Abstract

We present the verification of the normalisation of a binary decision diagram (BDD). The normalisation follows the original algorithm presented by Bryant in 1986 and transforms an ordered BDD in a reduced, ordered and shared BDD. The verification is based on Hoare logics.

BDD-Normalisation

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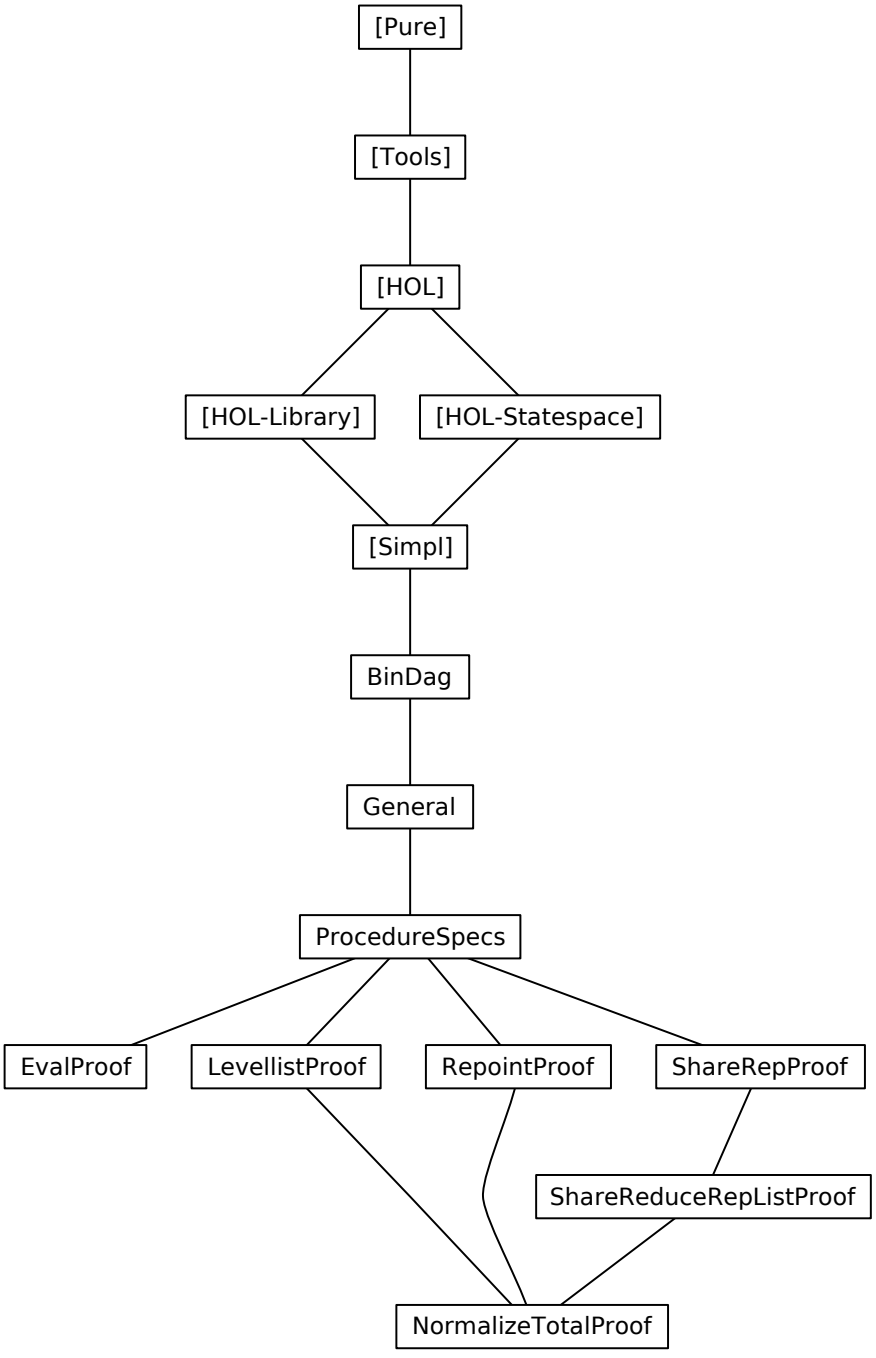
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BinDag
General
ProcedureSpecs
EvalProof
LevellistProof
RepointProof
ShareRepProof
ShareReduceRepListProof
NormalizeTotalProof

1 Introduction

In [1] we describe the partial correctness proofs for BDD normalisation. We extend this work to total correctness in these theories.

2 BDD Abstractions

theory *BinDag*
imports *Simpl.Simpl-Heap*
begin

datatype *dag* = *Tip* | *Node dag ref dag*

lemma [*simp*]: *Node lt a rt* ≠ *lt*
  by (*induct lt*) *auto*

lemma [*simp*]: *lt* ≠ *Node lt a rt*
  by (*induct lt*) *auto*

lemma [*simp*]: *Node lt a rt* ≠ *rt*
  by (*induct rt*) *auto*

lemma [*simp*]: *rt* ≠ *Node lt a rt*
  by (*induct rt*) *auto*

primrec *set-of*:: *dag* ⇒ *ref set* **where**
  set-of-Tip: *set-of Tip* = {}
  | *set-of-Node*: *set-of* (*Node lt a rt*) = {*a*} ∪ *set-of lt* ∪ *set-of rt*

primrec *DAG*:: *dag* ⇒ *bool* **where**
  DAG Tip = *True*
  | *DAG* (*Node l a r*) = (*a* ∉ *set-of l* ∧ *a* ∉ *set-of r* ∧ *DAG l* ∧ *DAG r*)

primrec *subdag*:: *dag* ⇒ *dag* ⇒ *bool* **where**
  subdag Tip t = *False*
  | *subdag* (*Node l a r*) *t* = (*t*=*l* ∨ *t*=*r* ∨ *subdag l t* ∨ *subdag r t*)

lemma *subdag-size*: *subdag t s* ⟹ *size s* < *size t*
  by (*induct t*) *auto*

lemma *subdag-neq*: *subdag t s* ⟹ *t*≠*s*
by (*induct t*) (*auto dest*: *subdag-size*)

lemma *subdag-trans* [*trans*]: *subdag t s* ⟹ *subdag s r* ⟹ *subdag t r*
by (*induct t*) *auto*

lemma *subdag-NodeD*:

subdag t (*Node lt a rt*) $\Longrightarrow$ *subdag t lt* $\wedge$ *subdag t rt*
by (*induct t*) *auto*

lemma *subdag-not-sym*: $\bigwedge t.\ [\![\mathit{subdag}\ s\ t;\ \mathit{subdag}\ t\ s]\!] \Longrightarrow P$
by (*induct s*) (*auto dest*: *subdag-NodeD*)

instantiation *dag* :: *order*
begin

definition
less-dag-def: $s < (t::dag) \longleftrightarrow \mathit{subdag}\ t\ s$

definition
le-dag-def: $s \leq (t::dag) \longleftrightarrow s{=}t \vee s < t$

lemma *le-dag-refl*: $(x::dag) \leq x$
by (*simp add*: *le-dag-def*)

lemma *le-dag-trans*:
fixes $x::dag$ **and** y **and** z
assumes *x-y*: $x \leq y$ **and** *y-z*: $y \leq z$
shows $x \leq z$
proof (*cases x=y*)
case *True* **with** *y-z* **show** *?thesis* **by** *simp*
next
case *False*
note *x-neq-y* = *this*
with *x-y* **have** *x-less-y*: $x < y$ **by** (*simp add*: *le-dag-def*)
show *?thesis*
proof (*cases y=z*)
case *True*
with *x-y* **show** *?thesis* **by** *simp*
next
case *False*
with *y-z* **have** $y < z$ **by** (*simp add*: *le-dag-def*)
with *x-less-y* **have** $x < z$
by (*auto simp add*: *less-dag-def intro*: *subdag-trans*)
thus *?thesis*
by (*simp add*: *le-dag-def*)
qed
qed

lemma *le-dag-antisym*:
fixes $x::dag$ **and** y
assumes *x-y*: $x \leq y$ **and** *y-x*: $y \leq x$
shows $x = y$
proof (*cases x=y*)
case *True* **thus** *?thesis* **.**
next

case *False*
with *x-y y-x* **show** *?thesis*
by (*auto simp add*: *less-dag-def le-dag-def intro*: *subdag-not-sym*)
qed

lemma *dag-less-le*:
fixes *x*::*dag* **and** *y*
shows $(x < y) = (x \leq y \wedge x \neq y)$
by (*auto simp add*: *less-dag-def le-dag-def dest*: *subdag-neq*)

instance
by *standard* (*auto simp add*: *dag-less-le le-dag-refl intro*: *le-dag-trans dest*: *le-dag-antisym*)

end

lemma *less-dag-Tip* [*simp*]: $\neg\ (x < Tip)$
by (*simp add*: *less-dag-def*)

lemma *less-dag-Node*: $x < (Node\ l\ a\ r) =$
$(x \leq l \vee x \leq r)$
by (*auto simp add*: *order-le-less less-dag-def*)

lemma *less-dag-Node'*: $x < (Node\ l\ a\ r) =$
$(x{=}l \vee x{=}r \vee x < l \vee x < r)$
by (*simp add*: *less-dag-def*)

lemma *less-Node-dag*: $(Node\ l\ a\ r) < x \Longrightarrow l < x \wedge r < x$
by (*auto simp add*: *less-dag-def dest*: *subdag-NodeD*)

lemma *less-dag-set-of*: $x < y \Longrightarrow set\text{-}of\ x \subseteq set\text{-}of\ y$
by (*unfold less-dag-def*, *induct y*, *auto*)

lemma *le-dag-set-of*: $x \leq y \Longrightarrow set\text{-}of\ x \subseteq set\text{-}of\ y$
apply (*unfold le-dag-def*)
apply (*erule disjE*)
apply *simp*
apply (*erule less-dag-set-of*)
done

lemma *DAG-less*: $DAG\ y \Longrightarrow x < y \Longrightarrow DAG\ x$
by (*induct y*) (*auto simp add*: *less-dag-Node'*)

lemma *less-DAG-set-of*:
assumes *x-less-y*: $x < y$
assumes *DAG-y*: $DAG\ y$
shows $set\text{-}of\ x \subset set\text{-}of\ y$
proof (*cases y*)
case *Tip* **with** *x-less-y* **show** *?thesis* **by** *simp*
next

case (*Node l a r*)
with *DAG-y* **obtain** *a*: $a \notin$ *set-of l* $a \notin$ *set-of r*
by *simp*
from *Node* **obtain** *l-less-y*: $l < y$ **and** *r-less-y*: $r < y$
by (*simp add*: *less-dag-Node*)
from *Node a* **obtain**
l-subset-y: *set-of l* $\subset$ *set-of y* **and**
r-subset-y: *set-of r* $\subset$ *set-of y*
by *auto*
from *Node x-less-y* **have** $x{=}l \vee x{=}r \vee x < l \vee x < r$
by (*simp add*: *less-dag-Node'*)
thus *?thesis*
proof (*elim disjE*)
assume $x{=}l$
with *l-subset-y* **show** *?thesis* **by** *simp*
next
assume $x{=}r$
with *r-subset-y* **show** *?thesis* **by** *simp*
next
assume $x < l$
hence *set-of x* $\subseteq$ *set-of l*
by (*rule less-dag-set-of*)
also note *l-subset-y*
finally show *?thesis* **.**
next
assume $x < r$
hence *set-of x* $\subseteq$ *set-of r*
by (*rule less-dag-set-of*)
also note *r-subset-y*
finally show *?thesis* **.**
qed
qed

lemma *in-set-of-decomp*:
$p \in$ *set-of t* $=$ $(\exists\, l\ r.\ t{=}(\mathit{Node}\ l\ p\ r) \vee \mathit{subdag}\ t\ (\mathit{Node}\ l\ p\ r))$
(**is** *?A* $=$ *?B*)
proof
assume *?A* **thus** *?B*
by (*induct t*) *auto*
next
assume *?B* **thus** *?A*
by (*induct t*) *auto*
qed

primrec *Dag*:: *ref* $\Rightarrow$ (*ref* $\Rightarrow$ *ref*) $\Rightarrow$ (*ref* $\Rightarrow$ *ref*) $\Rightarrow$ *dag* $\Rightarrow$ *bool*
where
Dag p l r Tip $=$ $(p = \mathit{Null})$ |
Dag p l r (*Node lt a rt*) $=$ $(p = a \wedge p \neq \mathit{Null} \wedge$

$$Dag\ (l\ p)\ l\ r\ lt \wedge Dag\ (r\ p)\ l\ r\ \ rt)$$

lemma *Dag-Null* [*simp*]: $Dag\ Null\ l\ r\ \ t = (t = Tip)$
by (*cases t*) *simp-all*

lemma *Dag-Ref* [*simp*]:
$p{\neq}Null \Longrightarrow Dag\ p\ l\ r\ \ t = (\exists\, lt\ rt.\ t{=}Node\ lt\ p\ rt \wedge$
$Dag\ (l\ p)\ l\ r\ lt \wedge Dag\ (r\ p)\ l\ r\ rt)$
by (*cases t*) *auto*

lemma *Null-notin-Dag* [*simp*, *intro*]: $\bigwedge p\ l\ r.\ Dag\ p\ l\ r\ t \Longrightarrow Null \notin set\text{-}of\ t$
by (*induct t*) *auto*

theorem *notin-Dag-update-l* [*simp*]:
$\bigwedge p.\ q \notin set\text{-}of\ t \Longrightarrow Dag\ p\ (l(q := y))\ r\ \ t = Dag\ p\ l\ r\ t$
by (*induct t*) *auto*

theorem *notin-Dag-update-r* [*simp*]:
$\bigwedge p.\ q \notin set\text{-}of\ t \Longrightarrow Dag\ p\ l\ (r(q := y))\ t = Dag\ p\ l\ r\ t$
by (*induct t*) *auto*

lemma *Dag-upd-same-l-lemma*: $\bigwedge p.\ p{\neq}Null \Longrightarrow \neg\ Dag\ p\ (l(p{:=}p))\ r\ t$
apply (*induct t*)
apply *simp*
apply (*simp* (*no-asm-simp*) *del*: *fun-upd-apply*)
apply (*simp* (*no-asm-simp*) *only*: *fun-upd-apply refl if-True*)
apply *blast*
done

lemma *Dag-upd-same-l* [*simp*]: $Dag\ p\ (l(p{:=}p))\ r\ t = (p{=}Null \wedge t{=}Tip)$
apply (*cases p*=*Null*)
apply *simp*
apply (*fast dest*: *Dag-upd-same-l-lemma*)
done

Dag-upd-same-l prevents $p \neq Null \Longrightarrow Dag\ p\ (l(p := p))\ r\ t = X$ from looping, because of *Dag-Ref* and *fun-upd-apply*.

lemma *Dag-upd-same-r-lemma*: $\bigwedge p.\ p{\neq}Null \Longrightarrow \neg\ Dag\ p\ l\ (r(p{:=}p))\ t$
apply (*induct t*)
apply *simp*
apply (*simp* (*no-asm-simp*) *del*: *fun-upd-apply*)
apply (*simp* (*no-asm-simp*) *only*: *fun-upd-apply refl if-True*)
apply *blast*
done

lemma *Dag-upd-same-r* [*simp*]: $Dag\ p\ l\ (r(p{:=}p))\ t = (p{=}Null \wedge t{=}Tip)$
apply (*cases p*=*Null*)
apply *simp*

apply (*fast dest*: *Dag-upd-same-r-lemma*)
done

lemma *Dag-update-l-new* [*simp*]: ⟦*set-of t* ⊆ *set alloc*⟧
⟹ *Dag p* (*l*(*new* (*set alloc*) := *x*)) *r t* = *Dag p l r t*
by (*rule notin-Dag-update-l*) *fastforce*

lemma *Dag-update-r-new* [*simp*]: ⟦*set-of t* ⊆ *set alloc*⟧
⟹ *Dag p l* (*r*(*new* (*set alloc*) := *x*)) *t* = *Dag p l r t*
by (*rule notin-Dag-update-r*) *fastforce*

lemma *Dag-update-lI* [*intro*]:
⟦*Dag p l r t*; *q* ∉ *set-of t*⟧ ⟹ *Dag p* (*l*(*q* := *y*)) *r t*
by *simp*

lemma *Dag-update-rI* [*intro*]:
⟦*Dag p l r t*; *q* ∉ *set-of t*⟧ ⟹ *Dag p l* (*r*(*q* := *y*)) *t*
by *simp*

lemma *Dag-unique*: ⋀ *p t2*. *Dag p l r t1* ⟹ *Dag p l r t2* ⟹ *t1*=*t2*
by (*induct t1*) *auto*

lemma *Dag-unique1*: *Dag p l r t* ⟹ ∃!*t*. *Dag p l r t*
by (*blast intro*: *Dag-unique*)

lemma *Dag-subdag*: ⋀ *p*. *Dag p l r t* ⟹ *subdag t s* ⟹ ∃ *q*. *Dag q l r s*
by (*induct t*) *auto*

lemma *Dag-root-not-in-subdag-l* [*simp*,*intro*]:
assumes *Dag* (*l p*) *l r t*
shows *p* ∉ *set-of t*
proof −
{
fix *lt rt*
assume *t*: *t* = *Node lt p rt*
from *assms* **have** *Dag* (*l p*) *l r lt*
by (*clarsimp simp only*: *t Dag.simps*)
with *assms* **have** *t*=*lt*
by (*rule Dag-unique*)
with *t* **have** *False*
by *simp*
}
moreover
{
fix *lt rt*
assume *subdag*: *subdag t* (*Node lt p rt*)
with *assms* **obtain** *q* **where** *Dag q l r* (*Node lt p rt*)
by (*rule Dag-subdag* [*elim-format*]) *iprover*
hence *Dag* (*l p*) *l r lt*

```
    by auto
  with assms have t=lt
    by (rule Dag-unique)
  moreover
  have subdag t lt
  proof −
    note subdag
    also have subdag (Node lt p rt) lt by simp
    finally show ?thesis .
  qed
  ultimately have False
    by (simp add: subdag-neq)
 }
 ultimately show ?thesis
  by (auto simp add: in-set-of-decomp)
qed

lemma Dag-root-not-in-subdag-r [simp, intro]:
 assumes Dag (r p) l r t
 shows p ∉ set-of t
proof −
 {
  fix lt rt
  assume t: t = Node lt p rt
  from assms have Dag (r p) l r rt
    by (clarsimp simp only: t Dag.simps)
  with assms have t=rt
    by (rule Dag-unique)
  with t have False
    by simp
 }
 moreover
 {
  fix lt rt
  assume subdag: subdag t (Node lt p rt)
  with assms obtain q where Dag q l r (Node lt p rt)
    by (rule Dag-subdag [elim-format]) iprover
  hence Dag (r p) l r rt
    by auto
  with assms have t=rt
    by (rule Dag-unique)
  moreover
  have subdag t rt
  proof −
    note subdag
    also have subdag (Node lt p rt) rt by simp
    finally show ?thesis .
  qed
  ultimately have False
```

by (*simp add*: *subdag-neq*)
}
ultimately show *?thesis*
by (*auto simp add*: *in-set-of-decomp*)
qed

lemma *Dag-is-DAG*: $\bigwedge p\ l\ r.\ Dag\ p\ l\ r\ t \Longrightarrow DAG\ t$
by (*induct t*) *auto*

lemma *heaps-eq-Dag-eq*:
$\bigwedge p.\ \forall x \in set\text{-}of\ t.\ l\ x = l'\ x \wedge r\ x = r'\ x$
$\Longrightarrow Dag\ p\ l\ r\ t = Dag\ p\ l'\ r'\ t$
by (*induct t*) *auto*

lemma *heaps-eq-DagI1*:
$[\![Dag\ p\ l\ r\ t;\ \forall x{\in}set\text{-}of\ t.\ l\ x = l'\ x \wedge r\ x = r'\ x]\!]$
$\Longrightarrow Dag\ p\ l'\ r'\ t$
by (*rule heaps-eq-Dag-eq* [*THEN iffD1*])

lemma *heaps-eq-DagI2*:
$[\![Dag\ p\ l'\ r'\ t;\ \forall x{\in}set\text{-}of\ t.\ l\ x = l'\ x \wedge r\ x = r'\ x]\!]$
$\Longrightarrow Dag\ p\ l\ r\ t$
by (*rule heaps-eq-Dag-eq* [*THEN iffD2*]) *auto*

lemma *Dag-unique-all-impl-simp* [*simp*]:
$Dag\ p\ l\ r\ t \Longrightarrow (\forall t.\ Dag\ p\ l\ r\ t \longrightarrow P\ t) = P\ t$
by (*auto dest*: *Dag-unique*)

lemma *Dag-unique-ex-conj-simp* [*simp*]:
$Dag\ p\ l\ r\ t \Longrightarrow (\exists t.\ Dag\ p\ l\ r\ t \wedge P\ t) = P\ t$
by (*auto dest*: *Dag-unique*)

lemma *Dags-eq-hp-eq*:
$\bigwedge p\ p'.\ [\![Dag\ p\ l\ r\ t;\ Dag\ p'\ l'\ r'\ t]\!] \Longrightarrow$
$p'{=}p \wedge (\forall no \in set\text{-}of\ t.\ l'\ no = l\ no \wedge r'\ no = r\ no)$
by (*induct t*) *auto*

definition *isDag* :: *ref* $\Rightarrow$ (*ref* $\Rightarrow$ *ref*) $\Rightarrow$ (*ref* $\Rightarrow$ *ref*) $\Rightarrow$ *bool*
where $isDag\ p\ l\ r = (\exists t.\ Dag\ p\ l\ r\ t)$

definition *dag* :: *ref* $\Rightarrow$ (*ref* $\Rightarrow$ *ref*) $\Rightarrow$ (*ref* $\Rightarrow$ *ref*) $\Rightarrow$ *dag*
where $dag\ p\ l\ r = (THE\ t.\ Dag\ p\ l\ r\ t)$

lemma *Dag-conv-isDag-dag*: $Dag\ p\ l\ r\ t = (isDag\ p\ l\ r \wedge t{=}dag\ p\ l\ r)$
apply (*simp add*: *isDag-def dag-def*)
apply (*rule iffI*)
apply (*rule conjI*)
apply *blast*

apply (*subst the1-equality*)
apply (*erule Dag-unique1*)
apply *assumption*
apply (*rule refl*)
apply *clarify*
apply (*rule theI*)
apply *assumption*
apply (*erule* (*1*) *Dag-unique*)
done

lemma *Dag-dag*: *Dag p l r t* $\Longrightarrow$ *dag p l r* = *t*
by (*simp add*: *Dag-conv-isDag-dag*)

end

3 General Lemmas on BDD Abstractions

theory *General* **imports** *BinDag* **begin**

definition *subdag-eq*:: *dag* $\Rightarrow$ *dag* $\Rightarrow$ *bool* **where**
subdag-eq t_1 t_2 = (t_1 = t_2 $\vee$ *subdag* t_1 t_2)

primrec *root* :: *dag* $\Rightarrow$ *ref*
where
root Tip = *Null* |
root (*Node l a r*) = *a*

fun *isLeaf* :: *dag* $\Rightarrow$ *bool* **where**
isLeaf Tip = *False*
| *isLeaf* (*Node Tip v Tip*) = *True*
| *isLeaf* (*Node* (*Node l* v_1 *r*) v_2 *Tip*) = *False*
| *isLeaf* (*Node Tip* v_1 (*Node l* v_2 *r*)) = *False*

datatype *bdt* = *Zero* | *One* | *Bdt-Node bdt nat bdt*

fun *bdt-fn* :: *dag* $\Rightarrow$ (*ref* $\Rightarrow$ *nat*) $\Rightarrow$ *bdt option* **where**
bdt-fn Tip = (λ*bdtvar* . *None*)
| *bdt-fn* (*Node Tip vref Tip*) =
(λ*bdtvar* .
(*if* (*bdtvar vref* = *0*)
then Some Zero
else (*if* (*bdtvar vref* = *1*)
then Some One
else None)))
| *bdt-fn* (*Node Tip vref* (*Node l vref1 r*)) = (λ*bdtvar* . *None*)
| *bdt-fn* (*Node* (*Node l vref1 r*) *vref Tip*) = (λ*bdtvar* . *None*)
| *bdt-fn* (*Node* (*Node l1 vref1 r1*) *vref* (*Node l2 vref2 r2*)) =
(λ*bdtvar* .

```
  (if (bdtvar vref = 0 ∨ bdtvar vref = 1)
   then None
   else
    (case (bdt-fn (Node l1 vref1 r1) bdtvar) of
     None ⇒ None
     |(Some b1) ⇒
       (case (bdt-fn (Node l2 vref2 r2) bdtvar) of
        None ⇒ None
       |(Some b2) ⇒ Some (Bdt-Node b1 (bdtvar vref) b2)))))
```

abbreviation *bdt == bdt-fn*

```
primrec eval :: bdt ⇒ bool list ⇒ bool
where
eval Zero env = False |
eval One env  = True |
eval (Bdt-Node l v r) env  = (if (env ! v) then eval r env else eval l env)
```

```
fun ordered-bdt:: bdt ⇒ bool where
ordered-bdt Zero = True
| ordered-bdt One = True
| ordered-bdt (Bdt-Node (Bdt-Node l1 v1 r1) v (Bdt-Node l2 v2 r2)) =
    ((v1 < v) ∧ (v2 < v) ∧
     (ordered-bdt (Bdt-Node l1 v1 r1)) ∧ (ordered-bdt (Bdt-Node l2 v2 r2)))
| ordered-bdt (Bdt-Node (Bdt-Node l1 v1 r1) v r) =
    ((v1 < v) ∧ (ordered-bdt (Bdt-Node l1 v1 r1)))
| ordered-bdt (Bdt-Node l v (Bdt-Node l2 v2 r2)) =
    ((v2 < v) ∧ (ordered-bdt (Bdt-Node l2 v2 r2)))
| ordered-bdt (Bdt-Node l v r) = True
```

```
fun ordered:: dag ⇒ (ref⇒nat) ⇒ bool where
ordered Tip = (λ var. True)
| ordered (Node (Node l1 v1 r1) v (Node l2 v2 r2)) =
    (λ var. (var v1 < var v ∧ var v2 < var v) ∧
        (ordered (Node l1 v1 r1) var) ∧ (ordered (Node l2 v2 r2) var))
| ordered (Node Tip v Tip) = (λ var. (True))
| ordered (Node Tip v r) =
    (λ var. (var (root r) < var v) ∧ (ordered r var))
| ordered (Node l v Tip) =
    (λ var. (var (root l) < var v) ∧ (ordered l var))
```

```
primrec max-var :: bdt ⇒ nat
where
max-var Zero = 0 |
```

max-var One = 1 |
max-var (Bdt-Node l v r) = max v (max (max-var l) (max-var r))

lemma *eval-zero*: ⟦*bdt (Node l v r) var = Some x*;
var (root (Node l v r)) = (0::nat)⟧ ⟹ *x = Zero*
apply (*cases l*)
apply (*cases r*)
apply *simp*
apply *simp*
apply (*cases r*)
apply *simp*
apply *simp*
done

lemma *bdt-Some-One-iff* [*simp*]:
(*bdt t var = Some One*) = (∃ *p. t = Node Tip p Tip* ∧ *var p = 1*)
apply (*induct t rule*: *bdt-fn.induct*)
apply (*auto split*: *option.splits*)
done

lemma *bdt-Some-Zero-iff* [*simp*]:
(*bdt t var = Some Zero*) = (∃ *p. t = Node Tip p Tip* ∧ *var p = 0*)
apply (*induct t rule*: *bdt-fn.induct*)
apply (*auto split*: *option.splits*)
done

lemma *bdt-Some-Node-iff* [*simp*]:
(*bdt t var = Some (Bdt-Node bdt1 v bdt2*)) =
(∃ *p l r. t = Node l p r* ∧ *bdt l var = Some bdt1* ∧ *bdt r var = Some bdt2* ∧
1 < v ∧ *var p = v*)
apply (*induct t rule*: *bdt-fn.induct*)
prefer *5*
apply (*fastforce split*: *if-splits option.splits*)
apply *auto*
done

lemma *balanced-bdt*:
⋀ *p bdt1.* ⟦ *Dag p low high t*; *bdt t var = Some bdt1*; *no* ∈ *set-of t*⟧
⟹ (*low no = Null*) = (*high no = Null*)
proof (*induct t*)
case *Tip*
then show *?case* **by** *simp*
next
case (*Node lt a rt*)
note *NN*= *this*
have *bdt1*: *bdt* (*Node lt a rt*) *var = Some bdt1* **by** *fact*
have *no-in-t*: *no* ∈ *set-of* (*Node lt a rt*) **by** *fact*
have *p-tree*: *Dag p low high* (*Node lt a rt*) **by** *fact*

```
from Node.prems obtain
  lt: Dag (low p) low high lt and
  rt: Dag (high p) low high rt
  by auto
show ?case
proof (cases lt)
  case (Node llt l rlt)
  note Nlt=this
  show ?thesis
  proof (cases rt)
    case (Node lrt r rrt)
    note Nrt=this
    from Nlt Nrt bdt1 obtain lbdt rbdt where
      lbdt-def: bdt lt var = Some lbdt and
      rbdt-def: bdt rt var = Some rbdt and
      bdt1-def: bdt1 = Bdt-Node lbdt (var a) rbdt
      by (auto split: if-split-asm option.splits)
    from no-in-t show ?thesis
    proof (simp, elim disjE)
      assume  no = a
      with p-tree Nlt Nrt show ?thesis
        by auto
    next
      assume no ∈ set-of lt
      with Node.hyps lbdt-def lt show ?thesis
        by simp
    next
      assume no ∈ set-of rt
      with Node.hyps rbdt-def rt show ?thesis
        by simp
    qed
  next
    case Tip
    with Nlt bdt1 show ?thesis
      by simp
  qed
next
  case Tip
  note ltTip=this
  show ?thesis
  proof (cases rt)
    case Tip
    with ltTip bdt1 no-in-t p-tree show ?thesis
      by auto
  next
    case (Node rlt r rrt)
    with bdt1 ltTip show ?thesis
      by simp
  qed
```

qed
qed

primrec *dag-map* :: (*ref* ⇒ *ref*) ⇒ *dag* ⇒ *dag*
where
dag-map f Tip = *Tip* |
dag-map f (*Node l a r*) = (*Node* (*dag-map f l*) (*f a*) (*dag-map f r*))

definition *wf-marking* :: *dag* ⇒ (*ref* ⇒ *bool*) ⇒ (*ref* ⇒ *bool*) ⇒ *bool* ⇒ *bool*
where
wf-marking t mark-old mark-new marked =
(*case t of Tip* ⇒ *mark-new* = *mark-old*
| (*Node lt p rt*) ⇒
(∀ *n*. *n* ∉ *set-of t* ⟶ *mark-new n* = *mark-old n*) ∧
(∀ *n*. *n* ∈ *set-of t* ⟶ *mark-new n* = *marked*))

definition *dag-in-levellist*:: *dag* ⇒ (*ref list list*) ⇒ (*ref* ⇒ *nat*) ⇒ *bool*
where *dag-in-levellist t levellist var* = (*t* ≠ *Tip* ⟶
(∀ *st*. *subdag-eq t st* ⟶ *root st* ∈ *set* (*levellist* ! (*var* (*root st*)))))

lemma *set-of-nn*: ⟦ *Dag p low high t*; *n* ∈ *set-of t*⟧ ⟹ *n* ≠ *Null*
apply (*induct t*)
apply *simp*
apply *auto*
done

lemma *subnodes-ordered* [*rule-format*]:
∀ *p*. *n* ∈ *set-of t* ⟶ *Dag p low high t* ⟶ *ordered t var* ⟶ *var n* <= *var p*
apply (*induct t*)
apply *simp*
apply (*intro allI*)
apply (*erule-tac x*=*low p* **in** *allE*)
apply (*erule-tac x*=*high p* **in** *allE*)
apply *clarsimp*
apply (*case-tac t1*)
apply (*case-tac t2*)
apply *simp*
apply *fastforce*
apply (*case-tac t2*)
apply *fastforce*
apply *fastforce*
done

lemma *ordered-set-of*:
⋀ *x*. ⟦*ordered t var*; *x* ∈ *set-of t*⟧ ⟹ *var x* <= *var* (*root t*)
apply (*induct t*)
apply *simp*

apply *simp*
apply (*elim disjE*)
apply *simp*
apply (*case-tac t1*)
apply *simp*
apply (*case-tac t2*)
apply *fastforce*
apply *fastforce*
apply (*case-tac t2*)
apply *simp*
apply (*case-tac t1*)
apply *fastforce*
apply *fastforce*
done

lemma *dag-setofD*: $\bigwedge$ *p low high n.* $[\![$ *Dag p low high t*; *n* $\in$ *set-of t* $]\!] \Longrightarrow$
(∃ *nt. Dag n low high nt*) ∧ (∀ *nt. Dag n low high nt* ⟶ *set-of nt* ⊆ *set-of t*)
apply (*induct t*)
apply *simp*
apply *auto*
apply *fastforce*
apply (*fastforce dest*: *Dag-unique*)
apply (*fastforce dest*: *Dag-unique*)
apply *blast*
apply *blast*
done

lemma *dag-setof-exD*: $[\![$*Dag p low high t*; *n* $\in$ *set-of t*$]\!] \Longrightarrow$ ∃ *nt. Dag n low high nt*
apply (*simp add*: *dag-setofD*)
done

lemma *dag-setof-subsetD*: $[\![$*Dag p low high t*; *n* $\in$ *set-of t*; *Dag n low high nt*$]\!] \Longrightarrow$ *set-of nt* ⊆ *set-of t*
apply (*simp add*: *dag-setofD*)
done

lemma *subdag-parentdag-low*: *not* <= *lt* ⟹ *not* <= (*Node lt p rt*)
apply (*cases not* = *lt*)
apply (*cases lt*)
apply *simp*
apply (*cases rt*)
apply *simp*
apply (*simp add*: *le-dag-def less-dag-def*)
apply (*simp add*: *le-dag-def less-dag-def*)
apply (*simp add*: *le-dag-def less-dag-def*)
apply (*simp add*: *le-dag-def less-dag-def*)
done

lemma *subdag-parentdag-high*: *not* <= *rt* ⟹ *not* <= (*Node lt p rt*)
apply (*cases not* = *rt*)
apply (*cases lt*)
apply *simp*
apply (*cases rt*)
apply *simp*
apply (*simp add*: *le-dag-def less-dag-def*)
apply (*simp add*: *le-dag-def less-dag-def*)
apply (*simp add*: *le-dag-def less-dag-def*)
apply (*simp add*: *le-dag-def less-dag-def*)
done

lemma *set-of-subdag*: ⋀ *p not no*.
⟦*Dag p low high t*; *Dag no low high not*; *no* ∈ *set-of t*⟧ ⟹ *not* <= *t*
proof (*induct t*)
case *Tip*
then show *?case* **by** *simp*
next
case (*Node lt po rt*)
note *rtNode=this*
from *Node.prems* **have** *ppo*: *p=po*
by *simp*
show *?case*
proof (*cases no* = *p*)
case *True*
with *ppo Node.prems* **have** *not* = (*Node lt po rt*)
by (*simp add*: *Dag-unique del*: *Dag-Ref*)
with *Node.prems ppo* **show** *?thesis* **by** (*simp add*: *subdag-eq-def*)
next
assume *no* ≠ *p*
with *Node.prems* **have** *no-in-ltorrt*: *no* ∈ *set-of lt* ∨ *no* ∈ *set-of rt*
by *simp*
show *?thesis*
proof (*cases no* ∈ *set-of lt*)
case *True*
from *Node.prems ppo* **have** *Dag* (*low po*) *low high lt*
by *simp*
with *Node.prems ppo True* **have** *not* <= *lt*
apply −
apply (*rule Node.hyps*)
apply *assumption*+
done
with *Node.prems no-in-ltorrt* **show** *?thesis*
apply −
apply (*rule subdag-parentdag-low*)
apply *simp*
done
next
assume *no* ∉ *set-of lt*

```
      with no-in-ltorrt have no-in-rt: no ∈ set-of rt
        by simp
      from Node.prems ppo have Dag (high po) low high rt
        by simp
      with Node.prems ppo no-in-rt have not <= rt
        apply −
        apply (rule Node.hyps)
        apply assumption+
        done
      with Node.prems no-in-rt show ?thesis
        apply −
        apply (rule subdag-parentdag-high)
        apply simp
        done
    qed
  qed
qed

lemma children-ordered: ⟦ordered (Node lt p rt) var⟧ ⟹
ordered lt var ∧ ordered rt var
proof (cases lt)
  case Tip
  note ltTip=this
  assume orderedNode: ordered (Node lt p rt) var
  show ?thesis
  proof (cases rt)
    case Tip
    note rtTip = this
    with ltTip show ?thesis
      by simp
  next
    case (Node lrt r rrt)
    with orderedNode ltTip show ?thesis
      by simp
  qed
next
  case (Node llt l rlt)
  note ltNode=this
  assume orderedNode: ordered (Node lt p rt) var
  show ?thesis
  proof (cases rt)
    case Tip
    note rtTip = this
    with orderedNode ltNode show ?thesis by simp
  next
    case (Node lrt r rrt)
    note rtNode = this
    with orderedNode ltNode show ?thesis by simp
  qed
```

qed

lemma *ordered-subdag*: ⟦*ordered t var*; *not* <= *t*⟧ ⟹ *ordered not var*
proof (*induct t*)
  case *Tip*
  then show *?thesis* **by** (*simp add*: *less-dag-def le-dag-def*)
next
  case (*Node lt p rt*)
  show *?case*
  proof (*cases not* = *Node lt p rt*)
    case *True*
    with *Node.prems* **show** *?thesis* **by** *simp*
  next
    assume *notnt*: *not* ≠ *Node lt p rt*
    with *Node.prems* **have** *notstltorrt*: *not* <= *lt* ∨ *not* <= *rt*
      apply −
      apply (*simp add*: *less-dag-def le-dag-def*)
      apply *fastforce*
      done
    from *Node.prems* **have** *ord-lt*: *ordered lt var*
      apply −
      apply (*drule children-ordered*)
      apply *simp*
      done
    from *Node.prems* **have** *ord-rt*: *ordered rt var*
      apply −
      apply (*drule children-ordered*)
      apply *simp*
      done
    show *?thesis*
    proof (*cases not* <= *lt*)
      case *True*
      with *ord-lt* **show** *?thesis*
        apply −
        apply (*rule Node.hyps*)
        apply *assumption*+
        done
    next
      assume ¬ *not* ≤ *lt*
      with *notstltorrt* **have** *notinrt*: *not* <= *rt*
        by *simp*
      from *Node.hyps* **have** *hyprt*: ⟦*ordered rt var*; *not* ≤ *rt*⟧ ⟹ *ordered not var*
by *simp*
      from *notinrt ord-rt* **show** *?thesis*
        apply −
        apply (*rule hyprt*)
        apply *assumption*+
        done
    qed

```
  qed
qed


lemma subdag-ordered:
⋀ not no p. ⟦ordered t var; Dag p low high t; Dag no low high not;
            no ∈ set-of t⟧ ⟹ ordered not var
proof (induct t)
  case Tip
  from Tip.prems show ?case by simp
next
  case (Node lt po rt)
  note nN=this
  show ?case
  proof (cases lt)
    case Tip
    note ltTip=this
    show ?thesis
    proof (cases rt)
      case Tip
      from Node.prems have ppo: p=po
        by simp
      from Tip ltTip Node.prems have no=p
        by simp
      with ppo Node.prems have not=(Node lt po rt)
        by (simp del: Dag-Ref add: Dag-unique)
      with Node.prems show ?thesis by simp
    next
      case (Node lrnot rn rrnot)
      from Node.prems ltTip Node have ord-rt: ordered rt var
        by simp
      from Node.prems have ppo: p=po
        by simp
      from Node.prems have ponN: po ≠ Null
        by auto
      with ppo ponN ltTip Node.prems have ∗: Dag (high po) low high rt
        by auto
      show ?thesis
      proof (cases no=po)
        case True
        with ppo Node.prems have not = Node lt po rt
          by (simp del: Dag-Ref add: Dag-unique)
        with Node.prems show ?thesis
          by simp
      next
        case False
        with Node.prems ltTip have no ∈ set-of rt
          by simp
        with ord-rt ∗ ‹Dag no low high not› show ?thesis
```

```
        by (rule Node.hyps)
    qed
  qed
 next
  case (Node llt l rlt)
  note ltNode=this
  show ?thesis
  proof (cases rt)
   case Tip
   from Node.prems Tip ltNode have ord-lt: ordered lt var
    by simp
   from Node.prems have ppo: p=po
    by simp
   from Node.prems have ponN: po ≠ Null
    by auto
   with ppo ponN Tip Node.prems ltNode have ∗: Dag (low po) low high lt
    by auto
   show ?thesis
   proof (cases no=po)
    case True
    with ppo Node.prems have not = (Node lt po rt)
     by (simp del: Dag-Ref add: Dag-unique)
    with Node.prems show ?thesis by simp
   next
    case False
    with Node.prems Tip have no ∈ set-of lt
     by simp
    with ord-lt ∗ ‹Dag no low high not› show ?thesis
     by (rule Node.hyps)
   qed
  next
   case (Node lrt r rrt)
   from Node.prems have ppo: p=po
    by simp
   from Node.prems Node ltNode have ord-lt: ordered lt var
    by simp
   from Node.prems Node ltNode have ord-rt: ordered rt var
    by simp
   from Node.prems have ponN: po ≠ Null
    by auto
   with ppo ponN Node Node.prems ltNode have lt-Dag: Dag (low po) low high
lt
    by auto
  with ppo ponN Node Node.prems ltNode have rt-Dag: Dag (high po) low high
rt
    by auto
   show ?thesis
   proof (cases no = po)
    case True
```

```
        with ppo Node.prems have not = (Node lt po rt)
          by (simp del: Dag-Ref add: Dag-unique)
        with Node.prems show ?thesis by simp
      next
        assume no ≠ po
        with Node.prems have no-in-lt-or-rt: no ∈ set-of lt ∨ no ∈ set-of rt
          by simp
        show ?thesis
        proof (cases no ∈ set-of lt)
          case True
          with ord-lt lt-Dag Node.prems show ?thesis
            apply −
            apply (rule Node.hyps)
            apply assumption+
            done
        next
          assume  no ∉ set-of lt
          with no-in-lt-or-rt have no-in-rt: no ∈ set-of rt
            by simp
          from Node.hyps have hyp2:
            ⋀p no not. ⟦ordered rt var; Dag p low high rt; Dag no low high not; no
∈ set-of rt⟧
            ⟹ ordered not var
            apply −
            apply assumption
            done
          from no-in-rt ord-rt rt-Dag Node.prems show ?thesis
            apply −
            apply (rule hyp2)
            apply assumption+
            done
        qed
      qed
    qed
  qed
qed

lemma elem-set-of: ⋀ x st. ⟦x ∈ set-of st; set-of st ⊆ set-of t⟧ ⟹ x ∈ set-of t
by blast




definition wf-ll :: dag ⇒ ref list list ⇒ (ref ⇒ nat) ⇒ bool
where
wf-ll t levellist var =
  ((∀ p. p ∈ set-of t ⟶ p ∈ set (levellist ! var p)) ∧
   (∀ k < length levellist. ∀ p ∈ set (levellist ! k). p ∈ set-of t ∧ var p = k))
```

definition *cong-eval* :: *bdt* ⇒ *bdt* ⇒ *bool* (**infix** ∼ *60*)
where *cong-eval bdt*$_1$ *bdt*$_2$ = (*eval bdt*$_1$ = *eval bdt*$_2$)

lemma *cong-eval-sym*: *l* ∼ *r* = *r* ∼ *l*
by (*auto simp add*: *cong-eval-def*)

lemma *cong-eval-trans*: ⟦ *l* ∼ *r*; *r* ∼ *t*⟧ ⟹ *l* ∼ *t*
by (*simp add*: *cong-eval-def*)

lemma *cong-eval-child-high*: *l* ∼ *r* ⟹ *r* ∼ (*Bdt-Node l v r*)
apply (*simp add*: *cong-eval-def*)
apply (*rule ext*)
apply *auto*
done

lemma *cong-eval-child-low*: *l* ∼ *r* ⟹ *l* ∼ (*Bdt-Node l v r*)
apply (*simp add*: *cong-eval-def*)
apply (*rule ext*)
apply *auto*
done

definition *null-comp* :: (*ref* ⇒ *ref*) ⇒ (*ref* ⇒ *ref*) ⇒ (*ref* ⇒ *ref*) (**infix** ∝ *60*)
where *null-comp a b* = (λ *p*. (*if* (*b p*) = *Null then Null else* ((*a* ∘ *b*) *p*)))

lemma *null-comp-not-Null* [*simp*]: *h q* ≠ *Null* ⟹ (*g* ∝ *h*) *q* = *g* (*h q*)
by (*simp add*: *null-comp-def*)

lemma *id-trans*: (*a* ∝ *id*) (*b* (*c p*)) = (*a* ∝ *b*) (*c p*)
by (*simp add*: *null-comp-def*)

definition *Nodes* :: *nat* ⇒ *ref list list* ⇒ *ref set*
where *Nodes i levellist* = (⋃*k*∈{*k*. *k* < *i*} . *set* (*levellist* ! *k*))

inductive-set *Dags* :: *ref set* ⇒ (*ref* ⇒ *ref*) ⇒ (*ref* ⇒ *ref*) ⇒ *dag set*
for *nodes low high*
where
DagsI: ⟦ *set-of t* ⊆ *nodes*; *Dag p low high t*; *t* ≠ *Tip*⟧
⟹ *t* ∈ *Dags nodes low high*

lemma *empty-Dags* [*simp*]: *Dags* {} *low high* = {}
apply *rule*
apply *rule*
apply (*erule Dags.cases*)
apply (*case-tac t*)
apply *simp*

apply *simp*
apply *rule*
done

definition *isLeaf-pt* :: *ref* ⇒ (*ref* ⇒ *ref*) ⇒ (*ref* ⇒ *ref*) ⇒ *bool*
where *isLeaf-pt p low high* = (*low p* = *Null* ∧ *high p* = *Null*)

definition *repNodes-eq* :: *ref* ⇒ *ref* ⇒ (*ref* ⇒ *ref*) ⇒ (*ref* ⇒ *ref*) ⇒ (*ref* ⇒ *ref*) ⇒ *bool*
where
repNodes-eq p q low high rep ==
(*rep* ∝ *high*) *p* = (*rep* ∝ *high*) *q* ∧ (*rep* ∝ *low*) *p* = (*rep* ∝ *low*) *q*

definition *isomorphic-dags-eq* :: *dag* ⇒ *dag* ⇒ (*ref* ⇒ *nat*) ⇒ *bool*
where
isomorphic-dags-eq st_1 st_2 *var* =
(∀ bdt_1 bdt_2. (*bdt* st_1 *var* = *Some* bdt_1 ∧ *bdt* st_2 *var* = *Some* bdt_2 ∧ (bdt_1 = bdt_2))
⟶ st_1 = st_2)

lemma *isomorphic-dags-eq-sym*: *isomorphic-dags-eq* st_1 st_2 *var* = *isomorphic-dags-eq* st_2 st_1 *var*
by (*auto simp add*: *isomorphic-dags-eq-def*)

definition *shared* :: *dag* ⇒ (*ref* ⇒ *nat*) ⇒ *bool*
where *shared t var* = (∀ st_1 st_2. (st_1 <= *t* ∧ st_2 <= *t*) ⟶ *isomorphic-dags-eq* st_1 st_2 *var*)

fun *reduced* :: *dag* ⇒ *bool* **where**
reduced Tip = *True*
| *reduced* (*Node Tip v Tip*) = *True*
| *reduced* (*Node l v r*) = (*l* ≠ *r* ∧ *reduced l* ∧ *reduced r*)

primrec *reduced-bdt* :: *bdt* ⇒ *bool*
where
reduced-bdt Zero = *True*
| *reduced-bdt One* = *True*
| *reduced-bdt* (*Bdt-Node lbdt v rbdt*) =
(*if lbdt* = *rbdt then False*
else (*reduced-bdt lbdt* ∧ *reduced-bdt rbdt*))

lemma *replicate-elem*: *i* < *n* ==> (*replicate n x* !*i*) = *x*

apply (*induct n*)
apply *simp*
apply (*cases i*)
apply *auto*
done

lemma *no-in-one-ll*:
⟦*wf-ll pret levellista var*; *i*<*length levellista*; *j* < *length levellista*;
no ∈ *set* (*levellista* ! *i*); *i*≠*j*⟧
⟹ *no* ∉ *set* (*levellista* ! *j*)
apply (*unfold wf-ll-def*)
apply (*erule conjE*)
apply (*rotate-tac 5*)
apply (*frule-tac x* = *i* **and** *?R*= *no* ∈ *set-of pret* ∧ *var no* = *i* **in** *allE*)
apply (*erule impE*)
apply *simp*
apply (*rotate-tac 6*)
apply (*erule-tac x*=*no* **in** *ballE*)
apply *assumption*
apply *simp*
apply (*cases no* ∉ *set* (*levellista* ! *j*))
apply *assumption*
apply (*erule-tac x*=*j* **in** *allE*)
apply (*erule impE*)
apply *assumption*
apply (*rotate-tac 7*)
apply (*erule-tac x*=*no* **in** *ballE*)
prefer *2*
apply *assumption*
apply (*elim conjE*)
apply (*thin-tac* ∀ *q*. *q* ∈ *set-of pret* ⟶ *q* ∈ *set* (*levellista* ! *var q*))
apply *fastforce*
done

lemma *nodes-in-wf-ll*:
⟦*wf-ll pret levellista var*; *i* < *length levellista*; *no* ∈ *set* (*levellista* ! *i*)⟧
⟹ *var no* = *i* ∧ *no* ∈ *set-of pret*
apply (*simp add*: *wf-ll-def*)
done

lemma *subelem-set-of-low*:
⋀ *p*. ⟦ *x* ∈ *set-of t*; *x* ≠ *Null*; *low x* ≠ *Null*; *Dag p low high t* ⟧
⟹ (*low x*) ∈ *set-of t*
proof (*induct t*)
case *Tip*
then show *?case* **by** *simp*
next
case (*Node lt po rt*)
note *tNode*=*this*

then have *ppo*: *p=po* **by** *simp*
show *?case*
proof (*cases x=p*)
  case *True*
  with *Node.prems* **have** *lxrootlt*: *low x = root lt*
  proof (*cases lt*)
    case *Tip*
    with *True Node.prems* **show** *?thesis*
      by *auto*
  next
    case (*Node llt l rlt*)
    with *Node.prems True* **show** *?thesis*
      by *auto*
  qed
  with *True Node.prems* **have** *low x ∈ set-of* (*Node lt p rt*)
  proof (*cases lt*)
    case *Tip*
    with *lxrootlt Node.prems* **show** *?thesis*
      by *simp*
  next
    case (*Node llt l rlt*)
    with *lxrootlt Node.prems* **show** *?thesis*
      by *simp*
  qed
  with *ppo* **show** *?thesis*
    by *simp*
next
  assume *xnp*: *x ≠ p*
  with *Node.prems* **have** *x ∈ set-of lt ∨ x ∈ set-of rt*
    by *simp*
  show *?thesis*
  proof (*cases x ∈ set-of lt*)
    case *True*
    note *xinlt=this*
    from *Node.prems* **have** *Dag* (*low p*) *low high lt*
      by *fastforce*
    with *Node.prems True* **have** *low x ∈ set-of lt*
      apply −
      apply (*rule Node.hyps*)
      apply *assumption+*
      done
    with *Node.prems* **show** *?thesis*
      by *auto*
  next
    assume *xnotinlt*: *x ∉ set-of lt*
    with *xnp Node.prems* **have** *xinrt*: *x ∈ set-of rt*
      by *simp*
    from *Node.prems* **have** *Dag* (*high p*) *low high rt*
      by *fastforce*

```
      with Node.prems xinrt have low x ∈ set-of rt
        apply −
        apply (rule Node.hyps)
        apply assumption+
        done
      with Node.prems show ?thesis
        by auto
    qed
  qed
qed

lemma subelem-set-of-high:
⋀ p. ⟦ x ∈ set-of t; x ≠ Null; high x ≠ Null; Dag p low high t ⟧
 ⟹ (high x) ∈ set-of t
proof (induct t)
  case Tip
  then show ?case by simp
next
  case (Node lt po rt)
  note tNode=this
  then have ppo: p=po by simp
  show ?case
  proof (cases x=p)
    case True
    with Node.prems have lxrootlt: high x = root rt
    proof (cases rt)
      case Tip
      with True Node.prems show ?thesis
        by auto
    next
      case (Node lrt l rrt)
      with Node.prems True show ?thesis
        by auto
    qed
    with True Node.prems have high x ∈ set-of (Node lt p rt)
    proof (cases rt)
      case Tip
      with lxrootlt Node.prems show ?thesis
        by simp
    next
      case (Node lrt l rrt)
      with lxrootlt Node.prems show ?thesis
        by simp
    qed
    with ppo show ?thesis
      by simp
  next
    assume xnp: x ≠ p
    with Node.prems have x ∈ set-of lt ∨ x ∈ set-of rt
```

```
    by simp
  show ?thesis
  proof (cases x ∈ set-of lt)
    case True
    note xinlt=this
    from Node.prems have Dag (low p) low high lt
      by fastforce
    with Node.prems True have high x ∈ set-of lt
      apply −
      apply (rule Node.hyps)
      apply assumption+
      done
    with Node.prems show ?thesis
      by auto
  next
    assume xnotinlt: x ∉ set-of lt
    with xnp Node.prems have xinrt: x ∈ set-of rt
      by simp
    from Node.prems have Dag (high p) low high rt
      by fastforce
    with Node.prems xinrt have high x ∈ set-of rt
      apply −
      apply (rule Node.hyps)
      apply assumption+
      done
    with Node.prems show ?thesis
      by auto
  qed
 qed
qed

lemma set-split: {k. k<(Suc n)} = {k. k<n} ∪ {n}
apply auto
done


lemma Nodes-levellist-subset-t:
⟦wf-ll t levellist var; i<= length levellist⟧ ⟹ Nodes i levellist ⊆ set-of t
proof (induct i)
 case 0
 show ?case by (simp add: Nodes-def)
next
 case (Suc n)
 from Suc.prems Suc.hyps have Nodesn-in-t: Nodes n levellist ⊆ set-of t
   by simp
 from Suc.prems have ∀ x ∈ set (levellist ! n). x ∈ set-of t
   apply −
   apply (rule ballI)
   apply (simp add: wf-ll-def)
```

```
    apply (erule conjE)
    apply (thin-tac ∀ q. q ∈ set-of t ⟶ q ∈ set (levellist ! var q))
    apply (erule-tac x=n in allE)
    apply (erule impE)
    apply simp
    apply fastforce
    done
  with Suc.prems have set (levellist ! n) ⊆ set-of t
    apply blast
    done
  with Suc.prems Nodesn-in-t show ?case
    apply (simp add: Nodes-def)
    apply (simp add: set-split)
    done
qed

lemma bdt-child:
⟦ bdt (Node (Node llt l rlt) p (Node lrt r rrt)) var = Some bdt1⟧
 ⟹ ∃ lbdt rbdt.  bdt (Node llt l rlt) var = Some lbdt ∧
                 bdt (Node lrt r rrt) var = Some rbdt
  by (simp split: option.splits)


lemma subbdt-ex-dag-def:
⋀ bdt1 p. ⟦Dag p low high t; bdt t var = Some bdt1; Dag no low high not;
no ∈ set-of t⟧ ⟹ ∃ bdt2.  bdt not var = Some bdt2
proof (induct t)
  case Tip
  then show ?case by simp
next
  case (Node lt po rt)
  note pNode=this
  with Node.prems have p-po: p=po by simp
  show ?case
  proof (cases no = po)
    case True
    note no-eq-po=this
     from p-po Node.prems no-eq-po have not = (Node lt po rt) by (simp add:
Dag-unique del: Dag-Ref)
    with Node.prems have bdt not var = Some bdt1 by (simp add: le-dag-def)
    then show ?thesis by simp
  next
    assume no ≠ po
    with Node.prems have no-in-lt-or-rt: no ∈ set-of lt ∨ no ∈ set-of rt by simp
    show ?thesis
    proof (cases no ∈ set-of lt)
      case True
      note no-in-lt=this
      from Node.prems p-po have lt-dag: Dag (low po) low high lt by simp
```

```
    from Node.prems have lbdt-def: ∃ lbdt. bdt lt var = Some lbdt
    proof (cases lt)
      case Tip
      with Node.prems no-in-lt show ?thesis by (simp add: le-dag-def)
    next
      case (Node llt l rlt)
      note lNode=this
      show ?thesis
      proof (cases rt)
        case Tip
        note rNode=this
        with lNode Node.prems show ?thesis by simp
      next
        case (Node lrt r rrt)
        note rNode=this
        with lNode Node.prems show ?thesis by (simp split: option.splits)
      qed
    qed
    then obtain lbdt where bdt lt var = Some lbdt..
    with Node.prems lt-dag no-in-lt show ?thesis
      apply −
      apply (rule Node.hyps)
      apply assumption+
      done
  next
    assume no ∉ set-of lt
    with no-in-lt-or-rt have no-in-rt: no ∈ set-of rt by simp
    from Node.prems p-po have rt-dag: Dag (high po) low high rt by simp
    from Node.hyps have hyp2: ⋀ rbdt. ⟦Dag (high po) low high rt; bdt rt var =
Some rbdt; Dag no low high not; no ∈ set-of rt⟧ ⟹ ∃ bdt2. bdt not var = Some
bdt2
      by simp
    from Node.prems have lbdt-def: ∃ rbdt. bdt rt var = Some rbdt
    proof (cases rt)
      case Tip
      with Node.prems no-in-rt show ?thesis by (simp add: le-dag-def)
    next
      case (Node lrt l rrt)
      note rNode=this
      show ?thesis
      proof (cases lt)
        case Tip
        note lTip=this
        with rNode Node.prems show ?thesis by simp
      next
        case (Node llt r rlt)
        note lNode=this
        with rNode Node.prems show ?thesis by (simp split: option.splits)
      qed
```

```
      qed
      then obtain rbdt where bdt rt var = Some rbdt..
      with Node.prems rt-dag no-in-rt show ?thesis
        apply −
        apply (rule hyp2)
        apply assumption+
        done
    qed
  qed
qed

lemma subbdt-ex:
⋀ bdt1. ⟦ (Node lst stp rst) <= t; bdt t var = Some bdt1⟧
 ⟹ ∃ bdt2.  bdt (Node lst stp rst) var = Some bdt2
proof (induct t)
  case Tip
  then show ?case by (simp add: le-dag-def)
next
  case (Node lt p rt)
  note pNode=this
  show ?case
  proof (cases Node lst stp rst = Node lt p rt)
    case True
    with Node.prems show ?thesis by simp
  next
    assume  Node lst stp rst ≠ Node lt p rt
     with Node.prems have Node lst stp rst < Node lt p rt apply (simp add:
le-dag-def) apply auto done
    then have in-ltrt: Node lst stp rst <= lt ∨ Node lst stp rst <= rt
      by (simp add: less-dag-Node)
    show ?thesis
    proof (cases Node lst stp rst <= lt)
      case True
      note in-lt=this
      from Node.prems have lbdt-def: ∃ lbdt. bdt lt var = Some lbdt
      proof (cases lt)
        case Tip
        with Node.prems in-lt show ?thesis by (simp add: le-dag-def)
      next
        case (Node llt l rlt)
        note lNode=this
        show ?thesis
        proof (cases rt)
          case Tip
          note rNode=this
          with lNode Node.prems show ?thesis by simp
        next
          case (Node lrt r rrt)
          note rNode=this
```

```
        with lNode Node.prems show ?thesis by (simp split: option.splits)
       qed
     qed
     then obtain lbdt where bdt lt var = Some lbdt..
     with Node.prems in-lt show ?thesis
      apply −
      apply (rule Node.hyps)
      apply assumption+
      done
   next
     assume ¬ Node lst stp rst ≤ lt
     with in-ltrt have in-rt: Node lst stp rst <= rt by simp
      from Node.hyps have hyp2: ⋀ rbdt. ⟦Node lst stp rst <= rt; bdt rt var =
Some rbdt⟧ ⟹ ∃ bdt2. bdt (Node lst stp rst) var = Some bdt2
       by simp
     from Node.prems have rbdt-def: ∃ rbdt. bdt rt var = Some rbdt
     proof (cases rt)
       case Tip
       with Node.prems in-rt show ?thesis by (simp add: le-dag-def)
     next
       case (Node lrt l rrt)
       note rNode=this
       show ?thesis
       proof (cases lt)
         case Tip
         note lNode=this
         with rNode Node.prems show ?thesis by simp
       next
         case (Node lrt r rrt)
         note lNode=this
         with rNode Node.prems show ?thesis by (simp split: option.splits)
       qed
     qed
     then obtain rbdt where bdt rt var = Some rbdt..
     with Node.prems in-rt show ?thesis
      apply −
      apply (rule hyp2)
      apply assumption+
      done
   qed
  qed
qed


lemma var-ordered-children:
⋀ p. ⟦ Dag p low high t; ordered t var; no ∈ set-of t;
       low no ≠ Null; high no ≠ Null⟧
      ⟹ var (low no) < var no ∧ var (high no) < var no
proof (induct t)
```

```
  case Tip
  then show ?case by simp
next
  case (Node lt po rt)
  then have ppo: p=po by simp
  show ?case
  proof (cases no = po)
    case True
    note no-po=this
    from Node.prems have var (low po) < var po ∧ var (high po) < var po
    proof (cases lt)
      case Tip
      note ltTip=this
      with Node.prems no-po ppo show ?thesis by simp
    next
      case (Node llt l rlt)
      note lNode=this
      show ?thesis
      proof (cases rt)
        case Tip
        note rTip=this
        with Node.prems no-po ppo show ?thesis by simp
      next
        case (Node lrt r rrt)
        note rNode=this
        with Node.prems ppo no-po lNode  show ?thesis by (simp del: Dag-Ref)
      qed
    qed
    with no-po show ?thesis by simp
  next
    assume  no ≠ po
    with Node.prems have no-in-ltrt: no ∈ set-of lt ∨ no ∈ set-of rt
      by simp
    show ?thesis
    proof (cases no ∈ set-of lt)
      case True
      note no-in-lt=this
      from Node.prems ppo have lt-dag: Dag (low po) low high lt
        by simp
      from Node.prems have ord-lt: ordered lt var
        apply −
        apply (drule children-ordered)
        apply simp
        done
      from no-in-lt lt-dag ord-lt Node.prems show ?thesis
        apply −
        apply (rule Node.hyps)
        apply assumption+
        done
```

```
    next
      assume  no ∉ set-of lt
      with no-in-ltrt have no-in-rt: no ∈ set-of rt by simp
      from Node.prems ppo have rt-dag: Dag (high po) low high rt by simp
      from Node.hyps have hyp2:  ⟦Dag (high po) low high rt; ordered rt var; no ∈
set-of rt; low no ≠ Null; high no ≠ Null⟧
                                ⟹ var (low no) < var no ∧ var (high no) < var no
        by simp
      from Node.prems have ord-rt: ordered rt var
        apply −
        apply (drule children-ordered)
        apply simp
        done
      from rt-dag ord-rt no-in-rt Node.prems show ?thesis
        apply −
        apply (rule hyp2)
        apply assumption+
        done
    qed
  qed
qed

lemma nort-null-comp:
assumes pret-dag: Dag p low high pret and
        prebdt-pret: bdt pret var = Some prebdt and
        nort-dag: Dag (repc no) (repb ∝ low) (repb ∝ high) nort and
        ord-pret: ordered pret var and
        wf-llb: wf-ll pret levellistb var and
        nbsll: nb < length levellistb and
        repbc-nc: ∀ nt. nt ∉ set (levellistb ! nb) ⟶ repb nt = repc nt and
        xsnb-in-pret: ∀ x ∈ set-of nort. var x < nb ∧ x ∈ set-of pret
shows ∀ x ∈ set-of nort. ((repc ∝ low) x = (repb ∝ low) x ∧
                          (repc ∝ high) x = (repb ∝ high) x)
proof (rule ballI)
  fix x
  assume x-in-nort: x ∈ set-of nort
  with nort-dag have xnN: x ≠ Null
    apply −
    apply (rule set-of-nn [rule-format])
    apply auto
    done
  from x-in-nort xsnb-in-pret have xsnb: var x <nb
    by simp
  from x-in-nort xsnb-in-pret have x-in-pret: x ∈ set-of pret
    by blast
  show  (repc ∝ low) x = (repb ∝ low) x ∧ (repc ∝ high) x = (repb ∝ high) x
  proof (cases (low x) ≠ Null)
    case True
    with pret-dag prebdt-pret x-in-pret have highnN: (high x) ≠ Null
```

apply −
apply (*drule balanced-bdt*)
apply *assumption*+
apply *simp*
done
from *x-in-pret ord-pret highnN True* **have** *children-var-smaller*: *var* (*low x*) < *var x* ∧ *var* (*high x*) < *var x*
apply −
apply (*rule var-ordered-children*)
apply (*rule pret-dag*)
apply (*rule ord-pret*)
apply (*rule x-in-pret*)
apply (*rule True*)
apply (*rule highnN*)
done
with *xsnb* **have** *lowxsnb*: *var* (*low x*) < *nb*
by *arith*
from *children-var-smaller xsnb* **have** *highxsnb*: *var* (*high x*) < *nb*
by *arith*
from *x-in-pret xnN True pret-dag* **have** *lowxinpret*: (*low x*) ∈ *set-of pret*
apply −
apply (*drule subelem-set-of-low*)
apply *assumption*
apply (*thin-tac x* ≠ *Null*)
apply *assumption*+
done
with *wf-llb* **have** *low x* ∈ *set* (*levellistb* ! (*var* (*low x*)))
by (*simp add*: *wf-ll-def*)
with *wf-llb nbsll lowxsnb* **have** *low x* ∉ *set* (*levellistb* ! *nb*)
apply −
apply (*rule-tac ?i*=(*var* (*low x*)) **and** *?j*=*nb* **in** *no-in-one-ll*)
apply *auto*
done
with *repbc-nc* **have** *repclow*: *repc* (*low x*) = *repb* (*low x*)
by *auto*
from *x-in-pret xnN highnN pret-dag* **have** *highxinpret*: (*high x*) ∈ *set-of pret*
apply −
apply (*drule subelem-set-of-high*)
apply *assumption*
apply (*thin-tac x* ≠ *Null*)
apply *assumption*+
done
with *wf-llb* **have** *high x* ∈ *set* (*levellistb* ! (*var* (*high x*)))
by (*simp add*: *wf-ll-def*)
with *wf-llb nbsll highxsnb* **have** *high x* ∉ *set* (*levellistb* ! *nb*)
apply −
apply (*rule-tac ?i*=(*var* (*high x*)) **and** *?j*=*nb* **in** *no-in-one-ll*)
apply *auto*
done

```
    with repbc-nc have repchigh: repc (high x) = repb (high x)
      by auto
    with repclow show ?thesis
      by (simp add: null-comp-def)
  next
    assume  ¬ low x ≠ Null
    then have lowxNull: low x = Null by simp
    with pret-dag x-in-pret prebdt-pret have highxNull: high x =Null
      apply −
      apply (drule balanced-bdt)
      apply simp
      apply simp
      apply simp
      done
    from lowxNull have repclowNull: (repc ∝ low) x = Null
      by (simp add: null-comp-def)
    from lowxNull have repblowNull: (repb ∝ low) x = Null
      by (simp add: null-comp-def)
    with repclowNull have lowxrepbc: (repc ∝ low) x = (repb ∝ low) x
      by simp
    from highxNull have repchighNull: (repc ∝ high) x = Null
      by (simp add: null-comp-def)
    from highxNull have (repb ∝ high) x = Null
      by (simp add: null-comp-def)
    with repchighNull have highxrepbc: (repc ∝ high) x = (repb ∝ high) x
      by simp
    with lowxrepbc show ?thesis
      by simp
  qed
qed

lemma wf-ll-Nodes-pret:
⟦wf-ll pret levellista var; nb < length levellista; x ∈ Nodes nb levellista⟧
 ⟹ x ∈ set-of pret ∧ var x < nb
  apply (simp add: wf-ll-def Nodes-def)
  apply (erule conjE)
  apply (thin-tac  ∀ q. q ∈ set-of pret ⟶ q ∈ set (levellista ! var q))
  apply (erule exE)
  apply (elim conjE)
  apply (erule-tac x=xa in allE)
  apply (erule impE)
  apply arith
  apply (erule-tac x=x in ballE)
  apply auto
  done

lemma bdt-Some-var1-One:
⋀ x. ⟦ bdt t var = Some x; var (root t) = 1⟧
 ⟹ x = One ∧ t = (Node Tip (root t) Tip)
```

```
proof (induct t)
  case Tip
  then show  ?case by simp
next
  case (Node lt p rt)
  note tNode = this
  show ?case
  proof (cases lt)
    case Tip
    note ltTip=this
    show ?thesis
    proof (cases rt)
      case Tip
      note rtTip = this
      with ltTip Node.prems show ?thesis by auto
    next
      case (Node lrt r rrt)
      note rtNode=this
      with Node.prems ltTip show ?thesis by auto
    qed
  next
    case (Node llt l rlt)
    note ltNode=this
    show ?thesis
    proof (cases rt)
      case Tip
      with ltNode Node.prems show ?thesis by auto
    next
      case (Node lrt r rrt)
      note rtNode=this
      with ltNode Node.prems show ?thesis by auto
    qed
  qed
qed

lemma bdt-Some-var0-Zero:
⋀ x. ⟦ bdt t var = Some x; var (root t) = 0⟧
⟹ x = Zero ∧ t = (Node Tip (root t) Tip)
proof (induct t)
  case Tip
  then show  ?case by simp
next
  case (Node lt p rt)
  note tNode = this
  show ?case
  proof (cases lt)
    case Tip
    note ltTip=this
    show ?thesis
```

```
    proof (cases rt)
      case Tip
      note rtTip = this
      with ltTip Node.prems show ?thesis by auto
    next
      case (Node lrt r rrt)
      note rtNode=this
      with Node.prems ltTip show ?thesis by auto
    qed
  next
    case (Node llt l rlt)
    note ltNode=this
    show ?thesis
    proof (cases rt)
      case Tip
      with ltNode Node.prems show ?thesis by auto
    next
      case (Node lrt r rrt)
      note rtNode=this
      with ltNode Node.prems show ?thesis by auto
    qed
  qed
qed

lemma reduced-children-parent:
⟦ reduced l; l= (Node llt lp rlt); reduced r; r=(Node lrt rp rrt);
  lp ≠ rp ⟧
 ⟹ reduced (Node l p r)
  by simp


lemma Nodes-subset: Nodes i levellista ⊆ Nodes (Suc i) levellista
  apply (simp add: Nodes-def)
  apply (simp add: set-split)
  done

lemma Nodes-levellist:
⟦ wf-ll pret levellista var; nb < length levellista; p ∈ Nodes nb levellista⟧
 ⟹ p ∉ set (levellista ! nb)
  apply (simp add: Nodes-def)
  apply (erule exE)
  apply (rule-tac i=x and j=nb in no-in-one-ll)
  apply auto
  done

lemma Nodes-var-pret:
 ⟦wf-ll pret levellista var; nb < length levellista; p ∈ Nodes nb levellista⟧
  ⟹ var p < nb ∧ p ∈ set-of pret
  apply (simp add: Nodes-def wf-ll-def)
```

```
  apply (erule conjE)
  apply (thin-tac ∀ q. q ∈ set-of pret ⟶ q ∈ set (levellista ! var q))
  apply (erule exE)
  apply (erule-tac x=x in allE)
  apply (erule impE)
  apply arith
  apply (erule-tac x=p in ballE)
  apply arith
  apply simp
  done

lemma Dags-root-in-Nodes:
assumes t-in-DagsSucnb: t ∈ Dags (Nodes (Suc nb) levellista) low high
shows ∃ p . Dag p low high t ∧ p ∈ Nodes (Suc nb) levellista
proof −
 from t-in-DagsSucnb obtain p where t-dag: Dag p low high t and t-subset-Nodes:
set-of t ⊆ Nodes (Suc nb) levellista and t-nTip: t≠ Tip
    by (fastforce elim: Dags.cases)
  from t-dag t-nTip have p≠Null by (cases t) auto
  with t-subset-Nodes t-dag have p ∈ Nodes (Suc nb) levellista
    by (cases t) auto
  with t-dag show ?thesis
    by auto
qed




lemma subdag-dag:
⋀ p. ⟦Dag p low high t; st <= t⟧ ⟹ ∃ stp. Dag stp low high st
proof (induct t)
  case Tip
  then show ?case
    by (simp add: less-dag-def le-dag-def)
next
  case (Node lt po rt)
  note t-Node=this
  with Node.prems have p-po: p=po
    by simp
  show ?case
  proof (cases st = Node lt po rt)
    case True
    note st-t=this
    with Node.prems show ?thesis
      by auto
  next
    assume st-nt: st ≠ Node lt po rt
    with Node.prems p-po have st-subdag-lt-rt: st<=lt ∨ st <=rt
      by (auto simp add:le-dag-def less-dag-def)
```

```
    from Node.prems p-po obtain lp rp where lt-dag: Dag lp low high lt and
rt-dag: Dag rp low high rt
     by auto
   show ?thesis
   proof (cases st<=lt)
     case True
     note st-lt=this
     with lt-dag show ?thesis
       apply−
       apply (rule Node.hyps)
       apply auto
       done
   next
     assume ¬ st ≤ lt
     with st-subdag-lt-rt have st-rt: st <= rt
       by simp
     from Node.hyps have rhyp: ⟦Dag rp low high rt; st ≤ rt⟧ ⟹ ∃ stp. Dag stp
low high st
       by simp
     from st-rt rt-dag show ?thesis
       apply −
       apply (rule rhyp)
       apply auto
       done
   qed
 qed
qed

lemma Dags-subdags:
assumes t-in-Dags: t ∈ Dags nodes low high and
  st-t: st <= t and
  st-nTip: st ≠ Tip
shows st ∈ Dags nodes low high
proof −
  from t-in-Dags obtain p where t-dag: Dag p low high t and t-subset-Nodes:
set-of t ⊆ nodes and t-nTip: t≠ Tip
    by (fastforce elim: Dags.cases)
  from st-t have set-of st ⊆ set-of t
    by (simp add: le-dag-set-of)
  with t-subset-Nodes have st-subset-fnctNodes: set-of st ⊆ nodes
    by blast
  from st-t t-dag obtain stp where Dag stp low high st
    apply −
    apply (drule subdag-dag)
    apply auto
    done
  with st-subset-fnctNodes st-nTip show ?thesis
    apply −
    apply (rule DagsI)
```

apply *auto*
done
qed

lemma *Dags-Nodes-cases*:
assumes *P-sym*: $\bigwedge$ *t1 t2. P t1 t2 var = P t2 t1 var* **and**
dags-in-lower-levels:
$\bigwedge$ *t1 t2.* $[\![$*t1* $\in$ *Dags* (*fnct* '(*Nodes n levellista*)) *low high*;
t2 $\in$ *Dags* (*fnct* '(*Nodes n levellista*)) *low high*$]\!]$
$\Longrightarrow$ *P t1 t2 var* **and**
dags-in-mixed-levels:
$\bigwedge$ *t1 t2.* $[\![$*t1* $\in$ *Dags* (*fnct* '(*Nodes n levellista*)) *low high*;
t2 $\in$ *Dags* (*fnct* '(*Nodes* (*Suc n*) *levellista*)) *low high*;
t2 $\notin$ *Dags* (*fnct* '(*Nodes n levellista*)) *low high*$]\!]$
$\Longrightarrow$ *P t1 t2 var* **and**
dags-in-high-level:
$\bigwedge$ *t1 t2.* $[\![$*t1* $\in$ *Dags* (*fnct* '(*Nodes* (*Suc n*) *levellista*)) *low high*;
t1 $\notin$ *Dags* (*fnct* '(*Nodes n levellista*)) *low high*;
t2 $\in$ *Dags* (*fnct* '(*Nodes* (*Suc n*) *levellista*)) *low high*;
t2 $\notin$ *Dags* (*fnct* '(*Nodes n levellista*)) *low high*$]\!]$
$\Longrightarrow$ *P t1 t2 var*
shows $\forall$ *t1 t2.* *t1* $\in$ *Dags* (*fnct* '(*Nodes* (*Suc n*) *levellista*)) *low high* $\wedge$
t2 $\in$ *Dags* (*fnct* '(*Nodes* (*Suc n*) *levellista*)) *low high*
$\longrightarrow$ *P t1 t2 var*
proof (*intro allI impI* , *elim conjE*)
fix *t1 t2*
assume *t1-in-higher-levels*: *t1* $\in$ *Dags* (*fnct* ' *Nodes* (*Suc n*) *levellista*) *low high*
assume *t2-in-higher-levels*: *t2* $\in$ *Dags* (*fnct* ' *Nodes* (*Suc n*) *levellista*) *low high*
show *P t1 t2 var*
proof (*cases t1* $\in$ *Dags* (*fnct* ' *Nodes n levellista*) *low high*)
case *True*
note *t1-in-ll* = *this*
show *?thesis*
proof (*cases t2* $\in$ *Dags* (*fnct* ' *Nodes n levellista*) *low high*)
case *True*
note *t2-in-ll=this*
with *t1-in-ll dags-in-lower-levels* **show** *?thesis*
by *simp*
next
assume *t2-notin-ll*: *t2* $\notin$ *Dags* (*fnct* ' *Nodes n levellista*) *low high*
with *t1-in-ll t2-in-higher-levels dags-in-mixed-levels* **show** *?thesis*
by *simp*
qed
next
assume *t1-notin-ll*: *t1* $\notin$ *Dags* (*fnct* ' *Nodes n levellista*) *low high*
show *?thesis*
proof (*cases t2* $\in$ *Dags* (*fnct* ' *Nodes n levellista*) *low high*)
case *True*

note *t2-in-ll=this*
with *dags-in-mixed-levels t1-in-higher-levels t1-notin-ll P-sym* **show** *?thesis*
by *auto*
next
assume *t2-notin-ll*: *t2* ∉ *Dags* (*fnct* ‘ *Nodes n levellista*) *low high*
with *t1-notin-ll t1-in-higher-levels t2-in-higher-levels dags-in-high-level* **show** *?thesis*
by *simp*
qed
qed
qed

lemma *Null-notin-Nodes*: ⟦*Dag p low high t*; *nb* <= *length levellista*; *wf-ll t levellista var*⟧ ⟹ *Null* ∉ *Nodes nb levellista*
apply (*simp add*: *Nodes-def wf-ll-def del*: *Dag-Ref*)
apply (*rule allI*)
apply (*rule impI*)
apply (*elim conjE*)
apply (*thin-tac* ∀ *q*. *P q* **for** *P*)
apply (*erule-tac x*=*x* **in** *allE*)
apply (*erule impE*)
apply *simp*
apply (*erule-tac x*=*Null* **in** *ballE*)
apply (*erule conjE*)
apply (*drule set-of-nn* [*rule-format*])
apply *auto*
done

lemma *Nodes-in-pret*: ⟦*wf-ll t levellista var*; *nb* <= *length levellista*⟧ ⟹ *Nodes nb levellista* ⊆ *set-of t*
apply −
apply *rule*
apply (*simp add*: *wf-ll-def Nodes-def*)
apply (*erule exE*)
apply (*elim conjE*)
apply (*thin-tac* ∀ *q*. *q* ∈ *set-of t* ⟶ *q* ∈ *set* (*levellista* ! *var q*))
apply (*erule-tac x*=*xa* **in** *allE*)
apply (*erule impE*)
apply *simp*
apply (*erule-tac x*=*x* **in** *ballE*)
apply *auto*
done

lemma *restrict-root-Node*:
⟦*t* ∈ *Dags* (*repc* ‘*Nodes* (*Suc nb*) *levellista*) (*repc* ∝ *low*) (*repc* ∝ *high*); *t* ∉ *Dags* (*repc* ‘*Nodes nb levellista*) (*repc* ∝ *low*) (*repc* ∝ *high*);

ordered t var; ∀ *no* ∈ *Nodes* (*Suc nb*) *levellista*. *var* (*repc no*) <= *var no* ∧ *repc* (*repc no*) = *repc no*; *wf-ll pret levellista var*; *nb* < *length levellista*;*repc* ‘*Nodes* (*Suc nb*) *levellista* ⊆ *Nodes* (*Suc nb*) *levellista*⟧
⟹ ∃ *q*. *Dag* (*repc q*) (*repc* ∝ *low*) (*repc* ∝ *high*) *t* ∧ *q* ∈ *set* (*levellista* ! *nb*)
proof (*elim Dags.cases*)
fix *p* **and** *ta* :: *dag*
assume *t-notin-DagsNodesnb*: *t* ∉ *Dags* (*repc* ‘ *Nodes nb levellista*) (*repc* ∝ *low*) (*repc* ∝ *high*)
assume *t-ta*: *t* = *ta*
assume *ta-in-repc-NodesSucnb*: *set-of ta* ⊆ *repc* ‘ *Nodes* (*Suc nb*) *levellista*
assume *ta-dag*: *Dag p* (*repc* ∝ *low*) (*repc* ∝ *high*) *ta*
assume *ta-nTip*: *ta* ≠ *Tip*
assume *ord-t*: *ordered t var*
assume *varrep-prop*: ∀ *no* ∈ *Nodes* (*Suc nb*) *levellista*. *var* (*repc no*) <= *var no* ∧ *repc* (*repc no*) = *repc no*
assume *wf-lla*: *wf-ll pret levellista var*
assume *nbslla*: *nb* < *length levellista*
assume *repcNodes-in-Nodes*: *repc* ‘*Nodes* (*Suc nb*) *levellista* ⊆ *Nodes* (*Suc nb*) *levellista*
from *ta-nTip ta-dag* **have** *p-nNull*: *p*≠ *Null*
by *auto*
with *ta-nTip ta-dag* **obtain** *lt rt* **where** *ta-Node*: *ta* = *Node lt p rt*
by *auto*
with *ta-nTip ta-dag* **have** *p-in-ta*: *p* ∈ *set-of ta*
by *auto*
with *ta-in-repc-NodesSucnb* **have** *p-in-repcNodes-Sucnb*: *p* ∈ *repc* ‘*Nodes* (*Suc nb*) *levellista*
by *auto*
show *?thesis*
proof (*cases p* ∈ *repc* ‘(*set* (*levellista* ! *nb*)))
case *True*
then obtain *q* **where**
p-repca: *p*=*repc q* **and**
a-in-llanb: *q* ∈ *set* (*levellista* ! *nb*)
by *auto*
with *ta-dag t-ta* **show** *?thesis*
apply −
apply (*rule-tac x*=*q* **in** *exI*)
apply *simp*
done
next
assume *p-notin-repc-llanb*: *p* ∉ *repc* ‘ *set* (*levellista* ! *nb*)
with *p-in-repcNodes-Sucnb* **have** *p-in-repc-Nodesnb*: *p* ∈ *repc* ‘*Nodes nb levellista*
apply −
apply (*erule imageE*)
apply *rule*
apply (*simp add*: *Nodes-def*)
apply (*simp add*: *Nodes-def*)

```
      apply (erule exE conjE)
      apply (case-tac xa=nb)
      apply simp
      apply (rule-tac x=xa in exI)
      apply auto
      done
    have t ∈ Dags (repc `Nodes nb levellista) (repc ∝ low) (repc ∝ high)
    proof −
      have set-of t ⊆ repc `Nodes nb levellista
      proof (rule)
        fix x :: ref
        assume x-in-t: x ∈ set-of t
        with ord-t have var x <= var (root t)
          apply −
          apply (rule ordered-set-of)
          apply auto
          done
        with t-ta ta-Node have varx-varp: var x <= var p
          by auto
       from p-in-repc-Nodesnb obtain k where ksnb: k < nb and p-in-repc-llak:
p ∈ repc `(set (levellista ! k))
          by (auto simp add: Nodes-def ImageE)
        then obtain q where p-repcq: p=repc q and q-in-llak: q ∈ set (levellista
! k)
          by auto
        from q-in-llak wf-lla nbslla ksnb have varqk: var q = k
          by (simp add: wf-ll-def)
      have Nodesnb-in-NodesSucnb: Nodes nb levellista ⊆ Nodes (Suc nb) levellista
          by (rule Nodes-subset)
        from q-in-llak ksnb have q ∈ Nodes nb levellista
          by (auto simp add: Nodes-def)
        with varrep-prop Nodesnb-in-NodesSucnb have var (repc q) <= var q
          by auto
        with varqk ksnb p-repcq have var p < nb
          by auto
        with varx-varp have varx-snb: var x < nb
          by auto
        from x-in-t t-ta ta-in-repc-NodesSucnb obtain a where
          x-repca: x= repc a and
          a-in-NodesSucnb: a ∈ Nodes (Suc nb) levellista
          by auto
        with varrep-prop have rx-x: repc x = x
          by auto
        have x ∈ set-of pret
        proof −
          from wf-lla nbslla have Nodes (Suc nb) levellista ⊆ set-of pret
            apply −
            apply (rule Nodes-in-pret)
            apply auto
```

```
          done
        with x-in-t t-ta ta-in-repc-NodesSucnb repcNodes-in-Nodes show ?thesis
          by auto
      qed
      with wf-lla have x ∈ set (levellista ! (var x))
        by (auto simp add: wf-ll-def)
      with varx-snb have x ∈ Nodes nb levellista
        by (auto simp add: Nodes-def)
      with rx-x show x ∈ repc ‘Nodes nb levellista
        apply −
        apply rule
        apply (subgoal-tac x=repc x)
        apply auto
        done
    qed
    with ta-nTip ta-dag t-ta show ?thesis
      apply −
      apply (rule DagsI)
      apply auto
      done
  qed
  with t-notin-DagsNodesnb show ?thesis
    by auto
 qed
qed
```

```
lemma same-bdt-var: ⟦bdt (Node lt1 p1 rt1) var = Some bdt1; bdt (Node lt2 p2
rt2) var = Some bdt1⟧
  ⟹ var p1 = var p2
proof (induct bdt1)
 case Zero
 then obtain var-p1: var p1 = 0 and var-p2: var p2 = 0
   by simp
 then show ?case
   by simp
next
 case One
 then obtain var-p1: var p1 = 1 and var-p2: var p2 = 1
   by simp
 then show ?case
   by simp
next
 case (Bdt-Node lbdt v rbdt)
 then obtain var-p1: var p1 = v and var-p2: var p2 = v
   by simp
```

```
  then show ?case by simp
qed

lemma bdt-Some-Leaf-var-le-1:
⟦Dag p low high t; bdt t var = Some x; isLeaf-pt p low high⟧
  ⟹ var p <= 1
proof (induct t)
  case Tip
  thus ?case by simp
next
  case (Node lt p rt)
  note tNode=this
  from Node.prems tNode show ?case
    apply (simp add: isLeaf-pt-def)
    apply (case-tac var p = 0)
    apply simp
    apply (case-tac var p = Suc 0)
    apply simp
    apply simp
    done
qed

lemma subnode-dag-cons:
⋀ p. ⟦Dag p low high t; no ∈ set-of t⟧ ⟹ ∃ not. Dag no low high not
proof (induct t)
  case Tip
  thus ?case by simp
next
  case (Node lt q rt)
  with Node.prems have q-p: p = q
    by simp
  from Node.prems have lt-dag: Dag (low p) low high lt
    by auto
  from Node.prems have rt-dag: Dag (high p) low high rt
    by auto
  show ?case
  proof (cases no ∈ set-of lt)
    case True
    with Node.hyps lt-dag show ?thesis
      by simp
  next
    assume no-notin-lt: no ∉ set-of lt
    show ?thesis
    proof (cases no=p)
      case True
      with Node.prems q-p show ?thesis
        by auto
    next
      assume no-neq-p: no ≠ p
```

```
    with Node.prems no-notin-lt have no-in-rt: no ∈ set-of rt
      by simp
    with rt-dag Node.hyps show ?thesis
      by auto
  qed
 qed
qed
```

```
lemma nodes-in-taken-in-takeSucn: no ∈ set (take n nodeslist) ⟹ no ∈ set (take
(Suc n) nodeslist)
proof −
 assume no-in-taken: no ∈ set (take n nodeslist)
 have set (take n nodeslist) ⊆ set (take (Suc n) nodeslist)
  apply −
  apply (rule set-take-subset-set-take)
  apply simp
  done
 with no-in-taken show ?thesis
  by blast
qed
```

```
lemma ind-in-higher-take: ⋀n k. ⟦n < k;  n < length xs⟧
 ⟹ xs ! n ∈ set (take k xs)
apply (induct xs)
apply simp
apply simp
apply (case-tac n)
apply simp
apply (case-tac k)
apply simp
apply simp
apply simp
apply (case-tac k)
apply simp
apply simp
done

lemma take-length-set: ⋀n. n=length xs ⟹ set (take n xs) = set xs
```

apply (*induct xs*)
apply (*auto simp add*: *take-Cons split*: *nat.splits*)
done

lemma *repNodes-eq-ext-rep*: ⟦*low no* ≠ *nodeslist*! *n*; *high no* ≠ *nodeslist* ! *n*;
low sn ≠ *nodeslist* ! *n*; *high sn* ≠ *nodeslist* ! *n*⟧
⟹ *repNodes-eq sn no low high repa* = *repNodes-eq sn no low high* (*repa*(*nodeslist*
! *n* := *repa* (*low* (*nodeslist* ! *n*))))
by (*simp add*: *repNodes-eq-def null-comp-def*)

lemma *filter-not-empty*: ⟦*x* ∈ *set xs*; *P x*⟧ ⟹ *filter P xs* ≠ []
by (*induct xs*) *auto*

lemma *x* ∈ *set* (*filter P xs*) ⟹ *P x*
by *auto*

lemma *hd-filter-in-list*: *filter P xs* ≠ [] ⟹ *hd* (*filter P xs*) ∈ *set xs*
by (*induct xs*) *auto*

lemma *hd-filter-in-filter*: *filter P xs* ≠ [] ⟹ *hd* (*filter P xs*) ∈ *set* (*filter P xs*)
by (*induct xs*) *auto*

lemma *hd-filter-prop*:
assumes *non-empty*: *filter P xs* ≠ []
shows *P* (*hd* (*filter P xs*))
proof −
from *non-empty* **have** *hd* (*filter P xs*) ∈ *set* (*filter P xs*)
by (*rule hd-filter-in-filter*)
thus *?thesis*
by *auto*
qed

lemma *index-elem*: *x* ∈ *set xs* ⟹ ∃ *i*<*length xs*. *x* = *xs* ! *i*
apply (*induct xs*)
apply *simp*
apply (*case-tac x*=*a*)
apply *auto*
done

lemma *filter-hd-P-rep-indep*:
⟦∀ *x*. *P x x*; ∀ *a b*. *P x a* ⟶ *P a b* ⟶ *P x b*; *filter* (*P x*) *xs* ≠ []⟧ ⟹
hd (*filter* (*P* (*hd* (*filter* (*P x*) *xs*))) *xs*) = *hd* (*filter* (*P x*) *xs*)
apply (*induct xs*)
apply *simp*
apply (*case-tac P x a*)
using [[*simp-depth-limit*=*2*]]
apply (*simp*)

apply *clarsimp*
apply (*fastforce dest*: *hd-filter-prop*)
done

lemma *take-Suc-not-last*:
$\bigwedge n.\ [\![x \in set\ (take\ (Suc\ n)\ xs);\ x \neq xs!n;\ n < length\ xs]\!] \Longrightarrow x \in set\ (take\ n\ xs)$
apply (*induct xs*)
apply *simp*
apply (*case-tac n*)
apply *simp*
using [[*simp-depth-limit*=*2*]]
apply *fastforce*
done

lemma *P-eq-list-filter*: $\forall x \in set\ xs.\ P\ x = Q\ x \Longrightarrow filter\ P\ xs = filter\ Q\ xs$
apply (*induct xs*)
apply *auto*
done

lemma *hd-filter-take-more*: $\bigwedge n\ m.[\![filter\ P\ (take\ n\ xs) \neq [];\ n \leq m]\!] \Longrightarrow$
$hd\ (filter\ P\ (take\ n\ xs)) = hd\ (filter\ P\ (take\ m\ xs))$
apply (*induct xs*)
apply *simp*
apply (*case-tac n*)
apply *simp*
apply (*case-tac m*)
apply *simp*
apply *clarsimp*
done

end

4 Definitions of Procedures

theory *ProcedureSpecs*
imports *General Simpl.Vcg*
begin

record *globals* =
var-' :: *ref* ⇒ *nat*
low-' :: *ref* ⇒ *ref*
high-' :: *ref* ⇒ *ref*
rep-' :: *ref* ⇒ *ref*
mark-' :: *ref* ⇒ *bool*
next-' ::*ref* ⇒ *ref*

record *'g bdd-state = 'g state +*
varval-' :: bool list
p-' :: ref
R-' :: bool
levellist-' :: ref list
nodeslist-' :: ref
node-':: ref
m-' :: bool
n-' :: nat

procedures
Eval (p, varval | R) =
IF (p→var = 0) THEN´R :==False
ELSE IF (p→var = 1) THEN´R :==True
ELSE IF (varval ! (p→var)) THEN CALL Eval (p→high,´varval,´R)
ELSE CALL Eval (p→low,´varval,´R)
FI
FI
FI

procedures
Levellist (p, m, levellist | levellist) =
IF (p ≠ Null)
THEN
IF (p →´mark ≠´m)
THEN
´levellist :== CALL Levellist (´p →´low,´m,´levellist);;
´levellist :== CALL Levellist (´p →´high,´m,´levellist);;
´p →´next :==´levellist ! (p→var);;
´levellist ! (p→var) :==´p;;
´p →´mark :==´m
FI
FI

procedures
ShareRep (nodeslist, p) =
IF (isLeaf-pt´p´low´high)
THEN´p →´rep :==´nodeslist
ELSE
WHILE (nodeslist ≠ Null) DO
IF (repNodes-eq´nodeslist´p´low´high´rep)

```
        THEN´p→´rep :==´nodeslist;;´nodeslist :== Null
        ELSE´nodeslist :==´nodeslist→´next
        FI
      OD
  FI
```

procedures

```
  ShareReduceRepList (nodeslist | ) =
´node :==´nodeslist;;
  WHILE (node ≠ Null) DO
    IF (¬ isLeaf-pt´node´low´high ∧
       ´node →´low →´rep =´node →´high →´rep )
    THEN´node →´rep :==´node →´low →´rep
    ELSE CALL ShareRep (nodeslist ,´node )
    FI;;
   ´node :=´node →´next
  OD
```

procedures

```
  Repoint (p|p) =
  IF (´p ≠ Null )
   THEN
   ´p :==´p →´rep;;
    IF (´p ≠ Null )
    THEN´p  →´low :== CALL Repoint (p  →´low);;
        ´p  →´high :== CALL Repoint (p →´high)
    FI
   FI
```

procedures

```
  Normalize (p|p) =
´levellist :==replicate (p→´var +1) Null;;
   ´levellist :== CALL Levellist (p, (¬´p→´mark) ,´levellist);;
   (n :==0;;
   WHILE (n < length´levellist) DO
     CALL ShareReduceRepList(levellist !´n);;
    ´n :=´n + 1
   OD);;
  ´p :== CALL Repoint (p)
```

end

5 Proof of Procedure Eval

theory *EvalProof* **imports** *ProcedureSpecs* **begin**

lemma (**in** *Eval-impl*) *Eval-modifies*:
shows $\forall\sigma.\ \Gamma\vdash\{\sigma\}$ *PROC Eval* (*p*,´*varval*,´*R*)
$\{t.\ t$ *may-not-modify-globals* $\sigma\}$
apply (*hoare-rule HoarePartial.ProcRec1*)
apply (*vcg spec*=*modifies*)
done

lemma (**in** *Eval-impl*) *Eval-spec*:
shows $\forall\sigma\ t$ *bdt1*. $\Gamma\vdash$
$\{\!|\sigma.$ *Dag´p´low´high* $t \wedge$ *bdt t´var* = *Some bdt1*$|\!\}$
´*R* :== *PROC Eval*(*p*,´*varval*)
$\{\!|R =$ *eval bdt1* $^{\sigma}$*varval* $|\!\}$
apply (*hoare-rule HoarePartial.ProcRec1*)
apply *vcg*
apply *clarsimp*
apply *safe*
apply (*case-tac bdt1*)
apply *simp*
apply *fastforce*
apply *fastforce*
apply *simp*
apply (*case-tac bdt1*)
apply *fastforce*
apply *fastforce*
apply *fastforce*
apply (*case-tac bdt1*)
apply *fastforce*
apply *fastforce*
apply *fastforce*
apply (*case-tac bdt1*)
apply *fastforce*
apply *fastforce*
apply *fastforce*
done

end

6 Proof of Procedure Levellist

theory *LevellistProof* **imports** *ProcedureSpecs Simpl.HeapList* **begin**

hide-const (**open**) *DistinctTreeProver.set-of tree.Node tree.Tip*

lemma (**in** *Levellist-impl*) *Levellist-modifies*:
shows $\forall\sigma.\ \Gamma\vdash\{\sigma\}$´*levellist* :== *PROC Levellist* (*p*,´*m*,´*levellist*)

```
        {t. t may-only-modify-globals σ in [mark,next]}
  apply (hoare-rule HoarePartial.ProcRec1)
  apply (vcg spec=modifies)
  done



lemma all-stop-cong: (∀ x. P x) = (∀ x. P x)
  by simp

lemma Dag-RefD:
  ⟦Dag p l r t; p≠Null⟧ ⟹
    ∃ lt rt. t=Node lt p rt ∧ Dag (l p) l r lt ∧ Dag (r p) l r rt
  by simp

lemma Dag-unique-ex-conjI:
  ⟦Dag p l r t;   P t⟧ ⟹ (∃ t. Dag p l r t ∧ P t)
  by simp


lemma dag-Null [simp]: dag Null l r = Tip
  by (simp add: dag-def)

definition first:: ref list ⇒ ref where
first ps = (case ps of [] ⇒ Null | (p#rs) ⇒ p)

lemma first-simps [simp]:
 first [] = Null
 first (r#rs) = r
by (simp-all add: first-def)

definition Levellist:: ref list ⇒ (ref ⇒ ref) ⇒ (ref list list) ⇒ bool where
Levellist hds next ll ⟷ (map first ll = hds) ∧
                        (∀ i < length hds. List (hds ! i) next (ll!i))

lemma Levellist-unique:
  assumes ll: Levellist hds next ll
  assumes ll': Levellist hds next ll'
  shows ll=ll'
proof −
  from ll have length ll = length hds
    by (clarsimp simp add: Levellist-def)
  moreover
  from ll' have length ll' = length hds
    by (clarsimp simp add: Levellist-def)
  ultimately have leq: length ll = length ll' by simp
  show ?thesis
  proof (rule nth-equalityI [OF leq, rule-format])
```

```
    fix i
    assume i < length ll
    with ll ll'
    show ll!i = ll'!i
      apply (clarsimp simp add: Levellist-def)
      apply (erule-tac x=i in allE)
      apply (erule-tac x=i in allE)
      apply simp
      by (erule List-unique)
  qed
qed

lemma Levellist-unique-ex-conj-simp [simp]:
Levellist hds next ll ⟹ (∃ ll. Levellist hds next ll ∧ P ll) = P ll
by (auto dest: Levellist-unique)


lemma in-set-concat-idx:
  x ∈ set (concat xss) ⟹ ∃ i < length xss. x ∈ set (xss!i)
apply (induct xss)
apply  simp
apply clarsimp
apply (erule disjE)
apply  (rule-tac x=0 in exI)
apply  simp
apply auto
done

definition wf-levellist :: dag ⇒ ref list list ⇒ ref list list ⇒
                    (ref ⇒ nat) ⇒ bool where
wf-levellist t levellist-old levellist-new var =
(case t of Tip ⇒ levellist-old = levellist-new
| (Node lt p rt) ⇒
  (∀ q. q ∈ set-of t ⟶ q ∈ set (levellist-new ! (var q))) ∧
  (∀ i ≤ var p. (∃ prx. (levellist-new ! i) = prx@(levellist-old ! i)
                  ∧ (∀ pt ∈ set prx. pt ∈ set-of t ∧ var pt = i))) ∧
  (∀ i. (var p) < i ⟶ (levellist-new ! i) = (levellist-old ! i)) ∧
  (length levellist-new = length levellist-old))

lemma wf-levellist-subset:
  assumes wf-ll: wf-levellist t ll ll' var
  shows set (concat ll') ⊆  set (concat ll) ∪ set-of t
proof (cases t)
  case Tip with wf-ll show ?thesis by (simp add: wf-levellist-def)
next
  case (Node lt p rt)
  show ?thesis
  proof −
    {
```

```
      fix n
      assume n ∈ set (concat ll')
      from in-set-concat-idx [OF this]
      obtain i where i-bound: i < length ll' and n-in: n ∈ set (ll' ! i)
        by blast
      have n ∈ set (concat ll) ∪ set-of t
      proof (cases i ≤ var p)
        case True
        with wf-ll obtain prx where
          ll'-ll: ll' ! i = prx @ ll ! i and
          prx: ∀ pt ∈ set prx. pt ∈ set-of t  and
          leq: length ll' = length ll
          apply (clarsimp simp add: wf-levellist-def Node)
          apply (erule-tac x=i in allE)
          apply clarsimp
          done
        show ?thesis
        proof (cases n ∈ set prx)
          case True
          with prx have n ∈ set-of t
            by simp
          thus ?thesis by simp
        next
          case False
          with n-in ll'-ll
          have n ∈ set (ll ! i)
            by simp
          with i-bound leq
          have n ∈ set (concat ll)
            by auto
          thus ?thesis by simp
        qed
      next
        case False
        with wf-ll obtain ll'!i = ll!i length ll' = length ll
          by (auto simp add: wf-levellist-def Node)
        with n-in i-bound
        have n ∈ set (concat ll)
          by auto
        thus ?thesis by simp
      qed
    }
    thus ?thesis by auto
  qed
qed
```

lemma *Levellist-ext-to-all*: $((\exists\, ll.\ Levellist\ hds\ next\ ll \wedge P\ ll) \longrightarrow Q)$
$=$
$(\forall\, ll.\ Levellist\ hds\ next\ ll \wedge P\ ll \longrightarrow Q)$
apply *blast*
done

lemma *Levellist-length*: *Levellist hds p ll* $\Longrightarrow$ *length ll* $=$ *length hds*
by (*auto simp add*: *Levellist-def*)

lemma *map-update*:
$\bigwedge i.\ i < length\ xss \Longrightarrow map\ f\ (xss[i := xs]) = (map\ f\ xss)\ [i := f\ xs]$
apply (*induct xss*)
apply *simp*
apply (*case-tac i*)
apply *simp*
apply *simp*
done

lemma (**in** *Levellist-impl*) *Levellist-spec-total'*:
shows $\forall\, ll\ \sigma\ t.\ \Gamma,\Theta\vdash_t$
$\{\!|\sigma.\ Dag'p'low'high\ t \wedge (p \neq Null \longrightarrow (p\rightarrow'var) < length'levellist) \wedge$
ordered t'var $\wedge$ *Levellist'levellist'next ll* $\wedge$
$(\forall\, n \in set\text{-}of\ t.$
$(if'mark\ n = 'm$
then $n \in set\ (ll\ !'var\ n) \wedge$
$(\forall\, nt\ p.\ Dag\ n'low'high\ nt \wedge p \in set\text{-}of\ nt$
$\longrightarrow 'mark\ p = 'm)$
else $n \notin set\ (concat\ ll)))|\!\}$
$'levellist :== PROC\ Levellist\ (p,'m,'levellist)$
$\{|\exists\, ll'.\ Levellist'levellist'next\ ll' \wedge wf\text{-}levellist\ t\ ll\ ll'\ ^\sigma var \wedge$
wf-marking t $^\sigma mark'mark\ ^\sigma m \wedge$
$(\forall\, p.\ p \notin set\text{-}of\ t \longrightarrow {}^\sigma next\ p = 'next\ p)$
$|\!\}$
apply (*hoare-rule HoareTotal.ProcRec1*
[**where** *r*=*measure* $(\lambda(s,p).\ size\ (dag\ ^s p\ ^s low\ ^s high))$])
apply *vcg*
apply (*rule conjI*)
apply *clarify*
apply (*rule conjI*)
apply *clarify*
apply (*clarsimp simp del*: *BinDag.set-of.simps split del*: *if-split*)
defer
apply (*rule impI*)
apply (*clarsimp simp del*: *BinDag.set-of.simps split del*: *if-split*)
defer
apply (*clarsimp simp add*: *wf-levellist-def wf-marking-def*)

apply (*simp only*: *Levellist-ext-to-all*)
proof −
fix *ll var low high mark next nexta p levellist m lt rt*
assume *pnN*: $p \neq Null$
assume *mark-p*: $mark\ p = (\neg\ m)$
assume *lt*: $Dag\ (low\ p)\ low\ high\ lt$
assume *rt*: $Dag\ (high\ p)\ low\ high\ rt$
from *pnN lt rt* **have** *Dag-p*: $Dag\ p\ low\ high\ (Node\ lt\ p\ rt)$ **by** *simp*
from *Dag-p rt*
have *size-rt-dec*: $size\ (dag\ (high\ p)\ low\ high) < size\ (dag\ p\ low\ high)$
by (*simp only*: *Dag-dag*) *simp*
from *Dag-p lt*
have *size-lt-dec*: $size\ (dag\ (low\ p)\ low\ high) < size\ (dag\ p\ low\ high)$
by (*simp only*: *Dag-dag*) *simp*
assume *ll*: *Levellist levellist next ll*

assume *marked-child-ll*:
$\forall\, n \in set\text{-}of\ (Node\ lt\ p\ rt).$
if $mark\ n = m$
then $n \in set\ (ll\ !\ var\ n) \wedge$
$(\forall\, nt\ p.\ Dag\ n\ low\ high\ nt \wedge p \in set\text{-}of\ nt \longrightarrow mark\ p = m)$
else $n \notin set\ (concat\ ll)$
with *mark-p* **have** *p-notin-ll*: $p \notin set\ (concat\ ll)$
by *auto*
assume *varsll'*: $var\ p < length\ levellist$
with *ll* **have** *varsll*: $var\ p < length\ ll$
by (*simp add*: *Levellist-length*)
assume *orderedt*: $ordered\ (Node\ lt\ p\ rt)\ var$
show $(low\ p \neq Null \longrightarrow var\ (low\ p) < length\ levellist) \wedge$
$ordered\ lt\ var \wedge$
$(\forall\, n \in set\text{-}of\ lt.$
if $mark\ n = m$
then $n \in set\ (ll\ !\ var\ n) \wedge$
$(\forall\, nt\ p.\ Dag\ n\ low\ high\ nt \wedge p \in set\text{-}of\ nt \longrightarrow mark\ p = m)$
else $n \notin set\ (concat\ ll)) \wedge$
$size\ (dag\ (low\ p)\ low\ high) < size\ (dag\ p\ low\ high) \wedge$
$(\forall\, marka\ nexta\ levellist\ lla.$
$Levellist\ levellist\ nexta\ lla \wedge$
$wf\text{-}levellist\ lt\ ll\ lla\ var \wedge wf\text{-}marking\ lt\ mark\ marka\ m \wedge$
$(\forall\, p.\ p \notin set\text{-}of\ lt \longrightarrow next\ p = nexta\ p) \longrightarrow$
$(high\ p \neq Null \longrightarrow var\ (high\ p) < length\ levellist) \wedge$
$ordered\ rt\ var \wedge$
$(\exists\, lla.\ Levellist\ levellist\ nexta\ lla \wedge$
$(\forall\, n \in set\text{-}of\ rt.$
if $marka\ n = m$
then $n \in set\ (lla\ !\ var\ n) \wedge$
$(\forall\, nt\ p.\ Dag\ n\ low\ high\ nt \wedge p \in set\text{-}of\ nt \longrightarrow$
$marka\ p = m)$
else $n \notin set\ (concat\ lla)) \wedge$

```
                    size (dag (high p) low high) < size (dag p low high) ∧
                    (∀ markb nextb levellist llb.
                        Levellist levellist nextb llb ∧
                        wf-levellist rt lla llb var ∧
                        wf-marking rt marka markb m ∧
                        (∀ p. p ∉ set-of rt ⟶ nexta p = nextb p) ⟶
                        (∃ ll'. Levellist (levellist[var p := p])
                                (nextb(p := levellist ! var p)) ll' ∧
                               wf-levellist (Node lt p rt) ll ll' var ∧
                               wf-marking (Node lt p rt) mark (markb(p := m)) m ∧
                               (∀ pa. pa ∉ set-of (Node lt p rt) ⟶
                                     next pa =
                                     (if pa = p then levellist ! var p
                                      else nextb pa))))))
  proof (cases lt)
    case Tip
    note lt-Tip = this
    show ?thesis
    proof (cases rt)
      case Tip
      show ?thesis
        using size-rt-dec Tip lt-Tip Tip lt rt
        apply clarsimp
      subgoal premises prems for marka nexta levellista lla markb nextb levellistb
llb
        proof −
          have lla: Levellist levellista nexta lla by fact
          have llb: Levellist levellistb nextb llb by fact
          have wfll-lt: wf-levellist Tip ll lla var
                        wf-marking Tip mark marka m by fact+

          then have ll-lla: ll = lla
            by (simp add: wf-levellist-def)

          moreover
          with wfll-lt lt-Tip lt have marka = mark
            by (simp add: wf-marking-def)
          moreover
          have wfll-rt:wf-levellist Tip lla llb var
                       wf-marking Tip marka markb m by fact+
          then have lla-llb: lla = llb
            by (simp add: wf-levellist-def)
          moreover
          with wfll-rt Tip rt have markb = marka
            by (simp add: wf-marking-def)
          moreover
          from varsll llb ll-lla lla-llb
          obtain var p < length levellistb var p < length llb
            by (simp add: Levellist-length)
```

```
      with llb pnN
      have llc: Levellist (levellistb[var p := p]) (nextb(p := levellistb ! var p))
                (llb[var p := p # llb ! var p])
        apply (clarsimp simp add: Levellist-def map-update)
        apply (erule-tac x=i in allE)
        apply clarsimp
        apply (subgoal-tac p ∉ set (llb ! i) )
        prefer 2
        using  p-notin-ll ll-lla lla-llb
        apply  simp
        apply (case-tac i=var p)
        apply  simp
        apply simp
        done
      ultimately
      show ?thesis
        using lt-Tip Tip varsll
        apply (clarsimp simp add: wf-levellist-def wf-marking-def)
      proof −
        fix i
        assume varsllb: var p < length llb
        assume i ≤ var p
        show ∃ prx. llb[var p := p#llb!var p]!i = prx @ llb!i ∧
                  (∀ pt∈set prx. pt = p ∧ var pt = i)
        proof (cases i = var p)
          case True
          with pnN lt rt varsllb lt-Tip Tip show ?thesis
            apply −
            apply (rule-tac x=[p] in exI)
            apply (simp add: subdag-eq-def)
            done
        next
          assume i ≠ var p
          with varsllb show ?thesis
            apply −
            apply (rule-tac x=[] in exI)
            apply (simp add: subdag-eq-def)
            done
        qed
      qed
    qed
  done
next
  case (Node dag1 a dag2)
  have rt-node: rt = Node dag1 a dag2 by fact
  with rt have high-p: high p = a
    by simp
  have s: ⋀nexta. (∀ p. next p = nexta p) = (next = nexta)
    by auto
```

show *?thesis*
using *size-rt-dec size-lt-dec rt-node lt-Tip Tip lt rt*
apply (*clarsimp simp del*: *set-of-Node split del*: *if-split simp add*: *s*)
subgoal premises *prems* **for** *marka levellista lla*
proof −
have *lla*: *Levellist levellista next lla* **by** *fact*
have *wfll-lt*:*wf-levellist Tip ll lla var*
wf-marking Tip mark marka m **by** *fact*+
from *this* **have** *ll-lla*: *ll* = *lla*
by (*simp add*: *wf-levellist-def*)
moreover
from *wfll-lt lt-Tip lt* **have** *marklrec*: *marka* = *mark*
by (*simp add*: *wf-marking-def*)
from *orderedt varsll lla ll-lla rt-node lt-Tip high-p*
have *var-highp-bound*: *var* (*high p*) < *length levellista*
by (*auto simp add*: *Levellist-length*)
from *orderedt high-p rt-node lt-Tip*
have *ordered-rt*: *ordered* (*Node dag1* (*high p*) *dag2*) *var*
by *simp*
from *high-p marklrec marked-child-ll lt rt lt-Tip rt-node ll-lla*
have *mark-rt*: (∀ *n*∈*set-of* (*Node dag1* (*high p*) *dag2*).
if marka n = *m*
then n ∈ *set* (*lla* ! *var n*) ∧
(∀ *nt p*. *Dag n low high nt* ∧ *p* ∈ *set-of nt* ⟶ *marka p* = *m*)
else n ∉ *set* (*concat lla*))
apply (*simp only*: *BinDag.set-of.simps*)
apply *clarify*
apply (*drule-tac x*=*n* **in** *bspec*)
apply *blast*
apply *assumption*
done
show *?thesis*
apply (*rule conjI*)
apply (*rule var-highp-bound*)
apply (*rule conjI*)
apply (*rule ordered-rt*)
apply (*rule conjI*)
apply (*rule mark-rt*)
apply *clarify*
apply *clarsimp*
subgoal premises *prems* **for** *markb nextb levellistb llb*
proof −
have *llb*: *Levellist levellistb nextb llb* **by** *fact*
have *wfll-rt*: *wf-levellist* (*Node dag1* (*high p*) *dag2*) *lla llb var* **by** *fact*
have *wfmarking-rt*: *wf-marking* (*Node dag1* (*high p*) *dag2*) *marka markb*
m **by** *fact*
from *wfll-rt varsll llb ll-lla*
obtain *var-p-bounds*: *var p* < *length levellistb var p* < *length llb*
by (*simp add*: *Levellist-length wf-levellist-def*)

with *p-notin-ll ll-lla wfll-rt*
have *p-notin-llb*: $\forall\, i < length\ llb.\ p \notin set\ (llb\ !\ i)$
apply −
apply (*intro allI impI*)
apply (*clarsimp simp add*: *wf-levellist-def*)
apply (*case-tac* $i \leq var\ (high\ p)$)
apply (*drule-tac* $x{=}i$ **in** *spec*)
using *orderedt rt-node lt-Tip high-p*
apply *clarsimp*
apply (*drule-tac* $x{=}i$ **in** *spec*)
apply (*drule-tac* $x{=}i$ **in** *spec*)
apply *clarsimp*
done
with *llb pnN var-p-bounds*
have *llc*: *Levellist* (*levellistb*[*var p* := *p*])
(*nextb*(*p* := *levellistb* ! *var p*))
(*llb*[*var p* := *p* # *llb* ! *var p*])
apply (*clarsimp simp add*: *Levellist-def map-update*)
apply (*erule-tac* $x{=}i$ **in** *allE*)
apply (*erule-tac* $x{=}i$ **in** *allE*)
apply *clarsimp*
apply (*case-tac* $i{=}var\ p$)
apply *simp*
apply *simp*
done
then show *?thesis*
apply *simp*
using *wfll-rt wfmarking-rt*
lt-Tip rt-node varsll orderedt lt rt pnN ll-lla marklrec
apply (*clarsimp simp add*: *wf-levellist-def wf-marking-def*)
apply (*intro conjI*)
apply (*rule allI*)
apply (*rule conjI*)
apply (*erule-tac* $x{=}q$ **in** *allE*)
apply (*case-tac var p* = *var q*)
apply *fastforce*
apply *fastforce*
apply (*case-tac var p* = *var q*)
apply *hypsubst-thin*
apply *fastforce*
apply *fastforce*
apply (*rule allI*)
apply (*rotate-tac 4*)
apply (*erule-tac* $x{=}i$ **in** *allE*)
apply (*case-tac* $i{=}var\ p$)
apply *simp*
apply (*case-tac var* $(high\ p) < i$)
apply *simp*
apply *simp*

```
            apply (erule exE)
            apply (rule-tac x=prx in exI)
            apply (intro conjI)
            apply  simp
            apply clarify
            apply (rotate-tac 15)
            apply (erule-tac x=pt in ballE)
            apply  fastforce
            apply fastforce
            done
          qed
        done
      qed
    done
  qed
next
  case (Node llt l rlt)
  have lt-Node: lt = Node llt l rlt by fact
  from orderedt lt varsll' lt-Node
  obtain ordered-lt:
    ordered lt var (low p ≠ Null ⟶ var (low p) < length levellist)
    by (cases rt) auto
  from lt lt-Node marked-child-ll
  have mark-lt: ∀ n∈set-of lt.
   if mark n = m
   then n ∈ set (ll ! var n) ∧
        (∀ nt p. Dag n low high nt ∧ p ∈ set-of nt ⟶ mark p = m)
   else n ∉ set (concat ll)
    apply (simp only: BinDag.set-of.simps)
    apply clarify
    apply (drule-tac x=n in bspec)
    apply  blast
    apply assumption
    done
  show ?thesis
    apply (intro conjI ordered-lt mark-lt size-lt-dec)
    apply (clarify)
    apply (simp add: size-rt-dec split del: if-split)
    apply (simp only: Levellist-ext-to-all)
    subgoal premises prems for marka nexta levellista lla
    proof −
      have lla: Levellist levellista nexta lla by fact
      have wfll-lt: wf-levellist lt ll lla var  by fact
      have wfmarking-lt:wf-marking lt mark marka m by fact
      from wfll-lt lt-Node
      have lla-eq-ll: length lla = length ll
        by (simp add: wf-levellist-def)
      with ll lla have lla-eq-ll': length levellista = length levellist
        by (simp add: Levellist-length)
```

with *orderedt rt lt-Node lt varsll'*
obtain *ordered-rt*:
  ordered rt var (*high p* ≠ *Null* ⟶ *var* (*high p*) < *length levellista*)
  by (*cases rt*) *auto*
from *wfll-lt lt-Node*
have *nodes-in-lla*: ∀ *q*. *q* ∈ *set-of lt* ⟶ *q* ∈ *set* (*lla* ! (*q*→*var*))
  by (*simp add*: *wf-levellist-def*)
from *wfll-lt lt-Node lt*
have *lla-st*: (∀ *i* ≤ (*low p*)→*var*.
            (∃ *prx*. (*lla* ! *i*) = *prx*@(*ll* ! *i*) ∧
                  (∀ *pt* ∈ *set prx*. *pt* ∈ *set-of lt* ∧ *pt*→*var* = *i*)))
  by (*simp add*: *wf-levellist-def*)
from *wfll-lt lt-Node lt*
have *lla-nc*: ∀ *i*. ((*low p*)→*var*) < *i* ⟶ (*lla* ! *i*) = (*ll* ! *i*)
  by (*simp add*: *wf-levellist-def*)
from *wfmarking-lt lt-Node lt*
have *mot-nc*: ∀ *n*. *n* ∉ *set-of lt* ⟶ *mark n* = *marka n*
  by (*simp add*: *wf-marking-def*)
from *wfmarking-lt lt-Node lt*
have *mit-marked*: ∀ *n*. *n* ∈ *set-of lt* ⟶ *marka n* = *m*
  by (*simp add*: *wf-marking-def*)
from *marked-child-ll nodes-in-lla mot-nc mit-marked lla-st*
have *mark-rt*: ∀ *n*∈*set-of rt*.
      if marka n = *m*
      then n ∈ *set* (*lla* ! *var n*) ∧
        (∀ *nt p*. *Dag n low high nt* ∧ *p* ∈ *set-of nt* ⟶ *marka p* = *m*)
      else n ∉ *set* (*concat lla*)
  apply −
  apply (*rule ballI*)
  apply (*drule-tac x*=*n* **in** *bspec*)
  apply (*simp*)
proof −
  fix *n*

  assume *nodes-in-lla*: ∀ *q*. *q* ∈ *set-of lt* ⟶ *q* ∈ *set* (*lla* ! *var q*)
  assume *mot-nc*: ∀ *n*. *n* ∉ *set-of lt* ⟶ *mark n* = *marka n*
  assume *mit-marked*: ∀ *n*. *n* ∈ *set-of lt* ⟶ *marka n* = *m*
  assume *marked-child-ll*: *if mark n* = *m*
   then n ∈ *set* (*ll* ! *var n*) ∧
      (∀ *nt p*. *Dag n low high nt* ∧ *p* ∈ *set-of nt* ⟶ *mark p* = *m*)
   else n ∉ *set* (*concat ll*)

  assume *lla-st*: ∀ *i*≤*var* (*low p*).
                ∃ *prx*. *lla* ! *i* = *prx* @ *ll* ! *i* ∧
                (∀ *pt*∈*set prx*. *pt* ∈ *set-of lt* ∧ *var pt* = *i*)

  assume *n-in-rt*: *n* ∈ *set-of rt*
  show *n-in-lla-marked*: *if marka n* = *m*
    then n ∈ *set* (*lla* ! *var n*) ∧

```
        (∀ nt p. Dag n low high nt ∧ p ∈ set-of nt ⟶ marka p = m)
    else n ∉ set (concat lla)
proof (cases n ∈ set-of lt)
  case True
  from True nodes-in-lla have n-in-ll: n ∈ set (lla ! var n)
    by simp
  moreover
  from True wfmarking-lt
  have marka n = m
    apply (cases lt)
    apply (auto simp add: wf-marking-def)
    done
  moreover
  {
    fix nt p
    assume Dag n low high nt
    with lt True have subset-nt-lt: set-of nt ⊆ set-of lt
      by (rule dag-setof-subsetD)
    moreover assume  p ∈ set-of nt
    ultimately have p ∈ set-of lt
      by blast
    with mit-marked have  marka p = m
      by simp
  }
  ultimately show ?thesis
    using n-in-rt
    apply clarsimp
    done
next
  assume n-notin-lt: n ∉ set-of lt
  show ?thesis
  proof (cases marka n = m)
    case True
    from n-notin-lt mot-nc have marka-eq-mark: mark n = marka n
      by simp
    from marka-eq-mark True have n-marked: mark n = m
      by simp
    from rt n-in-rt have nnN: n ≠ Null
      apply −
      apply (rule set-of-nn [rule-format])
      apply fastforce
      apply assumption
      done
    from marked-child-ll n-in-rt marka-eq-mark nnN n-marked
    have n-in-ll: n ∈ set (ll ! var n)
      by fastforce
    from marked-child-ll n-in-rt marka-eq-mark nnN n-marked lt rt
   have nt-mark: ∀ nt p. Dag n low high nt ∧ p ∈ set-of nt ⟶ mark p =
m
```

```
    by simp
  from nodes-in-lla n-in-ll lla-st
  have n-in-lla: n ∈ set (lla ! var n)
  proof (cases var (low p) < (var n))
    case True
    with lla-nc have (lla ! var n) = (ll ! var n)
      by fastforce
    with n-in-ll show ?thesis
      by fastforce
  next
    assume varnslp: ¬ var (low p) < var n
    with lla-st
    have ll-in-lla: ∃ prx. lla ! (var n) = prx @ ll ! (var n)
      apply −
      apply (erule-tac x=var n in allE)
      apply fastforce
      done
    with n-in-ll show ?thesis
      by fastforce
  qed
  {
    fix nt pt
    assume nt-Dag: Dag n low high nt
    assume pt-in-nt: pt ∈ set-of nt
    have  marka pt = m
    proof (cases pt ∈ set-of lt)
      case True
      with mit-marked show ?thesis
        by fastforce
    next
      assume pt-notin-lt:  pt ∉ set-of lt
      with mot-nc have mark pt = marka pt
        by fastforce
      with nt-mark nt-Dag pt-in-nt show ?thesis
        by fastforce
    qed
  }
  then have nt-marka:
    ∀ nt pt. Dag n low high nt ∧ pt ∈ set-of nt ⟶ marka pt = m
    by fastforce
  with n-in-lla nt-marka True show ?thesis
    by fastforce
next
  case False
  note n-not-marka = this
  with wfmarking-lt n-notin-lt
  have mark n ≠ m
    by (simp add: wf-marking-def lt-Node)
  with marked-child-ll
```

```
      have n-notin-ll: n ∉ set (concat ll)
        by simp
      show ?thesis
      proof (cases n ∈ set (concat lla))
        case False with n-not-marka show ?thesis by simp
      next
        case True
        with wf-levellist-subset [OF wfll-lt] n-notin-ll
        have n ∈ set-of lt
          by blast
        with n-notin-lt have False by simp
        thus ?thesis ..
      qed
    qed
  qed
qed
show ?thesis
  apply (intro conjI ordered-rt mark-rt)
  apply clarify
  subgoal premises prems for markb nextb levellistb llb
  proof −
    have llb: Levellist levellistb nextb llb by fact
    have wfll-rt: wf-levellist rt lla llb var by fact
    have wfmarking-rt: wf-marking rt marka markb m by fact
    show ?thesis
    proof (cases rt)
      case Tip
      from wfll-rt Tip have lla-llb: lla = llb
        by (simp add: wf-levellist-def)
      moreover
      from wfmarking-rt Tip rt have markb = marka
        by (simp add: wf-marking-def)
      moreover
      from wfll-lt varsll llb lla-llb
      obtain var-p-bounds: var p < length levellistb var p < length llb
        by (simp add: Levellist-length wf-levellist-def lt-Node Tip)
      with p-notin-ll lla-llb wfll-lt
      have p-notin-llb: ∀ i < length llb. p ∉ set (llb ! i)
        apply −
        apply (intro allI impI)
        apply (clarsimp simp add: wf-levellist-def lt-Node)
        apply (case-tac i ≤ var l)
        apply  (drule-tac x=i in spec)
        using  orderedt Tip lt-Node
        apply  clarsimp
        apply (drule-tac x=i in spec)
        apply (drule-tac x=i in spec)
        apply clarsimp
        done
```

```
  with llb pnN var-p-bounds
  have llc: Levellist (levellistb[var p := p])
                (nextb(p := levellistb ! var p))
                (llb[var p := p # llb ! var p])
    apply (clarsimp simp add: Levellist-def map-update)
    apply (erule-tac x=i in allE)
    apply (erule-tac x=i in allE)
    apply clarsimp
    apply (case-tac i=var p)
    apply  simp
    apply simp
    done
  ultimately show ?thesis
    using Tip lt-Node varsll orderedt lt rt pnN wfll-lt wfmarking-lt
    apply (clarsimp simp add: wf-levellist-def wf-marking-def)
    apply (intro conjI)
    apply  (rule allI)
    apply  (rule conjI)
    apply   (erule-tac x=q in allE)
    apply   (case-tac var p = var q)
    apply    fastforce
    apply   fastforce
    apply  (case-tac var p = var q)
    apply   hypsubst-thin
    apply   fastforce
    apply  fastforce
    apply (rule allI)
    apply (rotate-tac 4)
    apply (erule-tac x=i in allE)
    apply (case-tac i=var p)
    apply  simp
    apply (case-tac var (low p) < i)
    apply  simp
    apply simp
    apply (erule exE)
    apply (rule-tac x=prx in exI)
    apply (intro conjI)
    apply  simp
    apply clarify
    apply (rotate-tac 15)
    apply (erule-tac x=pt in ballE)
    apply  fastforce
    apply fastforce
    done
next
  case (Node lrt r rrt)
  have rt-Node: rt = Node lrt r rrt by fact
  from wfll-rt rt-Node
  have llb-eq-lla: length llb = length lla
```

by (*simp add*: *wf-levellist-def*)
with *llb lla*
have *llb-eq-lla'*: *length levellistb* = *length levellista*
by (*simp add*: *Levellist-length*)
from *wfll-rt rt-Node*
have *nodes-in-llb*: ∀ *q*. *q* ∈ *set-of rt* ⟶ *q* ∈ *set* (*llb* ! (*q*→*var*))
by (*simp add*: *wf-levellist-def*)
from *wfll-rt rt-Node rt*
have *llb-st*: (∀ *i* ≤ (*high p*)→*var*.
(∃ *prx*. (*llb* ! *i*) = *prx*@(*lla* ! *i*) ∧
(∀ *pt* ∈ *set prx*. *pt* ∈ *set-of rt* ∧ *pt*→*var* = *i*)))
by (*simp add*: *wf-levellist-def*)
from *wfll-rt rt-Node rt*
have *llb-nc*:
∀ *i*. ((*high p*)→*var*) < *i* ⟶ (*llb* ! *i*) = (*lla* ! *i*)
by (*simp add*: *wf-levellist-def*)
from *wfmarking-rt rt-Node rt*
have *mort-nc*: ∀ *n*. *n* ∉ *set-of rt* ⟶ *marka n* = *markb n*
by (*simp add*: *wf-marking-def*)
from *wfmarking-rt rt-Node rt*
have *mirt-marked*: ∀ *n*. *n* ∈ *set-of rt* ⟶ *markb n* = *m*
by (*simp add*: *wf-marking-def*)
with *p-notin-ll wfll-rt wfll-lt*
have *p-notin-llb*: ∀ *i* < *length llb*. *p* ∉ *set* (*llb* ! *i*)
apply −
apply (*intro allI impI*)
apply (*clarsimp simp add*: *wf-levellist-def lt-Node rt-Node*)
apply (*case-tac i* ≤ *var r*)
apply (*drule-tac x*=*i* **in** *spec*)
using *orderedt rt-Node lt-Node*
apply *clarsimp*
apply (*erule disjE*)
apply *clarsimp*
apply (*case-tac i* ≤ *var l*)
apply (*drule-tac x*=*i* **in** *spec*)
apply *clarsimp*
apply *clarsimp*
apply (*subgoal-tac llb* ! *i* = *lla* ! *i*)
prefer *2*
apply *clarsimp*
apply (*case-tac i* ≤ *var l*)
apply (*drule-tac x*=*i* **in** *spec*, *erule impE*, *assumption*)
apply *clarsimp*
using *orderedt rt-Node lt-Node*
apply *clarsimp*
apply *clarsimp*
done
from *wfll-lt wfll-rt varsll lla llb*
obtain *var-p-bounds*: *var p* < *length levellistb var p* < *length llb*

```
  by (simp add: Levellist-length wf-levellist-def lt-Node rt-Node)
with p-notin-llb llb pnN var-p-bounds
have llc: Levellist (levellistb[var p := p])
              (nextb(p := levellistb ! var p))
              (llb[var p := p # llb ! var p])
  apply (clarsimp simp add: Levellist-def map-update)
  apply (erule-tac x=i in allE)
  apply (erule-tac x=i in allE)
  apply clarsimp
  apply (case-tac i=var p)
  apply  simp
  apply simp
  done
then show ?thesis
proof (clarsimp)
  show wf-levellist (Node lt p rt) ll (llb[var p := p#llb ! var p]) var ∧
       wf-marking (Node lt p rt) mark (markb(p := m)) m
  proof −
    have nodes-in-upllb: ∀ q. q ∈ set-of (Node lt p rt)
    ⟶ q ∈ set (llb[var p :=p # llb ! var p] ! (var q))
      apply −
      apply (rule allI)
      apply (rule impI)
    proof −
      fix q
      assume q-in-t: q ∈ set-of (Node lt p rt)
      show q-in-upllb:
        q ∈ set (llb[var p :=p # llb ! var p] ! (var q))
      proof (cases q ∈ set-of rt)
        case True
        with nodes-in-llb have q-in-llb: q ∈ set (llb ! (var q))
          by fastforce
        from orderedt rt-Node lt-Node lt rt
        have ordered-rt: ordered rt var
          by fastforce
        from True rt ordered-rt rt-Node lt lt-Node have var q ≤ var r
          apply −
          apply (drule subnodes-ordered)
          apply fastforce
          apply fastforce
          apply fastforce
          done
        with orderedt rt lt rt-Node lt-Node have var q < var p
          by fastforce
        then have
          llb[var p :=p#llb ! var p] ! var q =
           llb ! var q
          by fastforce
        with q-in-llb show ?thesis
```

```
    by fastforce
next
  assume q-notin-rt: q ∉ set-of rt
  show q ∈ set (llb[var p :=p # llb ! var p] ! var q)
  proof (cases q ∈ set-of lt)
    case True
    assume q-in-lt: q ∈ set-of lt
    with nodes-in-lla have q-in-lla: q ∈ set (lla ! (var q))
      by fastforce
    from orderedt rt-Node lt-Node lt rt
    have ordered-lt: ordered lt var
      by fastforce
    from q-in-lt lt ordered-lt rt-Node rt lt-Node
    have var q ≤ var l
      apply −
      apply (drule subnodes-ordered)
      apply fastforce
      apply fastforce
      apply fastforce
      done
    with orderedt rt lt rt-Node lt-Node have qsp: var q < var p
      by fastforce
    then show ?thesis
    proof (cases var q ≤ var (high p))
      case True
      with llb-st
      have ∃ prx. (llb ! (var q)) = prx@(lla ! (var q))
        by fastforce
      with nodes-in-lla q-in-lla
      have q-in-llb: q ∈ set (llb ! (var q))
        by fastforce
      from qsp
      have llb[var p :=p#llb ! var p]!var q =
             llb ! (var q)
        by fastforce
      with q-in-llb show ?thesis
        by fastforce
    next
      assume ¬ var q ≤ var (high p)
      with llb-nc have llb ! (var q) = lla ! (var q)
        by fastforce
      with q-in-lla have q-in-llb: q ∈ set (llb ! (var q))
        by fastforce
      from qsp have
        llb[var p :=p # llb ! var p] ! var q = llb ! (var q)
        by fastforce
      with q-in-llb show ?thesis
        by fastforce
    qed
```

```
    next
      assume q-notin-lt: q ∉ set-of lt
      with q-notin-rt rt lt rt-Node lt-Node q-in-t have qp: q = p
        by fastforce
      with varsll lla-eq-ll llb-eq-lla have var p < length llb
        by fastforce
      with qp show ?thesis
        by simp
    qed
  qed
qed
have prx-ll-st: ∀ i ≤ var p.
 (∃ prx. llb[var p :=p#llb!var p]!i = prx@(ll!i) ∧
       (∀ pt ∈ set prx. pt ∈ set-of (Node lt p rt) ∧ var pt = i))
  apply −
  apply (rule allI)
  apply (rule impI)
proof −
  fix i
  assume isep: i ≤ var p
  show ∃ prx. llb[var p :=p#llb!var p]!i = prx@ll!i ∧
    (∀ pt∈set prx. pt ∈ set-of (Node lt p rt) ∧ var pt = i)
  proof (cases i = var p)
    case True
    with orderedt lt lt-Node rt rt-Node
    have lpsp: var (low p) < var p
      by fastforce
    with orderedt lt lt-Node rt rt-Node
    have hpsp: var (high p) < var p
      by fastforce
    with lpsp lla-nc
    have llall: lla ! var p = ll ! var p
      by fastforce
    with hpsp llb-nc have llb ! var p = ll ! var p
      by fastforce
    with llb-eq-lla lla-eq-ll isep True varsll lt rt show ?thesis
      apply −
      apply (rule-tac x=[p] in exI)
      apply (rule conjI)
      apply simp
      apply (rule ballI)
      apply fastforce
      done
  next
    assume inp:  i ≠ var p
    show ?thesis
    proof (cases var (low p) < i)
      case True
      with lla-nc have llall: lla ! i = ll ! i
```

```
    by fastforce
  assume vpsi: var (low p) < i
  show ?thesis
  proof (cases var (high p) < i)
    case True
    with llall llb-nc have llb ! i = ll ! i
      by fastforce
    with inp True vpsi varsll lt rt show ?thesis
      apply −
      apply (rule-tac x=[] in exI)
      apply (rule conjI)
      apply simp
      apply (rule ballI)
      apply fastforce
      done
  next
    assume isehp: ¬ var (high p) < i
    with vpsi lla-nc have lla-ll: lla ! i = ll ! i
      by fastforce
    with isehp llb-st
    have prx-lla: ∃ prx. llb ! i = prx @ lla ! i ∧
      (∀ pt∈set prx. pt ∈ set-of rt ∧ var pt = i)
      apply −
      apply (erule-tac x=i in allE)
      apply simp
      done
    with lla-ll inp rt show ?thesis
      apply −
      apply (erule exE)
      apply (rule-tac x=prx in exI)
      apply simp
      done
  qed
next
  assume iselp: ¬ var (low p) < i
  show ?thesis
  proof (cases var (high p) < i)
    case True
    with llb-nc have llb-ll: llb ! i = lla ! i
      by fastforce
    with iselp lla-st
    have prx-ll: ∃ prx. lla ! i = prx @ ll ! i ∧
      (∀ pt∈set prx. pt ∈ set-of lt ∧ var pt = i)
      apply −
      apply (erule-tac x=i in allE)
      apply simp
      done
    with llb-ll inp lt show ?thesis
      apply −
```

```
        apply (erule exE)
        apply (rule-tac x=prx in exI)
        apply simp
        done
      next
        assume isehp: ¬ var (high p) < i
        from iselp lla-st
        have prxl: ∃ prx. lla ! i = prx @ ll ! i ∧
          (∀ pt∈set prx. pt ∈ set-of lt ∧ var pt = i)
          by fastforce
        from isehp llb-st
        have prxh: ∃ prx. llb ! i = prx @ lla ! i ∧
          (∀ pt∈set prx. pt ∈ set-of rt ∧ var pt = i)
          by fastforce
        with prxl inp lt pnN rt show ?thesis
          apply −
          apply (elim exE)
          apply (rule-tac x=prxa @ prx in exI)
          apply simp
          apply (elim conjE)
          apply fastforce
          done
      qed
    qed
  qed
qed
have big-Nodes-nc: ∀ i. (p−>var) < i
  ⟶ (llb[var p :=p # llb ! var p]) ! i = ll ! i
  apply −
  apply (rule allI)
  apply (rule impI)
proof −
  fix i
  assume psi: var p < i
  with orderedt lt rt lt-Node rt-Node have lpsi: var (low p) < i
    by fastforce
  with lla-nc have lla-ll: lla ! i = ll ! i
    by fastforce
 from psi orderedt lt rt lt-Node rt-Node have hpsi: var (high p) < i
    by fastforce
  with llb-nc have llb-lla: llb ! i = lla ! i
    by fastforce
  from psi
  have upllb-llb: llb[var p :=p#llb!var p]!i = llb!i
    by fastforce
  from upllb-llb llb-lla lla-ll
  show llb[var p :=p # llb ! var p] ! i = ll ! i
    by fastforce
qed
```

```
  from lla-eq-ll llb-eq-lla
  have length-eq: length (llb[var p :=p # llb ! var p]) = length ll
    by fastforce
  from length-eq big-Nodes-nc prx-ll-st nodes-in-upllb
  have wf-ll-upllb:
    wf-levellist (Node lt p rt) ll (llb[var p :=p # llb ! var p]) var
    by (simp add: wf-levellist-def)
  have mark-nc:
    ∀ n. n ∉ set-of (Node lt p rt) ⟶ (markb(p:=m)) n = mark n
    apply −
    apply (rule allI)
    apply (rule impI)
  proof −
    fix n
    assume nnit: n ∉ set-of (Node lt p rt)
    with lt rt have nnilt: n ∉ set-of lt
      by fastforce
    from nnit lt rt have nnirt: n ∉ set-of rt
      by fastforce
    with nnilt mot-nc mort-nc have mb-eq-m: markb n = mark n
      by fastforce
    from nnit have n≠p
      by fastforce
    then have upmarkb-markb: (markb(p :=m)) n = markb n
      by fastforce
    with mb-eq-m show (markb(p :=m)) n = mark n
      by fastforce
  qed
  have mark-c: ∀ n. n ∈ set-of (Node lt p rt) ⟶ (markb(p :=m)) n
= m
    apply −
    apply (intro allI)
    apply (rule impI)
  proof −
    fix n
    assume nint: n ∈ set-of (Node lt p rt)
    show (markb(p :=m)) n = m
    proof (cases n=p)
      case True
      then show ?thesis
        by fastforce
    next
      assume nnp: n ≠ p
      show ?thesis
      proof (cases n ∈ set-of rt)
        case True
        with mirt-marked have markb n = m
          by fastforce
        with nnp show ?thesis
```

```
                    by fastforce
                  next
                    assume nninrt:  n ∉ set-of rt
                    with nint nnp have ninlt: n ∈ set-of lt
                      by fastforce
                    with mit-marked have marka-m: marka n = m
                      by fastforce
                    from mort-nc nninrt have marka n = markb n
                      by fastforce
                    with marka-m have markb n = m
                      by fastforce
                    with nnp show ?thesis
                      by fastforce
                  qed
                qed
              qed
              from mark-c mark-nc
              have wf-mark: wf-marking (Node lt p rt) mark (markb(p :=m)) m
                by (simp add: wf-marking-def)
              with wf-ll-upllb show ?thesis
                by fastforce
            qed
          qed
        qed
      qed
    done
  qed
done
qed
next
  fix var low high p lt rt and levellist and
    ll::ref list list and mark::ref ⇒ bool and next
  assume pnN: p ≠ Null
  assume ll: Levellist levellist next ll
  assume vpsll: var p < length levellist
  assume orderedt: ordered (Node lt p rt) var
  assume marked-child-ll: ∀ n∈set-of (Node lt p rt).
          if mark n = mark p
          then n ∈ set (ll ! var n) ∧
               (∀ nt pa. Dag n low high nt ∧ pa ∈ set-of nt ⟶ mark pa = mark p)
          else n ∉ set (concat ll)
  assume lt: Dag (low p) low high lt
  assume rt: Dag (high p) low high rt
  show wf-levellist (Node lt p rt) ll ll var ∧
       wf-marking (Node lt p rt) mark mark (mark p)
  proof −
    from marked-child-ll pnN lt rt have marked-st:
      (∀ pa. pa ∈ set-of (Node lt p rt) ⟶ mark pa = mark p)
      apply −
```

apply (*drule-tac x=p* **in** *bspec*)
apply *simp*
apply (*clarsimp*)
apply (*erule-tac x=*(*Node lt p rt*) **in** *allE*)
apply *simp*
done
have *nodest-in-ll*:
$\forall\, q.\ q \in$ *set-of* (*Node lt p rt*) $\longrightarrow q \in$ *set* (*ll* ! *var q*)
proof −
from *marked-child-ll pnN* **have** *pinll*: $p \in$ *set* (*ll* ! *var p*)
apply −
apply (*drule-tac x=p* **in** *bspec*)
apply *simp*
apply *fastforce*
done
from *marked-st marked-child-ll lt rt* **show** *?thesis*
apply −
apply (*rule allI*)
apply (*erule-tac x=q* **in** *allE*)
apply (*rule impI*)
apply (*erule impE*)
apply *assumption*
apply (*drule-tac x=q* **in** *bspec*)
apply *simp*
apply *fastforce*
done
qed
have *levellist-nc*: $\forall\ i \leq$ *var p*. ($\exists$ *prx*. *ll* ! *i* = *prx*@(*ll* ! *i*) $\wedge$
($\forall\ pt \in$ *set prx*. *pt* $\in$ *set-of* (*Node lt p rt*) $\wedge$ *var pt* = *i*))
apply −
apply (*rule allI*)
apply (*rule impI*)
apply (*rule-tac x=*[] **in** *exI*)
apply *fastforce*
done
have *ll-nc*: $\forall\ i$. (*var p*) $< i \longrightarrow$ *ll* ! *i* = *ll* ! *i*
by *fastforce*
have *length-ll*: *length ll* = *length ll*
by *fastforce*
with *ll-nc levellist-nc nodest-in-ll*
have *wf*: *wf-levellist* (*Node lt p rt*) *ll ll var*
by (*simp add*: *wf-levellist-def*)
have *m-nc*: $\forall\ n.\ n \notin$ *set-of* (*Node lt p rt*) $\longrightarrow$ *mark n* = *mark n*
by *fastforce*
from *marked-st* **have** $\forall\ n.\ n \in$ *set-of* (*Node lt p rt*) $\longrightarrow$ *mark n* = *mark p*
by *fastforce*
with *m-nc* **have** *wf-marking* (*Node lt p rt*) *mark mark* (*mark p*)
by (*simp add*: *wf-marking-def*)
with *wf* **show** *?thesis*

```
    by fastforce
  qed
qed

lemma allD: ∀ ll. P ll ⟹ P ll
  by blast

lemma replicate-spec: ⟦∀ i < n. xs ! i = x; n=length xs⟧
  ⟹ replicate (length xs) x = xs
apply hypsubst-thin
apply (induct xs)
apply  simp
apply force
done

lemma (in Levellist-impl) Levellist-spec-total:
shows ∀ σ  t. Γ,Θ⊢_t
       {|σ. Dag´p´low´high t ∧ (∀ i < length´levellist.´levellist ! i = Null) ∧
            length´levellist  =´p →´var + 1 ∧
            ordered t´var ∧ (∀ n ∈ set-of t.´mark n = (¬´m) )|}
        ´levellist :== PROC Levellist (p,´m,´levellist)
       {|∃ ll. Levellist´levellist´next ll ∧ wf-ll t ll ^σvar  ∧
        length´levellist = ^σp → ^σvar + 1 ∧
        wf-marking t ^σmark´mark ^σm ∧
        (∀ p. p ∉ set-of t ⟶ ^σnext p =´next p)|}
apply (hoare-rule HoareTotal.conseq)
apply  (rule-tac ll=replicate (^σp→^σvar + 1) [] in allD [OF Levellist-spec-total´])
apply (intro allI impI)
apply (rule-tac x=σ in exI)
apply (rule-tac x=t in exI)
apply (rule conjI)
apply  (clarsimp split:if-split-asm simp del: concat-replicate-trivial)
apply  (frule replicate-spec [symmetric])
apply   (simp)
apply  (clarsimp simp add: Levellist-def )
apply  (case-tac i)
apply   simp
apply  simp
apply (simp add: Collect-conv-if split:if-split-asm)
apply vcg-step
apply (elim exE conjE)
apply (rule-tac x=ll' in exI)
apply simp
apply (thin-tac ∀ p. p ∉ set-of t ⟶ next p = nexta p)
apply (simp add: wf-levellist-def wf-ll-def)
apply (case-tac t = Tip)
apply  simp
apply  (rule conjI)
apply   clarsimp
```

apply (*case-tac k*)
apply *simp*
apply *simp*
apply (*subgoal-tac length ll'=Suc* (*var Null*))
apply (*simp add*: *Levellist-length*)
apply *fastforce*
apply (*split dag.splits*)
apply *simp*
apply (*elim conjE*)
apply (*intro conjI*)
apply (*rule allI*)
apply (*erule-tac x=pa* **in** *allE*)
apply *clarify*
prefer *2*
apply (*simp add*: *Levellist-length*)
apply (*rule allI*)
apply (*rule impI*)
apply (*rule ballI*)
apply (*rotate-tac 11*)
apply (*erule-tac x=k* **in** *allE*)
apply (*rename-tac dag1 ref dag2 k pa*)
apply (*subgoal-tac k <= var ref*)
prefer *2*
apply (*subgoal-tac ref = p*)
apply *simp*
apply *clarify*
apply (*erule-tac ?P = Dag p low high* (*Node dag1 ref dag2*) **in** *rev-mp*)
apply (*simp* (*no-asm*))
apply (*rotate-tac 14*)
apply (*erule-tac x=k* **in** *allE*)
apply *clarify*
apply (*erule-tac x=k* **in** *allE*)
apply *clarify*
apply (*case-tac k*)
apply *simp*
apply *simp*
done

end

7 Proof of Procedure ShareRep

theory *ShareRepProof* **imports** *ProcedureSpecs Simpl.HeapList* **begin**

lemma (**in** *ShareRep-impl*) *ShareRep-modifies*:
shows $\forall \sigma.\ \Gamma \vdash \{\sigma\}$ *PROC ShareRep* (*nodeslist*,´*p*)
$\{t.\ t$ *may-only-modify-globals* σ *in* [*rep*]$\}$
apply (*hoare-rule HoarePartial.ProcRec1*)
apply (*vcg spec=modifies*)

done

lemma *hd-filter-cons*:
$\bigwedge$ *i*. ⟦ *P* (*xs* ! *i*) *p*; *i* < *length xs*; ∀ *no* ∈ *set* (*take i xs*). ¬ *P no p*; ∀ *a b*. *P a b* = *P b a*⟧
⟹ *xs* ! *i* = *hd* (*filter* (*P p*) *xs*)
apply (*induct xs*)
apply *simp*
apply (*case-tac P a p*)
apply *simp*
apply (*case-tac i*)
apply *simp*
apply *simp*
apply (*case-tac i*)
apply *simp*
apply *auto*
done

lemma (**in** *ShareRep-impl*) *ShareRep-spec-total*:
shows
∀ σ *ns*. Γ,Θ⊢$_t$
{|σ. *List´nodeslist´next ns* ∧
(∀ *no* ∈ *set ns*. *no* ≠ *Null* ∧
((*no*→*low* = *Null*) = (*no*→*high* = *Null*)) ∧
(*isLeaf-pt´p´low´high* ⟶ *isLeaf-pt no´low´high*) ∧
no→*var* =´*p*→*var*) ∧
´*p* ∈ *set ns*|}
PROC ShareRep (*nodeslist*,´*p*)
{| (σ*p* →´*rep* = *hd* (*filter* (λ *sn*. *repNodes-eq sn* σ*p* σ*low* σ*high* σ*rep*) *ns*)) ∧
(∀ *pt*. *pt* ≠ σ*p* ⟶ *pt*→σ*rep* = *pt*→*rep*) ∧
(σ*p*→*rep*→σ*var* = σ*p* → σ*var*)|}
apply (*hoare-rule HoareTotal.ProcNoRec1*)
apply (*hoare-rule anno*=
IF (*isLeaf-pt´p´low´high*)
THEN´*p* →´*rep* :==´*nodeslist*
ELSE
WHILE (*nodeslist* ≠ *Null*)
INV {|∃ *prx sfx*. *List´nodeslist´next sfx* ∧ *ns*=*prx*@*sfx* ∧
¬ *isLeaf-pt´p´low´high* ∧
(∀ *no* ∈ *set ns*. *no* ≠ *Null* ∧
((*no*→σ*low* = *Null*) = (*no*→σ*high* = *Null*)) ∧
(*isLeaf-pt* σ*p* σ*low* σ*high* ⟶ *isLeaf-pt no* σ*low* σ*high*) ∧
no→σ*var* = σ*p*→σ*var*) ∧
σ*p* ∈ *set ns* ∧
((∃ *pt* ∈ *set prx*. *repNodes-eq pt* σ*p* σ*low* σ*high* σ*rep*)
⟶´*rep* σ*p* = *hd* (*filter* (λ *sn*. *repNodes-eq sn* σ*p* σ*low* σ*high* σ*rep*) *prx*) ∧
(∀ *pt*. *pt* ≠ σ*p* ⟶ *pt*→σ*rep* = *pt*→*rep*)) ∧
((∀ *pt* ∈ *set prx*. ¬ *repNodes-eq pt* σ*p* σ*low* σ*high* σ*rep*) ⟶ σ*rep* =´*rep*) ∧

```
      ⦇nodeslist ≠ Null ⟶
         (∀ pt ∈ set prx. ¬ repNodes-eq pt ^σp ^σlow ^σhigh ^σrep)) ∧
      ⦇p = ^σp ∧´high = ^σhigh ∧´low = ^σlow)⦈
    VAR MEASURE (length (list´nodeslist´next))
    DO
     IF (repNodes-eq´nodeslist´p´low´high´rep)
     THEN´p→rep :==´nodeslist;;´nodeslist :== Null
     ELSE´nodeslist :==´nodeslist→next
     FI
    OD
  FI in HoareTotal.annotateI)
apply vcg
using [[simp-depth-limit = 2]]
apply  (rule conjI)
apply   clarify
apply   (simp (no-asm-use))
prefer 2
apply   clarify
apply   (rule-tac x=[] in exI)
apply   (rule-tac x=ns in exI)
apply   (simp (no-asm-use))
prefer 2
apply  clarify
apply  (rule conjI)
apply   clarify
apply   (rule conjI)
apply    (clarsimp simp add: List-list)
apply   (simp (no-asm-use))
apply   (rule conjI)
apply   assumption
prefer 2
apply   clarify
apply   (simp (no-asm-use))
apply   (rule conjI)
apply   (clarsimp simp add: List-list)
apply   (simp only: List-not-Null simp-thms triv-forall-equality)
apply   clarify
apply   (simp only: triv-forall-equality)
apply   (rename-tac sfx)
apply   (rule-tac x=prx@[nodeslist] in exI)
apply   (rule-tac x=sfx in exI)
apply   (rule conjI)
apply    assumption
apply   (rule conjI)
apply    simp
prefer 4
apply  (elim exE conjE)
apply  (simp (no-asm-use))
apply  hypsubst
```

using [[*simp-depth-limit* = *100*]]
proof −

fix *ns var low high rep next p nodeslist*
assume *ns*: *List nodeslist next ns*
assume *no-prop*: ∀ *no*∈*set ns*.
no ≠ *Null* ∧
(*low no* = *Null*) = (*high no* = *Null*) ∧
(*isLeaf-pt p low high* ⟶ *isLeaf-pt no low high*) ∧ *var no* = *var p*
assume *p-in-ns*: *p* ∈ *set ns*
assume *p-Leaf*: *isLeaf-pt p low high*
show *nodeslist* = *hd* [*sn*←*ns* . *repNodes-eq sn p low high rep*] ∧
var nodeslist = *var p*
proof −
from *p-in-ns no-prop* **have** *p-not-Null*: *p*≠*Null*
using [[*simp-depth-limit*=*2*]]
by *auto*
from *p-in-ns* **have** *ns* ≠ []
by (*cases ns*) *auto*
with *ns* **obtain** *ns′* **where** *ns′*: *ns* = *nodeslist*#*ns′*
by (*cases nodeslist*=*Null*) *auto*
with *no-prop p-Leaf* **obtain**
isLeaf-pt nodeslist low high **and**
var-eq: *var nodeslist* = *var p* **and**
nodeslist≠*Null*
using [[*simp-depth-limit*=*2*]]
by *auto*
with *p-not-Null p-Leaf* **have** *repNodes-eq nodeslist p low high rep*
by (*simp add*: *repNodes-eq-def isLeaf-pt-def null-comp-def*)
with *ns′ var-eq*
show *?thesis*
by *simp*
qed
next

fix *var*::*ref*⇒*nat* **and** *low high rep repa p prx sfx next*
assume *sfx*: *List Null next sfx*
assume *p-in-ns*: *p* ∈ *set* (*prx* @ *sfx*)
assume *no-props*: ∀ *no*∈*set* (*prx* @ *sfx*).
no ≠ *Null* ∧
(*low no* = *Null*) = (*high no* = *Null*) ∧
(*isLeaf-pt p low high* ⟶ *isLeaf-pt no low high*) ∧ *var no* = *var p*
assume *match-prx*: (∃ *pt*∈*set prx*. *repNodes-eq pt p low high rep*) ⟶
repa p = *hd* [*sn*←*prx* . *repNodes-eq sn p low high rep*] ∧
(∀ *pt*. *pt* ≠ *p* ⟶ *rep pt* = *repa pt*)
show *repa p* = *hd* [*sn*←*prx* @ *sfx* . *repNodes-eq sn p low high rep*] ∧
(∀ *pt*. *pt* ≠ *p* ⟶ *rep pt* = *repa pt*) ∧ *var* (*repa p*) = *var p*
proof −
from *sfx*

```
    have sfx-Nil: sfx=[]
      by simp
    with p-in-ns have ex-match: (∃ pt∈set prx. repNodes-eq pt p low high rep)
      apply −
      apply (rule-tac x=p in bexI)
      apply  (simp add: repNodes-eq-def)
      apply simp
      done
    hence not-empty: [sn←prx . repNodes-eq sn p low high rep] ≠ []
      apply −
      apply (erule bexE)
      apply (rule filter-not-empty)
      apply auto
      done
    from ex-match match-prx obtain
      found: repa p = hd [sn←prx . repNodes-eq sn p low high rep] and
      unmodif: ∀ pt. pt ≠ p ⟶ rep pt = repa pt
      by blast
    from hd-filter-in-list [OF not-empty] found
    have repa p ∈ set prx
      by simp
    with no-props
    have var (repa p) = var p
      using [[simp-depth-limit=2]]
      by simp
    with found unmodif sfx-Nil
    show ?thesis
      by simp
  qed
next

  fix var low high p repa next nodeslist prx sfx
  assume nodeslist-not-Null: nodeslist ≠ Null
  assume p-no-Leaf: ¬ isLeaf-pt p low high
  assume no-props: ∀ no∈set prx ∪ set (nodeslist # sfx).
          no ≠ Null ∧ (low no = Null) = (high no = Null) ∧ var no = var p
  assume p-in-ns: p ∈ set prx ∨ p ∈ set (nodeslist # sfx)
  assume match-prx: (∃ pt∈set prx. repNodes-eq pt p low high repa) ⟶
           repa p = hd [sn←prx . repNodes-eq sn p low high repa]
  assume nomatch-prx: ∀ pt∈set prx. ¬ repNodes-eq pt p low high repa
  assume nomatch-nodeslist: ¬ repNodes-eq nodeslist p low high repa
  assume sfx: List (next nodeslist) next sfx
  show (∀ no∈set prx ∪ set (nodeslist # sfx).
             no ≠ Null ∧ (low no = Null) = (high no = Null) ∧ var no = var p) ∧
       ((∃ pt∈set (prx @ [nodeslist]). repNodes-eq pt p low high repa) ⟶
          repa p = hd [sn←prx @ [nodeslist] . repNodes-eq sn p low high repa]) ∧
       (next nodeslist ≠ Null ⟶
          (∀ pt∈set (prx @ [nodeslist]). ¬ repNodes-eq pt p low high repa))
  proof −
```

```
    from nomatch-prx nomatch-nodeslist
    have ((∃ pt∈set (prx @ [nodeslist]). repNodes-eq pt p low high repa) ⟶
          repa p = hd [sn←prx @ [nodeslist] . repNodes-eq sn p low high repa])
      by auto
    moreover
    from nomatch-prx nomatch-nodeslist
    have (next nodeslist ≠ Null ⟶
          (∀ pt∈set (prx @ [nodeslist]). ¬ repNodes-eq pt p low high repa))
      by auto
    ultimately show ?thesis
      using no-props
      by (intro conjI)
  qed
next

  fix var low high p repa next nodeslist prx sfx
  assume nodeslist-not-Null: nodeslist ≠ Null
  assume sfx: List nodeslist next sfx
  assume p-not-Leaf: ¬ isLeaf-pt p low high
  assume no-props: ∀ no∈set prx ∪ set sfx.
          no ≠ Null ∧
          (low no = Null) = (high no = Null) ∧
          (isLeaf-pt p low high ⟶ isLeaf-pt no low high) ∧ var no = var p
  assume p-in-ns: p ∈ set prx ∨ p ∈ set sfx
  assume match-prx: (∃ pt∈set prx. repNodes-eq pt p low high repa) ⟶
        repa p = hd [sn←prx . repNodes-eq sn p low high repa]
  assume nomatch-prx: ∀ pt∈set prx. ¬ repNodes-eq pt p low high repa
  assume match: repNodes-eq nodeslist p low high repa
  show (∀ no∈set prx ∪ set sfx.
            no ≠ Null ∧
            (low no = Null) = (high no = Null) ∧
            (isLeaf-pt p low high ⟶ isLeaf-pt no low high) ∧ var no = var p) ∧
       (p ∈ set prx ∨ p ∈ set sfx) ∧
       ((∃ pt∈set prx ∪ set sfx. repNodes-eq pt p low high repa) ⟶
          nodeslist =
          hd ([sn←prx . repNodes-eq sn p low high repa] @
              [sn←sfx . repNodes-eq sn p low high repa])) ∧
       ((∀ pt∈set prx ∪ set sfx. ¬ repNodes-eq pt p low high repa) ⟶
          repa = repa(p := nodeslist))
  proof −
    from nodeslist-not-Null sfx
    obtain sfx′ where sfx′: sfx=nodeslist#sfx′
      by (cases nodeslist=Null) auto
    from nomatch-prx match sfx′
    have hd: hd ([sn←prx . repNodes-eq sn p low high repa] @
                 [sn←sfx . repNodes-eq sn p low high repa]) = nodeslist
      by simp
    from match sfx′
    have triv: ((∀ pt∈set prx ∪ set sfx. ¬ repNodes-eq pt p low high repa) ⟶
```

```
        repa = repa(p := nodeslist))
    by simp
  show ?thesis
    apply (rule conjI)
    apply (rule no-props)
    apply (intro conjI)
    apply  (rule p-in-ns)
    apply  (simp add: hd)
    apply (rule triv)
    done
 qed
qed

end
```

8 Proof of Procedure ShareReduceRepList

```
theory ShareReduceRepListProof imports ShareRepProof begin

lemma (in ShareReduceRepList-impl) ShareReduceRepList-modifies:
 shows ∀σ. Γ⊢{σ}  PROC ShareReduceRepList (nodeslist)
       {t. t may-only-modify-globals σ in [rep]}
 apply (hoare-rule HoarePartial.ProcRec1)
 apply (vcg spec=modifies)
 done

lemma hd-filter-app: ⟦filter P xs ≠ []; zs=xs@ys⟧ ⟹
      hd (filter P zs) =  hd (filter P xs)
 by (induct xs arbitrary: n m) auto


lemma (in ShareReduceRepList-impl) ShareReduceRepList-spec-total:
defines var-eq ≡ (λns var. (∀ no1 ∈ set ns. ∀ no2 ∈ set ns. no1→var = no2→var))
shows
 ∀σ ns. Γ⊢_t
  {|σ. List'nodeslist'next ns ∧
      (∀ no ∈ set ns.
           no ≠ Null ∧ ((no→'low = Null) = (no→'high = Null)) ∧
           no→'low ∉ set ns ∧ no→'high ∉ set ns ∧
           (isLeaf-pt no'low'high = (no→'var ≤ 1)) ∧
           (no→'low ≠ Null ⟶ (no→'low)→'rep ≠ Null) ∧
           ((rep ∝'low) no ∉ set ns)) ∧
       var-eq ns'var|}
   PROC  ShareReduceRepList (nodeslist)
  {|(∀ no. no ∉ set ns ⟶ no→^σrep = no→'rep)  ∧
   (∀ no ∈ set ns. no→'rep ≠ Null ∧
     (if ((rep ∝ ^σlow) no = (rep ∝ ^σhigh) no ∧ no→ ^σlow ≠ Null)
      then (no→'rep = (rep ∝ ^σlow) no )
      else ((no→'rep) ∈ set ns ∧ no→'rep→'rep = no→'rep ∧
```

```
            (∀ no1 ∈ set ns.
               ((rep ∝ σhigh) no1 = (rep ∝ σhigh) no ∧
               (rep ∝ σlow) no1 = (rep ∝ σlow) no) = (no→rep = no1→rep)))))}
apply (hoare-rule HoareTotal.ProcNoRec1)
apply (hoare-rule anno=
      'node :=='nodeslist;;
        WHILE (node ≠ Null )
        INV {∃ prx sfx. List'node'next sfx ∧
            List'nodeslist'next ns ∧ ns=prx@sfx ∧
            (∀ no ∈ set ns.
               no ≠ Null ∧ ((no→σlow = Null) = (no→σhigh = Null)) ∧
               no→σlow ∉ set ns ∧ no→σhigh ∉ set ns ∧
               (isLeaf-pt no σlow σhigh = (no→σvar ≤ 1)) ∧
               (no→σlow ≠ Null ⟶ (no→σlow)→σrep ≠ Null) ∧
               ((σrep ∝ σlow) no ∉ set ns)) ∧
            var-eq ns'var ∧
            (∀ no. no ∉ set prx ⟶ no→σrep = no →rep) ∧
            (∀ no ∈ set prx. no→rep ≠ Null ∧
             (if ((rep ∝ σlow) no = (rep ∝ σhigh) no ∧ no→σlow ≠ Null)
              then (no→rep = (rep ∝ σlow) no )
              else ((no→rep)=hd (filter (λsn. repNodes-eq sn no σlow σhigh'rep)
                                  prx) ∧
                   ((no→rep)→rep) = no→rep ∧
                   (∀ no1 ∈ set prx.
                      ((rep ∝ σhigh) no1 = (rep ∝ σhigh) no ∧
                       (rep ∝ σlow) no1 = (rep ∝ σlow) no) =
                       (no→rep = no1→rep))))) ∧
             'nodeslist= σnodeslist ∧'high=σhigh ∧'low=σlow ∧'var=σvar}
        VAR MEASURE (length (list'node'next))
        DO
        IF (¬ isLeaf-pt'node'low'high ∧
          'node →'low →'rep ='node →'high →'rep )
        THEN'node →'rep :=='node →'low →'rep
        ELSE CALL ShareRep (nodeslist ,'node)
        FI;;
       'node :='node →'next
        OD in HoareTotal.annotateI)
apply (vcg spec=spec-total)
apply  (rule-tac x=[] in exI)
apply  (rule-tac x=ns in exI)
using [[simp-depth-limit = 2]]
apply  (simp (no-asm-use))
prefer 2
using [[simp-depth-limit = 4]]
apply (clarsimp)
prefer 2
apply  (rule conjI)
apply  clarify
apply  (rule conjI)
```

apply (*clarsimp simp add*: *List-list*)
apply (*simp only*: *List-not-Null simp-thms triv-forall-equality*)
apply *clarify*
apply (*simp only*: *triv-forall-equality*)
apply (*rename-tac sfx*)
apply (*rule-tac x=prx@*[*node*] **in** *exI*)
apply (*rule-tac x=sfx* **in** *exI*)
apply (*rule conjI*)
apply *assumption*
apply (*rule conjI*)
apply (*simp* (*no-asm*))
apply (*rule conjI*)
apply (*assumption*)
prefer *2*
apply *clarify*
apply (*simp only*: *List-not-Null simp-thms triv-forall-equality*)
apply *clarify*
apply (*simp only*: *triv-forall-equality*)
apply (*rename-tac sfx*)
apply (*rule-tac x=prx@node#sfx* **in** *exI*)
apply (*rule conjI*)
apply *assumption*
apply (*rule conjI*)
apply (*rule ballI*)
apply (*frule-tac x=no* **in** *bspec*, *assumption*)
apply (*drule-tac x=node* **in** *bspec*)
apply (*simp* (*no-asm-use*))
apply (*elim conjE*)
apply (*rule conjI*)
apply *assumption*
apply (*rule conjI*)
apply *assumption*
apply (*unfold var-eq-def*)
apply (*drule-tac x=node* **in** *bspec*, *simp*)
apply (*drule-tac x=no* **in** *bspec*,*assumption*)
apply (*simp add*: *isLeaf-pt-def*)
apply (*rule conjI*)
apply (*simp* (*no-asm*))
apply (*clarify*)
apply (*rule conjI*)
apply (*subgoal-tac List node next* (*node#sfx*))
apply (*simp only*: *List-list*)
apply (*simp* (*no-asm*))
apply (*simp* (*no-asm-simp*))
apply (*rule-tac x=prx@*[*node*] **in** *exI*)
apply (*rule-tac x=sfx* **in** *exI*)
apply (*rule conjI*)
apply *assumption*
apply (*rule conjI*)

apply (*simp* (*no-asm*))
apply (*rule conjI*)
apply (*assumption*)
using [[*simp-depth-limit* = *100*]]
proof −
fix *var low high rep nodeslist ns repa next no*
assume *ns*: *List nodeslist next ns*
assume *no-in-ns*: *no* ∈ *set ns*
assume *while-inv*: ∀ *no*∈*set ns*.
repa no ≠ *Null* ∧
(*if* (*repa* ∝ *low*) *no* = (*repa* ∝ *high*) *no* ∧ *high no* ≠ *Null*
then repa no = (*repa* ∝ *low*) *no*
else repa no = *hd* [*sn*←*ns* . *repNodes-eq sn no low high repa*] ∧
repa (*repa no*) = *repa no* ∧
(∀ *no1*∈*set ns*.
((*repa* ∝ *high*) *no1* = (*repa* ∝ *high*) *no* ∧
(*repa* ∝ *low*) *no1* = (*repa* ∝ *low*) *no*) =
(*repa no* = *repa no1*)))
assume *pre*: ∀ *no*∈*set ns*.
no ≠ *Null* ∧
(*low no* = *Null*) = (*high no* = *Null*) ∧
low no ∉ *set ns* ∧
high no ∉ *set ns* ∧
isLeaf-pt no low high = (*var no* ≤ *Suc 0*) ∧
(*low no* ≠ *Null* ⟶ *rep* (*low no*) ≠ *Null*) ∧ (*rep* ∝ *low*) *no* ∉ *set ns*
assume *same-var*: ∀ *no1*∈*set ns*. ∀ *no2*∈*set ns*. *var no1* = *var no2*
assume *share-case*: (*repa* ∝ *low*) *no* = (*repa* ∝ *high*) *no* ⟶ *high no* = *Null*
assume *unmodif*: ∀ *no*. *no* ∉ *set ns* ⟶ *rep no* = *repa no*
show *hd* [*sn*←*ns* . *repNodes-eq sn no low high repa*] ∈ *set ns* ∧
repa (*hd* [*sn*←*ns* . *repNodes-eq sn no low high repa*]) =
hd [*sn*←*ns* . *repNodes-eq sn no low high repa*]
proof −
from *no-in-ns pre* **obtain**
no-nNull: *no* ≠ *Null* **and**
no-balanced: (*low no* = *Null*) = (*high no* = *Null*) **and**
isLeaf-var: *isLeaf-pt no low high* = (*var no* ≤ *Suc 0*)
by *blast*
have *repNodes-eq-same-node*: *repNodes-eq no no low high repa*
by (*simp add*: *repNodes-eq-def*)
from *no-in-ns* **have** *ns-nempty*: *ns* ≠ []
by *auto*
from *no-in-ns repNodes-eq-same-node*
have *repNodes-not-empty*: [*sn*←*ns* . *repNodes-eq sn no low high repa*] ≠ []
by (*rule filter-not-empty*)
then have *hd-term-in-ns*: *hd* [*sn*←*ns* . *repNodes-eq sn no low high repa*] ∈ *set ns*
by (*rule hd-filter-in-list*)
with *while-inv* **obtain**
repa-hd-nNull: *repa* (*hd* [*sn*←*ns* . *repNodes-eq sn no low high repa*]) ≠ *Null*

```
    by auto
  let ?hd = hd [sn←ns . repNodes-eq sn no low high repa]
  from hd-term-in-ns  pre obtain
    hd-nNull:  ?hd ≠ Null and
    hd-balanced:
      (low (hd [sn←ns . repNodes-eq sn no low high repa]) = Null) =
       (high (hd [sn←ns . repNodes-eq sn no low high repa]) = Null) and
    hd-isLeaf-var:
    isLeaf-pt (hd [sn←ns . repNodes-eq sn no low high repa]) low high =
    (var (hd [sn←ns . repNodes-eq sn no low high repa]) ≤ Suc 0)
    by blast
  have repa (hd [sn←ns . repNodes-eq sn no low high repa]) =
    hd [sn←ns . repNodes-eq sn no low high repa]
  proof (cases high no = Null)
    case True
    with no-balanced have low no = Null
      by simp
    with True have no-Leaf: isLeaf-pt no low high
      by (simp add: isLeaf-pt-def)
    with isLeaf-var have varno: var no <= 1
      by simp
    from same-var [rule-format, OF no-in-ns hd-term-in-ns] varno
    have var (hd [sn←ns . repNodes-eq sn no low high repa]) ≤ 1
      by simp
    with hd-isLeaf-var have
      isLeaf-pt (hd [sn←ns . repNodes-eq sn no low high repa]) low high
      by simp
    with while-inv hd-term-in-ns repNodes-not-empty show ?thesis
      apply (simp add: isLeaf-pt-def)
      apply (erule-tac x=
        hd [sn←ns . repNodes-eq sn no low high repa] in ballE)
      prefer 2
      apply simp
      apply (simp (no-asm-use) add: repNodes-eq-def)
      apply (rule filter-hd-P-rep-indep)
      apply   (simp (no-asm-simp))
      apply  (simp (no-asm-simp))
      apply assumption
      done
  next
    assume hno-nNull:  high no ≠ Null
    with share-case have repchildren-neq: (repa ∝ low) no ≠ (repa ∝ high) no
      by simp
    from repNodes-not-empty have
       repNodes-eq  (hd [sn←ns . repNodes-eq sn no low high repa]) no low high
repa
      by (rule hd-filter-prop)
    then
    have (repa ∝ low) (hd [sn←ns . repNodes-eq sn no low high repa]) =
```

```
          (repa ∝ low) no ∧
         (repa ∝ high) (hd [sn←ns . repNodes-eq sn no low high repa]) =
          (repa ∝ high) no
    by (simp add: repNodes-eq-def)
   with repchildren-neq have
    (repa ∝ low) (hd [sn←ns . repNodes-eq sn no low high repa])
    ≠ (repa ∝ high) (hd [sn←ns . repNodes-eq sn no low high repa])
    by simp
   with while-inv hd-term-in-ns repNodes-not-empty show ?thesis
    apply (simp add: isLeaf-pt-def)
    apply (erule-tac x=
      hd [sn←ns . repNodes-eq sn no low high repa] in ballE)
    prefer 2
    apply simp
    apply (simp (no-asm-use) add: repNodes-eq-def)
    apply (rule filter-hd-P-rep-indep)
    apply simp
    apply fastforce
    apply fastforce
    done
  qed
  with hd-term-in-ns
  show ?thesis
   by simp
 qed
next

 fix var low high rep nodeslist repa next node prx sfx
 assume ns: List nodeslist next (prx @ node # sfx)
 assume sfx: List (next node) next sfx
 assume node-not-Null: node ≠ Null
 assume nodes-balanced-ordered: ∀ no∈set (prx @ node # sfx).
        no ≠ Null ∧
        (low no = Null) = (high no = Null) ∧
        low no ∉ set (prx @ node # sfx) ∧
        high no ∉ set (prx @ node # sfx) ∧
        isLeaf-pt no low high = (var no ≤ (1::nat)) ∧
        (low no ≠ Null ⟶ rep (low no) ≠ Null) ∧
        (rep ∝ low) no ∉ set (prx @ node # sfx)
 assume all-nodes-same-var: ∀ no1∈set (prx @ node # sfx).
       ∀ no2∈set (prx @ node # sfx). var no1 = var no2
 assume rep-repa-nc: ∀ no. no ∉ set prx ⟶ rep no = repa no
 assume while-inv: ∀ no∈set prx.
        repa no ≠ Null ∧
        (if (repa ∝ low) no = (repa ∝ high) no ∧ low no ≠ Null
         then repa no = (repa ∝ low) no
         else repa no = hd [sn←prx . repNodes-eq sn no low high repa] ∧
              repa (repa no) = repa no ∧
              (∀ no1∈set prx.
```

((*repa* ∝ *high*) *no1* = (*repa* ∝ *high*) *no* ∧
(*repa* ∝ *low*) *no1* = (*repa* ∝ *low*) *no*) =
(*repa no* = *repa no1*)))
assume *not-Leaf*: ¬ *isLeaf-pt node low high*
assume *repchildren-eq-nln*: *repa* (*low node*) = *repa* (*high node*)
show (∀ *no*. *no* ∉ *set* (*prx* @ [*node*]) ⟶
rep no = (*repa*(*node* := *repa* (*high node*))) *no*) ∧
(∀ *no*∈*set* (*prx* @ [*node*]).
(*repa*(*node* := *repa* (*high node*))) *no* ≠ *Null* ∧
(*if* (*repa*(*node* := *repa* (*high node*)) ∝ *low*) *no* =
(*repa*(*node* := *repa* (*high node*)) ∝ *high*) *no* ∧
low no ≠ *Null*
then (*repa*(*node* := *repa* (*high node*))) *no* =
(*repa*(*node* := *repa* (*high node*)) ∝ *low*) *no*
else (*repa*(*node* := *repa* (*high node*))) *no* =
hd [*sn*←*prx* @ [*node*] .
repNodes-eq sn no low high
(*repa*(*node* := *repa* (*high node*)))] ∧
(*repa*(*node* := *repa* (*high node*)))
((*repa*(*node* := *repa* (*high node*))) *no*) =
(*repa*(*node* := *repa* (*high node*))) *no* ∧
(∀ *no1*∈*set* (*prx* @ [*node*]).
((*repa*(*node* := *repa* (*high node*)) ∝ *high*) *no1* =
(*repa*(*node* := *repa* (*high node*)) ∝ *high*) *no* ∧
(*repa*(*node* := *repa* (*high node*)) ∝ *low*) *no1* =
(*repa*(*node* := *repa* (*high node*)) ∝ *low*) *no*) =
((*repa*(*node* := *repa* (*high node*))) *no* =
(*repa*(*node* := *repa* (*high node*))) *no1*))))
(**is** *?NodesUnmodif* ∧ *?NodesModif*)
proof −
— This proof was originally conducted without the substitution *repa* (*low node*) = *repa* (*high node*) preformed. So don't be confused if we show everythin for *repa* (*low node*).
from *rep-repa-nc*
have *nodes-unmodif*: *?NodesUnmodif*
by *auto*
hence *rep-Sucna-nc*:
(∀ *no*. *no* ∉ *set* (*prx* @ [*node*])
⟶ *rep no* = (*repa*(*node* := *repa* (*low* (*node*)))) *no*)
by *auto*
have *nodes-modif*: *?NodesModif* (**is** ∀ *no*∈*set* (*prx* @ [*node*]). *?P no* ∧ *?Q no*)
proof (*rule ballI*)
fix *no*
assume *no-in-take-Sucna*: *no* ∈ *set* (*prx* @ [*node*])
show *?P no* ∧ *?Q no*
proof (*cases no* = *node*)
case *False*
note *no-noteq-nln=this*
with *no-in-take-Sucna*

have *no-in-take-n*: $no \in set\ prx$
by *auto*
with *no-in-take-n while-inv* **obtain**
repa-no-nNull: $repa\ no \neq Null$ **and**
repa-cases: $(if\ (repa \propto low)\ no = (repa \propto high)\ no \wedge low\ no \neq Null$
$then\ repa\ no = (repa \propto low)\ no$
$else\ \ repa\ no = hd\ [sn{\leftarrow}prx\ .\ repNodes\text{-}eq\ sn\ no\ low\ high\ repa]$
$\wedge\ repa\ (repa\ no) = repa\ no \wedge$
$(\forall\ no1{\in}set\ prx.\ ((repa \propto high)\ no1 = (repa \propto high)\ no$
$\wedge\ (repa \propto low)\ no1 = (repa \propto low)\ no)$
$= (repa\ no = repa\ no1)))$
using $[[simp\text{-}depth\text{-}limit = 2]]$
by *auto*
from *no-in-take-n*
have *no-in-nodeslist*: $no \in set\ (prx\ @\ node\ \#\ sfx)$
by *auto*
from *repa-no-nNull no-noteq-nln* **have** *ext-repa-nNull*: *?P no*
by *auto*
from *no-in-nodeslist nodes-balanced-ordered* **obtain**
nln-nNull: $node \neq Null$ **and**
nln-balanced-children: $(low\ node = Null) = (high\ node = Null)$ **and**
lnln-notin-nodeslist: $low\ node \notin set\ (prx\ @\ node\ \#\ sfx)$ **and**
hnln-notin-nodeslist: $high\ node \notin set\ (prx\ @\ node\ \#\ sfx)$ **and**
isLeaf-var-nln: $isLeaf\text{-}pt\ node\ low\ high = (var\ node \leq 1)$ **and**
node-nNull-rap-nNull-nln: $(low\ node \neq Null$
$\longrightarrow rep\ (low\ node) \neq Null)$ **and**
nln-varrep-le-var: $(rep \propto low)\ node \notin set\ (prx\ @\ node\ \#\ sfx)$
apply $(erule\text{-}tac\ x{=}node$ **in** $ballE)$
apply *auto*
done
from *no-in-nodeslist nodes-balanced-ordered no-in-take-Sucna*
obtain
no-nNull: $no \neq Null$ **and**
balanced-children: $(low\ no = Null) = (high\ no = Null)$ **and**
lno-notin-nodeslist: $low\ no \notin set\ (prx\ @\ node\ \#\ sfx)$ **and**
hno-notin-nodeslist: $high\ no \notin set\ (prx\ @\ node\ \#\ sfx)$ **and**
isLeaf-var-no: $isLeaf\text{-}pt\ no\ low\ high = (var\ no \leq 1)$ **and**
node-nNull-rep-nNull: $(low\ no \neq Null \longrightarrow rep\ (low\ no) \neq Null)$ **and**
varrep-le-var: $(rep \propto low)\ no \notin set\ (prx\ @\ node\ \#\ sfx)$
apply $-$
apply $(erule\text{-}tac\ x{=}no$ **in** $ballE)$
apply *auto*
done
from *lno-notin-nodeslist*
have *ext-rep-null-comp-low*:
$(repa\ (node := repa\ (low\ node)) \propto low)\ no = (repa \propto low)\ no$
by $(auto\ simp\ add{:}\ null\text{-}comp\text{-}def)$
from *hno-notin-nodeslist*
have *ext-rep-null-comp-high*:

```
  (repa (node := repa (low node)) ∝ high) no = (repa ∝ high) no
  by (auto simp add: null-comp-def)
have share-reduce-if: ?Q no
proof (cases (repa (node := repa (low node)) ∝ low) no =
    (repa(node := repa (low node)) ∝ high) no ∧ low no ≠ Null)
  case True
  then obtain
    red-case: (repa (node := repa (low node)) ∝ low) no =
    (repa(node := repa (low node)) ∝ high) no and
    lno-nNull: low no ≠ Null
    by simp
  from lno-nNull balanced-children have hno-nNull: high no ≠ Null
    by simp
  from True ext-rep-null-comp-low ext-rep-null-comp-high
  have repchildren-eq-no: (repa ∝ low) no = (repa ∝ high) no
    by simp
  with repa-cases lno-nNull have repa no = (repa ∝ low) no
    by auto
  with ext-rep-null-comp-low no-noteq-nln
  have (repa(node := repa (low node))) no =
    (repa (node := repa (low node)) ∝ low) no
    by simp
  with True repchildren-eq-nln show ?thesis
    by auto
next
  assume share-case-ext:
     ¬ ((repa(node := repa (low node)) ∝ low) no =
    (repa(node := repa (low node)) ∝ high) no ∧ low no ≠ Null)
  from not-Leaf isLeaf-var-nln
  have 1 < var node
    by simp
  with all-nodes-same-var
  have all-nodes-nl-Suc0-l-var: ∀ x ∈ set (prx @ node # sfx). 1 < var x
    using [[simp-depth-limit=1]]
    by auto
  with nodes-balanced-ordered
  have all-nodes-nl-noLeaf:
    ∀ x ∈ set (prx @ node # sfx). ¬ isLeaf-pt x low high
    apply −
    apply rule
    apply (drule-tac x=x in bspec,assumption)
    apply (drule-tac x=x in bspec,assumption)
    apply auto
    done
  from nodes-balanced-ordered
  have all-nodes-nl-balanced:
    ∀ x ∈ set (prx @ node # sfx). (low x = Null) = (high x = Null)
    apply −
    apply rule
```

apply (*drule-tac x=x* **in** *bspec,assumption*)
apply *auto*
done
from *all-nodes-nl-Suc0-l-var no-in-nodeslist*
have *Suc0-l-var-no*: *1 < var no*
by *auto*
with *isLeaf-var-no* **have** *no-nLeaf*: ¬ *isLeaf-pt no low high*
by *simp*
with *balanced-children* **have** *lno-nNull*: *low no ≠ Null*
by (*simp add*: *isLeaf-pt-def*)
with *balanced-children* **have** *hno-nNull*: *high no ≠ Null*
by *simp*
with *share-case-ext ext-rep-null-comp-low ext-rep-null-comp-high lno-nNull*

have *repchildren-neq-no*: (*repa ∝ low*) *no ≠* (*repa ∝ high*) *no*
by (*simp add*: *null-comp-def*)
with *repa-cases*
have *share-case-inv*:
repa no = hd [*sn←prx . repNodes-eq sn no low high repa*] ∧
repa (*repa no*) = *repa no* ∧
(∀ *no1∈set prx*. ((*repa ∝ high*) *no1* = (*repa ∝ high*) *no* ∧
(*repa ∝ low*) *no1* = (*repa ∝ low*) *no*) = (*repa no* = *repa no1*))
by *auto*
then have *repa-no*: *repa no = hd* [*sn←prx . repNodes-eq sn no low high repa*]
by *simp*
from *Suc0-l-var-no* **have** ∀ *x ∈ set* (*prx* @ *node # sfx*). *1 < var no*
by *auto*
from *no-in-take-n* **have** [*sn←prx . repNodes-eq sn no low high repa*] ≠ []
apply −
apply (*rule filter-not-empty*)
apply (*auto simp add*: *repNodes-eq-def*)
done
then have *repNodes-eq*
(*hd* [*sn←prx. repNodes-eq sn no low high repa*]) *no low high repa*
by (*rule hd-filter-prop*)
with *repa-no*
have *rep-children-eq-no-repa-no*:
(*repa ∝ low*) (*repa no*) = (*repa ∝ low*) *no* ∧
(*repa ∝ high*) (*repa no*) = (*repa ∝ high*) *no*
by (*simp add*: *repNodes-eq-def*)
from *lno-notin-nodeslist rep-repa-nc*
have *rep-repa-nc-low-no*: *rep* (*low no*) = *repa* (*low no*)
apply −
apply (*erule-tac x=low no* **in** *allE*)
apply *auto*
done
have ∀ *x ∈ set* (*prx* @ [*node*]).
repNodes-eq x no low high (*repa*(*node* := *repa* (*low node*))) =

```
  repNodes-eq x no low high repa
proof (rule ballI, unfold repNodes-eq-def)
  fix x
  assume x-in-take-Sucn:  x ∈ set (prx @ [node])
  hence x-in-nodeslist: x ∈ set (prx @ node # sfx)
    by auto
  with all-nodes-nl-noLeaf nodes-balanced-ordered
  have children-nNull-x: low x ≠ Null ∧ high x ≠ Null
    apply −
    apply (drule-tac x=x in bspec,assumption)
    apply (drule-tac x=x in bspec,assumption)
    apply (auto simp add: isLeaf-pt-def)
    done
  from x-in-nodeslist nodes-balanced-ordered
 have low x ∉ set (prx @ node # sfx) ∧ high x ∉ set (prx @ node # sfx)
    apply −
    apply (drule-tac x=x in bspec,assumption)
    apply auto
    done
  with lno-notin-nodeslist hno-notin-nodeslist
    children-nNull-x lno-nNull hno-nNull
  show ((repa(node := repa (low node)) ∝ high) x =
    (repa(node := repa (low node)) ∝ high) no ∧
    (repa(node := repa (low node)) ∝ low) x =
    (repa(node := repa (low node)) ∝ low) no) =
    ((repa ∝ high) x = (repa ∝ high) no ∧
    (repa ∝ low) x = (repa ∝ low) no)
    by (simp add: null-comp-def)
qed
then have filter-extrep-rep:
  [sn←(prx @ [node]). repNodes-eq sn no low high
                        (repa(node := repa (low node)))] =
  [sn←(prx @ [node]) . repNodes-eq sn no low high repa]
  by (rule P-eq-list-filter)
from no-in-take-n
have filter-n-notempty: [sn←prx. repNodes-eq sn no low high repa] ≠ []
  apply (rule filter-not-empty)
  apply (simp add: repNodes-eq-def)
  done
then have hd [sn←prx. repNodes-eq sn no low high repa] =
  hd [sn←prx@[node]. repNodes-eq sn no low high repa]
  by auto
with no-noteq-nln filter-extrep-rep repa-no
have ext-repa-no: (repa(node:= repa (low node))) no =
  hd [sn←prx@[node] . repNodes-eq sn no low high
  (repa(node := repa (low node)))]
  by simp
have (repa(node := repa (low node))) (repa no) = repa no
proof (cases repa no =  node)
```

case *True*
note *rno-nln=this*
from *rep-repa-nc-low-no rep-children-eq-no-repa-no lno-nNull node-nNull-rep-nNull*
have *low-rep-no-nNull*: $low\ (repa\ no) \neq Null$
apply (*simp add*: *null-comp-def*)
apply *auto*
done
with *nodes-balanced-ordered rno-nln*
have *high-rap-no-nNull*: $high\ (repa\ no) \neq Null$
apply −
apply (*drule-tac x=repa no* **in** *bspec*)
apply *auto*
done
with *low-rep-no-nNull rno-nln rep-children-eq-no-repa-no*
have $repa\ (low\ node) = (repa \propto low)\ no \wedge$
$repa\ (high\ node) = (repa \propto high)\ no$
by (*simp add*: *null-comp-def*)
with *repchildren-eq-nln* **have** $(repa \propto low)\ no = (repa \propto high)\ no$
by *simp*
with *repchildren-neq-no* **show** *?thesis*
by *simp*
next
assume *rno-not-nln*: $repa\ no \neq node$
from *share-case-inv* **have** $repa\ (repa\ no) = repa\ no$
by *auto*
with *rno-not-nln* **show** *?thesis*
by *simp*
qed
with *no-noteq-nln* **have** *ext-repa-ext-repa*:
$(repa(node := repa\ (low\ node)))$
$((repa(node := repa\ (low\ node)))\ no)$
$= (repa(node := repa\ (low\ node)))\ no$
by *simp*
have $(\forall\ no1 \in set\ (prx@[node]).$
$((repa(node := repa\ (low\ node)) \propto high)\ no1 =$
$(repa(node := repa\ (low\ node)) \propto high)\ no \wedge$
$(repa(node := repa\ (low\ node)) \propto low)\ no1 =$
$(repa(node := repa\ (low\ node)) \propto low)\ no) =$
$((repa(node := repa\ (low\ node)))\ no =$
$(repa(node := repa\ (low\ node)))\ no1))$
proof (*rule ballI*)
fix *no1*
assume *no1-in-take-Sucn*: $no1 \in set\ (prx@[node])$
hence *no1-in-nodeslist*: $no1 \in set\ (prx\ @\ node\ \#\ sfx)$
by *auto*
show $((repa(node := repa\ (low\ node)) \propto high)\ no1 =$
$(repa(node := repa\ (low\ node)) \propto high)\ no \wedge$
$(repa(node := repa\ (low\ node)) \propto low)\ no1 =$

```
    (repa(node := repa (low node)) ∝ low) no) =
    ((repa(node := repa (low node))) no =
     (repa(node := repa (low node))) no1)
proof (cases no1 = node)
  case True
  show ?thesis
  proof (rule, elim conjE)
    assume ext-repa-no-no1:
      (repa(node := repa (low node))) no =
        (repa(node := repa (low node))) no1
    with True no-noteq-nln
    have repa-no-repa-low-nln: repa no = repa (low node)
      by simp
    from filter-n-notempty
    have repa-no-in-take-n:
      hd [sn←prx. repNodes-eq sn no low high repa]
        ∈ set prx
      apply −
      apply (rule hd-filter-in-list)
      apply auto
      done
    with repa-no
    have repa-no-in-nodeslist: repa no ∈ set (prx @ node # sfx)
      by auto
    from lnln-notin-nodeslist rep-repa-nc
    have rep-repa-low-nln: rep (low node) = repa (low node)
      by auto
    from all-nodes-nl-noLeaf nln-balanced-children
    have low node ≠ Null
      by (auto simp add: isLeaf-pt-def)
    with rep-repa-low-nln lnln-notin-nodeslist lno-nNull
      nln-varrep-le-var
    have repa (low node) ∉ set (prx @ node # sfx)
      by (simp add: null-comp-def)
    with repa-no-repa-low-nln repa-no-in-nodeslist
    show (repa(node := repa (low node)) ∝ high) no1 =
      (repa(node := repa (low node)) ∝ high) no ∧
      (repa(node := repa (low node)) ∝ low) no1 =
      (repa(node := repa (low node)) ∝ low) no
      by simp
  next
    assume no-no1-high:
      (repa(node := repa (low node)) ∝ high) no1 =
      (repa(node := repa (low node)) ∝ high) no
    assume no-no1-low:
      (repa(node := repa (low node)) ∝ low) no1 =
      (repa(node := repa (low node)) ∝ low) no
    from True repchildren-eq-nln
    have repachildren-eq-no1:  repa (low no1) = repa (high no1)
```

```
      by simp
    from not-Leaf True nln-balanced-children
    have children-nNull-no1: (low no1) ≠ Null ∧ high no1 ≠ Null
      by (simp add: isLeaf-pt-def)
    with repachildren-eq-no1
    have repchildren-eq-no1: (repa ∝ low) no1 = (repa ∝ high) no1
      by (simp add: null-comp-def)
    from no-no1-low children-nNull-no1 lno-nNull
      lnln-notin-nodeslist lno-notin-nodeslist True
    have rep-low-eq-no-no1: (repa ∝ low) no1 = (repa ∝ low) no
      by (simp add: null-comp-def)
    from no-no1-high children-nNull-no1 hno-nNull
      hnln-notin-nodeslist hno-notin-nodeslist True
    have rep-high-eq-no-no1: (repa ∝ high) no1 = (repa ∝ high) no
      by (simp add: null-comp-def)
    with rep-low-eq-no-no1 repchildren-eq-no1
    have (repa ∝ low) no = (repa ∝ high) no
      by simp
    with repchildren-neq-no
    show (repa(node := repa (low node))) no =
      (repa(node := repa (low node))) no1
      by simp
  qed
next
  assume no1-neq-nln: no1 ≠ node
  from no1-in-nodeslist nodes-balanced-ordered
  have children-notin-nl-no1:
 low no1 ∉ set (prx @ node # sfx) ∧ high no1 ∉ set (prx @ node # sfx)
    apply −
    apply (drule-tac x=no1 in bspec,assumption)
    by auto
  from no1-neq-nln no1-in-take-Sucn
  have no1-in-take-n: no1 ∈ set prx
    by auto
  from no1-in-nodeslist all-nodes-nl-noLeaf all-nodes-nl-balanced
  have children-nNull-no1: (low no1) ≠ Null ∧ high no1 ≠ Null
    by  (auto simp add: isLeaf-pt-def)
  show ?thesis
  proof (rule, elim conjE)
    assume ext-repa-high-no1-no:
    (repa(node := repa (low node)) ∝ high) no1
      = (repa(node := repa (low node)) ∝ high) no
    assume ext-repa-low-no1-no:
      (repa(node := repa (low node)) ∝ low) no1
      = (repa(node := repa (low node)) ∝ low) no
    from children-nNull-no1 hno-nNull ext-repa-high-no1-no
      children-notin-nl-no1
      hno-notin-nodeslist
    have repa-high-no1-no: (repa ∝ high) no1 = (repa ∝ high) no
```

```
        by (simp add: null-comp-def)
      from children-nNull-no1 lno-nNull ext-repa-low-no1-no
        children-notin-nl-no1 lno-notin-nodeslist
      have repa-low-no1-no: (repa ∝ low) no1 = (repa ∝ low) no
        by (simp add: null-comp-def)
      from repchildren-neq-no repa-high-no1-no repa-low-no1-no
      have (repa ∝ low) no1 ≠ (repa ∝ high) no1
        by simp
      from no1-in-take-n share-case-inv repa-high-no1-no repa-low-no1-no
      have repa no = repa no1
        by auto
      with no-noteq-nln no1-neq-nln
      show  (repa(node := repa (low node))) no =
        (repa(node := repa (low node))) no1
        by simp
    next
      assume (repa(node := repa (low node))) no =
        (repa(node := repa (low node))) no1
      with no-noteq-nln no1-neq-nln
      have repa no = repa no1
        by simp
      with share-case-inv no1-in-take-n
      have ((repa ∝ high) no1 = (repa ∝ high) no ∧
        (repa ∝ low) no1 = (repa ∝ low) no)
        by auto
      with children-notin-nl-no1 children-nNull-no1 lno-notin-nodeslist
        hno-notin-nodeslist lno-nNull hno-nNull
      show (repa(node := repa (low node)) ∝ high) no1 =
      (repa(node := repa (low node)) ∝ high) no ∧
      (repa(node := repa (low node)) ∝ low) no1 =
      (repa(node := repa (low node)) ∝ low) no
        by (auto simp add: null-comp-def)
    qed
  qed
 qed
 from ext-repa-ext-repa ext-repa-no share-case-ext repchildren-eq-nln this
 show ?thesis
  using [[simp-depth-limit=4]]
  by auto
qed
with ext-repa-nNull show ?thesis
 by auto
next
 assume no-nln: no = node
 hence no-in-nodeslist: no ∈ set (prx @ node # sfx)
  by simp
 from no-nln not-Leaf no-in-nodeslist
  nodes-balanced-ordered [rule-format, OF this] obtain
  low-no-nNull: low no ≠ Null and
```

```
  high-no-nNull: high no ≠ Null and
  rep-low-no-nNull: rep (low no) ≠ Null and
  lno-notin-nl: low no ∉ set (prx @ node # sfx) and
  hno-notin-nl: high no ∉ set (prx @ node # sfx) and
  children-nNull-no: (low no ≠ Null) ∧ (high no ≠ Null)
  apply (unfold isLeaf-pt-def)
  apply blast
  done
then have low no ∉ set prx
  by auto
with rep-repa-nc no-nln rep-low-no-nNull
have (repa(node := repa (low node))) no ≠ Null
  by simp
moreover
have (if (repa(node := repa (low node)) ∝ low) no =
         (repa(node := repa (low node)) ∝ high) no ∧ low no ≠ Null
  then (repa(node := repa (low node))) no =
       (repa(node := repa (low node)) ∝ low) no
  else (repa(node := repa (low node))) no =
    hd [sn←prx@[node]. repNodes-eq sn no low high
        (repa(node := repa (low node)))] ∧
  (repa(node := repa (low node)))
    ((repa(node := repa (low node))) no) =
    (repa(node := repa (low node))) no ∧
  (∀ no1∈set (prx@[node]).
    ((repa(node := repa (low node)) ∝ high) no1 =
    (repa(node := repa (low node)) ∝ high) no ∧
    (repa(node := repa (low node)) ∝ low) no1 =
    (repa(node := repa (low node)) ∝ low) no) =
    ((repa(node := repa (low node))) no =
    (repa(node := repa (low node))) no1)))
proof (cases (repa(node := repa (low node)) ∝ low) no =
  (repa(node := repa (low node)) ∝ high) no ∧ low no ≠ Null)
  case True
  note red-case=this
  with children-nNull-no lno-notin-nl hno-notin-nl
  have (repa ∝ low) no = (repa ∝ high) no
    by (auto simp add: null-comp-def)
  from children-nNull-no lno-notin-nl
  have ext-repa-eq-repa-low: (repa(node := repa (low node)) ∝ low) no
    = (repa ∝ low) no
    by (auto simp add: null-comp-def)
  from children-nNull-no hno-notin-nl
  have ext-repa-eq-repa-high:
    (repa(node := repa (low node)) ∝ high) no
    = (repa ∝ high) no
    by (auto simp add: null-comp-def)
  from no-nln children-nNull-no
  have repa (low node) = (repa ∝ low) no
```

```
        by (simp add: null-comp-def)
      with red-case ext-repa-eq-repa-high ext-repa-eq-repa-low no-nln
      show ?thesis
        using [[simp-depth-limit=2]]
        by (auto simp del: null-comp-not-Null)
    next
      assume share-case: ¬ ((repa(node := repa (low node)) ∝ low) no
        = (repa(node := repa (low node)) ∝ high) no ∧ low no ≠ Null)
      with low-no-nNull have (repa(node := repa (low node)) ∝ low) no
        ≠ (repa(node := repa (low node)) ∝ high) no
        by simp
      with children-nNull-no lno-notin-nl hno-notin-nl
      have (repa ∝ low) no ≠ (repa ∝ high) no
        by (auto simp add: null-comp-def)
      with children-nNull-no have repa (low no) ≠ repa (high no)
        by (simp add: null-comp-def)
      with repchildren-eq-nln no-nln show ?thesis
        by simp
    qed
    ultimately show ?thesis
      using repchildren-eq-nln
      apply −
      apply (simp only:)
      apply (simp (no-asm))
      done
  qed
 qed
 from nodes-unmodif nodes-modif
 show ?thesis by iprover
qed
next
 fix var low high rep nodeslist repa next node prx sfx repb
 assume ns: List nodeslist next (prx @ node # sfx)
 assume sfx: List (next node) next sfx
 assume nodes-balanced-ordered: ∀ no∈set (prx @ node # sfx).
        no ≠ Null ∧
        (low no = Null) = (high no = Null) ∧
        low no ∉ set (prx @ node # sfx) ∧
        high no ∉ set (prx @ node # sfx) ∧
        isLeaf-pt no low high = (var no ≤ (1::nat)) ∧
        (low no ≠ Null ⟶ rep (low no) ≠ Null) ∧
        (rep ∝ low) no ∉ set (prx @ node # sfx)
 assume all-nodes-same-var: ∀ no1∈set (prx @ node # sfx).
        ∀ no2∈set (prx @ node # sfx). var no1 = var no2
 assume rep-repa-nc: ∀ no. no ∉ set prx ⟶ rep no = repa no
 assume while-inv: ∀ no∈set prx.
        repa no ≠ Null ∧
        (if (repa ∝ low) no = (repa ∝ high) no ∧ low no ≠ Null
         then repa no = (repa ∝ low) no
```

```
        else repa no = hd [sn←prx . repNodes-eq sn no low high repa] ∧
            repa (repa no) = repa no ∧
            (∀ no1∈set prx.
                ((repa ∝ high) no1 = (repa ∝ high) no ∧
                 (repa ∝ low) no1 = (repa ∝ low) no) =
                (repa no = repa no1)))
assume share-cond:
  ¬ (¬ isLeaf-pt node low high ∧ repa (low node) = repa (high node))
assume repb-node:
      repb node = hd [sn←prx @ node # sfx . repNodes-eq sn node low high repa]
assume repa-repb-nc: ∀ pt. pt ≠ node ⟶ repa pt = repb pt
assume var-repb-node: var (repb node) = var node
show (∀ no. no ∉ set (prx @ [node]) ⟶ rep no = repb no) ∧
       (∀ no∈set (prx @ [node]).
          repb no ≠ Null ∧
          (if (repb ∝ low) no = (repb ∝ high) no ∧ low no ≠ Null
           then repb no = (repb ∝ low) no
           else repb no =
               hd [sn←prx @ [node] . repNodes-eq sn no low high repb] ∧
               repb (repb no) = repb no ∧
               (∀ no1∈set (prx @ [node]).
                   ((repb ∝ high) no1 = (repb ∝ high) no ∧
                    (repb ∝ low) no1 = (repb ∝ low) no) =
                   (repb no = repb no1))))
proof −
  have rep-repb-nc: (∀ no. no ∉ set (prx @ [node]) ⟶ rep no = repb no)
  proof (intro allI impI)
    fix no
    assume no-notin-take-Sucn: no ∉ set (prx @ [node])
    with rep-repa-nc
    have rep-repa-nc-Sucn: rep no = repa no
      by auto
    from no-notin-take-Sucn have no ≠ node
      by auto
    with repa-repb-nc have repa no = repb no
      by auto
    with rep-repa-nc-Sucn show rep no = repb no
      by simp
  qed
  moreover
  have repb-no-share-def:
    (∀ no∈set (prx @ [node]).
    ¬ ((repb ∝ low) no = (repb ∝ high) no ∧ low no ≠ Null) ⟶
        repb no = hd [sn←(prx @ [node]) . repNodes-eq sn no low high repb])
  proof (intro ballI impI)
    fix no
    assume no-in-take-Sucn:  no ∈ set (prx @ [node])
   assume share-prop: ¬ ((repb ∝ low) no = (repb ∝ high) no ∧ low no ≠ Null)
    from share-prop have share-or:
```

```
  (repb ∝ low) no ≠ (repb ∝ high) no ∨ low no = Null
  using [[simp-depth-limit=2]]
  by simp
from no-in-take-Sucn have no-in-nl: no ∈ set (prx @ node # sfx)
  by auto
from nodes-balanced-ordered [rule-format, OF this] obtain
  no-nNull: no ≠ Null and
  balanced-no: (low no = Null) = (high no = Null) and
  lno-notin-nl: low no ∉ set (prx @ node # sfx) and
  hno-notin-nl: high no ∉ set (prx @ node # sfx) and
  isLeaf-var-no: isLeaf-pt no low high = (var no ≤ 1)
  by auto
have nodes-notin-nl-neq-nln: ∀ p. p ∉ set (prx @ node # sfx) ⟶ p ≠ node
  by auto
show  repb no = hd [sn←(prx @ [node]). repNodes-eq sn no low high repb]
proof (cases no = node)
  case False
  note no-notin-nl=this
  with no-in-take-Sucn have no-in-take-n: no ∈ set prx
    by auto
  from False repa-repb-nc have repb-repa-no: repb no = repa no
    by auto
  with while-inv [rule-format, OF no-in-take-n] no-in-take-n obtain
    repa-no-nNull: repa no ≠ Null and
    while-share-red-exp:
    (if (repa ∝ low) no = (repa ∝ high) no ∧ low no ≠ Null
        then repa no = (repa ∝ low) no
        else repa no = hd [sn←prx . repNodes-eq sn no low high repa] ∧
        repa (repa no) = repa no ∧
        (∀ no1∈set prx. ((repa ∝ high) no1 = (repa ∝ high) no ∧
        (repa ∝ low) no1 = (repa ∝ low) no) = (repa no = repa no1)))
    using [[simp-depth-limit = 2]]
    by auto
  from no-in-take-n
  have filter-take-n-notempty: [sn←prx.
    repNodes-eq sn no low high repa] ≠ []
    apply −
    apply (rule filter-not-empty)
    apply (auto simp add: repNodes-eq-def)
    done
  then have hd-term-n-Sucn:
    hd [sn←prx. repNodes-eq sn no low high repa]
      = hd [sn←prx@[node] . repNodes-eq sn no low high repa]
    by auto
  thus ?thesis
  proof (cases low no = Null)
    case True
    note lno-Null=this
    with balanced-no have hno-Null: high no = Null
```

by *simp*
from *lno-Null hno-Null* **have** *isLeaf-no*: *isLeaf-pt no low high*
by (*simp add*: *isLeaf-pt-def*)
from *True while-share-red-exp*
have *while-low-Null*:
repa no = *hd* [*sn*←*prx*. *repNodes-eq sn no low high repa*] ∧
repa (*repa no*) = *repa no* ∧
(∀ *no1*∈*set prx*. ((*repa* ∝ *high*) *no1* = (*repa* ∝ *high*) *no*
∧ (*repa* ∝ *low*) *no1* = (*repa* ∝ *low*) *no*) = (*repa no* = *repa no1*))
by *auto*
have *all-nodes-in-nl-Leafs*:
∀ *x* ∈ *set* (*prx* @ *node* # *sfx*). *isLeaf-pt x low high*
proof (*intro ballI*)
fix *x*
assume *x-in-nodeslist*: *x* ∈ *set* (*prx* @ *node* # *sfx*)
from *isLeaf-no isLeaf-var-no* **have** *var no* ≤ *1*
by *simp*
with *all-nodes-same-var* [*rule-format*, *OF x-in-nodeslist no-in-nl*]
have *var x* ≤ *1*
by *simp*
with *nodes-balanced-ordered* [*rule-format*, *OF x-in-nodeslist*]
show *isLeaf-pt x low high*
by (*auto simp add*: *isLeaf-pt-def*)
qed
have ∀ *x* ∈ *set* (*prx*@[*node*]). *repNodes-eq x no low high repb*
= *repNodes-eq x no low high repa*
proof (*rule ballI*)
fix *x*
assume *x-in-take-Sucn*: *x* ∈ *set* (*prx*@[*node*])
hence *x-in-nodeslist*: *x* ∈ *set* (*prx* @ *node* # *sfx*)
by *auto*
with *all-nodes-in-nl-Leafs* **have** *isLeaf-pt x low high*
by *auto*
with *isLeaf-no repa-repb-nc* **show** *repNodes-eq x no low high repb*
= *repNodes-eq x no low high repa*
by (*simp add*: *repNodes-eq-def null-comp-def isLeaf-pt-def*)
qed
then have [*sn*←(*prx*@[*node*]). *repNodes-eq sn no low high repa*]
= [*sn*←(*prx*@[*node*]) . *repNodes-eq sn no low high repb*]
apply −
apply (*rule P-eq-list-filter*)
apply *simp*
done
with *hd-term-n-Sucn while-low-Null repb-repa-no* **show** *?thesis*
by *auto*
next
assume *lno-nNull*: *low no* ≠ *Null*
with *balanced-no* **have** *hno-nNull*: *high no* ≠ *Null*
by *simp*

with *lno-nNull* **have** *no-nLeaf*: ¬ *isLeaf-pt no low high*
by (*simp add*: *isLeaf-pt-def*)
with *isLeaf-var-no* **have** *Sucn-s-varno*: *1* < *var no*
by *auto*
with *no-in-nl all-nodes-same-var*
have *all-nodes-nl-var*: ∀ *x* ∈ *set* (*prx* @ *node* # *sfx*). *1* < *var x*
apply −
apply (*rule ballI*)
apply (*drule-tac x*=*no* **in** *bspec*,*assumption*)
apply (*drule-tac x*=*x* **in** *bspec*,*assumption*)
apply *auto*
done
with *nodes-balanced-ordered*
have *all-nodes-nl-nLeaf*:
∀ *x* ∈ *set* (*prx* @ *node* # *sfx*). ¬ *isLeaf-pt x low high*
apply −
apply (*rule ballI*)
apply (*drule-tac x*=*x* **in** *bspec*,*assumption*)
apply (*drule-tac x*=*x* **in** *bspec*,*assumption*)
apply *auto*
done
from *lno-nNull share-or*
have *repbchildren-eq-no*: (*repb* ∝ *low*) *no* ≠ (*repb* ∝ *high*) *no*
by *simp*
with *lno-nNull hno-nNull lno-notin-nl hno-notin-nl repa-repb-nc*
nodes-notin-nl-neq-nln
have *repachildren-eq-no*: (*repa* ∝ *low*) *no* ≠ (*repa* ∝ *high*) *no*
using [[*simp-depth-limit*=*2*]]
by (*simp add*: *null-comp-def*)
with *while-share-red-exp*
have *repa-no-def*:
repa no = *hd* [*sn*←*prx* . *repNodes-eq sn no low high repa*]
by *auto*
with *no-notin-nl repa-repb-nc*
have *repb no* = *hd* [*sn*←*prx*. *repNodes-eq sn no low high repa*]
by *simp*
with *hd-term-n-Sucn*
have *repb-no-hd-term-repa*: *repb no* =
hd [*sn*←*prx*@[*node*] . *repNodes-eq sn no low high repa*]
by *simp*
have ∀ *x* ∈ *set* (*prx*@[*node*]).
repNodes-eq x no low high repa = *repNodes-eq x no low high repb*
proof (*intro ballI*)
fix *x*
assume *x-in-take-Sucn*: *x* ∈ *set* (*prx*@[*node*])
hence *x-in-nodeslist*: *x* ∈ *set* (*prx* @ *node* # *sfx*)
by *auto*
with *all-nodes-nl-nLeaf* **have** *x-nLeaf*: ¬ *isLeaf-pt x low high*
by *auto*

```
      from nodes-balanced-ordered [rule-format, OF x-in-nodeslist] obtain
        balanced-x: (low x = Null) = (high x = Null) and
        lx-notin-nl: low x ∉ set (prx @ node # sfx) and
        hx-notin-nl: high x ∉ set (prx @ node # sfx)
        by auto
      with nodes-notin-nl-neq-nln lno-notin-nl hno-notin-nl lno-nNull
        hno-nNull repa-repb-nc
      show  repNodes-eq x no low high repa = repNodes-eq x no low high repb
        by (simp add: repNodes-eq-def null-comp-def)
    qed
    then have  [sn←(prx@[node]). repNodes-eq sn no low high repa] =
      [sn←(prx@[node]). repNodes-eq sn no low high repb]
      apply −
      apply (rule P-eq-list-filter)
      apply auto
      done
    with repb-no-hd-term-repa show ?thesis
      by simp
  qed
next
  assume no-nln: no = node
  with repb-node have repb-no-def: repb no =
    hd [sn←(prx @ node # sfx). repNodes-eq sn node low high repa]
    by simp
  show ?thesis
  proof (cases isLeaf-pt no low high)
    case True
    note isLeaf-no=this
    have ∀ x ∈ set (prx @ node # sfx). repNodes-eq x no low high repb
      = repNodes-eq x no low high repa
    proof (rule ballI)
      fix x
      assume x-in-nodeslist: x ∈ set (prx @ node # sfx)
      have all-nodes-in-nl-Leafs:
        ∀ x ∈ set (prx @ node # sfx). isLeaf-pt x low high
      proof (intro ballI)
        fix x
        assume x-in-nodeslist:  x ∈ set (prx @ node # sfx)
        from isLeaf-no isLeaf-var-no have var no ≤ 1
          by simp
        with all-nodes-same-var [rule-format, OF x-in-nodeslist no-in-nl]
        have var x ≤ 1
          by simp
        with nodes-balanced-ordered [rule-format, OF x-in-nodeslist]
        show isLeaf-pt x low high
          by (auto simp add: isLeaf-pt-def)
      qed
      with x-in-nodeslist have isLeaf-pt x low high
        by auto
```

```
    with isLeaf-no repa-repb-nc
    show repNodes-eq x no low high repb = repNodes-eq x no low high repa
      by (simp add: repNodes-eq-def null-comp-def isLeaf-pt-def)
  qed
  with repb-no-def no-nln have repb-no-whole-nl: repb no =
    hd [sn← (prx @ node # sfx). repNodes-eq sn node low high repb]
    apply −
    apply (subgoal-tac
      [sn← (prx@node#sfx). repNodes-eq sn node low high repa]
      = [sn←(prx @ node # sfx) . repNodes-eq sn node low high repb])
    apply simp
    apply (rule P-eq-list-filter)
    apply auto
    done
  from no-in-take-Sucn no-nln
  have [sn← (prx@[node]). repNodes-eq sn node low high repb] ≠ []
    apply −
    apply (rule filter-not-empty)
    apply (auto simp add: repNodes-eq-def)
    done
  then
  have hd [sn←(prx@[node]). repNodes-eq sn node low high repb] =
       hd [sn←(prx @ node # sfx). repNodes-eq sn node low high repb]
    apply −
    apply (rule hd-filter-app [symmetric])
    apply auto
    done
  with repb-no-whole-nl no-nln show ?thesis
    by simp
next
  assume no-nLeaf: ¬ isLeaf-pt no low high
  with share-or balanced-no have (repb ∝ low) no ≠ (repb ∝ high) no
    using [[simp-depth-limit=2]]
    by (simp add: isLeaf-pt-def)
  from no-nLeaf share-cond no-nln have repa (low no) ≠ repa (high no)
    by auto
  with no-nLeaf balanced-no have (repa ∝ low) no ≠ (repa ∝ high) no
    by (simp add: null-comp-def isLeaf-pt-def)
  have ∀ x ∈ set (prx@node#sfx). repNodes-eq x no low high repb
    = repNodes-eq x no low high repa
  proof (rule ballI)
    fix x
    assume x-in-nodeslist: x ∈ set (prx@node#sfx)
    have all-nodes-in-nl-Leafs:
      ∀ x ∈ set (prx@node#sfx). ¬ isLeaf-pt x low high
    proof (intro ballI)
      fix x
      assume x-in-nodeslist: x ∈ set (prx@node#sfx)
      from no-nLeaf isLeaf-var-no have 1 < var no
```

```
      by simp
    with all-nodes-same-var [rule-format, OF x-in-nodeslist no-in-nl]
    have 1 < var x
      by auto
    with nodes-balanced-ordered [rule-format, OF x-in-nodeslist]
    show ¬ isLeaf-pt x low high
      apply (unfold isLeaf-pt-def)
      apply fastforce
      done
  qed
  with x-in-nodeslist have x-nLeaf: ¬ isLeaf-pt x low high
    by auto
  from nodes-balanced-ordered [rule-format, OF x-in-nodeslist]
  have (low x = Null) = (high x = Null)
       ∧ low x ∉ set (prx@node#sfx) ∧ high x ∉ set (prx@node#sfx)
    by auto
  with x-nLeaf balanced-no no-nLeaf repa-repb-nc
    nodes-notin-nl-neq-nln lno-notin-nl hno-notin-nl
  show repNodes-eq x no low high repb = repNodes-eq x no low high repa
    using [[simp-depth-limit=2]]
    by (simp add: repNodes-eq-def null-comp-def isLeaf-pt-def)
qed
with repb-no-def no-nln
have repb-no-whole-nl:
  repb no = hd [sn←(prx@node#sfx). repNodes-eq sn node low high repb]
  apply −
  apply (subgoal-tac
      [sn←(prx@node#sfx). repNodes-eq sn node low high repa]
      = [sn←(prx@node#sfx). repNodes-eq sn node low high repb])
  apply simp
  apply (rule P-eq-list-filter)
  apply auto
  done
from no-in-take-Sucn no-nln
have [sn←(prx@[node]) . repNodes-eq sn node low high repb] ≠ []
  apply −
  apply (rule filter-not-empty)
  apply (auto simp add: repNodes-eq-def)
  done
then have
  hd [sn← (prx@[node]) . repNodes-eq sn node low high repb] =
  hd [sn←(prx@node#sfx) . repNodes-eq sn node low high repb]
  apply −
  apply (rule hd-filter-app [symmetric])
  apply auto
  done
with repb-no-whole-nl no-nln show ?thesis
  by simp
qed
```

qed
qed
have *repb-no-red-def*: ($\forall$ *no*$\in$*set* (*prx*@[*node*]).(*repb* $\propto$ *low*) *no* = (*repb* $\propto$ *high*) *no*
$\wedge$ *low no* $\neq$ *Null* $\longrightarrow$ *repb no* = (*repb* $\propto$ *low*) *no*)
proof (*intro ballI impI*)
fix *no*
assume *no-in-take-Sucn*: *no* $\in$ *set* (*prx*@[*node*])
assume *red-cond-no*: (*repb* $\propto$ *low*) *no* = (*repb* $\propto$ *high*) *no* $\wedge$ *low no* $\neq$ *Null*
from *no-in-take-Sucn* **have** *no-in-nl*: *no* $\in$ *set* (*prx*@*node*#*sfx*)
by *auto*
from *nodes-balanced-ordered* [*rule-format*, *OF this*]**obtain**
no-nNull: *no* $\neq$ *Null* **and**
balanced-no: (*low no* = *Null*) = (*high no* = *Null*) **and**
lno-notin-nl: *low no* $\notin$ *set* (*prx*@*node*#*sfx*) **and**
hno-notin-nl: *high no* $\notin$ *set* (*prx*@*node*#*sfx*) **and**
isLeaf-var-no: *isLeaf-pt no low high* = (*var no* $\leq$ *1*)
by *auto*
have *nodes-notin-nl-neq-nln*: $\forall$ *p*. *p* $\notin$ *set* (*prx*@*node*#*sfx*) $\longrightarrow$ *p* $\neq$ *node*
by *auto*
show *repb no* = (*repb* $\propto$ *low*) *no*
proof (*cases no* = *node*)
case *False*
note *no-notin-nl*=*this*
with *no-in-take-Sucn* **have** *no-in-take-n*: *no* $\in$ *set prx*
by *auto*
from *False repa-repb-nc* **have** *repb-repa-no*: *repb no* = *repa no*
by *auto*
with *while-inv* [*rule-format*, *OF no-in-take-n*] **obtain**
repa-no-nNull: *repa no* $\neq$ *Null* **and**
while-share-red-exp:
(*if* (*repa* $\propto$ *low*) *no* = (*repa* $\propto$ *high*) *no* $\wedge$ *low no* $\neq$ *Null*
then repa no = (*repa* $\propto$ *low*) *no*
else repa no = *hd* [*sn*$\leftarrow$*prx*. *repNodes-eq sn no low high repa*] $\wedge$
repa (*repa no*) = *repa no* $\wedge$
($\forall$ *no1*$\in$*set prx*. ((*repa* $\propto$ *high*) *no1* = (*repa* $\propto$ *high*) *no* $\wedge$
(*repa* $\propto$ *low*) *no1* = (*repa* $\propto$ *low*) *no*) = (*repa no* = *repa no1*)))
using [[*simp-depth-limit*=*2*]]
by *auto*
from *red-cond-no nodes-notin-nl-neq-nln lno-notin-nl*
hno-notin-nl while-share-red-exp balanced-no repa-repb-nc
have *red-repa-no*: *repa no* = (*repa* $\propto$ *low*) *no*
by (*auto simp add*: *null-comp-def*)
from *red-cond-no nodes-notin-nl-neq-nln lno-notin-nl repa-repb-nc*
have (*repb* $\propto$ *low*) *no* = (*repa* $\propto$ *low*) *no*
by (*auto simp add*: *null-comp-def*)
with *red-repa-no no-notin-nl balanced-no repa-repb-nc*
have *repb no* = (*repb* $\propto$ *low*) *no*
by *auto*

```
    with red-cond-no show ?thesis
      by auto
  next
    assume no = node
    with share-cond
    have share-cond-pre:
      isLeaf-pt no low high ∨ repa (low no) ≠ repa (high no)
      by simp
    show ?thesis
    proof (cases isLeaf-pt no low high)
      case True
      with red-cond-no show ?thesis
        by (simp add: isLeaf-pt-def)
    next
      assume no-nLeaf: ¬ isLeaf-pt no low high
      with share-cond-pre
      have repa (low no) ≠ repa (high no)
        by simp
      with no-nLeaf lno-notin-nl hno-notin-nl nodes-notin-nl-neq-nln
        balanced-no repa-repb-nc
      have repb (low no) ≠ repb (high no)
        using [[simp-depth-limit=2]]
        by (auto simp add: isLeaf-pt-def)
      with no-nLeaf balanced-no have (repb ∝ low) no ≠ (repb ∝ high) no
        by (simp add: null-comp-def isLeaf-pt-def)
      with red-cond-no show ?thesis
        by simp
    qed
  qed
qed
have while-while: (∀ no∈set (prx@[node]).
      repb no ≠ Null ∧
      (if (repb ∝ low) no = (repb ∝ high) no ∧ low no ≠ Null
      then repb no = (repb ∝ low) no
      else repb no = hd [sn←(prx@[node]). repNodes-eq sn no low high repb] ∧
      repb (repb no) = repb no ∧
      (∀ no1∈set ((prx@[node])). ((repb ∝ high) no1 = (repb ∝ high) no
      ∧ (repb ∝ low) no1 = (repb ∝ low) no) = (repb no = repb no1))))
  (is ∀ no∈set (prx@[node]). ?P no ∧ ?Q no)
proof (intro ballI)
  fix no
  assume no-in-take-Sucn: no ∈ set (prx@[node])
  hence no-in-nl: no ∈ set (prx@node#sfx)
    by auto
  from nodes-balanced-ordered [rule-format, OF this] obtain
    no-nNull: no ≠ Null and
    balanced-no: (low no = Null) = (high no = Null) and
    lno-notin-nl: low no ∉ set (prx@node#sfx) and
    hno-notin-nl: high no ∉ set (prx@node#sfx) and
```

```
  isLeaf-var-no: isLeaf-pt no low high = (var no ≤ 1)
  by auto
from no-in-take-Sucn
have filter-take-Sucn-not-empty:
      [sn←(prx@[node]). repNodes-eq sn no low high repb] ≠ []
  apply −
  apply (rule filter-not-empty)
  apply (auto simp add: repNodes-eq-def)
  done
then have hd-filter-Sucn-in-Sucn:
      hd [sn←(prx@[node]). repNodes-eq sn no low high repb] ∈
      set (prx@[node])
  by (rule hd-filter-in-list)
have nodes-notin-nl-neq-nln: ∀ p. p ∉ set (prx@node#sfx) ⟶ p ≠ node
  by auto
show ?P no ∧ ?Q no
proof (cases no = node)
  case False
  note no-notin-nl=this
  with no-in-take-Sucn
  have no-in-take-n: no ∈ set prx
    by auto
  from False repa-repb-nc have repb-repa-no: repb no = repa no
    by auto
  with while-inv [rule-format, OF no-in-take-n] obtain
    repa-no-nNull: repa no ≠ Null and
    while-share-red-exp:
    (if (repa ∝ low) no = (repa ∝ high) no ∧ low no ≠ Null
       then repa no = (repa ∝ low) no
       else repa no = hd [sn←prx. repNodes-eq sn no low high repa] ∧
       repa (repa no) = repa no ∧
       (∀ no1∈set prx. ((repa ∝ high) no1 = (repa ∝ high) no ∧
       (repa ∝ low) no1 = (repa ∝ low) no) = (repa no = repa no1)))
    using [[simp-depth-limit=2]]
    by auto
  from repb-repa-no repa-no-nNull have repb-no-nNull: ?P no
    by simp
  have ?Q no
  proof (cases (repb ∝ low) no = (repb ∝ high) no ∧ low no ≠ Null)
    case True
    with no-in-take-Sucn repb-no-red-def show ?thesis
      by auto
  next
    assume share-case-repb:
      ¬ ((repb ∝ low) no = (repb ∝ high) no ∧ low no ≠ Null)
    with repb-no-share-def no-in-take-Sucn
    have repb-no-def: repb no = hd [sn← (prx@[node]).
      repNodes-eq sn no low high repb]
      by auto
```

with *share-case-repb*
have $(repb \propto low)\ no \neq (repb \propto high)\ no \vee low\ no = Null$
using [[*simp-depth-limit=2*]]
by *simp*
thus *?thesis*
proof (*cases low no = Null*)
case *True*
note *lno-Null=this*
with *balanced-no* **have** *hno-Null*: *high no = Null*
by *simp*
from *lno-Null hno-Null* **have** *isLeaf-no*: *isLeaf-pt no low high*
by (*simp add*: *isLeaf-pt-def*)
from *True while-share-red-exp*
have *while-low-Null*:
$repa\ no = hd\ [sn{\leftarrow}prx.\ repNodes\text{-}eq\ sn\ no\ low\ high\ repa] \wedge$
$repa\ (repa\ no) = repa\ no \wedge$
$(\forall\, no1{\in}set\ prx.\ ((repa \propto high)\ no1 = (repa \propto high)\ no$
$\wedge\ (repa \propto low)\ no1 = (repa \propto low)\ no) = (repa\ no = repa\ no1))$
by *auto*
from *no-in-take-n*
have $[sn{\leftarrow}prx.\ repNodes\text{-}eq\ sn\ no\ low\ high\ repa] \neq []$
apply −
apply (*rule filter-not-empty*)
apply (*auto simp add*: *repNodes-eq-def*)
done
then have *hd-term-n-Sucn*: $hd\ [sn{\leftarrow}prx.\ repNodes\text{-}eq\ sn\ no\ low\ high\ repa] =$
$hd\ [sn{\leftarrow}(prx@[node])\ .\ repNodes\text{-}eq\ sn\ no\ low\ high\ repa]$
apply −
apply (*rule hd-filter-app* [*symmetric*])
apply *auto*
done
have *all-nodes-in-nl-Leafs*:
$\forall\, x \in set\ (prx@node\#sfx).\ isLeaf\text{-}pt\ x\ low\ high$
proof (*intro ballI*)
fix *x*
assume *x-in-nodeslist*: $x \in set\ (prx@node\#sfx)$
from *isLeaf-no isLeaf-var-no* **have** $var\ no \leq 1$
by *simp*
with *all-nodes-same-var* [*rule-format*, *OF x-in-nodeslist no-in-nl*]
have $var\ x \leq 1$
by *simp*
with *nodes-balanced-ordered* [*rule-format*, *OF x-in-nodeslist*]
show *isLeaf-pt x low high*
by (*auto simp add*: *isLeaf-pt-def*)
qed
from *no-in-take-Sucn* **have**
filter-Sucn-no-notempty:
$[sn{\leftarrow}(prx@[node]).\ repNodes\text{-}eq\ sn\ no\ low\ high\ repb] \neq []$

apply −
apply (*rule filter-not-empty*)
apply (*auto simp add*: *repNodes-eq-def*)
done
then have *hd-term-in-take-Sucn*:
hd [*sn*←(*prx*@[*node*]) . *repNodes-eq sn no low high repb*]
∈ *set* (*prx*@[*node*])
by (*rule hd-filter-in-list*)
then have *hd-term-in-nl*:
hd [*sn*←(*prx*@[*node*]) . *repNodes-eq sn no low high repb*]
∈ *set* (*prx*@*node*#*sfx*)
by *auto*
with *all-nodes-in-nl-Leafs*
have *hd-term-Leaf*: *isLeaf-pt* (*hd* [*sn*← (*prx*@[*node*]).
repNodes-eq sn no low high repb]) *low high*
by *auto*
from *while-low-Null* **have** *repa* (*repa no*) = *repa no*
by *auto*
with *no-notin-nl repa-repb-nc*
have *repa-repb-no-repb*: *repa* (*repb no*) = *repb no*
by *auto*
have *repb-repb-no*: *repb* (*repb no*) = *repb no*
proof (*cases repb no* = *node*)
case *False*
with *repa-repb-nc repa-repb-no-repb* **show** *?thesis*
by *auto*
next
assume *repb-no-nln*: *repb no* = *node*
with *hd-term-Leaf isLeaf-no all-nodes-in-nl-Leafs*
have *nested-hd-repa-repb*:
hd [*sn*←(*prx*@*node*#*sfx*). *repNodes-eq sn*
(*hd* [*sn*←(*prx*@[*node*]) . *repNodes-eq sn no low high repb*])
low high repa] =
hd [*sn*←(*prx*@*node*#*sfx*). *repNodes-eq sn*
(*hd* [*sn*←(*prx*@[*node*]). *repNodes-eq sn no low high repb*])
low high repb]
by (*simp add*: *isLeaf-pt-def repNodes-eq-def null-comp-def*)
from *hd-term-in-take-Sucn*
have [*sn*←(*prx*@[*node*]). *repNodes-eq sn*
(*hd* [*sn*←(*prx*@[*node*]). *repNodes-eq sn no low high repb*])
low high repb] ≠ []
apply −
apply (*rule filter-not-empty*)
apply (*auto simp add*: *repNodes-eq-def*)
done
then have *hd* [*sn*←(*prx*@[*node*]). *repNodes-eq sn*
(*hd* [*sn*←(*prx*@[*node*]). *repNodes-eq sn no low high repb*])
low high repb] =
hd [*sn*←(*prx*@*node*#*sfx*). *repNodes-eq sn*

```
      ( hd [sn←(prx@[node]). repNodes-eq sn no low high repb])
          low high repb]
    apply −
    apply (rule hd-filter-app [symmetric])
    apply auto
    done
  then have hd-term-nodeslist-Sucn:
    hd [sn←(prx@node#sfx). repNodes-eq sn
      ( hd [sn←(prx@[node]). repNodes-eq sn no low high repb])
          low high repb] =
      hd [sn←(prx@[node]). repNodes-eq sn
      ( hd [sn←(prx@[node]). repNodes-eq sn no low high repb])
          low high repb]
    by simp
  from no-in-take-Sucn filter-Sucn-no-notempty
  have filter-filter: hd [sn←(prx@[node]). repNodes-eq sn
      (hd [sn←(prx@[node]). repNodes-eq sn no low high repb])
          low high repb] =
      hd [sn←(prx@[node]). repNodes-eq sn no low high repb]
    apply −
    apply (rule filter-hd-P-rep-indep)
    apply (auto simp add: repNodes-eq-def)
    done
  from repb-no-def repb-no-nln repb-node
  have repb (repb no) =  hd [sn←(prx@node#sfx). repNodes-eq sn
      ( hd [sn←(prx@[node]). repNodes-eq sn no low high repb])
          low high repa]
    by simp
  with nested-hd-repa-repb
  have repb (repb no) =  hd [sn←(prx@node#sfx). repNodes-eq sn
      (hd [sn←(prx@[node]). repNodes-eq sn no low high repb])
         low high repb]
    by simp
  with hd-term-nodeslist-Sucn
  have repb (repb no) =  hd [sn←(prx@[node]). repNodes-eq sn
      ( hd [sn←(prx@[node]). repNodes-eq sn no low high repb])
          low high repb]
    by simp
  with filter-filter
  have repb (repb no) = hd [sn←(prx@[node]).
    repNodes-eq sn no low high repb]
    by simp
  with repb-no-def show ?thesis
    by simp
qed
have two-nodes-repb: (∀ no1∈set (prx@[node]).
     ((repb ∝ high) no1 = (repb ∝ high) no
     ∧ (repb ∝ low) no1 = (repb ∝ low) no) = (repb no = repb no1))
proof (intro ballI)
```

fix *no1*
assume *no1-in-take-Sucn*: *no1* ∈ *set* (*prx*@[*node*])
then have *no1* ∈ *set* (*prx*@*node*#*sfx*) **by** *auto*
with *all-nodes-in-nl-Leafs*
have *isLeaf-no1*: *isLeaf-pt no1 low high*
by *auto*
with *isLeaf-no*
have *repbchildren-eq-no-no1*: (*repb* ∝ *high*) *no1* = (*repb* ∝ *high*) *no*
∧ (*repb* ∝ *low*) *no1* = (*repb* ∝ *low*) *no*
by (*simp add*: *null-comp-def isLeaf-pt-def*)
from *isLeaf-no1 isLeaf-no*
have *repachildren-eq-no-no1*: (*repa* ∝ *high*) *no1* = (*repa* ∝ *high*) *no*
∧ (*repa* ∝ *low*) *no1* = (*repa* ∝ *low*) *no*
by (*simp add*: *null-comp-def isLeaf-pt-def*)
from *while-low-Null*
have *while-low-same-rep*: (∀ *no1*∈*set prx*.
((*repa* ∝ *high*) *no1* = (*repa* ∝ *high*) *no*
∧ (*repa* ∝ *low*) *no1* = (*repa* ∝ *low*) *no*) = (*repa no* = *repa no1*))
by *auto*
show ((*repb* ∝ *high*) *no1* = (*repb* ∝ *high*) *no* ∧
(*repb* ∝ *low*) *no1* = (*repb* ∝ *low*) *no*) = (*repb no* = *repb no1*)
proof (*cases no1* = *node*)
case *False*
with *no1-in-take-Sucn* **have** *no1* ∈ *set prx*
by *auto*
with *while-low-same-rep repachildren-eq-no-no1*
have *repa no* = *repa no1*
by *auto*
with *repa-repb-nc no-notin-nl False repbchildren-eq-no-no1*
show *?thesis*
by *auto*
next
assume *no1-nln*: *no1* = *node*
hence *no1-in-take-Sucn*: *no1* ∈ *set* (*prx*@[*node*])
by *auto*
hence *no1-in-nl*: *no1* ∈ *set* (*prx*@*node*#*sfx*)
by *auto*
from *nodes-balanced-ordered* [*rule-format*, *OF this*] **have**
balanced-no1: (*low no1* = *Null*) = (*high no1* = *Null*)
by *auto*
with *no1-in-take-Sucn repb-no-share-def isLeaf-no1*
have *repb-no1*: *repb no1* = *hd* [*sn*←(*prx*@[*node*]).
repNodes-eq sn no1 low high repb]
by (*auto simp add*: *isLeaf-pt-def*)
from *balanced-no1 isLeaf-no1 isLeaf-no balanced-no*
have *repbchildren-eq-no1-no*: (*repb* ∝ *high*) *no1* = (*repb* ∝ *high*) *no*
∧ (*repb* ∝ *low*) *no1* = (*repb* ∝ *low*) *no*
by (*simp add*: *null-comp-def isLeaf-pt-def*)
have ∀ *x* ∈ *set* (*prx*@[*node*]). *repNodes-eq x no low high repb*

```
      =  repNodes-eq x no1 low high repb
    proof (intro ballI)
      fix x
      assume x-in-take-Sucn:  x ∈ set (prx@[node])
      with repbchildren-eq-no1-no show repNodes-eq x no low high repb
        = repNodes-eq x no1 low high repb
        by (simp add: repNodes-eq-def)
    qed
    then have  [sn←(prx@[node]). repNodes-eq sn no low high repb]
          = [sn←(prx@[node]). repNodes-eq sn no1 low high repb]
      by (rule P-eq-list-filter)
    with repb-no-def repb-no1 have repb-no-no1: repb no = repb no1
      by simp
    with repbchildren-eq-no1-no show ?thesis
      by simp
  qed
 qed
 with repb-repb-no repb-no-share-def no-in-take-Sucn share-case-repb
 show ?thesis
  using [[simp-depth-limit=4]]
  by auto
next
 assume lno-nNull: low no ≠ Null
 with share-case-repb
 have repbchildren-neq-no: (repb ∝ low) no ≠ (repb ∝ high) no
  by auto
 from balanced-no lno-nNull
 have hno-nNull: high no ≠ Null
  by simp
 with repbchildren-neq-no lno-nNull repa-repb-nc
  lno-notin-nl hno-notin-nl nodes-notin-nl-neq-nln
 have repachildren-neq-no: (repa ∝ low) no ≠ (repa ∝ high) no
  using [[simp-depth-limit=2]]
  by (auto simp add: null-comp-def)
 with while-share-red-exp
 have repa-while-inv: repa (repa no) = repa no
  ∧ (∀ no1∈set prx. ((repa ∝ high) no1 = (repa ∝ high) no
  ∧ (repa ∝ low) no1 = (repa ∝ low) no) = (repa no = repa no1))
  by auto
 from lno-nNull hno-nNull
 have no-nLeaf: ¬ isLeaf-pt no low high
  by (simp add: isLeaf-pt-def)
 have all-nodes-in-nl-nLeafs:
  ∀ x ∈ set (prx@node#sfx). ¬ isLeaf-pt x low high
 proof (intro ballI)
  fix x
  assume x-in-nodeslist:  x ∈ set (prx@node#sfx)
  from no-nLeaf isLeaf-var-no have 1 < var no
   by simp
```

with *all-nodes-same-var* [*rule-format*, *OF x-in-nodeslist no-in-nl*]
have *1* < *var x*
by *simp*
with *nodes-balanced-ordered* [*rule-format*, *OF x-in-nodeslist*]
show ¬ *isLeaf-pt x low high*
using [[*simp-depth-limit* = *2*]]
by (*auto simp add*: *isLeaf-pt-def*)
qed
have *repb-repb-no*: *repb* (*repb no*) = *repb no*
proof −
from *repa-while-inv no-notin-nl repa-repb-nc*
have *repa* (*repb no*) = *repb no*
by *simp*
from *hd-filter-Sucn-in-Sucn repb-no-def*
have *repb-no-in-take-Sucn*: *repb no* ∈ *set* (*prx*@[*node*])
by *simp*
hence *repb-no-in-nl*: *repb no* ∈ *set* (*prx*@*node*#*sfx*)
by *auto*
from *all-nodes-in-nl-nLeafs repb-no-in-nl*
have *repb-no-nLeaf*: ¬ *isLeaf-pt* (*repb no*) *low high*
by *auto*
from *nodes-balanced-ordered* [*rule-format*, *OF repb-no-in-nl*]
have (*low* (*repb no*) = *Null*) = (*high* (*repb no*) = *Null*)
∧ *low* (*repb no*) ∉ *set* (*prx*@*node*#*sfx*) ∧
high (*repb no*) ∉ *set* (*prx*@*node*#*sfx*)
by *auto*
from *filter-take-Sucn-not-empty*
have *repNodes-eq* (*hd* [*sn*←(*prx*@[*node*]).
repNodes-eq sn no low high repb]) *no low high repb*
by (*rule hd-filter-prop*)
with *repb-no-def* **have** *repNodes-eq* (*repb no*) *no low high repb*
by *simp*
then have (*repb* ∝ *low*) (*repb no*) = (*repb* ∝ *low*) *no*
∧ (*repb* ∝ *high*) (*repb no*) = (*repb* ∝ *high*) *no*
by (*simp add*: *repNodes-eq-def*)
with *repbchildren-neq-no* **have** (*repb* ∝ *low*) (*repb no*)
≠ (*repb* ∝ *high*) (*repb no*)
by *simp*
with *repb-no-in-take-Sucn repb-no-share-def*
have *repb-repb-no-double-hd*:
repb (*repb no*) = *hd* [*sn*←(*prx*@[*node*]).
repNodes-eq sn (*repb no*) *low high repb*]
by *auto*
from *filter-take-Sucn-not-empty*
have *hd* [*sn*←(*prx*@[*node*]).
repNodes-eq sn (*repb no*) *low high repb*] = *repb no*
apply (*simp only*: *repb-no-def*)
apply (*rule filter-hd-P-rep-indep*)
apply (*auto simp add*: *repNodes-eq-def*)

```
    done
  with repb-repb-no-double-hd show ?thesis
    by simp
qed
have (∀ no1∈set (prx@[node]).
    ((repb ∝ high) no1 = (repb ∝ high) no ∧
    (repb ∝ low) no1 = (repb ∝ low) no) = (repb no = repb no1))
proof (intro ballI)
  fix no1
  assume no1-in-take-Sucn: no1 ∈ set (prx@[node])
  hence no1-in-nl: no1 ∈ set (prx@node#sfx)
    by auto
  from all-nodes-in-nl-nLeafs no1-in-nl
  have no1-nLeaf: ¬ isLeaf-pt no1 low high
    by auto
  from nodes-balanced-ordered [rule-format, OF no1-in-nl]
  have no1-props: (low no1 = Null) = (high no1 = Null)
    ∧ low no1 ∉ set (prx@node#sfx) ∧ high no1 ∉ set (prx@node#sfx)
    by auto
  show ((repb ∝ high) no1 = (repb ∝ high) no
    ∧ (repb ∝ low) no1 = (repb ∝ low) no) = (repb no = repb no1)
  proof (cases no1 = node)
    case False
    note no1-neq-nln=this
    with no1-in-take-Sucn
    have no1-in-take-n: no1 ∈ set prx
      by auto
    with repa-while-inv have ((repa ∝ high) no1 = (repa ∝ high) no
      ∧ (repa ∝ low) no1 = (repa ∝ low) no) = (repa no = repa no1)
      by fastforce
    with no1-props no1-nLeaf no-nLeaf balanced-no lno-notin-nl
      hno-notin-nl nodes-notin-nl-neq-nln no-notin-nl
      no1-neq-nln repa-repb-nc
    show ?thesis
      using [[simp-depth-limit=1]]
      by (auto simp add: null-comp-def isLeaf-pt-def)
  next
    assume no1-nln:  no1 = node
    show ?thesis
    proof
      assume repbchildren-eq-no1-no:
        (repb ∝ high) no1 = (repb ∝ high) no
        ∧ (repb ∝ low) no1 = (repb ∝ low) no
      with repbchildren-neq-no
      have (repb ∝ high) no1 ≠ (repb ∝ low) no1
        by auto
      with repb-no-share-def no1-in-take-Sucn
      have repb-no1-def:  repb no1 = hd [sn←(prx@[node]).
        repNodes-eq sn no1 low high repb]
```

```
    by auto
  have filter-no1-eq-filter-no: [sn←(prx@[node]).
    repNodes-eq sn no1 low high repb] =
    [sn←(prx@[node]). repNodes-eq sn no low high repb]
  proof −
    have ∀ x ∈ set (prx@[node]).
      repNodes-eq x no1 low high repb =
      repNodes-eq x no low high repb
    proof (intro ballI)
      fix x
      assume x-in-take-Sucn: x ∈ set (prx@[node])
      with repbchildren-eq-no1-no
      show repNodes-eq x no1 low high repb =
        repNodes-eq x no low high repb
        by (simp add: repNodes-eq-def)
    qed
    then show ?thesis
      by (rule P-eq-list-filter)
  qed
  with repb-no1-def repb-no-def show  repb no = repb no1
    by simp
next
  assume repb-no-no1-eq: repb no = repb no1
  from no1-nln repb-node repb-no-def have repb-no1-def:
    repb no1 =
    hd [sn←(prx@node#sfx). repNodes-eq sn node low high repa]
    by auto
  with no1-nln repb-no-def repb-no-no1-eq
  have repb-Sucn-repa-nl-hd:  hd [sn←(prx@[node]).
    repNodes-eq sn no low high repb] =
    hd [sn←(prx@node#sfx). repNodes-eq sn no1 low high repa]
    by simp
  from filter-take-Sucn-not-empty
  have  hd [sn←(prx@[node]). repNodes-eq sn no low high repb]
    =  hd [sn←(prx@node#sfx) . repNodes-eq sn no low high repb]
    apply −
    apply (rule hd-filter-app [symmetric])
    apply auto
    done
  then have hd-Sucn-hd-whole-list:
    hd [sn←(prx@[node]) .
    repNodes-eq sn no low high repb] =
    hd [sn← (prx@node#sfx). repNodes-eq sn no low high repb]
    by simp
  have hd-nl-repb-repa:
    [sn← (prx@node#sfx). repNodes-eq sn no low high repb]
    = [sn←(prx@node#sfx). repNodes-eq sn no low high repa]
  proof −
    have ∀ x ∈ set (prx@node#sfx).
```

```
    repNodes-eq x no low high repb =
    repNodes-eq x no low high repa
  proof (intro ballI)
    fix x
    assume x-in-nl: x ∈ set (prx@node#sfx)
    from all-nodes-in-nl-nLeafs x-in-nl
    have x-nLeaf: ¬ isLeaf-pt x low high
      by auto
    from  nodes-balanced-ordered [rule-format, OF x-in-nl]
    have x-props: (low x = Null) = (high x = Null) ∧
      low x ∉ set (prx@node#sfx) ∧ high x ∉ set (prx@node#sfx)
      by auto
    with x-nLeaf lno-nNull hno-nNull lno-notin-nl hno-notin-nl
      nodes-notin-nl-neq-nln repa-repb-nc
    show repNodes-eq x no low high repb =
      repNodes-eq x no low high repa
      using [[simp-depth-limit=1]]
      by (simp add: repNodes-eq-def isLeaf-pt-def null-comp-def)
  qed
  then show ?thesis
    by (rule P-eq-list-filter)
qed
with repb-Sucn-repa-nl-hd hd-Sucn-hd-whole-list
have filter-nl-no-no1:
  hd [sn←(prx@node#sfx). repNodes-eq sn no low high repa]
  =  hd [sn←(prx@node#sfx). repNodes-eq sn no1 low high repa]
  by simp
from no-in-nl have filter-no-not-empty:
  [sn←(prx@node#sfx). repNodes-eq sn no low high repa] ≠ []
  apply −
  apply (rule filter-not-empty)
  apply (auto simp add: repNodes-eq-def)
  done
from no1-in-nl have filter-no1-not-empty:
  [sn←(prx@node#sfx). repNodes-eq sn no1 low high repa] ≠ []
  apply −
  apply (rule filter-not-empty)
  apply (auto simp add: repNodes-eq-def)
  done
from repb-no-def hd-Sucn-hd-whole-list hd-nl-repb-repa
have repb no =
  hd [sn←(prx@node#sfx). repNodes-eq sn no low high repa]
  by simp
with hd-filter-prop [OF filter-no-not-empty ]
have repNodes-no-repa: repNodes-eq (repb no) no low high repa
  by auto
from repb-no1-def no1-nln
have
  repb no1 = hd [sn←(prx@node#sfx). repNodes-eq sn no1
```

```
            low high repa]
            by simp
          with hd-filter-prop [OF filter-no1-not-empty ]
          have repNodes-eq (repb no1) no1 low high repa
            by auto
          with filter-nl-no-no1 repNodes-no-repa repb-no-no1-eq
          have (repa ∝ high) no1 =
            (repa ∝ high) no ∧ (repa ∝ low) no1 = (repa ∝ low) no
            by (simp add: repNodes-eq-def)
          with hno-nNull no1-props no1-nLeaf lno-nNull lno-notin-nl
            hno-notin-nl nodes-notin-nl-neq-nln repa-repb-nc
          show (repb ∝ high) no1 =
            (repb ∝ high) no ∧ (repb ∝ low) no1 = (repb ∝ low) no
            using [[simp-depth-limit=1]]
            by (auto simp add: isLeaf-pt-def null-comp-def)
        qed
      qed
    qed
    with repb-repb-no repb-no-share-def share-case-repb no-in-take-Sucn
    show ?thesis
      using [[simp-depth-limit=1]]
      by auto
  qed
 qed
 with repb-no-nNull show ?thesis
  by simp
next
 assume no-nln: no = node
 with repb-node have repb-no-def:
  repb no = hd [sn←(prx@node#sfx). repNodes-eq sn no low high repa]
  by simp
 from no-nln have no ∈ set (prx@node#sfx)
  by auto
 then have filter-nl-repa-not-empty:
  [sn←(prx@node#sfx). repNodes-eq sn no low high repa] ≠ []
  apply −
  apply (rule filter-not-empty)
  apply (auto simp add: repNodes-eq-def)
  done
 then have hd-filter-nl-in-nl:
       hd [sn←(prx@node#sfx). repNodes-eq sn no low high repa] ∈ set
(prx@node#sfx)
  by (rule hd-filter-in-list)
 with repb-no-def
 have repb-no-in-nodeslist: repb no ∈ set (prx@node#sfx)
  by simp
 from nodes-balanced-ordered [rule-format,OF this]
 have repb-no-nNull: repb no ≠ Null
  by auto
```

from *share-cond no-nln* **have** *share-cond-or*:
isLeaf-pt no low high ∨ *repa* (*low no*) ≠ *repa* (*high no*)
by *auto*
have *share-reduce-if*: (*if* (*repb* ∝ *low*) *no* = (*repb* ∝ *high*) *no* ∧ *low no* ≠ *Null*
then repb no = (*repb* ∝ *low*) *no*
else repb no = *hd* [*sn*←(*prx*@[*node*]). *repNodes-eq sn no low high repb*] ∧
repb (*repb no*) = *repb no*
∧ (∀ *no1*∈*set* (*prx*@[*node*]). ((*repb* ∝ *high*) *no1* = (*repb* ∝ *high*) *no*
∧ (*repb* ∝ *low*) *no1* = (*repb* ∝ *low*) *no*) = (*repb no* = *repb no1*)))
proof (*cases isLeaf-pt no low high*)
case *True*
note *isLeaf-no*=*this*
then have *lno-Null*: *low no* = *Null* **by** (*simp add*: *isLeaf-pt-def*)
from *isLeaf-no no-in-take-Sucn repb-no-share-def*
have *repb-no-repb-def*: *repb no*
= *hd* [*sn*←(*prx*@[*node*]). *repNodes-eq sn no low high repb*]
by (*auto simp add*: *isLeaf-pt-def*)
from *isLeaf-no nodes-balanced-ordered* [*rule-format*, *OF no-in-nl*]
have *var-no*: *var no* ≤ *1*
by *auto*
have *all-nodes-nl-var-l-1*: ∀ *x* ∈ *set* (*prx*@*node*#*sfx*). *var x* ≤ *1*
proof (*intro ballI*)
fix *x*
assume *x-in-nl*: *x* ∈ *set* (*prx*@*node*#*sfx*)
from *all-nodes-same-var* [*rule-format*, *OF x-in-nl no-in-nl*] *var-no*
show *var x* ≤ *1*
by *auto*
qed
have *all-nodes-nl-Leafs*: ∀ *x* ∈ *set* (*prx*@*node*#*sfx*). *isLeaf-pt x low high*
proof (*intro ballI*)
fix *x*
assume *x-in-nl*: *x* ∈ *set* (*prx*@*node*#*sfx*)
with *all-nodes-nl-var-l-1* **have** *var x* ≤ *1*
by *auto*
with *nodes-balanced-ordered* [*rule-format*, *OF x-in-nl*]
show *isLeaf-pt x low high*
by *auto*
qed
have *repb-repb-no*: *repb* (*repb no*) = *repb no*
proof −
from *repb-no-share-def no-in-take-Sucn lno-Null*
have *repb-no-def*: *repb no* =
hd [*sn*←(*prx*@[*node*]). *repNodes-eq sn no low high repb*]
by *auto*
with *hd-filter-Sucn-in-Sucn*
have *repb-no-in-take-Sucn*: *repb no* ∈ *set* (*prx*@[*node*])
by *simp*
hence *repb-no-in-nl*: *repb no* ∈ *set* (*prx*@[*node*])

```
    by auto
  with all-nodes-nl-Leafs
  have repb-no-Leaf: isLeaf-pt (repb no) low high
    by auto
  with repb-no-in-take-Sucn repb-no-share-def
  have repb-repb-no-def: repb (repb no) =
    hd [sn←(prx@[node]). repNodes-eq sn (repb no) low high repb]
    by (auto simp add: isLeaf-pt-def)
  from filter-take-Sucn-not-empty
  show ?thesis
    apply (simp only: repb-repb-no-def )
    apply (simp only: repb-no-def)
    apply (rule filter-hd-P-rep-indep)
    apply (auto simp add: repNodes-eq-def)
    done
qed
have two-nodes-repb: (∀ no1∈set (prx@[node]).
     ((repb ∝ high) no1 = (repb ∝ high) no ∧
     (repb ∝ low) no1 = (repb ∝ low) no) = (repb no = repb no1))
proof (intro ballI)
  fix no1
  assume no1-in-take-Sucn: no1 ∈ set (prx@[node])
  from no1-in-take-Sucn
  have no1 ∈ set (prx@node#sfx)
    by auto
  with all-nodes-nl-Leafs
  have isLeaf-no1: isLeaf-pt no1 low high
    by auto
  with repb-no-share-def no1-in-take-Sucn
  have repb-no1-def: repb no1 =
       hd [sn←(prx@[node]). repNodes-eq sn no1 low high repb]
    by (auto simp add: isLeaf-pt-def)
  show ((repb ∝ high) no1 = (repb ∝ high) no
       ∧ (repb ∝ low) no1 = (repb ∝ low) no) = (repb no = repb no1)
  proof
   assume repbchildren-eq-no1-no: (repb ∝ high) no1 = (repb ∝ high) no
        ∧ (repb ∝ low) no1 = (repb ∝ low) no
   have [sn←(prx@[node]). repNodes-eq sn no1 low high repb]
        = [sn←(prx@[node]). repNodes-eq sn no low high repb]
   proof −
     have ∀ x ∈ set (prx@[node]).
          repNodes-eq x no1 low high repb = repNodes-eq x no low high repb
     proof (intro ballI)
       fix x
       assume x-in-take-Sucn:  x ∈ set (prx@[node])
       with repbchildren-eq-no1-no
      show  repNodes-eq x no1 low high repb = repNodes-eq x no low high
repb
         by (simp add: repNodes-eq-def)
```

```
            qed
            then show ?thesis
              by (rule P-eq-list-filter)
          qed
          with repb-no1-def repb-no-repb-def
          show repb no = repb no1
            by simp
        next
          assume repb-no-no1: repb no = repb no1
          with isLeaf-no isLeaf-no1
          show (repb ∝ high) no1 = (repb ∝ high) no
            ∧ (repb ∝ low) no1 = (repb ∝ low) no
            by (simp add: null-comp-def isLeaf-pt-def)
        qed
      qed
    with repb-repb-no lno-Null no-in-take-Sucn repb-no-share-def show ?thesis
      by auto
  next
    assume no-nLeaf: ¬ isLeaf-pt no low high
    with balanced-no obtain
      lno-nNull: low no ≠ Null and
      hno-nNull: high no ≠ Null
      by (simp add: isLeaf-pt-def)
    from no-nLeaf nodes-balanced-ordered [rule-format, OF no-in-nl]
    have var-no: 1 < var no
      by auto
    have all-nodes-nl-var-l-1: ∀ x ∈ set (prx@node#sfx). 1 < var x
    proof (intro ballI)
      fix x
      assume x-in-nl: x ∈ set (prx@node#sfx)
      with all-nodes-same-var [rule-format, OF x-in-nl no-in-nl] var-no
      show 1 < var x
        by simp
    qed
     have all-nodes-nl-nLeafs: ∀ x ∈ set (prx@node#sfx). ¬ isLeaf-pt x low
high
    proof (intro ballI)
      fix x
      assume x-in-nl: x ∈ set (prx@node#sfx)
      with all-nodes-nl-var-l-1 have 1 < var x
        by auto
     with nodes-balanced-ordered [rule-format, OF x-in-nl] show ¬ isLeaf-pt
x low high
        by auto
    qed
    from no-nLeaf share-cond-or
    have repachildren-neq-no: repa (low no) ≠ repa (high no)
      by auto
    with lno-nNull hno-nNull
```

have (*repa* ∝ *low*) *no* ≠ (*repa* ∝ *high*) *no*
by (*simp add*: *null-comp-def*)
with *repa-repb-nc lno-notin-nl hno-notin-nl*
nodes-notin-nl-neq-nln lno-nNull hno-nNull
have *repbchildren-neq-no*: (*repb* ∝ *low*) *no* ≠ (*repb* ∝ *high*) *no*
using [[*simp-depth-limit=1*]]
by (*auto simp add*: *null-comp-def*)
have *repb-repb-no*: *repb* (*repb no*) = *repb no*
proof −
from *repb-no-share-def no-in-take-Sucn repbchildren-neq-no*
have *repb-no-def*: *repb no* =
hd [*sn*←(*prx*@[*node*]). *repNodes-eq sn no low high repb*]
by *auto*
from *filter-take-Sucn-not-empty*
have *repNodes-eq* (*repb no*) *no low high repb*
apply (*simp only*: *repb-no-def*)
apply (*rule hd-filter-prop*)
apply *simp*
done
with *repbchildren-neq-no*
have *repbchildren-neq-repb-no*: (*repb* ∝ *low*) (*repb no*) ≠ (*repb* ∝ *high*)
(*repb no*)
by (*simp add*: *repNodes-eq-def*)
from *filter-take-Sucn-not-empty*
have *repb no* ∈ *set* (*prx*@[*node*])
apply (*simp only*: *repb-no-def*)
apply (*rule hd-filter-in-list*)
apply *simp*
done
with *repbchildren-neq-repb-no repb-no-share-def*
have *repb-repb-no-def*: *repb* (*repb no*) =
hd [*sn*←(*prx*@[*node*]) . *repNodes-eq sn* (*repb no*) *low high repb*]
by *auto*
from *filter-take-Sucn-not-empty* **show** *?thesis*
apply (*simp only*: *repb-repb-no-def*)
apply (*simp only*: *repb-no-def*)
apply (*rule filter-hd-P-rep-indep*)
apply (*auto simp add*: *repNodes-eq-def*)
done
qed
have *two-nodes-repb*: (∀ *no1*∈*set* (*prx*@[*node*]).
((*repb* ∝ *high*) *no1* = (*repb* ∝ *high*) *no* ∧
(*repb* ∝ *low*) *no1* = (*repb* ∝ *low*) *no*) = (*repb no* = *repb no1*))
(**is** (∀ *no1*∈*set* (*prx*@[*node*]). *?P no1*))
proof (*intro ballI*)
fix *no1*
assume *no1-in-take-Sucn*: *no1* ∈ *set* (*prx*@[*node*])
hence *no1-in-nodeslist*: *no1* ∈ *set* (*prx*@*node*#*sfx*)
by *auto*

```
with all-nodes-nl-nLeafs
have no1-nLeaf: ¬ isLeaf-pt no1 low high
  by auto
show ?P no1
proof
 assume repbchildren-eq-no1-no: (repb ∝ high) no1 = (repb ∝ high) no
   ∧ (repb ∝ low) no1 = (repb ∝ low) no
  with repbchildren-neq-no have (repb ∝ high) no1 ≠ (repb ∝ low) no1
   by auto
  with no1-in-take-Sucn repb-no-share-def have repb-no1-def: repb no1
=
   hd [sn←(prx@[node]). repNodes-eq sn no1 low high repb]
   by auto
  from repb-no-share-def no-in-take-Sucn repbchildren-neq-no
  have repb-no-def: repb no =
   hd [sn←(prx@[node]). repNodes-eq sn no low high repb]
   by auto
  have [sn←(prx@[node]). repNodes-eq sn no1 low high repb] =
   [sn←(prx@[node]). repNodes-eq sn no low high repb]
  proof −
   have ∀ x ∈ set (prx@[node]).
    repNodes-eq x no1 low high repb = repNodes-eq x no low high repb
   proof (intro ballI)
    fix x
    assume x-in-take-Sucn: x ∈ set (prx@[node])
    with repbchildren-eq-no1-no
   show repNodes-eq x no1 low high repb = repNodes-eq x no low high
repb
     by (simp add: repNodes-eq-def)
   qed
   then show ?thesis
    by (rule P-eq-list-filter)
  qed
  with repb-no-def repb-no1-def show repb no = repb no1
   by simp
next
  assume repb-no-no1: repb no = repb no1
  from repb-no-share-def no-in-take-Sucn repbchildren-neq-no
  have repb-no-def: repb no =
   hd [sn←(prx@[node]). repNodes-eq sn no low high repb]
   by auto
  from filter-take-Sucn-not-empty
  have repb no ∈ set (prx@[node])
   apply (simp only: repb-no-def)
   apply (rule hd-filter-in-list)
   apply simp
   done
  then have repb-no-in-nl: repb no ∈ set (prx@node#sfx)
   by auto
```

from *filter-take-Sucn-not-empty*
have *repNodes-repb-no*: *repNodes-eq* (*repb no*) *no low high repb*
apply (*simp only*: *repb-no-def*)
apply (*rule hd-filter-prop*)
apply *simp*
done
show (*repb* ∝ *high*) *no1* = (*repb* ∝ *high*) *no*
∧ (*repb* ∝ *low*) *no1* = (*repb* ∝ *low*) *no*
proof (*cases* (*repb* ∝ *low*) *no1* = (*repb* ∝ *high*) *no1*)
case *True*
note *red-cond*=*this*
from *no1-in-nodeslist all-nodes-nl-nLeafs*
have *no1-nLeaf*: ¬ *isLeaf-pt no1 low high*
by *auto*
from *nodes-balanced-ordered* [*rule-format*, *OF no1-in-nodeslist*]
have *no1-props*: (*low no1* ∉ *set* (*prx*@*node*#*sfx*))
∧ (*high no1* ∉ *set* (*prx*@*node*#*sfx*)) ∧(*low no1* = *Null*) = (*high no1* = *Null*)
∧ ((*rep* ∝ *low*) *no1* ∉ *set* (*prx*@*node*#*sfx*))
by *auto*
with *red-cond no1-nLeaf no1-in-take-Sucn repb-no-red-def*
have *repb-no1-def*: *repb no1* = (*repb* ∝ *low*) *no1*
by (*auto simp add*: *isLeaf-pt-def*)
with *no1-nLeaf no1-props* **have** *repb no1* = *repb* (*low no1*)
by (*simp add*: *null-comp-def isLeaf-pt-def*)
from *no1-props no1-nLeaf* **have** *rep* (*low no1*) ∉ *set* (*prx*@*node*#*sfx*)
by (*auto simp add*: *isLeaf-pt-def null-comp-def*)
with *rep-repb-nc no1-props*
have *repb* (*low no1*) ∉ *set* (*prx*@*node*#*sfx*)
by *auto*
with *repb-no1-def repb-no-no1 no1-props no1-nLeaf*
have *repb no* ∉ *set* (*prx*@*node*#*sfx*)
by (*simp add*: *isLeaf-pt-def null-comp-def*)
with *repb-no-in-nl* **show** *?thesis*
by *simp*
next
assume (*repb* ∝ *low*) *no1* ≠ (*repb* ∝ *high*) *no1*
with *repb-no-share-def no1-in-take-Sucn*
have *repb-no1-def*: *repb no1* =
hd [*sn*←(*prx*@[*node*]). *repNodes-eq sn no1 low high repb*]
by *auto*
from *no1-in-take-Sucn*
have [*sn*←(*prx*@[*node*]). *repNodes-eq sn no1 low high repb*] ≠ []
apply −
apply (*rule filter-not-empty*)
apply (*auto simp add*: *repNodes-eq-def*)
done
then
have *repNodes-repb-no1*: *repNodes-eq* (*repb no1*) *no1 low high repb*

apply (*simp only*: *repb-no1-def*)
apply (*rule hd-filter-prop*)
apply *simp*
done
with *repNodes-repb-no repb-no-no1*
have *repNodes-eq no1 no low high repb*
by (*simp add*: *repNodes-eq-def*)
then show *?thesis*
by (*simp add*: *repNodes-eq-def*)
qed
qed
qed
with *repb-repb-no repb-no-share-def no-in-take-Sucn repbchildren-neq-no*
show *?thesis*
using [[*simp-depth-limit=2*]]
by *fastforce*
qed
with *repb-no-nNull* **show** *?thesis*
by *simp*
qed
qed
with *rep-repb-nc* **show** *?thesis*
by (*intro conjI*)
qed
qed

end

9 Proof of Procedure Repoint

theory *RepointProof* **imports** *ProcedureSpecs* **begin**

hide-const (**open**) *DistinctTreeProver.set-of tree.Node tree.Tip*

lemma (**in** *Repoint-impl*) *Repoint-modifies*:
shows $\forall \sigma.\ \Gamma\vdash\{\sigma\}'p :==$ *PROC Repoint* (*p*)
$\{t.\ t$ *may-only-modify-globals* σ *in* [*low*,*high*]$\}$
apply (*hoare-rule HoarePartial.ProcRec1*)
apply (*vcg spec=modifies*)
done

lemma *low-high-exchange-dag*:
assumes *pt-same*: $\forall\, pt.\ pt \notin$ *set-of lt* $\longrightarrow$ *low pt* = *lowa pt* $\wedge$ *high pt* = *higha pt*
assumes *pt-changed*: $\forall\, pt \in$ *set-of lt. lowa pt* = (*rep* $\propto$ *low*) *pt* $\wedge$
higha pt = (*rep* $\propto$ *high*) *pt*
assumes *rep-pt*: $\forall\, pt \in$ *set-of rt. rep pt* = *pt*
shows $\bigwedge q.$ *Dag q* (*rep* $\propto$ *low*) (*rep* $\propto$ *high*) *rt* $\Longrightarrow$
Dag q (*rep* $\propto$ *lowa*) (*rep* $\propto$ *higha*) *rt*
using *rep-pt*

```
proof (induct rt)
  case Tip thus ?case by simp
next
  case (Node lrt q' rrt)
  have Dag q (rep ∝ low) (rep ∝ high) (Node lrt q' rrt) by fact
  then obtain
    q': q = q'  and
    q-notNull: q ≠ Null and
    lrt: Dag ((rep ∝ low) q) (rep ∝ low) (rep ∝ high) lrt and
    rrt: Dag ((rep ∝ high) q) (rep ∝ low) (rep ∝ high) rrt
    by auto
  have rlowa-rlow: ((rep ∝ lowa) q) = ((rep ∝ low) q)
  proof (cases q ∈ set-of lt)
    case True
    note q-in-lt=this
    with pt-changed have lowa-q: lowa q = (rep ∝ low) q
      by simp
    thus (rep ∝ lowa) q = (rep ∝ low) q
    proof (cases low q = Null)
      case True
      with lowa-q have lowa q = Null
        by (simp add: null-comp-def)
      with True show ?thesis
        by (simp add: null-comp-def)
    next
      assume lq-nNull: low q ≠ Null
      show ?thesis
      proof (cases (rep ∝ low) q = Null)
        case True
        with lowa-q have lowa q = Null
          by simp
        with True show ?thesis
          by (simp add: null-comp-def)
      next
        assume rlq-nNull: (rep ∝ low) q ≠ Null
        with lrt lowa-q have lowa q ∈ set-of lrt
          by auto
        with Node.prems Node have lowa q ∈ set-of (Node lrt q' rrt)
          by simp
        with Node.prems have rep (lowa q) = lowa q
          by auto
        with lowa-q rlq-nNull show ?thesis
          by (simp add: null-comp-def)
      qed
    qed
  next
    assume q-notin-lt:  q ∉ set-of lt
    with pt-same have low q = lowa q
      by auto
```

```
  thus ?thesis
    by (simp add: null-comp-def)
qed
have rhigha-rhigh: ((rep ∝ higha) q) = ((rep ∝ high) q)
proof (cases q ∈ set-of lt)
  case True
  note q-in-lt=this
  with pt-changed have higha-q: higha q = (rep ∝ high) q
    by simp
  thus ?thesis
  proof (cases high q = Null)
    case True
    with higha-q have higha q = Null
      by (simp add: null-comp-def)
    with True show ?thesis
      by (simp add: null-comp-def)
  next
    assume hq-nNull: high q ≠ Null
    show ?thesis
    proof (cases (rep ∝ high) q = Null)
      case True
      with higha-q have higha q = Null
        by simp
      with True show ?thesis
        by (simp add: null-comp-def)
    next
      assume rhq-nNull: (rep ∝ high) q ≠ Null
      with rrt higha-q have higha q ∈ set-of rrt
        by auto
      with Node.prems Node have higha q ∈ set-of (Node lrt q' rrt)
        by simp
      with Node.prems have rep (higha q) = higha q
        by auto
      with higha-q rhq-nNull show ?thesis
        by (simp add: null-comp-def)
    qed
  qed
next
  assume q-notin-lt:  q ∉ set-of lt
  with pt-same have high q = higha q
    by auto
  thus ?thesis
    by (simp add: null-comp-def)
qed
with rrt have rhigha-mixed-dag:
  Dag ((rep ∝ higha) q) (rep ∝ low) (rep ∝ high) rrt
  by simp
from lrt rlowa-rlow have rlowa-mixed-dag:
   Dag ((rep ∝ lowa) q) (rep ∝ low) (rep ∝ high) lrt
```

by *simp*
from *Node.prems* **have** *lrt-rep-eq*: $\forall\, pt{\in}$*set-of lrt*. *rep pt* = *pt*
by *simp*
from *Node.prems* **have** *rrt-rep-eq*: $\forall\, pt{\in}$*set-of rrt*. *rep pt* = *pt*
by *simp*
from *rlowa-mixed-dag lrt-rep-eq* **have** *lowa-lrt*:
Dag ((*rep* $\propto$ *lowa*) *q*) (*rep* $\propto$ *lowa*) (*rep* $\propto$ *higha*) *lrt*
apply −
apply (*rule Node.hyps*)
apply *auto*
done
from *rhigha-mixed-dag rrt-rep-eq* **have** *higha-rrt*:
Dag ((*rep* $\propto$ *higha*) *q*) (*rep* $\propto$ *lowa*) (*rep* $\propto$ *higha*) *rrt*
apply −
apply (*rule Node.hyps*)
apply *auto*
done
with *lowa-lrt q′ q-notNull*
show *Dag q* (*rep* $\propto$ *lowa*) (*rep* $\propto$ *higha*) (*Node lrt q′ rrt*)
by *simp*
qed

lemma (**in** *Repoint-impl*) *Repoint-spec′*:
shows
$\forall\,\sigma$. $\Gamma\vdash$ $\{\sigma\}$
´*p* :== *PROC Repoint* (*p*)
$\{\!|\forall$ *rept*. ((*Dag* (($^{\sigma}$*rep* $\propto$ *id*) $^{\sigma}$*p*) ($^{\sigma}$*rep* $\propto$ $^{\sigma}$*low*) ($^{\sigma}$*rep* $\propto$ $^{\sigma}$*high*) *rept*)
$\wedge$ ($\forall$ *no* $\in$ *set-of rept*. $^{\sigma}$*rep no* = *no*))
$\longrightarrow$ *Dag*´*p*´*low*´*high rept* $\wedge$
($\forall\, pt$. *pt* $\notin$ *set-of rept* $\longrightarrow$ $^{\sigma}$*low pt* =´*low pt* $\wedge$ $^{\sigma}$*high pt* =´*high pt*)$|\!\}$
apply (*hoare-rule HoarePartial.ProcRec1*)
apply *vcg*
apply (*rule conjI*)
apply *clarify*
prefer *2*
apply (*intro impI allI*)
apply (*simp add*: *null-comp-def*)
apply (*rule conjI*)
prefer *2*
apply (*clarsimp*)
apply *clarify*
proof −
fix *low high p rep lowa higha pa lowb highb pb rept*
assume *p-nNull*: *p* $\neq$ *Null*
assume *rp-nNull*: *rep p* $\neq$ *Null*

assume *rec-spec-lrept*:
$\forall$ *rept*. *Dag* ((*rep* $\propto$ *id*) (*low* (*rep p*))) (*rep* $\propto$ *low*) (*rep* $\propto$ *high*) *rept*
$\wedge$ ($\forall$ *no*$\in$*set-of rept*. *rep no* = *no*)
$\longrightarrow$ *Dag pa lowa higha rept* $\wedge$
($\forall$ *pt*. *pt* $\notin$ *set-of rept* $\longrightarrow$ *low pt* = *lowa pt* $\wedge$ *high pt* = *higha pt*)
assume *rec-spec-rrept*:
$\forall$ *rept*. *Dag* ((*rep* $\propto$ *id*) (*higha* (*rep p*))) (*rep* $\propto$ *lowa*(*rep p* := *pa*)) (*rep* $\propto$ *higha*) *rept*
$\wedge$ ($\forall$ *no*$\in$*set-of rept*. *rep no* = *no*)
$\longrightarrow$ *Dag pb lowb highb rept* $\wedge$
($\forall$ *pt*. *pt* $\notin$ *set-of rept* $\longrightarrow$ (*lowa*(*rep p* := *pa*)) *pt* = *lowb pt* $\wedge$ *higha pt* = *highb pt*)
assume *rept-dag*: *Dag* ((*rep* $\propto$ *id*) *p*) (*rep* $\propto$ *low*) (*rep* $\propto$ *high*) *rept*
assume *rno-rept*: $\forall$ *no*$\in$*set-of rept*. *rep no* = *no*
show *Dag* (*rep p*) *lowb* (*highb*(*rep p* := *pb*)) *rept* $\wedge$
($\forall$ *pt*. *pt* $\notin$ *set-of rept* $\longrightarrow$ *low pt* = *lowb pt* $\wedge$ *high pt* = (*highb*(*rep p* := *pb*)) *pt*)
proof −
from *rp-nNull rept-dag p-nNull* **obtain** *lrept rrept* **where**
rept-def: *rept* = *Node lrept* (*rep p*) *rrept*
by *auto*
with *rept-dag p-nNull* **have** *lrept-dag*:
Dag ((*rep* $\propto$ *low*) (*rep p*)) (*rep* $\propto$ *low*) (*rep* $\propto$ *high*) *lrept*
by *simp*
from *rept-def rept-dag p-nNull* **have** *rrept-dag*:
Dag ((*rep* $\propto$ *high*) (*rep p*)) (*rep* $\propto$ *low*) (*rep* $\propto$ *high*) *rrept*
by *simp*
from *rno-rept rept-def* **have** *rno-lrept*: $\forall$ *no* $\in$ *set-of lrept*. *rep no* = *no*
by *auto*
from *rno-rept rept-def* **have** *rno-rrept*: $\forall$ *no* $\in$ *set-of rrept*. *rep no* = *no*
by *auto*
have *repoint-post-low*:
Dag pa lowa higha lrept $\wedge$
($\forall$ *pt*. *pt* $\notin$ *set-of lrept* $\longrightarrow$ *low pt* = *lowa pt* $\wedge$ *high pt* = *higha pt*)
proof −
from *lrept-dag* **have** *Dag* ((*rep* $\propto$ *id*) (*low* (*rep p*))) (*rep* $\propto$ *low*) (*rep* $\propto$ *high*) *lrept*
by (*simp add*: *id-trans*)
with *rec-spec-lrept rno-lrept* **show** *?thesis*
apply −
apply (*erule-tac x*=*lrept* **in** *allE*)
apply (*erule impE*)
apply *simp*
apply *assumption*
done
qed
hence *low-lowa-nc*: ($\forall$ *pt*. *pt* $\notin$ *set-of lrept* $\longrightarrow$ *low pt* = *lowa pt* $\wedge$ *high pt* = *higha pt*)
by *simp*
from *lrept-dag repoint-post-low* **obtain**

pa-def: *pa* = (*rep* ∝ *low*) (*rep p*) **and**
lowa-higha-def: (∀ *no* ∈ *set-of lrept*. *lowa no* = (*rep* ∝ *low*) *no* ∧ *higha no* = (*rep* ∝ *high*) *no*)
apply −
apply (*drule Dags-eq-hp-eq*)
apply *auto*
done
from *rept-dag* **have** *rept-DAG*: *DAG rept*
by (*rule Dag-is-DAG*)
with *rept-def* **have** *rp-notin-lrept*: *rep p* ∉ *set-of lrept*
by *simp*
from *rept-DAG rept-def* **have** *rp-notin-rrept*: *rep p* ∉ *set-of rrept*
by *simp*
have *Dag* ((*rep* ∝ *id*) (*higha* (*rep p*))) (*rep* ∝ *lowa*(*rep p* := *pa*)) (*rep* ∝ *higha*) *rrept*
proof −
from *low-lowa-nc rp-notin-lrept* **have** (*rep* ∝ *high*) (*rep p*) = (*rep* ∝ *higha*) (*rep p*)
by (*auto simp add*: *null-comp-def*)
with *rrept-dag* **have** *higha-mixed-rrept*:
Dag ((*rep* ∝ *id*) (*higha* (*rep p*))) (*rep* ∝ *low*) (*rep* ∝ *high*) *rrept*
by (*simp add*: *id-trans*)
thm *low-high-exchange-dag*
with *low-lowa-nc lowa-higha-def rno-rrept* **have** *lowa-higha-rrept*:
Dag ((*rep* ∝ *id*) (*higha* (*rep p*))) (*rep* ∝ *lowa*) (*rep* ∝ *higha*) *rrept*
apply −
apply (*rule low-high-exchange-dag*)
apply *auto*
done
have *Dag* ((*rep* ∝ *id*) (*higha* (*rep p*))) (*rep* ∝ *lowa*) (*rep* ∝ *higha*) *rrept* =
Dag ((*rep* ∝ *id*) (*higha* (*rep p*))) (*rep* ∝ *lowa*(*rep p* := *pa*)) (*rep* ∝ *higha*) *rrept*
proof −
have ∀ *no* ∈ *set-of rrept*. (*rep* ∝ *lowa*) *no* = (*rep* ∝ *lowa*(*rep p* := *pa*)) *no* ∧
(*rep* ∝ *higha*) *no* = (*rep* ∝ *higha*) *no*
proof
fix *no*
assume *no-in-rrept*: *no* ∈ *set-of rrept*
with *rp-notin-rrept* **have** *no* ≠ *rep p*
by *blast*
thus (*rep* ∝ *lowa*) *no* = (*rep* ∝ *lowa*(*rep p* := *pa*)) *no* ∧
(*rep* ∝ *higha*) *no* = (*rep* ∝ *higha*) *no*
by (*simp add*: *null-comp-def*)
qed
thus *?thesis*
by (*rule heaps-eq-Dag-eq*)
qed
with *lowa-higha-rrept* **show** *?thesis*
by *simp*

qed
with *rec-spec-rrept rno-rrept* **have** *repoint-rrept*: *Dag pb lowb highb rrept* ∧
(∀ *pt*. *pt* ∉ *set-of rrept* ⟶
(*lowa*(*rep p* := *pa*)) *pt* = *lowb pt* ∧ *higha pt* = *highb pt*)
apply −
apply (*erule-tac x*=*rrept* **in** *allE*)
apply (*erule impE*)
apply *simp*
apply *assumption*
done
then have *ab-nc*: (∀ *pt*. *pt* ∉ *set-of rrept* ⟶
(*lowa*(*rep p* := *pa*)) *pt* = *lowb pt* ∧ *higha pt* = *highb pt*)
by *simp*
from *repoint-rrept rrept-dag* **obtain**
pb-def: *pb* = ((*rep* ∝ *high*) (*rep p*)) **and**
lowb-highb-def: (∀ *no* ∈ *set-of rrept*. *lowb no* = (*rep* ∝ *low*) *no* ∧ *highb no* = (*rep* ∝ *high*) *no*)
apply −
apply (*drule Dags-eq-hp-eq*)
apply *auto*
done
have *rept-end-dag*: *Dag* (*rep p*) *lowb* (*highb*(*rep p* := *pb*)) *rept*
proof −
have ∀ *no* ∈ *set-of rept*. *lowb no* = (*rep* ∝ *low*) *no* ∧ (*highb*(*rep p* := *pb*)) *no* = (*rep* ∝ *high*) *no*
proof
fix *no*
assume *no-in-rept*: *no* ∈ *set-of rept*
show *lowb no* = (*rep* ∝ *low*) *no* ∧ (*highb*(*rep p* := *pb*)) *no* = (*rep* ∝ *high*) *no*
proof (*cases no* ∈ *set-of rrept*)
case *True*
with *lowb-highb-def pb-def* **show** *?thesis*
by *simp*
next
assume *no-notin-rrept*: *no* ∉ *set-of rrept*
show *?thesis*
proof (*cases no* ∈ *set-of lrept*)
case *True*
with *no-notin-rrept rp-notin-lrept ab-nc*
have *ab-nc-no*: *lowa no* = *lowb no* ∧ *higha no* = *highb no*
apply −
apply (*erule-tac x*=*no* **in** *allE*)
apply (*erule impE*)
apply *simp*
apply (*subgoal-tac no* ≠ *rep p*)
apply *simp*
apply *blast*
done

```
          from lowa-higha-def True have
            lowa no = (rep ∝ low) no ∧ higha no = (rep ∝ high) no
            by auto
          with ab-nc-no have lowb no = (rep ∝ low) no ∧ highb no =(rep ∝ high)
no
            by simp
          with rp-notin-lrept True show ?thesis
            apply (subgoal-tac no ≠ rep p)
            apply simp
            apply blast
            done
        next
          assume no-notin-lrept:  no ∉ set-of lrept
          with no-in-rept rept-def no-notin-rrept have no-rp: no = rep p
            by simp
          with rp-notin-lrept low-lowa-nc have a-nc:
            low no = lowa no ∧ high no = higha no
            by auto
          from rp-notin-rrept no-rp ab-nc have
            (lowa(rep p := pa)) no = lowb no ∧ higha no = highb no
            by auto
            with a-nc pa-def no-rp have (rep ∝ low) no = lowb no ∧ high no =
highb no
            by auto
          with pb-def no-rp show ?thesis
            by simp
        qed
      qed
    qed
    with rept-dag have  Dag (rep p) lowb (highb(rep p := pb)) rept =
      Dag (rep p) (rep ∝ low) (rep ∝ high) rept
      apply −
      thm heaps-eq-Dag-eq
      apply (rule heaps-eq-Dag-eq)
      apply auto
      done
    with rept-dag p-nNull show ?thesis
      by simp
  qed
  have (∀ pt. pt ∉ set-of rept ⟶ low pt = lowb pt ∧ high pt = (highb(rep p :=
pb)) pt)
  proof (intro allI impI)
    fix pt
    assume pt-notin-rept: pt ∉ set-of rept
    with rept-def obtain
      pt-notin-lrept: pt ∉ set-of lrept and
      pt-notin-rrept: pt ∉ set-of rrept and
      pt-neq-rp: pt ≠ rep p
      by simp
```

```
      with low-lowa-nc ab-nc show low pt = lowb pt ∧ high pt = (highb(rep p :=
pb)) pt
        by auto
    qed
    with rept-end-dag show ?thesis
      by simp
  qed
qed

lemma (in Repoint-impl) Repoint-spec:
shows
  ∀ σ rept. Γ⊢ {|σ. Dag ((rep ∝ id)´p) (rep ∝´low) (rep ∝´high) rept
  ∧ (∀ no ∈ set-of rept.´rep no = no) |}
 ´p :== PROC Repoint (p)
  {|Dag´p´low´high rept ∧
  (∀ pt. pt ∉ set-of rept ⟶ ^σlow pt =´low pt ∧ ^σhigh pt =´high pt)|}
apply (hoare-rule HoarePartial.ProcRec1)
apply vcg
apply (rule conjI)
prefer 2
apply  (clarsimp simp add: null-comp-def)
apply clarify
apply (rule conjI)
prefer 2
apply  clarsimp
apply clarify
proof −
  fix rept low high rep p
  assume rept-dag: Dag ((rep ∝ id) p) (rep ∝ low) (rep ∝ high) rept
  assume rno-rept: ∀ no∈set-of rept. rep no = no
  assume p-nNull: p ≠ Null
  assume rp-nNull:  rep p ≠ Null
  show ∃ lrept.
            Dag ((rep ∝ id) (low (rep p))) (rep ∝ low) (rep ∝ high) lrept ∧
            (∀ no∈set-of lrept. rep no = no) ∧
            (∀ lowa higha pa.
                Dag pa lowa higha lrept ∧
                (∀ pt. pt ∉ set-of lrept ⟶
                      low pt = lowa pt ∧ high pt = higha pt) ⟶
                (∃ rrept.
                    Dag ((rep ∝ id) (higha (rep p))) (rep ∝ lowa(rep p := pa))
                     (rep ∝ higha) rrept ∧
                    (∀ no∈set-of rrept. rep no = no) ∧
                    (∀ lowb highb pb.
                        Dag pb lowb highb rrept ∧
                        (∀ pt. pt ∉ set-of rrept ⟶
                              (lowa(rep p := pa)) pt = lowb pt ∧
                              higha pt = highb pt) ⟶
                        Dag (rep p) lowb (highb(rep p := pb)) rept ∧
```

```
                    (∀ pt. pt ∉ set-of rept ⟶
                         low pt = lowb pt ∧
                         high pt = (highb(rep p := pb)) pt))))
proof −
  from rp-nNull rept-dag p-nNull obtain lrept rrept where
    rept-def: rept = Node lrept (rep p) rrept
    by auto
  with rept-dag p-nNull have lrept-dag:
    Dag ((rep ∝ low) (rep p)) (rep ∝ low) (rep ∝ high) lrept
    by simp
  from rept-def rept-dag p-nNull have rrept-dag:
    Dag ((rep ∝ high) (rep p)) (rep ∝ low) (rep ∝ high) rrept
    by simp
  from rno-rept rept-def have rno-lrept: ∀ no ∈ set-of lrept. rep no = no
    by auto
  from rno-rept rept-def have rno-rrept: ∀ no ∈ set-of rrept. rep no = no
    by auto
  show ?thesis
    apply (rule-tac x=lrept in exI)
    apply (rule conjI)
    apply (simp add: id-trans lrept-dag)
    apply (rule conjI)
    apply (rule rno-lrept)
    apply clarify
    subgoal premises prems for lowa higha pa
    proof −
      have lrepta: Dag pa lowa higha lrept by fact
      have low-lowa-nc:
        ∀ pt. pt ∉ set-of lrept ⟶ low pt = lowa pt ∧ high pt = higha pt by fact
      from lrept-dag lrepta obtain
        pa-def: pa = (rep ∝ low) (rep p) and
        lowa-higha-def: ∀ no ∈ set-of lrept.
        lowa no = (rep ∝ low) no ∧ higha no = (rep ∝ high) no
        apply −
        apply (drule Dags-eq-hp-eq)
        apply auto
        done
      from rept-dag have rept-DAG: DAG rept
        by (rule Dag-is-DAG)
      with rept-def have rp-notin-lrept: rep p ∉ set-of lrept
        by simp
      from rept-DAG rept-def have rp-notin-rrept: rep p ∉ set-of rrept
        by simp
      have rrepta: Dag ((rep ∝ id) (higha (rep p)))
                       (rep ∝ lowa(rep p := pa)) (rep ∝ higha) rrept
      proof −
        from low-lowa-nc rp-notin-lrept
        have (rep ∝ high) (rep p) = (rep ∝ higha) (rep p)
          by (auto simp add: null-comp-def)
```

```
    with rrept-dag have higha-mixed-rrept:
      Dag ((rep ∝ id) (higha (rep p))) (rep ∝ low) (rep ∝ high) rrept
      by (simp add: id-trans)
    thm low-high-exchange-dag
    with low-lowa-nc lowa-higha-def rno-rrept
    have lowa-higha-rrept:
       Dag ((rep ∝ id) (higha (rep p))) (rep ∝ lowa) (rep ∝ higha) rrept
      apply −
      apply (rule low-high-exchange-dag)
      apply auto
      done
   have Dag ((rep ∝ id) (higha (rep p))) (rep ∝ lowa) (rep ∝ higha) rrept =
         Dag ((rep ∝ id) (higha (rep p)))
               (rep ∝ lowa(rep p := pa)) (rep ∝ higha) rrept
    proof −
      have ∀ no ∈ set-of rrept.
              (rep ∝ lowa) no = (rep ∝ lowa(rep p := pa)) no ∧
              (rep ∝ higha) no = (rep ∝ higha) no
      proof
        fix no
        assume no-in-rrept: no ∈ set-of rrept
        with rp-notin-rrept have no ≠ rep p
          by blast
        thus (rep ∝ lowa) no = (rep ∝ lowa(rep p := pa)) no ∧
          (rep ∝ higha) no = (rep ∝ higha) no
          by (simp add: null-comp-def)
      qed
      thus ?thesis
        by (rule heaps-eq-Dag-eq)
    qed
    with lowa-higha-rrept show ?thesis
      by simp
  qed
  show ?thesis
    apply (rule-tac x=rrept in exI)
    apply (rule conjI)
    apply  (rule rrepta)
    apply (rule conjI)
    apply  (rule rno-rrept)
    apply clarify
    subgoal premises prems for lowb highb pb
    proof −
      have rreptb: Dag pb lowb highb rrept by fact
      have ab-nc: ∀ pt. pt ∉ set-of rrept ⟶
                    (lowa(rep p := pa)) pt = lowb pt ∧ higha pt = highb pt by
fact
      from rreptb rrept-dag obtain
        pb-def: pb = ((rep ∝ high) (rep p)) and
        lowb-highb-def: ∀ no ∈ set-of rrept.
```

```
                    lowb no = (rep ∝ low) no ∧ highb no = (rep ∝ high) no
      apply −
      apply (drule Dags-eq-hp-eq)
      apply auto
      done
    have rept-end-dag: Dag (rep p) lowb (highb(rep p := pb)) rept
    proof −
      have ∀ no ∈ set-of rept.
            lowb no = (rep ∝ low) no ∧ (highb(rep p := pb)) no = (rep ∝
high) no
      proof
        fix no
        assume no-in-rept: no ∈ set-of rept
        show lowb no = (rep ∝ low) no ∧
             (highb(rep p := pb)) no = (rep ∝ high) no
        proof (cases no ∈ set-of rrept)
          case True
          with lowb-highb-def pb-def show ?thesis
            by simp
        next
          assume no-notin-rrept: no ∉ set-of rrept
          show ?thesis
          proof (cases no ∈ set-of lrept)
            case True
            with no-notin-rrept rp-notin-lrept ab-nc
            have ab-nc-no: lowa no = lowb no ∧ higha no = highb no
              apply −
              apply (erule-tac x=no in allE)
              apply (erule impE)
              apply simp
              apply (subgoal-tac no ≠ rep p)
              apply simp
              apply blast
              done
            from lowa-higha-def True have
              lowa no = (rep ∝ low) no ∧ higha no = (rep ∝ high) no
              by auto
            with ab-nc-no
            have lowb no = (rep ∝ low) no ∧ highb no =(rep ∝ high) no
              by simp
            with rp-notin-lrept True show ?thesis
              apply (subgoal-tac no ≠ rep p)
              apply simp
              apply blast
              done
          next
            assume no-notin-lrept: no ∉ set-of lrept
            with no-in-rept rept-def no-notin-rrept have no-rp: no = rep p
              by simp
```

```
                with rp-notin-lrept low-lowa-nc
                have a-nc: low no = lowa no ∧ high no = higha no
                  by auto
                from rp-notin-rrept no-rp ab-nc
                have (lowa(rep p := pa)) no = lowb no ∧ higha no = highb no
                  by auto
                with a-nc pa-def no-rp
                have (rep ∝ low) no = lowb no ∧ high no = highb no
                  by auto
                with pb-def no-rp show ?thesis
                  by simp
              qed
            qed
          qed
          with rept-dag
          have Dag (rep p) lowb (highb(rep p := pb)) rept =
               Dag (rep p) (rep ∝ low) (rep ∝ high) rept
            apply −
            apply (rule heaps-eq-Dag-eq)
            apply auto
            done
          with rept-dag p-nNull show ?thesis
            by simp
        qed
        have (∀ pt. pt ∉ set-of rept ⟶ low pt = lowb pt ∧
                  high pt = (highb(rep p := pb)) pt)
        proof (intro allI impI)
          fix pt
          assume pt-notin-rept: pt ∉ set-of rept
          with rept-def obtain
            pt-notin-lrept: pt ∉ set-of lrept and
            pt-notin-rrept: pt ∉ set-of rrept and
            pt-neq-rp: pt ≠ rep p
            by simp
          with low-lowa-nc ab-nc
          show low pt = lowb pt ∧ high pt = (highb(rep p := pb)) pt
            by auto
        qed
        with rept-end-dag show ?thesis
          by simp
      qed
    done
  qed
 done
qed
qed

lemma (in Repoint-impl) Repoint-spec-total:
shows
```

```
 ∀σ rept. Γ⊢t {|σ. Dag ((rep ∝ id)´p) (rep ∝´low) (rep ∝´high) rept
 ∧ (∀ no ∈ set-of rept.´rep no = no) |}
´p :== PROC Repoint (p)
 {|Dag´p´low´high rept ∧
 (∀ pt. pt ∉ set-of rept ⟶ ᵒlow pt =´low pt ∧ ᵒhigh pt =´high pt)|}

apply (hoare-rule HoareTotal.ProcRec1
        [where r=measure (λ(s,p). size
                  (dag ((ˢrep ∝ id) ˢp) (ˢrep ∝ ˢlow) (ˢrep ∝ ˢhigh)))])
apply vcg
apply (rule conjI)
prefer 2
apply  (clarsimp simp add: null-comp-def)
apply clarify
apply (rule conjI)
prefer 2
apply  clarsimp
apply clarify
proof −
  fix rept low high rep p
  assume rept-dag: Dag ((rep ∝ id) p) (rep ∝ low) (rep ∝ high) rept
  assume rno-rept: ∀ no∈set-of rept. rep no = no
  assume p-nNull: p ≠ Null
  assume rp-nNull:  rep p ≠ Null
  show ∃ lrept.
            Dag ((rep ∝ id) (low (rep p))) (rep ∝ low) (rep ∝ high) lrept ∧
            (∀ no∈set-of lrept. rep no = no) ∧
            size (dag ((rep ∝ id) (low (rep p))) (rep ∝ low) (rep ∝ high))
            < size (dag ((rep ∝ id) p) (rep ∝ low) (rep ∝ high)) ∧
            (∀ lowa higha pa.
                Dag pa lowa higha lrept ∧
                (∀ pt. pt ∉ set-of lrept ⟶
                      low pt = lowa pt ∧ high pt = higha pt) ⟶
                (∃ rrept.
                    Dag ((rep ∝ id) (higha (rep p))) (rep ∝ lowa(rep p := pa))
                     (rep ∝ higha) rrept ∧
                    (∀ no∈set-of rrept. rep no = no) ∧
                    size (dag ((rep ∝ id) (higha (rep p)))
                          (rep ∝ lowa(rep p := pa)) (rep ∝ higha))
                    < size (dag ((rep ∝ id) p) (rep ∝ low) (rep ∝ high)) ∧
                    (∀ lowb highb pb.
                        Dag pb lowb highb rrept ∧
                        (∀ pt. pt ∉ set-of rrept ⟶
                              (lowa(rep p := pa)) pt = lowb pt ∧
                              higha pt = highb pt) ⟶
                        Dag (rep p) lowb (highb(rep p := pb)) rept ∧
                        (∀ pt. pt ∉ set-of rept ⟶
                              low pt = lowb pt ∧
                              high pt = (highb(rep p := pb)) pt))))
```

proof −
from *rp-nNull rept-dag p-nNull* **obtain** *lrept rrept* **where**
rept-def: *rept* = *Node lrept* (*rep p*) *rrept*
by *auto*
with *rept-dag p-nNull* **have** *lrept-dag*:
Dag ((*rep* ∝ *low*) (*rep p*)) (*rep* ∝ *low*) (*rep* ∝ *high*) *lrept*
by *simp*
from *rept-def rept-dag p-nNull* **have** *rrept-dag*:
Dag ((*rep* ∝ *high*) (*rep p*)) (*rep* ∝ *low*) (*rep* ∝ *high*) *rrept*
by *simp*
from *rno-rept rept-def* **have** *rno-lrept*: ∀ *no* ∈ *set-of lrept*. *rep no* = *no*
by *auto*
from *rno-rept rept-def* **have** *rno-rrept*: ∀ *no* ∈ *set-of rrept*. *rep no* = *no*
by *auto*
show *?thesis*
apply (*rule-tac x*=*lrept* **in** *exI*)
apply (*rule conjI*)
apply (*simp add*: *id-trans lrept-dag*)
apply (*rule conjI*)
apply (*rule rno-lrept*)
apply (*rule conjI*)
using *rept-dag rept-def*
apply (*simp only*: *Dag-dag*)
apply (*clarsimp simp add*: *id-trans Dag-dag*)
apply *clarify*
subgoal premises *prems* **for** *lowa higha pa*
proof −
have *lrepta*: *Dag pa lowa higha lrept* **by** *fact*
have *low-lowa-nc*:
∀ *pt*. *pt* ∉ *set-of lrept* ⟶ *low pt* = *lowa pt* ∧ *high pt* = *higha pt* **by** *fact*
from *lrept-dag lrepta* **obtain**
pa-def: *pa* = (*rep* ∝ *low*) (*rep p*) **and**
lowa-higha-def: ∀ *no* ∈ *set-of lrept*.
lowa no = (*rep* ∝ *low*) *no* ∧ *higha no* = (*rep* ∝ *high*) *no*
apply −
apply (*drule Dags-eq-hp-eq*)
apply *auto*
done
from *rept-dag* **have** *rept-DAG*: *DAG rept*
by (*rule Dag-is-DAG*)
with *rept-def* **have** *rp-notin-lrept*: *rep p* ∉ *set-of lrept*
by *simp*
from *rept-DAG rept-def* **have** *rp-notin-rrept*: *rep p* ∉ *set-of rrept*
by *simp*
have *rrepta*: *Dag* ((*rep* ∝ *id*) (*higha* (*rep p*)))
(*rep* ∝ *lowa*(*rep p* := *pa*)) (*rep* ∝ *higha*) *rrept*
proof −
from *low-lowa-nc rp-notin-lrept*
have (*rep* ∝ *high*) (*rep p*) = (*rep* ∝ *higha*) (*rep p*)

```
    by (auto simp add: null-comp-def)
  with rrept-dag have higha-mixed-rrept:
    Dag ((rep ∝ id) (higha (rep p))) (rep ∝ low) (rep ∝ high) rrept
    by (simp add: id-trans)
  thm low-high-exchange-dag
  with low-lowa-nc lowa-higha-def rno-rrept
  have lowa-higha-rrept:
     Dag ((rep ∝ id) (higha (rep p))) (rep ∝ lowa) (rep ∝ higha) rrept
    apply −
    apply (rule low-high-exchange-dag)
    apply auto
    done
 have Dag ((rep ∝ id) (higha (rep p))) (rep ∝ lowa) (rep ∝ higha) rrept =
       Dag ((rep ∝ id) (higha (rep p)))
             (rep ∝ lowa(rep p := pa)) (rep ∝ higha) rrept
  proof −
    have ∀ no ∈ set-of rrept.
            (rep ∝ lowa) no = (rep ∝ lowa(rep p := pa)) no ∧
            (rep ∝ higha) no = (rep ∝ higha) no
    proof
      fix no
      assume no-in-rrept: no ∈ set-of rrept
      with rp-notin-rrept have no ≠ rep p
        by blast
      thus (rep ∝ lowa) no = (rep ∝ lowa(rep p := pa)) no ∧
        (rep ∝ higha) no = (rep ∝ higha) no
        by (simp add: null-comp-def)
    qed
    thus ?thesis
      by (rule heaps-eq-Dag-eq)
  qed
  with lowa-higha-rrept show ?thesis
    by simp
qed
show ?thesis
  apply (rule-tac x=rrept in exI)
  apply (rule conjI)
  apply  (rule rrepta)
  apply (rule conjI)
  apply  (rule rno-rrept)
  apply (rule conjI)
  using rept-dag rept-def rrepta
  apply  (simp only: Dag-dag)
  apply  (clarsimp simp add: id-trans Dag-dag)
  apply clarify
  subgoal premises prems for lowb highb pb
  proof −
    have rreptb: Dag pb lowb highb rrept by fact
    have ab-nc: ∀ pt. pt ∉ set-of rrept ⟶
```

```
                    (lowa(rep p := pa)) pt = lowb pt ∧ higha pt = highb pt by
fact
          from rreptb rrept-dag obtain
            pb-def: pb = ((rep ∝ high) (rep p)) and
            lowb-highb-def: ∀ no ∈ set-of rrept.
                        lowb no = (rep ∝ low) no ∧ highb no = (rep ∝ high) no
            apply −
            apply (drule Dags-eq-hp-eq)
            apply auto
            done
          have rept-end-dag: Dag (rep p) lowb (highb(rep p := pb)) rept
          proof −
            have ∀ no ∈ set-of rept.
                  lowb no = (rep ∝ low) no ∧ (highb(rep p := pb)) no = (rep ∝
high) no
            proof
              fix no
              assume no-in-rept: no ∈ set-of rept
              show lowb no = (rep ∝ low) no ∧
                   (highb(rep p := pb)) no = (rep ∝ high) no
              proof (cases no ∈ set-of rrept)
                case True
                with lowb-highb-def pb-def show ?thesis
                  by simp
              next
                assume no-notin-rrept: no ∉ set-of rrept
                show ?thesis
                proof (cases no ∈ set-of lrept)
                  case True
                  with no-notin-rrept rp-notin-lrept ab-nc
                  have ab-nc-no: lowa no = lowb no ∧ higha no = highb no
                    apply −
                    apply (erule-tac x=no in allE)
                    apply (erule impE)
                    apply simp
                    apply (subgoal-tac no ≠ rep p)
                    apply simp
                    apply blast
                    done
                  from lowa-higha-def True have
                    lowa no = (rep ∝ low) no ∧ higha no = (rep ∝ high) no
                    by auto
                  with ab-nc-no
                  have lowb no = (rep ∝ low) no ∧ highb no =(rep ∝ high) no
                    by simp
                  with rp-notin-lrept True show ?thesis
                    apply (subgoal-tac no ≠ rep p)
                    apply simp
                    apply blast
```

```
            done
          next
            assume no-notin-lrept:  no ∉ set-of lrept
            with no-in-rept rept-def no-notin-rrept have no-rp: no = rep p
              by simp
            with rp-notin-lrept low-lowa-nc
            have a-nc: low no = lowa no ∧ high no = higha no
              by auto
            from rp-notin-rrept no-rp ab-nc
            have (lowa(rep p := pa)) no = lowb no ∧ higha no = highb no
              by auto
            with a-nc pa-def no-rp
            have (rep ∝ low) no = lowb no ∧ high no = highb no
              by auto
            with pb-def no-rp show ?thesis
              by simp
          qed
        qed
      qed
      with rept-dag
      have Dag (rep p) lowb (highb(rep p := pb)) rept =
           Dag (rep p) (rep ∝ low) (rep ∝ high) rept
        apply −
        apply (rule heaps-eq-Dag-eq)
        apply auto
        done
      with rept-dag p-nNull show ?thesis
        by simp
    qed
    have (∀ pt. pt ∉ set-of rept ⟶ low pt = lowb pt ∧
              high pt = (highb(rep p := pb)) pt)
    proof (intro allI impI)
      fix pt
      assume pt-notin-rept: pt ∉ set-of rept
      with rept-def obtain
        pt-notin-lrept: pt ∉ set-of lrept and
        pt-notin-rrept: pt ∉ set-of rrept and
        pt-neq-rp: pt ≠ rep p
        by simp
      with low-lowa-nc ab-nc
      show low pt = lowb pt ∧ high pt = (highb(rep p := pb)) pt
        by auto
    qed
    with rept-end-dag show ?thesis
      by simp
  qed
 done
qed
done
```

qed
qed

end

10 Proof of Procedure Normalize

theory *NormalizeTotalProof* **imports** *LevellistProof ShareReduceRepListProof RepointProof* **begin**

hide-const (**open**) *DistinctTreeProver.set-of tree.Node tree.Tip*

lemma (**in** *Normalize-impl*) *Normalize-modifies*:
shows
$\forall \sigma.\ \Gamma\vdash\{\sigma\}\acute{p} :== PROC\ Normalize\ (p)$
$\{t.\ t\ may\text{-}only\text{-}modify\text{-}globals\ \sigma\ in\ [rep,mark,low,high,next]\}$
apply (*hoare-rule HoarePartial.ProcRec1*)
apply (*vcg spec=modifies*)
done

lemma (**in** *Normalize-impl*) *Normalize-spec*:
shows $\forall \sigma\ pret\ prebdt.\ \Gamma\vdash_t$
$\{\!|\sigma.\ Dag\acute{p}low\acute{}high\ pret \wedge ordered\ pret\acute{}var \wedge$
$\acute{p} \neq Null \wedge (\forall n.\ n \in set\text{-}of\ pret \longrightarrow \acute{}mark\ n = \acute{}mark\acute{p}) \wedge$
$bdt\ pret\acute{}var = Some\ prebdt|\!\}$
$\acute{p} :== PROC\ Normalize(p)$
$\{\!|(\forall pt.\ pt \notin set\text{-}of\ pret$
$\longrightarrow {}^{\sigma}rep\ pt = \acute{}rep\ pt \wedge {}^{\sigma}low\ pt = \acute{}low\ pt \wedge {}^{\sigma}high\ pt = \acute{}high\ pt \wedge$
${}^{\sigma}mark\ pt = \acute{}mark\ pt \wedge {}^{\sigma}next\ pt = \acute{}next\ pt) \wedge$
$(\exists postt.\ Dag\acute{p}low\acute{}high\ postt \wedge reduced\ postt \wedge$
$shared\ postt\ {}^{\sigma}var \wedge ordered\ postt\ {}^{\sigma}var \wedge$
$set\text{-}of\ postt \subseteq set\text{-}of\ pret \wedge$
$(\exists postbdt.\ \ bdt\ postt\ {}^{\sigma}var = Some\ postbdt \wedge prebdt \sim postbdt)) \wedge$
$(\forall\ no.\ no \in set\text{-}of\ pret \longrightarrow \acute{}mark\ no = (\neg\ {}^{\sigma}mark\ {}^{\sigma}p))\ |\!\}$
apply (*hoare-rule HoareTotal.ProcNoRec1*)
apply (*hoare-rule anno=*
$\acute{}levellist :== replicate\ (p \rightarrow \acute{}var + 1)\ Null;;$
$\acute{}levellist :== CALL\ Levellist\ (p,\ (\neg \acute{p} \rightarrow \acute{}mark)\ ,\acute{}levellist);;$
$(ANNO\ (\tau,ll).\ \{\!|\tau.\ Levellist\acute{}levellist\acute{}next\ ll \wedge$
$Dag\ {}^{\sigma}p\ {}^{\sigma}low\ {}^{\sigma}high\ pret \wedge ordered\ pret\ {}^{\sigma}var \wedge {}^{\sigma}p \neq Null \wedge$
$(bdt\ pret\ {}^{\sigma}var\ = Some\ prebdt) \wedge$
$wf\text{-}ll\ pret\ ll\ {}^{\sigma}var \wedge$
$length\acute{}levellist = (({}^{\sigma}p \rightarrow {}^{\sigma}var) + 1) \wedge$
$wf\text{-}marking\ pret\ {}^{\sigma}mark\acute{}mark\ (\neg\ {}^{\sigma}mark\ {}^{\sigma}p) \wedge$
$(\forall pt.\ pt \notin set\text{-}of\ pret \longrightarrow {}^{\sigma}next\ pt = \acute{}next\ pt) \wedge$
$\acute{}low = {}^{\sigma}low \wedge \acute{}high = {}^{\sigma}high\ \wedge \acute{p} = {}^{\sigma}p \wedge \acute{}rep = {}^{\sigma}rep \wedge$
$\acute{}var = {}^{\sigma}var|\!\}$
$\acute{n} :== 0;;$
$WHILE\ (n < length\acute{}levellist)$

```
  INV {|Levellist´levellist´next ll ∧
       Dag ^σp ^σlow ^σhigh pret ∧ ordered pret ^σvar ∧ ^σp ≠ Null ∧
       (bdt pret ^σvar  = Some prebdt) ∧ wf-ll pret ll ^σvar  ∧
       length ^τlevellist = ((^σp → ^σvar) + 1) ∧
       wf-marking pret ^σmark ^τmark (¬ ^σmark ^σp) ∧
       ^τlow = ^σlow ∧ ^τhigh = ^σhigh  ∧  ^τp = ^σp ∧ ^τrep = ^σrep  ∧ ^τvar = ^σvar ∧
      ´n ≤ length  ^τlevellist ∧
       (∀ pt i. (pt ∉ set-of pret ∨ (n <= i ∧ pt ∈ set (ll ! i) ∧
                i <length ^τlevellist )
                ⟶ ^σrep pt =´rep pt)) ∧
      ´rep ‘ Nodes´n ll ⊆ Nodes´n ll ∧
       (∀ no ∈ Nodes´n ll.
          no→´rep→^σvar <= no→^σvar ∧
          (∃ not nort. Dag (rep no) (rep ∝ ^σlow ) (rep ∝ ^σhigh ) nort ∧
             Dag no ^σlow ^σhigh not ∧ reduced nort ∧
             ordered nort ^σvar ∧ set-of nort ⊆´rep ‘ Nodes´n ll ∧
             (∀  no ∈ set-of nort.´rep no = no) ∧
             (∃ nobdt norbdt. bdt not ^σvar = Some nobdt ∧
                bdt nort ^σvar = Some norbdt ∧ nobdt ∼ norbdt))) ∧
       (∀ t1 t2.
          t1∈Dags (rep ‘(Nodes´n ll))(rep ∝ ^σlow )(rep ∝ ^σhigh)∧
          t2∈Dags (rep ‘(Nodes´n ll))(rep ∝ ^σlow )(rep ∝ ^σhigh)
          ⟶
          isomorphic-dags-eq t1 t2  ^σvar) ∧
      ´levellist = ^τlevellist ∧´next = ^τnext ∧´mark = ^τmark ∧´low = ^σlow ∧
      ´high = ^σhigh  ∧´p = ^σp ∧ ´var = ^σvar |}
  VAR MEASURE (length´levellist −´n)
  DO
  CALL ShareReduceRepList(levellist !´n);;
 ´n :=´n + 1
  OD
  {|(∃ postnormt. Dag (rep ^σp) (rep ∝ ^σlow ) (rep ∝ ^σhigh ) postnormt ∧
    reduced postnormt ∧ shared postnormt ^σvar ∧
    ordered postnormt ^σvar ∧ set-of postnormt ⊆ set-of pret ∧
     (∃ postnormbdt. bdt postnormt ^σvar = Some postnormbdt ∧ prebdt ∼ post-
normbdt) ∧
    (∀  no ∈ set-of postnormt. (rep no = no))) ∧
    ordered pret ^σvar  ∧ ^σp ≠ Null ∧
    (∀  pt. pt ∉ set-of pret ⟶ ^σrep pt =´rep pt) ∧
   ´levellist = ^τlevellist ∧´next = ^τnext ∧´mark = ^τmark ∧´low = ^σlow ∧´high = ^σ
high ∧
   ´p = ^σp ∧ (∀  no. no ∈ set-of pret ⟶´mark no = (¬ ^σmark ^σp)) |})
  ;;
 ´p :== CALL Repoint (p)
  in HoareTotal.annotateI)
 apply (vcg spec=spec-total)
 prefer 2

 apply    (simp add: Nodes-def null-comp-def)
```

apply (*rule-tac x=pret* **in** *exI*)
apply *clarify*
apply (*rule conjI*)
apply *clarsimp*
apply (*case-tac i*)
apply *simp*
apply *simp*
apply (*rule conjI*)
apply *simp*
apply (*rule conjI*)
apply *simp*
apply (*rule conjI*)
apply *simp*
apply *clarify*
apply (*simp* (*no-asm-use*) *only*: *simp-thms*)
apply (*rule-tac x=ll* **in** *exI*)
apply (*rule conjI*)
apply *assumption*
apply *clarify*
apply (*simp only*: *simp-thms triv-forall-equality True-implies-equals*)
apply (*rule-tac x=postnormt* **in** *exI*)
apply (*rule conjI*)
apply *simp*
apply (*rule conjI*)
apply *simp*
apply *clarify*
apply (*simp* (*no-asm-simp*))
prefer *2*

apply *clarify*
apply (*simp only*: *simp-thms triv-forall-equality True-implies-equals*)
apply (*rule-tac x=ll*!*n* **in** *exI*)
apply (*rule conjI*)
apply (*simp add*: *Levellist-def*)
prefer *3*

apply (*clarify*)
apply (*simp* (*no-asm-use*) *only*: *simp-thms triv-forall-equality True-implies-equals*)

proof −
— End of while (invariant + false condition) to end of inner SPEC
fix *var p rep mark vara lowa higha pa levellista repa marka nexta varb ll nb pret prebdt* **and** *low* :: *ref* ⇒ *ref* **and** *high* :: *ref* ⇒ *ref* **and** *repb* :: *ref* ⇒ *ref*
assume *ll*: *Levellist levellista nexta ll*
assume *wf-lla*: *wf-ll pret ll var*
assume *length-lla*: *length levellista* = *var p* + *1*
assume *ord-pret*: *ordered pret var*

assume *pnN*: $p \neq Null$
assume *rep-repb-nc*:
$\forall\, pt\ i.\ pt \notin set\text{-}of\ pret \vee nb \leq i \wedge pt \in set\ (ll\ !\ i) \wedge$
$i < length\ levellista$
$\longrightarrow rep\ pt = repb\ pt$

assume *wf-marking-prop*: $wf\text{-}marking\ pret\ mark\ marka\ (\neg\ mark\ p)$
assume *pret-dag*: $Dag\ p\ low\ high\ pret$
assume *prebdt*: $bdt\ pret\ var = Some\ prebdt$
assume *not-nbslla*: $\neg\ nb < length\ levellista$
assume *nb-le-lla*: $nb \leq length\ levellista$

assume *normalize-prop*: $\forall\, no{\in}Nodes\ nb\ ll.$
$var\ (repb\ no) \leq var\ no \wedge$
$(\exists\, not\ nort.\ Dag\ (repb\ no)\ (repb \propto low)\ (repb \propto high)\ nort \wedge$
$Dag\ no\ low\ high\ not \wedge reduced\ nort \wedge ordered\ nort\ var \wedge$
$set\text{-}of\ nort \subseteq repb\ `\ Nodes\ nb\ ll \wedge$
$(\forall\, no{\in}set\text{-}of\ nort.\ repb\ no = no) \wedge$
$(\exists\, nobdt\ norbdt.\ bdt\ not\ var = Some\ nobdt \wedge$
$bdt\ nort\ var = Some\ norbdt \wedge nobdt \sim norbdt))$
assume *repbNodes-in-Nodes*: $repb\ `\ Nodes\ nb\ ll \subseteq Nodes\ nb\ ll$
assume *shared-mult-dags*:
$\forall\, t1\ t2.\ t1 \in Dags\ (repb\ `\ Nodes\ nb\ ll)\ (repb \propto low)\ (repb \propto high) \wedge$
$t2 \in Dags\ (repb\ `\ Nodes\ nb\ ll)\ (repb \propto low)\ (repb \propto high)$
$\longrightarrow isomorphic\text{-}dags\text{-}eq\ t1\ t2\ var$
show $(\exists\, postnormt.\ Dag\ (repb\ p)\ (repb \propto low)\ (repb \propto high)\ postnormt \wedge$
$reduced\ postnormt \wedge shared\ postnormt\ var \wedge$
$ordered\ postnormt\ var \wedge set\text{-}of\ postnormt \subseteq set\text{-}of\ pret \wedge$
$(\exists\, postnormbdt.$
$bdt\ postnormt\ var = Some\ postnormbdt \wedge prebdt \sim postnormbdt) \wedge$
$(\forall\ \ no \in set\text{-}of\ postnormt.\ repb\ no = no)) \wedge$
$ordered\ pret\ var \wedge p \neq Null \wedge$
$(\forall\, pt.\ pt \notin set\text{-}of\ pret \longrightarrow rep\ pt = repb\ pt) \wedge$
$(\forall\, no.\ no \in set\text{-}of\ pret \longrightarrow marka\ no = (\neg\ mark\ p))$

proof −
from *ll* **have** *length-ll-eq*: $length\ levellista = length\ ll$
by (*simp add*: *Levellist-length*)
from *rep-repb-nc* **have** *rep-nc-post*: $\forall\, pt.\ pt \notin set\text{-}of\ pret \longrightarrow rep\ pt = repb\ pt$
by *auto*
from *pnN pret-dag* **obtain** *lt rt* **where** *pret-def*: $pret = Node\ lt\ p\ rt$
by *auto*
from *wf-marking-prop pret-def*
have *marking-inverted*: $(\forall\, no.\ no \in set\text{-}of\ pret \longrightarrow marka\ no = (\neg\ mark\ p))$
by (*simp add*: *wf-marking-def*)
from *not-nbslla nb-le-lla* **have** *nb-length-lla*: $nb = length\ levellista$
by *simp*
with *length-lla* **have** *varp-s-nb*: $var\ p < nb$
by *simp*

from *pret-def* **have** *p-in-pret*: $p \in$ *set-of pret*
by *simp*
with *wf-lla* **have** $p \in$ *set* (*ll* ! (*var p*))
by (*simp add*: *wf-ll-def*)
with *varp-s-nb* **have** *p-in-Nodes*: $p \in$ *Nodes nb ll*
by (*auto simp add*: *Nodes-def*)
with *normalize-prop* **obtain** *not nort* **where**
varrepno-varno: *var* (*repb p*) $\leq$ *var p* **and**
nort-dag: *Dag* (*repb p*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*) *nort* **and**
not-dag: *Dag p low high not* **and**
red-nort: *reduced nort* **and**
ord-nort: *ordered nort var* **and**
nort-in-repNodes: *set-of nort* $\subseteq$ *repb* ‘ *Nodes nb ll* **and**
nort-repb: ($\forall$ *no*$\in$*set-of nort*. *repb no* = *no*) **and**
bdt-prop: $\exists$ *nobdt norbdt*. *bdt not var* = *Some nobdt* $\wedge$ *bdt nort var* = *Some norbdt* $\wedge$
nobdt $\sim$ *norbdt*
by *auto*

from *wf-lla nb-length-lla* **have** *Nodes-in-pret*: *Nodes nb ll* $\subseteq$ *set-of pret*
apply −
apply (*rule Nodes-in-pret*)
apply (*auto simp add*: *length-ll-eq*)
done
from *pret-dag wf-lla nb-length-lla* **have** *Null* $\notin$ *Nodes nb ll*
apply −
apply (*rule Null-notin-Nodes*)
apply (*auto simp add*: *length-ll-eq*)
done
with *p-in-Nodes repbNodes-in-Nodes* **have** *rp-nNull*: *repb p* $\neq$ *Null*
by *auto*
with *nort-dag* **have** *nort-nTip*: *nort*$\neq$ *Tip*
by *auto*
have $\exists$ *postnormt*. *Dag* (*repb p*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*) *postnormt* $\wedge$
reduced postnormt $\wedge$ *shared postnormt var* $\wedge$
ordered postnormt var $\wedge$ *set-of postnormt* $\subseteq$ *set-of pret* $\wedge$
($\exists$ *postnormbdt*.
bdt postnormt var = *Some postnormbdt* $\wedge$ *prebdt* $\sim$ *postnormbdt*) $\wedge$
($\forall$ *no* $\in$ *set-of postnormt*. *repb no* = *no*)
proof (*rule-tac x*=*nort* **in** *exI*)
from *nort-in-repNodes repbNodes-in-Nodes Nodes-in-pret*
have *nort-in-pret*: *set-of nort* $\subseteq$ *set-of pret*
by *blast*
from *not-dag pret-dag* **have** *not-pret*: *not* = *pret*
by (*simp add*: *Dag-unique*)
with *bdt-prop prebdt* **have** *pret-bdt-prop*:
$\exists$ *postnormbdt*.
bdt nort var = *Some postnormbdt* $\wedge$ *prebdt* $\sim$ *postnormbdt*
by *auto*

from *shared-mult-dags* **have** *shared nort var*
proof (*auto simp add*: *shared-def isomorphic-dags-eq-def*)
fix *st1 st2 bdt1*
assume *shared-imp*:
$\forall$ *t1 t2*. *t1*$\in$*Dags* (*repb* ‘ *Nodes nb ll*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*) $\wedge$
t2 $\in$ *Dags* (*repb* ‘ *Nodes nb ll*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*)
$\longrightarrow$
($\exists$ *bdt1*. *bdt t1 var* = *Some bdt1* $\wedge$ *bdt t2 var* = *Some bdt1*) $\longrightarrow$ *t1* = *t2*
assume *st1-nort*: *st1* $\leq$ *nort*
assume *st2-nort*: *st2* $\leq$ *nort*
assume *bdt-st1*: *bdt st1 var* = *Some bdt1*
assume *bdt-st2*: *bdt st2 var* = *Some bdt1*
from *nort-in-repNodes nort-dag nort-nTip*
have *nort-in-DagsrNodes*:
nort $\in$ *Dags* (*repb* ‘(*Nodes nb ll*)) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*)
apply −
apply (*rule DagsI*)
apply *auto*
done
show *st1* = *st2*
proof (*cases st1*)
case *Tip*
note *st1-Tip*=*this*
with *bdt-st1 bdt-st2* **show** *?thesis*
by *auto*
next
case (*Node lst1 st1p rst1*)
note *st1-Node*=*this*
then have *st1-nTip*: *st1* $\neq$ *Tip*
by *simp*
show *?thesis*
proof (*cases st2*)
case *Tip*
with *bdt-st1 bdt-st2* **show** *?thesis*
by *auto*
next
case (*Node lst2 st2p rst2*)
note *st2-Node*=*this*
then have *st2-nTip*: *st2* $\neq$ *Tip*
by *simp*
from *nort-in-DagsrNodes st1-nort ord-nort wf-lla st1-nTip*
have *st1-in-Dags*:
st1 $\in$ *Dags* (*repb* ‘ *Nodes nb ll*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*)
apply −
apply (*rule Dags-subdags*)
apply *auto*
done
from *nort-in-DagsrNodes st2-nort ord-nort wf-lla st2-nTip*
have *st2-in-Dags*:

st2 ∈ *Dags* (*repb* ‘ *Nodes nb ll*) (*repb* ∝ *low*) (*repb* ∝ *high*)
apply −
apply (*rule Dags-subdags*)
apply *auto*
done
from *st1-in-Dags st2-in-Dags bdt-st1 bdt-st2 shared-imp* **show** *st1=st2*
by *simp*
qed
qed
qed
with *nort-dag red-nort ord-nort nort-in-pret pret-bdt-prop nort-repb*
show *Dag* (*repb p*) (*repb* ∝ *low*) (*repb* ∝ *high*) *nort* ∧
reduced nort ∧ *shared nort var* ∧ *ordered nort var* ∧
set-of nort ⊆ *set-of pret* ∧
(∃ *postnormbdt*.
bdt nort var = *Some postnormbdt* ∧ *prebdt* ∼ *postnormbdt*) ∧
(∀ *no* ∈ *set-of nort*. *repb no* = *no*)
apply −
apply (*intro conjI*)
apply *assumption*+
done
qed
with *wf-lla length-lla ord-pret pnN rep-nc-post marking-inverted*
show *?thesis*
by *simp*
qed
next
— From postcondition inner SPEC to final postcondition
fix *var low high p rep levellist marka next*
nexta lowb highb pb levellista ll repa pret prebdt
and *mark*::*ref*⇒*bool* **and** *postnormt postnormbdt*
assume *ll*: *Levellist levellista nexta ll*
assume *repoint-spec*:
Dag pb lowb highb postnormt
∀ *pt*. *pt* ∉ *set-of postnormt* ⟶ *low pt* = *lowb pt* ∧ *high pt* = *highb pt*
assume *pret-dag*: *Dag p low high pret*
assume *ord-pret*: *ordered pret var*
assume *pnN*: *p* ≠ *Null*
assume *onemark-pret*:
∀ *n*. *n* ∈ *set-of pret* ⟶ *mark n* = *mark p* (**is** ∀ *n*. *?in-pret n* ⟶ *?eq-mark-p n*)
assume *pret-bdt*: *bdt pret var* = *Some prebdt*

assume *wf-ll*: *wf-ll pret ll var* **and**
length-ll:*length levellist* =*var p* + *1* **and**
wf-marking-ll: *wf-marking pret mark marka* (¬ *mark p*)
assume
postnormt-dag: *Dag* (*repa p*) (*repa* ∝ *low*) (*repa* ∝ *high*) *postnormt* **and**
reduced-postnormt: *reduced postnormt* **and**
shared-postnormt: *shared postnormt var* **and**

ordered-postnormt: *ordered postnormt var* **and**
subset-pret: *set-of postnormt* $\subseteq$ *set-of pret***and**
sim-bdt: *bdt postnormt var* = *Some postnormbdt prebdt* $\sim$ *postnormbdt* **and**
postdag-repa: $\forall$ *no* $\in$ *set-of postnormt*. *repa no* = *no* **and**
rep-eq: $\forall$ *pt*. *pt* $\notin$ *set-of pret* $\longrightarrow$ *rep pt* = *repa pt* **and**
pret-marked: $\forall$ *no*. *no* $\in$ *set-of pret* $\longrightarrow$ *marka no* = ($\neg$ *mark p*)
assume *unmodif-next*: $\forall$ *p*. *p* $\notin$ *set-of pret* $\longrightarrow$ *next p* = *nexta p*
show ($\forall$ *pt*. *pt* $\notin$ *set-of pret*
$\longrightarrow$ *low pt* = *lowb pt* $\wedge$ *high pt* = *highb pt* $\wedge$
mark pt = *marka pt*)

proof −
from *ll* **have** *length-ll-eq*: *length levellista* = *length ll*
by (*simp add*: *Levellist-length*)
from *repoint-spec pnN subset-pret*
have *repoint-nc*: ($\forall$ *pt*. *pt* $\notin$ *set-of pret*
$\longrightarrow$ *low pt* = *lowb pt* $\wedge$ *high pt* = *highb pt*) $\wedge$ *Dag pb lowb highb postnormt*
by *auto*
then have *lowhigh-b-eq*: $\forall$ *pt*. *pt* $\notin$ *set-of pret*
$\longrightarrow$ *low pt* = *lowb pt* $\wedge$ *high pt* = *highb pt*
by *fastforce*
from *wf-marking-ll pret-dag pnN*
have *mark-b-eq*: $\forall$ *pt*. *pt* $\notin$ *set-of pret* $\longrightarrow$ *mark pt* = *marka pt*
apply −
apply (*simp add*: *wf-marking-def*)
apply (*split dag.splits*)
apply *simp*
apply (*rule allI*)
apply (*rule impI*)
apply (*elim conjE*)
apply (*erule-tac x*=*pt* **in** *allE*)
apply *fastforce*
done
with *lowhigh-b-eq rep-eq unmodif-next*
have *pret-nc*: $\forall$ *pt*. *pt* $\notin$ *set-of pret*
$\longrightarrow$ *rep pt* = *repa pt* $\wedge$ *low pt* = *lowb pt* $\wedge$ *high pt* = *highb pt* $\wedge$
mark pt = *marka pt* $\wedge$ *next pt* = *nexta pt*
by *blast*

from *pret-nc*
show *?thesis*
by *fastforce*
qed
next
— invariant to invariant
fix *var low high p rep mark pret prebdt levellist ll next marka n repc*
and *repb* :: *ref* $\Rightarrow$ *ref*
assume *ll*: *Levellist levellist next ll*
assume *pret-dag*: *Dag p low high pret*

assume *ord-pret*: *ordered pret var*
assume *pnN*: $p \neq Null$
assume *prebdt-pret*: *bdt pret var* = *Some prebdt*
assume *wf-ll*: *wf-ll pret ll var*
assume *lll*: *length levellist* = *var p* + *1*
assume *n-Suc-var-p*: $n < var\ p + 1$
assume *wf-marking-m-ma*: *wf-marking pret mark marka* ($\neg$ *mark p*)

assume *rep-nc*: $\forall\, pt\ i.$
$pt \notin set\text{-}of\ pret \vee n \leq i \wedge pt \in set\ (ll\ !\ i) \wedge i < var\ p + 1 \longrightarrow$
rep pt = *repb pt*
assume *repbNodes-in-Nodes*: *repb* ‘ *Nodes n ll* $\subseteq$ *Nodes n ll*
assume
normalize-prop: $\forall\, no \in Nodes\ n\ ll.$
var (*repb no*) $\leq$ *var no* $\wedge$
($\exists\, not\ nort.$ *Dag* (*repb no*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*) *nort* $\wedge$
Dag no low high not $\wedge$ *reduced nort* $\wedge$ *ordered nort var* $\wedge$
set-of nort $\subseteq$ *repb* ‘ *Nodes n ll* $\wedge$
($\forall\, no \in set\text{-}of\ nort.$ *repb no* = *no*) $\wedge$
($\exists\, nobdt.$ *bdt not var* = *Some nobdt* $\wedge$
($\exists\, norbdt.$ *bdt nort var* = *Some norbdt* $\wedge$
nobdt $\sim$ *norbdt*)))
assume
isomorphic-dags-eq:
$\forall\, t1\ t2.$ *t1* $\in$ *Dags* (*repb* ‘ *Nodes n ll*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*)$\wedge$
t2 $\in$ *Dags* (*repb* ‘ *Nodes n ll*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*)
$\longrightarrow$ *isomorphic-dags-eq t1 t2 var*
show ($\forall\, no \in set\ (ll\ !\ n).$
$no \neq Null \wedge$
(*low no* = *Null*) = (*high no* = *Null*) $\wedge$
low no $\notin$ *set* (*ll* ! *n*) $\wedge$
high no $\notin$ *set* (*ll* ! *n*) $\wedge$
isLeaf-pt no low high = (*var no* $\leq$ *1*) $\wedge$
(*low no* $\neq$ *Null* $\longrightarrow$ *repb* (*low no*) $\neq$ *Null*) $\wedge$ (*repb* $\propto$ *low*) *no* $\notin$ *set* (*ll* ! *n*)) $\wedge$
($\forall\, no1 \in set\ (ll\ !\ n).\ \forall\, no2 \in set\ (ll\ !\ n).$ *var no1* = *var no2*) $\wedge$
($\forall\, repa.$ ($\forall\, no.$ *no* $\notin$ *set* (*ll* ! *n*) $\longrightarrow$ *repb no* = *repa no*) $\wedge$
($\forall\, no \in set\ (ll\ !\ n).$
repa no $\neq$ *Null* $\wedge$
(*if* (*repa* $\propto$ *low*) *no* = (*repa* $\propto$ *high*) *no* $\wedge$ *low no* $\neq$ *Null*
then repa no = (*repa* $\propto$ *low*) *no*
else repa no $\in$ *set* (*ll* ! *n*) $\wedge$
repa (*repa no*) = *repa no* $\wedge$
($\forall\, no1 \in set\ (ll\ !\ n).$
((*repa* $\propto$ *high*) *no1* = (*repa* $\propto$ *high*) *no* $\wedge$
(*repa* $\propto$ *low*) *no1* = (*repa* $\propto$ *low*) *no*) =
(*repa no* = *repa no1*)))) $\longrightarrow$
var p + *1* $-$ (*n* + *1*) $<$ *var p* + *1* $-$ *n* $\wedge$

```
                n + 1 ≤ var p + 1 ∧
                (∀ pt i. pt ∉ set-of pret ∨ (n + 1 ≤ i ∧ pt ∈ set (ll ! i) ∧ i < var p
+ 1) ⟶
                    rep pt = repa pt) ∧
                repa ‘ Nodes (n + 1) ll ⊆ Nodes (n + 1) ll ∧
                (∀ no∈Nodes (n + 1) ll.
                   var (repa no) ≤ var no ∧
                   (∃ not nort.
                      Dag (repa no) (repa ∝ low) (repa ∝ high) nort ∧
                      Dag no low high not ∧
                      reduced nort ∧
                      ordered nort var ∧
                      set-of nort ⊆ repa ‘ Nodes (n + 1) ll ∧
                      (∀ no∈set-of nort. repa no = no) ∧
                      (∃ nobdt.
                         bdt not var = Some nobdt ∧
                       (∃ norbdt. bdt nort var = Some norbdt ∧ nobdt ∼ norbdt))))
∧
                (∀ t1 t2.
                  t1 ∈ Dags (repa ‘ Nodes (n + 1) ll) (repa ∝ low) (repa ∝ high) ∧
                   t2 ∈ Dags (repa ‘ Nodes (n + 1) ll) (repa ∝ low) (repa ∝ high)
⟶
                   isomorphic-dags-eq t1 t2 var))
  proof −
    from ll have length-ll-eq: length levellist = length ll
      by (simp add: Levellist-length)
    from n-Suc-var-p lll have nsll: n < length levellist by simp
    hence nseqll: n ≤ length levellist by simp
    have srrl-precond: (∀ no ∈ set (ll ! n).
      no ≠ Null ∧
      (low no = Null) = (high no = Null) ∧
      low no ∉ set (ll ! n) ∧
      high no ∉ set (ll ! n) ∧
      isLeaf-pt no low high = (var no ≤ 1) ∧
      (low no ≠ Null ⟶ repb (low no) ≠ Null) ∧
      (repb ∝ low) no ∉ set (ll ! n))
    proof (intro ballI)
      fix no
      assume no-in-lln: no ∈ set (ll ! n)
      with wf-ll nsll have no-in-pret-var: no ∈ set-of pret ∧ var no = n
        by (simp add: wf-ll-def length-ll-eq)
      with pret-dag have no-nNull: no ≠ Null
        apply −
        apply (rule set-of-nn)
        apply auto
        done
      from pret-dag prebdt-pret no-in-pret-var
      have balanced-no: (low no = Null) = (high no = Null)
        apply −
```

```
  apply (erule conjE)
  apply (rule-tac p=p and low=low in balanced-bdt)
  apply auto
  done
have low-no-notin-lln: low no ∉ set (ll ! n)
proof (cases low no = Null)
  case True
  note lno-Null=this
  with balanced-no have hno-Null: high no = Null
    by simp
  show ?thesis
  proof (cases low no ∈ set (ll ! n))
    case True
    with wf-ll nsll have low no ∈ set-of pret ∧ var (low no) = n
      by (auto simp add: wf-ll-def length-ll-eq)
    with pret-dag have low no ≠ Null
      apply −
      apply (rule set-of-nn)
      apply auto
      done
    with lno-Null show ?thesis
      by simp
  next
    assume lno-notin-lln: low no ∉ set (ll ! n)
    then show ?thesis
      by simp
  qed
next
  assume lno-nNull: low no ≠ Null
  with balanced-no have hno-nNull: high no ≠ Null
    by simp
  with lno-nNull pret-dag ord-pret no-in-pret-var
 have var-children-smaller: var (low no) < var no ∧ var (high no) < var no
    apply −
    apply (rule var-ordered-children)
    apply auto
    done
  show ?thesis
  proof (cases low no ∈ set (ll ! n))
    case True
    with wf-ll nsll have low no ∈ set-of pret ∧ var (low no) = n
      by (simp add: wf-ll-def length-ll-eq)
    with var-children-smaller no-in-pret-var show ?thesis
      by simp
  next
    assume low no ∉ set (ll ! n)
    thus ?thesis
      by simp
  qed
```

qed
have *high-no-notin-lln*: *high no* ∉ *set* (*ll* ! *n*)
proof (*cases high no* = *Null*)
case *True*
note *hno-Null=this*
with *balanced-no* **have** *lno-Null*: *low no* = *Null*
by *simp*
show *?thesis*
proof (*cases high no* ∈ *set* (*ll* ! *n*))
case *True*
with *wf-ll nsll* **have** *high no* ∈ *set-of pret* ∧ *var* (*high no*) = *n*
by (*auto simp add*: *wf-ll-def length-ll-eq*)
with *pret-dag* **have** *high no* ≠ *Null*
apply −
apply (*rule set-of-nn*)
apply *auto*
done
with *hno-Null* **show** *?thesis*
by *simp*
next
assume *hno-notin-lln*: *high no* ∉ *set* (*ll* ! *n*)
then show *?thesis*
by *simp*
qed
next
assume *hno-nNull*: *high no* ≠ *Null*
with *balanced-no* **have** *lno-nNull*: *low no* ≠ *Null*
by *simp*
with *hno-nNull pret-dag ord-pret no-in-pret-var*
have *var-children-smaller*: *var* (*low no*) < *var no* ∧ *var* (*high no*) < *var no*
apply −
apply (*rule var-ordered-children*)
apply *auto*
done
show *?thesis*
proof (*cases high no* ∈ *set* (*ll* ! *n*))
case *True*
with *wf-ll nsll* **have** *high no* ∈ *set-of pret* ∧ *var* (*high no*) = *n*
by (*simp add*: *wf-ll-def length-ll-eq*)
with *var-children-smaller no-in-pret-var* **show** *?thesis*
by *simp*
next
assume *high no* ∉ *set* (*ll* ! *n*)
thus *?thesis*
by *simp*
qed
qed
from *no-in-pret-var pret-dag no-nNull* **obtain** *not* **where**
no-dag-ex: *Dag no low high not*

```
  apply −
  apply (rotate-tac 2)
  apply (drule subnode-dag-cons)
  apply (auto simp del: Dag-Ref)
  done
with pret-dag prebdt-pret no-in-pret-var obtain nobdt
  where nobdt-ex:
  bdt not var = Some nobdt
  apply −
  apply (drule subbdt-ex-dag-def)
  apply auto
  done
have isLeaf-var: isLeaf-pt no low high = (var no ≤ 1)
proof
  assume no-isLeaf: isLeaf-pt no low high
  from nobdt-ex no-dag-ex no-isLeaf show var no ≤ 1
    apply −
    apply (rule bdt-Some-Leaf-var-le-1)
    apply auto
    done
next
  assume varno-le-1: var no ≤ 1
  show isLeaf-pt no low high
  proof (cases var no = 0)
    case True
    with nobdt-ex no-nNull no-dag-ex have not = Node Tip no Tip
      apply −
      apply (drule bdt-Some-var0-Zero)
      apply auto
      done
    with no-dag-ex show isLeaf-pt no low high
      by (simp add: isLeaf-pt-def)
  next
    assume var no ≠ 0
    with varno-le-1 have var no = 1
      by simp
    with nobdt-ex no-nNull no-dag-ex have not = Node Tip no Tip
      apply −
      apply (drule bdt-Some-var1-One)
      apply auto
      done
    with no-dag-ex show isLeaf-pt no low high
      by (simp add: isLeaf-pt-def)
  qed
qed
have repb-low-nNull: (low no ≠ Null ⟶ repb (low no) ≠ Null)
proof
  assume lno-nNull: low no ≠ Null
 with no-nNull no-in-pret-var pret-dag have lno-in-pret: low no ∈ set-of pret
```

```
    apply −
    apply (rule-tac low=low in subelem-set-of-low)
    apply auto
    done
  from lno-nNull balanced-no have hno-nNull: high no ≠ Null
    by simp
  with lno-nNull pret-dag ord-pret no-in-pret-var
 have var-children-smaller: var (low no) < var no ∧ var (high no) < var no
    apply −
    apply (rule var-ordered-children)
    apply auto
    done
  with no-in-pret-var have var-lno-l-n: var (low no) <n
    by simp
  with wf-ll lno-in-pret nsll have low no ∈ set (ll ! (var (low no)))
    by (simp add: wf-ll-def length-ll-eq)
  with lno-in-pret var-lno-l-n have low no ∈ Nodes n ll
    apply (simp add: Nodes-def)
    apply (rule-tac x=var (low no) in exI)
    apply simp
    done
  hence repb (low no) ∈ repb ‘ Nodes n ll
    by simp
  with repbNodes-in-Nodes have repb-lno-in-Nodes:
    repb (low no) ∈ Nodes n ll
    by blast
  from pret-dag wf-ll nsll have Null ∉ Nodes n ll
    apply −
    apply (rule Null-notin-Nodes)
    apply (auto simp add: length-ll-eq)
    done
  with repb-lno-in-Nodes show repb (low no) ≠ Null
    by auto
qed
have Null-notin-lln: Null ∉ set (ll ! n)
proof (cases Null ∈ set (ll ! n))
  case True
  with wf-ll nsll have Null ∈ set-of pret ∧ var (Null) = n
    by (simp add: wf-ll-def length-ll-eq)
  with pret-dag have Null ≠ Null
    apply −
    apply (rule set-of-nn)
    apply auto
    done
  thus ?thesis
    by auto
next
  assume Null ∉ set (ll ! n)
  thus ?thesis
```

by *simp*
qed
have (*repb* ∝ *low*) *no* ∉ *set* (*ll* ! *n*)
proof (*cases low no* = *Null*)
case *True*
with *Null-notin-lln* **show** *?thesis*
by (*simp add*: *null-comp-def*)
next
assume *lno-nNull*: *low no* ≠ *Null*
with *no-nNull no-in-pret-var pret-dag* **have** *lno-in-pret*: *low no* ∈ *set-of pret*
apply −
apply (*rule-tac low*=*low* **in** *subelem-set-of-low*)
apply *auto*
done
from *lno-nNull* **have** *propto-eq-comp*: (*repb* ∝ *low*) *no* = *repb* (*low no*)
by (*simp add*: *null-comp-def*)
from *lno-nNull balanced-no* **have** *hno-nNull*: *high no* ≠ *Null*
by *simp*
with *lno-nNull pret-dag ord-pret no-in-pret-var*
have *var-children-smaller*: *var* (*low no*) < *var no* ∧ *var* (*high no*) < *var no*
apply −
apply (*rule var-ordered-children*)
apply *auto*
done
with *no-in-pret-var* **have** *var-lno-l-n*: *var* (*low no*) <*n*
by *simp*
with *wf-ll lno-in-pret nsll* **have** *low no* ∈ *set* (*ll* ! (*var* (*low no*)))
by (*simp add*: *wf-ll-def length-ll-eq*)
with *lno-in-pret var-lno-l-n* **have** *lno-in-Nodes-n*: *low no* ∈ *Nodes n ll*
apply (*simp add*: *Nodes-def*)
apply (*rule-tac x*=*var* (*low no*) **in** *exI*)
apply *simp*
done
hence *repb* (*low no*) ∈ *repb* ‘ *Nodes n ll*
by *simp*
with *repbNodes-in-Nodes* **have** *repb-lno-in-Nodes*:
repb (*low no*) ∈ *Nodes n ll*
by *blast*
with *lno-in-Nodes-n normalize-prop* **have** *var* (*repb* (*low no*)) ≤ *var* (*low no*)
by *auto*
with *var-lno-l-n* **have** *var-rep-lno-l-n*: *var* (*repb* (*low no*)) < *n*
by *simp*
with *repb-lno-in-Nodes* **have** ∃ *k* < *n*. *repb* (*low no*) ∈ *set* (*ll* ! *k*)
by (*auto simp add*: *Nodes-def*)
with *wf-ll propto-eq-comp nsll* **show** (*repb* ∝ *low*) *no* ∉ *set* (*ll* ! *n*)
apply −
apply (*erule exE*)
apply (*rule-tac i*=*k* **and** *j*=*n* **in** *no-in-one-ll*)

apply (*auto simp add*: *length-ll-eq*)
done
qed
with *no-nNull balanced-no low-no-notin-lln high-no-notin-lln isLeaf-var repb-low-nNull*
show $no \neq Null \wedge$
$(low\ no = Null) = (high\ no = Null) \wedge$
$low\ no \notin set\ (ll\ !\ n) \wedge high\ no \notin set\ (ll\ !\ n) \wedge$
$isLeaf\text{-}pt\ no\ low\ high = (var\ no \leq 1) \wedge$
$(low\ no \neq Null \longrightarrow repb\ (low\ no) \neq Null) \wedge$
$(repb \propto low)\ no \notin set\ (ll\ !\ n)$
by *auto*
qed
have *all-nodes-same-var*: $\forall no1 \in set\ (ll\ !\ n).\ \forall no2 \in set\ (ll\ !\ n).\ var\ no1 = var\ no2$
proof (*intro ballI impI*)
fix *no1 no2*
assume $no1 \in set\ (ll\ !\ n)$
with *wf-ll nsll* **have** *var-lln-i*: $var\ no1 = n$
by (*simp add*: *wf-ll-def length-ll-eq*)
assume $no2 \in set\ (ll\ !\ n)$
with *wf-ll nsll* **have** $var\ no2 = n$
by (*simp add*: *wf-ll-def length-ll-eq*)
with *var-lln-i* **show** $var\ no1 = var\ no2$
by *simp*
qed
have $(\forall repa.\ (\forall no.\ no \notin set\ (ll\ !\ n) \longrightarrow repb\ no = repa\ no) \wedge$
$(\forall no{\in}set\ (ll\ !\ n).$
$repa\ no \neq Null \wedge$
$(if\ (repa \propto low)\ no = (repa \propto high)\ no \wedge low\ no \neq Null$
$then\ repa\ no = (repa \propto low)\ no$
$else\ repa\ no \in set\ (ll\ !\ n) \wedge$
$repa\ (repa\ no) = repa\ no \wedge$
$(\forall no1{\in}set\ (ll\ !\ n).$
$((repa \propto high)\ no1 = (repa \propto high)\ no \wedge$
$(repa \propto low)\ no1 = (repa \propto low)\ no) =$
$(repa\ no = repa\ no1)))) \longrightarrow$
$var\ p + 1 - (n + 1) < var\ p + 1 - n \wedge$
$n + 1 \leq var\ p + 1 \wedge$
$(\forall pt\ i.\ pt \notin set\text{-}of\ pret \vee (n + 1 \leq i \wedge pt \in set\ (ll\ !\ i) \wedge i < var\ p + 1) \longrightarrow$
$rep\ pt = repa\ pt) \wedge$
$repa\ `\ Nodes\ (n + 1)\ ll \subseteq Nodes\ (n + 1)\ ll \wedge$
$(\forall no{\in}Nodes\ (n + 1)\ ll.$
$var\ (repa\ no) \leq var\ no \wedge$
$(\exists not\ nort.$
$Dag\ (repa\ no)\ (repa \propto low)\ (repa \propto high)\ nort \wedge$
$Dag\ no\ low\ high\ not \wedge$
$reduced\ nort \wedge$

```
                    ordered nort var ∧
                    set-of nort ⊆ repa ‘ Nodes (n + 1) ll ∧
                    (∀ no∈set-of nort. repa no = no) ∧
                    (∃ nobdt.
                        bdt not var = Some nobdt ∧
                     (∃ norbdt. bdt nort var = Some norbdt ∧ nobdt ∼ norbdt))))
∧
            (∀ t1 t2.
              t1 ∈ Dags (repa ‘ Nodes (n + 1) ll) (repa ∝ low) (repa ∝ high) ∧
               t2 ∈ Dags (repa ‘ Nodes (n + 1) ll) (repa ∝ low) (repa ∝ high)
⟶
                isomorphic-dags-eq t1 t2 var))
  (is (∀ repc. ?srrl-post repc ⟶ ?norm-inv repc) )
 proof (intro allI impI, elim conjE)
  fix repc
  assume repbc-nc: ∀ no. no ∉ set (ll ! n) ⟶ repb no = repc no
  assume rep-prop:  ∀ no∈set (ll ! n).
   repc no ≠ Null ∧
        (if (repc ∝ low) no = (repc ∝ high) no ∧ low no ≠ Null
         then repc no = (repc ∝ low) no
         else repc no ∈ set (ll ! n) ∧
              repc (repc no) = repc no ∧
              (∀ no1∈set (ll ! n).
                  ((repc ∝ high) no1 = (repc ∝ high) no ∧
                   (repc ∝ low) no1 = (repc ∝ low) no) =
                  (repc no = repc no1)))
  show ?norm-inv repc
  proof −
   from n-Suc-var-p have termi: var p + 1 − (n + 1) < var p + 1 − n
    by arith
   from wf-ll repbc-nc nsll
   have Nodes-n-rep-nc: ∀ p. p ∈ Nodes n ll ⟶ repb p = repc p
    apply −
    apply (rule allI)
    apply (rule impI)
    apply (simp add: Nodes-def)
    apply (erule exE)
    apply (erule-tac x=p in allE)
    apply (drule-tac i=x and j=n in no-in-one-ll)
    apply (auto simp add: length-ll-eq)
    done
   from repbNodes-in-Nodes
   have Nodes-n-rep-in-Nodesn:
    ∀ p. p ∈ Nodes n ll ⟶ repb p ∈ Nodes n ll
    by auto
   from wf-ll nsll have Nodes n ll ⊆ set-of pret
    apply −
    apply (rule Nodes-in-pret)
    apply (auto simp add: length-ll-eq)
```

done
with *Nodes-n-rep-in-Nodesn*
have *Nodes-n-rep-in-pret*: $\forall\, p.\ p \in Nodes\ n\ ll \longrightarrow repb\ p \in set\text{-}of\ pret$
apply −
apply (*intro allI impI*)
apply *blast*
done
have *Nodes-repbc-Dags-eq*: $\forall\, p\ t.\ p \in Nodes\ n\ ll$
$\longrightarrow Dag\ (repb\ p)\ (repb \propto low)\ (repb \propto high)\ t =$
$Dag\ (repc\ p)\ (repc \propto low)\ (repc \propto high)\ t$
proof (*intro allI impI*)
fix *p t*
assume *p-in-Nodes*: $p \in Nodes\ n\ ll$
then have *repp-nc*: $repb\ p = repc\ p$
by (*rule Nodes-n-rep-nc* [*rule-format*])
from *p-in-Nodes normalize-prop* **obtain** *nort* **where**
nort-repb-dag: $Dag\ (repb\ p)\ (repb \propto low)\ (repb \propto high)\ nort$ **and**
nort-in-repbNodes: $set\text{-}of\ nort \subseteq repb\ `\ Nodes\ n\ ll$
apply −
apply (*erule-tac x=p* **in** *ballE*)
prefer *2*
apply *auto*
done
from *nort-in-repbNodes repbNodes-in-Nodes*
have *nort-in-Nodesn*: $set\text{-}of\ nort \subseteq Nodes\ n\ ll$
by *blast*
from *pret-dag wf-ll nsll* **have** $Null \notin Nodes\ n\ ll$
apply −
apply (*rule Null-notin-Nodes*)
apply (*auto simp add*: *length-ll-eq*)
done
with *p-in-Nodes repbNodes-in-Nodes* **have** *repp-nNull*: $repb\ p \neq Null$
by *auto*
from *nort-repb-dag repp-nc*
have *nort-repbc-dag*: $Dag\ (repc\ p)\ (repb \propto low)\ (repb \propto high)\ nort$
by *simp*
from *nort-in-Nodesn* **have** $\forall\, x \in set\text{-}of\ nort.\ x \in Nodes\ n\ ll$
apply −
apply (*rule ballI*)
apply *blast*
done
with *wf-ll nsll* **have** $\forall\, x \in set\text{-}of\ nort.\ x \in set\text{-}of\ pret \wedge var\ x < n$
apply −
apply (*rule ballI*)
apply (*rule wf-ll-Nodes-pret*)
apply (*auto simp add*: *length-ll-eq*)
done
with *pret-dag prebdt-pret nort-repbc-dag ord-pret wf-ll nsll repbc-nc*
have

```
    ∀ x ∈ set-of nort. (repc ∝ low) x = (repb ∝ low) x ∧
    (repc ∝ high) x = (repb ∝ high) x
    apply −
    apply (rule nort-null-comp)
    apply (auto simp add: length-ll-eq)
    done
  with nort-repbc-dag repp-nc
  have Dag (repc p) (repb ∝ low) (repb ∝ high) nort =
    Dag (repc p) (repc ∝ low) (repc ∝ high) nort
    apply −
    apply (rule heaps-eq-Dag-eq)
    apply (rule ballI)
    apply (erule-tac x=x in ballE)
    apply (elim conjE)
    apply (rule conjI)
    apply auto
    done
  with nort-repbc-dag repp-nc show
    Dag (repb p) (repb ∝ low) (repb ∝ high) t =
    Dag (repc p) (repc ∝ low) (repc ∝ high) t
    apply auto
    apply (rotate-tac 2)
    apply (frule-tac Dag-unique)
    apply (rotate-tac 1)
    apply simp
    apply simp
    apply (frule Dag-unique)
    apply (rotate-tac 3)
    apply simp
    apply simp
    done
qed
from rep-prop have repbc-changes: ∀ no∈set (ll ! n).
  repc no ≠ Null ∧
  (if (repc ∝ low) no = (repc ∝ high) no ∧ low no ≠ Null
  then repc no = (repc ∝ low) no
  else repc no ∈ set (ll ! n) ∧ repc (repc no) = repc no ∧
  (∀ no1∈set (ll ! n). ((repc ∝ high) no1 = (repc ∝ high) no ∧
  (repc ∝ low) no1 = (repc ∝ low) no) = (repc no = repc no1)))
  by blast
from nsll lll  have n-var-prop: n + 1 <= var p + 1
  by simp
from rep-nc have Sucn-repb-nc: (∀ pt. pt ∉ set-of pret ∨
  (∃ i. n + 1 ≤ i ∧ pt ∈ set (ll ! i) ∧ i < var p + 1)
  ⟶ rep pt = repb pt)
  apply −
  apply (intro allI impI)
  apply (erule-tac x=pt in allE)
  apply auto
```

```
  apply (rule-tac x=i in exI)
  apply auto
  done
have repc-nc:
  (∀ pt. pt ∉ set-of pret ∨
  (∃ i. n + 1 ≤ i ∧ pt ∈ set (ll ! i) ∧ i < var p + 1)
  ⟶ rep pt = repc pt)
proof (intro allI impI)
  fix pt
  assume pt-notin-lower-ll: pt ∉ set-of pret ∨
    (∃ i. n + 1 ≤ i ∧ pt ∈ set (ll ! i) ∧ i < var p + 1)
  show rep pt = repc pt
  proof (cases pt ∉ set-of pret)
    case True
    with wf-ll nsll have pt ∉ set (ll ! n)
      apply (simp add: wf-ll-def length-ll-eq)
      apply (case-tac pt ∈ set (ll ! n))
      apply (subgoal-tac pt ∈ set-of pret)
      apply (auto)
      done
    with repbc-nc have repb pt = repc pt
      by auto
    with Sucn-repb-nc True show ?thesis
      by auto
  next
    assume pt-in-pret: ¬ pt ∉ set-of pret
    with pt-notin-lower-ll have pt-in-higher-ll:
      ∃ i. n + 1 ≤ i ∧ pt ∈ set (ll ! i) ∧ i < var p + 1
      by simp
    with nsll wf-ll lll have pt-notin-lln: pt ∉ set (ll ! n)
      apply -
      apply (erule exE)
      apply (rule-tac i=i and j=n in no-in-one-ll)
      apply (auto simp add: length-ll-eq)
      done
    with repbc-nc have repb pt = repc pt
      by auto
    with Sucn-repb-nc pt-in-higher-ll show ?thesis
      by auto
  qed
qed
from wf-ll nsll
have Nodesn-notin-lln: ∀ no ∈ Nodes n ll. no ∉ set (ll ! n)
  apply (simp add: Nodes-def)
  apply clarify
  apply (drule no-in-one-ll)
  apply (auto simp add: length-ll-eq)
  done
with repbc-nc
```

have *Nodesn-repnc*: $\forall$ *no* $\in$ *Nodes n ll*. *repb no* = *repc no*
apply −
apply (*rule ballI*)
apply (*erule-tac x=no* **in** *allE*)
apply *simp*
done
then have *repbNodes-repcNodes*:
repb ‘(*Nodes n ll*) = *repc* ‘(*Nodes n ll*)
apply −
apply *rule*
apply *blast*
apply *rule*
apply (*erule imageE*)
apply (*erule-tac x=xa* **in** *ballE*)
prefer *2*
apply *simp*
apply *rule*
apply *auto*
done
have *repcNodes-in-Nodes*:
repc ‘ *Nodes* (*n* + *1*) *ll* $\subseteq$ *Nodes* (*n* + *1*) *ll*
proof
fix *x*
assume *x-in-repcNodesSucn*: *x* $\in$ *repc* ‘ *Nodes* (*n* + *1*) *ll*
show *x* $\in$ *Nodes* (*n* + *1*) *ll*
proof (*cases x* $\in$ *repc* ‘*Nodes n ll*)
case *True*
with *repbNodes-repcNodes repbNodes-in-Nodes* **have** *x* $\in$ *Nodes n ll*
by *auto*
with *Nodes-subset* **show** *?thesis*
by *auto*
next
assume *x* $\notin$ *repc* ‘*Nodes n ll*
with *x-in-repcNodesSucn* **have** *x-in-repclln*: *x* $\in$ *repc* ‘*set* (*ll* ! *n*)
apply (*auto simp add*: *Nodes-def*)
apply (*case-tac k*<*n*)
apply *auto*
apply (*case-tac k* = *n*)
apply *simp*
apply *arith*
done
from *x-in-repclln* **show** *?thesis*
proof (*elim imageE*)
fix *y*
assume *x-repcy*: *x* = *repc y*
assume *y-in-repclln*: *y* $\in$ *set* (*ll* ! *n*)
from *rep-prop y-in-repclln* **obtain**
repcy-nNull: *repc y* $\neq$ *Null* **and**
red-prop: (*repc* $\propto$ *low*) *y* = (*repc* $\propto$ *high*) *y* $\wedge$

low y $\neq$ *Null* $\longrightarrow$ *repc y* = (*repc* $\propto$ *high*) *y* **and**
share-prop: ((*repc* $\propto$ *low*) *y* = (*repc* $\propto$ *high*) *y* $\longrightarrow$ *low y* = *Null*)
$\longrightarrow$ *repc y* $\in$ *set* (*ll* ! *n*) $\wedge$ *repc* (*repc y*) = *repc y* $\wedge$
($\forall$ *no1*$\in$*set* (*ll* ! *n*).
((*repc* $\propto$ *high*) *no1* = (*repc* $\propto$ *high*) *y* $\wedge$
(*repc* $\propto$ *low*) *no1* = (*repc* $\propto$ *low*) *y*) = (*repc y* = *repc no1*))
using [[*simp-depth-limit* = *4*]]
by *auto*
from *wf-ll nsll y-in-repclln* **obtain**
y-in-pret: *y* $\in$ *set-of pret* **and**
vary-n: *var y* = *n*
by (*auto simp add*: *wf-ll-def length-ll-eq*)
from *y-in-pret pret-dag* **have** *y-nNull*: *y* $\neq$ *Null*
apply −
apply (*rule set-of-nn*)
apply *auto*
done
show *x* $\in$ *Nodes* (*n* + *1*) *ll*
proof (*cases low y* = *Null*)
case *True*
from *pret-dag prebdt-pret True y-in-pret*
have *highy-Null*: *high y* = *Null*
apply −
apply (*drule balanced-bdt*)
apply *auto*
done
with *share-prop True* **obtain**
repcy-in-llb: *repc y* $\in$ *set* (*ll* ! *n*) **and**
rry-ry: *repc* (*repc y*) = *repc y* **and**
y-other-node-prop: $\forall$ *no1*$\in$*set* (*ll* ! *n*).
((*repc* $\propto$ *high*) *no1* = (*repc* $\propto$ *high*) *y* $\wedge$
(*repc* $\propto$ *low*) *no1* = (*repc* $\propto$ *low*) *y*) = (*repc y* = *repc no1*)
by *simp*
from *repcy-in-llb x-repcy* **show** *?thesis*
by (*auto simp add*: *Nodes-def*)
next
assume *lowy-nNull*: *low y* $\neq$ *Null*
with *pret-dag prebdt-pret y-in-pret*
have *highy-nNull*: *high y* $\neq$ *Null*
apply −
apply (*drule balanced-bdt*)
apply *auto*
done
show *?thesis*
proof (*cases* (*repc* $\propto$ *low*) *y* = (*repc* $\propto$ *high*) *y*)
case *True*
with *red-prop lowy-nNull* **have** *repc y* = (*repc* $\propto$ *high*) *y*
by *auto*
with *highy-nNull* **have** *red-repc-y*: *repc y* = *repc* (*high y*)

```
      by (simp add: null-comp-def)
    from pret-dag ord-pret y-in-pret lowy-nNull  highy-nNull

    have var (low y) < var y ∧ var (high y) < var y
      apply −
      apply (rule var-ordered-children)
      apply auto
      done
    with  vary-n have varhighy: var (high y) < n
      by auto
    from y-in-pret y-nNull highy-nNull pret-dag
    have high y ∈ set-of pret
      apply −
      apply (drule subelem-set-of-high)
      apply auto
      done
    with wf-ll varhighy have high y ∈ Nodes n ll
      by (auto simp add: wf-ll-def Nodes-def)
    with red-repc-y have repc y ∈ repc ‘Nodes n ll
      by simp
    with x-repcy have x ∈ repc ‘Nodes n ll
      by simp
    with repbNodes-repcNodes repbNodes-in-Nodes
    have x ∈ Nodes n ll
      by auto
    with Nodes-subset show ?thesis
      by auto
  next
    assume (repc ∝ low) y ≠ (repc ∝ high) y
    with share-prop obtain
      repcy-in-llbn: repc y ∈ set (ll ! n) and
      rry-ry: repc (repc y) = repc y and
      y-other-node-share: ∀ no1∈set (ll ! n).
      ((repc ∝ high) no1 = (repc ∝ high) y ∧
      (repc ∝ low) no1 = (repc ∝ low) y) = (repc y = repc no1)
      by auto
    with repcy-in-llbn  x-repcy have x ∈ set (ll ! n)
      by auto
    then show ?thesis
      by (auto simp add: Nodes-def)
  qed
 qed
 qed
qed
qed
have (∀ no∈Nodes (n + 1) ll.
  var (repc no) ≤ var no ∧
  (∃ not nort. Dag (repc no) (repc ∝ low) (repc ∝ high) nort ∧
  Dag no low high not ∧
```

```
  reduced nort ∧ ordered nort var ∧
  set-of nort ⊆ repc ‘ Nodes (n + 1) ll ∧
  (∀ no∈set-of nort. repc no = no) ∧
  (∃ nobdt. bdt not var = Some nobdt ∧
  (∃ norbdt. bdt nort var = Some norbdt ∧ nobdt ∼ norbdt))))
  (is ∀ no∈Nodes (n + 1) ll. ?Q i no)
proof (intro ballI)
  fix no
  assume no-in-Nodes: no ∈ Nodes (n + 1) ll
  from wf-ll no-in-Nodes nsll  have no-in-pret: no ∈ set-of pret
    apply (simp add: wf-ll-def Nodes-def length-ll-eq)
    apply (erule conjE)
    apply (thin-tac ∀ q. q ∈ set-of pret ⟶ q ∈ set (ll ! var q))
    apply (erule exE)
    apply (erule-tac x=x in allE)
    apply (erule impE)
    apply arith
    apply (erule-tac x=no in ballE)
    apply auto
    done
  from pret-dag no-in-pret have nonNull: no≠ Null
    apply −
    apply (rule set-of-nn [rule-format])
    apply auto
    done
  show ?Q i no
  proof (cases no ∈ Nodes n ll)
    case True
    note no-in-Nodesn=this
    with wf-ll nsll  no-in-Nodes
    have no-notin-llbn: no ∉ set (ll ! n)
      apply −
      apply (simp add: Nodes-def length-ll-eq)
      apply (elim exE)
      apply (drule-tac ?i=xa and ?j=n in no-in-one-ll)
      apply arith
      apply simp
      apply auto
      done
    with repbc-nc have repb-no-eq-repc-no: repb no = repc no
      by simp
    from repbc-nc no-in-Nodes no-notin-llbn normalize-prop True
    have varrep-eq-var: var (repc no) ≤ var no
      apply −
      apply (erule-tac x=no in ballE)
      prefer 2
      apply simp
      apply (erule-tac x=no in allE)
      apply simp
```

```
      done
    moreover
    from True normalize-prop no-in-Nodes obtain not nort where
      nort-dag: Dag (repb no) (repb ∝ low) (repb ∝ high) nort and
      ord-nort: ordered nort var and
      subset-nort-not: set-of nort ⊆ repb `(Nodes n ll) and
      not-dag: Dag no low high not and
      red-nort: reduced nort and
      nort-repb: (∀ no∈set-of nort. repb no = no) and
      bdt-prop: ∃ nobdt norbdt. bdt not var = Some nobdt ∧
      bdt nort var = Some norbdt ∧ nobdt ∼ norbdt
      by blast
    moreover
        from Nodesn-notin-lln repbc-nc nort-repb subset-nort-not repbN-
odes-in-Nodes
    have nort-repc:
      (∀ no∈set-of nort. repc no = no)
      apply auto
      apply (subgoal-tac no ∈ Nodes n ll)
      apply fastforce
      apply blast
      done
    moreover
   from nort-dag have nortnodesnN: (∀ no. no ∈ set-of nort ⟶ no ≠ Null)
      apply −
      apply (rule allI)
      apply (rule impI)
      apply (rule set-of-nn)
      apply auto
      done
    moreover
    have Dag (repc no) (repc ∝ low) (repc ∝ high) nort
    proof −
      from no-notin-llbn repbc-nc have repbc-no: repc no = repb no
        by fastforce
      with nort-dag
      have nortrepbc-dag: Dag (repc no) (repb ∝ low) (repb ∝ high) nort
        by simp
      from wf-ll nseqll have Nodes n ll ⊆ set-of pret
        apply −
        apply (rule Nodes-levellist-subset-t)
        apply assumption+
        apply (simp add: length-ll-eq)
        done
      with repbNodes-in-Nodes subset-nort-not
      have subset-nort-pret: set-of nort ⊆ set-of pret
        by simp
      have vxsn-in-pret: ∀ x ∈ set-of nort. var x < n ∧ x ∈ set-of pret
      proof (rule ballI)
```

```
    fix x
    assume x-in-nort: x ∈ set-of nort
    from x-in-nort subset-nort-not repbNodes-in-Nodes
    have x ∈ Nodes n ll
      by blast
    with wf-ll nsll  have xsn: var x < n
      apply (simp add: wf-ll-def Nodes-def length-ll-eq)
      apply (erule conjE)
      apply (thin-tac  ∀ q. q ∈ set-of pret ⟶ q ∈ set (ll ! var q))
      apply (erule exE)
      apply (erule-tac x=xa in allE)
      apply (erule impE)
      apply arith
      apply (erule-tac x=x in ballE)
      apply auto
      done
    from x-in-nort subset-nort-pret have x-in-pret: x ∈ set-of pret
      by blast
    with xsn show var x < n ∧ x ∈ set-of pret by simp
  qed
  with pret-dag prebdt-pret nortrepbc-dag ord-pret wf-ll  nsll
    repbc-nc
  have ∀  x ∈ set-of nort. ((repc ∝ low) x = (repb ∝ low) x ∧
    (repc ∝ high) x = (repb ∝ high) x)
    apply −
    apply (rule nort-null-comp)
    apply (auto simp add: length-ll-eq)
    done
  with nort-dag
  have Dag (repc no) (repc ∝ low) (repc ∝ high) nort =
    Dag (repc no) (repb ∝ low) (repb ∝ high) nort
    apply −
    apply (rule heaps-eq-Dag-eq)
    apply simp
    done
  with nortrepbc-dag show ?thesis
    by simp
qed
moreover
have set-of nort ⊆ repc ‘(Nodes (n + 1) ll)
proof −
  have Nodesn-in-NodesSucn: Nodes n ll ⊆ Nodes (n + 1) ll
    by (simp add: Nodes-def set-split)
  then have repbNodesn-in-repbNodesSucn:
    repb ‘(Nodes n ll) ⊆ repb ‘(Nodes (n + 1) ll)
    by blast
  from wf-ll nsll
  have Nodes-n-notin-lln: ∀ no ∈ Nodes n ll. no ∉ set (ll ! n)
    apply (simp add: Nodes-def length-ll-eq)
```

```
    apply clarify
    apply (drule no-in-one-ll)
    apply auto
    done
  with repbc-nc  have ∀ no ∈ Nodes n ll. repb no = repc no
    apply −
    apply (rule ballI)
    apply (erule-tac x=no in allE)
    apply simp
    done
  then have repbNodes-repcNodes:
    repb `(Nodes n ll) = repc `(Nodes n ll)
    apply −
    apply rule
    apply blast
    apply rule
    apply (erule imageE)
    apply (erule-tac x=xa in ballE)
    prefer 2
    apply simp
    apply rule
    apply auto
    done
  from Nodesn-in-NodesSucn
  have repc `(Nodes n ll) ⊆ repc `(Nodes (n + 1) ll)
    by blast
 with repbNodes-repcNodes subset-nort-not repbNodesn-in-repbNodesSucn

  show ?thesis by simp
 qed
 ultimately show ?thesis
  by blast
next
 assume  no ∉ Nodes n ll
 with no-in-Nodes  have no-in-llbn: no ∈ set (ll ! n)
  apply (simp add: Nodes-def)
  apply (erule exE)
  apply (erule-tac x=x in allE)
  apply (case-tac x<n)
  apply simp
  apply simp
  apply (elim conjE)
  apply (case-tac x=n)
  apply simp
  apply arith
  done
 with wf-ll  nsll have varno: var no = n
  by (simp add: wf-ll-def length-ll-eq)
 from repbc-changes no-in-llbn
```

have *repbcno-changes*: *repc no* $\neq$ *Null* $\wedge$
(($repc \propto low$) *no* = ($repc \propto high$) *no* $\wedge$ *low no* $\neq$ *Null*
$\longrightarrow$ *repc no* = ($repc \propto high$) *no*) $\wedge$
((($repc \propto low$) *no* = ($repc \propto high$) *no* $\longrightarrow$ *low no* = *Null*)
$\longrightarrow$ *repc no* $\in$ *set* (*ll* ! *n*) $\wedge$ *repc* (*repc no*) = *repc no* $\wedge$
($\forall$ *no1*$\in$*set* (*ll* ! *n*). (($repc \propto high$) *no1* = ($repc \propto high$) *no* $\wedge$
($repc \propto low$) *no1* = ($repc \propto low$) *no*) = (*repc no* = *repc no1*)))
(**is** *?rnonN* $\wedge$ *?repreduce* $\wedge$ *?repshare*)
using [[*simp-depth-limit=4*]]
by (*simp split*: *if-split*)
then obtain
rnonN: *?rnonN* **and**
repreduce: *?repreduce* **and**
repshare: *?repshare*
by *blast*
have *repcn-normalize*: *var* (*repc no*) $\leq$ *var no* $\wedge$
($\exists$ *not nort. Dag* (*repc no*) ($repc \propto low$) ($repc \propto high$) *nort* $\wedge$
Dag no low high not $\wedge$ *reduced nort* $\wedge$ *ordered nort var* $\wedge$
set-of nort $\subseteq$ *repc* ‘ *Nodes* (*n* + *1*) *ll* $\wedge$
($\forall$ *no*$\in$*set-of nort. repc no* = *no*) $\wedge$
($\exists$ *nobdt. bdt not var* = *Some nobdt* $\wedge$
($\exists$ *norbdt. bdt nort var* = *Some norbdt* $\wedge$ *nobdt* $\sim$ *norbdt*)))
(**is** *?varrep* $\wedge$ *?repcn-prop*
is *?varrep* $\wedge$
($\exists$ *not nort. ?nort-dag nort* $\wedge$ *?not-dag not* $\wedge$ *?red nort* $\wedge$
?ord nort $\wedge$ *?nort-in-Nodes nort* $\wedge$ *?repcno-no-n nort* $\wedge$ *?bdt-equ not nort*))
proof (*cases high no* = *Null*)
case *True*
note *highnoNull=this*
with *pret-dag prebdt-pret no-in-pret*
have *lownoNull*: *low no* = *Null*
apply −
apply (*drule balanced-bdt*)
apply *assumption+*
apply *simp*
done
with *repshare* **have** *repcnoinlln*:*repc no* $\in$ *set* (*ll* ! *n*)
by *simp*
with *wf-ll nsll* **have** *varrno-n*: *var* (*repc no*) = *n*
by (*simp add*: *wf-ll-def length-ll-eq*)
with *varno* **have** *varrep*: *?varrep*
by *simp*
from *wf-ll nsll no-in-llbn varrno-n*
have *varrno-varno*: *var* (*repc no*) = *var no*
by (*simp add*: *wf-ll-def length-ll-eq*)
from *wf-ll nsll repcnoinlln*
have *rno-in-pret*: *repc no* $\in$ *set-of pret*
by (*simp add*: *wf-ll-def length-ll-eq*)

from *repcnoinlln repshare lownoNull*
have *reprep-eq-rep*: *repc* (*repc no*) = *repc no*
by *simp*
with *repcnoinlln repshare lownoNull*
have *repchildreneq*:
((*repc* ∝ *high*) (*repc no*) = (*repc* ∝ *high*) *no* ∧
(*repc* ∝ *low*) (*repc no*) = (*repc* ∝ *low*) *no*)
by *simp*
have *repcn-prop*: *?repcn-prop*
apply −
apply (*rule-tac x*=(*Node Tip no Tip*) **in** *exI*)
apply (*rule-tac x*=(*Node Tip* (*repc no*) *Tip*) **in** *exI*)
apply (*intro conjI*)
apply *simp*
prefer *3*
apply *simp*
prefer *3*
apply *simp*
proof −
from *pret-dag pnN rno-in-pret* **have** *rnonN*: *repc no* ≠ *Null*
apply (*case-tac repc no* = *Null*)
apply *auto*
done
from *highnoNull repchildreneq*
have *rhighNull*: (*repc* ∝ *high*) (*repc no*) = *Null*
by (*simp add*: *null-comp-def*)
from *lownoNull repchildreneq*
have *rlowNull*: (*repc* ∝ *low*) (*repc no*) = *Null*
by (*simp add*: *null-comp-def*)
with *rhighNull rnonN*
show *repc no* ≠ *Null* ∧ (*repc* ∝ *low*) (*repc no*) = *Null* ∧
(*repc* ∝ *high*) (*repc no*) = *Null*
by *simp*
next
from *nonNull lownoNull highnoNull*
show *?not-dag* (*Node Tip no Tip*)
by *simp*
next
from *no-in-Nodes*
show *set-of* (*Node Tip* (*repc no*) *Tip*) ⊆ *repc* ‘ *Nodes* (*n* + *1*) *ll*
by *simp*
next
show ∀ *no*∈*set-of* (*Node Tip* (*repc no*) *Tip*). *repc no* = *no*
proof
fix *pt*
assume *pt-in-repcLeaf*: *pt* ∈ *set-of* (*Node Tip* (*repc no*) *Tip*)
with *reprep-eq-rep* **show** *repc pt* = *pt*
by *simp*
qed

```
      next
        show ?bdt-equ (Node Tip no Tip) (Node Tip (repc no) Tip)
        proof (cases var no = 0)
          case True
          note vno-Null=this
         then have nobdt: bdt (Node Tip no Tip) var = Some Zero by simp
          from varrep  vno-Null have varrno: var (repc no) = 0 by simp
          then have norbdt: bdt (Node Tip (repc no) Tip) var = Some Zero
by simp
          from nobdt norbdt vno-Null varrno show ?thesis
            by (simp add: cong-eval-def)
        next
          assume vno-not-Null: var no ≠ 0
          show ?thesis
          proof (cases var no = 1)
            case True
            note vno-One=this
          then have nobdt: bdt (Node Tip no Tip) var = Some One by simp
            from varrno-varno vno-One
            have bdt (Node Tip (repc no) Tip) var = Some One by simp
            with nobdt show ?thesis by (auto simp add: cong-eval-def)
          next
            assume vno-nOne: var no ≠ 1
            with vno-not-Null have onesvno: 1 < var no by simp
            from nonNull lownoNull highnoNull
            have no-dag: Dag no low high (Node Tip no Tip)
              by simp
             with pret-dag no-in-pret have not-in-pret: (Node Tip no Tip) ≤
pret
              by (metis set-of-subdag)
            with prebdt-pret have ∃ bdt2. bdt (Node Tip no Tip) var = Some
bdt2
              by (metis subbdt-ex)
            with onesvno show ?thesis
              by simp
          qed
        qed
      qed
      with varrep reprep-eq-rep show ?thesis by simp
    next
      assume hno-nNull: high no ≠ Null
      with pret-dag prebdt-pret no-in-pret   have lno-nNull: low no ≠ Null
        by (metis balanced-bdt)


      from no-in-pret nonNull hno-nNull pret-dag
      have hno-in-pret: high no ∈ set-of pret
        by (metis subelem-set-of-high)
      with wf-ll
```

have *hno-in-ll*: *high no* ∈ *set* (*ll* ! (*var* (*high no*)))
by (*simp add*: *wf-ll-def*)
from *pret-dag ord-pret no-in-pret lno-nNull hno-nNull*

have *varhnos-varno*: *var* (*high no*) < *var no*
by (*metis var-ordered-children*)
with *varno* **have** *varhnos-n*: *var* (*high no*) < *n* **by** *simp*
with *hno-in-ll* **have** *hno-in-Nodesn*: *high no* ∈ *Nodes n ll*
apply (*simp add*: *Nodes-def*)
apply (*rule-tac x*=*var* (*high no*) **in** *exI*)
apply *simp*
done
from *wf-ll nsll hno-in-ll varhnos-n*
have *high no* ∉ *set* (*ll* ! *n*)
apply −
apply (*rule no-in-one-ll*)
apply (*auto simp add*: *length-ll-eq*)
done
with *repbc-nc* **have** *repb-repc-high*: *repb* (*high no*) = *repc* (*high no*) **by** *simp*
with *normalize-prop hno-in-Nodesn varhnos-varno varno*
have *high-normalize*: *var* (*repc* (*high no*)) ≤ *var* (*high no*) ∧
(∃ *not nort*. *Dag* (*repc* (*high no*)) (*repb* ∝ *low*) (*repb* ∝ *high*) *nort* ∧
Dag (*high no*) *low high not* ∧ *reduced nort* ∧
ordered nort var ∧ *set-of nort* ⊆ *repb* ‘(*Nodes n ll*) ∧
(∀ *no*∈*set-of nort*. *repb no* = *no*) ∧
(∃ *nobdt norbdt*. *bdt not var* = *Some nobdt* ∧ *bdt nort var* =
Some norbdt ∧ *nobdt* ∼ *norbdt*))
(**is** *?varrep-high* ∧
(∃ *not nort*. *?repbchigh-dag nort* ∧ *?high-dag not* ∧
?redhigh nort ∧ *?ordhigh nort* ∧ *?rephigh-in-Nodes nort* ∧
?repbno-no nort ∧ *?highdd-prop not nort*)
is *?varrep-high* ∧ *?not-nort-prop*)
apply *simp*
apply (*erule-tac x*=*high no* **in** *ballE*)
apply (*simp del*: *Dag-Ref*)
apply *simp*
done
then have *varrep-high*: *?varrep-high* **by** *simp*
from *varhnos-n varrep-high* **have** *varrephno-s-n*:
var (*repc* (*high no*)) < *n*
by *simp*
from *Nodes-subset*
have *Nodes n ll* ⊆ *Nodes* (*Suc n*) *ll*
by *auto*
with *hno-in-Nodesn repcNodes-in-Nodes*
have *repc* (*high no*) ∈ *Nodes* (*Suc n*) *ll*
apply *simp*
apply *blast*

done
with *wf-ll nsll* **have** *repc* (*high no*) ∈ *set-of pret*
apply (*simp add*: *wf-ll-def Nodes-def length-ll-eq*)
apply (*elim conjE exE*)
apply (*thin-tac* ∀ *q*. *q* ∈ *set-of pret* ⟶ *q* ∈ *set* (*ll* ! *var q*))
apply (*erule-tac x*=*x* **in** *allE*)
apply (*erule impE*)
apply *simp*
apply (*erule-tac x*=*repc* (*high no*) **in** *ballE*)
apply *auto*
done
with *wf-ll varrephno-s-n*
have ∃ *k*<*n*. *repc* (*high no*) ∈ *set* (*ll* ! *k*)
by (*auto simp add*: *wf-ll-def*)
with *wf-ll nsll* **have** *repc* (*high no*) ∉ *set* (*ll* ! *n*)
apply −
apply (*erule exE*)
apply (*rule-tac i*=*k* **and** *j*=*n* **in** *no-in-one-ll*)
apply (*auto simp add*: *length-ll-eq*)
done
with *repbc-nc*
have *repbchigh-idem*: *repb* (*repc* (*high no*)) = *repc* (*repc* (*high no*))
by *auto*
from *high-normalize*
have *not-nort-prop-high*: *?not-nort-prop* **by** (*simp del*: *Dag-Ref*)
from *not-nort-prop-high* **obtain** *hnot* **where** *high-dag*: *?high-dag hnot*
by *auto*
from *wf-ll nsll*
have ∀ *no* ∈ *Nodes n ll*. *no* ∉ *set* (*ll* ! *n*)
apply (*simp add*: *Nodes-def length-ll-eq*)
apply *clarify*
apply (*drule no-in-one-ll*)
apply *auto*
done
with *repbc-nc* **have** ∀ *no* ∈ *Nodes n ll*. *repb no* = *repc no*
apply −
apply (*rule ballI*)
apply (*erule-tac x*=*no* **in** *allE*)
apply *simp*
done
then
have *repbNodes-repcNodes*:
repb ‘(*Nodes n ll*) = *repc* ‘(*Nodes n ll*)
apply −
apply *rule*
apply *blast*
apply *rule*
apply (*erule imageE*)
apply (*erule-tac x*=*xa* **in** *ballE*)

prefer *2*
apply *simp*
apply *rule*
apply *auto*
done
then have *repcNodes-repbNodes*:
repc ‘(Nodes n ll) = repb ‘(Nodes n ll)
by *simp*
from *pret-dag nsll wf-ll*
have *null-notin-Nodesn*: *Null ∉ Nodes n ll*
apply −
apply (*rule Null-notin-Nodes*)
apply (*auto simp add*: *length-ll-eq*)
done
from *hno-in-Nodesn* **have** *repc* (*high no*) ∈ *repc ‘*(*Nodes n ll*)
by *blast*
with *repbNodes-in-Nodes repcNodes-repbNodes*
have *repc* (*high no*) ∈ *Nodes n ll*
apply *simp*
apply *blast*
done
with *null-notin-Nodesn* **have** *rhn-nNull*: *repc* (*high no*) ≠ *Null*
by *auto*

from *no-in-pret nonNull lno-nNull pret-dag*
have *lno-in-pret*: *low no* ∈ *set-of pret*
by (*rule subelem-set-of-low*)
with *wf-ll*
have *lno-in-ll*: *low no* ∈ *set* (*ll* ! (*var* (*low no*)))
by (*simp add*: *wf-ll-def*)
from *pret-dag ord-pret no-in-pret lno-nNull hno-nNull*
have *varlnos-varno*: *var* (*low no*) < *var no*
apply −
apply (*drule var-ordered-children*)
apply *assumption*+
apply *auto*
done
with *varno* **have** *varlnos-n*: *var* (*low no*) < *n* **by** *simp*
with *lno-in-ll* **have** *lno-in-Nodesn*: *low no* ∈ *Nodes n ll*
apply (*simp add*: *Nodes-def*)
apply (*rule-tac x*=*var* (*low no*) **in** *exI*)
apply *simp*
done
from *varlnos-n wf-ll nsll lno-in-ll*
have *low no* ∉ *set* (*ll* ! *n*)
apply −
apply (*rule no-in-one-ll*)

apply (*auto simp add*: *length-ll-eq*)
done
with *repbc-nc* **have** *repb-repc-low*: *repb* (*low no*) = *repc* (*low no*) **by** *simp*
with *normalize-prop lno-in-Nodesn varlnos-varno varno*
have *low-normalize*: *var* (*repc* (*low no*)) ≤ *var* (*low no*) ∧
(∃ *not nort*. *Dag* (*repc* (*low no*)) (*repb* ∝ *low*) (*repb* ∝ *high*) *nort* ∧
Dag (*low no*) *low high not* ∧ *reduced nort* ∧ *ordered nort var* ∧
set-of nort ⊆ *repb* ‘(*Nodes n ll*) ∧
(∀ *no*∈*set-of nort*. *repb no* = *no*) ∧
(∃ *nobdt norbdt*. *bdt not var* = *Some nobdt* ∧ *bdt nort var* = *Some norbdt* ∧
nobdt ∼ *norbdt*))
(**is** *?varrep-low* ∧
(∃ *not nort*. *?repbclow-dag nort* ∧ *?low-dag not* ∧ *?redhigh nort* ∧
?ordhigh nort ∧ *?replow-in-Nodes nort* ∧ *?low-repno-no nort*
∧ *?lowdd-prop not nort*)
is *?varrep-low* ∧ *?not-nort-prop-low*)
apply *simp*
apply (*erule-tac x*=*low no* **in** *ballE*)
apply (*simp del*: *Dag-Ref*)
apply *simp*
done
then have *varrep-low*: *?varrep-low* **by** *simp*
from *low-normalize* **have** *not-nort-prop-low*: *?not-nort-prop-low*
by (*simp del*: *Dag-Ref*)
from *lno-in-Nodesn* **have** *repc* (*low no*) ∈ *repc* ‘(*Nodes n ll*)
by *blast*
with *repbNodes-in-Nodes repcNodes-repbNodes*
have *repc* (*low no*) ∈ *Nodes n ll*
apply *simp*
apply *blast*
done
with *null-notin-Nodesn* **have** *rln-nNull*: *repc* (*low no*) ≠ *Null*
by *auto*

show *?thesis*
proof (*cases repc* (*low no*) = *repc* (*high no*))
case *True*
note *red-case*=*this*
with *repreduce lno-nNull hno-nNull*
have *rno-eq-hrno*: *repc no* = *repc* (*high no*)
by (*simp add*: *null-comp-def*)
from *varhnos-varno rno-eq-hrno varrep-high* **have** *varrep*: *?varrep* **by** *simp*
from *not-nort-prop-high not-nort-prop-low* **have** *repcn-prop*: *?repcn-prop*
apply −
apply (*elim exE*)

```
    apply (rename-tac rnot lnot rnort lnort )
    apply (rule-tac x=(Node lnot no rnot) in exI)
    apply (rule-tac x=rnort in exI)
    apply (elim conjE)
    apply (intro conjI)
    prefer 7
    apply (elim exE)
  apply (rename-tac rnot lnot rnort lnort rnobdt lnobdt rnorbdt lnorbdt)
    apply (elim conjE)
    apply (case-tac  Suc 0 < var no)
    apply (rule-tac x=(Bdt-Node lnobdt (var no) rnobdt) in exI)
    apply (rule conjI)
    prefer 2
    apply (rule-tac x=rnorbdt in exI)
    apply (rule conjI)
   proof −
    fix rnot lnot rnort lnort
    assume highnort-dag:
      Dag (repc (high no)) (repb ∝ low) (repb ∝ high) rnort
    assume ord-nort:  ordered rnort var
    assume rnort-in-repNodes:  set-of rnort ⊆  repb ‘ Nodes n ll
    from rnort-in-repNodes repbNodes-in-Nodes
    have nort-in-Nodes: set-of rnort ⊆ Nodes n ll
      by blast
    from varhnos-n varrep-high
    have vrhnos-n: var (repc (high no)) < n by simp
    from rhn-nNull highnort-dag
    have ∃ lno rno. rnort = Node lno (repc (high no)) rno by fastforce
     with highnort-dag rhn-nNull have root rnort = repc (high no) by
auto
   with ord-nort have ∀ x ∈ set-of rnort. var x <= var (repc (high no))
      apply −
      apply (rule ballI)
      apply (drule ordered-set-of)
      apply auto
      done
    with vrhnos-n have vxsn: ∀ x ∈ set-of rnort. var x < n
      by fastforce
    from nort-in-Nodes have ∀ x ∈ set-of rnort. x ∈ Nodes n ll
      by auto
    with wf-ll nsll
    have x-in-pret: ∀ x ∈ set-of rnort. x ∈ set-of pret
      apply −
      apply (rule ballI)
      apply (drule wf-ll-Nodes-pret)
      apply (auto simp add: length-ll-eq)
      done
    from vxsn x-in-pret
    have vxsn-in-nort: ∀ x ∈ set-of rnort. var x <n ∧ x ∈ set-of pret
```

by *auto*
with *pret-dag prebdt-pret highnort-dag ord-pret wf-ll nsll repbc-nc*
have $\forall\, x \in$ *set-of rnort*. (*repc* $\propto$ *low*) $x =$ (*repb* $\propto$ *low*) $x \wedge$ (*repc* $\propto$ *high*) $x =$ (*repb* $\propto$ *high*) x
apply $-$
apply (*rule nort-null-comp*)
apply (*auto simp add*: *length-ll-eq*)
done
with *rno-eq-hrno*
have *Dag* (*repc no*) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*) *rnort* = *Dag* (*repc no*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*) *rnort*
apply $-$
apply (*rule heaps-eq-Dag-eq*)
apply *simp*
done
with *highnort-dag rno-eq-hrno*
show *Dag* (*repc no*) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*) *rnort* **by** *simp*
next
fix *rnot lnot rnort lnort*
assume *lnot-dag*: *Dag* (*low no*) *low high lnot*
assume *rnot-dag*: *Dag* (*high no*) *low high rnot*
with *lnot-dag nonNull*
show *Dag no low high* (*Node lnot no rnot*) **by** *simp*
next
fix *rnot lnot rnort lnort*
assume *reduced rnort*
then show *reduced rnort* **by** *simp*
next
fix *rnort*
assume *ordered rnort var*
then show *ordered rnort var* **by** *simp*
next
fix *rnort*
assume *rnort-in-Nodes*: *set-of rnort* $\subseteq$ *repb* ‘ *Nodes n ll*
have *Nodes n ll* $\subseteq$ *Nodes* (*n* + *1*) *ll*
by (*simp add*: *Nodes-def set-split*)
then have *repc* ‘ *Nodes n ll* $\subseteq$ *repc* ‘ *Nodes* (*n* + *1*) *ll*
by *blast*
with *rnort-in-Nodes repbNodes-repcNodes*
show *set-of rnort* $\subseteq$ *repc* ‘ *Nodes* (*n* + *1*) *ll*
by (*simp add*: *Nodes-def*)
next
fix *rnort rnorbdt*
assume *bdt rnort var* = *Some rnorbdt*
then show *bdt rnort var* = *Some rnorbdt* **by** *simp*
next
fix *rnot lnot rnort lnort rnobdt lnobdt rnorbdt lnorbdt*
assume *rcongeval*: *rnobdt* $\sim$ *rnorbdt*

assume *lnort-dag*:
Dag (*repc* (*low no*)) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*) *lnort*
assume *rnort-dag*:
Dag (*repc* (*high no*)) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*) *rnort*
assume *lnorbdt-def*: *bdt lnort var* = *Some lnorbdt*
assume *rnorbdt-def*: *bdt rnort var* = *Some rnorbdt*
assume *lcongeval*:*lnobdt* $\sim$ *lnorbdt*
from *red-case lnort-dag rnort-dag*
have *lnort-rnort*: *lnort* = *rnort*
by (*simp add*: *Dag-unique del*: *Dag-Ref*)
with *lnorbdt-def lcongeval rnorbdt-def*
have *lnobdt-rnorbdt*: *lnobdt* $\sim$ *rnorbdt* **by** *simp*
with *rcongeval* **have** *lnobdt* $\sim$ *rnobdt*
apply −
apply (*rule cong-eval-trans*)
apply (*auto simp add*: *cong-eval-sym*)
done
then have *Bdt-Node lnobdt* (*var no*) *rnobdt* $\sim$ *rnobdt*
apply −
apply (*simp add*: *cong-eval-sym* [*rule-format*])
apply (*rule cong-eval-child-high*)
apply *assumption*
done
with *rcongeval* **show** *Bdt-Node lnobdt* (*var no*) *rnobdt* $\sim$ *rnorbdt*
apply −
apply (*rotate-tac 1*)
apply (*rule cong-eval-trans*)
apply *auto*
done
next
fix *lnot rnot lnobdt rnobdt*
assume *lnot-dag*: *Dag* (*low no*) *low high lnot*
assume *rnot-dag*: *Dag* (*high no*) *low high rnot*
assume *lnobdt-def*: *bdt lnot var* = *Some lnobdt*
assume *rnobdt-def*: *bdt rnot var* = *Some rnobdt*
assume *onesvarno*: *Suc 0* < *var no*
with *rnobdt-def lnot-dag rnot-dag lnobdt-def*
show *bdt* (*Node lnot no rnot*) *var* =
Some (*Bdt-Node lnobdt* (*var no*) *rnobdt*)
by *simp*
next
fix *rnot lnot rnort lnort rnobdt lnobdt rnorbdt lnorbdt*
assume *lnobdt-def*: *bdt lnot var* = *Some lnobdt*
assume *rnobdt-def*: *bdt rnot var* = *Some rnobdt*
assume *rnorbdt-def*: *bdt rnort var* = *Some rnorbdt*
assume *cong-rno-rnor*: *rnobdt* $\sim$ *rnorbdt*
assume *lnot-dag*: *Dag* (*low no*) *low high lnot*
assume *rnot-dag*: *Dag* (*high no*) *low high rnot*
assume ¬ *Suc 0* < *var no*

```
then have varnoseq1: var no = 0 ∨ var no = 1 by auto
show ∃ nobdt. bdt (Node lnot no rnot) var = Some nobdt ∧
  (∃ norbdt. bdt rnort var = Some norbdt ∧ nobdt ∼ norbdt)
proof (cases var no = 0)
  case True
  note vnoNull=this
  with pret-dag ord-pret no-in-pret lno-nNull hno-nNull
  show ?thesis
    apply −
    apply (drule var-ordered-children)
    apply auto
    done
next
  assume var no ≠ 0
  with varnoseq1 have vnoOne: var no = 1 by simp
  from pret-dag  ord-pret no-in-pret lno-nNull hno-nNull
    vnoOne
  have vlvrNull: var (low no) = 0 ∧ var (high no) = 0
    apply −
    apply (drule var-ordered-children)
    apply auto
    done
  then have vlNull: var (low no) = 0 by simp
  from vlvrNull have vrNull: var (high no) = 0 by simp
  from lnobdt-def lnot-dag vlNull  lno-nNull
  have lnobdt-Zero: lnobdt = Zero
    apply −
    apply (drule bdt-Some-var0-Zero)
    apply auto
    done
  from rnobdt-def rnot-dag vrNull  hno-nNull
  have rnobdt-Zero: rnobdt = Zero
    apply −
    apply (drule bdt-Some-var0-Zero)
    apply auto
    done
   from lnobdt-Zero lnobdt-def have bdt lnot var = Some Zero by
simp
  with lnot-dag vlNull
  have lnot-Node: lnot = (Node Tip (low no) Tip)
    by auto
  from rnobdt-Zero rnobdt-def rnot-dag vrNull
  have rnot-Node: rnot = (Node Tip (high no) Tip)
    by auto
  from pret-dag no-in-pret obtain not where
    not-ex: Dag no low high not
    apply −
    apply (drule dag-setof-exD)
    apply auto
```

```
      done
    with pret-dag no-in-pret have not-ex-in-pret: not <= pret
      apply -
      apply (rule set-of-subdag)
      apply auto
      done
    from not-ex lnot-dag rnot-dag   nonNull
    have not-def: not = (Node lnot no rnot)
      by (simp add: Dag-unique del: Dag-Ref)
    with not-ex-in-pret prebdt-pret
   have nobdt-ex: ∃ nobdt. bdt (Node lnot no rnot) var = Some nobdt
      apply -
      apply (rule subbdt-ex)
      apply auto
      done
    then obtain nobdt where
      nobdt-def: bdt (Node lnot no rnot) var = Some nobdt by auto
    from not-def have root not = no by simp
    with nobdt-def vnoOne not-def have not = (Node Tip no Tip)
      apply -
      apply (drule bdt-Some-var1-One)
      apply auto
      done
    with not-def have rnot = Tip by simp
    with rnot-Node show ?thesis by simp
  qed
 next
  fix rnot lnot rnort lnort
  assume rnort-in-repb-Nodesn: set-of rnort ⊆ repb ‘ Nodes n ll
  assume rnort-repb-no: ∀ no∈set-of rnort. repb no = no
  from repbNodes-in-Nodes rnort-in-repb-Nodesn
  have rnort-in-Nodesn: set-of rnort ⊆ Nodes n ll
    by blast
  show ∀ no∈set-of rnort. repc no = no
  proof
    fix pt
    assume pt-in-rnort:  pt ∈ set-of rnort
    with rnort-in-Nodesn have pt ∈ Nodes n ll
      by blast
    with Nodesn-notin-lln have pt ∉ set (ll ! n)
      by auto
    with repbc-nc have repb pt = repc pt
      by auto
    with rnort-repb-no pt-in-rnort show repc pt = pt
      by auto
  qed
 qed
 with varrep show ?thesis by simp
next
```

assume *share-case-cond*: *repc* (*low no*) ≠ *repc* (*high no*)
with *lno-nNull hno-nNull*
have *share-case-cond-propto*: (*repc* ∝ *low*) *no* ≠ (*repc* ∝ *high*) *no*
by (*simp add*: *null-comp-def*)
with *repshare* **obtain**
rno-in-llbn: *repc no* ∈ *set* (*ll* ! *n*) **and**
rrno-eq-rno: *repc* (*repc no*) = *repc no* **and**
twonodes-in-llbn-prop: (∀ *no1*∈*set* (*ll* ! *n*).
((*repc* ∝ *high*) *no1* = (*repc* ∝ *high*) *no* ∧
(*repc* ∝ *low*) *no1* = (*repc* ∝ *low*) *no*) = (*repc no* = *repc no1*))
by *auto*
from *wf-ll rno-in-llbn nsll*
have *varrepno-n*: *var* (*repc no*) = *n*
by (*simp add*: *wf-ll-def length-ll-eq*)
with *varno* **have** *varrep*: *?varrep*
by *simp*
from *not-nort-prop-high not-nort-prop-low* **have** *repcn-prop*: *?repcn-prop*
apply−
apply (*elim exE*)
apply (*rename-tac rnot lnot rnort lnort*)
apply (*rule-tac x*=*Node lnot no rnot* **in** *exI*)
apply (*rule-tac x*=*Node lnort* (*repc no*) *rnort* **in** *exI*)
apply (*elim conjE*)
apply (*intro conjI*)
prefer *7*
apply (*elim exE*)
apply (*rename-tac rnot lnot rnort lnort rnobdt lnobdt rnorbdt lnorbdt*)
apply (*elim conjE*)
apply (*case-tac Suc 0* < *var no*)
apply (*rule-tac x*=(*Bdt-Node lnobdt* (*var no*) *rnobdt*) **in** *exI*)
apply (*rule conjI*)
prefer *2*
apply (*rule-tac x*=(*Bdt-Node lnorbdt* (*var* (*repc no*)) *rnorbdt*) **in** *exI*)
apply (*rule conjI*)
proof −
fix *rnot lnot rnort lnort*
assume *rnort-dag*:
Dag (*repc* (*high no*)) (*repb* ∝ *low*) (*repb* ∝ *high*) *rnort*
assume *lnort-dag*:
Dag (*repc* (*low no*)) (*repb* ∝ *low*) (*repb* ∝ *high*) *lnort*
assume *rnort-in-repNodes*: *set-of rnort* ⊆ *repb* ‘ *Nodes n ll*
assume *lnort-in-repNodes*: *set-of lnort* ⊆ *repb* ‘ *Nodes n ll*
from *rnort-in-repNodes repbNodes-in-Nodes*
have *rnort-in-Nodes*: *set-of rnort* ⊆ *Nodes n ll*
by *simp*
from *lnort-in-repNodes repbNodes-in-Nodes*
have *lnort-in-Nodes*: *set-of lnort* ⊆ *Nodes n ll*
by *simp*

from *rnort-in-Nodes*
have *rnortx-in-Nodes*: $\forall\ x \in$ *set-of rnort*. $x \in$ *Nodes n ll*
by *blast*
with *wf-ll nsll*
have $\forall\ x \in$ *set-of rnort*. $x \in$ *set-of pret* $\wedge$ *var* $x < n$
apply $-$
apply (*rule ballI*)
apply (*rule wf-ll-Nodes-pret*)
apply (*auto simp add*: *length-ll-eq*)
done
with *pret-dag prebdt-pret rnort-dag ord-pret wf-ll nsll repbc-nc*
have $\forall x \in$ *set-of rnort*. (*repc* $\propto$ *low*) $x =$ (*repb* $\propto$ *low*) $x \wedge$ (*repc* $\propto$ *high*) $x =$ (*repb* $\propto$ *high*) x
apply $-$
apply (*rule nort-null-comp*)
apply (*auto simp add*: *length-ll-eq*)
done
then have *Dag* (*repc* (*high no*)) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*) *rnort* =
Dag (*repc* (*high no*)) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*) *rnort*
apply $-$
apply (*rule heaps-eq-Dag-eq*)
apply *assumption*
done
with *rnort-dag*
have *rnort-dag-repc*:
Dag (*repc* (*high no*)) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*) *rnort*
by *simp*
from *lnort-in-Nodes*
have *lnortx-in-Nodes*: $\forall x \in$ *set-of lnort*. $x \in$ *Nodes n ll*
by *blast*
with *wf-ll nsll*
have $\forall\ x \in$ *set-of lnort*. $x \in$ *set-of pret* $\wedge$ *var* $x < n$
apply $-$
apply (*rule ballI*)
apply (*rule wf-ll-Nodes-pret*)
apply (*auto simp add*: *length-ll-eq*)
done
with *pret-dag prebdt-pret lnort-dag ord-pret wf-ll nsll repbc-nc*
have $\forall\ x \in$ *set-of lnort*. (*repc* $\propto$ *low*) $x =$ (*repb* $\propto$ *low*) $x \wedge$ (*repc* $\propto$ *high*) $x =$ (*repb* $\propto$ *high*) x
apply $-$
apply (*rule nort-null-comp*)
apply (*auto simp add*: *length-ll-eq*)
done
then have
Dag (*repc* (*low no*)) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*) *lnort* =
Dag (*repc* (*low no*)) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*) *lnort*

apply −
apply (*rule heaps-eq-Dag-eq*)
apply *assumption*
done
with *lnort-dag*
have *lnort-dag-repc*:
Dag (*repc* (*low no*)) (*repc* ∝ *low*) (*repc* ∝ *high*) *lnort*
by *simp*
from *lno-nNull hno-nNull*
have *propto-comp*: (*repc* ∝ *low*) *no* = *repc* (*low no*) ∧
(*repc* ∝ *high*) *no* = *repc* (*high no*)
by (*simp add*: *null-comp-def*)
from *rno-in-llbn twonodes-in-llbn-prop rrno-eq-rno*
have (*repc* ∝ *high*) (*repc no*) = (*repc* ∝ *high*) *no* ∧
(*repc* ∝ *low*) (*repc no*) = (*repc* ∝ *low*) *no*
by *simp*
with *propto-comp lnort-dag-repc rnort-dag-repc lno-nNull hno-nNull*
rnonN
show *Dag*(*repc no*)(*repc* ∝ *low*)(*repc* ∝ *high*)(*Node lnort* (*repc no*)
rnort)
by *auto*
next
fix *rnot lnot rnort lnort*
assume *rnot-dag*: *Dag* (*high no*) *low high rnot*
assume *lnot-dag*: *Dag* (*low no*) *low high lnot*
with *rnot-dag nonNull*
show *Dag no low high* (*Node lnot no rnot*)
by *simp*
next
fix *rnort lnort*
assume *rnort-dag*:
Dag (*repc* (*high no*)) (*repb* ∝ *low*) (*repb* ∝ *high*) *rnort*
assume *lnort-dag*:
Dag (*repc* (*low no*)) (*repb* ∝ *low*) (*repb* ∝ *high*) *lnort*
assume *red-rnort*: *reduced rnort*
assume *red-lnort*: *reduced lnort*
from *rhn-nNull rnort-dag* **obtain** *lrnort rrnort* **where**
rnort-Node: *rnort* = (*Node lrnort* (*repc* (*high no*)) *rrnort*)
by *auto*
from *rln-nNull lnort-dag* **obtain** *llnort rlnort* **where**
lnort-Node: *lnort* = (*Node llnort* (*repc* (*low no*)) *rlnort*)
by *auto*
from *twonodes-in-llbn-prop rrno-eq-rno rno-in-llbn hno-nNull*
lno-nNull
have ((*repc* ∝ *high*) (*repc no*)) = *repc* (*high no*) ∧
((*repc* ∝ *low*) (*repc no*)) = *repc* (*low no*)
apply −
apply (*erule-tac x*=*repc no* **in** *ballE*)
apply (*auto simp add*: *null-comp-def*)

```
    done
  with share-case-cond
  have ((repc ∝ high) (repc no)) ≠ ((repc ∝ low) (repc no))
    by auto
  with red-lnort red-rnort rnort-Node lnort-Node share-case-cond
  show reduced (Node lnort (repc no) rnort)
    apply −
    apply (rule-tac lp=repc (low no) and rp=repc (high no) and
      llt=llnort and rlt = rlnort and lrt=lrnort and rrt=rrnort
      in reduced-children-parent)
    apply auto
    done
next
  fix lnort rnort
  assume lnort-dag:
    Dag (repc (low no)) (repb ∝ low) (repb ∝ high) lnort
  assume ord-lnort: ordered lnort var
  assume rnort-dag:
    Dag (repc (high no)) (repb ∝ low) (repb ∝ high) rnort
  assume ord-rnort:  ordered rnort var
  assume lnort-in-repNodes: set-of lnort ⊆ repb 'Nodes n ll
  assume rnort-in-repNodes: set-of rnort ⊆ repb 'Nodes n ll
  from lnort-in-repNodes repbNodes-in-Nodes
  have lnort-in-Nodes: set-of lnort ⊆ Nodes n ll
    by simp
  from rnort-in-repNodes repbNodes-in-Nodes
  have rnort-in-Nodes: set-of rnort ⊆ Nodes n ll
    by simp

  from rhn-nNull rnort-dag obtain lrnort rrnort where
    rnort-Node: rnort = (Node lrnort (repc (high no)) rrnort)
    by auto
  from rln-nNull lnort-dag obtain llnort rlnort where
    lnort-Node: lnort = (Node llnort (repc (low no)) rlnort)
    by auto
  from lnort-dag rln-nNull lnort-in-Nodes
  have repc (low no) ∈ set-of lnort
    by auto
  with lnort-in-Nodes
  have repc (low no) ∈ Nodes n ll
    by blast
  with wf-ll nsll
  have vrlno-sn: var (repc (low no)) < n
    apply −
    apply (drule wf-ll-Nodes-pret)
    apply (auto simp add: length-ll-eq)
    done
  from rnort-dag rhn-nNull rnort-in-Nodes
  have repc (high no) ∈ set-of rnort
```

by *auto*
with *rnort-in-Nodes*
have *repc* (*high no*) ∈ *Nodes n ll*
by *blast*
with *wf-ll nsll* **have** *vrhno-sn*: *var* (*repc* (*high no*)) < *n*
apply −
apply (*drule wf-ll-Nodes-pret*)
apply (*auto simp add*: *length-ll-eq*)
done
with *varrepno-n vrlno-sn lnort-dag ord-lnort rnort-dag rnort-Node lnort-Node ord-rnort*
show *ordered* (*Node lnort* (*repc no*) *rnort*) *var*
by *auto*
next
fix *lnort rnort*
assume *lnort-in-Nodes*: *set-of lnort* ⊆ *repb 'Nodes n ll*
assume *rnort-in-Nodes*: *set-of rnort* ⊆ *repb 'Nodes n ll*
from *lnort-in-Nodes repbNodes-repcNodes*
have *lnort-in-repcNodes*: *set-of lnort* ⊆ *repc 'Nodes n ll*
by *simp*
from *rnort-in-Nodes repbNodes-repcNodes*
have *rnort-in-repcNodes*: *set-of rnort* ⊆ *repc 'Nodes n ll*
by *simp*
have *nNodessubset*: *Nodes n ll* ⊆ *Nodes* (*n+1*) *ll*
by (*simp add*: *Nodes-subset*)
then have *repc-Nodes-subset*:
repc 'Nodes n ll ⊆ *repc 'Nodes* (*n+1*) *ll*
by *blast*
from *no-in-Nodes* **have** *repc no* ∈ *repc 'Nodes* (*n+1*) *ll*
by *blast*
with *repc-Nodes-subset lnort-in-repcNodes rnort-in-repcNodes*
show *set-of* (*Node lnort* (*repc no*) *rnort*) ⊆
repc 'Nodes (*n* + *1*) *ll*
apply *simp*
apply *blast*
done
next
fix *rnot lnot rnort lnort rnobdt lnobdt rnorbdt lnorbdt*
assume *lnobdt-def*: *bdt lnot var* = *Some lnobdt*
assume *rnobdt-def*: *bdt rnot var* = *Some rnobdt*
assume *rnorbdt-def*: *bdt rnort var* = *Some rnorbdt*
assume *cong-rno-rnor*: *rnobdt* ∼ *rnorbdt*
assume *lnot-dag*: *Dag* (*low no*) *low high lnot*
assume *rnot-dag*: *Dag* (*high no*) *low high rnot*
assume ¬ *Suc 0* < *var no*
then have *varnoseq1*: *var no* = *0* ∨ *var no* = *1* **by** *auto*
show ∃ *nobdt*. *bdt* (*Node lnot no rnot*) *var* = *Some nobdt* ∧
(∃ *norbdt*. *bdt* (*Node lnort* (*repc no*) *rnort*) *var* = *Some norbdt* ∧
nobdt ∼ *norbdt*)

proof (*cases var no = 0*)
case *True*
note *vnoNull=this*
with *pret-dag ord-pret no-in-pret lno-nNull hno-nNull*
show *?thesis*
apply −
apply (*drule var-ordered-children*)
apply *auto*
done
next
assume *var no ≠ 0*
with *varnoseq1* **have** *vnoOne*: *var no = 1* **by** *simp*
from *pret-dag ord-pret no-in-pret lno-nNull hno-nNull vnoOne*
have *vlvrNull*: *var* (*low no*) = *0* ∧ *var* (*high no*) = *0*
apply −
apply (*drule var-ordered-children*)
apply *auto*
done
then have *vlNull*: *var* (*low no*) = *0* **by** *simp*
from *vlvrNull* **have** *vrNull*: *var* (*high no*) = *0* **by** *simp*
from *lnobdt-def lnot-dag vlNull lno-nNull*
have *lnobdt-Zero*: *lnobdt = Zero*
apply −
apply (*drule bdt-Some-var0-Zero*)
apply *auto*
done
from *rnobdt-def rnot-dag vrNull hno-nNull*
have *rnobdt-Zero*: *rnobdt = Zero*
apply −
apply (*drule bdt-Some-var0-Zero*)
apply *auto*
done
from *lnobdt-Zero lnobdt-def*
have *bdt lnot var = Some Zero* **by** *simp*
with *lnot-dag vlNull*
have *lnot-Node*: *lnot* = (*Node Tip* (*low no*) *Tip*)
by *auto*
from *rnobdt-Zero rnobdt-def rnot-dag vrNull*
have *rnot-Node*: *rnot* = (*Node Tip* (*high no*) *Tip*)
by *auto*
from *pret-dag no-in-pret* **obtain** *not*
where *not-ex*: *Dag no low high not*
apply −
apply (*drule dag-setof-exD*)
apply *auto*
done
with *pret-dag no-in-pret* **have** *not-ex-in-pret*: *not <= pret*
apply −

```
    apply (rule set-of-subdag)
    apply auto
    done
  from not-ex lnot-dag rnot-dag   nonNull
  have not-def: not = (Node lnot no rnot)
    by (simp add: Dag-unique del: Dag-Ref)
  with not-ex-in-pret prebdt-pret
 have nobdt-ex: ∃ nobdt. bdt (Node lnot no rnot) var = Some nobdt
    apply -
    apply (rule subbdt-ex)
    apply auto
    done
  then obtain nobdt where
    nobdt-def: bdt (Node lnot no rnot) var = Some nobdt by auto
  from not-def have root not = no by simp
  with nobdt-def vnoOne not-def
  have not = (Node Tip no Tip)
    apply -
    apply (drule bdt-Some-var1-One)
    apply auto
    done
  with not-def have rnot = Tip by simp
  with rnot-Node show ?thesis by simp
 qed
next
 fix lnot rnot lnobdt rnobdt
 assume lnot-dag: Dag (low no) low high lnot
 assume rnot-dag:  Dag (high no) low high rnot
 assume lnobdt-def:  bdt lnot var = Some lnobdt
 assume rnobdt-def:  bdt rnot var = Some rnobdt
 assume onesvarno:  Suc 0 < var no
 with rnobdt-def lnot-dag rnot-dag lnobdt-def
 show bdt (Node lnot no rnot) var =
   Some (Bdt-Node lnobdt (var no) rnobdt) by simp
next
 fix rnot lnot rnort lnort rnobdt lnobdt rnorbdt lnorbdt
 assume rnort-dag:
   Dag (repc (high no)) (repb ∝ low) (repb ∝ high) rnort
 assume lnort-dag:
   Dag (repc (low no)) (repb ∝ low) (repb ∝ high) lnort
 assume rnorbdt-def:  bdt rnort var = Some rnorbdt
 assume lnorbdt-def: bdt lnort var = Some lnorbdt
 assume varno-bOne: Suc 0 < var no
 with varno have Suc 0 < n by simp
 with varrepno-n have Suc 0 < var (repc no) by simp
 with rnorbdt-def lnorbdt-def
 show bdt (Node lnort (repc no) rnort) var =
   Some (Bdt-Node lnorbdt (var (repc no)) rnorbdt)
   by simp
```

next
fix *rnobdt lnobdt rnorbdt lnorbdt*
assume *lcong-eval*: *lnobdt* $\sim$ *lnorbdt*
assume *rcong-eval*: *rnobdt* $\sim$ *rnorbdt*
from *varno varrepno-n* **have** *var* (*repc no*) = *var no* **by** *simp*
with *lcong-eval rcong-eval*
show *Bdt-Node lnobdt* (*var no*) *rnobdt* $\sim$
Bdt-Node lnorbdt (*var* (*repc no*)) *rnorbdt*
apply (*unfold cong-eval-def*)
apply (*rule ext*)
by *simp*
next
fix *rnot lnot rnort lnort*
assume *lnort-repb*: $\forall$ *no*$\in$*set-of lnort*. *repb no* = *no*
assume *rnort-repb*: $\forall$ *no*$\in$*set-of rnort*. *repb no* = *no*
assume *rnort-in-repb-Nodesn*: *set-of rnort* $\subseteq$ *repb* ‘ *Nodes n ll*
assume *lnort-in-repb-Nodesn*: *set-of lnort* $\subseteq$ *repb* ‘ *Nodes n ll*
from *repbNodes-in-Nodes rnort-in-repb-Nodesn*
have *rnort-in-Nodesn*: *set-of rnort* $\subseteq$ *Nodes n ll*
by *blast*
from *repbNodes-in-Nodes lnort-in-repb-Nodesn*
have *lnort-in-Nodesn*: *set-of lnort* $\subseteq$ *Nodes n ll*
by *blast*
have *rnort-repc*: $\forall$ *no*$\in$*set-of rnort*. *repc no* = *no*
proof
fix *pt*
assume *pt-in-rnort*: *pt* $\in$ *set-of rnort*
with *rnort-in-Nodesn* **have** *pt* $\in$ *Nodes n ll*
by *blast*
with *Nodesn-notin-lln* **have** *pt* $\notin$ *set* (*ll* ! *n*)
by *auto*
with *repbc-nc* **have** *repb pt* = *repc pt*
by *auto*
with *rnort-repb pt-in-rnort* **show** *repc pt* = *pt*
by *auto*
qed
have *lnort-repc*: $\forall$ *no*$\in$*set-of lnort*. *repc no* = *no*
proof
fix *pt*
assume *pt-in-lnort*: *pt* $\in$ *set-of lnort*
with *lnort-in-Nodesn* **have** *pt* $\in$ *Nodes n ll*
by *blast*
with *Nodesn-notin-lln* **have** *pt* $\notin$ *set* (*ll* ! *n*)
by *auto*
with *repbc-nc* **have** *repb pt* = *repc pt*
by *auto*
with *lnort-repb pt-in-lnort* **show** *repc pt* = *pt*
by *auto*
qed

```
        show ∀ no∈set-of (Node lnort (repc no) rnort). repc no = no
        proof
          fix pt
          assume pt-in-rept: pt ∈ set-of (Node lnort (repc no) rnort)
          show repc pt = pt
          proof (cases pt ∈ set-of lnort)
            case True
            with lnort-repc show ?thesis
              by auto
          next
            assume pt-notin-lnort: pt ∉ set-of lnort
            show ?thesis
            proof (cases pt ∈ set-of rnort)
              case True
              with rnort-repc show ?thesis
                by auto
            next
              assume pt-notin-rnort: pt ∉ set-of rnort
              with pt-notin-lnort pt-in-rept have pt = repc no
                by simp
              with rrno-eq-rno show repc pt = pt
                by simp
            qed
          qed
        qed

        with varrep rrno-eq-rno show ?thesis by simp
      qed
    qed
    with rnonN show ?thesis by simp
  qed
qed
note while-while-prop=this
from wf-ll nsll
have ∀ no ∈ Nodes n ll. no ∉ set (ll ! n)
  apply (simp add: Nodes-def length-ll-eq)
  apply clarify
  apply (drule no-in-one-ll)
  apply auto
  done
with repbc-nc  have ∀ no ∈ Nodes n ll. repb no = repc no
  apply −
  apply (rule ballI)
  apply (erule-tac x=no in allE)
  apply simp
  done
then have repbNodes-repcNodes:
  repb `(Nodes n ll) = repc `(Nodes n ll)
```

```
  apply −
  apply rule
  apply blast
  apply rule
  apply (erule imageE)
  apply (erule-tac x=xa in ballE)
  prefer 2
  apply simp
  apply rule
  apply auto
  done
then have repcNodes-repbNodes:
  repc ‘(Nodes n ll) = repb ‘(Nodes n ll)
  by simp
have repbc-dags-eq:
  Dags (repc ‘ Nodes n ll) (repc ∝ low) (repc ∝ high) =
  Dags (repb  ‘ Nodes n ll) (repb ∝ low) (repb ∝ high)
  apply −
  apply rule
  apply rule
  apply (erule Dags.cases)
  apply (rule DagsI)
  prefer 4
  apply rule
  apply (erule Dags.cases)
  apply (rule DagsI)
proof −
  fix x p t
  assume t-in-repcNodes: set-of t ⊆ repc ‘ Nodes n ll
  assume x-t: x=t
  with t-in-repcNodes repcNodes-repbNodes
  show set-of x ⊆ repb ‘ Nodes n ll
    by simp
next
  fix x p t
  assume t-in-repcNodes: set-of t ⊆ repc ‘ Nodes n ll
  assume t-dag: Dag p (repc ∝ low) (repc ∝ high) t
  assume t-nTip:  t ≠ Tip
  assume x-t: x=t
  from t-nTip t-dag have p ≠ Null
    apply −
    apply (case-tac p=Null)
    apply auto
    done
  with t-nTip t-dag obtain lt rt where t-Node: t=Node lt p rt
    by auto
  from t-in-repcNodes t-dag t-nTip t-Node obtain q where
    rq-p: repc q = p and q-in-Nodes: q ∈ Nodes n ll
    apply simp
```

```
    apply (elim conjE)
    apply (erule imageE)
    apply auto
    done
  from q-in-Nodes have repb q = repc q
    by (rule Nodes-n-rep-nc [rule-format])
  with rq-p have repbq-p: repb q = p by simp
  from q-in-Nodes
  have Dag (repb q) (repb ∝ low) (repb ∝ high) t =
    Dag (repc q) (repc ∝ low) (repc ∝ high) t
    by (rule Nodes-repbc-Dags-eq [rule-format])
  with t-dag rq-p have Dag (repb q) (repb ∝ low) (repb ∝ high) t by simp
  with repbq-p x-t show Dag p (repb ∝ low) (repb ∝ high) x
    by simp
next
  fix x p t
  assume t-in-repcNodes: set-of t ⊆ repb ‘ Nodes n ll
  assume x-t: x=t
  with t-in-repcNodes repbNodes-repcNodes
  show set-of x ⊆ repc ‘ Nodes n ll
    by simp
next
  fix x p t
  assume t-in-repcNodes: set-of t ⊆ repb ‘ Nodes n ll
  assume t-dag: Dag p (repb ∝ low) (repb ∝ high) t
  assume t-nTip:  t ≠ Tip
  assume x-t: x=t
  from t-nTip t-dag have p ≠ Null
    apply −
    apply (case-tac p=Null)
    apply auto
    done
  with t-nTip t-dag obtain lt rt where t-Node: t=Node lt p rt
    by auto
  from t-in-repcNodes t-dag t-nTip t-Node obtain q where
    rq-p: repb q = p and q-in-Nodes: q ∈ Nodes n ll
    apply simp
    apply (elim conjE)
    apply (erule imageE)
    apply auto
    done
  from q-in-Nodes have repb q = repc q
    by (rule Nodes-n-rep-nc [rule-format])
  with rq-p have repbq-p: repc q = p by simp
  from q-in-Nodes
  have Dag (repb q) (repb ∝ low) (repb ∝ high) t =
    Dag (repc q) (repc ∝ low) (repc ∝ high) t
    by (rule Nodes-repbc-Dags-eq [rule-format])
  with t-dag rq-p have Dag (repc q) (repc ∝ low) (repc ∝ high) t by simp
```

with *repbq-p x-t* **show** *Dag p* (*repc* $\propto$ *low*) (*repc* $\propto$ *high*) *x*
by *simp*
next
fix *x p* **and** *t* :: *dag*
assume *x-t*: $x = t$
assume *t-nTip*: $t \neq Tip$
with *x-t* **show** $x \neq Tip$ **by** *simp*
next
fix *x p* **and** *t* :: *dag*
assume *x-t*: $x = t$
assume *t-nTip*: $t \neq Tip$
with *x-t* **show** $x \neq Tip$ **by** *simp*
qed
from *pret-dag wf-ll nsll*
have *null-notin-Nodes-Suc-n*: *Null* $\notin$ *Nodes* (*Suc n*) *ll*
by − (*rule Null-notin-Nodes,auto simp add*: *length-ll-eq*)
{ fix *t1 t2*
assume *t1-in-DagsNodesn*:
t1 $\in$ *Dags* (*repc* ‘ *Nodes n ll*) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*)
assume *t2-notin-DagsNodesn*:
t2 $\notin$ *Dags* (*repc* ‘ *Nodes n ll*) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*)
assume *t2-in-DagsNodesSucn*:
t2 $\in$ *Dags* (*repc* ‘ *Nodes* (*Suc n*) *ll*) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*)
assume *isomorphic-dags-eq-asm*:
$\forall$ *t1 t2*. *t1* $\in$ *Dags* (*repb* ‘ *Nodes n ll*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*)
$\wedge$ *t2* $\in$ *Dags* (*repb* ‘ *Nodes n ll*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*)
$\longrightarrow$ *isomorphic-dags-eq t1 t2 var*
assume *repbc-Dags*:
Dags (*repc* ‘ *Nodes n ll*) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*) =
Dags (*repb* ‘ *Nodes n ll*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*)
from *t1-in-DagsNodesn repbc-Dags*
have *t1-repb-subnode*:
t1 $\in$ *Dags* (*repb* ‘ *Nodes n ll*) (*repb* $\propto$ *low*) (*repb* $\propto$ *high*)
by *simp*
from *t2-in-DagsNodesSucn*
have *t2-in-DagsNodesSucn*:
t2 $\in$ *Dags* (*repc* ‘ *Nodes* (*Suc n*) *ll*) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*)
by *simp*
from *repbNodes-in-Nodes repbNodes-repcNodes*
have *repcNodesn-in-Nodesn*: *repc* ‘*Nodes n ll* $\subseteq$ *Nodes n ll*
by *simp*
from *t1-in-DagsNodesn* **obtain** *q* **where**
Dag-q-Nodes-n:
Dag (*repc q*) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*) *t1* $\wedge$ *q* $\in$ *Nodes n ll*
proof (*elim Dags.cases*)
fix *p t*
assume *t1-t*: *t1* = *t*
assume *t-in-repcNodesn*: *set-of t* $\subseteq$ *repc* ‘ *Nodes n ll*
assume *t-dag*: *Dag p* (*repc* $\propto$ *low*) (*repc* $\propto$ *high*) *t*

assume *t-nTip*: $t \neq Tip$
assume *obtain-prop*: $\bigwedge q.\ Dag\ (repc\ q)\ (repc \propto low)\ (repc \propto high)\ t1 \wedge q \in Nodes\ n\ ll \Longrightarrow ?thesis$
from *t-nTip t-dag* **have** $p \neq Null$
apply −
apply (*case-tac p=Null*)
apply *auto*
done
with *t-nTip t-dag* **obtain** *lt rt* **where** *t-Node*: *t=Node lt p rt*
by *auto*
from *t-in-repcNodesn t-dag t-nTip t-Node* **obtain** *k* **where**
rk-p: *repc k = p* **and** *k-in-Nodes*: $k \in Nodes\ n\ ll$
apply *simp*
apply (*elim conjE*)
apply (*erule imageE*)
apply *auto*
done
with *t1-t t-dag obtain-prop rk-p k-in-Nodes* **show** *?thesis*
by *auto*
qed
with *wf-ll nsll* **have** *varq-sn*: $(var\ q < n)$
apply (*simp add*: *Nodes-def*)
apply (*elim conjE*)
apply (*erule exE*)
apply (*simp add*: *wf-ll-def length-ll-eq*)
apply (*elim conjE*)
apply (*thin-tac* $\forall q.\ q \in set\text{-}of\ pret \longrightarrow q \in set\ (ll\ !\ var\ q)$)
apply (*erule-tac x=x* **in** *allE*)
apply *auto*
done
from *Dag-q-Nodes-n* **have** *q-in-Nodesn*: $q \in Nodes\ n\ ll$
by *simp*
then have $\exists\ k<n.\ q \in set\ (ll\ !\ k)$
by (*simp add*: *Nodes-def*)
with *wf-ll nsll* **have** $q \notin set\ (ll\ !\ n)$
apply −
apply (*erule exE*)
apply (*rule-tac i=k* **and** *j=n* **in** *no-in-one-ll*)
apply (*auto simp add*: *length-ll-eq*)
done
with *repbc-nc* **have** *repbc-q*: *repc q = repb q*
apply −
apply (*erule-tac x=q* **in** *allE*)
apply *auto*
done
from *normalize-prop q-in-Nodesn* **have** $var\ (repb\ q) <= var\ q$
apply −
apply (*erule-tac x=q* **in** *ballE*)
apply *auto*

done
with *repbc-q* **have** *var-repc-q*: *var* (*repc q*) <= *var q*
by *simp*
with *varq-sn* **have** *var-repc-q-n*: *var* (*repc q*) < *n*
by *simp*

from *Nodes-subset Dag-q-Nodes-n while-while-prop*
have *ord-t1-var-rep*: *ordered t1 var* ∧ *var* (*repc q*) <= *var q*
apply (*elim conjE*)
apply (*erule-tac x=q* **in** *ballE*)
apply *auto*
done
then have *ord-t1*: *ordered t1 var* **by** *simp*
from *ord-t1-var-rep* **have** *varrep-q*: *var* (*repc q*) <= *var q* **by** *simp*
from *t2-in-DagsNodesSucn* **have** *ord-t2*: *ordered t2 var*
proof (*elim Dags.cases*)
fix *p t*
assume *t-in-repcNodes*: *set-of t* ⊆ *repc* ‘ *Nodes* (*Suc n*) *ll*
assume *t-nTip*: *t* ≠ *Tip*
assume *t2t*: *t2* = *t*
assume *t-dag*: *Dag p* (*repc* ∝ *low*) (*repc* ∝ *high*) *t*
from *t-in-repcNodes* **have** *x-in-repcNodesSucn*:
∀ *x*. *x* ∈ *set-of t* ⟶ *x* ∈ *repc* ‘ *Nodes* (*Suc n*) *ll*
apply −
apply *auto*
done
from *t-nTip t-dag* **have** *p* ≠ *Null*
apply −
apply (*case-tac p=Null*)
apply *auto*
done
with *t-nTip t-dag* **obtain** *lt rt* **where** *t-Node*: *t=Node lt p rt*
by *auto*
then have *p* ∈ *set-of t*
by *auto*
with *x-in-repcNodesSucn* **have** *p* ∈ *repc* ‘ *Nodes* (*Suc n*) *ll*
by *simp*
then obtain *a* **where** *repca-p*: *p=repc a* **and**
a-in-NodesSucn: *a* ∈ *Nodes* (*Suc n*) *ll*
by *auto*
with *repca-p while-while-prop t-dag t2t* **show** *?thesis*
apply −
apply (*erule-tac x=a* **in** *ballE*)
apply (*elim conjE exE*)
apply (*subgoal-tac nort* = *t*)
prefer *2*
apply (*simp add*: *Dag-unique*)
apply *auto*
done

qed
from *while-while-prop* **have** *while-prop-part*:
$\forall$ *no* $\in$ *Nodes* (*Suc n*) *ll*.
var (*repc no*) <= *var no*
by *auto*
from *while-while-prop* **have** *rep-rep-nort*:
$\forall$ *no*$\in$*Nodes* (*n* + *1*) *ll*. ($\exists$ *nort*. *Dag* (*repc no*) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*) *nort* $\wedge$
($\forall$ *no*$\in$*set-of nort*. *repc no* = *no*))
by *auto*
from *repcNodes-in-Nodes null-notin-Nodes-Suc-n*
have $\forall$ *no* $\in$ *Nodes* (*n*+*1*) *ll*. *repc no* $\neq$ *Null*
by *auto*
with *rep-rep-nort* **have** $\forall$ *no* $\in$ *Nodes* (*n*+*1*) *ll*. *repc* (*repc no*) = (*repc no*)
apply −
apply (*rule ballI*)
apply (*erule-tac x*=*no* **in** *ballE*)
prefer *2*
apply *simp*
apply (*erule-tac x*=*no* **in** *ballE*)
apply (*erule exE*)
apply (*subgoal-tac repc no* $\in$ *set-of nort*)
apply (*elim conjE*)
apply (*erule-tac x*=*repc no* **in** *ballE*)
apply *simp*
apply *simp*
apply (*simp*)
apply (*elim conjE*)
apply (*thin-tac* $\forall$ *no*$\in$*set-of nort*. *repc no* = *no*)
apply *auto*
done
with *t2-in-DagsNodesSucn t2-notin-DagsNodesn ord-t2 while-prop-part*
wf-ll nsll repcNodes-in-Nodes **obtain** *a* **where**
t2-repc-dag: *Dag* (*repc a*) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*) *t2* **and**
a-in-lln: *a* $\in$ *set* (*ll* ! *n*)
apply −
apply (*drule restrict-root-Node*)
apply (*auto simp add*: *length-ll-eq*)
done
with *wf-ll nsll* **have** *a-in-pret*: *a* $\in$ *set-of pret*
by (*simp add*: *wf-ll-def length-ll-eq*)
from *wf-ll nsll a-in-lln* **have** *vara-n*: *var a* = *n*
by (*simp add*: *wf-ll-def length-ll-eq*)
from *a-in-lln rep-prop* **obtain**
repp-nNull: *repc a* $\neq$ *Null* **and**
repp-reduce: (*repc* $\propto$ *low*) *a* = (*repc* $\propto$ *high*) *a* $\wedge$ *low a* $\neq$ *Null*
$\longrightarrow$ *repc a* = (*repc* $\propto$ *high*) *a* **and**
repp-share: ((*repc* $\propto$ *low*) *a* = (*repc* $\propto$ *high*) *a* $\longrightarrow$ *low a* = *Null*)
$\longrightarrow$ *repc a* $\in$ *set* (*ll* ! *n*) $\wedge$

repc (*repc a*) = *repc a* ∧
(∀ *no1*∈*set* (*ll* ! *n*). ((*repc* ∝ *high*) *no1* = (*repc* ∝ *high*) *a* ∧
(*repc* ∝ *low*) *no1* = (*repc* ∝ *low*) *a*) = (*repc a* = *repc no1*))
using [[*simp-depth-limit*=*4*]]
by *auto*
from *t2-repc-dag a-in-lln repp-nNull* **obtain** *lt2 rt2* **where**
t2-Node: *t2* = (*Node lt2* (*repc a*) *rt2*)
by *auto*
have *isomorphic-dags-eq t1 t2 var*
proof (*cases* (*repc* ∝ *low*) *a* = (*repc* ∝ *high*) *a* ∧ *low a* ≠ *Null*)
case *True*
note *red*=*this*
then have *red-case*: (*repc* ∝ *low*) *a* = (*repc* ∝ *high*) *a*
by *simp*
from *red* **have** *low-nNull*: *low a* ≠ *Null*
by *simp*
with *pret-dag prebdt-pret a-in-pret* **have** *highp-nNull*: *high a* ≠ *Null*
apply −
apply (*drule balanced-bdt*)
apply *auto*
done
from *pret-dag ord-pret a-in-pret low-nNull highp-nNull*
have *var-children-smaller*: *var* (*low a*) < *var a* ∧ *var* (*high a*) < *var a*
apply −
apply (*rule var-ordered-children*)
apply *auto*
done
from *pret-dag a-in-pret* **have** *a-nNull*: *a* ≠ *Null*
apply −
apply (*rule set-of-nn* [*rule-format*])
apply *auto*
done
with *a-in-pret highp-nNull pret-dag* **have** *high a* ∈ *set-of pret*
apply −
apply (*drule subelem-set-of-high*)
apply *auto*
done
with *wf-ll* **have** *high a* ∈ *set* (*ll* ! (*var* (*high a*)))
by (*simp add*: *wf-ll-def*)
with *a-in-lln t2-repc-dag var-children-smaller vara-n*
have ∃ *k*<*n*. (*high a*) ∈ *set* (*ll* ! *k*)
by *auto*
then have *higha-in-Nodesn*: (*high a*) ∈ *Nodes n ll*
by (*simp add*: *Nodes-def*)
then have *rhigha-in-rNodesn*: *repc* (*high a*) ∈ *repc* ‘ *Nodes n ll*
by *simp*
from *higha-in-Nodesn normalize-prop* **obtain** *rt* **where**
rt-dag: *Dag* (*repb* (*high a*)) (*repb* ∝ *low*) (*repb* ∝ *high*) *rt* **and**
rt-in-repbNort: *set-of rt* ⊆ *repb* ‘*Nodes n ll*

apply −
apply (*erule-tac x=high a* **in** *ballE*)
apply *auto*
done
from *rt-in-repbNort repbNodes-repcNodes*
have *rt-in-repcNodesn*: *set-of rt* ⊆ *repc 'Nodes n ll*
by *blast*
from *rt-dag higha-in-Nodesn*
have *repcrt-dag*: *Dag* (*repc* (*high a*)) (*repc* ∝ *low*) (*repc* ∝ *high*) *rt*
apply −
apply (*drule Nodes-repbc-Dags-eq* [*rule-format*])
apply *auto*
done
have *rt-nTip*: *rt* ≠ *Tip*
proof −
have *repc* (*high a*) ≠ *Null*
proof −
note *rhigha-in-rNodesn*
also have *repc 'Nodes n ll* ⊆ *repc 'Nodes* (*Suc n*) *ll*
using *Nodes-subset*
by *blast*
also have . . . ⊆ *Nodes* (*Suc n*) *ll*
using *repcNodes-in-Nodes*
by *simp*
finally show *?thesis*
using *null-notin-Nodes-Suc-n*
by *auto*
qed
with *repcrt-dag* **show** *?thesis* **by** *auto*
qed
from *a-nNull a-in-pret low-nNull pret-dag* **have** *low a* ∈ *set-of pret*
apply −
apply (*drule subelem-set-of-low*)
apply *auto*
done
with *wf-ll* **have** *low a* ∈ *set* (*ll* ! (*var* (*low a*)))
by (*simp add*: *wf-ll-def*)
with *a-in-lln t2-repc-dag var-children-smaller vara-n*
have ∃ *k*<*n*. (*low a*) ∈ *set* (*ll* ! *k*)
by *auto*
then have (*low a*) ∈ *Nodes n ll*
by (*simp add*: *Nodes-def*)
then have *rlow-in-rNodesn*: *repc* (*low a*) ∈ *repc ' Nodes n ll*
by *simp*
show *?thesis*
proof −
from *repp-reduce low-nNull highp-nNull red-case*
have *repc-p-def*: *repc a* = *repc* (*high a*)
by (*simp add*: *null-comp-def*)

```
          with rt-in-repcNodesn repcrt-dag rhigha-in-rNodesn a-in-lln t2-repc-dag
            repc-p-def  rt-nTip
           have t2-in-Dags-Nodesn:
            t2 ∈ Dags (repc ‘ Nodes n ll) (repc ∝ low) (repc ∝ high)
            apply −
            apply (rule DagsI)
            apply simp
            apply (subgoal-tac t2=rt)
            apply (auto simp add: Dag-unique)
            done
              from t1-in-DagsNodesn t2-in-Dags-Nodesn repbc-dags-eq isomor-
phic-dags-eq-asm
           show shared-t1-t2: isomorphic-dags-eq t1 t2 var
            apply −
            apply (erule-tac x=t1 in allE)
            apply (erule-tac x=t2 in allE)
            apply simp
            done
         qed
        next
         assume share:  ¬ ((repc ∝ low) a = (repc ∝ high) a ∧ low a ≠ Null)
         then
         have share: (repc ∝ low) a ≠ (repc ∝ high) a ∨ low a = Null
          using [[simp-depth-limit=1]]
          by simp
         with repp-share obtain
          ra-in-llbn: repc a ∈ set (ll ! n) and
          rra-ra: repc (repc a) = repc a and
          two-nodes-share: (∀ no1∈set (ll ! n).
          ((repc ∝ high) no1 = (repc ∝ high) a ∧
          (repc ∝ low) no1 = (repc ∝ low) a) = (repc a = repc no1))
          using [[simp-depth-limit=3]]
          by simp
         from wf-ll ra-in-llbn  nsll have var-repc-a-n: var (repc a) = n
          by (auto simp add: wf-ll-def length-ll-eq)
         show ?thesis
         proof (auto simp add: isomorphic-dags-eq-def)
          fix bdt1
          assume bdt-t1: bdt t1 var = Some bdt1
          assume bdt-t2: bdt t2 var = Some bdt1
          show t1 = t2
          proof (cases t1)
           case Tip
           with bdt-t1 show ?thesis
            by simp
          next
           case (Node lt1 p1 rt1)
           note t1-Node=this
           with Dag-q-Nodes-n have p1=(repc q)
```

```
        by simp
      with t2-Node bdt-t1 bdt-t2 t1-Node have var (repc q) = var (repc a)
        apply −
        apply (rule same-bdt-var)
        apply auto
        done
      with var-repc-q-n var-repc-a-n show ?thesis
        by simp
    qed
  qed
 qed }
note mixed-Dags-case = this
from repbc-dags-eq isomorphic-dags-eq
have dags-shared:
 ∀ t1 t2. t1 ∈ Dags (repc ‘ Nodes (Suc n) ll)(repc ∝ low)(repc ∝ high)∧
 t2 ∈ Dags (repc ‘ Nodes (Suc n) ll) (repc ∝ low) (repc ∝ high)
 ⟶ isomorphic-dags-eq t1 t2 var
 apply −
 apply (rule Dags-Nodes-cases)
 apply (rule isomorphic-dags-eq-sym)
proof −
 fix t1 t2
 assume t1-in-Dagsn:
  t1 ∈ Dags (repc ‘ Nodes n ll) (repc ∝ low) (repc ∝ high)
 assume t2-in-Dagsn:
  t2 ∈ Dags (repc ‘ Nodes n ll) (repc ∝ low) (repc ∝ high)
 assume isomorphic-dags-eq-asm:
  ∀ t1 t2. t1 ∈ Dags (repb ‘ Nodes n ll) (repb ∝ low) (repb ∝ high) ∧
  t2 ∈ Dags (repb ‘ Nodes n ll) (repb ∝ low) (repb ∝ high)
  ⟶ isomorphic-dags-eq t1 t2 var
 assume repb-repc-Dags:
  Dags (repc ‘ Nodes n ll) (repc ∝ low) (repc ∝ high) =
  Dags (repb ‘ Nodes n ll) (repb ∝ low) (repb ∝ high)
 with t1-in-Dagsn t2-in-Dagsn isomorphic-dags-eq-asm
 show isomorphic-dags-eq t1 t2 var by simp
next
 fix t1 t2
 assume t1-in-DagsNodesn:
  t1 ∈ Dags (repc ‘ Nodes n ll) (repc ∝ low) (repc ∝ high)
 assume t2-notin-DagsNodesn:
  t2 ∉ Dags (repc ‘ Nodes n ll) (repc ∝ low) (repc ∝ high)
 assume t2-in-DagsNodesSucn:
  t2 ∈ Dags (repc ‘ Nodes (Suc n) ll) (repc ∝ low) (repc ∝ high)
 assume isomorphic-dags-eq-asm:
  ∀ t1 t2. t1 ∈ Dags (repb ‘ Nodes n ll) (repb ∝ low) (repb ∝ high) ∧
  t2 ∈ Dags (repb ‘ Nodes n ll) (repb ∝ low) (repb ∝ high)
  ⟶ isomorphic-dags-eq t1 t2 var
 assume repbc-Dags:
  Dags (repc ‘ Nodes n ll) (repc ∝ low) (repc ∝ high) =
```

Dags (*repb* ‘ *Nodes n ll*) (*repb* ∝ *low*) (*repb* ∝ *high*)
from *t1-in-DagsNodesn t2-notin-DagsNodesn t2-in-DagsNodesSucn isomorphic-dags-eq-asm repbc-Dags*
show *isomorphic-dags-eq t1 t2 var*
apply −
apply (*rule mixed-Dags-case*)
apply *auto*
done
next
fix *t1 t2*
assume *t1-in-DagsNodesSucn*:
t1 ∈ *Dags* (*repc* ‘ *Nodes* (*Suc n*) *ll*) (*repc* ∝ *low*) (*repc* ∝ *high*)
assume *t1-notin-DagsNodesn*:
t1 ∉ *Dags* (*repc* ‘ *Nodes n ll*) (*repc* ∝ *low*) (*repc* ∝ *high*)
assume *t2-in-DagsNodesSucn*:
t2 ∈ *Dags* (*repc* ‘ *Nodes* (*Suc n*) *ll*) (*repc* ∝ *low*) (*repc* ∝ *high*)
assume *t2-notin-DagsNodesn*:
t2 ∉ *Dags* (*repc* ‘ *Nodes n ll*) (*repc* ∝ *low*) (*repc* ∝ *high*)

from *t1-in-DagsNodesSucn* **have** *ord-t1*: *ordered t1 var*
proof (*elim Dags.cases*)
fix *p t*
assume *t-in-repcNodes*: *set-of t* ⊆ *repc* ‘ *Nodes* (*Suc n*) *ll*
assume *t-nTip*: *t* ≠ *Tip*
assume *t2t*: *t1* = *t*
assume *t-dag*: *Dag p* (*repc* ∝ *low*) (*repc* ∝ *high*) *t*
from *t-in-repcNodes*
have *x-in-repcNodesSucn*:
∀ *x*. *x* ∈ *set-of t* ⟶ *x* ∈ *repc* ‘ *Nodes* (*Suc n*) *ll*
apply −
apply *auto*
done
from *t-nTip t-dag* **have** *p* ≠ *Null*
apply −
apply (*case-tac p=Null*)
apply *auto*
done
with *t-nTip t-dag* **obtain** *lt rt* **where** *t-Node*: *t=Node lt p rt*
by *auto*
then have *p* ∈ *set-of t*
by *auto*
with *x-in-repcNodesSucn* **have** *p* ∈ *repc* ‘ *Nodes* (*Suc n*) *ll*
by *simp*
then obtain *a* **where**
repca-p: *p=repc a* **and** *a-in-NodesSucn*: *a* ∈ *Nodes* (*Suc n*) *ll*
by *auto*
with *repca-p while-while-prop t-dag t2t* **show** *?thesis*

```
      apply −
      apply (erule-tac x=a in ballE)
      apply (elim conjE exE)
      apply (subgoal-tac nort = t)
      prefer 2
      apply (simp add: Dag-unique)
      apply auto
      done
  qed
  from while-while-prop
  have while-prop-part: ∀ no ∈ Nodes (Suc n) ll.
    var (repc no) <= var no
    by auto
  from while-while-prop have rep-rep-nort:
    ∀ no∈Nodes (n + 1) ll.
      (∃ nort. Dag (repc no) (repc ∝ low) (repc ∝ high) nort ∧
      (∀ no∈set-of nort. repc no = no))
    by auto
  from repcNodes-in-Nodes null-notin-Nodes-Suc-n
  have ∀ no ∈ Nodes (n+1) ll. repc no ≠ Null
    by auto
  with rep-rep-nort
  have rep-rep-no: ∀ no ∈ Nodes (n+1) ll. repc (repc no) = (repc no)
    apply −
    apply (rule ballI)
    apply (erule-tac x=no in ballE)
    prefer 2
    apply simp
    apply (erule-tac x=no in ballE)
    apply (erule exE)
    apply (subgoal-tac repc no ∈ set-of nort)
    apply (elim conjE)
    apply (erule-tac x=repc no in ballE)
    apply simp
    apply simp
    apply (simp)
    apply (elim conjE)
    apply (thin-tac ∀ no∈set-of nort. repc no = no)
    apply auto
    done
    with t1-in-DagsNodesSucn t1-notin-DagsNodesn ord-t1 while-prop-part
wf-ll
    nsll repcNodes-in-Nodes obtain a1 where
    t1-repc-dag: Dag (repc a1) (repc ∝ low) (repc ∝ high) t1 and
    a1-in-lln: a1 ∈ set (ll ! n)
    apply −
    apply (drule restrict-root-Node)
    apply (auto simp add: length-ll-eq)
    done
```

with *wf-ll nsll* **have** *a1-in-pret*: $a1 \in$ *set-of pret*
by (*simp add*: *wf-ll-def length-ll-eq*)
from *wf-ll nsll a1-in-lln* **have** *vara1-n*: *var a1* = *n*
by (*simp add*: *wf-ll-def length-ll-eq*)
from *a1-in-lln rep-prop* **obtain**
repa1-nNull: *repc a1* $\neq$ *Null* **and**
repa1-reduce: (*repc* $\propto$ *low*) *a1* = (*repc* $\propto$ *high*) *a1* $\wedge$ *low a1* $\neq$ *Null*
$\longrightarrow$ *repc a1* = (*repc* $\propto$ *high*) *a1* **and**
repa1-share: ((*repc* $\propto$ *low*) *a1* = (*repc* $\propto$ *high*) *a1* $\longrightarrow$ *low a1* = *Null*)
$\longrightarrow$ *repc a1* $\in$ *set* (*ll* ! *n*) $\wedge$ *repc* (*repc a1*) = *repc a1* $\wedge$
($\forall$ *no1*$\in$*set* (*ll* ! *n*). ((*repc* $\propto$ *high*) *no1* = (*repc* $\propto$ *high*) *a1* $\wedge$
(*repc* $\propto$ *low*) *no1* = (*repc* $\propto$ *low*) *a1*) = (*repc a1* = *repc no1*))
using [[*simp-depth-limit*=*4*]]
by *auto*
from *t1-repc-dag a1-in-lln repa1-nNull* **obtain** *lt1 rt1* **where**
t1-Node: *t1* = (*Node lt1* (*repc a1*) *rt1*)
by *auto*

from *t2-in-DagsNodesSucn* **have** *ord-t2*: *ordered t2 var*
proof (*elim Dags.cases*)
fix *p t*
assume *t-in-repcNodes*: *set-of t* $\subseteq$ *repc* ‘ *Nodes* (*Suc n*) *ll*
assume *t-nTip*: *t* $\neq$ *Tip*
assume *t2t*: *t2* = *t*
assume *t-dag*: *Dag p* (*repc* $\propto$ *low*) (*repc* $\propto$ *high*) *t*
from *t-in-repcNodes*
have *x-in-repcNodesSucn*:
$\forall$ *x*. *x* $\in$ *set-of t* $\longrightarrow$ *x* $\in$ *repc* ‘ *Nodes* (*Suc n*) *ll*
apply −
apply *auto*
done
from *t-nTip t-dag* **have** *p* $\neq$ *Null*
apply −
apply (*case-tac p*=*Null*)
apply *auto*
done
with *t-nTip t-dag* **obtain** *lt rt* **where** *t-Node*: *t*=*Node lt p rt*
by *auto*
then have *p* $\in$ *set-of t*
by *auto*
with *x-in-repcNodesSucn* **have** *p* $\in$ *repc* ‘ *Nodes* (*Suc n*) *ll*
by *simp*
then obtain *a* **where**
repca-p: *p*=*repc a* **and** *a-in-NodesSucn*: *a* $\in$ *Nodes* (*Suc n*) *ll*
by *auto*
with *repca-p while-while-prop t-dag t2t* **show** *?thesis*

apply −
apply (*erule-tac x=a* **in** *ballE*)
apply (*elim conjE exE*)
apply (*subgoal-tac nort = t*)
prefer *2*
apply (*simp add*: *Dag-unique*)
apply *auto*
done
qed
from *rep-rep-no t2-in-DagsNodesSucn t2-notin-DagsNodesn ord-t2 while-prop-part wf-ll*
nsll repcNodes-in-Nodes **obtain** *a2* **where**
t2-repc-dag: *Dag* (*repc a2*) (*repc* ∝ *low*) (*repc* ∝ *high*) *t2* **and**
a2-in-lln: *a2* ∈ *set* (*ll* ! *n*)
apply −
apply (*drule restrict-root-Node*)
apply (*auto simp add*: *length-ll-eq*)
done
with *wf-ll nsll* **have** *a2-in-pret*: *a2* ∈ *set-of pret*
by (*simp add*: *wf-ll-def length-ll-eq*)
from *wf-ll nsll a2-in-lln* **have** *vara2-n*: *var a2* = *n*
by (*simp add*: *wf-ll-def length-ll-eq*)
from *a2-in-lln rep-prop* **obtain**
repa2-nNull: *repc a2* ≠ *Null* **and**
repa2-reduce: (*repc* ∝ *low*) *a2* = (*repc* ∝ *high*) *a2* ∧ *low a2* ≠ *Null*
⟶ *repc a2* = (*repc* ∝ *high*) *a2* **and**
repa2-share: ((*repc* ∝ *low*) *a2* = (*repc* ∝ *high*) *a2* ⟶ *low a2* = *Null*)
⟶ *repc a2* ∈ *set* (*ll* ! *n*) ∧ *repc* (*repc a2*) = *repc a2* ∧
(∀ *no1*∈*set* (*ll* ! *n*). ((*repc* ∝ *high*) *no1* = (*repc* ∝ *high*) *a2* ∧
(*repc* ∝ *low*) *no1* = (*repc* ∝ *low*) *a2*) = (*repc a2* = *repc no1*))
using [[*simp-depth-limit* = *4*]]
by *auto*
from *t2-repc-dag a2-in-lln repa2-nNull* **obtain** *lt2 rt2* **where**
t2-Node: *t2* = (*Node lt2* (*repc a2*) *rt2*)
by *auto*
show *isomorphic-dags-eq t1 t2 var*
proof (*cases* (*repc* ∝ *low*) *a1* = (*repc* ∝ *high*) *a1* ∧ *low a1* ≠ *Null*)
case *True*
note *t1-red-cond*=*this*
with *t1-red-cond* **have** *t1-red-case*: (*repc* ∝ *low*) *a1* = (*repc* ∝ *high*) *a1*
by *simp*
from *t1-red-cond* **have** *lowa1-nNull*: *low a1* ≠ *Null*
by *simp*
with *pret-dag prebdt-pret a1-in-pret* **have** *higha1-nNull*: *high a1* ≠ *Null*
apply −
apply (*drule balanced-bdt*)
apply *auto*
done
from *pret-dag ord-pret a1-in-pret lowa1-nNull higha1-nNull*

have *var-children-smaller-a1*: *var* (*low a1*) < *var a1* ∧ *var* (*high a1*) < *var a1*
apply −
apply (*rule var-ordered-children*)
apply *auto*
done
from *pret-dag a1-in-pret* **have** *a1-nNull*: *a1* ≠ *Null*
apply −
apply (*rule set-of-nn* [*rule-format*])
apply *auto*
done
with *a1-in-pret higha1-nNull pret-dag* **have** *high a1* ∈ *set-of pret*
apply −
apply (*drule subelem-set-of-high*)
apply *auto*
done
with *wf-ll* **have** *high a1* ∈ *set* (*ll* ! (*var* (*high a1*)))
by (*simp add*: *wf-ll-def*)
with *a1-in-lln t1-repc-dag var-children-smaller-a1 vara1-n*
have ∃ *k*<*n*. (*high a1*) ∈ *set* (*ll* ! *k*)
by *auto*
then have *higha1-in-Nodesn*: (*high a1*) ∈ *Nodes n ll*
by (*simp add*: *Nodes-def*)
then have *rhigha1-in-rNodesn*: *repc* (*high a1*) ∈ *repc* ‘ *Nodes n ll*
by *simp*
from *higha1-in-Nodesn normalize-prop* **obtain** *rt1* **where**
rt1-dag: *Dag* (*repb* (*high a1*)) (*repb* ∝ *low*) (*repb* ∝ *high*) *rt1* **and**
rt1-in-repbNort: *set-of rt1* ⊆ *repb* ‘*Nodes n ll*
apply −
apply (*erule-tac x*=*high a1* **in** *ballE*)
apply *auto*
done
from *rt1-in-repbNort repbNodes-repcNodes*
have *rt1-in-repcNodesn*: *set-of rt1* ⊆ *repc* ‘*Nodes n ll*
by *blast*
from *rt1-dag higha1-in-Nodesn*
have *repcrt1-dag*: *Dag* (*repc* (*high a1*)) (*repc* ∝ *low*) (*repc* ∝ *high*) *rt1*
apply −
apply (*drule Nodes-repbc-Dags-eq* [*rule-format*])
apply *auto*
done
have *rt1-nTip*: *rt1* ≠ *Tip*
proof −
have *repc* (*high a1*) ≠ *Null*
proof −
note *rhigha1-in-rNodesn*
also have *repc* ‘*Nodes n ll* ⊆ *repc* ‘*Nodes* (*Suc n*) *ll*
using *Nodes-subset*
by *blast*

```
    also have ... ⊆ Nodes (Suc n) ll
      using repcNodes-in-Nodes
      by simp
    finally show ?thesis
      using null-notin-Nodes-Suc-n
      by auto
  qed
  with repcrt1-dag show ?thesis by auto
qed
from repa1-reduce lowa1-nNull higha1-nNull t1-red-case
have repc-a1-def: repc a1 = repc (high a1)
  by (simp add: null-comp-def)
with rt1-in-repcNodesn repcrt1-dag rhigha1-in-rNodesn a1-in-lln
  t1-repc-dag repc-a1-def rt1-nTip
have t1-in-Dags-Nodesn:
  t1 ∈ Dags (repc ‘ Nodes n ll) (repc ∝ low) (repc ∝ high)
  apply −
  apply (rule DagsI)
  apply simp
  apply (subgoal-tac t1=rt1)
  apply (auto simp add: Dag-unique)
  done
show ?thesis
proof (cases (repc ∝ low) a2 = (repc ∝ high) a2 ∧ low a2 ≠ Null)
  case True
  note t2-red-cond=this
 with t2-red-cond have t2-red-case: (repc ∝ low) a2 = (repc ∝ high) a2
    by simp
  from t2-red-cond have lowa2-nNull: low a2 ≠ Null
    by simp
 with pret-dag prebdt-pret a2-in-pret have higha2-nNull: high a2 ≠ Null
    apply −
    apply (drule balanced-bdt)
    apply auto
    done
  from pret-dag ord-pret a2-in-pret lowa2-nNull higha2-nNull
  have var-children-smaller-a2:
    var (low a2) < var a2 ∧ var (high a2) < var a2
    apply −
    apply (rule var-ordered-children)
    apply auto
    done
  from pret-dag a2-in-pret have a2-nNull: a2 ≠ Null
    apply −
    apply (rule set-of-nn [rule-format])
    apply auto
    done
  with a2-in-pret higha2-nNull pret-dag have high a2 ∈ set-of pret
    apply −
```

```
  apply (drule subelem-set-of-high)
  apply auto
  done
with wf-ll have high a2 ∈ set (ll ! (var (high a2)))
  by (simp add: wf-ll-def)
with a2-in-lln t2-repc-dag var-children-smaller-a2 vara2-n
have ∃ k<n. (high a2) ∈ set (ll ! k)
  by auto
then have higha2-in-Nodesn: (high a2) ∈ Nodes n ll
  by (simp add: Nodes-def)
then have rhigha2-in-rNodesn: repc (high a2) ∈ repc ‘ Nodes n ll
  by simp
from higha2-in-Nodesn normalize-prop obtain rt2 where
  rt2-dag: Dag (repb (high a2)) (repb ∝ low) (repb ∝ high) rt2 and
  rt2-in-repbNort: set-of rt2 ⊆ repb ‘Nodes n ll
  apply −
  apply (erule-tac x=high a2 in ballE)
  apply auto
  done
from rt2-in-repbNort repbNodes-repcNodes
have rt2-in-repcNodesn: set-of rt2 ⊆ repc ‘Nodes n ll
  by blast
from rt2-dag higha2-in-Nodesn
have repcrt2-dag: Dag (repc (high a2)) (repc ∝ low) (repc ∝ high) rt2
  apply −
  apply (drule Nodes-repbc-Dags-eq [rule-format])
  apply auto
  done
have rt2-nTip: rt2 ≠ Tip
proof −
  have repc (high a2) ≠ Null
  proof −
    note rhigha2-in-rNodesn
    also have repc ‘Nodes n ll ⊆ repc ‘Nodes (Suc n) ll
      using Nodes-subset
      by blast
    also have ... ⊆ Nodes (Suc n) ll
      using repcNodes-in-Nodes
      by simp
    finally show ?thesis
      using null-notin-Nodes-Suc-n
      by auto
  qed
  with repcrt2-dag show ?thesis by auto
qed
from repa2-reduce lowa2-nNull higha2-nNull t2-red-case
have repc-a2-def: repc a2 = repc (high a2)
  by (simp add: null-comp-def)
with rt2-in-repcNodesn repcrt2-dag rhigha2-in-rNodesn a2-in-lln
```

```
        t2-repc-dag repc-a2-def  rt2-nTip
      have t2-in-Dags-Nodesn:
        t2 ∈ Dags (repc ‘ Nodes n ll) (repc ∝ low) (repc ∝ high)
        apply −
        apply (rule DagsI)
        apply simp
        apply (subgoal-tac t2=rt2)
        apply (auto simp add: Dag-unique)
        done
            from isomorphic-dags-eq  t1-in-Dags-Nodesn  t2-in-Dags-Nodesn
repbc-dags-eq
      show ?thesis
        by auto
    next
      assume t2-share-cond:
        ¬ ((repc ∝ low) a2 = (repc ∝ high) a2 ∧ low a2 ≠ Null)
      from t1-in-Dags-Nodesn t2-notin-DagsNodesn t2-in-DagsNodesSucn
        isomorphic-dags-eq repbc-dags-eq
      show ?thesis
        apply −
        apply (rule mixed-Dags-case)
        apply auto
        done
    qed
  next
    assume t1-share-cond:
      ¬((repc ∝ low) a1 = (repc ∝ high) a1 ∧ low a1 ≠ Null)
    with repa1-share obtain
      repca1-in-llbn: repc a1 ∈ set (ll ! n) and
      reprepa1: repc (repc a1) = repc a1 and
      twonodes-llbn-a1:
      (∀ no1∈set (ll ! n). ((repc ∝ high) no1 = (repc ∝ high) a1 ∧
      (repc ∝ low) no1 = (repc ∝ low) a1) = (repc a1 = repc no1))
      using [[simp-depth-limit=2]]
      by auto
    show ?thesis
    proof (cases (repc ∝ low) a2 = (repc ∝ high) a2 ∧ low a2 ≠ Null)
      case True
      note t2-red-cond=this
     with t2-red-cond have t2-red-case: (repc ∝ low) a2 = (repc ∝ high) a2
        by simp
      from t2-red-cond have lowa2-nNull: low a2 ≠ Null
        by simp
     with pret-dag prebdt-pret a2-in-pret have higha2-nNull: high a2 ≠ Null
        apply −
        apply (drule balanced-bdt)
        apply auto
        done
      from pret-dag ord-pret a2-in-pret lowa2-nNull higha2-nNull
```

have *var-children-smaller-a2*:
var (*low a2*) < *var a2* ∧ *var* (*high a2*) < *var a2*
apply −
apply (*rule var-ordered-children*)
apply *auto*
done
from *pret-dag a2-in-pret* **have** *a2-nNull*: *a2* ≠ *Null*
apply −
apply (*rule set-of-nn* [*rule-format*])
apply *auto*
done
with *a2-in-pret higha2-nNull pret-dag* **have** *high a2* ∈ *set-of pret*
apply −
apply (*drule subelem-set-of-high*)
apply *auto*
done
with *wf-ll*
have *high a2* ∈ *set* (*ll* ! (*var* (*high a2*)))
by (*simp add*: *wf-ll-def*)
with *a2-in-lln t2-repc-dag var-children-smaller-a2 vara2-n*
have ∃ *k*<*n*. (*high a2*) ∈ *set* (*ll* ! *k*)
by *auto*
then have *higha2-in-Nodesn*: (*high a2*) ∈ *Nodes n ll*
by (*simp add*: *Nodes-def*)
then have *rhigha2-in-rNodesn*: *repc* (*high a2*) ∈ *repc* ‘ *Nodes n ll*
by *simp*
from *higha2-in-Nodesn normalize-prop* **obtain** *rt2* **where**
rt2-dag: *Dag* (*repb* (*high a2*)) (*repb* ∝ *low*) (*repb* ∝ *high*) *rt2* **and**
rt2-in-repbNort: *set-of rt2* ⊆ *repb* ‘*Nodes n ll*
apply −
apply (*erule-tac x*=*high a2* **in** *ballE*)
apply *auto*
done
from *rt2-in-repbNort repbNodes-repcNodes*
have *rt2-in-repcNodesn*: *set-of rt2* ⊆ *repc* ‘*Nodes n ll*
by *blast*
from *rt2-dag higha2-in-Nodesn*
have *repcrt2-dag*: *Dag* (*repc* (*high a2*)) (*repc* ∝ *low*) (*repc* ∝ *high*) *rt2*
apply −
apply (*drule Nodes-repbc-Dags-eq* [*rule-format*])
apply *auto*
done
have *rt2-nTip*: *rt2* ≠ *Tip*
proof −
have *repc* (*high a2*) ≠ *Null*
proof −
note *rhigha2-in-rNodesn*
also have *repc* ‘*Nodes n ll* ⊆ *repc* ‘*Nodes* (*Suc n*) *ll*
using *Nodes-subset*

```
      by blast
    also have ... ⊆ Nodes (Suc n) ll
      using repcNodes-in-Nodes
      by simp
    finally show ?thesis
      using null-notin-Nodes-Suc-n
      by auto
  qed
  with repcrt2-dag show ?thesis by auto
qed
from repa2-reduce lowa2-nNull higha2-nNull t2-red-case
have repc-a2-def: repc a2 = repc (high a2)
  by (simp add: null-comp-def)
with rt2-in-repcNodesn repcrt2-dag rhigha2-in-rNodesn a2-in-lln
  t2-repc-dag repc-a2-def rt2-nTip
have t2-in-Dags-Nodesn:
  t2 ∈ Dags (repc ‘ Nodes n ll) (repc ∝ low) (repc ∝ high)
  apply −
  apply (rule DagsI)
  apply simp
  apply (subgoal-tac t2=rt2)
  apply (auto simp add: Dag-unique)
  done
from t2-in-Dags-Nodesn t1-notin-DagsNodesn t1-in-DagsNodesSucn
  isomorphic-dags-eq repbc-dags-eq
have isomorphic-dags-eq t2 t1 var
  apply −
  apply (rule mixed-Dags-case)
  apply auto
  done
then show ?thesis
  by (simp add: isomorphic-dags-eq-sym)
next
assume t2-shared-cond:
  ¬ ((repc ∝ low) a2 = (repc ∝ high) a2 ∧ low a2 ≠ Null)
with repa2-share obtain
  repca2-in-llbn: repc a2 ∈ set (ll ! n) and
  reprepa2: repc (repc a2) = repc a2 and
  twonodes-llbn-a2: (∀ no1∈set (ll ! n).
  ((repc ∝ high) no1 = (repc ∝ high) a2 ∧
  (repc ∝ low) no1 = (repc ∝ low) a2) = (repc a2 = repc no1))
  using [[simp-depth-limit=2]]
  by auto
from twonodes-llbn-a2 a1-in-lln
have share-a1-a2:
  ((repc ∝ high) a1 = (repc ∝ high) a2 ∧
  (repc ∝ low) a1 = (repc ∝ low) a2) = (repc a2 = repc a1)
  by auto
from twonodes-llbn-a1 repca1-in-llbn reprepa1
```

```
    have children-repc-eq-a1: (repc ∝ high) (repc a1) = (repc ∝ high) a1 ∧
      (repc ∝ low) (repc a1) = (repc ∝ low) a1
      by auto
   from twonodes-llbn-a2 repca2-in-llbn reprepa2
    have children-repc-eq-a2: (repc ∝ high) (repc a2) = (repc ∝ high) a2 ∧
      (repc ∝ low) (repc a2) = (repc ∝ low) a2
      by auto
   from  t1-Node t2-Node show ?thesis
   proof (clarsimp simp add: isomorphic-dags-eq-def)
     fix bdt1
     assume t1-bdt: bdt (Node lt1 (repc a1) rt1) var = Some bdt1
     assume t2-bdt: bdt (Node lt2 (repc a2) rt2) var = Some bdt1
     show lt1 = lt2 ∧ repc a1 = repc a2 ∧ rt1 = rt2
     proof (cases bdt1)
       case Zero
       with t1-bdt t1-Node obtain
         lt1-Tip: lt1 = Tip and
         rt1-Tip: rt1 = Tip
         by simp
       from Zero t2-bdt t2-Node obtain
         lt2-Tip: lt2 = Tip and
         rt2-Tip: rt2 = Tip
         by simp
         from t1-repc-dag t1-Node lt1-Tip have (repc ∝ low) (repc a1) =
Null
         by simp
       with children-repc-eq-a1
       have repc-low-a1-Null: (repc ∝ low) a1 = Null
         by simp
       from t1-repc-dag t1-Node rt1-Tip
       have (repc ∝ high) (repc a1) = Null
         by simp
       with children-repc-eq-a1
       have repc-high-a1-Null: (repc ∝ high) a1 = Null
         by simp
         from t2-repc-dag t2-Node lt2-Tip have (repc ∝ low) (repc a2) =
Null
         by simp
       with children-repc-eq-a2
       have repc-low-a2-Null: (repc ∝ low) a2 = Null
         by simp
       from t2-repc-dag t2-Node rt2-Tip
       have (repc ∝ high) (repc a2) = Null
         by simp
       with children-repc-eq-a2
       have repc-high-a2-Null: (repc ∝ high) a2 = Null
         by simp
       with share-a1-a2   repc-low-a1-Null repc-high-a1-Null
         repc-low-a2-Null repc-high-a2-Null
```

```
    have repc a2 = repc a1
      by auto
    with lt1-Tip rt1-Tip lt2-Tip rt2-Tip show ?thesis
      by auto
  next
    case One
    with t1-bdt t1-Node obtain
      lt1-Tip: lt1 = Tip and
      rt1-Tip: rt1 = Tip
      by simp
    from One t2-bdt t2-Node obtain
      lt2-Tip: lt2 = Tip and
      rt2-Tip: rt2 = Tip
      by simp
      from t1-repc-dag t1-Node lt1-Tip have (repc ∝ low) (repc a1) =
Null
      by simp
    with children-repc-eq-a1
    have repc-low-a1-Null: (repc ∝ low) a1 = Null
      by simp
     from t1-repc-dag t1-Node rt1-Tip have (repc ∝ high) (repc a1) =
Null
      by simp
    with children-repc-eq-a1
    have repc-high-a1-Null: (repc ∝ high) a1 = Null
      by simp
      from t2-repc-dag t2-Node lt2-Tip have (repc ∝ low) (repc a2) =
Null
      by simp
    with children-repc-eq-a2
    have repc-low-a2-Null: (repc ∝ low) a2 = Null
      by simp
     from t2-repc-dag t2-Node rt2-Tip have (repc ∝ high) (repc a2) =
Null
      by simp
    with children-repc-eq-a2
    have repc-high-a2-Null: (repc ∝ high) a2 = Null
      by simp
    with share-a1-a2   repc-low-a1-Null repc-high-a1-Null
      repc-low-a2-Null repc-high-a2-Null
    have repc a2 = repc a1
      by auto
    with lt1-Tip rt1-Tip lt2-Tip rt2-Tip show ?thesis
      by auto
  next
    case (Bdt-Node lbdt v rbdt)
    note bdt-Node-case=this
    with t1-bdt t1-Node obtain
      lbdt-def-lt1: bdt lt1 var = Some lbdt and
```

rbdt-def-rt1: *bdt rt1 var* = *Some rbdt*
by *auto*
from *t2-bdt bdt-Node-case t2-Node* **obtain**
lbdt-def-lt2: *bdt lt2 var* = *Some lbdt* **and**
rbdt-def-rt2: *bdt rt2 var* = *Some rbdt*
by *auto*
from *lbdt-def-lt1 t1-Node t1-repc-dag children-repc-eq-a1*
have (*repc* ∝ *low*) *a1* ≠ *Null*
by *auto*
then have *low-a1-nNull*: (*low a1*) ≠ *Null*
by (*auto simp*: *null-comp-def*)
from *rbdt-def-rt1 t1-Node t1-repc-dag children-repc-eq-a1*
have (*repc* ∝ *high*) *a1* ≠ *Null*
by *auto*
then have *high-a1-nNull*: (*high a1*) ≠ *Null*
by (*auto simp*: *null-comp-def*)
from *lbdt-def-lt2 t2-Node t2-repc-dag children-repc-eq-a2*
have (*repc* ∝ *low*) *a2* ≠ *Null*
by *auto*
then have *low-a2-nNull*: (*low a2*) ≠ *Null*
by (*auto simp*: *null-comp-def*)
from *rbdt-def-rt2 t2-Node t2-repc-dag children-repc-eq-a2*
have (*repc* ∝ *high*) *a2* ≠ *Null*
by *auto*
then have *high-a2-nNull*: (*high a2*) ≠ *Null*
by (*auto simp*: *null-comp-def*)

from *pret-dag ord-pret a1-in-pret low-a1-nNull high-a1-nNull*
have *var-children-smaller-a1*:
var (*low a1*) < *var a1* ∧ *var* (*high a1*) < *var a1*
apply −
apply (*rule var-ordered-children*)
apply *auto*
done
from *pret-dag a1-in-pret* **have** *a1-nNull*: *a1* ≠ *Null*
apply −
apply (*rule set-of-nn* [*rule-format*])
apply *auto*
done

with *a1-in-pret high-a1-nNull pret-dag* **have** *high a1* ∈ *set-of pret*
apply −
apply (*drule subelem-set-of-high*)
apply *auto*
done
with *wf-ll*
have *high a1* ∈ *set* (*ll* ! (*var* (*high a1*)))

```
    by (simp add: wf-ll-def)
  with a1-in-lln t1-repc-dag var-children-smaller-a1 vara1-n
  have ∃ k<n. (high a1) ∈ set (ll ! k)
    by auto
  then have higha1-in-Nodesn: (high a1) ∈ Nodes n ll
    by (simp add: Nodes-def)
  then have rhigha1-in-rNodesn:
    repc (high a1) ∈ repc ‘ Nodes n ll
    by simp
  from higha1-in-Nodesn normalize-prop obtain rt1h where
   rt1-dag: Dag (repb (high a1)) (repb ∝ low) (repb ∝ high) rt1h and
    rt1-in-repbNort: set-of rt1h ⊆ repb ‘Nodes n ll
    apply −
    apply (erule-tac x=high a1 in ballE)
    apply auto
    done
  from rt1-in-repbNort repbNodes-repcNodes
  have rt1-in-repcNodesn: set-of rt1h ⊆ repc ‘Nodes n ll
    by blast
  from rt1-dag higha1-in-Nodesn
  have repcrt1-dag:
    Dag (repc (high a1)) (repc ∝ low) (repc ∝ high) rt1h
    apply −
    apply (drule Nodes-repbc-Dags-eq [rule-format])
    apply auto
    done
  from t1-Node t1-repc-dag high-a1-nNull children-repc-eq-a1
  have Dag (repc (high a1)) (repc ∝ low) (repc ∝ high) rt1
    by (auto simp add: null-comp-def)
with repcrt1-dag have rt1h-rt1: rt1h = rt1 by (simp add: Dag-unique)
  have rt1-nTip: rt1 ≠ Tip
  proof −
    have repc (high a1) ≠ Null
    proof −
      note rhigha1-in-rNodesn
      also have
        repc ‘Nodes n ll ⊆ repc ‘Nodes (Suc n) ll
        using Nodes-subset
        by blast
      also have ... ⊆ Nodes (Suc n) ll
        using repcNodes-in-Nodes
        by simp
      finally show ?thesis
        using null-notin-Nodes-Suc-n
        by auto
    qed
    with repcrt1-dag rt1h-rt1 show ?thesis by auto
  qed
  with rt1-in-repcNodesn repcrt1-dag rhigha1-in-rNodesn a1-in-lln
```

t1-repc-dag rt1h-rt1
have *rt1-in-Dags-Nodesn*:
rt1 ∈ *Dags* (*repc* ‘ *Nodes n ll*) (*repc* ∝ *low*) (*repc* ∝ *high*)
apply −
apply (*rule DagsI*)
apply *auto*
done

from *a1-nNull a1-in-pret low-a1-nNull pret-dag*
have *low a1* ∈ *set-of pret*
apply −
apply (*drule subelem-set-of-low*)
apply *auto*
done
with *wf-ll* **have**
low a1 ∈ *set* (*ll* ! (*var* (*low a1*))) **by** (*simp add*: *wf-ll-def*)
with *a1-in-lln t1-repc-dag var-children-smaller-a1 vara1-n*
have ∃ *k*<*n*. (*low a1*) ∈ *set* (*ll* ! *k*)
by *auto*
then have *lowa1-in-Nodesn*: (*low a1*) ∈ *Nodes n ll*
by (*simp add*: *Nodes-def*)
then have *rlowa1-in-rNodesn*: *repc* (*low a1*) ∈ *repc* ‘ *Nodes n ll*
by *simp*
from *lowa1-in-Nodesn normalize-prop* **obtain** *lt1h* **where**
lt1-dag: *Dag* (*repb* (*low a1*)) (*repb* ∝ *low*) (*repb* ∝ *high*) *lt1h* **and**
lt1-in-repbNort: *set-of lt1h* ⊆ *repb* ‘*Nodes n ll*
apply −
apply (*erule-tac x*=*low a1* **in** *ballE*)
apply *auto*
done
from *lt1-in-repbNort repbNodes-repcNodes*
have *lt1-in-repcNodesn*: *set-of lt1h* ⊆ *repc* ‘*Nodes n ll*
by *blast*
from *lt1-dag lowa1-in-Nodesn*
have *repclt1-dag*: *Dag* (*repc* (*low a1*)) (*repc* ∝ *low*) (*repc* ∝ *high*) *lt1h*
apply −
apply (*drule Nodes-repbc-Dags-eq* [*rule-format*])
apply *auto*
done
from *t1-Node t1-repc-dag low-a1-nNull children-repc-eq-a1*
have *Dag* (*repc* (*low a1*)) (*repc* ∝ *low*) (*repc* ∝ *high*) *lt1*
by (*auto simp add*: *null-comp-def*)
with *repclt1-dag* **have** *lt1h-lt1*: *lt1h* = *lt1* **by** (*simp add*: *Dag-unique*)
have *lt1-nTip*: *lt1* ≠ *Tip*
proof −
have *repc* (*low a1*) ≠ *Null*
proof −

note *rlowa1-in-rNodesn*
also have
repc 'Nodes n ll $\subseteq$ *repc 'Nodes* (*Suc n*) *ll*
using *Nodes-subset*
by *blast*
also have ... $\subseteq$ *Nodes* (*Suc n*) *ll*
using *repcNodes-in-Nodes*
by *simp*
finally show *?thesis*
using *null-notin-Nodes-Suc-n*
by *auto*
qed
with *repclt1-dag lt1h-lt1* **show** *?thesis* **by** *auto*
qed
with *lt1-in-repcNodesn repclt1-dag rlowa1-in-rNodesn a1-in-lln t1-repc-dag lt1h-lt1*
have *lt1-in-Dags-Nodesn*:
lt1 $\in$ *Dags* (*repc ' Nodes n ll*) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*)
apply $-$
apply (*rule DagsI*)
apply *auto*
done

from *pret-dag ord-pret a2-in-pret low-a2-nNull high-a2-nNull*
have *var-children-smaller-a2*:
var (*low a2*) $<$ *var a2* $\wedge$ *var* (*high a2*) $<$ *var a2*
apply $-$
apply (*rule var-ordered-children*)
apply *auto*
done
from *pret-dag a2-in-pret* **have** *a2-nNull*: *a2* $\neq$ *Null*
apply $-$
apply (*rule set-of-nn* [*rule-format*])
apply *auto*
done

with *a2-in-pret high-a2-nNull pret-dag* **have** *high a2* $\in$ *set-of pret*
apply $-$
apply (*drule subelem-set-of-high*)
apply *auto*
done
with *wf-ll* **have** *high a2* $\in$ *set* (*ll* ! (*var* (*high a2*)))
by (*simp add*: *wf-ll-def*)
with *a2-in-lln t2-repc-dag var-children-smaller-a2 vara2-n*
have $\exists$ *k*$<$*n*. (*high a2*) $\in$ *set* (*ll* ! *k*)
by *auto*

then have *higha2-in-Nodesn*: (*high a2*) ∈ *Nodes n ll*
by (*simp add*: *Nodes-def*)
then have *rhigha2-in-rNodesn*:
repc (*high a2*) ∈ *repc* ‘ *Nodes n ll*
by *simp*
from *higha2-in-Nodesn normalize-prop* **obtain** *rt2h* **where**
rt2-dag: *Dag* (*repb* (*high a2*)) (*repb* ∝ *low*) (*repb* ∝ *high*) *rt2h* **and**
rt2-in-repbNort: *set-of rt2h* ⊆ *repb* ‘*Nodes n ll*
apply −
apply (*erule-tac x*=*high a2* **in** *ballE*)
apply *auto*
done
from *rt2-in-repbNort repbNodes-repcNodes*
have *rt2-in-repcNodesn*: *set-of rt2h* ⊆ *repc* ‘*Nodes n ll*
by *blast*
from *rt2-dag higha2-in-Nodesn*
have *repcrt2-dag*:
Dag (*repc* (*high a2*)) (*repc* ∝ *low*) (*repc* ∝ *high*) *rt2h*
apply −
apply (*drule Nodes-repbc-Dags-eq* [*rule-format*])
apply *auto*
done
from *t2-Node t2-repc-dag high-a2-nNull children-repc-eq-a2*
have *Dag* (*repc* (*high a2*)) (*repc* ∝ *low*) (*repc* ∝ *high*) *rt2*
by (*auto simp add*: *null-comp-def*)
with *repcrt2-dag* **have** *rt2h-rt2*: *rt2h* = *rt2* **by** (*simp add*: *Dag-unique*)
have *rt2-nTip*: *rt2* ≠ *Tip*
proof −
have *repc* (*high a2*) ≠ *Null*
proof −
note *rhigha2-in-rNodesn*
also have
repc ‘*Nodes n ll* ⊆ *repc* ‘*Nodes* (*Suc n*) *ll*
using *Nodes-subset*
by *blast*
also have . . . ⊆ *Nodes* (*Suc n*) *ll*
using *repcNodes-in-Nodes*
by *simp*
finally show *?thesis*
using *null-notin-Nodes-Suc-n*
by *auto*
qed
with *repcrt2-dag rt2h-rt2* **show** *?thesis* **by** *auto*
qed
with *rt2-in-repcNodesn repcrt2-dag rhigha2-in-rNodesn a2-in-lln*
t2-repc-dag rt2h-rt2
have *rt2-in-Dags-Nodesn*:
rt2 ∈ *Dags* (*repc* ‘ *Nodes n ll*) (*repc* ∝ *low*) (*repc* ∝ *high*)
apply −

apply (*rule DagsI*)
apply *auto*
done

from *a2-nNull a2-in-pret low-a2-nNull pret-dag*
have *low a2* ∈ *set-of pret*
apply −
apply (*drule subelem-set-of-low*)
apply *auto*
done
with *wf-ll* **have** *low a2* ∈ *set* (*ll* ! (*var* (*low a2*)))
by (*simp add*: *wf-ll-def*)
with *a2-in-lln t2-repc-dag var-children-smaller-a2 vara2-n*
have ∃ *k*<*n*. (*low a2*) ∈ *set* (*ll* ! *k*)
by *auto*
then have *lowa2-in-Nodesn*: (*low a2*) ∈ *Nodes n ll*
by (*simp add*: *Nodes-def*)
then have *rlowa2-in-rNodesn*: *repc* (*low a2*) ∈ *repc* ‘ *Nodes n ll*
by *simp*
from *lowa2-in-Nodesn normalize-prop* **obtain** *lt2h* **where**
lt2-dag: *Dag* (*repb* (*low a2*)) (*repb* ∝ *low*) (*repb* ∝ *high*) *lt2h* **and**
lt2-in-repbNort: *set-of lt2h* ⊆ *repb* ‘*Nodes n ll*
apply −
apply (*erule-tac x*=*low a2* **in** *ballE*)
apply *auto*
done
from *lt2-in-repbNort repbNodes-repcNodes*
have *lt2-in-repcNodesn*: *set-of lt2h* ⊆ *repc* ‘*Nodes n ll*
by *blast*
from *lt2-dag lowa2-in-Nodesn*
have *repclt2-dag*: *Dag* (*repc* (*low a2*)) (*repc* ∝ *low*) (*repc* ∝ *high*) *lt2h*
apply −
apply (*drule Nodes-repbc-Dags-eq* [*rule-format*])
apply *auto*
done
from *t2-Node t2-repc-dag low-a2-nNull children-repc-eq-a2*
have *Dag* (*repc* (*low a2*)) (*repc* ∝ *low*) (*repc* ∝ *high*) *lt2*
by (*auto simp add*: *null-comp-def*)
with *repclt2-dag* **have** *lt2h-lt2*: *lt2h* = *lt2* **by** (*simp add*: *Dag-unique*)
have *lt2-nTip*: *lt2* ≠ *Tip*
proof −
have *repc* (*low a2*) ≠ *Null*
proof −
note *rlowa2-in-rNodesn*
also have
repc ‘*Nodes n ll* ⊆ *repc* ‘*Nodes* (*Suc n*) *ll*
using *Nodes-subset*

by *blast*
also have $\ldots \subseteq$ *Nodes* (*Suc n*) *ll*
using *repcNodes-in-Nodes*
by *simp*
finally show *?thesis*
using *null-notin-Nodes-Suc-n*
by *auto*
qed
with *repclt2-dag lt2h-lt2* **show** *?thesis* **by** *auto*
qed
with *lt2-in-repcNodesn repclt2-dag rlowa2-in-rNodesn a2-in-lln t2-repc-dag lt2h-lt2*
have *lt2-in-Dags-Nodesn*:
lt2 $\in$ *Dags* (*repc* ‘ *Nodes n ll*) (*repc* $\propto$ *low*) (*repc* $\propto$ *high*)
apply −
apply (*rule DagsI*)
apply *auto*
done

from *isomorphic-dags-eq lt1-in-Dags-Nodesn lt2-in-Dags-Nodesn repbc-dags-eq*
have *shared-lt1-lt2*: *isomorphic-dags-eq lt1 lt2 var*
by *auto*
from *isomorphic-dags-eq rt1-in-Dags-Nodesn rt2-in-Dags-Nodesn repbc-dags-eq*
have *shared-rt1-rt2*: *isomorphic-dags-eq rt1 rt2 var*
by *auto*

from *shared-lt1-lt2 lbdt-def-lt1 lbdt-def-lt2* **have** *lt1-lt2*: *lt1* = *lt2*
by (*auto simp add*: *isomorphic-dags-eq-def*)
then have *root-lt1-lt2*: *root lt1* = *root lt2*
by *auto*
from *lt1-nTip t1-repc-dag t1-Node* **have** (*repc* $\propto$ *low*) (*repc a1*) $\neq$ *Null*
by *auto*
with *lt1-nTip t1-repc-dag t1-Node* **obtain** *llt1 lt1p rlt1* **where**
lt1-Node: *lt1* = *Node llt1 lt1p rlt1*
by *auto*
with *t1-repc-dag t1-Node children-repc-eq-a1 lt1-nTip*
have *root-lt1*: *root lt1* = (*repc* $\propto$ *low*) *a1*
by *auto*
from *lt2-nTip t2-repc-dag t2-Node* **have** (*repc* $\propto$ *low*) (*repc a2*) $\neq$ *Null*
by *auto*
with *lt2-nTip t2-repc-dag t2-Node* **obtain** *llt2 lt2p rlt2* **where**
lt2-Node: *lt2* = *Node llt2 lt2p rlt2*
by *auto*
with *t2-repc-dag t2-Node children-repc-eq-a2 lt2-nTip*

```
                have root-lt2: root lt2 = (repc ∝ low) a2
                  by auto
                from root-lt1-lt2 root-lt2 root-lt1
                have low-a1-a2: (repc ∝ low) a1 = (repc ∝ low) a2
                  by auto
               from shared-rt1-rt2 rbdt-def-rt1 rbdt-def-rt2 have rt1-rt2: rt1 = rt2
                  by (auto simp add: isomorphic-dags-eq-def)
                then have root-rt1-rt2: root rt1 = root rt2
                  by auto
                from rt1-nTip t1-repc-dag t1-Node have (repc ∝ high) (repc a1) ≠
Null
                  by auto
                with rt1-nTip t1-repc-dag t1-Node obtain lrt1 rt1p rrt1 where
                  rt1-Node: rt1 = Node lrt1 rt1p rrt1
                  by auto
                with t1-repc-dag t1-Node children-repc-eq-a1 rt1-nTip
                have root-rt1: root rt1 = (repc ∝ high) a1
                  by auto
                from rt2-nTip t2-repc-dag t2-Node
                have (repc ∝ high) (repc a2) ≠ Null
                  by auto
                with rt2-nTip t2-repc-dag t2-Node obtain lrt2 rt2p rrt2 where
                  rt2-Node: rt2 = Node lrt2 rt2p rrt2
                  by auto
                with t2-repc-dag t2-Node children-repc-eq-a2 rt2-nTip
                have root-rt2: root rt2 = (repc ∝ high) a2
                  by auto
                from root-rt1-rt2 root-rt2 root-rt1
                have high-a1-a2: (repc ∝ high) a1 = (repc ∝ high) a2
                  by auto
                from low-a1-a2 high-a1-a2 share-a1-a2
                have repc a1 = repc a2
                  by auto
                with lt1-lt2 rt1-rt2 show ?thesis
                  by auto
              qed
            qed
          qed
        qed
      qed
    from termi dags-shared while-while-prop repcNodes-in-Nodes repc-nc n-var-prop

      wf-marking-m-ma
    show ?thesis
      by auto
  qed
qed
with srrl-precond all-nodes-same-var
show ?thesis
```

```
    apply −
    apply (intro conjI)
    apply assumption+
    done
  qed
qed

end
```

References

[1] V. Ortner and N. Schirmer. Verification of BDD normalization. In J. Hurd and T. Melham, editors, *Theorem Proving in Higher Order Logics, 18th International Conference, TPHOLs 2005, Oxford, UK, August 2005*, volume 3603 of *LNCS*, pages 261–277. Springer, 2005.