Axioms Systems for Category Theory in Free Logic

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1 Introduction

This document provides a concise overview on the core results of our previous work [2, 3, 1] on the exploration of axiom systems for category theory. Extending the previous studies we include one further axiomatic theory in our experiments. This additional theory has been suggested by Mac Lane [5] in 1948. We show that the axioms proposed by Mac Lane are equivalent to the ones studied in [3], which includes an axioms set suggested by Scott [6] in the 1970s and another axioms set proposed by Freyd and Scedrov [4] in 1990, which we slightly modified in [3] to remedy a minor technical issue.

The explanations given below are minimal, for more details we refer to the referenced papers, in particular, to [3].

2 Embedding of Free Logic in HOL

We introduce a shallow semantical embedding of free logic [3] in Isabelle/HOL. Definite description is omitted, since it is not needed in the studies below and also since the definition provided in [1] introduces the here undesired commitment that at least one non-existing element of type i is a priori given. We here want to consider this an optional condition.

typedecl i — Type for individuals
consts *fExistence*:: $i \Rightarrow bool$ (E) — Existence/definedness predicate in free logic

abbreviation *fNot* $(\neg)$ **where** $\neg\varphi \equiv \neg\varphi$
abbreviation *fImpl* (**infixr** $\rightarrow$ *13*) **where** $\varphi \rightarrow \psi \equiv \varphi \longrightarrow \psi$
abbreviation *fId* (**infixr** $=$ *25*) **where** $l = r \equiv l = r$
abbreviation *fAll* $(\forall)$ **where** $\forall\,\Phi \equiv \forall x.\ E\ x \longrightarrow \Phi\ x$
abbreviation *fAllBi* (**binder** $\forall$ $[8]9$) **where** $\forall x.\ \varphi\ x \equiv \forall\,\varphi$
abbreviation *fOr* (**infixr** $\vee$ *21*) **where** $\varphi \vee \psi \equiv (\neg\varphi) \rightarrow \psi$
abbreviation *fAnd* (**infixr** $\wedge$ *22*) **where** $\varphi \wedge \psi \equiv \neg(\neg\varphi \vee \neg\psi)$
abbreviation *fImpli* (**infixr** $\leftarrow$ *13*) **where** $\varphi \leftarrow \psi \equiv \psi \rightarrow \varphi$
abbreviation *fEquiv* (**infixr** $\leftrightarrow$ *15*) **where** $\varphi \leftrightarrow \psi \equiv (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
abbreviation *fEx* $(\exists)$ **where** $\exists\,\Phi \equiv \neg(\forall\,(\lambda y.\ \neg(\Phi\ y)))$
abbreviation *fExiBi* (**binder** $\exists$ $[8]9$) **where** $\exists x.\ \varphi\ x \equiv \exists\,\varphi$

3 Some Basic Notions in Category Theory

Morphisms in the category are modeled as objects of type i. We introduce three partial functions, *dom* (domain), *cod* (codomain), and morphism composition $(\cdot)$.

For composition we assume set-theoretical composition here (i.e., functional composition from right to left).

consts
domain:: $i \Rightarrow i$ (*dom* - $[108]$ *109*)
codomain:: $i \Rightarrow i$ (*cod* - $[110]$ *111*)
composition:: $i \Rightarrow i \Rightarrow i$ (**infix** $\cdot$ *110*)

— Kleene Equality
abbreviation *KlEq* (**infixr** $\cong$ *56*) **where** $x \cong y \equiv (E\ x \vee E\ y) \rightarrow x = y$
— Existing Identity
abbreviation *ExId* (**infixr** $\simeq$ *56*) **where** $x \simeq y \equiv (E\ x \wedge E\ y \wedge x = y)$

— Identity-morphism: see also p. 4. of [4].
abbreviation *ID* $i \equiv (\forall x.\ E(i\cdot x) \rightarrow i\cdot x \cong x) \wedge (\forall x.\ E(x\cdot i) \rightarrow x\cdot i \cong x)$
— Identity-morphism: Mac Lane's definition, the same as ID except for notion of equality.
abbreviation *IDMcL* $\varrho \equiv (\forall \alpha.\ E(\varrho\cdot\alpha) \rightarrow \varrho\cdot\alpha = \alpha) \wedge (\forall \beta.\ E(\beta\cdot\varrho) \rightarrow \beta\cdot\varrho = \beta)$

— The two notions of identity-morphisms are obviously equivalent.
lemma *IDPredicates*: *ID* $\equiv$ *IDMcL* $\langle proof \rangle$

4 The Axioms Sets studied by Benzmüller and Scott [3]

4.1 AxiomsSet1

AxiomsSet1 generalizes the notion of a monoid by introducing a partial, strict binary composition operation "$\cdot$". The existence of left and right identity elements is addressed in axioms C_i and D_i. The notions of *dom* (domain) and *cod* (codomain) abstract from their common meaning in the context of sets. In category theory we work with just a single type of objects (the type i in our setting) and therefore identity morphisms are employed to suitably characterize their meanings.

locale *AxiomsSet1* =
assumes
S_i: $E(x \cdot y) \rightarrow (E\ x \wedge E\ y)$ **and**
E_i: $E(x \cdot y) \leftarrow (E\ x \wedge E\ y \wedge (\exists z.\ z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$ **and**
A_i: $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$ **and**
C_i: $\forall y. \exists i.\ ID\ i \wedge i \cdot y \cong y$ **and**
D_i: $\forall x. \exists j.\ ID\ j \wedge x \cdot j \cong x$
begin
lemma *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $\exists x.\ \neg(E\ x)$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $(\exists x.\ \neg(E\ x)) \wedge (\exists x.\ (E\ x))$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩

lemma E_i*Impl*: $E(x \cdot y) \rightarrow (E\ x \wedge E\ y \wedge (\exists z.\ z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$ ⟨*proof*⟩
lemma UC_i: $\forall y. \exists i.\ ID\ i \wedge i \cdot y \cong y \wedge (\forall j.(ID\ j \wedge j \cdot y \cong y) \rightarrow i \cong j)$ ⟨*proof*⟩
lemma UD_i: $\forall x. \exists j.\ ID\ j \wedge x \cdot j \cong x \wedge (\forall i.(ID\ i \wedge x \cdot i \cong x) \rightarrow j \cong i)$ ⟨*proof*⟩
lemma $(\exists C\ D.\ (\forall y.\ ID\ (C\ y) \wedge (C\ y) \cdot y \cong y) \wedge (\forall x.\ ID\ (D\ x) \wedge x \cdot (D\ x) \cong x) \wedge \neg(D = C))$
nitpick [*satisfy*] ⟨*proof*⟩
lemma $(\exists x.\ E\ x) \wedge (\exists C\ D.\ (\forall y.\ ID(C\ y) \wedge (C\ y) \cdot y \cong y) \wedge (\forall x.\ ID(D\ x) \wedge x \cdot (D\ x) \cong x) \wedge \neg(D = C))$
nitpick [*satisfy*] ⟨*proof*⟩
end

4.2 AxiomsSet2

AxiomsSet2 is developed from AxiomsSet1 by Skolemization of the existentially quantified variables i and j in axioms C_i and D_i. We can argue semantically that every model of AxiomsSet1 has such functions. Hence, we get a conservative extension of AxiomsSet1. The strictness axiom S is extended, so that strictness is now also postulated for the new Skolem functions *dom* and *cod*.

locale *AxiomsSet2* =
assumes
S_{ii}: $(E(x \cdot y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$ **and**
E_{ii}: $E(x \cdot y) \leftarrow (E\ x \wedge E\ y \wedge (\exists z.\ z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$ **and**
A_{ii}: $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$ **and**
C_{ii}: $E\ y \rightarrow (ID(cod\ y) \wedge (cod\ y) \cdot y \cong y)$ **and**
D_{ii}: $E\ x \rightarrow (ID(dom\ x) \wedge x \cdot (dom\ x) \cong x)$
begin
lemma *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $\exists x.\ \neg(E\ x)$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $(\exists x.\ \neg(E\ x)) \wedge (\exists x.\ (E\ x))$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩

lemma E_{ii}*Impl*: $E(x \cdot y) \rightarrow (E\ x \wedge E\ y \wedge (\exists z.\ z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$ ⟨*proof*⟩
lemma *domTotal*: $E\ x \rightarrow E(dom\ x)$ ⟨*proof*⟩
lemma *codTotal*: $E\ x \rightarrow E(cod\ x)$ ⟨*proof*⟩
end

4.2.1 AxiomsSet2 entails AxiomsSet1

context *AxiomsSet2*
begin
lemma S_i: $E(x \cdot y) \rightarrow (E\ x \wedge E\ y)$ ⟨*proof*⟩
lemma E_i: $E(x \cdot y) \leftarrow (E\ x \wedge E\ y \wedge (\exists z.\ z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$ ⟨*proof*⟩

lemma A_i: $x\cdot(y\cdot z) \cong (x\cdot y)\cdot z$ ⟨*proof*⟩
lemma C_i: $\forall\, y.\exists\, i.\ ID\ i \wedge i\cdot y \cong y$ ⟨*proof*⟩
lemma D_i: $\forall\, x.\exists\, j.\ ID\ j \wedge x\cdot j \cong x$ ⟨*proof*⟩
end

4.2.2 AxiomsSet1 entails AxiomsSet2 (by semantic means)

By semantic means (Skolemization).

4.3 AxiomsSet3

In AxiomsSet3 the existence axiom E_{ii} from AxiomsSet2 is simplified by taking advantage of the two new Skolem functions *dom* and *cod*.

The left-to-right direction of existence axiom E_{iii} is implied.

locale *AxiomsSet3* =
assumes
S_{iii}: $(E(x\cdot y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x\ ) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$ **and**
E_{iii}: $E(x\cdot y) \leftarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$ **and**
A_{iii}: $x\cdot(y\cdot z) \cong (x\cdot y)\cdot z$ **and**
C_{iii}: $E\ y \rightarrow (ID(cod\ y) \wedge (cod\ y)\cdot y \cong y)$ **and**
D_{iii}: $E\ x \rightarrow (ID(dom\ x) \wedge x\cdot(dom\ x) \cong x)$
begin
lemma *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $\exists\, x.\ \neg(E\ x)$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $(\exists\, x.\ \neg(E\ x)) \wedge (\exists\, x.\ (E\ x))$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩

lemma $E_{iii}Impl$: $E(x\cdot y) \rightarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$ ⟨*proof*⟩
end

4.3.1 AxiomsSet3 entails AxiomsSet2

context *AxiomsSet3*
begin
lemma S_{ii}: $(E(x\cdot y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x\ ) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$ ⟨*proof*⟩
lemma E_{ii}: $E(x\cdot y) \leftarrow (E\ x \wedge E\ y \wedge (\exists\, z.\ z\cdot z \cong z \wedge x\cdot z \cong x \wedge z\cdot y \cong y))$ ⟨*proof*⟩
lemma A_{ii}: $x\cdot(y\cdot z) \cong (x\cdot y)\cdot z$ ⟨*proof*⟩
lemma C_{ii}: $E\ y \rightarrow (ID(cod\ y) \wedge (cod\ y)\cdot y \cong y)$ ⟨*proof*⟩
lemma D_{ii}: $E\ x \rightarrow (ID(dom\ x) \wedge x\cdot(dom\ x) \cong x)$ ⟨*proof*⟩
end

4.3.2 AxiomsSet2 entails AxiomsSet3

context *AxiomsSet2*
begin
lemma S_{iii}: $(E(x\cdot y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$ ⟨*proof*⟩
lemma E_{iii}: $E(x\cdot y) \leftarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$ ⟨*proof*⟩
lemma A_{iii}: $x\cdot(y\cdot z) \cong (x\cdot y)\cdot z$ ⟨*proof*⟩
lemma C_{iii}: $E\ y \rightarrow (ID(cod\ y) \wedge (cod\ y)\cdot y \cong y)$ ⟨*proof*⟩
lemma D_{iii}: $E\ x \rightarrow (ID(dom\ x) \wedge x\cdot(dom\ x) \cong x)$ ⟨*proof*⟩
end

4.4 The Axioms Set AxiomsSet4

AxiomsSet4 simplifies the axioms C_{iii} and D_{iii}. However, as it turned out, these simplifications also require the existence axiom E_{iii} to be strengthened into an equivalence.

locale *AxiomsSet4* =
assumes
S_{iv}: $(E(x\cdot y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$ **and**

E_{iv}: $E(x{\cdot}y) \leftrightarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$ **and**
A_{iv}: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ **and**
C_{iv}: $(cod\ y){\cdot}y \cong y$ **and**
D_{iv}: $x{\cdot}(dom\ x) \cong x$
begin
lemma *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $\exists x.\ \neg(E\ x)$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $(\exists x.\ \neg(E\ x)) \wedge (\exists x.\ (E\ x))$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
end

4.4.1 AxiomsSet4 entails AxiomsSet3

context *AxiomsSet4*
begin
lemma S_{iii}: $(E(x{\cdot}y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$ ⟨*proof*⟩
lemma E_{iii}: $E(x{\cdot}y) \leftarrow (dom\ x \cong cod\ y \wedge (E(cod\ y)))$ ⟨*proof*⟩
lemma A_{iii}: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ ⟨*proof*⟩
lemma C_{iii}: $E\ y \rightarrow (ID(cod\ y) \wedge (cod\ y){\cdot}y \cong y)$ ⟨*proof*⟩
lemma D_{iii}: $E\ x \rightarrow (ID(dom\ x) \wedge x{\cdot}(dom\ x) \cong x)$ ⟨*proof*⟩
end

4.4.2 AxiomsSet3 entails AxiomsSet4

context *AxiomsSet3*
begin
lemma S_{iv}: $(E(x{\cdot}y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x\ ) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$ ⟨*proof*⟩
lemma E_{iv}: $E(x{\cdot}y) \leftrightarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$ ⟨*proof*⟩
lemma A_{iv}: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ ⟨*proof*⟩
lemma C_{iv}: $(cod\ y){\cdot}y \cong y$ ⟨*proof*⟩
lemma D_{iv}: $x{\cdot}(dom\ x) \cong x$ ⟨*proof*⟩
end

4.5 AxiomsSet5

AxiomsSet5 has been proposed by Scott [6] in the 1970s. This set of axioms is equivalent to the axioms set presented by Freyd and Scedrov in their textbook “Categories, Allegories” [4] when encoded in free logic, corrected/adapted and further simplified, see Section 5.

locale *AxiomsSet5* =
assumes
S1: $E(dom\ x) \rightarrow E\ x$ **and**
S2: $E(cod\ y) \rightarrow E\ y$ **and**
S3: $E(x{\cdot}y) \leftrightarrow dom\ x \simeq cod\ y$ **and**
S4: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ **and**
S5: $(cod\ y){\cdot}y \cong y$ **and**
S6: $x{\cdot}(dom\ x) \cong x$
begin
lemma *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $\exists x.\ \neg(E\ x)$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $(\exists x.\ \neg(E\ x)) \wedge (\exists x.\ (E\ x))$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
end

4.5.1 AxiomsSet5 entails AxiomsSet4

context *AxiomsSet5*
begin
lemma S_{iv}: $(E(x{\cdot}y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x\ ) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$ ⟨*proof*⟩
lemma E_{iv}: $E(x{\cdot}y) \leftrightarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$ ⟨*proof*⟩
lemma A_{iv}: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ ⟨*proof*⟩
lemma C_{iv}: $(cod\ y){\cdot}y \cong y$ ⟨*proof*⟩
lemma D_{iv}: $x{\cdot}(dom\ x) \cong x$ ⟨*proof*⟩

end

4.5.2 AxiomsSet4 entails AxiomsSet5

context *AxiomsSet4*
begin
lemma *S1*: $E(dom\ x) \rightarrow E\ x$ ⟨*proof*⟩
lemma *S2*: $E(cod\ y) \rightarrow E\ y$ ⟨*proof*⟩
lemma *S3*: $E(x{\cdot}y) \leftrightarrow dom\ x \simeq cod\ y$ ⟨*proof*⟩
lemma *S4*: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ ⟨*proof*⟩
lemma *S5*: $(cod\ y){\cdot}y \cong y$ ⟨*proof*⟩
lemma *S6*: $x{\cdot}(dom\ x) \cong x$ ⟨*proof*⟩
end

5 The Axioms Sets by Freyd and Scedrov [4]

5.1 AxiomsSet6

The axioms by Freyd and Scedrov [4] in our notation, when being corrected (cf. the modification in axiom A1).

Freyd and Scedrov employ a different notation for *dom x* and *cod x*. They denote these operations by $\Box x$ and $x\Box$. Moreover, they employ diagrammatic composition instead of the set-theoretic definition (functional composition from right to left) used so far. We leave it to the reader to verify that their axioms corresponds to the axioms presented here modulo an appropriate conversion of notation.

locale *AxiomsSet6* =
assumes
A1: $E(x{\cdot}y) \leftrightarrow dom\ x \simeq cod\ y$ **and**
A2a: $cod(dom\ x) \cong dom\ x$ **and**
A2b: $dom(cod\ y) \cong cod\ y$ **and**
A3a: $x{\cdot}(dom\ x) \cong x$ **and**
A3b: $(cod\ y){\cdot}y \cong y$ **and**
A4a: $dom(x{\cdot}y) \cong dom((dom\ x){\cdot}y)$ **and**
A4b: $cod(x{\cdot}y) \cong cod(x{\cdot}(cod\ y))$ **and**
A5: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$
begin
lemma *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $\exists x.\ \neg(E\ x)$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $(\exists x.\ \neg(E\ x)) \wedge (\exists x.\ (E\ x))$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
end

5.1.1 AxiomsSet6 entails AxiomsSet5

context *AxiomsSet6*
begin
lemma *S1*: $E(dom\ x) \rightarrow E\ x$ ⟨*proof*⟩
lemma *S2*: $E(cod\ y) \rightarrow E\ y$ ⟨*proof*⟩
lemma *S3*: $E(x{\cdot}y) \leftrightarrow dom\ x \simeq cod\ y$ ⟨*proof*⟩
lemma *S4*: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ ⟨*proof*⟩
lemma *S5*: $(cod\ y){\cdot}y \cong y$ ⟨*proof*⟩
lemma *S6*: $x{\cdot}(dom\ x) \cong x$ ⟨*proof*⟩

lemma *A4aRedundant*: $dom(x{\cdot}y) \cong dom((dom\ x){\cdot}y)$ ⟨*proof*⟩
lemma *A4bRedundant*: $cod(x{\cdot}y) \cong cod(x{\cdot}(cod\ y))$ ⟨*proof*⟩
lemma *A2aRedundant*: $cod(dom\ x) \cong dom\ x$ ⟨*proof*⟩
lemma *A2bRedundant*: $dom(cod\ y) \cong cod\ y$ ⟨*proof*⟩
end

5.1.2 AxiomsSet5 entails AxiomsSet6

context *AxiomsSet5*
begin
lemma *A1*: $E(x \cdot y) \leftrightarrow dom\ x \simeq cod\ y$ ⟨*proof*⟩
lemma *A2*: $cod(dom\ x) \cong dom\ x$ ⟨*proof*⟩
lemma *A2b*: $dom(cod\ y) \cong cod\ y$ ⟨*proof*⟩
lemma *A3a*: $x \cdot (dom\ x) \cong x$ ⟨*proof*⟩
lemma *A3b*: $(cod\ y) \cdot y \cong y$ ⟨*proof*⟩
lemma *A4a*: $dom(x \cdot y) \cong dom((dom\ x) \cdot y)$ ⟨*proof*⟩
lemma *A4b*: $cod(x \cdot y) \cong cod(x \cdot (cod\ y))$ ⟨*proof*⟩
lemma *A5*: $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$ ⟨*proof*⟩
end

5.2 AxiomsSet7 (technically flawed)

The axioms by Freyd and Scedrov in our notation, without the suggested correction of axiom A1. This axioms set is technically flawed when encoded in our given context. It leads to a constricted inconsistency.

locale *AxiomsSet7* =
assumes
A1: $E(x \cdot y) \leftrightarrow dom\ x \cong cod\ y$ **and**
A2a: $cod(dom\ x) \cong dom\ x$ **and**
A2b: $dom(cod\ y) \cong cod\ y$ **and**
A3a: $x \cdot (dom\ x) \cong x$ **and**
A3b: $(cod\ y) \cdot y \cong y$ **and**
A4a: $dom(x \cdot y) \cong dom((dom\ x) \cdot y)$ **and**
A4b: $cod(x \cdot y) \cong cod(x \cdot (cod\ y))$ **and**
A5: $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
begin
lemma *True* **nitpick** [*satisfy*] ⟨*proof*⟩

lemma *InconsistencyAutomatic*: $(\exists x.\ \neg(E\ x)) \rightarrow False$ ⟨*proof*⟩
lemma $\forall x.\ E\ x$ ⟨*proof*⟩

lemma *InconsistencyInteractive*:
assumes *NEx*: $\exists x.\ \neg(E\ x)$ **shows** *False*
⟨*proof*⟩
end

5.3 AxiomsSet7orig (technically flawed)

The axioms by Freyd and Scedrov in their original notation, without the suggested correction of axiom A1.

We present the constricted inconsistency argument from above once again, but this time in the original notation of Freyd and Scedrov.

locale *AxiomsSet7orig* =
fixes
source:: $i \Rightarrow i$ (□- [*108*] *109*) **and**
target:: $i \Rightarrow i$ (-□ [*110*] *111*) **and**
compositionF:: $i \Rightarrow i \Rightarrow i$ (**infix** · *110*)
assumes
A1: $E(x \cdot y) \leftrightarrow (x\square \cong \square y)$ **and**
A2a: $((\square x)\square) \cong \square x$ **and**
A2b: $\square(x\square) \cong \square x$ **and**
A3a: $(\square x) \cdot x \cong x$ **and**
A3b: $x \cdot (x\square) \cong x$ **and**
A4a: $\square(x \cdot y) \cong \square(x \cdot (\square y))$ **and**
A4b: $(x \cdot y)\square \cong ((x\square) \cdot y)\square$ **and**

$A5$: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$
begin
lemma *True* **nitpick** [*satisfy*] ⟨*proof*⟩

lemma *InconsistencyAutomatic*: $(\exists\, x.\ \neg(E\ x)) \rightarrow \mathit{False}$ ⟨*proof*⟩
lemma $\forall\, x.\ E\ x$ ⟨*proof*⟩

lemma *InconsistencyInteractive*:
assumes *NEx*: $\exists\, x.\ \neg(E\ x)$ **shows** *False*
⟨*proof*⟩
end

5.4 AxiomsSet8 (algebraic reading, still technically flawed)

The axioms by Freyd and Scedrov in our notation again, but this time we adopt an algebraic reading of the free variables, meaning that they range over existing morphisms only.

locale *AxiomsSet8* =
assumes
$B1$: $\forall\, x.\forall\, y.\ E(x{\cdot}y) \leftrightarrow \mathit{dom}\ x \cong \mathit{cod}\ y$ **and**
$B2a$: $\forall\, x.\ \mathit{cod}(\mathit{dom}\ x) \cong \mathit{dom}\ x$ **and**
$B2b$: $\forall\, y.\ \mathit{dom}(\mathit{cod}\ y) \cong \mathit{cod}\ y$ **and**
$B3a$: $\forall\, x.\ x{\cdot}(\mathit{dom}\ x) \cong x$ **and**
$B3b$: $\forall\, y.\ (\mathit{cod}\ y){\cdot}y \cong y$ **and**
$B4a$: $\forall\, x.\forall\, y.\ \mathit{dom}(x{\cdot}y) \cong \mathit{dom}((\mathit{dom}\ x){\cdot}y)$ **and**
$B4b$: $\forall\, x.\forall\, y.\ \mathit{cod}(x{\cdot}y) \cong \mathit{cod}(x{\cdot}(\mathit{cod}\ y))$ **and**
$B5$: $\forall\, x.\forall\, y.\forall\, z.\ x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$
begin
lemma *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $\exists\, x.\ \neg(E\ x)$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $(\exists\, x.\ \neg(E\ x)) \wedge (\exists\, x.\ (E\ x))$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
end

None of the axioms in AxiomsSet5 are implied.

context *AxiomsSet8*
begin
lemma $S1$: $E(\mathit{dom}\ x) \rightarrow E\ x$ **nitpick** ⟨*proof*⟩
lemma $S2$: $E(\mathit{cod}\ y) \rightarrow E\ y$ **nitpick** ⟨*proof*⟩
lemma $S3$: $E(x{\cdot}y) \leftrightarrow \mathit{dom}\ x \simeq \mathit{cod}\ y$ **nitpick** ⟨*proof*⟩
lemma $S4$: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ **nitpick** ⟨*proof*⟩
lemma $S5$: $(\mathit{cod}\ y){\cdot}y \cong y$ **nitpick** ⟨*proof*⟩
lemma $S6$: $x{\cdot}(\mathit{dom}\ x) \cong x$ **nitpick** ⟨*proof*⟩
end

5.5 AxiomsSet8Strict (algebraic reading)

The situation changes when strictness conditions are postulated. Note that in the algebraic framework of Freyd and Scedrov such conditions have to be assumed as given in the logic, while here we can explicitly encode them as axioms.

locale *AxiomsSet8Strict* = *AxiomsSet8* +
assumes
$B0a$: $E(x{\cdot}y) \rightarrow (E\ x \wedge E\ y)$ **and**
$B0b$: $E(\mathit{dom}\ x) \rightarrow E\ x$ **and**
$B0c$: $E(\mathit{cod}\ x) \rightarrow E\ x$
begin
lemma *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $\exists\, x.\ \neg(E\ x)$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $(\exists\, x.\ \neg(E\ x)) \wedge (\exists\, x.\ (E\ x))$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
end

5.5.1 AxiomsSet8Strict entails AxiomsSet5

context *AxiomsSet8Strict*
begin
lemma *S1*: $E(dom\ x) \rightarrow E\ x$ ⟨*proof*⟩
lemma *S2*: $E(cod\ y) \rightarrow E\ y$ ⟨*proof*⟩
lemma *S3*: $E(x{\cdot}y) \leftrightarrow dom\ x \simeq cod\ y$ ⟨*proof*⟩
lemma *S4*: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ ⟨*proof*⟩
lemma *S5*: $(cod\ y){\cdot}y \cong y$ ⟨*proof*⟩
lemma *S6*: $x{\cdot}(dom\ x) \cong x$ ⟨*proof*⟩
end

5.5.2 AxiomsSet5 entails AxiomsSet8Strict

context *AxiomsSet5*
begin
lemma *B0a*: $E(x{\cdot}y) \rightarrow (E\ x \wedge E\ y)$ ⟨*proof*⟩
lemma *B0b*: $E(dom\ x) \rightarrow E\ x$ ⟨*proof*⟩
lemma *B0c*: $E(cod\ x) \rightarrow E\ x$ ⟨*proof*⟩
lemma *B1*: $\forall x.\forall y.\ E(x{\cdot}y) \leftrightarrow dom\ x \cong cod\ y$ ⟨*proof*⟩
lemma *B2a*: $\forall x.\ cod(dom\ x) \cong dom\ x$ ⟨*proof*⟩
lemma *B2b*: $\forall y.\ dom(cod\ y) \cong cod\ y$ ⟨*proof*⟩
lemma *B3a*: $\forall x.\ x{\cdot}(dom\ x) \cong x$ ⟨*proof*⟩
lemma *B3b*: $\forall y.\ (cod\ y){\cdot}y \cong y$ ⟨*proof*⟩
lemma *B4a*: $\forall x.\forall y.\ dom(x{\cdot}y) \cong dom((dom\ x){\cdot}y)$ ⟨*proof*⟩
lemma *B4b*: $\forall x.\forall y.\ cod(x{\cdot}y) \cong cod(x{\cdot}(cod\ y))$ ⟨*proof*⟩
lemma *B5*: $\forall x.\forall y.\forall z.\ x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ ⟨*proof*⟩
end

5.5.3 AxiomsSet8Strict is Redundant

AxiomsSet8Strict is redundant: either the B2-axioms can be omitted or the B4-axioms.

context *AxiomsSet8Strict*
begin
lemma *B2aRedundant*: $\forall x.\ cod(dom\ x) \cong dom\ x$ ⟨*proof*⟩
lemma *B2bRedundant*: $\forall y.\ dom(cod\ y) \cong cod\ y$ ⟨*proof*⟩
lemma *B4aRedundant*: $\forall x.\forall y.\ dom(x{\cdot}y) \cong dom((dom\ x){\cdot}y)$ ⟨*proof*⟩
lemma *B4bRedundant*: $\forall x.\forall y.\ cod(x{\cdot}y) \cong cod(x{\cdot}(cod\ y))$ ⟨*proof*⟩
end

6 The Axioms Sets of Mac Lane [5]

We analyse the axioms set suggested by Mac Lane [5] already in 1948. As for the theory by Freyd and Scedrov above, which was developed much later, we need to assume strictness of composition to show equivalence to our previous axiom sets. Note that his complicated conditions on existence of compositions proved to be unnecessary, as we show. It shows it is hard to think about partial operations.

locale *AxiomsSetMcL* =
assumes
$C_0 : E(x{\cdot}y) \rightarrow (E\ x \wedge E\ y)$ **and**
$C_1 : \forall \gamma\ \beta\ \alpha.\ (E(\gamma{\cdot}\beta) \wedge E((\gamma{\cdot}\beta){\cdot}\alpha)) \rightarrow E(\beta{\cdot}\alpha)$ **and**
$C_1' : \forall \gamma\ \beta\ \alpha.\ (E(\beta{\cdot}\alpha) \wedge E(\gamma{\cdot}(\beta{\cdot}\alpha))) \rightarrow E(\gamma{\cdot}\beta)$ **and**
$C_2 : \forall \gamma\ \beta\ \alpha.\ (E(\gamma{\cdot}\beta) \wedge E(\beta{\cdot}\alpha)) \rightarrow (E((\gamma{\cdot}\beta){\cdot}\alpha) \wedge E(\gamma{\cdot}(\beta{\cdot}\alpha)) \wedge ((\gamma{\cdot}\beta){\cdot}\alpha) = (\gamma{\cdot}(\beta{\cdot}\alpha)))$ **and**
$C_3 : \forall \gamma.\ \exists\ eD.\ IDMcL(eD) \wedge E(\gamma{\cdot}eD)$ **and**
$C_4 : \forall \gamma.\ \exists\ eR.\ IDMcL(eR) \wedge E(eR{\cdot}\gamma)$
begin
lemma *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $\exists x.\ \neg(E\ x)$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $(\exists x.\ \neg(E\ x)) \wedge (\exists x.\ (E\ x))$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩

end

Remember that IDMcL was defined on p. 2 and proved equivalent to ID.

6.1 AxiomsSetMcL entails AxiomsSet1

context *AxiomsSetMcL*
begin
lemma S_i: $E(x{\cdot}y) \rightarrow (E\ x \wedge E\ y)$ ⟨*proof*⟩
lemma E_i: $E(x{\cdot}y) \leftarrow (E\ x \wedge E\ y \wedge (\exists\, z.\ z{\cdot}z \cong z \wedge x{\cdot}z \cong x \wedge z{\cdot}y \cong y))$ ⟨*proof*⟩
lemma A_i: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ ⟨*proof*⟩
lemma C_i: $\forall\, y.\exists\, i.\ ID\ i \wedge i{\cdot}y \cong y$ ⟨*proof*⟩
lemma D_i: $\forall\, x.\exists\, j.\ ID\ j \wedge x{\cdot}j \cong x$ ⟨*proof*⟩
end

6.2 AxiomsSet1 entails AxiomsSetMcL

context *AxiomsSet1*
begin
lemma C_0 : $E(x{\cdot}y) \rightarrow (E\ x \wedge E\ y)$ ⟨*proof*⟩
lemma C_1 : $\forall\, \gamma\ \beta\ \alpha.\ (E(\gamma{\cdot}\beta) \wedge E((\gamma{\cdot}\beta){\cdot}\alpha)) \rightarrow E(\beta{\cdot}\alpha)$ ⟨*proof*⟩
lemma $C_1{}'$: $\forall\, \gamma\ \beta\ \alpha.\ (E(\beta{\cdot}\alpha) \wedge E(\gamma{\cdot}(\beta{\cdot}\alpha))) \rightarrow E(\gamma{\cdot}\beta)$ ⟨*proof*⟩
lemma C_2 : $\forall\, \gamma\ \beta\ \alpha.\ (E(\gamma{\cdot}\beta) \wedge E(\beta{\cdot}\alpha)) \rightarrow (E((\gamma{\cdot}\beta){\cdot}\alpha) \wedge E(\gamma{\cdot}(\beta{\cdot}\alpha)) \wedge ((\gamma{\cdot}\beta){\cdot}\alpha) = (\gamma{\cdot}(\beta{\cdot}\alpha)))$ ⟨*proof*⟩
lemma C_3 : $\forall\, \gamma.\ \exists\, eD.\ IDMcL(eD) \wedge E(\gamma{\cdot}eD)$ ⟨*proof*⟩
lemma C_4 : $\forall\, \gamma.\ \exists\, eR.\ IDMcL(eR) \wedge E(eR{\cdot}\gamma)$ ⟨*proof*⟩
end

6.3 Skolemization of the Axioms of Mac Lane

Mac Lane employs diagrammatic composition instead of the set-theoretic definition as used in our axiom sets. As we have seen above, this is not a problem as long as composition is the only primitive. But when adding the Skolem terms *dom* and *cod* care must be taken and we should actually transform all axioms into a common form. Below we address this (in a minimal way) by using *dom* in axiom C_3s and *cod* in axiom C_4s, which is opposite of what Mac Lane proposed. For this axioms set we then show equivalence to AxiomsSet1/2/5.

locale *SkolemizedAxiomsSetMcL* =
assumes
C_0s : $(E(x{\cdot}y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$ **and**
C_1s : $\forall\, \gamma\ \beta\ \alpha.\ (E(\gamma{\cdot}\beta) \wedge E((\gamma{\cdot}\beta){\cdot}\alpha)) \rightarrow E(\beta{\cdot}\alpha)$ **and**
$C_1{}'s$: $\forall\, \gamma\ \beta\ \alpha.\ (E(\beta{\cdot}\alpha) \wedge E(\gamma{\cdot}(\beta{\cdot}\alpha))) \rightarrow E(\gamma{\cdot}\beta)$ **and**
C_2s : $\forall\, \gamma\ \beta\ \alpha.\ (E(\gamma{\cdot}\beta) \wedge E(\beta{\cdot}\alpha)) \rightarrow (E((\gamma{\cdot}\beta){\cdot}\alpha) \wedge E(\gamma{\cdot}(\beta{\cdot}\alpha)) \wedge ((\gamma{\cdot}\beta){\cdot}\alpha) = (\gamma{\cdot}(\beta{\cdot}\alpha)))$ **and**
C_3s : $\forall\, \gamma.\ IDMcL(dom\ \gamma) \wedge E(\gamma{\cdot}(dom\ \gamma))$ **and**
C_4s : $\forall\, \gamma.\ IDMcL(cod\ \gamma) \wedge E((cod\ \gamma){\cdot}\gamma)$
begin
lemma *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $\exists\, x.\ \neg(E\ x)$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
lemma assumes $(\exists\, x.\ \neg(E\ x)) \wedge (\exists\, x.\ (E\ x))$ **shows** *True* **nitpick** [*satisfy*] ⟨*proof*⟩
end

6.4 SkolemizedAxiomsSetMcL entails AxiomsSetMcL and AxiomsSet1-5

context *SkolemizedAxiomsSetMcL*
begin
lemma C_0 : $E(x{\cdot}y) \rightarrow (E\ x \wedge E\ y)$ ⟨*proof*⟩
lemma C_1 : $\forall\, \gamma\ \beta\ \alpha.\ (E(\gamma{\cdot}\beta) \wedge E((\gamma{\cdot}\beta){\cdot}\alpha)) \rightarrow E(\beta{\cdot}\alpha)$ ⟨*proof*⟩
lemma $C_1{}'$: $\forall\, \gamma\ \beta\ \alpha.\ (E(\beta{\cdot}\alpha) \wedge E(\gamma{\cdot}(\beta{\cdot}\alpha))) \rightarrow E(\gamma{\cdot}\beta)$ ⟨*proof*⟩
lemma C_2 : $\forall\, \gamma\ \beta\ \alpha.\ (E(\gamma{\cdot}\beta) \wedge E(\beta{\cdot}\alpha)) \rightarrow (E((\gamma{\cdot}\beta){\cdot}\alpha) \wedge E(\gamma{\cdot}(\beta{\cdot}\alpha)) \wedge ((\gamma{\cdot}\beta){\cdot}\alpha) = (\gamma{\cdot}(\beta{\cdot}\alpha)))$ ⟨*proof*⟩
lemma C_3 : $\forall\, \gamma.\ \exists\, eD.\ IDMcL(eD) \wedge E(\gamma{\cdot}eD)$ ⟨*proof*⟩
lemma C_4 : $\forall\, \gamma.\ \exists\, eR.\ IDMcL(eR) \wedge E(eR{\cdot}\gamma)$ ⟨*proof*⟩

lemma S_i: $E(x{\cdot}y) \rightarrow (E\ x \wedge E\ y)$ ⟨*proof*⟩
lemma E_i: $E(x{\cdot}y) \leftarrow (E\ x \wedge E\ y \wedge (\exists\, z.\ z{\cdot}z \cong z \wedge x{\cdot}z \cong x \wedge z{\cdot}y \cong y))$ ⟨*proof*⟩
lemma A_i: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ ⟨*proof*⟩
lemma C_i: $\forall\, y.\exists\, i.\ ID\ i \wedge i{\cdot}y \cong y$ ⟨*proof*⟩
lemma D_i: $\forall\, x.\exists\, j.\ ID\ j \wedge x{\cdot}j \cong x$ ⟨*proof*⟩

lemma S_{ii}: $(E(x{\cdot}y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x\ ) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$ ⟨*proof*⟩
lemma E_{ii}: $E(x{\cdot}y) \leftarrow (E\ x \wedge E\ y \wedge (\exists\, z.\ z{\cdot}z \cong z \wedge x{\cdot}z \cong x \wedge z{\cdot}y \cong y))$ ⟨*proof*⟩
lemma A_{ii}: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ ⟨*proof*⟩
lemma C_{ii}: $E\ y \rightarrow (ID(cod\ y) \wedge (cod\ y){\cdot}y \cong y)$ ⟨*proof*⟩
lemma D_{ii}: $E\ x \rightarrow (ID(dom\ x) \wedge x{\cdot}(dom\ x) \cong x)$ ⟨*proof*⟩

lemma *S1*: $E(dom\ x) \rightarrow E\ x$ ⟨*proof*⟩
lemma *S2*: $E(cod\ y) \rightarrow E\ y$ ⟨*proof*⟩
lemma *S3*: $E(x{\cdot}y) \leftrightarrow dom\ x \simeq cod\ y$ ⟨*proof*⟩
lemma *S4*: $x{\cdot}(y{\cdot}z) \cong (x{\cdot}y){\cdot}z$ ⟨*proof*⟩
lemma *S5*: $(cod\ y){\cdot}y \cong y$ ⟨*proof*⟩
lemma *S6*: $x{\cdot}(dom\ x) \cong x$ ⟨*proof*⟩
end

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