\ id\ (ctor\_fold\_F\ s)$

**unfolding**  $fun\_eq\_iff\ o\_apply\ F.ctor\_fold\ \mathbf{by}\ simp$

**lemma**  $G\_ctor\_o\_rec: ctor\_rec\_G\ s\ o\ ctor\_G = s\ o\ map\_pre\_G\ id\ (BNF\_Def.convolve\ id\ (ctor\_rec\_G\ s))$

**unfolding**  $fun\_eq\_iff\ o\_apply\ G.ctor\_rec\ \mathbf{by}\ simp$

**lemma**  $F\_ctor\_o\_rec: ctor\_rec\_F\ s\ o\ ctor\_F = s\ o\ map\_pre\_F\ id\ (BNF\_Def.convolve\ id\ (ctor\_rec\_F\ s))$

**unfolding**  $fun\_eq\_iff\ o\_apply\ F.ctor\_rec\ \mathbf{by}\ simp$

**lemma**  $n2m\_ctor\_fold\_G:$

$n2m\_ctor\_fold\_G\ s1\ s2\ o\ ctor\_G = s1\ o\ map\_pre\_G_M\ id\ id\ (n2m\_ctor\_fold\_G\_F\ s1\ s2)\ o\ BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf$

**unfolding**  $n2m\_ctor\_fold\_G\_def\ n2m\_ctor\_fold\_G\_F\_def$

$map\_pre\_G\_def\ map\_pre\_F\_def\ map\_pre\_G_M\_def\ map\_pre\_GF_M\_def$

$G\_ctor\_o\_fold\ id\_apply\ comp\_id\ id\_comp\ comp\_assoc$

$rewriteL\_comp\_comp[OF\ type\_copy\_map\_comp0\_undo[OF\ BNF\_Composition.type\_definition\_id\_bnf\_UNIV\ BNF\_Composition.type\_definition\_id\_bnf\_UNIV\ BNF\_Composition.type\_definition\_id\_bnf\_UNIV\ pre\_G_M.map\_comp0[unfolding\ map\_pre\_G_M\_def]]]$

$F.ctor\_fold\_o\_map$

$rewriteL\_comp\_comp[OF\ type\_copy\_Rep\_o\_Abs[OF\ BNF\_Composition.type\_definition\_id\_bnf\_UNIV]]\ ..$

**lemma**  $n2m\_ctor\_fold\_G\_F:$

$n2m\_ctor\_fold\_G\_F\ s1\ s2\ o\ ctor\_F = s2\ o\ map\_pre\_GF_M\ id\ (n2m\_ctor\_fold\_G\ s1\ s2)\ (n2m\_ctor\_fold\_G\_F\ s1\ s2)\ o\ BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf$

**unfolding**  $n2m\_ctor\_fold\_G\_F\_def\ map\_pre\_F\_def\ map\_pre\_G_M\_def\ map\_pre\_GF_M\_def$

$F\_ctor\_o\_fold\ id\_apply\ comp\_id\ id\_comp\ comp\_assoc$

$rewriteL\_comp\_comp[OF\ F0.map\_comp0[symmetric]]$

$rewriteL\_comp\_comp[OF\ type\_copy\_Rep\_o\_Abs[OF\ BNF\_Composition.type\_definition\_id\_bnf\_UNIV]]\ ..$

### 7.3.2 Recursors

**definition**  $n2m\_ctor\_rec\_G ::$

$(('a\ GF \times 'c, 'a)\ GF_M\_pre\_G_M \Rightarrow 'b) \Rightarrow (('a\ GF \times 'c, 'a\ GF \times 'b)\ GF_M\_pre\_GF_M \Rightarrow 'c) \Rightarrow 'a\ G \Rightarrow 'b$

**where**  $n2m\_ctor\_rec\_G\ s1\ s2 =$

$ctor\_rec\_G\ (s1\ o$

$map\_pre\_G_M\ id\ (id :: unit \Rightarrow unit)$

$(BNF\_Def.convolve\ (map\_F\ fst)\ (ctor\_rec\_F\ (s2\ o\ map\_pre\_GF_M\ (id :: unit \Rightarrow unit)\ id\ (map\_prod\ (map\_F\ fst)\ id)\ o\ BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf)))\ o$

$BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf)$

**definition**  $n2m\_ctor\_rec\_G\_F ::$

$(('a\ GF \times 'c, 'a)\ GF_M\_pre\_G_M \Rightarrow 'b) \Rightarrow (('a\ GF \times 'c, 'a\ GF \times 'b)\ GF_M\_pre\_GF_M \Rightarrow 'c) \Rightarrow 'a\ GF \Rightarrow 'c$

**where**  $n2m\_ctor\_rec\_G\_F\ s1\ s2 = ctor\_rec\_F\ (s2\ o\ map\_pre\_GF_M\ (id :: unit \Rightarrow unit)\ (BNF\_Def.convolve\ id\ (n2m\_ctor\_rec\_G\ s1\ s2)))\ id\ o\ BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf)$

**lemma**  $n2m\_ctor\_rec\_G:$

$n2m\_ctor\_rec\_G\ s1\ s2\ o\ ctor\_G = s1\ o\ map\_pre\_G_M\ id\ id\ (BNF\_Def.convolve\ id\ (n2m\_ctor\_rec\_G\_F\ s1\ s2))$   
 $o\ BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf$   
**unfolding**  $n2m\_ctor\_rec\_G\_def\ n2m\_ctor\_rec\_G\_F\_def$   
 $map\_pre\_G\_def\ map\_pre\_F\_def\ map\_pre\_G_M\_def\ map\_pre\_GF_M\_def$   
 $G\_ctor\_o\_rec$   
 $id\_apply\ comp\_id\ id\_comp\ comp\_assoc\ map\_prod.comp\ map\_prod.id$   
 $fst\_convolve\ map\_prod.o\_convolve\ convolve\_o$   
 $rewriteL\_comp\_comp[OF\ G0.map\_comp0[symmetric]]$   
 $rewriteL\_comp\_comp[OF\ F0.map\_comp0[symmetric]]$   
 $F.map\_comp0[symmetric]\ F.map\_id0$   
 $F.ctor\_rec\_o\_map$   
 $rewriteL\_comp\_comp[OF\ type\_copy\_Rep\_o\_Abs[OF\ BNF\_Composition.type\_definition\_id\_bnf\_UNIV]]\ ..$

**lemma**  $n2m\_ctor\_rec\_G\_F$ :

$n2m\_ctor\_rec\_G\_F\ s1\ s2\ o\ ctor\_F = s2\ o\ map\_pre\_GF_M\ id\ (BNF\_Def.convolve\ id\ (n2m\_ctor\_rec\_G\ s1\ s2))$   
 $(BNF\_Def.convolve\ id\ (n2m\_ctor\_rec\_G\_F\ s1\ s2))\ o\ BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf$   
**unfolding**  $n2m\_ctor\_rec\_G\_F\_def\ map\_pre\_F\_def\ map\_pre\_G_M\_def\ map\_pre\_GF_M\_def$   
 $F\_ctor\_o\_rec\ id\_apply\ comp\_id\ id\_comp\ comp\_assoc$   
 $rewriteL\_comp\_comp[OF\ F0.map\_comp0[symmetric]]$   
 $rewriteL\_comp\_comp[OF\ type\_copy\_Rep\_o\_Abs[OF\ BNF\_Composition.type\_definition\_id\_bnf\_UNIV]]\ ..$

### 7.3.3 Induction

**lemma**  $n2m\_rel\_induct\_G\_G\_F$ :

**assumes**  $IH1: \forall x\ y. BNF\_Def.vimage2p\ (BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf)\ (BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf)\ (rel\_pre\_G_M\ P\ R\ S)\ x\ y \longrightarrow R\ (ctor\_G\ x)\ (ctor\_G\ y)$   
**and**  $IH2: \forall x\ y. BNF\_Def.vimage2p\ (BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf)\ (BNF\_Composition.id\_bnf\ o\ BNF\_Composition.id\_bnf)\ (rel\_pre\_GF_M\ P\ R\ S)\ x\ y \longrightarrow S\ (ctor\_F\ x)\ (ctor\_F\ y)$   
**shows**  $rel\_G\ P \leq R \wedge rel\_F\ (rel\_G\ P) \leq S$   
**apply** (rule context\_conjI)  
**apply** (rule G.ctor\_rel\_induct[unfolded rel\_pre\_G\_def id\_apply vimage2p\_def o\_apply])  
**apply** (erule mp[OF spec2[OF IH1], OF vimage2p\_mono[OF \_ pre\_G\_M.rel\_mono], unfolded vimage2p\_def o\_apply rel\_pre\_G\_M\_def type\_definition.Abs\_inverse[OF BNF\_Composition.type\_definition\_id\_bnf\_UNIV\_UNIV\_I]])  
**apply** (rule order\_refl)  
**apply** (rule order\_refl)  
**apply** (rule F.ctor\_rel\_induct[unfolded rel\_pre\_F\_def id\_apply vimage2p\_def o\_apply])  
**apply** (erule mp[OF spec2[OF IH2], OF vimage2p\_mono[OF \_ pre\_GF\_M.rel\_mono], unfolded vimage2p\_def o\_apply rel\_pre\_GF\_M\_def type\_definition.Abs\_inverse[OF BNF\_Composition.type\_definition\_id\_bnf\_UNIV\_UNIV\_I]])  
**apply** (rule F.ctor\_rel\_induct[unfolded rel\_pre\_F\_def id\_apply vimage2p\_def o\_apply])  
**apply** (erule mp[OF spec2[OF IH2], OF vimage2p\_mono[OF \_ pre\_GF\_M.rel\_mono], unfolded vimage2p\_def o\_apply rel\_pre\_GF\_M\_def type\_definition.Abs\_inverse[OF BNF\_Composition.type\_definition\_id\_bnf\_UNIV\_UNIV\_I]])  
**apply** (rule order\_refl)  
**apply** assumption  
**apply** (rule order\_refl)  
**done**

**lemmas**  $n2m\_ctor\_induct\_G\_G\_F = spec[OF\ spec\ [OF$

$n2m\_rel\_induct\_G\_G\_F[of\ (=)\ BNF\_Def.Grp\ (Collect\ R)\ id\ BNF\_Def.Grp\ (Collect\ S)\ id\ \mathbf{for}\ R\ S,$   
 $unfolded\ G.rel\_eq\ F.rel\_eq\ eq\_le\_Grp\_id\_iff\ all\_simps(1,2)[symmetric]]],$   
 $unfolded\ eq\_alt\ pre\_G_M.rel\_Grp\ pre\_GF_M.rel\_Grp\ pre\_G_M.map\_id0\ pre\_GF_M.map\_id0,$   
 $unfolded\ vimage2p\_comp\ vimage2p\_id\ comp\_apply\ comp\_id\ Grp\_id\_mono\_subst$   
 $type\_copy\_vimage2p\_Grp\_Rep[OF\ BNF\_Composition.type\_definition\_id\_bnf\_UNIV]$   
 $type\_copy\_Abs\_o\_Rep[OF\ BNF\_Composition.type\_definition\_id\_bnf\_UNIV]$   
 $eqTrueI[OF\ subset\_UNIV]\ simp\_thms(22)$   
 $atomize\_conjL[symmetric]\ atomize\_imp[symmetric],$   
 $unfolded\ subset\_iff\ mem\_Collect\_eq]$

## 8 Mutual View on Nested Coatypes

**bnf-axiomatization** ('a, 'b) coF0

**bnf-axiomatization** ('a, 'b) coG0

## 8.1 Nested definition

**codatatype**  $'a$   $coF = CcoF ('a, 'a coF) coF0$   
**codatatype**  $'a$   $coG = CcoG ('a, ('a coG) coF) coG0$

**type-synonym**  $('b, 'c) coF\_pre\_coF = ('c, 'b) coF0$   
**type-synonym**  $('c, 'a) coG\_pre\_coG = ('a, 'c coF) coG0$

**term**  $dtor\_unfold\_coF :: ('b \Rightarrow ('b, 'c) coF\_pre\_coF) \Rightarrow 'b \Rightarrow 'c coF$   
**term**  $dtor\_unfold\_coG :: ('c \Rightarrow ('c, 'a) coG\_pre\_coG) \Rightarrow 'c \Rightarrow 'a coG$   
**term**  $dtor\_corec\_coF :: ('b \Rightarrow ('c coF + 'b, 'c) coF\_pre\_coF) \Rightarrow 'b \Rightarrow 'c coF$   
**term**  $dtor\_corec\_coG :: ('c \Rightarrow ('a coG + 'c, 'a) coG\_pre\_coG) \Rightarrow 'c \Rightarrow 'a coG$   
**thm**  $coF.dtor\_rel\_coinduct$   
**thm**  $coG.dtor\_rel\_coinduct[unfolded rel\_pre\_coG\_def id\_apply]$

## 8.2 Isomorphic Mutual Definition

**codatatype**  $'a$   $coG_M = CcoG ('a, 'a coGcoF_M) coG0$   
**and**  $'a$   $coGcoF_M = CcoF ('a coG_M, 'a coGcoF_M) coF0$

**type-synonym**  $('b, 'c) coGcoF_M\_pre\_coGcoF_M = ('c, 'b) coF0$   
**type-synonym**  $('c, 'a) coG_M\_pre\_coG_M = ('a, 'c) coG0$

**term**  $dtor\_unfold\_coG_M :: ('b \Rightarrow ('c, 'a) coG_M\_pre\_coG_M) \Rightarrow ('c \Rightarrow ('c, 'b) coGcoF_M\_pre\_coGcoF_M) \Rightarrow 'b \Rightarrow 'a coG_M$   
**term**  $dtor\_unfold\_coGcoF_M :: ('b \Rightarrow ('c, 'a) coG_M\_pre\_coG_M) \Rightarrow ('c \Rightarrow ('c, 'b) coGcoF_M\_pre\_coGcoF_M) \Rightarrow 'c \Rightarrow 'a coGcoF_M$   
**term**  $dtor\_corec\_coG_M :: ('b \Rightarrow ('a coGcoF_M + 'c, 'a) coG_M\_pre\_coG_M) \Rightarrow ('c \Rightarrow ('a coGcoF_M + 'c, 'a coG_M + 'b) coGcoF_M\_pre\_coGcoF_M) \Rightarrow 'b \Rightarrow 'a coG_M$   
**term**  $dtor\_corec\_coGcoF_M :: ('b \Rightarrow ('a coGcoF_M + 'c, 'a) coG_M\_pre\_coG_M) \Rightarrow ('c \Rightarrow ('a coGcoF_M + 'c, 'a coG_M + 'b) coGcoF_M\_pre\_coGcoF_M) \Rightarrow 'c \Rightarrow 'a coGcoF_M$   
**thm**  $coG_M\_coGcoF_M.dtor\_rel\_coinduct[unfolded rel\_pre\_coG_M\_def rel\_pre\_coGcoF_M\_def]$

## 8.3 Mutualization

### 8.3.1 Coiterators

**definition**  $n2m\_dtor\_unfold\_coG :: ('b \Rightarrow ('c, 'a) coG_M\_pre\_coG_M) \Rightarrow ('c \Rightarrow ('c, 'b) coGcoF_M\_pre\_coGcoF_M) \Rightarrow 'b \Rightarrow 'a coG$

**where**  $n2m\_dtor\_unfold\_coG s1 s2 = dtor\_unfold\_coG (BNF\_Composition.id\_bnf o BNF\_Composition.id\_bnf o map\_pre\_coG_M id (id :: unit \Rightarrow unit) (dtor\_unfold\_coF (BNF\_Composition.id\_bnf o BNF\_Composition.id\_bnf o s2))) o s1$

**definition**  $n2m\_dtor\_unfold\_coG\_coF :: ('b \Rightarrow ('c, 'a) coG_M\_pre\_coG_M) \Rightarrow ('c \Rightarrow ('c, 'b) coGcoF_M\_pre\_coGcoF_M) \Rightarrow 'c \Rightarrow 'a coG coF$

**where**  $n2m\_dtor\_unfold\_coG\_coF s1 s2 = dtor\_unfold\_coF (BNF\_Composition.id\_bnf o BNF\_Composition.id\_bnf o map\_pre\_coGcoF_M (id :: unit \Rightarrow unit) (n2m\_dtor\_unfold\_coG s1 s2) id o s2)$

**lemma**  $coG\_dtor\_o\_unfold: dtor\_coG o dtor\_unfold\_coG s = map\_pre\_coG id (dtor\_unfold\_coG s) o s$   
**unfolding**  $fun\_eq\_iff o\_apply coG.dtor\_unfold$  **by**  $simp$

**lemma**  $coF\_dtor\_o\_unfold: dtor\_coF o dtor\_unfold\_coF s = map\_pre\_coF id (dtor\_unfold\_coF s) o s$   
**unfolding**  $fun\_eq\_iff o\_apply coF.dtor\_unfold$  **by**  $simp$

**lemma**  $coG\_dtor\_o\_corec: dtor\_coG o dtor\_corec\_coG s = map\_pre\_coG id (case\_sum id (dtor\_corec\_coG s)) o s$

**unfolding**  $fun\_eq\_iff o\_apply coG.dtor\_corec$  **by**  $simp$

**lemma**  $coF\_dtor\_o\_corec: dtor\_coF o dtor\_corec\_coF s = map\_pre\_coF id (case\_sum id (dtor\_corec\_coF s)) o s$

**unfolding**  $fun\_eq\_iff o\_apply coF.dtor\_corec$  **by**  $simp$

**lemma**  $n2m\_dtor\_unfold\_coG:$

$dtor\_coG o n2m\_dtor\_unfold\_coG s1 s2 = BNF\_Composition.id\_bnf o BNF\_Composition.id\_bnf o map\_pre\_coG_M id id (n2m\_dtor\_unfold\_coG\_coF s1 s2) o s1$

**unfolding**  $n2m\_dtor\_unfold\_coG\_def n2m\_dtor\_unfold\_coG\_coF\_def$   
 $map\_pre\_coG\_def map\_pre\_coF\_def map\_pre\_coG_M\_def map\_pre\_coGcoF_M\_def$

```

coG_dtor_o_unfold_id_apply_comp_id_id_comp_comp_assoc
rewriteL_comp_comp[OF type_copy_map_comp0_undo[OF BNF_Composition.type_definition_id_bnf_UNIV
BNF_Composition.type_definition_id_bnf_UNIV BNF_Composition.type_definition_id_bnf_UNIV pre_coG_M.map_comp0[unfo
map_pre_coG_M_def]]]
coF.dtor_unfold_o_map
rewriteL_comp_comp[OF type_copy_Rep_o_Abs[OF BNF_Composition.type_definition_id_bnf_UNIV]] ..

```

**lemma**  $n2m\_dtor\_unfold\_coG\_coF$ :

```

dtor_coF o n2m_dtor_unfold_coG_coF s1 s2 = BNF_Composition.id_bnf o BNF_Composition.id_bnf o map_pre_coGcoF_M
id (n2m_dtor_unfold_coG s1 s2) (n2m_dtor_unfold_coG_coF s1 s2) o s2
unfolding n2m_dtor_unfold_coG_coF_def map_pre_coF_def map_pre_coG_M_def map_pre_coGcoF_M_def
coF_dtor_o_unfold_id_apply_comp_id_id_comp_comp_assoc
rewriteL_comp_comp[OF coF0.map_comp0[symmetric]]
rewriteL_comp_comp[OF type_copy_Rep_o_Abs[OF BNF_Composition.type_definition_id_bnf_UNIV]] ..

```

### 8.3.2 Corecursors

**definition**  $n2m\_dtor\_corec\_coG$  ::

```

('b ⇒ ('a coG coF + 'c, 'a) coG_M_pre_coG_M) ⇒ ('c ⇒ ('a coG coF + 'c, 'a coG + 'b) coGcoF_M_pre_coGcoF_M)
⇒ 'b ⇒ 'a coG

```

```

where n2m_dtor_corec_coG s1 s2 =
dtor_corec_coG (BNF_Composition.id_bnf o BNF_Composition.id_bnf o
map_pre_coG_M id (id :: unit ⇒ unit)
(case_sum (map_coF Inl) (dtor_corec_coF (BNF_Composition.id_bnf o BNF_Composition.id_bnf o
map_pre_coGcoF_M (id :: unit ⇒ unit) id (map_sum (map_coF Inl) id) o s2))) o
s1)

```

**definition**  $n2m\_dtor\_corec\_coG\_coF$  ::

```

('b ⇒ ('a coG coF + 'c, 'a) coG_M_pre_coG_M) ⇒ ('c ⇒ ('a coG coF + 'c, 'a coG + 'b) coGcoF_M_pre_coGcoF_M)
⇒ 'c ⇒ 'a coG coF

```

```

where n2m_dtor_corec_coG_coF s1 s2 = dtor_corec_coF (BNF_Composition.id_bnf o BNF_Composition.id_bnf
o map_pre_coGcoF_M (id :: unit ⇒ unit) (case_sum id (n2m_dtor_corec_coG s1 s2)) id o s2)

```

**lemma**  $n2m\_dtor\_corec\_coG$ :

```

dtor_coG o n2m_dtor_corec_coG s1 s2 = BNF_Composition.id_bnf o BNF_Composition.id_bnf o map_pre_coG_M
id id (case_sum id (n2m_dtor_corec_coG_coF s1 s2)) o s1
unfolding n2m_dtor_corec_coG_def n2m_dtor_corec_coG_coF_def
map_pre_coG_def map_pre_coF_def map_pre_coG_M_def map_pre_coGcoF_M_def
coG_dtor_o_corec
id_apply_comp_id_id_comp_comp_assoc[symmetric] map_sum.comp map_sum.id
case_sum_o_inj(1) case_sum_o_map_sum_o_case_sum
rewriteR_comp_comp[OF coG0.map_comp0[symmetric]]
rewriteR_comp_comp[OF coF0.map_comp0[symmetric]]
coF.map_comp0[symmetric] coF.map_id0
coF.dtor_corec_o_map
rewriteR_comp_comp[OF type_copy_Rep_o_Abs[OF BNF_Composition.type_definition_id_bnf_UNIV]] ..

```

**lemma**  $n2m\_dtor\_corec\_coG\_coF$ :

```

dtor_coF o n2m_dtor_corec_coG_coF s1 s2 = BNF_Composition.id_bnf o BNF_Composition.id_bnf o map_pre_coGcoF_M
id (case_sum id (n2m_dtor_corec_coG s1 s2)) (case_sum id (n2m_dtor_corec_coG_coF s1 s2)) o s2
unfolding n2m_dtor_corec_coG_coF_def map_pre_coF_def map_pre_coG_M_def map_pre_coGcoF_M_def
coF_dtor_o_corec_id_apply_comp_id_id_comp_comp_assoc
rewriteL_comp_comp[OF coF0.map_comp0[symmetric]]
rewriteL_comp_comp[OF type_copy_Rep_o_Abs[OF BNF_Composition.type_definition_id_bnf_UNIV]] ..

```

### 8.3.3 Coinduction

**lemma**  $n2m\_rel\_coinduct\_coG\_coG\_coF$ :

```

assumes CIH1: ∀ x y. R x y → BNF_Def.vimage2p (BNF_Composition.id_bnf o BNF_Composition.id_bnf)
(BNF_Composition.id_bnf o BNF_Composition.id_bnf) (rel_pre_coG_M P R S) (dtor_coG x) (dtor_coG y)
and CIH2: ∀ x y. S x y → BNF_Def.vimage2p (BNF_Composition.id_bnf o BNF_Composition.id_bnf)
(BNF_Composition.id_bnf o BNF_Composition.id_bnf) (rel_pre_coGcoF_M P R S) (dtor_coF x) (dtor_coF y)
shows R ≤ rel_coG P ∧ S ≤ rel_coF (rel_coG P)
apply (rule context_conjI)

```

```

apply (rule coG.dtor_rel_coinduct[unfolded rel_pre_coG_def id_apply vimage2p_def o_apply])
apply (erule mp[OF spec2[OF CIH1], THEN vimage2p_mono[OF _ pre_coG_M.rel_mono], unfolded vimage2p_def
o_apply rel_pre_coG_M_def type_definition.Abs_inverse[OF BNF_Composition.type_definition_id_bnf_UNIV_UNIV_I]])
apply (rule order_refl)
apply (rule order_refl)
apply (rule coF.dtor_rel_coinduct[unfolded rel_pre_coF_def id_apply vimage2p_def o_apply])
apply (erule mp[OF spec2[OF CIH2], unfolded vimage2p_def o_apply rel_pre_coGcoF_M_def type_definition.Abs_inverse[OF
BNF_Composition.type_definition_id_bnf_UNIV_UNIV_I]])

```

```

apply (rule coF.dtor_rel_coinduct[unfolded rel_pre_coF_def id_apply vimage2p_def o_apply])
apply (erule mp[OF spec2[OF CIH2], THEN vimage2p_mono[OF _ pre_coGcoF_M.rel_mono], unfolded vimage2p_def o_apply rel_pre_coGcoF_M_def type_definition.Abs_inverse[OF BNF_Composition.type_definition_id_bnf_UNIV_UNIV_I]])
apply (rule order_refl)
apply assumption
apply (rule order_refl)
done

```

```

lemmas n2m_ctor_induct_coG_coG_coF = spec[OF spec[OF spec[OF spec[OF
n2m_rel_coinduct_coG_coG_coF[of _ (=),
unfolded coG.rel_eq coF.rel_eq le_fun_def le_bool_def all_simps(1,2)[symmetric]]]]]]]]

```