

Axioms Systems for Category Theory in Free Logic

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1 Introduction

This document provides a concise overview on the core results of our previous work [2, 3, 1] on the exploration of axiom systems for category theory. Extending the previous studies we include one further axiomatic theory in our experiments. This additional theory has been suggested by Mac Lane [5] in 1948. We show that the axioms proposed by Mac Lane are equivalent to the ones studied in [3], which includes an axioms set suggested by Scott [6] in the 1970s and another axioms set proposed by Freyd and Scedrov [4] in 1990, which we slightly modified in [3] to remedy a minor technical issue. The explanations given below are minimal, for more details we refer to the referenced papers, in particular, to [3].

2 Embedding of Free Logic in HOL

We introduce a shallow semantical embedding of free logic [3] in Isabelle/HOL. Definite description is omitted, since it is not needed in the studies below and also since the definition provided in [1] introduces the here undesired commitment that at least one non-existing element of type i is a priori given. We here want to consider this an optional condition.

typedecl i — Type for individuals

consts $fExistence:: i \Rightarrow bool$ ($\langle E \rangle$) — Existence/definedness predicate in free logic

abbreviation $fNot$ ($\langle \neg \rangle$) **where** $\neg \varphi \equiv \neg \varphi$
abbreviation $fImpl$ (**infixr** $\langle \rightarrow \rangle$ 13) **where** $\varphi \rightarrow \psi \equiv \varphi \longrightarrow \psi$
abbreviation fId (**infixr** $\langle = \rangle$ 25) **where** $l = r \equiv l = r$
abbreviation $fAll$ ($\langle \forall \rangle$) **where** $\forall \Phi \equiv \forall x. E x \longrightarrow \Phi x$
abbreviation $fAllBi$ (**binder** $\langle \forall \rangle$ [8]9) **where** $\forall x. \varphi x \equiv \forall \varphi$
abbreviation fOr (**infixr** $\langle \vee \rangle$ 21) **where** $\varphi \vee \psi \equiv (\neg \varphi) \rightarrow \psi$
abbreviation $fAnd$ (**infixr** $\langle \wedge \rangle$ 22) **where** $\varphi \wedge \psi \equiv \neg(\neg \varphi \vee \neg \psi)$
abbreviation $fImpli$ (**infixr** $\langle \leftarrow \rangle$ 13) **where** $\varphi \leftarrow \psi \equiv \psi \rightarrow \varphi$
abbreviation $fEquiv$ (**infixr** $\langle \leftrightarrow \rangle$ 15) **where** $\varphi \leftrightarrow \psi \equiv (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
abbreviation fEx ($\langle \exists \rangle$) **where** $\exists \Phi \equiv \neg(\forall (\lambda y. \neg(\Phi y)))$
abbreviation $fExiBi$ (**binder** $\langle \exists \rangle$ [8]9) **where** $\exists x. \varphi x \equiv \exists \varphi$

3 Some Basic Notions in Category Theory

Morphisms in the category are modeled as objects of type i . We introduce three partial functions, dom (domain), cod (codomain), and morphism composition (\cdot) .

For composition we assume set-theoretical composition here (i.e., functional composition from right to left).

consts

$domain:: i \Rightarrow i$ ($\langle dom \rightarrow \rangle$ [108] 109)

$codomain:: i \Rightarrow i$ ($\langle cod \rightarrow \rangle$ [110] 111)

$composition:: i \Rightarrow i \Rightarrow i$ (**infix** $\langle \cdot \rangle$ 110)

— Kleene Equality

abbreviation $KLEq$ (**infixr** $\langle \cong \rangle$ 56) **where** $x \cong y \equiv (E x \vee E y) \rightarrow x = y$

— Existing Identity

abbreviation $ExId$ (**infixr** $\langle \simeq \rangle$ 56) **where** $x \simeq y \equiv (E x \wedge E y \wedge x = y)$

— Identity-morphism: see also p. 4. of [4].

abbreviation ID $i \equiv (\forall x. E(i \cdot x) \rightarrow i \cdot x \cong x) \wedge (\forall x. E(x \cdot i) \rightarrow x \cdot i \cong x)$

— Identity-morphism: Mac Lane's definition, the same as ID except for notion of equality.

abbreviation $IDMcL$ $\varrho \equiv (\forall \alpha. E(\varrho \cdot \alpha) \rightarrow \varrho \cdot \alpha = \alpha) \wedge (\forall \beta. E(\beta \cdot \varrho) \rightarrow \beta \cdot \varrho = \beta)$

— The two notions of identity-morphisms are obviously equivalent.

lemma $IDPredicates: ID \equiv IDMcL$ $\langle proof \rangle$

4 The Axioms Sets studied by Benz Müller and Scott [3]

4.1 AxiomsSet1

AxiomsSet1 generalizes the notion of a monoid by introducing a partial, strict binary composition operation “ \cdot ”. The existence of left and right identity elements is addressed in axioms C_i and D_i . The notions of *dom* (domain) and *cod* (codomain) abstract from their common meaning in the context of sets. In category theory we work with just a single type of objects (the type i in our setting) and therefore identity morphisms are employed to suitably characterize their meanings.

locale *AxiomsSet1* =

assumes

$S_i: E(x \cdot y) \rightarrow (E x \wedge E y)$ **and**

$E_i: E(x \cdot y) \leftarrow (E x \wedge E y \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$ **and**

$A_i: x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$ **and**

$C_i: \forall y. \exists i. ID i \wedge i \cdot y \cong y$ **and**

$D_i: \forall x. \exists j. ID j \wedge x \cdot j \cong x$

begin

lemma *True nitpick* [satisfy] <proof>

lemma **assumes** $\exists x. \neg(E x)$ **shows** *True nitpick* [satisfy] <proof>

lemma **assumes** $(\exists x. \neg(E x)) \wedge (\exists x. (E x))$ **shows** *True nitpick* [satisfy] <proof>

lemma *E_iImpl*: $E(x \cdot y) \rightarrow (E x \wedge E y \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$ <proof>

lemma *UC_i*: $\forall y. \exists i. ID i \wedge i \cdot y \cong y \wedge (\forall j. (ID j \wedge j \cdot y \cong y) \rightarrow i \cong j)$ <proof>

lemma *UD_i*: $\forall x. \exists j. ID j \wedge x \cdot j \cong x \wedge (\forall i. (ID i \wedge x \cdot i \cong x) \rightarrow j \cong i)$ <proof>

lemma $(\exists C D. (\forall y. ID(C y) \wedge (C y) \cdot y \cong y) \wedge (\forall x. ID(D x) \wedge x \cdot (D x) \cong x) \wedge \neg(D = C))$

nitpick [satisfy] <proof>

lemma $(\exists x. E x) \wedge (\exists C D. (\forall y. ID(C y) \wedge (C y) \cdot y \cong y) \wedge (\forall x. ID(D x) \wedge x \cdot (D x) \cong x) \wedge \neg(D = C))$

nitpick [satisfy] <proof>

end

4.2 AxiomsSet2

AxiomsSet2 is developed from AxiomsSet1 by Skolemization of the existentially quantified variables i and j in axioms C_i and D_i . We can argue semantically that every model of AxiomsSet1 has such functions. Hence, we get a conservative extension of AxiomsSet1. The strictness axiom S is extended, so that strictness is now also postulated for the new Skolem functions *dom* and *cod*.

locale *AxiomsSet2* =

assumes

$S_{ii}: (E(x \cdot y) \rightarrow (E x \wedge E y)) \wedge (E(\text{dom } x) \rightarrow E x) \wedge (E(\text{cod } y) \rightarrow E y)$ **and**

$E_{ii}: E(x \cdot y) \leftarrow (E x \wedge E y \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$ **and**

$A_{ii}: x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$ **and**

$C_{ii}: E y \rightarrow (ID(\text{cod } y) \wedge (\text{cod } y) \cdot y \cong y)$ **and**

$D_{ii}: E x \rightarrow (ID(\text{dom } x) \wedge x \cdot (\text{dom } x) \cong x)$

begin

lemma *True nitpick* [satisfy] <proof>

lemma **assumes** $\exists x. \neg(E x)$ **shows** *True nitpick* [satisfy] <proof>

lemma **assumes** $(\exists x. \neg(E x)) \wedge (\exists x. (E x))$ **shows** *True nitpick* [satisfy] <proof>

lemma *E_{ii}Impl*: $E(x \cdot y) \rightarrow (E x \wedge E y \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$ <proof>

lemma *domTotal*: $E x \rightarrow E(\text{dom } x)$ <proof>

lemma *codTotal*: $E x \rightarrow E(\text{cod } x)$ <proof>

end

4.2.1 AxiomsSet2 entails AxiomsSet1

context *AxiomsSet2*

begin

lemma $S_i: E(x \cdot y) \rightarrow (E x \wedge E y)$ <proof>

lemma $E_i: E(x \cdot y) \leftarrow (E x \wedge E y \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$ <proof>

```

lemma  $A_i$ :  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$   $\langle$ proof $\rangle$ 
lemma  $C_i$ :  $\forall y. \exists i. ID\ i \wedge i \cdot y \cong y$   $\langle$ proof $\rangle$ 
lemma  $D_i$ :  $\forall x. \exists j. ID\ j \wedge x \cdot j \cong x$   $\langle$ proof $\rangle$ 
end

```

4.2.2 AxiomsSet1 entails AxiomsSet2 (by semantic means)

By semantic means (Skolemization).

4.3 AxiomsSet3

In AxiomsSet3 the existence axiom E_{ii} from AxiomsSet2 is simplified by taking advantage of the two new Skolem functions dom and cod .

The left-to-right direction of existence axiom E_{iii} is implied.

```

locale AxiomsSet3 =
  assumes
     $S_{iii}$ :  $(E(x \cdot y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$  and
     $E_{iii}$ :  $E(x \cdot y) \leftarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$  and
     $A_{iii}$ :  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$  and
     $C_{iii}$ :  $E\ y \rightarrow (ID(cod\ y) \wedge (cod\ y) \cdot y \cong y)$  and
     $D_{iii}$ :  $E\ x \rightarrow (ID(dom\ x) \wedge x \cdot (dom\ x) \cong x)$ 
  begin
    lemma True nitpick [satisfy]  $\langle$ proof $\rangle$ 
    lemma assumes  $\exists x. \neg(E\ x)$  shows True nitpick [satisfy]  $\langle$ proof $\rangle$ 
    lemma assumes  $(\exists x. \neg(E\ x)) \wedge (\exists x. (E\ x))$  shows True nitpick [satisfy]  $\langle$ proof $\rangle$ 

    lemma  $E_{iii}Impl$ :  $E(x \cdot y) \rightarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$   $\langle$ proof $\rangle$ 
  end

```

4.3.1 AxiomsSet3 entails AxiomsSet2

```

context AxiomsSet3
begin
  lemma  $S_{ii}$ :  $(E(x \cdot y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$   $\langle$ proof $\rangle$ 
  lemma  $E_{ii}$ :  $E(x \cdot y) \leftarrow (E\ x \wedge E\ y \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$   $\langle$ proof $\rangle$ 
  lemma  $A_{ii}$ :  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$   $\langle$ proof $\rangle$ 
  lemma  $C_{ii}$ :  $E\ y \rightarrow (ID(cod\ y) \wedge (cod\ y) \cdot y \cong y)$   $\langle$ proof $\rangle$ 
  lemma  $D_{ii}$ :  $E\ x \rightarrow (ID(dom\ x) \wedge x \cdot (dom\ x) \cong x)$   $\langle$ proof $\rangle$ 
end

```

4.3.2 AxiomsSet2 entails AxiomsSet3

```

context AxiomsSet2
begin
  lemma  $S_{iii}$ :  $(E(x \cdot y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$   $\langle$ proof $\rangle$ 
  lemma  $E_{iii}$ :  $E(x \cdot y) \leftarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$   $\langle$ proof $\rangle$ 
  lemma  $A_{iii}$ :  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$   $\langle$ proof $\rangle$ 
  lemma  $C_{iii}$ :  $E\ y \rightarrow (ID(cod\ y) \wedge (cod\ y) \cdot y \cong y)$   $\langle$ proof $\rangle$ 
  lemma  $D_{iii}$ :  $E\ x \rightarrow (ID(dom\ x) \wedge x \cdot (dom\ x) \cong x)$   $\langle$ proof $\rangle$ 
end

```

4.4 The Axioms Set AxiomsSet4

AxiomsSet4 simplifies the axioms C_{iii} and D_{iii} . However, as it turned out, these simplifications also require the existence axiom E_{iii} to be strengthened into an equivalence.

```

locale AxiomsSet4 =
  assumes
     $S_{iv}$ :  $(E(x \cdot y) \rightarrow (E\ x \wedge E\ y)) \wedge (E(dom\ x) \rightarrow E\ x) \wedge (E(cod\ y) \rightarrow E\ y)$  and

```

```

Eiv:  $E(x \cdot y) \leftrightarrow (\text{dom } x \cong \text{cod } y \wedge E(\text{cod } y))$  and
Aiv:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$  and
Civ:  $(\text{cod } y) \cdot y \cong y$  and
Div:  $x \cdot (\text{dom } x) \cong x$ 
begin
  lemma True nitpick [satisfy] ⟨proof⟩
  lemma assumes  $\exists x. \neg(E x)$  shows True nitpick [satisfy] ⟨proof⟩
  lemma assumes  $(\exists x. \neg(E x)) \wedge (\exists x. (E x))$  shows True nitpick [satisfy] ⟨proof⟩
end

```

4.4.1 AxiomsSet4 entails AxiomsSet3

```

context AxiomsSet4
begin
  lemma Siii:  $(E(x \cdot y) \rightarrow (E x \wedge E y)) \wedge (E(\text{dom } x) \rightarrow E x) \wedge (E(\text{cod } y) \rightarrow E y)$  ⟨proof⟩
  lemma Eiii:  $E(x \cdot y) \leftarrow (\text{dom } x \cong \text{cod } y \wedge (E(\text{cod } y)))$  ⟨proof⟩
  lemma Aiii:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$  ⟨proof⟩
  lemma Ciii:  $E y \rightarrow (ID(\text{cod } y) \wedge (\text{cod } y) \cdot y \cong y)$  ⟨proof⟩
  lemma Diii:  $E x \rightarrow (ID(\text{dom } x) \wedge x \cdot (\text{dom } x) \cong x)$  ⟨proof⟩
end

```

4.4.2 AxiomsSet3 entails AxiomsSet4

```

context AxiomsSet3
begin
  lemma Siv:  $(E(x \cdot y) \rightarrow (E x \wedge E y)) \wedge (E(\text{dom } x) \rightarrow E x) \wedge (E(\text{cod } y) \rightarrow E y)$  ⟨proof⟩
  lemma Eiv:  $E(x \cdot y) \leftrightarrow (\text{dom } x \cong \text{cod } y \wedge E(\text{cod } y))$  ⟨proof⟩
  lemma Aiv:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$  ⟨proof⟩
  lemma Civ:  $(\text{cod } y) \cdot y \cong y$  ⟨proof⟩
  lemma Div:  $x \cdot (\text{dom } x) \cong x$  ⟨proof⟩
end

```

4.5 AxiomsSet5

AxiomsSet5 has been proposed by Scott [6] in the 1970s. This set of axioms is equivalent to the axioms set presented by Freyd and Scedrov in their textbook “Categories, Allegories” [4] when encoded in free logic, corrected/adapted and further simplified, see Section 5.

```

locale AxiomsSet5 =
assumes
  S1:  $E(\text{dom } x) \rightarrow E x$  and
  S2:  $E(\text{cod } y) \rightarrow E y$  and
  S3:  $E(x \cdot y) \leftrightarrow \text{dom } x \simeq \text{cod } y$  and
  S4:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$  and
  S5:  $(\text{cod } y) \cdot y \cong y$  and
  S6:  $x \cdot (\text{dom } x) \cong x$ 
begin
  lemma True nitpick [satisfy] ⟨proof⟩
  lemma assumes  $\exists x. \neg(E x)$  shows True nitpick [satisfy] ⟨proof⟩
  lemma assumes  $(\exists x. \neg(E x)) \wedge (\exists x. (E x))$  shows True nitpick [satisfy] ⟨proof⟩
end

```

4.5.1 AxiomsSet5 entails AxiomsSet4

```

context AxiomsSet5
begin
  lemma Siv:  $(E(x \cdot y) \rightarrow (E x \wedge E y)) \wedge (E(\text{dom } x) \rightarrow E x) \wedge (E(\text{cod } y) \rightarrow E y)$  ⟨proof⟩
  lemma Eiv:  $E(x \cdot y) \leftrightarrow (\text{dom } x \cong \text{cod } y \wedge E(\text{cod } y))$  ⟨proof⟩
  lemma Aiv:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$  ⟨proof⟩
  lemma Civ:  $(\text{cod } y) \cdot y \cong y$  ⟨proof⟩
  lemma Div:  $x \cdot (\text{dom } x) \cong x$  ⟨proof⟩

```

end

4.5.2 AxiomsSet4 entails AxiomsSet5

```
context AxiomsSet4
begin
  lemma S1:  $E(\text{dom } x) \rightarrow E x$   $\langle$ proof $\rangle$ 
  lemma S2:  $E(\text{cod } y) \rightarrow E y$   $\langle$ proof $\rangle$ 
  lemma S3:  $E(x \cdot y) \leftrightarrow \text{dom } x \simeq \text{cod } y$   $\langle$ proof $\rangle$ 
  lemma S4:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$   $\langle$ proof $\rangle$ 
  lemma S5:  $(\text{cod } y) \cdot y \cong y$   $\langle$ proof $\rangle$ 
  lemma S6:  $x \cdot (\text{dom } x) \cong x$   $\langle$ proof $\rangle$ 
end
```

5 The Axioms Sets by Freyd and Scedrov [4]

5.1 AxiomsSet6

The axioms by Freyd and Scedrov [4] in our notation, when being corrected (cf. the modification in axiom A1).

Freyd and Scedrov employ a different notation for $\text{dom } x$ and $\text{cod } x$. They denote these operations by $\square x$ and $x \square$. Moreover, they employ diagrammatic composition instead of the set-theoretic definition (functional composition from right to left) used so far. We leave it to the reader to verify that their axioms corresponds to the axioms presented here modulo an appropriate conversion of notation.

```
locale AxiomsSet6 =
  assumes
    A1:  $E(x \cdot y) \leftrightarrow \text{dom } x \simeq \text{cod } y$  and
    A2a:  $\text{cod}(\text{dom } x) \cong \text{dom } x$  and
    A2b:  $\text{dom}(\text{cod } y) \cong \text{cod } y$  and
    A3a:  $x \cdot (\text{dom } x) \cong x$  and
    A3b:  $(\text{cod } y) \cdot y \cong y$  and
    A4a:  $\text{dom}(x \cdot y) \cong \text{dom}((\text{dom } x) \cdot y)$  and
    A4b:  $\text{cod}(x \cdot y) \cong \text{cod}(x \cdot (\text{cod } y))$  and
    A5:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$ 
  begin
    lemma True nitpick [satisfy]  $\langle$ proof $\rangle$ 
    lemma assumes  $\exists x. \neg(E x)$  shows True nitpick [satisfy]  $\langle$ proof $\rangle$ 
    lemma assumes  $(\exists x. \neg(E x)) \wedge (\exists x. (E x))$  shows True nitpick [satisfy]  $\langle$ proof $\rangle$ 
  end
```

5.1.1 AxiomsSet6 entails AxiomsSet5

```
context AxiomsSet6
begin
  lemma S1:  $E(\text{dom } x) \rightarrow E x$   $\langle$ proof $\rangle$ 
  lemma S2:  $E(\text{cod } y) \rightarrow E y$   $\langle$ proof $\rangle$ 
  lemma S3:  $E(x \cdot y) \leftrightarrow \text{dom } x \simeq \text{cod } y$   $\langle$ proof $\rangle$ 
  lemma S4:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$   $\langle$ proof $\rangle$ 
  lemma S5:  $(\text{cod } y) \cdot y \cong y$   $\langle$ proof $\rangle$ 
  lemma S6:  $x \cdot (\text{dom } x) \cong x$   $\langle$ proof $\rangle$ 

  lemma A4aRedundant:  $\text{dom}(x \cdot y) \cong \text{dom}((\text{dom } x) \cdot y)$   $\langle$ proof $\rangle$ 
  lemma A4bRedundant:  $\text{cod}(x \cdot y) \cong \text{cod}(x \cdot (\text{cod } y))$   $\langle$ proof $\rangle$ 
  lemma A2aRedundant:  $\text{cod}(\text{dom } x) \cong \text{dom } x$   $\langle$ proof $\rangle$ 
  lemma A2bRedundant:  $\text{dom}(\text{cod } y) \cong \text{cod } y$   $\langle$ proof $\rangle$ 
end
```

5.1.2 AxiomsSet5 entails AxiomsSet6

```

context AxiomsSet5
begin
lemma A1:  $E(x \cdot y) \leftrightarrow \text{dom } x \cong \text{cod } y$  <proof>
lemma A2:  $\text{cod}(\text{dom } x) \cong \text{dom } x$  <proof>
lemma A2b:  $\text{dom}(\text{cod } y) \cong \text{cod } y$  <proof>
lemma A3a:  $x \cdot (\text{dom } x) \cong x$  <proof>
lemma A3b:  $(\text{cod } y) \cdot y \cong y$  <proof>
lemma A4a:  $\text{dom}(x \cdot y) \cong \text{dom}((\text{dom } x) \cdot y)$  <proof>
lemma A4b:  $\text{cod}(x \cdot y) \cong \text{cod}(x \cdot (\text{cod } y))$  <proof>
lemma A5:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$  <proof>
end

```

5.2 AxiomsSet7 (technically flawed)

The axioms by Freyd and Scedrov in our notation, without the suggested correction of axiom A1. This axioms set is technically flawed when encoded in our given context. It leads to a constricted inconsistency.

```

locale AxiomsSet7 =
assumes
  A1:  $E(x \cdot y) \leftrightarrow \text{dom } x \cong \text{cod } y$  and
  A2a:  $\text{cod}(\text{dom } x) \cong \text{dom } x$  and
  A2b:  $\text{dom}(\text{cod } y) \cong \text{cod } y$  and
  A3a:  $x \cdot (\text{dom } x) \cong x$  and
  A3b:  $(\text{cod } y) \cdot y \cong y$  and
  A4a:  $\text{dom}(x \cdot y) \cong \text{dom}((\text{dom } x) \cdot y)$  and
  A4b:  $\text{cod}(x \cdot y) \cong \text{cod}(x \cdot (\text{cod } y))$  and
  A5:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$ 
begin
lemma True nitpick [satisfy] <proof>

lemma InconsistencyAutomatic:  $(\exists x. \neg(E x)) \rightarrow \text{False}$  <proof>
lemma  $\forall x. E x$  <proof>

lemma InconsistencyInteractive:
  assumes  $NEx: \exists x. \neg(E x)$  shows False
  <proof>
end

```

5.3 AxiomsSet7orig (technically flawed)

The axioms by Freyd and Scedrov in their original notation, without the suggested correction of axiom A1.

We present the constricted inconsistency argument from above once again, but this time in the original notation of Freyd and Scedrov.

```

locale AxiomsSet7orig =
fixes
  source::  $i \Rightarrow i$  ( $\langle \square \rightarrow$  [108] 109) and
  target::  $i \Rightarrow i$  ( $\langle \leftarrow \square \rangle$  [110] 111) and
  compositionF::  $i \Rightarrow i \Rightarrow i$  (infix  $\langle \cdot \rangle$  110)
assumes
  A1:  $E(x \cdot y) \leftrightarrow (x \square \cong \square y)$  and
  A2a:  $((\square x) \square) \cong \square x$  and
  A2b:  $\square(x \square) \cong \square x$  and
  A3a:  $(\square x) \cdot x \cong x$  and
  A3b:  $x \cdot (x \square) \cong x$  and
  A4a:  $\square(x \cdot y) \cong \square(x \cdot (\square y))$  and
  A4b:  $(x \cdot y) \square \cong ((x \square) \cdot y) \square$  and

```

```

A5:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$ 
begin
lemma True nitpick [satisfy] ⟨proof⟩

lemma InconsistencyAutomatic:  $(\exists x. \neg(E x)) \rightarrow \text{False}$  ⟨proof⟩
lemma  $\forall x. E x$  ⟨proof⟩

lemma InconsistencyInteractive:
  assumes NE $x$ :  $\exists x. \neg(E x)$  shows False
  ⟨proof⟩
end

```

5.4 AxiomsSet8 (algebraic reading, still technically flawed)

The axioms by Freyd and Scedrov in our notation again, but this time we adopt an algebraic reading of the free variables, meaning that they range over existing morphisms only.

```

locale AxiomsSet8 =
  assumes
    B1:  $\forall x. \forall y. E(x \cdot y) \leftrightarrow \text{dom } x \cong \text{cod } y$  and
    B2a:  $\forall x. \text{cod}(\text{dom } x) \cong \text{dom } x$  and
    B2b:  $\forall y. \text{dom}(\text{cod } y) \cong \text{cod } y$  and
    B3a:  $\forall x. x \cdot (\text{dom } x) \cong x$  and
    B3b:  $\forall y. (\text{cod } y) \cdot y \cong y$  and
    B4a:  $\forall x. \forall y. \text{dom}(x \cdot y) \cong \text{dom}((\text{dom } x) \cdot y)$  and
    B4b:  $\forall x. \forall y. \text{cod}(x \cdot y) \cong \text{cod}(x \cdot (\text{cod } y))$  and
    B5:  $\forall x. \forall y. \forall z. x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$ 
  begin
    lemma True nitpick [satisfy] ⟨proof⟩
    lemma assumes  $\exists x. \neg(E x)$  shows True nitpick [satisfy] ⟨proof⟩
    lemma assumes  $(\exists x. \neg(E x)) \wedge (\exists x. (E x))$  shows True nitpick [satisfy] ⟨proof⟩
  end

```

None of the axioms in AxiomsSet5 are implied.

```

context AxiomsSet8
begin
  lemma S1:  $E(\text{dom } x) \rightarrow E x$  nitpick ⟨proof⟩
  lemma S2:  $E(\text{cod } y) \rightarrow E y$  nitpick ⟨proof⟩
  lemma S3:  $E(x \cdot y) \leftrightarrow \text{dom } x \simeq \text{cod } y$  nitpick ⟨proof⟩
  lemma S4:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$  nitpick ⟨proof⟩
  lemma S5:  $(\text{cod } y) \cdot y \cong y$  nitpick ⟨proof⟩
  lemma S6:  $x \cdot (\text{dom } x) \cong x$  nitpick ⟨proof⟩
end

```

5.5 AxiomsSet8Strict (algebraic reading)

The situation changes when strictness conditions are postulated. Note that in the algebraic framework of Freyd and Scedrov such conditions have to be assumed as given in the logic, while here we can explicitly encode them as axioms.

```

locale AxiomsSet8Strict = AxiomsSet8 +
  assumes
    B0a:  $E(x \cdot y) \rightarrow (E x \wedge E y)$  and
    B0b:  $E(\text{dom } x) \rightarrow E x$  and
    B0c:  $E(\text{cod } x) \rightarrow E x$ 
  begin
    lemma True nitpick [satisfy] ⟨proof⟩
    lemma assumes  $\exists x. \neg(E x)$  shows True nitpick [satisfy] ⟨proof⟩
    lemma assumes  $(\exists x. \neg(E x)) \wedge (\exists x. (E x))$  shows True nitpick [satisfy] ⟨proof⟩
  end

```


5.5.1 AxiomsSet8Strict entails AxiomsSet5

```

context AxiomsSet8Strict
begin
  lemma S1:  $E(\text{dom } x) \rightarrow E x$  <proof>
  lemma S2:  $E(\text{cod } y) \rightarrow E y$  <proof>
  lemma S3:  $E(x \cdot y) \leftrightarrow \text{dom } x \cong \text{cod } y$  <proof>
  lemma S4:  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$  <proof>
  lemma S5:  $(\text{cod } y) \cdot y \cong y$  <proof>
  lemma S6:  $x \cdot (\text{dom } x) \cong x$  <proof>
end

```

5.5.2 AxiomsSet5 entails AxiomsSet8Strict

```

context AxiomsSet5
begin
  lemma B0a:  $E(x \cdot y) \rightarrow (E x \wedge E y)$  <proof>
  lemma B0b:  $E(\text{dom } x) \rightarrow E x$  <proof>
  lemma B0c:  $E(\text{cod } x) \rightarrow E x$  <proof>
  lemma B1:  $\forall x. \forall y. E(x \cdot y) \leftrightarrow \text{dom } x \cong \text{cod } y$  <proof>
  lemma B2a:  $\forall x. \text{cod}(\text{dom } x) \cong \text{dom } x$  <proof>
  lemma B2b:  $\forall y. \text{dom}(\text{cod } y) \cong \text{cod } y$  <proof>
  lemma B3a:  $\forall x. x \cdot (\text{dom } x) \cong x$  <proof>
  lemma B3b:  $\forall y. (\text{cod } y) \cdot y \cong y$  <proof>
  lemma B4a:  $\forall x. \forall y. \text{dom}(x \cdot y) \cong \text{dom}((\text{dom } x) \cdot y)$  <proof>
  lemma B4b:  $\forall x. \forall y. \text{cod}(x \cdot y) \cong \text{cod}(x \cdot (\text{cod } y))$  <proof>
  lemma B5:  $\forall x. \forall y. \forall z. x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$  <proof>
end

```

5.5.3 AxiomsSet8Strict is Redundant

AxiomsSet8Strict is redundant: either the B2-axioms can be omitted or the B4-axioms.

```

context AxiomsSet8Strict
begin
  lemma B2aRedundant:  $\forall x. \text{cod}(\text{dom } x) \cong \text{dom } x$  <proof>
  lemma B2bRedundant:  $\forall y. \text{dom}(\text{cod } y) \cong \text{cod } y$  <proof>
  lemma B4aRedundant:  $\forall x. \forall y. \text{dom}(x \cdot y) \cong \text{dom}((\text{dom } x) \cdot y)$  <proof>
  lemma B4bRedundant:  $\forall x. \forall y. \text{cod}(x \cdot y) \cong \text{cod}(x \cdot (\text{cod } y))$  <proof>
end

```

6 The Axioms Sets of Mac Lane [5]

We analyse the axioms set suggested by Mac Lane [5] already in 1948. As for the theory by Freyd and Scedrov above, which was developed much later, we need to assume strictness of composition to show equivalence to our previous axiom sets. Note that his complicated conditions on existence of compositions proved to be unnecessary, as we show. It shows it is hard to think about partial operations.

```

locale AxiomsSetMcL =
  assumes
    C0 :  $E(x \cdot y) \rightarrow (E x \wedge E y)$  and
    C1 :  $\forall \gamma \beta \alpha. (E(\gamma \cdot \beta) \wedge E((\gamma \cdot \beta) \cdot \alpha)) \rightarrow E(\beta \cdot \alpha)$  and
    C1' :  $\forall \gamma \beta \alpha. (E(\beta \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha))) \rightarrow E(\gamma \cdot \beta)$  and
    C2 :  $\forall \gamma \beta \alpha. (E(\gamma \cdot \beta) \wedge E(\beta \cdot \alpha)) \rightarrow (E((\gamma \cdot \beta) \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha)) \wedge ((\gamma \cdot \beta) \cdot \alpha) = (\gamma \cdot (\beta \cdot \alpha)))$  and
    C3 :  $\forall \gamma. \exists eD. \text{IDMcL}(eD) \wedge E(\gamma \cdot eD)$  and
    C4 :  $\forall \gamma. \exists eR. \text{IDMcL}(eR) \wedge E(eR \cdot \gamma)$ 
  begin
    lemma True nitpick [satisfy] <proof>
    lemma assumes  $\exists x. \neg(E x)$  shows True nitpick [satisfy] <proof>
    lemma assumes  $(\exists x. \neg(E x)) \wedge (\exists x. (E x))$  shows True nitpick [satisfy] <proof>
  end

```

end

Remember that IDMcL was defined on p. 1 and proved equivalent to ID.

6.1 AxiomsSetMcL entails AxiomsSet1

```

context AxiomsSetMcL
begin
  lemma  $S_i$ :  $E(x \cdot y) \rightarrow (E x \wedge E y)$   $\langle proof \rangle$ 
  lemma  $E_i$ :  $E(x \cdot y) \leftarrow (E x \wedge E y \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$   $\langle proof \rangle$ 
  lemma  $A_i$ :  $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$   $\langle proof \rangle$ 
  lemma  $C_i$ :  $\forall y. \exists i. ID i \wedge i \cdot y \cong y$   $\langle proof \rangle$ 
  lemma  $D_i$ :  $\forall x. \exists j. ID j \wedge x \cdot j \cong x$   $\langle proof \rangle$ 
end

```

6.2 AxiomsSet1 entails AxiomsSetMcL

```

context AxiomsSet1
begin
  lemma  $C_0$ :  $E(x \cdot y) \rightarrow (E x \wedge E y)$   $\langle proof \rangle$ 
  lemma  $C_1$ :  $\forall \gamma \beta \alpha. (E(\gamma \cdot \beta) \wedge E((\gamma \cdot \beta) \cdot \alpha)) \rightarrow E(\beta \cdot \alpha)$   $\langle proof \rangle$ 
  lemma  $C_1'$ :  $\forall \gamma \beta \alpha. (E(\beta \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha))) \rightarrow E(\gamma \cdot \beta)$   $\langle proof \rangle$ 
  lemma  $C_2$ :  $\forall \gamma \beta \alpha. (E(\gamma \cdot \beta) \wedge E(\beta \cdot \alpha)) \rightarrow (E((\gamma \cdot \beta) \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha)) \wedge ((\gamma \cdot \beta) \cdot \alpha) = (\gamma \cdot (\beta \cdot \alpha)))$   $\langle proof \rangle$ 
  lemma  $C_3$ :  $\forall \gamma. \exists eD. IDMcL(eD) \wedge E(\gamma \cdot eD)$   $\langle proof \rangle$ 
  lemma  $C_4$ :  $\forall \gamma. \exists eR. IDMcL(eR) \wedge E(eR \cdot \gamma)$   $\langle proof \rangle$ 
end

```

6.3 Skolemization of the Axioms of Mac Lane

Mac Lane employs diagrammatic composition instead of the set-theoretic definition as used in our axiom sets. As we have seen above, this is not a problem as long as composition is the only primitive. But when adding the Skolem terms *dom* and *cod* care must be taken and we should actually transform all axioms into a common form. Below we address this (in a minimal way) by using *dom* in axiom C_3 s and *cod* in axiom C_4 s, which is opposite of what Mac Lane proposed. For this axioms set we then show equivalence to AxiomsSet1/2/5.

```

locale SkolemizedAxiomsSetMcL =
  assumes
     $C_0s$ :  $(E(x \cdot y) \rightarrow (E x \wedge E y)) \wedge (E(dom x) \rightarrow E x) \wedge (E(cod y) \rightarrow E y)$  and
     $C_1s$ :  $\forall \gamma \beta \alpha. (E(\gamma \cdot \beta) \wedge E((\gamma \cdot \beta) \cdot \alpha)) \rightarrow E(\beta \cdot \alpha)$  and
     $C_1's$ :  $\forall \gamma \beta \alpha. (E(\beta \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha))) \rightarrow E(\gamma \cdot \beta)$  and
     $C_2s$ :  $\forall \gamma \beta \alpha. (E(\gamma \cdot \beta) \wedge E(\beta \cdot \alpha)) \rightarrow (E((\gamma \cdot \beta) \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha)) \wedge ((\gamma \cdot \beta) \cdot \alpha) = (\gamma \cdot (\beta \cdot \alpha)))$  and
     $C_3s$ :  $\forall \gamma. IDMcL(dom \gamma) \wedge E(\gamma \cdot (dom \gamma))$  and
     $C_4s$ :  $\forall \gamma. IDMcL(cod \gamma) \wedge E((cod \gamma) \cdot \gamma)$ 
  begin
    lemma True nitpick [satisfy]  $\langle proof \rangle$ 
    lemma assumes  $\exists x. \neg(E x)$  shows True nitpick [satisfy]  $\langle proof \rangle$ 
    lemma assumes  $(\exists x. \neg(E x)) \wedge (\exists x. (E x))$  shows True nitpick [satisfy]  $\langle proof \rangle$ 
  end

```

6.4 SkolemizedAxiomsSetMcL entails AxiomsSetMcL and AxiomsSet1-5

```

context SkolemizedAxiomsSetMcL
begin
  lemma  $C_0$ :  $E(x \cdot y) \rightarrow (E x \wedge E y)$   $\langle proof \rangle$ 
  lemma  $C_1$ :  $\forall \gamma \beta \alpha. (E(\gamma \cdot \beta) \wedge E((\gamma \cdot \beta) \cdot \alpha)) \rightarrow E(\beta \cdot \alpha)$   $\langle proof \rangle$ 
  lemma  $C_1'$ :  $\forall \gamma \beta \alpha. (E(\beta \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha))) \rightarrow E(\gamma \cdot \beta)$   $\langle proof \rangle$ 
  lemma  $C_2$ :  $\forall \gamma \beta \alpha. (E(\gamma \cdot \beta) \wedge E(\beta \cdot \alpha)) \rightarrow (E((\gamma \cdot \beta) \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha)) \wedge ((\gamma \cdot \beta) \cdot \alpha) = (\gamma \cdot (\beta \cdot \alpha)))$   $\langle proof \rangle$ 
  lemma  $C_3$ :  $\forall \gamma. \exists eD. IDMcL(eD) \wedge E(\gamma \cdot eD)$   $\langle proof \rangle$ 
  lemma  $C_4$ :  $\forall \gamma. \exists eR. IDMcL(eR) \wedge E(eR \cdot \gamma)$   $\langle proof \rangle$ 
end

```

lemma $S_i: E(x \cdot y) \rightarrow (E x \wedge E y)$ *<proof>*
lemma $E_i: E(x \cdot y) \leftarrow (E x \wedge E y \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$ *<proof>*
lemma $A_i: x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$ *<proof>*
lemma $C_i: \forall y. \exists i. ID i \wedge i \cdot y \cong y$ *<proof>*
lemma $D_i: \forall x. \exists j. ID j \wedge x \cdot j \cong x$ *<proof>*

lemma $S_{ii}: (E(x \cdot y) \rightarrow (E x \wedge E y)) \wedge (E(dom x) \rightarrow E x) \wedge (E(cod y) \rightarrow E y)$ *<proof>*
lemma $E_{ii}: E(x \cdot y) \leftarrow (E x \wedge E y \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$ *<proof>*
lemma $A_{ii}: x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$ *<proof>*
lemma $C_{ii}: E y \rightarrow (ID(cod y) \wedge (cod y) \cdot y \cong y)$ *<proof>*
lemma $D_{ii}: E x \rightarrow (ID(dom x) \wedge x \cdot (dom x) \cong x)$ *<proof>*

lemma $S1: E(dom x) \rightarrow E x$ *<proof>*
lemma $S2: E(cod y) \rightarrow E y$ *<proof>*
lemma $S3: E(x \cdot y) \leftrightarrow dom x \simeq cod y$ *<proof>*
lemma $S4: x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$ *<proof>*
lemma $S5: (cod y) \cdot y \cong y$ *<proof>*
lemma $S6: x \cdot (dom x) \cong x$ *<proof>*
end

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