

Automated Stateful Protocol Verification

Andreas V. Hess*
Achim D. Brucker†

Sebastian Mödersheim*
Anders Schlichtkrull

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*DTU Compute, Technical University of Denmark, Lyngby, Denmark
`{avhe, samo, andschl}@dtu.dk`

† Department of Computer Science, University of Exeter, Exeter, UK
`a.brucker@exeter.ac.uk`

Abstract

In protocol verification we observe a wide spectrum from fully automated methods to interactive theorem proving with proof assistants like Isabelle/HOL. In this AFP entry, we present a fully-automated approach for verifying stateful security protocols, i.e., protocols with mutable state that may span several sessions. The approach supports reachability goals like secrecy and authentication. We also include a simple user-friendly transaction-based protocol specification language that is embedded into Isabelle.

Keywords: Fully automated verification, stateful security protocols

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1 Introduction

In protocol verification we observe a wide spectrum from fully automated methods to interactive theorem proving with proof assistants like Isabelle/HOL. The latter provide overwhelmingly high assurance of the correctness, which automated methods often cannot: due to their complexity, bugs in such automated verification tools are likely and thus the risk of erroneously verifying a flawed protocol is non-negligible. There are a few works that try to combine advantages from both ends of the spectrum: a high degree of automation and assurance.

Inspired by [1], we present here a first step towards achieving this for a more challenging class of protocols, namely those that work with a mutable long-term state. To our knowledge this is the first approach that achieves fully automated verification of stateful protocols in an LCF-style theorem prover. The approach also includes a simple user-friendly transaction-based protocol specification language embedded into Isabelle, and can also leverage a number of existing results such as soundness of a typed model (see, e.g., [2–4]) and compositionality (see, e.g., [2, 5]). The Isabelle formalization extends the AFP entry on stateful protocol composition and typing [6].

The rest of this document is automatically generated from the formalization in Isabelle/HOL, i.e., all content is checked by Isabelle. Overall, the structure of this document follows the theory dependencies (see Figure 1.1): We start with the formal framework for verifying stateful security protocols (chapter 2). We continue with the setup for supporting the high-level protocol specifications language for security protocols (the Trac format) and the implementation of the fully automated proof tactics (chapter 3). Finally, we present examples (chapter 4).

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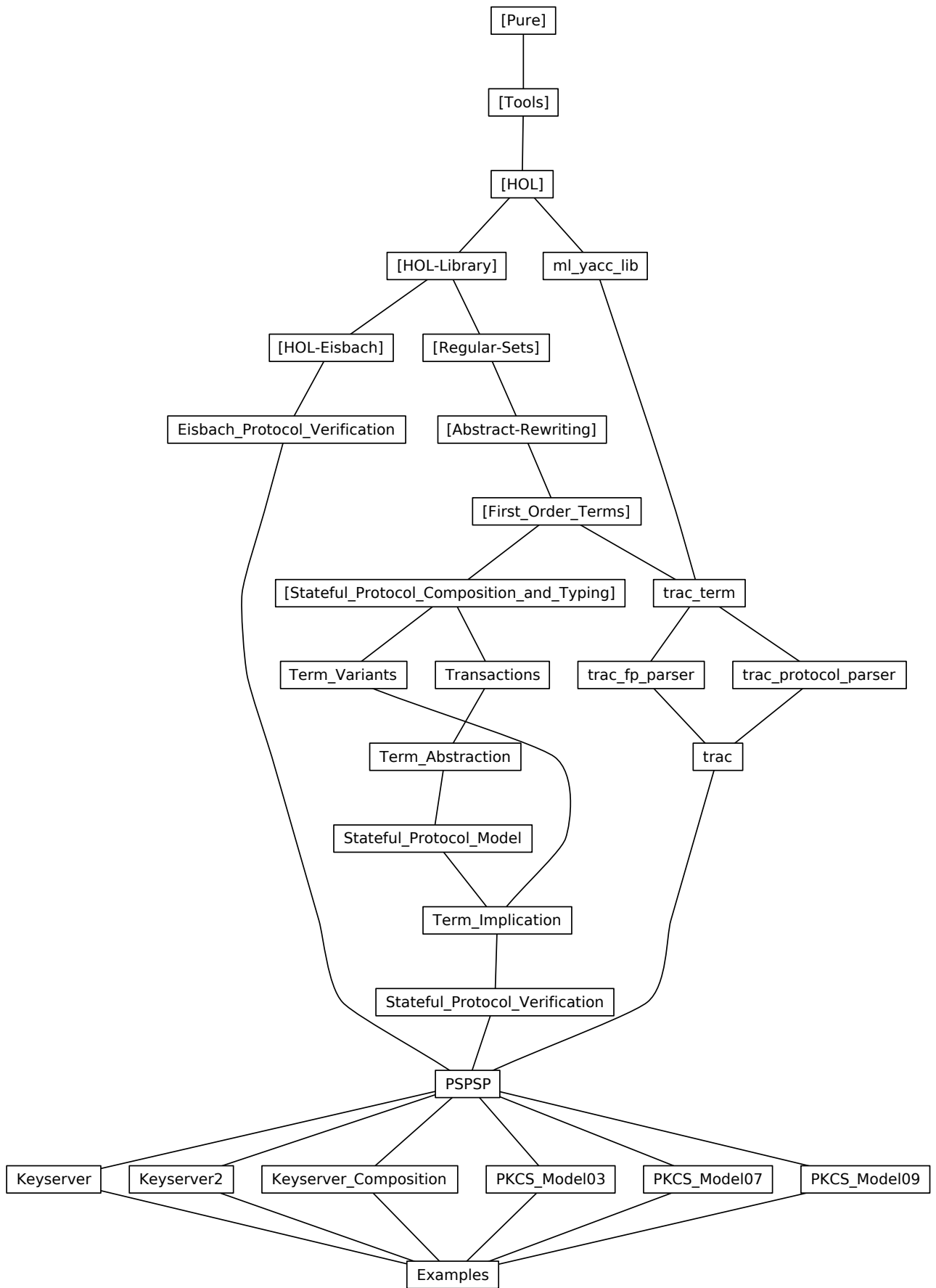


Figure 1.1: The Dependency Graph of the Isabelle Theories.

2 Stateful Protocol Verification

2.1 Protocol Transactions (Transactions)

```
theory Transactions
  imports
    Stateful_Protocol_Composition_and_Typing.Typed_Model
    Stateful_Protocol_Composition_and_Typing.Labeled_Stateful_Strands
begin
```

2.1.1 Definitions

```
datatype 'b prot_atom =
  is_Atom: Atom 'b
| Value
| SetType
| AttackType
| Bottom
| OccursSecType

datatype ('a,'b,'c) prot_fun =
  Fu (the_Fu: 'a)
| Set (the_Set: 'c)
| Val (the_Val: "nat × bool")
| Abs (the_Abs: "'c set")
| Pair
| Attack nat
| PubConstAtom 'b nat
| PubConstSetType nat
| PubConstAttackType nat
| PubConstBottom nat
| PubConstOccursSecType nat
| OccursFact
| OccursSec

definition "is_Fun_Set t ≡ is_Fun t ∧ args t = [] ∧ is_Set (the_Fun t)"

abbreviation occurs where
  "occurs t ≡ Fun OccursFact [Fun OccursSec [], t]"

type_synonym ('a,'b,'c) prot_term_type = "'((('a,'b,'c) prot_fun,'b prot_atom) term_type)"

type_synonym ('a,'b,'c) prot_var = "'('a,'b,'c) prot_term_type × nat"

type_synonym ('a,'b,'c) prot_term = "'((('a,'b,'c) prot_fun,('a,'b,'c) prot_var) term)"
type_synonym ('a,'b,'c) prot_terms = "'('a,'b,'c) prot_term set"

type_synonym ('a,'b,'c) prot_subst = "'((('a,'b,'c) prot_fun, ('a,'b,'c) prot_var) subst)"

type_synonym ('a,'b,'c,'d) prot_strand_step =
  "'((('a,'b,'c) prot_fun, ('a,'b,'c) prot_var, 'd) labeled_stateful_strand_step)"
type_synonym ('a,'b,'c,'d) prot_strand = "'('a,'b,'c,'d) prot_strand_step list"
type_synonym ('a,'b,'c,'d) prot_constr = "'('a,'b,'c,'d) prot_strand_step list"

datatype ('a,'b,'c,'d) prot_transaction =
  Transaction
  (transaction_fresh: "'('a,'b,'c) prot_var list")
```

```

(transaction_receive: "('a,'b,'c,'d) prot_strand")
(transaction_selects: "('a,'b,'c,'d) prot_strand")
(transaction_checks: "('a,'b,'c,'d) prot_strand")
(transaction_updates: "('a,'b,'c,'d) prot_strand")
(transaction_send:    "('a,'b,'c,'d) prot_strand")

```

definition transaction_strand where

```

"transaction_strand T ≡
  transaction_receive T@transaction_selects T@transaction_checks T@
  transaction_updates T@transaction_send T"

```

fun transaction_proj where

```

"transaction_proj l (Transaction A B C D E F) = (
  let f = proj l
  in Transaction A (f B) (f C) (f D) (f E) (f F))"

```

fun transaction_star_proj where

```

"transaction_star_proj (Transaction A B C D E F) = (
  let f = filter is_LabelS
  in Transaction A (f B) (f C) (f D) (f E) (f F))"

```

abbreviation fv_transaction where

```

"fv_transaction T ≡ fvlsst (transaction_strand T)"

```

abbreviation bvars_transaction where

```

"bvars_transaction T ≡ bvarslsst (transaction_strand T)"

```

abbreviation vars_transaction where

```

"vars_transaction T ≡ varslsst (transaction_strand T)"

```

abbreviation trms_transaction where

```

"trms_transaction T ≡ trmslsst (transaction_strand T)"

```

abbreviation setops_transaction where

```

"setops_transaction T ≡ setopssst (unlabel (transaction_strand T))"

```

definition wellformed_transaction where

```

"wellformed_transaction T ≡
  list_all is_Receive (unlabel (transaction_receive T)) ∧
  list_all is_Assignment (unlabel (transaction_selects T)) ∧
  list_all is_Check (unlabel (transaction_checks T)) ∧
  list_all is_Update (unlabel (transaction_updates T)) ∧
  list_all is_Send (unlabel (transaction_send T)) ∧
  set (transaction_fresh T) ⊆ fvlsst (transaction_updates T) ∪ fvlsst (transaction_send T) ∧
  set (transaction_fresh T) ∩ fvlsst (transaction_receive T) = {} ∧
  set (transaction_fresh T) ∩ fvlsst (transaction_selects T) = {} ∧
  fv_transaction T ∩ bvars_transaction T = {} ∧
  fvlsst (transaction_checks T) ⊆ fvlsst (transaction_receive T) ∪ fvlsst (transaction_selects T) ∧
  fvlsst (transaction_updates T) ∪ fvlsst (transaction_send T) - set (transaction_fresh T)
  ⊆ fvlsst (transaction_receive T) ∪ fvlsst (transaction_selects T) ∧
  (∀ x ∈ set (unlabel (transaction_selects T)).
    is_Equality x → fv (the_rhs x) ⊆ fvlsst (transaction_receive T))"

```

type_synonym ('a,'b,'c,'d) prot = "('a,'b,'c,'d) prot_transaction list"

abbreviation Var_Value_term ("_v") where

```

"<n>v ≡ Var (Var Value, n)::('a,'b,'c) prot_term"
```

abbreviation Fun_Fu_term ("_t") where

```

"<f T>t ≡ Fun (Fu f) T::('a,'b,'c) prot_term"
```

abbreviation Fun_Fu_const_term ("_c") where

```

"<c>c ≡ Fun (Fu c) []::('a,'b,'c) prot_term"
```

abbreviation *Fun_Set_const_term* (" $\langle _ \rangle_s$ ") **where**
 " $\langle f \rangle_s \equiv \text{Fun } (\text{Set } f) [] :: ('a, 'b, 'c) \text{ prot_term}$ "

abbreviation *Fun_Abs_const_term* (" $\langle _ \rangle_a$ ") **where**
 " $\langle a \rangle_a \equiv \text{Fun } (\text{Abs } a) [] :: ('a, 'b, 'c) \text{ prot_term}$ "

abbreviation *Fun_Attack_const_term* ("attack $\langle _ \rangle$ ") **where**
 "attack $\langle n \rangle \equiv \text{Fun } (\text{Attack } n) [] :: ('a, 'b, 'c) \text{ prot_term}$ "

abbreviation *prot_transaction1* ("transaction₁ _ _ new _ _") **where**
 "transaction₁ (S1::('a, 'b, 'c, 'd) prot_strand) S2 new (B::('a, 'b, 'c) prot_term list) S3 S4
 $\equiv \text{Transaction } (\text{map the_Var } B) S1 [] S2 S3 S4$ "

abbreviation *prot_transaction2* ("transaction₂ _ _ _") **where**
 "transaction₂ (S1::('a, 'b, 'c, 'd) prot_strand) S2 S3 S4
 $\equiv \text{Transaction } [] S1 [] S2 S3 S4$ "

2.1.2 Lemmata

lemma *prot_atom_UNIV*:

"(UNIV::'b prot_atom set) = range Atom \cup {Value, SetType, AttackType, Bottom, OccursSecType}"
 $\langle \text{proof} \rangle$

instance *prot_atom::(finite) finite*
 $\langle \text{proof} \rangle$

instantiation *prot_atom::(enum) enum*

begin

definition "enum_prot_atom == map Atom enum_class.enum@[Value, SetType, AttackType, Bottom, OccursSecType]"

definition "enum_all_prot_atom P == list_all P (map Atom enum_class.enum@[Value, SetType, AttackType, Bottom, OccursSecType])"

definition "enum_ex_prot_atom P == list_ex P (map Atom enum_class.enum@[Value, SetType, AttackType, Bottom, OccursSecType])"

instance

$\langle \text{proof} \rangle$

end

lemma *wellformed_transaction_cases*:

assumes "wellformed_transaction T"

shows

"(l,x) \in set (transaction_receive T) $\implies \exists t. x = \text{receive}\langle t \rangle$ " (is "?A \implies ?A'")

"(l,x) \in set (transaction_selects T) \implies

($\exists t s. x = \langle t := s \rangle$) \vee ($\exists t s. x = \text{select}\langle t, s \rangle$)" (is "?B \implies ?B'")

"(l,x) \in set (transaction_checks T) \implies

($\exists t s. x = \langle t == s \rangle$) \vee ($\exists t s. x = \langle t \text{ in } s \rangle$) \vee ($\exists X F G. x = \forall X \langle \vee \neq : F \vee \notin : G \rangle$)" (is "?C \implies

?C'")

"(l,x) \in set (transaction_updates T) \implies

($\exists t s. x = \text{insert}\langle t, s \rangle$) \vee ($\exists t s. x = \text{delete}\langle t, s \rangle$)" (is "?D \implies ?D'")

"(l,x) \in set (transaction_send T) $\implies \exists t. x = \text{send}\langle t \rangle$ " (is "?E \implies ?E'")

$\langle \text{proof} \rangle$

lemma *wellformed_transaction_unlabel_cases*:

assumes "wellformed_transaction T"

shows

"x \in set (unlabel (transaction_receive T)) $\implies \exists t. x = \text{receive}\langle t \rangle$ " (is "?A \implies ?A'")

"x \in set (unlabel (transaction_selects T)) \implies

($\exists t s. x = \langle t := s \rangle$) \vee ($\exists t s. x = \text{select}\langle t, s \rangle$)" (is "?B \implies ?B'")

"x \in set (unlabel (transaction_checks T)) \implies

($\exists t s. x = \langle t == s \rangle$) \vee ($\exists t s. x = \langle t \text{ in } s \rangle$) \vee ($\exists X F G. x = \forall X \langle \vee \neq : F \vee \notin : G \rangle$)" (is "?C \implies ?C'")

(is "?C \implies ?C'")

"x \in set (unlabel (transaction_updates T)) \implies

```

      ( $\exists t s. x = \text{insert}\langle t, s \rangle$ )  $\vee$  ( $\exists t s. x = \text{delete}\langle t, s \rangle$ )" (is "?D  $\implies$  ?D'")
      "x  $\in$  set (unlabel (transaction_send T))  $\implies$   $\exists t. x = \text{send}\langle t \rangle$ " (is "?E  $\implies$  ?E'")
<proof>

```

```

lemma transaction_strand_subsets[simp]:
  "set (transaction_receive T)  $\subseteq$  set (transaction_strand T)"
  "set (transaction_selects T)  $\subseteq$  set (transaction_strand T)"
  "set (transaction_checks T)  $\subseteq$  set (transaction_strand T)"
  "set (transaction_updates T)  $\subseteq$  set (transaction_strand T)"
  "set (transaction_send T)  $\subseteq$  set (transaction_strand T)"
  "set (unlabel (transaction_receive T))  $\subseteq$  set (unlabel (transaction_strand T))"
  "set (unlabel (transaction_selects T))  $\subseteq$  set (unlabel (transaction_strand T))"
  "set (unlabel (transaction_checks T))  $\subseteq$  set (unlabel (transaction_strand T))"
  "set (unlabel (transaction_updates T))  $\subseteq$  set (unlabel (transaction_strand T))"
  "set (unlabel (transaction_send T))  $\subseteq$  set (unlabel (transaction_strand T))"
<proof>

```

```

lemma transaction_strand_subst_subsets[simp]:
  "set (transaction_receive T  $\cdot$ lsst  $\vartheta$ )  $\subseteq$  set (transaction_strand T  $\cdot$ lsst  $\vartheta$ )"
  "set (transaction_selects T  $\cdot$ lsst  $\vartheta$ )  $\subseteq$  set (transaction_strand T  $\cdot$ lsst  $\vartheta$ )"
  "set (transaction_checks T  $\cdot$ lsst  $\vartheta$ )  $\subseteq$  set (transaction_strand T  $\cdot$ lsst  $\vartheta$ )"
  "set (transaction_updates T  $\cdot$ lsst  $\vartheta$ )  $\subseteq$  set (transaction_strand T  $\cdot$ lsst  $\vartheta$ )"
  "set (transaction_send T  $\cdot$ lsst  $\vartheta$ )  $\subseteq$  set (transaction_strand T  $\cdot$ lsst  $\vartheta$ )"
  "set (unlabel (transaction_receive T  $\cdot$ lsst  $\vartheta$ ))  $\subseteq$  set (unlabel (transaction_strand T  $\cdot$ lsst  $\vartheta$ ))"
  "set (unlabel (transaction_selects T  $\cdot$ lsst  $\vartheta$ ))  $\subseteq$  set (unlabel (transaction_strand T  $\cdot$ lsst  $\vartheta$ ))"
  "set (unlabel (transaction_checks T  $\cdot$ lsst  $\vartheta$ ))  $\subseteq$  set (unlabel (transaction_strand T  $\cdot$ lsst  $\vartheta$ ))"
  "set (unlabel (transaction_updates T  $\cdot$ lsst  $\vartheta$ ))  $\subseteq$  set (unlabel (transaction_strand T  $\cdot$ lsst  $\vartheta$ ))"
  "set (unlabel (transaction_send T  $\cdot$ lsst  $\vartheta$ ))  $\subseteq$  set (unlabel (transaction_strand T  $\cdot$ lsst  $\vartheta$ ))"
<proof>

```

```

lemma transaction_dual_subst_unfold:
  "unlabel (duallsst (transaction_strand T  $\cdot$ lsst  $\vartheta$ )) =
  unlabel (duallsst (transaction_receive T  $\cdot$ lsst  $\vartheta$ ))@
  unlabel (duallsst (transaction_selects T  $\cdot$ lsst  $\vartheta$ ))@
  unlabel (duallsst (transaction_checks T  $\cdot$ lsst  $\vartheta$ ))@
  unlabel (duallsst (transaction_updates T  $\cdot$ lsst  $\vartheta$ ))@
  unlabel (duallsst (transaction_send T  $\cdot$ lsst  $\vartheta$ ))"
<proof>

```

```

lemma trms_transaction_unfold:
  "trmslsst transaction T =
  trmslsst (transaction_receive T)  $\cup$  trmslsst (transaction_selects T)  $\cup$ 
  trmslsst (transaction_checks T)  $\cup$  trmslsst (transaction_updates T)  $\cup$ 
  trmslsst (transaction_send T)"
<proof>

```

```

lemma trms_transaction_subst_unfold:
  "trmslsst (transaction_strand T  $\cdot$ lsst  $\vartheta$ ) =
  trmslsst (transaction_receive T  $\cdot$ lsst  $\vartheta$ )  $\cup$  trmslsst (transaction_selects T  $\cdot$ lsst  $\vartheta$ )  $\cup$ 
  trmslsst (transaction_checks T  $\cdot$ lsst  $\vartheta$ )  $\cup$  trmslsst (transaction_updates T  $\cdot$ lsst  $\vartheta$ )  $\cup$ 
  trmslsst (transaction_send T  $\cdot$ lsst  $\vartheta$ )"
<proof>

```

```

lemma vars_transaction_unfold:
  "varslsst transaction T =
  varslsst (transaction_receive T)  $\cup$  varslsst (transaction_selects T)  $\cup$ 
  varslsst (transaction_checks T)  $\cup$  varslsst (transaction_updates T)  $\cup$ 
  varslsst (transaction_send T)"
<proof>

```

```

lemma vars_transaction_subst_unfold:
  "varslsst (transaction_strand T  $\cdot$ lsst  $\vartheta$ ) =
  varslsst (transaction_receive T  $\cdot$ lsst  $\vartheta$ )  $\cup$  varslsst (transaction_selects T  $\cdot$ lsst  $\vartheta$ )  $\cup$ 
  varslsst (transaction_checks T  $\cdot$ lsst  $\vartheta$ )  $\cup$  varslsst (transaction_updates T  $\cdot$ lsst  $\vartheta$ )  $\cup$ 
  varslsst (transaction_send T  $\cdot$ lsst  $\vartheta$ )"

```

```

varslsst (transaction_checks T ·lsst ∅) ∪ varslsst (transaction_updates T ·lsst ∅) ∪
varslsst (transaction_send T ·lsst ∅)"
⟨proof⟩

lemma fv_transaction_unfold:
  "fv_transaction T =
  fvlsst (transaction_receive T) ∪ fvlsst (transaction_selects T) ∪
  fvlsst (transaction_checks T) ∪ fvlsst (transaction_updates T) ∪
  fvlsst (transaction_send T)"
⟨proof⟩

lemma fv_transaction_subst_unfold:
  "fvlsst (transaction_strand T ·lsst ∅) =
  fvlsst (transaction_receive T ·lsst ∅) ∪ fvlsst (transaction_selects T ·lsst ∅) ∪
  fvlsst (transaction_checks T ·lsst ∅) ∪ fvlsst (transaction_updates T ·lsst ∅) ∪
  fvlsst (transaction_send T ·lsst ∅)"
⟨proof⟩

lemma fv_wellformed_transaction_unfold:
  assumes "wellformed_transaction T"
  shows "fv_transaction T =
  fvlsst (transaction_receive T) ∪ fvlsst (transaction_selects T) ∪ set (transaction_fresh T)"
⟨proof⟩

lemma bvars_transaction_unfold:
  "bvars_transaction T =
  bvarslsst (transaction_receive T) ∪ bvarslsst (transaction_selects T) ∪
  bvarslsst (transaction_checks T) ∪ bvarslsst (transaction_updates T) ∪
  bvarslsst (transaction_send T)"
⟨proof⟩

lemma bvars_transaction_subst_unfold:
  "bvarslsst (transaction_strand T ·lsst ∅) =
  bvarslsst (transaction_receive T ·lsst ∅) ∪ bvarslsst (transaction_selects T ·lsst ∅) ∪
  bvarslsst (transaction_checks T ·lsst ∅) ∪ bvarslsst (transaction_updates T ·lsst ∅) ∪
  bvarslsst (transaction_send T ·lsst ∅)"
⟨proof⟩

lemma bvars_wellformed_transaction_unfold:
  assumes "wellformed_transaction T"
  shows "bvars_transaction T = bvarslsst (transaction_checks T)" (is ?A)
  and "bvarslsst (transaction_receive T) = {}" (is ?B)
  and "bvarslsst (transaction_selects T) = {}" (is ?C)
  and "bvarslsst (transaction_updates T) = {}" (is ?D)
  and "bvarslsst (transaction_send T) = {}" (is ?E)
⟨proof⟩

lemma transaction_strand_memberD[dest]:
  assumes "x ∈ set (transaction_strand T)"
  shows "x ∈ set (transaction_receive T) ∨ x ∈ set (transaction_selects T) ∨
  x ∈ set (transaction_checks T) ∨ x ∈ set (transaction_updates T) ∨
  x ∈ set (transaction_send T)"
⟨proof⟩

lemma transaction_strand_unlabel_memberD[dest]:
  assumes "x ∈ set (unlabel (transaction_strand T))"
  shows "x ∈ set (unlabel (transaction_receive T)) ∨ x ∈ set (unlabel (transaction_selects T)) ∨
  x ∈ set (unlabel (transaction_checks T)) ∨ x ∈ set (unlabel (transaction_updates T)) ∨
  x ∈ set (unlabel (transaction_send T))"
⟨proof⟩

lemma wellformed_transaction_strand_memberD[dest]:
  assumes "wellformed_transaction T" and "(l,x) ∈ set (transaction_strand T)"

```

shows

```
"x = receive⟨t⟩ ⇒ (l,x) ∈ set (transaction_receive T)" (is "?A ⇒ ?A'")
"x = select⟨t,s⟩ ⇒ (l,x) ∈ set (transaction_selects T)" (is "?B ⇒ ?B'")
"x = ⟨t == s⟩ ⇒ (l,x) ∈ set (transaction_checks T)" (is "?C ⇒ ?C'")
"x = ⟨t in s⟩ ⇒ (l,x) ∈ set (transaction_checks T)" (is "?D ⇒ ?D'")
"x = ∀X(∀≠: F ∨ ∉: G) ⇒ (l,x) ∈ set (transaction_checks T)" (is "?E ⇒ ?E'")
"x = insert⟨t,s⟩ ⇒ (l,x) ∈ set (transaction_updates T)" (is "?F ⇒ ?F'")
"x = delete⟨t,s⟩ ⇒ (l,x) ∈ set (transaction_updates T)" (is "?G ⇒ ?G'")
"x = send⟨t⟩ ⇒ (l,x) ∈ set (transaction_send T)" (is "?H ⇒ ?H'")
```

⟨proof⟩

lemma wellformed_transaction_strand_unlabel_memberD[dest]:

assumes "wellformed_transaction T" and "x ∈ set (unlabel (transaction_strand T))"

shows

```
"x = receive⟨t⟩ ⇒ x ∈ set (unlabel (transaction_receive T))" (is "?A ⇒ ?A'")
"x = select⟨t,s⟩ ⇒ x ∈ set (unlabel (transaction_selects T))" (is "?B ⇒ ?B'")
"x = ⟨t == s⟩ ⇒ x ∈ set (unlabel (transaction_checks T))" (is "?C ⇒ ?C'")
"x = ⟨t in s⟩ ⇒ x ∈ set (unlabel (transaction_checks T))" (is "?D ⇒ ?D'")
"x = ∀X(∀≠: F ∨ ∉: G) ⇒ x ∈ set (unlabel (transaction_checks T))" (is "?E ⇒ ?E'")
"x = insert⟨t,s⟩ ⇒ x ∈ set (unlabel (transaction_updates T))" (is "?F ⇒ ?F'")
"x = delete⟨t,s⟩ ⇒ x ∈ set (unlabel (transaction_updates T))" (is "?G ⇒ ?G'")
"x = send⟨t⟩ ⇒ x ∈ set (unlabel (transaction_send T))" (is "?H ⇒ ?H'")
```

⟨proof⟩

lemma wellformed_transaction_send_receive_trm_cases:

assumes T: "wellformed_transaction T"

shows "t ∈ trms_{l_{sst}} (transaction_receive T) ⇒ receive⟨t⟩ ∈ set (unlabel (transaction_receive T))"
and "t ∈ trms_{l_{sst}} (transaction_send T) ⇒ send⟨t⟩ ∈ set (unlabel (transaction_send T))"

⟨proof⟩

lemma wellformed_transaction_send_receive_subst_trm_cases:

assumes T: "wellformed_transaction T"

shows "t ∈ trms_{l_{sst}} (transaction_receive T) ·_{set} ∅ ⇒ receive⟨t⟩ ∈ set (unlabel (transaction_receive T ·_{l_{sst}} ∅))"
and "t ∈ trms_{l_{sst}} (transaction_send T) ·_{set} ∅ ⇒ send⟨t⟩ ∈ set (unlabel (transaction_send T ·_{l_{sst}} ∅))"

⟨proof⟩

lemma wellformed_transaction_send_receive_fv_subset:

assumes T: "wellformed_transaction T"

shows "t ∈ trms_{l_{sst}} (transaction_receive T) ⇒ fv t ⊆ fv_transaction T" (is "?A ⇒ ?A'")
and "t ∈ trms_{l_{sst}} (transaction_send T) ⇒ fv t ⊆ fv_transaction T" (is "?B ⇒ ?B'")

⟨proof⟩

lemma dual_wellformed_transaction_ident_cases[dest]:

"list_all is_Assignment (unlabel S) ⇒ dual_{l_{sst}} S = S"

"list_all is_Check (unlabel S) ⇒ dual_{l_{sst}} S = S"

"list_all is_Update (unlabel S) ⇒ dual_{l_{sst}} S = S"

⟨proof⟩

lemma wellformed_transaction_wf_{sst}:

fixes T: "('a, 'b, 'c, 'd) prot_transaction"

assumes T: "wellformed_transaction T"

shows "wf'_{sst} (set (transaction_fresh T)) (unlabel (dual_{l_{sst}} (transaction_strand T)))" (is ?A)

and "fv_transaction T ∩ bvars_transaction T = {}" (is ?B)

and "set (transaction_fresh T) ∩ bvars_transaction T = {}" (is ?C)

⟨proof⟩

lemma dual_wellformed_transaction_ident_cases'[dest]:

assumes "wellformed_transaction T"

shows "dual_{l_{sst}} (transaction_selects T) = transaction_selects T"

"dual_{l_{sst}} (transaction_checks T) = transaction_checks T"

"dual_{l_{sst}} (transaction_updates T) = transaction_updates T"

⟨proof⟩

```

lemma dual_transaction_strand:
  assumes "wellformed_transaction T"
  shows "duallsst (transaction_strand T) =
        duallsst (transaction_receive T)@transaction_selects T@transaction_checks T@
        transaction_updates T@duallsst (transaction_send T)"
⟨proof⟩

lemma dual_unlabel_transaction_strand:
  assumes "wellformed_transaction T"
  shows "unlabel (duallsst (transaction_strand T)) =
        (unlabel (duallsst (transaction_receive T)))@(unlabel (transaction_selects T))@
        (unlabel (transaction_checks T))@(unlabel (transaction_updates T))@
        (unlabel (duallsst (transaction_send T)))"
⟨proof⟩

lemma dual_transaction_strand_subst:
  assumes "wellformed_transaction T"
  shows "duallsst (transaction_strand T ·lsst δ) =
        (duallsst (transaction_receive T)@transaction_selects T@transaction_checks T@
        transaction_updates T@duallsst (transaction_send T)) ·lsst δ"
⟨proof⟩

lemma dual_transaction_ik_is_transaction_send:
  assumes "wellformed_transaction T"
  shows "iksst (unlabel (duallsst (transaction_strand T))) = trmssst (unlabel (transaction_send T))"
  (is "?A = ?B")
⟨proof⟩

lemma dual_transaction_ik_is_transaction_send':
  fixes δ::"('a,'b,'c) prot_subst"
  assumes "wellformed_transaction T"
  shows "iksst (unlabel (duallsst (transaction_strand T ·lsst δ))) =
        trmssst (unlabel (transaction_send T)) ·set δ" (is "?A = ?B")
⟨proof⟩

lemma dbsst_transaction_prefix_eq:
  assumes T: "wellformed_transaction T"
  and S: "prefix S (transaction_receive T@transaction_selects T@transaction_checks T)"
  shows "dblsst A = dblsst (A@duallsst (S ·lsst δ))"
⟨proof⟩

lemma dblsst_duallsst_set_ex:
  assumes "d ∈ set (db'lsst (duallsst A ·lsst ϑ) I D)"
  "∀t u. insert⟨t,u⟩ ∈ set (unlabel A) ⟶ (∃s. u = Fun (Set s) [])"
  "∀t u. delete⟨t,u⟩ ∈ set (unlabel A) ⟶ (∃s. u = Fun (Set s) [])"
  "∀d ∈ set D. ∃s. snd d = Fun (Set s) []"
  shows "∃s. snd d = Fun (Set s) []"
⟨proof⟩

lemma is_Fun_SetE[elim]:
  assumes t: "is_Fun_Set t"
  obtains s where "t = Fun (Set s) []"
⟨proof⟩

lemma Fun_Set_InSet_iff:
  "(u = ⟨a: Var x ∈ Fun (Set s) []⟩) ⟷
  (is_InSet u ∧ is_Var (the_elem_term u) ∧ is_Fun_Set (the_set_term u) ∧
  the_Set (the_Fun (the_set_term u)) = s ∧ the_Var (the_elem_term u) = x ∧ the_check u = a)"
  (is "?A ⟷ ?B")
⟨proof⟩

lemma Fun_Set_NotInSet_iff:

```

2 Stateful Protocol Verification

```

"(u = ⟨Var x not in Fun (Set s) []⟩) ↔
(is_NegChecks u ∧ bvarssstp u = [] ∧ the_eqs u = [] ∧ length (the_ins u) = 1 ∧
is_Var (fst (hd (the_ins u))) ∧ is_Fun_Set (snd (hd (the_ins u)))) ∧
the_Set (the_Fun (snd (hd (the_ins u)))) = s ∧ the_Var (fst (hd (the_ins u))) = x"
⟨proof⟩

```

```

lemma is_Fun_Set_exi: "is_Fun_Set x ↔ (∃s. x = Fun (Set s) [])"
⟨proof⟩

```

```

lemma is_Fun_Set_subst:
  assumes "is_Fun_Set S'"
  shows "is_Fun_Set (S' · σ)"
⟨proof⟩

```

```

lemma is_Update_in_transaction_updates:
  assumes tu: "is_Update t"
  assumes t: "t ∈ set (unlabel (transaction_strand TT))"
  assumes vt: "wellformed_transaction TT"
  shows "t ∈ set (unlabel (transaction_updates TT))"
⟨proof⟩

```

```

lemma transaction_fresh_vars_subset:
  assumes "wellformed_transaction T"
  shows "set (transaction_fresh T) ⊆ fv_transaction T"
⟨proof⟩

```

```

lemma transaction_fresh_vars_notin:
  assumes T: "wellformed_transaction T"
  and x: "x ∈ set (transaction_fresh T)"
  shows "x ∉ fvlsst (transaction_receive T)" (is ?A)
  and "x ∉ fvlsst (transaction_selects T)" (is ?B)
  and "x ∉ fvlsst (transaction_checks T)" (is ?C)
  and "x ∉ varslsst (transaction_receive T)" (is ?D)
  and "x ∉ varslsst (transaction_selects T)" (is ?E)
  and "x ∉ varslsst (transaction_checks T)" (is ?F)
  and "x ∉ bvarslsst (transaction_receive T)" (is ?G)
  and "x ∉ bvarslsst (transaction_selects T)" (is ?H)
  and "x ∉ bvarslsst (transaction_checks T)" (is ?I)
⟨proof⟩

```

```

lemma transaction_proj_member:
  assumes "T ∈ set P"
  shows "transaction_proj n T ∈ set (map (transaction_proj n) P)"
⟨proof⟩

```

```

lemma transaction_strand_proj:
  "transaction_strand (transaction_proj n T) = proj n (transaction_strand T)"
⟨proof⟩

```

```

lemma transaction_proj_fresh_eq:
  "transaction_fresh (transaction_proj n T) = transaction_fresh T"
⟨proof⟩

```

```

lemma transaction_proj_trms_subset:
  "trms_transaction (transaction_proj n T) ⊆ trms_transaction T"
⟨proof⟩

```

```

lemma transaction_proj_vars_subset:
  "vars_transaction (transaction_proj n T) ⊆ vars_transaction T"
⟨proof⟩

```


end

2.2 Term Abstraction (Term_Abstraction)

```
theory Term_Abstraction
  imports Transactions
begin
```

2.2.1 Definitions

```
fun to_abs ("α₀") where
  "α₀ [] _ = {}"
| "α₀ ((Fun (Val m) [], Fun (Set s) S)#D) n =
  (if m = n then insert s (α₀ D n) else α₀ D n)"
| "α₀ (_#D) n = α₀ D n"

fun abs_apply_term (infixl ".α" 67) where
  "Var x .α α = Var x"
| "Fun (Val n) T .α α = Fun (Abs (α n)) (map (λt. t .α α) T)"
| "Fun f T .α α = Fun f (map (λt. t .α α) T)"

definition abs_apply_list (infixl ".αlist" 67) where
  "M .αlist α ≡ map (λt. t .α α) M"

definition abs_apply_terms (infixl ".αset" 67) where
  "M .αset α ≡ (λt. t .α α) ` M"

definition abs_apply_pairs (infixl ".αpairs" 67) where
  "F .αpairs α ≡ map (λ(s,t). (s .α α, t .α α)) F"

definition abs_apply_strand_step (infixl ".αstp" 67) where
  "s .αstp α ≡ (case s of
    (l, send⟨t⟩) ⇒ (l, send⟨t .α α⟩)
  | (l, receive⟨t⟩) ⇒ (l, receive⟨t .α α⟩)
  | (l, ⟨ac: t ≐ t'⟩) ⇒ (l, ⟨ac: (t .α α) ≐ (t' .α α)⟩)
  | (l, insert⟨t, t'⟩) ⇒ (l, insert⟨t .α α, t' .α α⟩)
  | (l, delete⟨t, t'⟩) ⇒ (l, delete⟨t .α α, t' .α α⟩)
  | (l, ⟨ac: t ∈ t'⟩) ⇒ (l, ⟨ac: (t .α α) ∈ (t' .α α)⟩)
  | (l, ∀X⟨∇≠: F ∇≠: F'⟩) ⇒ (l, ∀X⟨∇≠: (F .αpairs α) ∇≠: (F' .αpairs α)⟩))"

definition abs_apply_strand (infixl ".αst" 67) where
  "S .αst α ≡ map (λx. x .αstp α) S"
```

2.2.2 Lemmata

```
lemma to_abs_alt_def:
  "α₀ D n = {s. ∃S. (Fun (Val n) [], Fun (Set s) S) ∈ set D}"
⟨proof⟩

lemma abs_term_apply_const[simp]:
  "is_Val f ⇒ Fun f [] .α a = Fun (Abs (a (the_Val f))) []"
  "¬is_Val f ⇒ Fun f [] .α a = Fun f []"
⟨proof⟩

lemma abs_fv: "fv (t .α a) = fv t"
⟨proof⟩

lemma abs_eq_if_no_Val:
  assumes "∀f ∈ funs_term t. ¬is_Val f"
  shows "t .α a = t .α b"
⟨proof⟩
```

lemma `abs_list_set_is_set_abs_set`: " $\text{set } (M \cdot_{\alpha_{list}} \alpha) = (\text{set } M) \cdot_{\alpha_{set}} \alpha$ "
 $\langle \text{proof} \rangle$

lemma `abs_set_empty[simp]`: " $\{\} \cdot_{\alpha_{set}} \alpha = \{\}$ "
 $\langle \text{proof} \rangle$

lemma `abs_in`:
 assumes " $t \in M$ "
 shows " $t \cdot_{\alpha} \alpha \in M \cdot_{\alpha_{set}} \alpha$ "
 $\langle \text{proof} \rangle$

lemma `abs_set_union`: " $(A \cup B) \cdot_{\alpha_{set}} a = (A \cdot_{\alpha_{set}} a) \cup (B \cdot_{\alpha_{set}} a)$ "
 $\langle \text{proof} \rangle$

lemma `abs_subterms`: " $\text{subterms } (t \cdot_{\alpha} \alpha) = \text{subterms } t \cdot_{\alpha_{set}} \alpha$ "
 $\langle \text{proof} \rangle$

lemma `abs_subterms_in`: " $s \in \text{subterms } t \implies s \cdot_{\alpha} a \in \text{subterms } (t \cdot_{\alpha} a)$ "
 $\langle \text{proof} \rangle$

lemma `abs_ik_append`: " $(\text{ik}_{sst} (A@B) \cdot_{set} I) \cdot_{\alpha_{set}} a = (\text{ik}_{sst} A \cdot_{set} I) \cdot_{\alpha_{set}} a \cup (\text{ik}_{sst} B \cdot_{set} I) \cdot_{\alpha_{set}} a$ "
 $\langle \text{proof} \rangle$

lemma `to_abs_in`:
 assumes " $(\text{Fun } (\text{Val } n) [], \text{Fun } (\text{Set } s) []) \in \text{set } D$ "
 shows " $s \in \alpha_0 D n$ "
 $\langle \text{proof} \rangle$

lemma `to_abs_empty_iff_notin_db`:
" $\text{Fun } (\text{Val } n) [] \cdot_{\alpha} \alpha_0 D = \text{Fun } (\text{Abs } \{\}) [] \longleftrightarrow (\nexists s S. (\text{Fun } (\text{Val } n) [], \text{Fun } (\text{Set } s) S) \in \text{set } D)$ "
 $\langle \text{proof} \rangle$

lemma `to_abs_list_insert`:
 assumes " $\text{Fun } (\text{Val } n) [] \neq t$ "
 shows " $\alpha_0 D n = \alpha_0 (\text{List.insert } (t,s) D) n$ "
 $\langle \text{proof} \rangle$

lemma `to_abs_list_insert'`:
" $\text{insert } s (\alpha_0 D n) = \alpha_0 (\text{List.insert } (\text{Fun } (\text{Val } n) [], \text{Fun } (\text{Set } s) S) D) n$ "
 $\langle \text{proof} \rangle$

lemma `to_abs_list_remove_all`:
 assumes " $\text{Fun } (\text{Val } n) [] \neq t$ "
 shows " $\alpha_0 D n = \alpha_0 (\text{List.removeAll } (t,s) D) n$ "
 $\langle \text{proof} \rangle$

lemma `to_abs_list_remove_all'`:
" $\alpha_0 D n - \{s\} = \alpha_0 (\text{filter } (\lambda d. \nexists S. d = (\text{Fun } (\text{Val } n) [], \text{Fun } (\text{Set } s) S)) D) n$ "
 $\langle \text{proof} \rangle$

lemma `to_abs_db_sst_append`:
 assumes " $\forall u s. \text{insert}(u, s) \in \text{set } B \implies \text{Fun } (\text{Val } n) [] \neq u \cdot \mathcal{I}$ "
 and " $\forall u s. \text{delete}(u, s) \in \text{set } B \implies \text{Fun } (\text{Val } n) [] \neq u \cdot \mathcal{I}$ "
 shows " $\alpha_0 (\text{db}'_{sst} A \mathcal{I} D) n = \alpha_0 (\text{db}'_{sst} (A@B) \mathcal{I} D) n$ "
 $\langle \text{proof} \rangle$

lemma `to_abs_neq_imp_db_update`:
 assumes " $\alpha_0 (\text{db}_{sst} A \mathcal{I}) n \neq \alpha_0 (\text{db}_{sst} (A@B) \mathcal{I}) n$ "
 shows " $\exists u s. u \cdot \mathcal{I} = \text{Fun } (\text{Val } n) [] \wedge (\text{insert}(u,s) \in \text{set } B \vee \text{delete}(u,s) \in \text{set } B)$ "
 $\langle \text{proof} \rangle$

lemma `abs_term_subst_eq`:
 fixes $\delta \vartheta :: "(('a, 'b, 'c) \text{prot_fun}, ('d, 'e) \text{prot_atom}) \text{term} \times \text{nat}) \text{subst}$ "

```

assumes "∀x ∈ fv t. δ x ·α a = ∅ x ·α b"
and "⊥n T. Fun (Val n) T ∈ subterms t"
shows "t · δ ·α a = t · ∅ ·α b"
⟨proof⟩

lemma abs_term_subst_eq':
  fixes δ ∅::"('a,'b,'c) prot_fun, ('d,'e prot_atom) term × nat) subst"
  assumes "∀x ∈ fv t. δ x ·α a = ∅ x"
  and "⊥n T. Fun (Val n) T ∈ subterms t"
  shows "t · δ ·α a = t · ∅"
⟨proof⟩

lemma abs_val_in_funs_term:
  assumes "f ∈ funs_term t" "is_Val f"
  shows "Abs (α (the_Val f)) ∈ funs_term (t ·α α)"
⟨proof⟩

end

```

2.3 Stateful Protocol Model (Stateful_Protocol_Model)

```

theory Stateful_Protocol_Model
  imports Stateful_Protocol_Composition_and_Typing.Stateful_Compositionality
          Transactions_Term_Abstraction
begin

```

2.3.1 Locale Setup

```

locale stateful_protocol_model =
  fixes arity_f::"'fun ⇒ nat"
  and arity_s::"'sets ⇒ nat"
  and public_f::"'fun ⇒ bool"
  and Ana_f::"'fun ⇒ (('fun,'atom::finite,'sets) prot_fun, nat) term list × nat list)"
  and Γ_f::"'fun ⇒ 'atom option"
  and label_witness1::"'lbl"
  and label_witness2::"'lbl"
  assumes Ana_f_assm1: "∀f. let (K, M) = Ana_f f in (∀k ∈ subtermsset (set K).
    is_Fun k → (is_Fu (the_Fun k)) ∧ length (args k) = arity_f (the_Fu (the_Fun k)))"
  and Ana_f_assm2: "∀f. let (K, M) = Ana_f f in ∀i ∈ fvset (set K) ∪ set M. i < arity_f f"
  and public_f_assm: "∀f. arity_f f > (0::nat) → public_f f"
  and Γ_f_assm: "∀f. arity_f f = (0::nat) → Γ_f f ≠ None"
  and label_witness_assm: "label_witness1 ≠ label_witness2"
begin

```

```

lemma Ana_f_assm1_alt:
  assumes "Ana_f f = (K,M)" "k ∈ subtermsset (set K)"
  shows "(∃x. k = Var x) ∨ (∃h T. k = Fun (Fu h) T ∧ length T = arity_f h)"
⟨proof⟩

```

```

lemma Ana_f_assm2_alt:
  assumes "Ana_f f = (K,M)" "i ∈ fvset (set K) ∪ set M"
  shows "i < arity_f f"
⟨proof⟩

```

2.3.2 Definitions

```

fun arity where
  "arity (Fu f) = arity_f f"
| "arity (Set s) = arity_s s"
| "arity (Val _) = 0"
| "arity (Abs _) = 0"
| "arity Pair = 2"

```

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```

| "arity (Attack _) = 0"
| "arity OccursFact = 2"
| "arity OccursSec = 0"
| "arity (PubConstAtom _ _) = 0"
| "arity (PubConstSetType _) = 0"
| "arity (PubConstAttackType _) = 0"
| "arity (PubConstBottom _) = 0"
| "arity (PubConstOccursSecType _) = 0"

fun public where
  "public (Fu f) = publicf f"
| "public (Set s) = (aritys s > 0)"
| "public (Val n) = snd n"
| "public (Abs _) = False"
| "public Pair = True"
| "public (Attack _) = False"
| "public OccursFact = True"
| "public OccursSec = False"
| "public (PubConstAtom _ _) = True"
| "public (PubConstSetType _) = True"
| "public (PubConstAttackType _) = True"
| "public (PubConstBottom _) = True"
| "public (PubConstOccursSecType _) = True"

fun Ana where
  "Ana (Fun (Fu f) T) = (
    if arityf f = length T ∧ arityf f > 0
    then let (K,M) = Anaf f in (K ·list (!) T, map ((!) T) M)
    else ([, []])"
| "Ana _ = ([, []]"

definition Γv where
  "Γv v ≡ (
    if (∀t ∈ subterms (fst v).
      case t of (TComp f T) ⇒ arity f > 0 ∧ arity f = length T | _ ⇒ True)
    then fst v
    else TAtom Bottom)"

fun Γ where
  "Γ (Var v) = Γv v"
| "Γ (Fun f T) = (
  if arity f = 0
  then case f of
    (Fu g) ⇒ TAtom (case Γf g of Some a ⇒ Atom a | None ⇒ Bottom)
  | (Val _) ⇒ TAtom Value
  | (Abs _) ⇒ TAtom Value
  | (Set _) ⇒ TAtom SetType
  | (Attack _) ⇒ TAtom AttackType
  | OccursSec ⇒ TAtom OccursSecType
  | (PubConstAtom a _) ⇒ TAtom (Atom a)
  | (PubConstSetType _) ⇒ TAtom SetType
  | (PubConstAttackType _) ⇒ TAtom AttackType
  | (PubConstBottom _) ⇒ TAtom Bottom
  | (PubConstOccursSecType _) ⇒ TAtom OccursSecType
  | _ ⇒ TAtom Bottom
  else TComp f (map Γ T))"

lemma Γ_consts_simps[simp]:
  "arityf g = 0 ⇒ Γ (Fun (Fu g) []) = TAtom (case Γf g of Some a ⇒ Atom a | None ⇒ Bottom)"
  "Γ (Fun (Val n) []) = TAtom Value"
  "Γ (Fun (Abs b) []) = TAtom Value"
  "aritys s = 0 ⇒ Γ (Fun (Set s) []) = TAtom SetType"
  "Γ (Fun (Attack x) []) = TAtom AttackType"

```

```

"Γ (Fun OccursSec []) = TAtom OccursSecType"
"Γ (Fun (PubConstAtom a t) []) = TAtom (Atom a)"
"Γ (Fun (PubConstSetType t) []) = TAtom SetType"
"Γ (Fun (PubConstAttackType t) []) = TAtom AttackType"
"Γ (Fun (PubConstBottom t) []) = TAtom Bottom"
"Γ (Fun (PubConstOccursSecType t) []) = TAtom OccursSecType"
⟨proof⟩

```

```

lemma Γ_Set_simps[simp]:
  "aritys s ≠ 0 ⇒ Γ (Fun (Set s) T) = TComp (Set s) (map Γ T)"
  "Γ (Fun (Set s) T) = TAtom SetType ∨ Γ (Fun (Set s) T) = TComp (Set s) (map Γ T)"
  "Γ (Fun (Set s) T) ≠ TAtom Value"
  "Γ (Fun (Set s) T) ≠ TAtom (Atom a)"
  "Γ (Fun (Set s) T) ≠ TAtom AttackType"
  "Γ (Fun (Set s) T) ≠ TAtom OccursSecType"
  "Γ (Fun (Set s) T) ≠ TAtom Bottom"
⟨proof⟩

```

2.3.3 Locale Interpretations

```

lemma Ana_Fu_cases:
  assumes "Ana (Fun f T) = (K,M)"
    and "f = Fu g"
    and "Anaf g = (K',M')"
  shows "(K,M) = (if arityf g = length T ∧ arityf g > 0
    then (K' ·list (!) T, map (!! T) M')
    else ([], []))" (is ?A)
    and "(K,M) = (K' ·list (!) T, map (!! T) M') ∨ (K,M) = ([], [])" (is ?B)
⟨proof⟩

```

```

lemma Ana_Fu_intro:
  assumes "arityf f = length T" "arityf f > 0"
    and "Anaf f = (K',M')"
  shows "Ana (Fun (Fu f) T) = (K' ·list (!) T, map (!! T) M')"
⟨proof⟩

```

```

lemma Ana_Fu_elim:
  assumes "Ana (Fun f T) = (K,M)"
    and "f = Fu g"
    and "Anaf g = (K',M')"
    and "(K,M) ≠ ([], [])"
  shows "arityf g = length T" (is ?A)
    and "(K,M) = (K' ·list (!) T, map (!! T) M')" (is ?B)
⟨proof⟩

```

```

lemma Ana_nonempty_inv:
  assumes "Ana t ≠ ([], [])"
  shows "∃ f T. t = Fun (Fu f) T ∧ arityf f = length T ∧ arityf f > 0 ∧
    (∃ K M. Anaf f = (K, M) ∧ Ana t = (K ·list (!) T, map (!! T) M))"
⟨proof⟩

```

```

lemma assm1:
  assumes "Ana t = (K,M)"
  shows "fvset (set K) ⊆ fv t"
⟨proof⟩

```

```

lemma assm2:
  assumes "Ana t = (K,M)"
  and "∧g S'. Fun g S' ⊆ t ⇒ length S' = arity g"
  and "k ∈ set K"
  and "Fun f T' ⊆ k"
  shows "length T' = arity f"
⟨proof⟩

```

```

lemma assm4:
  assumes "Ana (Fun f T) = (K, M)"
  shows "set M  $\subseteq$  set T"
  <proof>

lemma assm5: "Ana t = (K,M)  $\implies$  K  $\neq$  []  $\vee$  M  $\neq$  []  $\implies$  Ana (t  $\cdot$   $\delta$ ) = (K  $\cdot$ list  $\delta$ , M  $\cdot$ list  $\delta$ )"
  <proof>

sublocale intruder_model arity public Ana
  <proof>

adhoc_overloading INTRUDER_SYNTH intruder_synth
adhoc_overloading INTRUDER_DEDUCT intruder_deduct

lemma assm6: "arity c = 0  $\implies$   $\exists$ a.  $\forall$ X.  $\Gamma$  (Fun c X) = TAtom a" <proof>

lemma assm7: "0 < arity f  $\implies$   $\Gamma$  (Fun f T) = TComp f (map  $\Gamma$  T)" <proof>

lemma assm8: "infinite {c.  $\Gamma$  (Fun c []::('fun,'atom,'sets) prot_term) = TAtom a  $\wedge$  public c}"
  (is "?P a")
  <proof>

lemma assm9: "TComp f T  $\sqsubseteq$   $\Gamma$  t  $\implies$  arity f > 0"
  <proof>

lemma assm10: "wftrm ( $\Gamma$  (Var x))"
  <proof>

lemma assm11: "arity f > 0  $\implies$  public f" <proof>

lemma assm12: " $\Gamma$  (Var ( $\tau$ , n)) =  $\Gamma$  (Var ( $\tau$ , m))" <proof>

lemma assm13: "arity c = 0  $\implies$  Ana (Fun c T) = ([], [])" <proof>

lemma assm14:
  assumes "Ana (Fun f T) = (K,M)"
  shows "Ana (Fun f T  $\cdot$   $\delta$ ) = (K  $\cdot$ list  $\delta$ , M  $\cdot$ list  $\delta$ )"
  <proof>

sublocale labeled_stateful_typed_model' arity public Ana  $\Gamma$  Pair label_witness1 label_witness2
  <proof>

```

2.3.4 Minor Lemmata

```

lemma  $\Gamma_v\_TAtom[simp]$ : " $\Gamma_v$  (TAtom a, n) = TAtom a"
  <proof>

lemma  $\Gamma_v\_TAtom'$ :
  assumes "a  $\neq$  Bottom"
  shows " $\Gamma_v$  ( $\tau$ , n) = TAtom a  $\iff$   $\tau$  = TAtom a"
  <proof>

lemma  $\Gamma_v\_TAtom\_inv$ :
  " $\Gamma_v$  x = TAtom (Atom a)  $\implies$   $\exists$ m. x = (TAtom (Atom a), m)"
  " $\Gamma_v$  x = TAtom Value  $\implies$   $\exists$ m. x = (TAtom Value, m)"
  " $\Gamma_v$  x = TAtom SetType  $\implies$   $\exists$ m. x = (TAtom SetType, m)"
  " $\Gamma_v$  x = TAtom AttackType  $\implies$   $\exists$ m. x = (TAtom AttackType, m)"
  " $\Gamma_v$  x = TAtom OccursSecType  $\implies$   $\exists$ m. x = (TAtom OccursSecType, m)"
  <proof>

lemma  $\Gamma_v\_TAtom''$ :
  "(fst x = TAtom (Atom a)) = ( $\Gamma_v$  x = TAtom (Atom a))" (is "?A = ?A'")

```

```

"(fst x = TAtom Value) = ( $\Gamma_v$  x = TAtom Value)" (is "?B = ?B'")
"(fst x = TAtom SetType) = ( $\Gamma_v$  x = TAtom SetType)" (is "?C = ?C'")
"(fst x = TAtom AttackType) = ( $\Gamma_v$  x = TAtom AttackType)" (is "?D = ?D'")
"(fst x = TAtom OccursSecType) = ( $\Gamma_v$  x = TAtom OccursSecType)" (is "?E = ?E'")
<proof>

```

```

lemma  $\Gamma_v$ _Var_image:
  " $\Gamma_v$  ' X =  $\Gamma$  ' Var ' X"
<proof>

```

```

lemma  $\Gamma$ _Fu_const:
  assumes "arityf g = 0"
  shows " $\exists$  a.  $\Gamma$  (Fun (Fu g) T) = TAtom (Atom a)"
<proof>

```

```

lemma Fun_Value_type_inv:
  fixes T:: "('fun, 'atom, 'sets) prot_term list"
  assumes " $\Gamma$  (Fun f T) = TAtom Value"
  shows " $(\exists n. f = \text{Val } n) \vee (\exists bs. f = \text{Abs } bs)$ "
<proof>

```

```

lemma abs_ $\Gamma$ : " $\Gamma$  t =  $\Gamma$  (t  $\cdot_\alpha$   $\alpha$ )"
<proof>

```

```

lemma Anaf_keys_not_pubval_terms:
  assumes "Anaf f = (K, T)"
  and "k  $\in$  set K"
  and "g  $\in$  funs_term k"
  shows " $\neg$ is_Val g"
<proof>

```

```

lemma Anaf_keys_not_abs_terms:
  assumes "Anaf f = (K, T)"
  and "k  $\in$  set K"
  and "g  $\in$  funs_term k"
  shows " $\neg$ is_Abs g"
<proof>

```

```

lemma Anaf_keys_not_pairs:
  assumes "Anaf f = (K, T)"
  and "k  $\in$  set K"
  and "g  $\in$  funs_term k"
  shows "g  $\neq$  Pair"
<proof>

```

```

lemma Ana_Fu_keys_funs_term_subset:
  fixes K:: "('fun, 'atom, 'sets) prot_term list"
  assumes "Ana (Fun (Fu f) S) = (K, T)"
  and "Anaf f = (K', T')"
  shows " $\bigcup$  (funs_term ' set K)  $\subseteq$   $\bigcup$  (funs_term ' set K')  $\cup$  funs_term (Fun (Fu f) S)"
<proof>

```

```

lemma Ana_Fu_keys_not_pubval_terms:
  fixes k:: "('fun, 'atom, 'sets) prot_term"
  assumes "Ana (Fun (Fu f) S) = (K, T)"
  and "Anaf f = (K', T')"
  and "k  $\in$  set K"
  and " $\forall g \in$  funs_term (Fun (Fu f) S). is_Val g  $\longrightarrow$   $\neg$ public g"
  shows " $\forall g \in$  funs_term k. is_Val g  $\longrightarrow$   $\neg$ public g"
<proof>

```

```

lemma Ana_Fu_keys_not_abs_terms:
  fixes k:: "('fun, 'atom, 'sets) prot_term"

```

```

assumes "Ana (Fun (Fu f) S) = (K, T)"
and "Anaf f = (K', T')"
and "k ∈ set K"
and "∀g ∈ funs_term (Fun (Fu f) S). ¬is_Abs g"
shows "∀g ∈ funs_term k. ¬is_Abs g"
⟨proof⟩

```

```

lemma Ana_Fu_keys_not_pairs:
fixes k:: "('fun, 'atom, 'sets) prot_term"
assumes "Ana (Fun (Fu f) S) = (K, T)"
and "Anaf f = (K', T')"
and "k ∈ set K"
and "∀g ∈ funs_term (Fun (Fu f) S). g ≠ Pair"
shows "∀g ∈ funs_term k. g ≠ Pair"
⟨proof⟩

```

```

lemma deduct_occurs_in_ik:
fixes t:: "('fun, 'atom, 'sets) prot_term"
assumes t: "M ⊢ occurs t"
and M: "∀s ∈ subtermsset M. OccursFact ∉ ⋃ (funs_term ' set (snd (Ana s)))"
      "∀s ∈ subtermsset M. OccursSec ∉ ⋃ (funs_term ' set (snd (Ana s)))"
      "Fun OccursSec [] ∉ M"
shows "occurs t ∈ M"
⟨proof⟩

```

```

lemma wellformed_transaction_sem_receives:
fixes T:: "('fun, 'atom, 'sets, 'lbl) prot_transaction"
assumes T_valid: "wellformed_transaction T"
and I: "strand_sem_stateful IK DB (unlabel (duallsst (transaction_strand T ·lsst ∅))) I"
and s: "receive⟨t⟩ ∈ set (unlabel (transaction_receive T ·lsst ∅))"
shows "IK ⊢ t · I"
⟨proof⟩

```

```

lemma wellformed_transaction_sem_selects:
assumes T_valid: "wellformed_transaction T"
and I: "strand_sem_stateful IK DB (unlabel (duallsst (transaction_strand T ·lsst ∅))) I"
and "select⟨t,u⟩ ∈ set (unlabel (transaction_selects T ·lsst ∅))"
shows "(t · I, u · I) ∈ DB"
⟨proof⟩

```

```

lemma wellformed_transaction_sem_pos_checks:
assumes T_valid: "wellformed_transaction T"
and I: "strand_sem_stateful IK DB (unlabel (duallsst (transaction_strand T ·lsst ∅))) I"
and "⟨t in u⟩ ∈ set (unlabel (transaction_checks T ·lsst ∅))"
shows "(t · I, u · I) ∈ DB"
⟨proof⟩

```

```

lemma wellformed_transaction_sem_neg_checks:
assumes T_valid: "wellformed_transaction T"
and I: "strand_sem_stateful IK DB (unlabel (duallsst (transaction_strand T ·lsst ∅))) I"
and "NegChecks X [] [(t,u)] ∈ set (unlabel (transaction_checks T ·lsst ∅))"
shows "∀δ. subst_domain δ = set X ∧ ground (subst_range δ) ⟶ (t · δ · I, u · δ · I) ∉ DB" (is ?A)
and "X = [] ⟹ (t · I, u · I) ∉ DB" (is "?B ⟹ ?B'")
⟨proof⟩

```

```

lemma wellformed_transaction_fv_in_receives_or_selects:
assumes T: "wellformed_transaction T"
and x: "x ∈ fv_transaction T" "x ∉ set (transaction_fresh T)"
shows "x ∈ fvlsst (transaction_receive T) ∪ fvlsst (transaction_selects T)"
⟨proof⟩

```

```

lemma dual_transaction_ik_is_transaction_send'':
fixes δ I:: "('a, 'b, 'c) prot_subst"

```



```

assumes "wellformed_transaction T"
shows "(iksst (unlabel (duallsst (transaction_strand T ·lsst δ))) ·set I) ·aset a =
      (trmssst (unlabel (transaction_send T)) ·set δ ·set I) ·aset a" (is "?A = ?B")

```

⟨proof⟩

```

lemma while_prot_terms_fun_mono:
  "mono (λM'. M ∪ ⋃ (subterms ' M') ∪ ⋃ ((set ∘ fst ∘ Ana) ' M'))"

```

⟨proof⟩

```

lemma while_prot_terms_SMP_overapprox:
  fixes M: "('fun, 'atom, 'sets) prot_terms"
  assumes N_supset: "M ∪ ⋃ (subterms ' N) ∪ ⋃ ((set ∘ fst ∘ Ana) ' N) ⊆ N"
  and Value_vars_only: "∀x ∈ fvset N. Γv x = TAtom Value"
  shows "SMP M ⊆ {a · δ | a δ. a ∈ N ∧ wtsubst δ ∧ wftrms (subst_range δ)}"

```

⟨proof⟩

2.3.5 The Protocol Transition System, Defined in Terms of the Reachable Constraints

```

definition transaction_fresh_subst where
  "transaction_fresh_subst σ T A ≡
    subst_domain σ = set (transaction_fresh T) ∧
    (∀t ∈ subst_range σ. ∃n. t = Fun (Val (n, False)) []) ∧
    (∀t ∈ subst_range σ. t ∉ subtermsset (trmslsst A)) ∧
    (∀t ∈ subst_range σ. t ∉ subtermsset (trms_transaction T)) ∧
    inj_on σ (subst_domain σ)"

```

```

definition transaction_renaming_subst where
  "transaction_renaming_subst α P A ≡
    ∃n ≥ max_var_set (⋃ (vars_transaction ' set P) ∪ varslsst A). α = var_rename n"

```

```

definition constraint_model where
  "constraint_model I A ≡
    constr_sem_stateful I (unlabel A) ∧
    interpretationsubst I ∧
    wftrms (subst_range I)"

```

```

definition welltyped_constraint_model where
  "welltyped_constraint_model I A ≡ wtsubst I ∧ constraint_model I A"

```

```

lemma constraint_model_prefix:
  assumes "constraint_model I (A@B)"
  shows "constraint_model I A"

```

⟨proof⟩

```

lemma welltyped_constraint_model_prefix:
  assumes "welltyped_constraint_model I (A@B)"
  shows "welltyped_constraint_model I A"

```

⟨proof⟩

```

lemma constraint_model_Val_is_Value_term:
  assumes "welltyped_constraint_model I A"
  and "t · I = Fun (Val n) []"
  shows "t = Fun (Val n) [] ∨ (∃m. t = Var (TAtom Value, m))"

```

⟨proof⟩

The set of symbolic constraints reachable in any symbolic run of the protocol P.

σ instantiates the fresh variables of transaction T with fresh terms. α is a variable-renaming whose range consists of fresh variables.

```

inductive_set reachable_constraints::
  "('fun, 'atom, 'sets, 'lbl) prot ⇒ ('fun, 'atom, 'sets, 'lbl) prot_constr set"
  for P: "('fun, 'atom, 'sets, 'lbl) prot"
where

```

```

init:
"[] ∈ reachable_constraints P"
/ step:
"[[A ∈ reachable_constraints P;
  T ∈ set P;
  transaction_fresh_subst σ T A;
  transaction_renaming_subst α P A
]] ⇒ A@duallsst (transaction_strand T ·lsst σ os α) ∈ reachable_constraints P"

```

2.3.6 Admissible Transactions

definition `admissible_transaction_checks` where

```

"admissible_transaction_checks T ≡
  ∀x ∈ set (unlabel (transaction_checks T)).
    is_Check x ∧
    (is_InSet x →
      is_Var (the_elem_term x) ∧ is_Fun_Set (the_set_term x) ∧
      fst (the_Var (the_elem_term x)) = TAtom Value) ∧
    (is_NegChecks x →
      bvarssstp x = [] ∧
      ((the_eqs x = [] ∧ length (the_ins x) = 1) ∨
       (the_ins x = [] ∧ length (the_eqs x) = 1)) ∧
      (is_NegChecks x ∧ the_eqs x = [] → (let h = hd (the_ins x) in
        is_Var (fst h) ∧ is_Fun_Set (snd h) ∧
        fst (the_Var (fst h)) = TAtom Value)))"

```

definition `admissible_transaction_selects` where

```

"admissible_transaction_selects T ≡
  ∀x ∈ set (unlabel (transaction_selects T)).
    is_InSet x ∧ the_check x = Assign ∧ is_Var (the_elem_term x) ∧ is_Fun_Set (the_set_term x) ∧
    fst (the_Var (the_elem_term x)) = TAtom Value"

```

definition `admissible_transaction_updates` where

```

"admissible_transaction_updates T ≡
  ∀x ∈ set (unlabel (transaction_updates T)).
    is_Update x ∧ is_Var (the_elem_term x) ∧ is_Fun_Set (the_set_term x) ∧
    fst (the_Var (the_elem_term x)) = TAtom Value"

```

definition `admissible_transaction_terms` where

```

"admissible_transaction_terms T ≡
  wftrms' arity (trmslsst (transaction_strand T)) ∧
  (∀f ∈ ⋃ (funs_term ' trms_transaction T).
    ¬is_Val f ∧ ¬is_Abs f ∧ ¬is_PubConstSetType f ∧ f ≠ Pair ∧
    ¬is_PubConstAttackType f ∧ ¬is_PubConstBottom f ∧ ¬is_PubConstOccursSecType f) ∧
  (∀r ∈ set (unlabel (transaction_strand T)).
    (∃f ∈ ⋃ (funs_term ' (trmssstp r)). is_Attack f) →
    (let t = the_msg r in is_Send r ∧ is_Fun t ∧ is_Attack (the_Fun t) ∧ args t = []))"

```

definition `admissible_transaction_occurs_checks` where

```

"admissible_transaction_occurs_checks T ≡ (
  (∀x ∈ fv_transaction T - set (transaction_fresh T). fst x = TAtom Value →
    receive⟨occurs (Var x)⟩ ∈ set (unlabel (transaction_receive T))) ∧
  (∀x ∈ set (transaction_fresh T). fst x = TAtom Value →
    send⟨occurs (Var x)⟩ ∈ set (unlabel (transaction_send T))) ∧
  (∀r ∈ set (unlabel (transaction_receive T)). is_Receive r →
    (OccursFact ∈ funs_term (the_msg r) ∨ OccursSec ∈ funs_term (the_msg r)) →
    (∃x ∈ fv_transaction T - set (transaction_fresh T).
      fst x = TAtom Value ∧ the_msg r = occurs (Var x))) ∧
  (∀r ∈ set (unlabel (transaction_send T)). is_Send r →
    (OccursFact ∈ funs_term (the_msg r) ∨ OccursSec ∈ funs_term (the_msg r)) →
    (∃x ∈ set (transaction_fresh T).
      fst x = TAtom Value ∧ the_msg r = occurs (Var x)))
)"

```

definition `admissible_transaction` where

```
"admissible_transaction T ≡ (
  wellformed_transaction T ∧
  distinct (transaction_fresh T) ∧
  list_all (λx. fst x = TAtom Value) (transaction_fresh T) ∧
  (∀x ∈ varslsst (transaction_strand T). is_Var (fst x) ∧ (the_Var (fst x) = Value)) ∧
  bvarslsst (transaction_strand T) = {} ∧
  (∀x ∈ fv_transaction T - set (transaction_fresh T).
    ∀y ∈ fv_transaction T - set (transaction_fresh T).
      x ≠ y → (Var x ≠ Var y) ∈ set (unlabel (transaction_checks T)) ∨
              (Var y ≠ Var x) ∈ set (unlabel (transaction_checks T))) ∧
  admissible_transaction_selects T ∧
  admissible_transaction_checks T ∧
  admissible_transaction_updates T ∧
  admissible_transaction_terms T ∧
  admissible_transaction_occurs_checks T
)"
```

lemma `transaction_no_bvars`:

```
assumes "admissible_transaction T"
shows "fv_transaction T = vars_transaction T"
and "bvars_transaction T = {}"
<proof>
```

lemma `transactions_fv_bvars_disj`:

```
assumes "∀T ∈ set P. admissible_transaction T"
shows "(⋃T ∈ set P. fv_transaction T) ∩ (⋃T ∈ set P. bvars_transaction T) = {}"
<proof>
```

lemma `transaction_bvars_no_Value_type`:

```
assumes "admissible_transaction T"
and "x ∈ bvars_transaction T"
shows "¬TAtom Value ⊆ Γv x"
<proof>
```

lemma `transaction_receive_deduct`:

```
assumes Tadm: "admissible_transaction T"
and I: "constraint_model I (A@duallsst (transaction_strand T ·lsst σ ∘s α))"
and σ: "transaction_fresh_subst σ T A"
and α: "transaction_renaming_subst α P A"
and t: "receive⟨t⟩ ∈ set (unlabel (transaction_receive T ·lsst σ ∘s α))"
shows "iklsst A ·set I ⊢ t · I"
<proof>
```

lemma `transaction_checks_db`:

```
assumes T: "admissible_transaction T"
and I: "constraint_model I (A@duallsst (transaction_strand T ·lsst σ ∘s α))"
and σ: "transaction_fresh_subst σ T A"
and α: "transaction_renaming_subst α P A"
shows "⟨Var (TAtom Value, n) in Fun (Set s) []⟩ ∈ set (unlabel (transaction_checks T))
  ⇒ (α (TAtom Value, n) · I, Fun (Set s) []) ∈ set (dblsst A I)"
(is "?A ⇒ ?B")
and "⟨Var (TAtom Value, n) not in Fun (Set s) []⟩ ∈ set (unlabel (transaction_checks T))
  ⇒ (α (TAtom Value, n) · I, Fun (Set s) []) ∉ set (dblsst A I)"
(is "?C ⇒ ?D")
<proof>
```

lemma `transaction_selects_db`:

```
assumes T: "admissible_transaction T"
and I: "constraint_model I (A@duallsst (transaction_strand T ·lsst σ ∘s α))"
and σ: "transaction_fresh_subst σ T A"
and α: "transaction_renaming_subst α P A"
```

```

shows "select⟨Var (TAtom Value, n), Fun (Set s) []⟩ ∈ set (unlabel (transaction_selects T))
      ⇒ (α (TAtom Value, n) ·  $\mathcal{I}$ , Fun (Set s) []) ∈ set (dblsst A  $\mathcal{I}$ )"
(is "?A ⇒ ?B")
⟨proof⟩

```

```

lemma transactions_have_no_Value_consts:
  assumes "admissible_transaction T"
  and "t ∈ subtermsset (trmslsst (transaction_strand T))"
  shows "∄a T. t = Fun (Val a) T" (is ?A)
  and "∄a T. t = Fun (Abs a) T" (is ?B)
⟨proof⟩

```

```

lemma transactions_have_no_Value_consts':
  assumes "admissible_transaction T"
  and "t ∈ trmslsst (transaction_strand T)"
  shows "∄a T. Fun (Val a) T ∈ subterms t"
  and "∄a T. Fun (Abs a) T ∈ subterms t"
⟨proof⟩

```

```

lemma transactions_have_no_PubConsts:
  assumes "admissible_transaction T"
  and "t ∈ subtermsset (trmslsst (transaction_strand T))"
  shows "∄a T. t = Fun (PubConstSetType a) T" (is ?A)
  and "∄a T. t = Fun (PubConstAttackType a) T" (is ?B)
  and "∄a T. t = Fun (PubConstBottom a) T" (is ?C)
  and "∄a T. t = Fun (PubConstOccursSecType a) T" (is ?D)
⟨proof⟩

```

```

lemma transactions_have_no_PubConsts':
  assumes "admissible_transaction T"
  and "t ∈ trmslsst (transaction_strand T)"
  shows "∄a T. Fun (PubConstSetType a) T ∈ subterms t"
  and "∄a T. Fun (PubConstAttackType a) T ∈ subterms t"
  and "∄a T. Fun (PubConstBottom a) T ∈ subterms t"
  and "∄a T. Fun (PubConstOccursSecType a) T ∈ subterms t"
⟨proof⟩

```

```

lemma transaction_inserts_are_Value_vars:
  assumes T_valid: "wellformed_transaction T"
  and "admissible_transaction_updates T"
  and "insert⟨t,s⟩ ∈ set (unlabel (transaction_strand T))"
  shows "∃n. t = Var (TAtom Value, n)"
  and "∃u. s = Fun (Set u) []"
⟨proof⟩

```

```

lemma transaction_deletes_are_Value_vars:
  assumes T_valid: "wellformed_transaction T"
  and "admissible_transaction_updates T"
  and "delete⟨t,s⟩ ∈ set (unlabel (transaction_strand T))"
  shows "∃n. t = Var (TAtom Value, n)"
  and "∃u. s = Fun (Set u) []"
⟨proof⟩

```

```

lemma transaction_selects_are_Value_vars:
  assumes T_valid: "wellformed_transaction T"
  and "admissible_transaction_selects T"
  and "select⟨t,s⟩ ∈ set (unlabel (transaction_strand T))"
  shows "∃n. t = Var (TAtom Value, n) ∧ (TAtom Value, n) ∉ set (transaction_fresh T)" (is ?A)
  and "∃u. s = Fun (Set u) []" (is ?B)
⟨proof⟩

```

```

lemma transaction_inset_checks_are_Value_vars:
  assumes T_valid: "wellformed_transaction T"

```

```

and "admissible_transaction_checks T"
and "{t in s} ∈ set (unlabel (transaction_strand T))"
shows "∃n. t = Var (TAtom Value, n) ∧ (TAtom Value, n) ∉ set (transaction_fresh T)" (is ?A)
and "∃u. s = Fun (Set u) []" (is ?B)
⟨proof⟩

```

```

lemma transaction_notinset_checks_are_Value_vars:
assumes T_valid: "wellformed_transaction T"
and "admissible_transaction_checks T"
and "∀X⟨∀≠: F ∨≠: G⟩ ∈ set (unlabel (transaction_strand T))"
and "(t,s) ∈ set G"
shows "∃n. t = Var (TAtom Value, n) ∧ (TAtom Value, n) ∉ set (transaction_fresh T)" (is ?A)
and "∃u. s = Fun (Set u) []" (is ?B)
⟨proof⟩

```

```

lemma admissible_transaction_strand_step_cases:
assumes T_adm: "admissible_transaction T"
shows "r ∈ set (unlabel (transaction_receive T)) ⇒ ∃t. r = receive⟨t⟩"
(is "?A ⇒ ?A'")
and "r ∈ set (unlabel (transaction_selects T)) ⇒
  ∃x s. r = select⟨Var x, Fun (Set s) []⟩ ∧
  fst x = TAtom Value ∧ x ∈ fv_transaction T - set (transaction_fresh T)"
(is "?B ⇒ ?B'")
and "r ∈ set (unlabel (transaction_checks T)) ⇒
  (∃x s. (r = ⟨Var x in Fun (Set s) []⟩ ∨ r = ⟨Var x not in Fun (Set s) []⟩) ∧
  fst x = TAtom Value ∧ x ∈ fv_transaction T - set (transaction_fresh T)) ∨
  (∃s t. r = ⟨s == t⟩ ∨ r = ⟨s != t⟩)"
(is "?C ⇒ ?C'")
and "r ∈ set (unlabel (transaction_updates T)) ⇒
  ∃x s. (r = insert⟨Var x, Fun (Set s) []⟩ ∨ r = delete⟨Var x, Fun (Set s) []⟩) ∧
  fst x = TAtom Value"
(is "?D ⇒ ?D'")
and "r ∈ set (unlabel (transaction_send T)) ⇒ ∃t. r = send⟨t⟩"
(is "?E ⇒ ?E'")
⟨proof⟩

```

```

lemma transaction_Value_vars_are_fv:
assumes "admissible_transaction T"
and "x ∈ vars_transaction T"
and "Γv x = TAtom Value"
shows "x ∈ fv_transaction T"
⟨proof⟩

```

```

lemma protocol_transaction_vars_TAtom_typed:
assumes P: "admissible_transaction T"
shows "∀x ∈ vars_transaction T. Γv x = TAtom Value ∨ (∃a. Γv x = TAtom (Atom a))"
and "∀x ∈ fv_transaction T. Γv x = TAtom Value ∨ (∃a. Γv x = TAtom (Atom a))"
and "∀x ∈ set (transaction_fresh T). Γv x = TAtom Value"
⟨proof⟩

```

```

lemma protocol_transactions_no_pubconsts:
assumes "admissible_transaction T"
shows "Fun (Val (n,True)) S ∉ subtermsset (trms_transaction T)"
⟨proof⟩

```

```

lemma protocol_transactions_no_abss:
assumes "admissible_transaction T"
shows "Fun (Abs n) S ∉ subtermsset (trms_transaction T)"
⟨proof⟩

```

```

lemma admissible_transaction_strand_sem_fv_ineq:
assumes T_adm: "admissible_transaction T"
and I: "strand_sem_stateful IK DB (unlabel (duallsst (transaction_strand T ·lsst ∅))) I"

```

```

    and x: "x ∈ fv_transaction T - set (transaction_fresh T)"
    and y: "y ∈ fv_transaction T - set (transaction_fresh T)"
    and x_not_y: "x ≠ y"
  shows "∅ x ·  $\mathcal{I}$  ≠ ∅ y ·  $\mathcal{I}$ "
⟨proof⟩

lemma admissible_transactions_wf_trms:
  assumes "admissible_transaction T"
  shows "wf_trms (trms_transaction T)"
⟨proof⟩

lemma admissible_transaction_no_Ana_Attack:
  assumes "admissible_transaction_terms T"
  and "t ∈ subterms_set (trms_transaction T)"
  shows "attack(n) ∉ set (snd (Ana t))"
⟨proof⟩

lemma admissible_transaction_occurs_fv_types:
  assumes "admissible_transaction T"
  and "x ∈ vars_transaction T"
  shows "∃ a.  $\Gamma$  (Var x) = TAtom a ∧  $\Gamma$  (Var x) ≠ TAtom OccursSecType"
⟨proof⟩

lemma admissible_transaction_Value_vars:
  assumes T: "admissible_transaction T"
  and x: "x ∈ fv_transaction T"
  shows " $\Gamma_v$  x = TAtom Value"
⟨proof⟩

```

2.3.7 Lemmata: Renaming and Fresh Substitutions

```

lemma transaction_renaming_subst_is_renaming:
  fixes  $\alpha$ :: "('fun, 'atom, 'sets) prot_subst"
  assumes "transaction_renaming_subst  $\alpha$  P A"
  shows "∃ m.  $\alpha$  ( $\tau$ , n) = Var ( $\tau$ , n + Suc m)"
⟨proof⟩

lemma transaction_renaming_subst_is_renaming':
  fixes  $\alpha$ :: "('fun, 'atom, 'sets) prot_subst"
  assumes "transaction_renaming_subst  $\alpha$  P A"
  shows "∃ y.  $\alpha$  x = Var y"
⟨proof⟩

lemma transaction_renaming_subst_vars_disj:
  fixes  $\alpha$ :: "('fun, 'atom, 'sets) prot_subst"
  assumes "transaction_renaming_subst  $\alpha$  P A"
  shows "fv_set ( $\alpha$  ' ( $\bigcup$  (vars_transaction ' set P))) ∩ ( $\bigcup$  (vars_transaction ' set P)) = {}" (is ?A)
  and "fv_set ( $\alpha$  ' vars_lsst A) ∩ vars_lsst A = {}" (is ?B)
  and "T ∈ set P ⇒ vars_transaction T ∩ range_vars  $\alpha$  = {}" (is "T ∈ set P ⇒ ?C1")
  and "T ∈ set P ⇒ bvars_transaction T ∩ range_vars  $\alpha$  = {}" (is "T ∈ set P ⇒ ?C2")
  and "T ∈ set P ⇒ fv_transaction T ∩ range_vars  $\alpha$  = {}" (is "T ∈ set P ⇒ ?C3")
  and "vars_lsst A ∩ range_vars  $\alpha$  = {}" (is ?D1)
  and "bvars_lsst A ∩ range_vars  $\alpha$  = {}" (is ?D2)
  and "fv_lsst A ∩ range_vars  $\alpha$  = {}" (is ?D3)
⟨proof⟩

lemma transaction_renaming_subst_wt:
  fixes  $\alpha$ :: "('fun, 'atom, 'sets) prot_subst"
  assumes "transaction_renaming_subst  $\alpha$  P A"
  shows "wt_subst  $\alpha$ "
⟨proof⟩

lemma transaction_renaming_subst_is_wf_trm:

```

```

fixes  $\alpha::$ "('fun,'atom,'sets) prot_subst"
assumes "transaction_renaming_subst  $\alpha$  P A"
shows "wftrm ( $\alpha$  v)"
⟨proof⟩

lemma transaction_renaming_subst_range_wf_trms:
  fixes  $\alpha::$ "('fun,'atom,'sets) prot_subst"
  assumes "transaction_renaming_subst  $\alpha$  P A"
  shows "wftrms (subst_range  $\alpha$ )"
⟨proof⟩

lemma transaction_renaming_subst_range_notin_vars:
  fixes  $\alpha::$ "('fun,'atom,'sets) prot_subst"
  assumes "transaction_renaming_subst  $\alpha$  P A"
  shows " $\exists y. \alpha x = \text{Var } y \wedge y \notin \bigcup (\text{vars\_transaction } \text{' set } P) \cup \text{vars}_{lsst} \mathcal{A}$ "
⟨proof⟩

lemma transaction_renaming_subst_var_obtain:
  fixes  $\alpha::$ "('fun,'atom,'sets) prot_subst"
  assumes x: "x  $\in$  fvsst (S ·sst  $\alpha$ )"
  and  $\alpha$ : "transaction_renaming_subst  $\alpha$  P A"
  shows " $\exists y. \alpha y = \text{Var } x$ "
⟨proof⟩

lemma transaction_fresh_subst_is_wf_trm:
  fixes  $\sigma::$ "('fun,'atom,'sets) prot_subst"
  assumes "transaction_fresh_subst  $\sigma$  T A"
  shows "wftrm ( $\sigma$  v)"
⟨proof⟩

lemma transaction_fresh_subst_wt:
  fixes  $\sigma::$ "('fun,'atom,'sets) prot_subst"
  assumes "transaction_fresh_subst  $\sigma$  T A"
  and " $\forall x \in \text{set } (\text{transaction\_fresh } T). \Gamma_v x = \text{TAtom Value}$ "
  shows "wtsubst  $\sigma$ "
⟨proof⟩

lemma transaction_fresh_subst_domain:
  fixes  $\sigma::$ "('fun,'atom,'sets) prot_subst"
  assumes "transaction_fresh_subst  $\sigma$  T A"
  shows "subst_domain  $\sigma = \text{set } (\text{transaction\_fresh } T)"$ "
⟨proof⟩

lemma transaction_fresh_subst_range_wf_trms:
  fixes  $\sigma::$ "('fun,'atom,'sets) prot_subst"
  assumes "transaction_fresh_subst  $\sigma$  T A"
  shows "wftrms (subst_range  $\sigma$ )"
⟨proof⟩

lemma transaction_fresh_subst_range_fresh:
  fixes  $\sigma::$ "('fun,'atom,'sets) prot_subst"
  assumes "transaction_fresh_subst  $\sigma$  T A"
  shows " $\forall t \in \text{subst\_range } \sigma. t \notin \text{subterms}_{set} (\text{trms}_{lsst} \mathcal{A})$ "
  and " $\forall t \in \text{subst\_range } \sigma. t \notin \text{subterms}_{set} (\text{trms}_{lsst} (\text{transaction\_strand } T))$ "
⟨proof⟩

lemma transaction_fresh_subst_sends_to_val:
  fixes  $\sigma::$ "('fun,'atom,'sets) prot_subst"
  assumes "transaction_fresh_subst  $\sigma$  T A"
  and "y  $\in$  set (transaction_fresh T)"
  obtains n where " $\sigma y = \text{Fun } (\text{Val } n) []$ " " $\text{Fun } (\text{Val } n) [] \in \text{subst\_range } \sigma$ "
⟨proof⟩

```

```

lemma transaction_fresh_subst_sends_to_val':
  fixes  $\sigma \alpha :: ('fun, 'atom, 'sets) prot\_subst$ 
  assumes "transaction_fresh_subst  $\sigma T \mathcal{A}$ "
    and "y  $\in$  set (transaction_fresh T)"
  obtains n where " $(\sigma \circ_s \alpha) y \cdot \mathcal{I} = \text{Fun (Val n) []}$ " "Fun (Val n) []  $\in$  subst_range  $\sigma$ "
<proof>

```

```

lemma transaction_fresh_subst_grounds_domain:
  fixes  $\sigma :: ('fun, 'atom, 'sets) prot\_subst$ 
  assumes "transaction_fresh_subst  $\sigma T \mathcal{A}$ "
    and "y  $\in$  set (transaction_fresh T)"
  shows "fv ( $\sigma y$ ) = {}"
<proof>

```

```

lemma transaction_fresh_subst_transaction_renaming_subst_range:
  fixes  $\sigma \alpha :: ('fun, 'atom, 'sets) prot\_subst$ 
  assumes "transaction_fresh_subst  $\sigma T \mathcal{A}$ " "transaction_renaming_subst  $\alpha P \mathcal{A}$ "
  shows "x  $\in$  set (transaction_fresh T)  $\implies \exists n. (\sigma \circ_s \alpha) x = \text{Fun (Val (n, False)) []}$ "
    and "x  $\notin$  set (transaction_fresh T)  $\implies \exists y. (\sigma \circ_s \alpha) x = \text{Var y}$ "
<proof>

```

```

lemma transaction_fresh_subst_transaction_renaming_subst_range':
  fixes  $\sigma \alpha :: ('fun, 'atom, 'sets) prot\_subst$ 
  assumes "transaction_fresh_subst  $\sigma T \mathcal{A}$ " "transaction_renaming_subst  $\alpha P \mathcal{A}$ "
    and "t  $\in$  subst_range ( $\sigma \circ_s \alpha$ )"
  shows " $(\exists n. t = \text{Fun (Val (n, False)) []}) \vee (\exists x. t = \text{Var x})$ "
<proof>

```

```

lemma transaction_fresh_subst_transaction_renaming_subst_range'':
  fixes  $\sigma \alpha :: ('fun, 'atom, 'sets) prot\_subst$ 
  assumes s: "transaction_fresh_subst  $\sigma T \mathcal{A}$ " "transaction_renaming_subst  $\alpha P \mathcal{A}$ "
    and y: "y  $\in$  fv ( $(\sigma \circ_s \alpha) x$ )"
  shows " $\sigma x = \text{Var x}$ "
    and " $\alpha x = \text{Var y}$ "
    and " $(\sigma \circ_s \alpha) x = \text{Var y}$ "
<proof>

```

```

lemma transaction_fresh_subst_transaction_renaming_subst_vars_subset:
  fixes  $\sigma \alpha :: ('fun, 'atom, 'sets) prot\_subst$ 
  assumes  $\sigma$ : "transaction_fresh_subst  $\sigma T \mathcal{A}$ "
    and  $\alpha$ : "transaction_renaming_subst  $\alpha P \mathcal{A}$ "
  shows " $\bigcup (fv\_transaction \text{ ' set } P) \subseteq \text{subst\_domain } (\sigma \circ_s \alpha)$ " (is ?A)
    and "fvlsst  $\mathcal{A} \subseteq \text{subst\_domain } (\sigma \circ_s \alpha)$ " (is ?B)
    and " $T' \in \text{set } P \implies fv\_transaction T' \subseteq \text{subst\_domain } (\sigma \circ_s \alpha)$ " (is " $T' \in \text{set } P \implies ?C$ ")
    and " $T' \in \text{set } P \implies fv_{l_{sst}} (\text{transaction\_strand } T' \cdot l_{sst} (\sigma \circ_s \alpha)) \subseteq \text{range\_vars } (\sigma \circ_s \alpha)$ "
      (is " $T' \in \text{set } P \implies ?D$ ")
<proof>

```

```

lemma transaction_fresh_subst_transaction_renaming_subst_vars_disj:
  fixes  $\sigma \alpha :: ('fun, 'atom, 'sets) prot\_subst$ 
  assumes  $\sigma$ : "transaction_fresh_subst  $\sigma T \mathcal{A}$ "
    and  $\alpha$ : "transaction_renaming_subst  $\alpha P \mathcal{A}$ "
  shows "fvset ( $(\sigma \circ_s \alpha) \text{ ' } (\bigcup (\text{vars\_transaction \text{ ' set } } P))) \cap (\bigcup (\text{vars\_transaction \text{ ' set } } P)) = \{\}$ "
    (is ?A)
    and "x  $\in \bigcup (\text{vars\_transaction \text{ ' set } } P) \implies fv ((\sigma \circ_s \alpha) x) \cap (\bigcup (\text{vars\_transaction \text{ ' set } } P)) = \{\}$ "
    (is "?B'  $\implies$  ?B")
    and " $T' \in \text{set } P \implies \text{vars\_transaction } T' \cap \text{range\_vars } (\sigma \circ_s \alpha) = \{\}$ " (is " $T' \in \text{set } P \implies ?C1$ ")
    and " $T' \in \text{set } P \implies \text{bvars\_transaction } T' \cap \text{range\_vars } (\sigma \circ_s \alpha) = \{\}$ " (is " $T' \in \text{set } P \implies ?C2$ ")
    and " $T' \in \text{set } P \implies fv\_transaction T' \cap \text{range\_vars } (\sigma \circ_s \alpha) = \{\}$ " (is " $T' \in \text{set } P \implies ?C3$ ")
    and "varslsst  $\mathcal{A} \cap \text{range\_vars } (\sigma \circ_s \alpha) = \{\}$ " (is ?D1)
    and "bvarslsst  $\mathcal{A} \cap \text{range\_vars } (\sigma \circ_s \alpha) = \{\}$ " (is ?D2)
    and "fvlsst  $\mathcal{A} \cap \text{range\_vars } (\sigma \circ_s \alpha) = \{\}$ " (is ?D3)
<proof>

```



```

lemma transaction_fresh_subst_transaction_renaming_subst_trms:
  fixes  $\sigma \alpha :: ('fun, 'atom, 'sets) prot\_subst$ 
  assumes "transaction_fresh_subst  $\sigma T \mathcal{A}$ " "transaction_renaming_subst  $\alpha P \mathcal{A}$ "
    and "bvarslsst  $S \cap subst\_domain \sigma = \{\}$ "
    and "bvarslsst  $S \cap subst\_domain \alpha = \{\}$ "
  shows "subtermsset (trmslsst ( $S \cdot_{l_{sst}} (\sigma \circ_s \alpha)$ )) = subtermsset (trmslsst  $S$ )  $\cdot_{set} (\sigma \circ_s \alpha)$ "
  <proof>

lemma transaction_fresh_subst_transaction_renaming_wt:
  fixes  $\sigma \alpha :: ('fun, 'atom, 'sets) prot\_subst$ 
  assumes "transaction_fresh_subst  $\sigma T \mathcal{A}$ " "transaction_renaming_subst  $\alpha P \mathcal{A}$ "
    and " $\forall x \in set (transaction\_fresh T). \Gamma_v x = TAtom Value$ "
  shows "wtsubst ( $\sigma \circ_s \alpha$ )"
  <proof>

lemma transaction_fresh_subst_transaction_renaming_fv:
  fixes  $\sigma \alpha :: ('fun, 'atom, 'sets) prot\_subst$ 
  assumes  $\sigma$ : "transaction_fresh_subst  $\sigma T \mathcal{A}$ "
    and  $\alpha$ : "transaction_renaming_subst  $\alpha P \mathcal{A}$ "
    and  $x$ : " $x \in fv_{l_{sst}} (dual_{l_{sst}} (transaction\_strand T \cdot_{l_{sst}} \sigma \circ_s \alpha))$ "
  shows " $\exists y \in fv\_transaction T - set (transaction\_fresh T). (\sigma \circ_s \alpha) y = Var x$ "
  <proof>

lemma transaction_fresh_subst_transaction_renaming_subst_occurs_fact_send_receive:
  fixes  $t :: ('fun, 'atom, 'sets) prot\_term$ 
  assumes  $\sigma$ : "transaction_fresh_subst  $\sigma T \mathcal{A}$ "
    and  $\alpha$ : "transaction_renaming_subst  $\alpha P \mathcal{A}$ "
    and  $T$ : "wellformed_transaction  $T$ "
  shows "send{occurs  $t$ }  $\in set (unlabel (transaction\_strand T \cdot_{l_{sst}} \sigma \circ_s \alpha))$ "
     $\implies \exists s. send\{occurs s\} \in set (unlabel (transaction\_send T)) \wedge t = s \cdot \sigma \circ_s \alpha$ "
    (is "?A  $\implies ?A'$ ")
    and "receive{occurs  $t$ }  $\in set (unlabel (transaction\_strand T \cdot_{l_{sst}} \sigma \circ_s \alpha))$ "
     $\implies \exists s. receive\{occurs s\} \in set (unlabel (transaction\_receive T)) \wedge t = s \cdot \sigma \circ_s \alpha$ "
    (is "?B  $\implies ?B'$ ")
  <proof>

lemma transaction_fresh_subst_proj:
  assumes "transaction_fresh_subst  $\sigma T \mathcal{A}$ "
  shows "transaction_fresh_subst  $\sigma (transaction\_proj n T) (proj n \mathcal{A})$ "
  <proof>

lemma transaction_renaming_subst_proj:
  assumes "transaction_renaming_subst  $\alpha P \mathcal{A}$ "
  shows "transaction_renaming_subst  $\alpha (map (transaction\_proj n) P) (proj n \mathcal{A})$ "
  <proof>

lemma protocol_transaction_wf_subst:
  fixes  $\sigma \alpha :: ('fun, 'atom, 'sets) prot\_subst$ 
  assumes  $T$ : " $wf'_{sst} (set (transaction\_fresh T)) (unlabel (dual_{l_{sst}} (transaction\_strand T)))$ "
    and  $\sigma$ : "transaction_fresh_subst  $\sigma T \mathcal{A}$ "
    and  $\alpha$ : "transaction_renaming_subst  $\alpha P \mathcal{A}$ "
  shows " $wf'_{sst} \{\} (unlabel (dual_{l_{sst}} (transaction\_strand T \cdot_{l_{sst}} \sigma \circ_s \alpha)))$ "
  <proof>

```

2.3.8 Lemmata: Reachable Constraints

```

lemma reachable_constraints_wf_trms:
  assumes " $\forall T \in set P. wf_{trms} (trms\_transaction T)$ "
    and " $\mathcal{A} \in reachable\_constraints P$ "
  shows " $wf_{trms} (trms_{l_{sst}} \mathcal{A})$ "
  <proof>

```

```

lemma reachable_constraints_TAtom_types:
  assumes "A ∈ reachable_constraints P"
  and "∀T ∈ set P. ∀x ∈ set (transaction_fresh T). Γv x = TAtom Value"
  shows "Γv ' fvlsst A ⊆ (∪T ∈ set P. Γv ' fv_transaction T)" (is "?A A")
  and "Γv ' bvarslsst A ⊆ (∪T ∈ set P. Γv ' bvars_transaction T)" (is "?B A")
  and "Γv ' varslsst A ⊆ (∪T ∈ set P. Γv ' vars_transaction T)" (is "?C A")
⟨proof⟩

```

```

lemma reachable_constraints_no_bvars:
  assumes A: "A ∈ reachable_constraints P"
  and P: "∀T ∈ set P. bvarslsst (transaction_strand T) = {}"
  shows "bvarslsst A = {}"
⟨proof⟩

```

```

lemma reachable_constraints_fv_bvars_disj:
  assumes A_reach: "A ∈ reachable_constraints P"
  and P: "∀S ∈ set P. admissible_transaction S"
  shows "fvlsst A ∩ bvarslsst A = {}"
⟨proof⟩

```

```

lemma reachable_constraints_vars_TAtom_typed:
  assumes A_reach: "A ∈ reachable_constraints P"
  and P: "∀T ∈ set P. admissible_transaction T"
  and x: "x ∈ varslsst A"
  shows "Γv x = TAtom Value ∨ (∃a. Γv x = TAtom (Atom a))"
⟨proof⟩

```

```

lemma reachable_constraints_Value_vars_are_fv:
  assumes A_reach: "A ∈ reachable_constraints P"
  and P: "∀T ∈ set P. admissible_transaction T"
  and x: "x ∈ varslsst A"
  and "Γv x = TAtom Value"
  shows "x ∈ fvlsst A"
⟨proof⟩

```

```

lemma reachable_constraints_subterms_subst:
  assumes A_reach: "A ∈ reachable_constraints P"
  and I: "welltyped_constraint_model I A"
  and P: "∀T ∈ set P. admissible_transaction T"
  shows "subtermsset (trmslsst (A ·lsst I)) = (subtermsset (trmslsst A)) ·set I"
⟨proof⟩

```

```

lemma reachable_constraints_val_funs_private:
  assumes A_reach: "A ∈ reachable_constraints P"
  and P: "∀T ∈ set P. admissible_transaction T"
  and f: "f ∈ ∪ (funs_term ' trmslsst A)"
  shows "is_Val f ⇒ ¬public f"
  and "¬is_Abs f"
⟨proof⟩

```

```

lemma reachable_constraints_occurs_fact_ik_case:
  assumes A_reach: "A ∈ reachable_constraints P"
  and P: "∀T ∈ set P. admissible_transaction T"
  and occ: "occurs t ∈ iklsst A"
  shows "∃n. t = Fun (Val (n,False)) []"
⟨proof⟩

```

```

lemma reachable_constraints_occurs_fact_send_ex:
  assumes A_reach: "A ∈ reachable_constraints P"
  and P: "∀T ∈ set P. admissible_transaction T"
  and x: "Γv x = TAtom Value" "x ∈ fvlsst A"

  shows "send⟨occurs (Var x)⟩ ∈ set (unlabel A)"

```

<proof>

```

lemma reachable_constraints_dblsst_set_args_empty:
  assumes A: "A ∈ reachable_constraints P"
    and PP: "list_all wellformed_transaction P"
    and admissible_transaction_updates:
      "let f = (λT. ∀x ∈ set (unlabel (transaction_updates T)).
        is_Update x ∧ is_Var (the_elem_term x) ∧ is_Fun_Set (the_set_term x) ∧
        fst (the_Var (the_elem_term x)) = TAtom Value)
      in list_all f P"
    and d: "(t, s) ∈ set (dblsst A I)"
  shows "∃ss. s = Fun (Set ss) []"
<proof>

```

```

lemma reachable_constraints_occurs_fact_ik_ground:
  assumes A_reach: "A ∈ reachable_constraints P"
    and P: "∀T ∈ set P. admissible_transaction T"
    and t: "occurs t ∈ iklsst A"
  shows "fv (occurs t) = {}"
<proof>

```

```

lemma reachable_constraints_occurs_fact_ik_funs_terms:
  fixes A:: "('fun, 'atom, 'sets, 'lbl) prot_constr"
  assumes A_reach: "A ∈ reachable_constraints P"
    and I: "welltyped_constraint_model I A"
    and P: "∀T ∈ set P. admissible_transaction T"
  shows "∀s ∈ subtermsset (iklsst A ·set I). OccursFact ∉ ∪ (funs_term ' set (snd (Ana s)))" (is "?A A")
    and "∀s ∈ subtermsset (iklsst A ·set I). OccursSec ∉ ∪ (funs_term ' set (snd (Ana s)))" (is "?B A")
    and "Fun OccursSec [] ∉ iklsst A ·set I" (is "?C A")
    and "∀x ∈ varslsst A. I x ≠ Fun OccursSec []" (is "?D A")
<proof>

```

```

lemma reachable_constraints_occurs_fact_ik_subst_aux:
  assumes A_reach: "A ∈ reachable_constraints P"
    and I: "welltyped_constraint_model I A"
    and P: "∀T ∈ set P. admissible_transaction T"
    and t: "t ∈ iklsst A" "t · I = occurs s"
  shows "∃u. t = occurs u"
<proof>

```

```

lemma reachable_constraints_occurs_fact_ik_subst:
  assumes A_reach: "A ∈ reachable_constraints P"
    and I: "welltyped_constraint_model I A"
    and P: "∀T ∈ set P. admissible_transaction T"
    and t: "occurs t ∈ iklsst A ·set I"
  shows "occurs t ∈ iklsst A"
<proof>

```

```

lemma reachable_constraints_occurs_fact_send_in_ik:
  assumes A_reach: "A ∈ reachable_constraints P"
    and I: "welltyped_constraint_model I A"
    and P: "∀T ∈ set P. admissible_transaction T"
    and x: "send(occurs (Var x)) ∈ set (unlabel A)"
  shows "occurs (I x) ∈ iklsst A"
<proof>

```

```

lemma reachable_constraints_fv_bvars_subset:
  assumes A: "A ∈ reachable_constraints P"
  shows "bvarslsst A ⊆ (∪ T ∈ set P. bvars_transaction T)"
<proof>

```

```

lemma reachable_constraints_fv_disj:
  assumes A: "A ∈ reachable_constraints P"

```

shows "fv_{l_{sst}} A \cap ($\bigcup T \in \text{set } P. \text{bvars_transaction } T$) = {}"
 <proof>

lemma reachable_constraints_fv_bvars_disj:
 assumes P: " $\forall T \in \text{set } P. \text{wellformed_transaction } T$ "
 and A: "A \in reachable_constraints P"
 shows "fv_{l_{sst}} A \cap bvars_{l_{sst}} A = {}"
 <proof>

lemma reachable_constraints_wf:
 assumes P:
 " $\forall T \in \text{set } P. \text{wellformed_transaction } T$ "
 " $\forall T \in \text{set } P. \text{wf}_{trms}$ ' arity (trms_transaction T)"
 and A: "A \in reachable_constraints P"
 shows "wf_{sst} (unlabel A)"
 and "wf_{trms} (trms_{l_{sst}} A)"
 <proof>

lemma reachable_constraints_no_Ana_Attack:
 assumes A: "A \in reachable_constraints P"
 and P: " $\forall T \in \text{set } P. \text{admissible_transaction } T$ "
 and t: "t \in subterms_{set} (ik_{l_{sst}} A)"
 shows "attack(n) \notin set (snd (Ana t))"
 <proof>

lemma constraint_model_Value_term_is_Val:
 assumes A_reach: "A \in reachable_constraints P"
 and I: "welltyped_constraint_model I A"
 and P: " $\forall T \in \text{set } P. \text{admissible_transaction } T$ "
 and x: " $\Gamma_v x = \text{TAtom Value}$ " "x \in fv_{l_{sst}} A"
 shows " $\exists n. I x = \text{Fun (Val (n,False)) []}$ "
 <proof>

lemma constraint_model_Value_term_is_Val':
 assumes A_reach: "A \in reachable_constraints P"
 and I: "welltyped_constraint_model I A"
 and P: " $\forall T \in \text{set } P. \text{admissible_transaction } T$ "
 and x: "(TAtom Value, m) \in fv_{l_{sst}} A"
 shows " $\exists n. I (\text{TAtom Value}, m) = \text{Fun (Val (n,False)) []}$ "
 <proof>

lemma constraint_model_Value_var_in_constr_prefix:
 assumes A_reach: "A \in reachable_constraints P"
 and I: "welltyped_constraint_model I A"
 and P: " $\forall T \in \text{set } P. \text{admissible_transaction } T$ "
 shows " $\forall x \in \text{fv}_{l_{sst}} A. \Gamma_v x = \text{TAtom Value}$
 $\rightarrow (\exists B. \text{prefix } B A \wedge x \notin \text{fv}_{l_{sst}} B \wedge I x \in \text{subterms}_{set} (\text{trms}_{l_{sst}} B))$ " (is "?P A")
 <proof>

lemma admissible_transaction_occurs_checks_prop:
 assumes A_reach: "A \in reachable_constraints P"
 and I: "welltyped_constraint_model I A"
 and P: " $\forall T \in \text{set } P. \text{admissible_transaction } T$ "
 and f: "f \in \bigcup (funs_term ' (I ' fv_{l_{sst}} A))"
 shows "is_Val f \implies \neg public f"
 and " \neg is_Abs f"
 <proof>

lemma admissible_transaction_occurs_checks_prop':
 assumes A_reach: "A \in reachable_constraints P"
 and I: "welltyped_constraint_model I A"
 and P: " $\forall T \in \text{set } P. \text{admissible_transaction } T$ "

```

  and f: "f ∈ ∪ (funs_term ' (I ' fvlsst A))"
  shows "∄n. f = Val (n,True)"
  and "∄n. f = Abs n"
⟨proof⟩

```

lemma transaction_var_becomes_Val:

```

  assumes A_reach: "A@duallsst (transaction_strand T ·lsst σ ∘s α) ∈ reachable_constraints P"
  and I: "welltyped_constraint_model I (A@duallsst (transaction_strand T ·lsst σ ∘s α))"
  and σ: "transaction_fresh_subst σ T A"
  and α: "transaction_renaming_subst α P A"
  and P: "∀T ∈ set P. admissible_transaction T"
  and T: "T ∈ set P"
  and x: "x ∈ fv_transaction T" "fst x = TAtom Value"
  shows "∃n. Fun (Val (n,False)) [] = (σ ∘s α) x · I"
⟨proof⟩

```

lemma reachable_constraints_SMP_subset:

```

  assumes A: "A ∈ reachable_constraints P"
  and P: "∀T ∈ set P. ∀x ∈ set (transaction_fresh T). Γv x = TAtom Value"
  shows "SMP (trmslsst A) ⊆ SMP (∪T ∈ set P. trms_transaction T)" (is "?A A")
  and "SMP (pair'setopssst (unlabel A)) ⊆ SMP (∪T ∈ set P. pair'setops_transaction T)" (is "?B A")
⟨proof⟩

```

lemma reachable_constraints_no_Pair_fun:

```

  assumes A: "A ∈ reachable_constraints P"
  and P: "∀T ∈ set P. admissible_transaction T"
  shows "Pair ∉ ∪ (funs_term ' SMP (trmslsst A))"
⟨proof⟩

```

lemma reachable_constraints_setops_form:

```

  assumes A: "A ∈ reachable_constraints P"
  and P: "∀T ∈ set P. admissible_transaction T"
  and t: "t ∈ pair ' setopssst (unlabel A)"
  shows "∃c s. t = pair (c, Fun (Set s) []) ∧ Γ c = TAtom Value"
⟨proof⟩

```

lemma reachable_constraints_setops_type:

```

  fixes t:: "('fun,'atom,'sets) prot_term"
  assumes A: "A ∈ reachable_constraints P"
  and P: "∀T ∈ set P. admissible_transaction T"
  and t: "t ∈ pair ' setopssst (unlabel A)"
  shows "Γ t = TComp Pair [TAtom Value, TAtom SetType]"
⟨proof⟩

```

lemma reachable_constraints_setops_same_type_if_unifiable:

```

  assumes A: "A ∈ reachable_constraints P"
  and P: "∀T ∈ set P. admissible_transaction T"
  shows "∀s ∈ pair ' setopssst (unlabel A). ∀t ∈ pair ' setopssst (unlabel A).
    (∃δ. Unifier δ s t) → Γ s = Γ t"
  (is "?P A")
⟨proof⟩

```

lemma reachable_constraints_setops_unifiable_if_wt_instance_unifiable:

```

  assumes A: "A ∈ reachable_constraints P"
  and P: "∀T ∈ set P. admissible_transaction T"
  shows "∀s ∈ pair ' setopssst (unlabel A). ∀t ∈ pair ' setopssst (unlabel A).
    (∃σ ∅ ρ. wtsubst σ ∧ wtsubst ∅ ∧ wftrms (subst_range σ) ∧ wftrms (subst_range ∅) ∧
      Unifier ρ (s · σ) (t · ∅))
    → (∃δ. Unifier δ s t)"
⟨proof⟩

```

lemma reachable_constraints_tfr:

```

  assumes M:

```

```

    "M  $\equiv \bigcup T \in \text{set } P. \text{trms\_transaction } T$ "
    "has_all_wt_instances_of  $\Gamma M N$ "
    "finite N"
    "tfrset N"
    "wftrms N"
  and P:
    " $\forall T \in \text{set } P. \text{admissible\_transaction } T$ "
    " $\forall T \in \text{set } P. \text{list\_all tfr}_{sstp} (\text{unlabel } (\text{transaction\_strand } T))$ "
  and A: "A  $\in \text{reachable\_constraints } P$ "
  shows "tfrsst (unlabel A)"
<proof>

lemma reachable_constraints_tfr':
  assumes M:
    "M  $\equiv \bigcup T \in \text{set } P. \text{trms\_transaction } T \cup \text{pair' Pair ' setops\_transaction } T$ "
    "has_all_wt_instances_of  $\Gamma M N$ "
    "finite N"
    "tfrset N"
    "wftrms N"
  and P:
    " $\forall T \in \text{set } P. \forall x \in \text{set } (\text{transaction\_fresh } T). \Gamma_v x = \text{TAtom Value}$ "
    " $\forall T \in \text{set } P. \text{wf}_{trms}' \text{arity } (\text{trms\_transaction } T)$ "
    " $\forall T \in \text{set } P. \text{list\_all tfr}_{sstp} (\text{unlabel } (\text{transaction\_strand } T))$ "
  and A: "A  $\in \text{reachable\_constraints } P$ "
  shows "tfrsst (unlabel A)"
<proof>

lemma reachable_constraints_typing_condsst:
  assumes M:
    "M  $\equiv \bigcup T \in \text{set } P. \text{trms\_transaction } T \cup \text{pair' Pair ' setops\_transaction } T$ "
    "has_all_wt_instances_of  $\Gamma M N$ "
    "finite N"
    "tfrset N"
    "wftrms N"
  and P:
    " $\forall T \in \text{set } P. \text{wellformed\_transaction } T$ "
    " $\forall T \in \text{set } P. \text{wf}_{trms}' \text{arity } (\text{trms\_transaction } T)$ "
    " $\forall T \in \text{set } P. \forall x \in \text{set } (\text{transaction\_fresh } T). \Gamma_v x = \text{TAtom Value}$ "
    " $\forall T \in \text{set } P. \text{list\_all tfr}_{sstp} (\text{unlabel } (\text{transaction\_strand } T))$ "
  and A: "A  $\in \text{reachable\_constraints } P$ "
  shows "typing_condsst (unlabel A)"
<proof>

context
begin
private lemma reachable_constraints_par_complsst_aux:
  fixes P
  defines "Ts  $\equiv \text{concat } (\text{map } \text{transaction\_strand } P)$ "
  assumes P_fresh_wf: " $\forall T \in \text{set } P. \forall x \in \text{set } (\text{transaction\_fresh } T). \Gamma_v x = \text{TAtom Value}$ "
    (is " $\forall T \in \text{set } P. \text{?fresh\_wf } T$ ")
  and A: "A  $\in \text{reachable\_constraints } P$ "
  shows " $\forall b \in \text{set } (\text{dual}_{lsst} A). \exists a \in \text{set } Ts. \exists \delta. b = a \cdot_{lsstp} \delta \wedge$ 
     $\text{wt}_{subst} \delta \wedge \text{wf}_{trms} (\text{subst\_range } \delta) \wedge$ 
     $(\forall t \in \text{subst\_range } \delta. (\exists x. t = \text{Var } x) \vee (\exists c. t = \text{Fun } c []))$ "
    (is " $\forall b \in \text{set } (\text{dual}_{lsst} A). \exists a \in \text{set } Ts. \text{?P } b a$ ")
<proof>

lemma reachable_constraints_par_complsst:
  fixes P
  defines "f  $\equiv \lambda M. \{t \cdot \delta \mid t \delta. t \in M \wedge \text{wt}_{subst} \delta \wedge \text{wf}_{trms} (\text{subst\_range } \delta) \wedge \text{fv } (t \cdot \delta) = \{\}\}$ "
    and "Ts  $\equiv \text{concat } (\text{map } \text{transaction\_strand } P)$ "
  assumes P_pc: "comp_par_complsst public arity Ana  $\Gamma$  Pair Ts M S"
    and P_wf: " $\forall T \in \text{set } P. \forall x \in \text{set } (\text{transaction\_fresh } T). \Gamma_v x = \text{TAtom Value}$ "

```

```

    and A: "A ∈ reachable_constraints P"
    shows "par_complsst A ((f (set S)) - {m. intruder_synth {} m})"
  <proof>
end

lemma reachable_constraints_par_comp_constr:
  fixes P f S
  defines "f ≡ λM. {t · δ | t δ. t ∈ M ∧ wtsubst δ ∧ wftrms (subst_range δ) ∧ fv (t · δ) = {}}"
    and "Ts ≡ concat (map transaction_strand P)"
    and "Sec ≡ (f (set S)) - {m. intruder_synth {} m}"
    and "M ≡ ⋃ T ∈ set P. trms_transaction T ∪ pair' Pair ' setops_transaction T"
  assumes M:
    "has_all_wt_instances_of Γ M N"
    "finite N"
    "tfrset N"
    "wftrms N"
  and P:
    "∀ T ∈ set P. wellformed_transaction T"
    "∀ T ∈ set P. wftrms' arity (trms_transaction T)"
    "∀ T ∈ set P. ∀ x ∈ set (transaction_fresh T). Γv x = TAtom Value"
    "∀ T ∈ set P. list_all tfrsstp (unlabel (transaction_strand T))"
    "comp_par_complsst public arity Ana Γ Pair Ts M_fun S"
  and A: "A ∈ reachable_constraints P"
  and I: "constraint_model I A"
  shows "∃ Iτ. welltyped_constraint_model Iτ A ∧
    ((∀ n. welltyped_constraint_model Iτ (proj n A)) ∨
    (∃ A'. prefix A' A ∧ strand_leakslsst A' Sec Iτ))"
  <proof>
end

end

end

```

2.4 Term Variants (Term_Variants)

```

theory Term_Variants
  imports Stateful_Protocol_Composition_and_Typing.Intruder_Deduction
begin

fun term_variants where
  "term_variants P (Var x) = [Var x]"
| "term_variants P (Fun f T) = (
  let S = product_lists (map (term_variants P) T)
  in map (Fun f) S@concat (map (λg. map (Fun g) S) (P f)))"

inductive term_variants_pred where
  term_variants_Var:
    "term_variants_pred P (Var x) (Var x)"
| term_variants_P:
  "[length T = length S; ∧i. i < length T ⇒ term_variants_pred P (T ! i) (S ! i); g ∈ set (P f)]
  ⇒ term_variants_pred P (Fun f T) (Fun g S)"
| term_variants_Fun:
  "[length T = length S; ∧i. i < length T ⇒ term_variants_pred P (T ! i) (S ! i)]
  ⇒ term_variants_pred P (Fun f T) (Fun f S)"

lemma term_variants_pred_inv:
  assumes "term_variants_pred P (Fun f T) (Fun h S)"
  shows "length T = length S"
    and "∧i. i < length T ⇒ term_variants_pred P (T ! i) (S ! i)"
    and "f ≠ h ⇒ h ∈ set (P f)"
  <proof>

```

```

lemma term_variants_pred_inv':
  assumes "term_variants_pred P (Fun f T) t"
  shows "is_Fun t"
    and "length T = length (args t)"
    and " $\bigwedge i. i < \text{length } T \implies \text{term\_variants\_pred } P (T ! i) (\text{args } t ! i)$ "
    and " $f \neq \text{the\_Fun } t \implies \text{the\_Fun } t \in \text{set } (P f)$ "
    and " $P \equiv (\lambda_. []) (g := [h]) \implies f \neq \text{the\_Fun } t \implies f = g \wedge \text{the\_Fun } t = h$ "
  <proof>

```

```

lemma term_variants_pred_inv'':
  assumes "term_variants_pred P t (Fun f T)"
  shows "is_Fun t"
    and "length T = length (args t)"
    and " $\bigwedge i. i < \text{length } T \implies \text{term\_variants\_pred } P (\text{args } t ! i) (T ! i)$ "
    and " $f \neq \text{the\_Fun } t \implies f \in \text{set } (P (\text{the\_Fun } t))$ "
    and " $P \equiv (\lambda_. []) (g := [h]) \implies f \neq \text{the\_Fun } t \implies f = h \wedge \text{the\_Fun } t = g$ "
  <proof>

```

```

lemma term_variants_pred_inv_Var:
  "term_variants_pred P (Var x) t  $\longleftrightarrow$  t = Var x"
  "term_variants_pred P t (Var x)  $\longleftrightarrow$  t = Var x"
  <proof>

```

```

lemma term_variants_pred_inv_const:
  "term_variants_pred P (Fun c []) t  $\longleftrightarrow$  (( $\exists g \in \text{set } (P c). t = \text{Fun } g []$ )  $\vee$  (t = Fun c []))"
  <proof>

```

```

lemma term_variants_pred_refl: "term_variants_pred P t t"
  <proof>

```

```

lemma term_variants_pred_refl_inv:
  assumes st: "term_variants_pred P s t"
  and P: " $\forall f. \forall g \in \text{set } (P f). f = g$ "
  shows "s = t"
  <proof>

```

```

lemma term_variants_pred_const:
  assumes "b  $\in$  set (P a)"
  shows "term_variants_pred P (Fun a []) (Fun b [])"
  <proof>

```

```

lemma term_variants_pred_const_cases:
  "P a  $\neq$  []  $\implies$  term_variants_pred P (Fun a []) t  $\longleftrightarrow$ 
    (t = Fun a []  $\vee$  ( $\exists b \in \text{set } (P a). t = \text{Fun } b []$ ))"
  "P a = []  $\implies$  term_variants_pred P (Fun a []) t  $\longleftrightarrow$  t = Fun a []"
  <proof>

```

```

lemma term_variants_pred_param:
  assumes "term_variants_pred P t s"
  and fg: " $f = g \vee g \in \text{set } (P f)$ "
  shows "term_variants_pred P (Fun f (S@t#T)) (Fun g (S@s#T))"
  <proof>

```

```

lemma term_variants_pred_Cons:
  assumes t: "term_variants_pred P t s"
  and T: "term_variants_pred P (Fun f T) (Fun f S)"
  and fg: " $f = g \vee g \in \text{set } (P f)$ "
  shows "term_variants_pred P (Fun f (t#T)) (Fun g (s#S))"
  <proof>

```

```

lemma term_variants_pred_dense:
  fixes P Q: "'a set" and fs gs: "'a list"
  defines "P_fs x  $\equiv$  if x  $\in$  P then fs else []"

```



```

    and "P_gs x ≡ if x ∈ P then gs else []"
    and "Q_fs x ≡ if x ∈ Q then fs else []"
    assumes ut: "term_variants_pred P_fs u t"
    and g: "g ∈ Q" "g ∈ set gs"
    shows "∃s. term_variants_pred P_gs u s ∧ term_variants_pred Q_fs s t"
  <proof>

lemma term_variants_pred_dense':
  assumes ut: "term_variants_pred ((λ_. []) (a := [b])) u t"
  shows "∃s. term_variants_pred ((λ_. []) (a := [c])) u s ∧
        term_variants_pred ((λ_. []) (c := [b])) s t"
  <proof>

lemma term_variants_pred_eq_case:
  fixes t s: "('a,'b) term"
  assumes "term_variants_pred P t s" "∀f ∈ funs_term t. P f = []"
  shows "t = s"
  <proof>

lemma term_variants_pred_subst:
  assumes "term_variants_pred P t s"
  shows "term_variants_pred P (t · δ) (s · δ)"
  <proof>

lemma term_variants_pred_subst':
  fixes t s: "('a,'b) term" and δ: "('a,'b) subst"
  assumes "term_variants_pred P (t · δ) s"
    and "∀x ∈ fv t ∪ fv s. (∃y. δ x = Var y) ∨ (∃f. δ x = Fun f [] ∧ P f = [])"
  shows "∃u. term_variants_pred P t u ∧ s = u · δ"
  <proof>

lemma term_variants_pred_iff_in_term_variants:
  fixes t: "('a,'b) term"
  shows "term_variants_pred P t s ↔ s ∈ set (term_variants P t)"
    (is "?A t s ↔ ?B t s")
  <proof>

lemma term_variants_pred_finite:
  "finite {s. term_variants_pred P t s}"
  <proof>

lemma term_variants_pred_fv_eq:
  assumes "term_variants_pred P s t"
  shows "fv s = fv t"
  <proof>

lemma (in intruder_model) term_variants_pred_wf_trms:
  assumes "term_variants_pred P s t"
    and "∧f g. g ∈ set (P f) ⇒ arity f = arity g"
    and "wf_trm s"
  shows "wf_trm t"
  <proof>

lemma term_variants_pred_funs_term:
  assumes "term_variants_pred P s t"
    and "f ∈ funs_term t"
  shows "f ∈ funs_term s ∨ (∃g ∈ funs_term s. f ∈ set (P g))"
  <proof>
end

```

2.5 Term Implication (Term_Implication)

```
theory Term_Implication
  imports Stateful_Protocol_Model Term_Variants
begin
```

2.5.1 Single Term Implications

```
definition timpl_apply_term ("⟨_ --> _⟩⟨_⟩") where
  "⟨a --> b⟩⟨t⟩ ≡ term_variants ((λ_. []) (Abs a := [Abs b])) t"
```

```
definition timpl_apply_terms ("⟨_ --> _⟩⟨_⟩set") where
  "⟨a --> b⟩⟨M⟩set ≡ ⋃ ((set o timpl_apply_term a b) ` M)"
```

```
lemma timpl_apply_Fun:
  assumes "∧i. i < length T ⇒ S ! i ∈ set ⟨a --> b⟩⟨T ! i⟩"
  and "length T = length S"
  shows "Fun f S ∈ set ⟨a --> b⟩⟨Fun f T⟩"
⟨proof⟩
```

```
lemma timpl_apply_Abs:
  assumes "∧i. i < length T ⇒ S ! i ∈ set ⟨a --> b⟩⟨T ! i⟩"
  and "length T = length S"
  shows "Fun (Abs b) S ∈ set ⟨a --> b⟩⟨Fun (Abs a) T⟩"
⟨proof⟩
```

```
lemma timpl_apply_refl: "t ∈ set ⟨a --> b⟩⟨t⟩"
⟨proof⟩
```

```
lemma timpl_apply_const: "Fun (Abs b) [] ∈ set ⟨a --> b⟩⟨Fun (Abs a) []⟩"
⟨proof⟩
```

```
lemma timpl_apply_const':
  "c = a ⇒ set ⟨a --> b⟩⟨Fun (Abs c) []⟩ = {Fun (Abs b) [], Fun (Abs c) []}"
  "c ≠ a ⇒ set ⟨a --> b⟩⟨Fun (Abs c) []⟩ = {Fun (Abs c) []}"
⟨proof⟩
```

```
lemma timpl_apply_term_subst:
  "s ∈ set ⟨a --> b⟩⟨t⟩ ⇒ s · δ ∈ set ⟨a --> b⟩⟨t · δ⟩"
⟨proof⟩
```

```
lemma timpl_apply_inv:
  assumes "Fun h S ∈ set ⟨a --> b⟩⟨Fun f T⟩"
  shows "length T = length S"
  and "∧i. i < length T ⇒ S ! i ∈ set ⟨a --> b⟩⟨T ! i⟩"
  and "f ≠ h ⇒ f = Abs a ∧ h = Abs b"
⟨proof⟩
```

```
lemma timpl_apply_inv':
  assumes "s ∈ set ⟨a --> b⟩⟨Fun f T⟩"
  shows "∃g S. s = Fun g S"
⟨proof⟩
```

```
lemma timpl_apply_term_Var_iff:
  "Var x ∈ set ⟨a --> b⟩⟨t⟩ ↔ t = Var x"
⟨proof⟩
```

2.5.2 Term Implication Closure

```
inductive_set timpl_closure for t TI where
```

```
  FP: "t ∈ timpl_closure t TI"
```

```
  / TI: "[u ∈ timpl_closure t TI; (a,b) ∈ TI; term_variants_pred ((λ_. []) (Abs a := [Abs b])) u s]
  ⇒ s ∈ timpl_closure t TI"
```

definition "timpl_closure_set M TI \equiv ($\bigcup t \in M. \text{timpl_closure } t \text{ TI}$)"

inductive_set timpl_closure'_step for TI where

"[(a,b) \in TI; term_variants_pred (($\lambda_. []$)(Abs a := [Abs b])) t s]
 \implies (t,s) \in timpl_closure'_step TI"

definition "timpl_closure' TI \equiv (timpl_closure'_step TI)*"

definition comp_timpl_closure where

"comp_timpl_closure FP TI \equiv
 let f = $\lambda X. FP \cup (\bigcup x \in X. \bigcup (a,b) \in TI. \text{set } \langle a \dashrightarrow b \rangle \langle x \rangle)$
 in while ($\lambda X. f X \neq X$) f {}"

definition comp_timpl_closure_list where

"comp_timpl_closure_list FP TI \equiv
 let f = $\lambda X. \text{remdups } (\text{concat } (\text{map } (\lambda x. \text{concat } (\text{map } (\lambda (a,b). \langle a \dashrightarrow b \rangle \langle x \rangle) \text{TI})) X))$
 in while ($\lambda X. \text{set } (f X) \neq \text{set } X$) f FP"

lemma timpl_closure_setI:

"t \in M \implies t \in timpl_closure_set M TI"
 <proof>

lemma timpl_closure_set_empty_timpls:

"timpl_closure t {} = {t}" (is "?A = ?B")
 <proof>

lemmas timpl_closure_set_is_timpl_closure_union = meta_eq_to_obj_eq[OF timpl_closure_set_def]

lemma term_variants_pred_eq_case_Abs:

fixes a b
 defines "P \equiv ($\lambda_. []$)(Abs a := [Abs b])"
 assumes "term_variants_pred P t s" " $\forall f \in \text{funs_term } s. \neg \text{is_Abs } f$ "
 shows "t = s"
 <proof>

lemma timpl_closure'_step_inv:

assumes "(t,s) \in timpl_closure'_step TI"
 obtains a b where "(a,b) \in TI" "term_variants_pred (($\lambda_. []$)(Abs a := [Abs b])) t s"
 <proof>

lemma timpl_closure_mono:

assumes "TI \subseteq TI'"
 shows "timpl_closure t TI \subseteq timpl_closure t TI'"
 <proof>

lemma timpl_closure_set_mono:

assumes "M \subseteq M'" "TI \subseteq TI'"
 shows "timpl_closure_set M TI \subseteq timpl_closure_set M' TI'"
 <proof>

lemma timpl_closure_idem:

"timpl_closure_set (timpl_closure t TI) TI = timpl_closure t TI" (is "?A = ?B")
 <proof>

lemma timpl_closure_set_idem:

"timpl_closure_set (timpl_closure_set M TI) TI = timpl_closure_set M TI"
 <proof>

lemma timpl_closure_set_mono_timpl_closure_set:

assumes N: "N \subseteq timpl_closure_set M TI"
 shows "timpl_closure_set N TI \subseteq timpl_closure_set M TI"
 <proof>

```

lemma timpl_closure_is_timpl_closure':
  "s ∈ timpl_closure t TI ↔ (t,s) ∈ timpl_closure' TI"
  <proof>

lemma timpl_closure'_mono:
  assumes "TI ⊆ TI'"
  shows "timpl_closure' TI ⊆ timpl_closure' TI'"
  <proof>

lemma timpl_closure_ton_is_timpl_closure:
  "timpl_closure_set {t} TI = timpl_closure t TI"
  <proof>

lemma timpl_closure'_timpls_trancl_subset:
  "timpl_closure' (c+) ⊆ timpl_closure' c"
  <proof>

lemma timpl_closure'_timpls_trancl_subset':
  "timpl_closure' {(a,b) ∈ c+. a ≠ b} ⊆ timpl_closure' c"
  <proof>

lemma timpl_closure_set_timpls_trancl_subset:
  "timpl_closure_set M (c+) ⊆ timpl_closure_set M c"
  <proof>

lemma timpl_closure_set_timpls_trancl_subset':
  "timpl_closure_set M {(a,b) ∈ c+. a ≠ b} ⊆ timpl_closure_set M c"
  <proof>

lemma timpl_closure'_timpls_trancl_supset':
  "timpl_closure' c ⊆ timpl_closure' {(a,b) ∈ c+. a ≠ b}"
  <proof>

lemma timpl_closure'_timpls_trancl_supset:
  "timpl_closure' c ⊆ timpl_closure' (c+)"
  <proof>

lemma timpl_closure'_timpls_trancl_eq:
  "timpl_closure' (c+) = timpl_closure' c"
  <proof>

lemma timpl_closure'_timpls_trancl_eq':
  "timpl_closure' {(a,b) ∈ c+. a ≠ b} = timpl_closure' c"
  <proof>

lemma timpl_closure'_timpls_rtrancl_subset:
  "timpl_closure' (c*) ⊆ timpl_closure' c"
  <proof>

lemma timpl_closure'_timpls_rtrancl_supset:
  "timpl_closure' c ⊆ timpl_closure' (c*)"
  <proof>

lemma timpl_closure'_timpls_rtrancl_eq:
  "timpl_closure' (c*) = timpl_closure' c"
  <proof>

lemma timpl_closure_timpls_trancl_eq:
  "timpl_closure t (c+) = timpl_closure t c"
  <proof>

lemma timpl_closure_set_timpls_trancl_eq:

```

```

"timpl_closure_set M (c+) = timpl_closure_set M c"
⟨proof⟩

lemma timpl_closure_set_timpls_trancl_eq':
  "timpl_closure_set M {(a,b) ∈ c+. a ≠ b} = timpl_closure_set M c"
⟨proof⟩

lemma timpl_closure_Var_in_iff:
  "Var x ∈ timpl_closure t TI ↔ t = Var x" (is "?A ↔ ?B")
⟨proof⟩

lemma timpl_closure_set_Var_in_iff:
  "Var x ∈ timpl_closure_set M TI ↔ Var x ∈ M"
⟨proof⟩

lemma timpl_closure_Var_inv:
  assumes "t ∈ timpl_closure (Var x) TI"
  shows "t = Var x"
⟨proof⟩

lemma timpls_Un_mono: "mono (λX. FP ∪ (∪x ∈ X. ∪(a,b) ∈ TI. set ⟨a --> b⟩⟨x⟩))"
⟨proof⟩

lemma timpl_closure_set_lfp:
  fixes M TI
  defines "f ≡ λX. M ∪ (∪x ∈ X. ∪(a,b) ∈ TI. set ⟨a --> b⟩⟨x⟩)"
  shows "lfp f = timpl_closure_set M TI"
⟨proof⟩

lemma timpl_closure_set_supset:
  assumes "∀t ∈ FP. t ∈ closure"
  and "∀t ∈ closure. ∀(a,b) ∈ TI. ∀s ∈ set ⟨a --> b⟩⟨t⟩. s ∈ closure"
  shows "timpl_closure_set FP TI ⊆ closure"
⟨proof⟩

lemma timpl_closure_set_supset':
  assumes "∀t ∈ FP. ∀(a,b) ∈ TI. ∀s ∈ set ⟨a --> b⟩⟨t⟩. s ∈ FP"
  shows "timpl_closure_set FP TI ⊆ FP"
⟨proof⟩

lemma timpl_closure'_param:
  assumes "(t,s) ∈ timpl_closure' c"
  and fg: "f = g ∨ (∃a b. (a,b) ∈ c ∧ f = Abs a ∧ g = Abs b)"
  shows "(Fun f (S@t#T), Fun g (S@s#T)) ∈ timpl_closure' c"
⟨proof⟩

lemma timpl_closure'_param':
  assumes "(t,s) ∈ timpl_closure' c"
  shows "(Fun f (S@t#T), Fun f (S@s#T)) ∈ timpl_closure' c"
⟨proof⟩

lemma timpl_closure_FunI:
  assumes IH: "∧i. i < length T ⇒ (T ! i, S ! i) ∈ timpl_closure' c"
  and len: "length T = length S"
  and fg: "f = g ∨ (∃a b. (a,b) ∈ c+ ∧ f = Abs a ∧ g = Abs b)"
  shows "(Fun f T, Fun g S) ∈ timpl_closure' c"
⟨proof⟩

lemma timpl_closure_FunI':
  assumes IH: "∧i. i < length T ⇒ (T ! i, S ! i) ∈ timpl_closure' c"
  and len: "length T = length S"
  shows "(Fun f T, Fun f S) ∈ timpl_closure' c"
⟨proof⟩

```

lemma `timpl_closure_FunI2`:

fixes `f g::('a, 'b, 'c) prot_fun`
 assumes `IH: "∧i. i < length T ⇒ ∃u. (T!i, u) ∈ timpl_closure' c ∧ (S!i, u) ∈ timpl_closure' c"`
 and `len: "length T = length S"`
 and `fg: "f = g ∨ (∃a b d. (a, d) ∈ c+ ∧ (b, d) ∈ c+ ∧ f = Abs a ∧ g = Abs b)"`
 shows `"∃h U. (Fun f T, Fun h U) ∈ timpl_closure' c ∧ (Fun g S, Fun h U) ∈ timpl_closure' c"`
`<proof>`

lemma `timpl_closure_FunI3`:

fixes `f g::('a, 'b, 'c) prot_fun`
 assumes `IH: "∧i. i < length T ⇒ ∃u. (T!i, u) ∈ timpl_closure' c ∧ (S!i, u) ∈ timpl_closure' c"`
 and `len: "length T = length S"`
 and `fg: "f = g ∨ (∃a b d. (a, d) ∈ c ∧ (b, d) ∈ c ∧ f = Abs a ∧ g = Abs b)"`
 shows `"∃h U. (Fun f T, Fun h U) ∈ timpl_closure' c ∧ (Fun g S, Fun h U) ∈ timpl_closure' c"`
`<proof>`

lemma `timpl_closure_fv_eq`:

assumes `"s ∈ timpl_closure t T"`
 shows `"fv s = fv t"`
`<proof>`

lemma (in `stateful_protocol_model`) `timpl_closure_subst`:

assumes `t: "wftrm t" "∀x ∈ fv t. ∃a. Γv x = TAtom (Atom a)"`
 and `δ: "wtsubst δ" "wftrms (subst_range δ)"`
 shows `"timpl_closure (t · δ) T = timpl_closure t T ·set δ"`
`<proof>`

lemma (in `stateful_protocol_model`) `timpl_closure_subst_subset`:

assumes `t: "t ∈ M"`
 and `M: "wftrms M" "∀x ∈ fvset M. ∃a. Γv x = TAtom (Atom a)"`
 and `δ: "wtsubst δ" "wftrms (subst_range δ)" "ground (subst_range δ)" "subst_domain δ ⊆ fvset M"`
 and `Msupset: "timpl_closure t T ⊆ M"`
 shows `"timpl_closure (t · δ) T ⊆ M ·set δ"`
`<proof>`

lemma (in `stateful_protocol_model`) `timpl_closure_set_subst_subset`:

assumes `M: "wftrms M" "∀x ∈ fvset M. ∃a. Γv x = TAtom (Atom a)"`
 and `δ: "wtsubst δ" "wftrms (subst_range δ)" "ground (subst_range δ)" "subst_domain δ ⊆ fvset M"`
 and `Msupset: "timpl_closure_set M T ⊆ M"`
 shows `"timpl_closure_set (M ·set δ) T ⊆ M ·set δ"`
`<proof>`

lemma `timpl_closure_set_Union`:

`"timpl_closure_set (∪Ms) T = (∪M ∈ Ms. timpl_closure_set M T)"`
`<proof>`

lemma `timpl_closure_set_Union_subst_set`:

assumes `"s ∈ timpl_closure_set (∪{M ·set δ | δ. P δ}) T"`
 shows `"∃δ. P δ ∧ s ∈ timpl_closure_set (M ·set δ) T"`
`<proof>`

lemma `timpl_closure_set_Union_subst_singleton`:

assumes `"s ∈ timpl_closure_set {t · δ | δ. P δ} T"`
 shows `"∃δ. P δ ∧ s ∈ timpl_closure_set {t · δ} T"`
`<proof>`

lemma `timpl_closure'_inv`:

assumes `"(s, t) ∈ timpl_closure' TI"`
 shows `"(∃x. s = Var x ∧ t = Var x) ∨ (∃f g S T. s = Fun f S ∧ t = Fun g T ∧ length S = length T)"`
`<proof>`

lemma `timpl_closure'_inv'`:

```

assumes "(s, t) ∈ timpl_closure' TI"
shows "(∃x. s = Var x ∧ t = Var x) ∨
  (∃f g S T. s = Fun f S ∧ t = Fun g T ∧ length S = length T ∧
    (∀i < length T. (S ! i, T ! i) ∈ timpl_closure' TI) ∧
    (f ≠ g → is_Abs f ∧ is_Abs g ∧ (the_Abs f, the_Abs g) ∈ TI+))"
  (is "?A s t ∨ ?B s t (timpl_closure' TI)")
<proof>

```

```

lemma timpl_closure'_inv'':
  assumes "(Fun f S, Fun g T) ∈ timpl_closure' TI"
  shows "length S = length T"
  and "∧i. i < length T ⇒ (S ! i, T ! i) ∈ timpl_closure' TI"
  and "f ≠ g ⇒ is_Abs f ∧ is_Abs g ∧ (the_Abs f, the_Abs g) ∈ TI+"
<proof>

```

```

lemma timpl_closure_Fun_inv:
  assumes "s ∈ timpl_closure (Fun f T) TI"
  shows "∃g S. s = Fun g S"
<proof>

```

```

lemma timpl_closure_Fun_inv':
  assumes "Fun g S ∈ timpl_closure (Fun f T) TI"
  shows "length S = length T"
  and "∧i. i < length S ⇒ S ! i ∈ timpl_closure (T ! i) TI"
  and "f ≠ g ⇒ is_Abs f ∧ is_Abs g ∧ (the_Abs f, the_Abs g) ∈ TI+"
<proof>

```

```

lemma timpl_closure_Fun_not_Var[simp]:
  "Fun f T ∉ timpl_closure (Var x) TI"
<proof>

```

```

lemma timpl_closure_Var_not_Fun[simp]:
  "Var x ∉ timpl_closure (Fun f T) TI"
<proof>

```

```

lemma (in stateful_protocol_model) timpl_closure_wf_trms:
  assumes m: "wftrms m"
  shows "wftrms (timpl_closure m TI)"
<proof>

```

```

lemma (in stateful_protocol_model) timpl_closure_set_wf_trms:
  assumes M: "wftrms M"
  shows "wftrms (timpl_closure_set M TI)"
<proof>

```

```

lemma timpl_closure_Fu_inv:
  assumes "t ∈ timpl_closure (Fun (Fu f) T) TI"
  shows "∃S. length S = length T ∧ t = Fun (Fu f) S"
<proof>

```

```

lemma timpl_closure_Fu_inv':
  assumes "Fun (Fu f) T ∈ timpl_closure t TI"
  shows "∃S. length S = length T ∧ t = Fun (Fu f) S"
<proof>

```

```

lemma timpl_closure_no_Abs_eq:
  assumes "t ∈ timpl_closure s TI"
  and "∀f ∈ funs_term t. ¬is_Abs f"
  shows "t = s"
<proof>

```

```

lemma timpl_closure_set_no_Abs_in_set:
  assumes "t ∈ timpl_closure_set FP TI"

```

```

    and "∀ f ∈ funs_term t. ¬is_Abs f"
    shows "t ∈ FP"
  ⟨proof⟩

lemma timpl_closure_funs_term_subset:
  "⋃ (funs_term ' (timpl_closure t TI)) ⊆ funs_term t ∪ Abs ' snd ' TI"
  (is "?A ⊆ ?B ∪ ?C")
  ⟨proof⟩

lemma timpl_closure_set_funs_term_subset:
  "⋃ (funs_term ' (timpl_closure_set FP TI)) ⊆ ⋃ (funs_term ' FP) ∪ Abs ' snd ' TI"
  ⟨proof⟩

lemma funs_term_OCC_TI_subset:
  defines "absc ≡ λa. Fun (Abs a) []"
  assumes OCC1: "∀ t ∈ FP. ∀ f ∈ funs_term t. is_Abs f → f ∈ Abs ' OCC"
    and OCC2: "snd ' TI ⊆ OCC"
  shows "∀ t ∈ timpl_closure_set FP TI. ∀ f ∈ funs_term t. is_Abs f → f ∈ Abs ' OCC" (is ?A)
    and "∀ t ∈ absc ' OCC. ∀ (a,b) ∈ TI. ∀ s ∈ set ⟨a --> b⟩t. s ∈ absc ' OCC" (is ?B)
  ⟨proof⟩

lemma (in stateful_protocol_model) intruder_synth_timpl_closure_set:
  fixes M:: "('fun,'atom,'sets) prot_terms" and t:: "('fun,'atom,'sets) prot_term"
  assumes "M ⊢c t"
    and "s ∈ timpl_closure t TI"
  shows "timpl_closure_set M TI ⊢c s"
  ⟨proof⟩

lemma (in stateful_protocol_model) intruder_synth_timpl_closure':
  fixes M:: "('fun,'atom,'sets) prot_terms" and t:: "('fun,'atom,'sets) prot_term"
  assumes "timpl_closure_set M TI ⊢c t"
    and "s ∈ timpl_closure t TI"
  shows "timpl_closure_set M TI ⊢c s"
  ⟨proof⟩

lemma timpl_closure_set_absc_subset_in:
  defines "absc ≡ λa. Fun (Abs a) []"
  assumes A: "timpl_closure_set (absc ' A) TI ⊆ absc ' A"
    and a: "a ∈ A" "(a,b) ∈ TI+"
  shows "b ∈ A"
  ⟨proof⟩

```

2.5.3 Composition-only Intruder Deduction Modulo Term Implication Closure of the Intruder Knowledge

```

context stateful_protocol_model
begin

fun in_trancl where
  "in_trancl TI a b = (
    if (a,b) ∈ set TI then True
    else list_ex (λ(c,d). c = a ∧ in_trancl (removeAll (c,d) TI) d b) TI)"

definition in_rtrancl where
  "in_rtrancl TI a b ≡ a = b ∨ in_trancl TI a b"

declare in_trancl.simps[simp del]

fun timpls_transformable_to where
  "timpls_transformable_to TI (Var x) (Var y) = (x = y)"
| "timpls_transformable_to TI (Fun f T) (Fun g S) = (
  (f = g ∨ (is_Abs f ∧ is_Abs g ∧ (the_Abs f, the_Abs g) ∈ set TI)) ∧
  list_all2 (timpls_transformable_to TI) T S)"

```



```

| "timpls_transformable_to _ _ _ = False"

fun timpls_transformable_to' where
  "timpls_transformable_to' TI (Var x) (Var y) = (x = y)"
| "timpls_transformable_to' TI (Fun f T) (Fun g S) = (
  (f = g ∨ (is_Abs f ∧ is_Abs g ∧ in_trancl TI (the_Abs f) (the_Abs g))) ∧
  list_all2 (timpls_transformable_to' TI) T S)"
| "timpls_transformable_to' _ _ _ = False"

fun equal_mod_timpls where
  "equal_mod_timpls TI (Var x) (Var y) = (x = y)"
| "equal_mod_timpls TI (Fun f T) (Fun g S) = (
  (f = g ∨ (is_Abs f ∧ is_Abs g ∧
    ((the_Abs f, the_Abs g) ∈ set TI ∨
     (the_Abs g, the_Abs f) ∈ set TI ∨
     (∃ ti ∈ set TI. (the_Abs f, snd ti) ∈ set TI ∧ (the_Abs g, snd ti) ∈ set TI)))) ∧
  list_all2 (equal_mod_timpls TI) T S)"
| "equal_mod_timpls _ _ _ = False"

fun intruder_synth_mod_timpls where
  "intruder_synth_mod_timpls M TI (Var x) = List.member M (Var x)"
| "intruder_synth_mod_timpls M TI (Fun f T) = (
  (list_ex (λt. timpls_transformable_to TI t (Fun f T)) M) ∨
  (public f ∧ length T = arity f ∧ list_all (intruder_synth_mod_timpls M TI) T))"

fun intruder_synth_mod_timpls' where
  "intruder_synth_mod_timpls' M TI (Var x) = List.member M (Var x)"
| "intruder_synth_mod_timpls' M TI (Fun f T) = (
  (list_ex (λt. timpls_transformable_to' TI t (Fun f T)) M) ∨
  (public f ∧ length T = arity f ∧ list_all (intruder_synth_mod_timpls' M TI) T))"

fun intruder_synth_mod_eq_timpls where
  "intruder_synth_mod_eq_timpls M TI (Var x) = (Var x ∈ M)"
| "intruder_synth_mod_eq_timpls M TI (Fun f T) = (
  (∃ t ∈ M. equal_mod_timpls TI t (Fun f T)) ∨
  (public f ∧ length T = arity f ∧ list_all (intruder_synth_mod_eq_timpls M TI) T))"

definition analyzed_closed_mod_timpls where
  "analyzed_closed_mod_timpls M TI ≡
  let f = list_all (intruder_synth_mod_timpls M TI);
      g = λt. if f (fst (Ana t)) then f (snd (Ana t))
            else ∀ s ∈ comp_timpl_closure {t} (set TI). case Ana s of (K,R) ⇒ f K → f R
  in list_all g M"

definition analyzed_closed_mod_timpls' where
  "analyzed_closed_mod_timpls' M TI ≡
  let f = list_all (intruder_synth_mod_timpls' M TI);
      g = λt. if f (fst (Ana t)) then f (snd (Ana t))
            else ∀ s ∈ comp_timpl_closure {t} (set TI). case Ana s of (K,R) ⇒ f K → f R
  in list_all g M"

definition analyzed_closed_mod_timpls_alt where
  "analyzed_closed_mod_timpls_alt M TI timpl_cl_witness ≡
  let f = λR. ∀ r ∈ set R. intruder_synth_mod_timpls M TI r;
      N = {t ∈ set M. f (fst (Ana t))};
      N' = set M - N
  in (∀ t ∈ N. f (snd (Ana t))) ∧
  (N' ≠ {} → (N' ∪ (∪ x ∈ timpl_cl_witness. ∪ (a,b) ∈ set TI. set ⟨a → b⟩⟨x⟩) ⊆ timpl_cl_witness))
  ∧
  (∀ s ∈ timpl_cl_witness. case Ana s of (K,R) ⇒ f K → f R)"

lemma in_trancl_closure_iff_in_trancl_fun:
  "(a,b) ∈ (set TI)+ ↔ in_trancl TI a b" (is "?A TI a b ↔ ?B TI a b")

```

<proof>

lemma *in_rtrancl_closure_iff_in_rtrancl_fun*:

"(a,b) ∈ (set TI)* \longleftrightarrow in_rtrancl TI a b"

<proof>

lemma *in_trancl_mono*:

assumes "set TI \subseteq set TI'"

and "in_trancl TI a b"

shows "in_trancl TI' a b"

<proof>

lemma *equal_mod_timpls_refl*:

"equal_mod_timpls TI t t"

<proof>

lemma *equal_mod_timpls_inv_Var*:

"equal_mod_timpls TI (Var x) t \implies t = Var x" (is "?A \implies ?C")

"equal_mod_timpls TI t (Var x) \implies t = Var x" (is "?B \implies ?C")

<proof>

lemma *equal_mod_timpls_inv*:

assumes "equal_mod_timpls TI (Fun f T) (Fun g S)"

shows "length T = length S"

and " $\bigwedge i. i < \text{length } T \implies \text{equal_mod_timpls } TI (T ! i) (S ! i)$ "

and " $f \neq g \implies (\text{is_Abs } f \wedge \text{is_Abs } g \wedge ($
 $(\text{the_Abs } f, \text{the_Abs } g) \in \text{set } TI \vee (\text{the_Abs } g, \text{the_Abs } f) \in \text{set } TI \vee$
 $(\exists ti \in \text{set } TI. (\text{the_Abs } f, \text{snd } ti) \in \text{set } TI \wedge$
 $(\text{the_Abs } g, \text{snd } ti) \in \text{set } TI)))$ "

<proof>

lemma *equal_mod_timpls_inv'*:

assumes "equal_mod_timpls TI (Fun f T) t"

shows "is_Fun t"

and "length T = length (args t)"

and " $\bigwedge i. i < \text{length } T \implies \text{equal_mod_timpls } TI (T ! i) (\text{args } t ! i)$ "

and " $f \neq \text{the_Fun } t \implies (\text{is_Abs } f \wedge \text{is_Abs } (\text{the_Fun } t) \wedge ($
 $(\text{the_Abs } f, \text{the_Abs } (\text{the_Fun } t)) \in \text{set } TI \vee$
 $(\text{the_Abs } (\text{the_Fun } t), \text{the_Abs } f) \in \text{set } TI \vee$
 $(\exists ti \in \text{set } TI. (\text{the_Abs } f, \text{snd } ti) \in \text{set } TI \wedge$
 $(\text{the_Abs } (\text{the_Fun } t), \text{snd } ti) \in \text{set } TI)))$ "

and " $\neg \text{is_Abs } f \implies f = \text{the_Fun } t$ "

<proof>

lemma *equal_mod_timpls_if_term_variants*:

fixes s t::"('a, 'b, 'c) prot_fun, 'd) term" and a b::"'c set"

defines "P $\equiv (\lambda_. []) (\text{Abs } a := [\text{Abs } b])$ "

assumes st: "term_variants_pred P s t"

and ab: "(a,b) ∈ set TI"

shows "equal_mod_timpls TI s t"

<proof>

lemma *equal_mod_timpls_mono*:

assumes "set TI \subseteq set TI'"

and "equal_mod_timpls TI s t"

shows "equal_mod_timpls TI' s t"

<proof>

lemma *equal_mod_timpls_refl_minus_eq*:

"equal_mod_timpls TI s t \longleftrightarrow equal_mod_timpls (filter ($\lambda(a,b). a \neq b$) TI) s t"

(is "?A \longleftrightarrow ?B")

<proof>

```

lemma timpls_transformable_to_refl:
  "timpls_transformable_to TI t t" (is ?A)
  "timpls_transformable_to' TI t t" (is ?B)
<proof>

lemma timpls_transformable_to_inv_Var:
  "timpls_transformable_to TI (Var x) t  $\implies$  t = Var x" (is "?A  $\implies$  ?C")
  "timpls_transformable_to TI t (Var x)  $\implies$  t = Var x" (is "?B  $\implies$  ?C")
  "timpls_transformable_to' TI (Var x) t  $\implies$  t = Var x" (is "?A'  $\implies$  ?C")
  "timpls_transformable_to' TI t (Var x)  $\implies$  t = Var x" (is "?B'  $\implies$  ?C")
<proof>

lemma timpls_transformable_to_inv:
  assumes "timpls_transformable_to TI (Fun f T) (Fun g S)"
  shows "length T = length S"
    and " $\bigwedge i. i < \text{length } T \implies \text{timpls\_transformable\_to } TI (T ! i) (S ! i)"$ "
    and " $f \neq g \implies (\text{is\_Abs } f \wedge \text{is\_Abs } g \wedge (\text{the\_Abs } f, \text{the\_Abs } g) \in \text{set } TI)"$ "
<proof>

lemma timpls_transformable_to'_inv:
  assumes "timpls_transformable_to' TI (Fun f T) (Fun g S)"
  shows "length T = length S"
    and " $\bigwedge i. i < \text{length } T \implies \text{timpls\_transformable\_to' } TI (T ! i) (S ! i)"$ "
    and " $f \neq g \implies (\text{is\_Abs } f \wedge \text{is\_Abs } g \wedge \text{in\_trancl } TI (\text{the\_Abs } f) (\text{the\_Abs } g))"$ "
<proof>

lemma timpls_transformable_to_inv':
  assumes "timpls_transformable_to TI (Fun f T) t"
  shows "is_Fun t"
    and "length T = length (args t)"
    and " $\bigwedge i. i < \text{length } T \implies \text{timpls\_transformable\_to } TI (T ! i) (\text{args } t ! i)"$ "
    and " $f \neq \text{the\_Fun } t \implies ($ 
       $\text{is\_Abs } f \wedge \text{is\_Abs } (\text{the\_Fun } t) \wedge (\text{the\_Abs } f, \text{the\_Abs } (\text{the\_Fun } t)) \in \text{set } TI)"$ 
    and " $\neg \text{is\_Abs } f \implies f = \text{the\_Fun } t"$ "
<proof>

lemma timpls_transformable_to'_inv':
  assumes "timpls_transformable_to' TI (Fun f T) t"
  shows "is_Fun t"
    and "length T = length (args t)"
    and " $\bigwedge i. i < \text{length } T \implies \text{timpls\_transformable\_to' } TI (T ! i) (\text{args } t ! i)"$ "
    and " $f \neq \text{the\_Fun } t \implies ($ 
       $\text{is\_Abs } f \wedge \text{is\_Abs } (\text{the\_Fun } t) \wedge \text{in\_trancl } TI (\text{the\_Abs } f) (\text{the\_Abs } (\text{the\_Fun } t)))"$ 
    and " $\neg \text{is\_Abs } f \implies f = \text{the\_Fun } t"$ "
<proof>

lemma timpls_transformable_to_size_eq:
  fixes s t::"('b, 'c, 'a) prot_fun, 'd) term"
  shows "timpls_transformable_to TI s t  $\implies$  size s = size t" (is "?A  $\implies$  ?C")
    and "timpls_transformable_to' TI s t  $\implies$  size s = size t" (is "?B  $\implies$  ?C")
<proof>

lemma timpls_transformable_to_if_term_variants:
  fixes s t::"('a, 'b, 'c) prot_fun, 'd) term" and a b::"'c set"
  defines "P  $\equiv$  ( $\lambda\_.$  []) (Abs a := [Abs b])"
  assumes st: "term_variants_pred P s t"
    and ab: "(a,b)  $\in$  set TI"
  shows "timpls_transformable_to TI s t"
<proof>

lemma timpls_transformable_to'_if_term_variants:
  fixes s t::"('a, 'b, 'c) prot_fun, 'd) term" and a b::"'c set"
  defines "P  $\equiv$  ( $\lambda\_.$  []) (Abs a := [Abs b])"

```

2 Stateful Protocol Verification

```

assumes st: "term_variants_pred P s t"
    and ab: "(a,b) ∈ (set TI)+"
shows "timpls_transformable_to' TI s t"
⟨proof⟩

```

```

lemma timpls_transformable_to_trans:
  assumes TI_trancl: "∀(a,b) ∈ (set TI)+. a ≠ b → (a,b) ∈ set TI"
    and st: "timpls_transformable_to TI s t"
    and tu: "timpls_transformable_to TI t u"
  shows "timpls_transformable_to TI s u"
⟨proof⟩

```

```

lemma timpls_transformable_to'_trans:
  assumes st: "timpls_transformable_to' TI s t"
    and tu: "timpls_transformable_to' TI t u"
  shows "timpls_transformable_to' TI s u"
⟨proof⟩

```

```

lemma timpls_transformable_to_mono:
  assumes "set TI ⊆ set TI'"
    and "timpls_transformable_to TI s t"
  shows "timpls_transformable_to TI' s t"
⟨proof⟩

```

```

lemma timpls_transformable_to'_mono:
  assumes "set TI ⊆ set TI'"
    and "timpls_transformable_to' TI s t"
  shows "timpls_transformable_to' TI' s t"
⟨proof⟩

```

```

lemma timpls_transformable_to_refl_minus_eq:
  "timpls_transformable_to TI s t ↔ timpls_transformable_to (filter (λ(a,b). a ≠ b) TI) s t"
  (is "?A ↔ ?B")
⟨proof⟩

```

```

lemma timpls_transformable_to_iff_in_timpl_closure:
  assumes "set TI' = {(a,b) ∈ (set TI)+. a ≠ b}"
  shows "timpls_transformable_to TI' s t ↔ t ∈ timpl_closure s (set TI)" (is "?A s t ↔ ?B s t")
⟨proof⟩

```

```

lemma timpls_transformable_to'_iff_in_timpl_closure:
  "timpls_transformable_to' TI s t ↔ t ∈ timpl_closure s (set TI)" (is "?A s t ↔ ?B s t")
⟨proof⟩

```

```

lemma equal_mod_timpls_iff_ex_in_timpl_closure:
  assumes "set TI' = {(a,b) ∈ TI+. a ≠ b}"
  shows "equal_mod_timpls TI' s t ↔ (∃u. u ∈ timpl_closure s TI ∧ u ∈ timpl_closure t TI)"
  (is "?A s t ↔ ?B s t")
⟨proof⟩

```

context

begin

```

private inductive timpls_transformable_to_pred where
  Var: "timpls_transformable_to_pred A (Var x) (Var x)"
| Fun: "[¬is_Abs f; length T = length S;
  ∧i. i < length T ⇒ timpls_transformable_to_pred A (T ! i) (S ! i)]
  ⇒ timpls_transformable_to_pred A (Fun f T) (Fun f S)"
| Abs: "b ∈ A ⇒ timpls_transformable_to_pred A (Fun (Abs a) []) (Fun (Abs b) [])"

```

```

private lemma timpls_transformable_to_pred_inv_Var:
  assumes "timpls_transformable_to_pred A (Var x) t"

```

```

shows "t = Var x"
<proof> lemma timpls_transformable_to_pred_inv:
  assumes "timpls_transformable_to_pred A (Fun f T) t"
  shows "is_Fun t"
    and "length T = length (args t)"
    and " $\bigwedge i. i < \text{length } T \implies \text{timpls\_transformable\_to\_pred } A \ (T \ ! \ i) \ (\text{args } t \ ! \ i)"$ "
    and " $\neg \text{is\_Abs } f \implies f = \text{the\_Fun } t$ "
    and " $\text{is\_Abs } f \implies (\text{is\_Abs } (\text{the\_Fun } t) \wedge \text{the\_Abs } (\text{the\_Fun } t) \in A)"$ "
<proof> lemma timpls_transformable_to_pred_finite_aux1:
  assumes f: " $\neg \text{is\_Abs } f$ "
  shows "{s. timpls_transformable_to_pred A (Fun f T) s}  $\subseteq$ 
    ( $\lambda S. \text{Fun } f \ S$ ) ' {S. length T = length S  $\wedge$ 
      ( $\forall s \in \text{set } S. \exists t \in \text{set } T. \text{timpls\_transformable\_to\_pred } A \ t \ s$ )}"
    (is "?B  $\subseteq$  ?C")
<proof> lemma timpls_transformable_to_pred_finite_aux2:
  "{s. timpls_transformable_to_pred A (Fun (Abs a) []) s}  $\subseteq$  ( $\lambda b. \text{Fun } (\text{Abs } b) \ []$ ) ' A" (is "?B  $\subseteq$  ?C")
<proof> lemma timpls_transformable_to_pred_finite:
  fixes t::"((fun,atom,sets) prot_fun, 'a) term"
  assumes A: "finite A"
    and t: "wf_trm t"
  shows "finite {s. timpls_transformable_to_pred A t s}"
<proof> lemma timpls_transformable_to_pred_if_timpls_transformable_to:
  assumes s: "timpls_transformable_to TI t s"
    and t: "wf_trm t" " $\forall f \in \text{funs\_term } t. \text{is\_Abs } f \implies \text{the\_Abs } f \in A$ "
  shows "timpls_transformable_to_pred (A  $\cup$  fst ' (set TI)+  $\cup$  snd ' (set TI)+) t s"
<proof> lemma timpls_transformable_to_pred_if_timpls_transformable_to':
  assumes s: "timpls_transformable_to' TI t s"
    and t: "wf_trm t" " $\forall f \in \text{funs\_term } t. \text{is\_Abs } f \implies \text{the\_Abs } f \in A$ "
  shows "timpls_transformable_to_pred (A  $\cup$  fst ' (set TI)+  $\cup$  snd ' (set TI)+) t s"
<proof> lemma timpls_transformable_to_pred_if_equal_mod_timpls:
  assumes s: "equal_mod_timpls TI t s"
    and t: "wf_trm t" " $\forall f \in \text{funs\_term } t. \text{is\_Abs } f \implies \text{the\_Abs } f \in A$ "
  shows "timpls_transformable_to_pred (A  $\cup$  fst ' (set TI)+  $\cup$  snd ' (set TI)+) t s"
<proof>

lemma timpls_transformable_to_finite:
  assumes t: "wf_trm t"
  shows "finite {s. timpls_transformable_to TI t s}" (is ?P)
    and "finite {s. timpls_transformable_to' TI t s}" (is ?Q)
<proof>

lemma equal_mod_timpls_finite:
  assumes t: "wf_trm t"
  shows "finite {s. equal_mod_timpls TI t s}"
<proof>

end

lemma intruder_synth_mod_timpls_is_synth_timpl_closure_set:
  fixes t::"((fun,atom,sets) prot_fun, 'a) term" and TI TI'
  assumes "set TI' = {(a,b)  $\in$  (set TI)+. a  $\neq$  b}"
  shows "intruder_synth_mod_timpls M TI' t  $\longleftrightarrow$  timpl_closure_set (set M) (set TI)  $\vdash_c$  t"
    (is "?C t  $\longleftrightarrow$  ?D t")
<proof>

lemma intruder_synth_mod_timpls'_is_synth_timpl_closure_set:
  fixes t::"((fun,atom,sets) prot_fun, 'a) term" and TI
  shows "intruder_synth_mod_timpls' M TI t  $\longleftrightarrow$  timpl_closure_set (set M) (set TI)  $\vdash_c$  t"
    (is "?A t  $\longleftrightarrow$  ?B t")
<proof>

lemma intruder_synth_mod_eq_timpls_is_synth_timpl_closure_set:
  fixes t::"((fun,atom,sets) prot_fun, 'a) term" and TI

```

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```

defines "cl  $\equiv \lambda TI. \{(a,b) \in TI^+. a \neq b\}$ "
shows "set TI' =  $\{(a,b) \in (\text{set } TI)^+. a \neq b\} \implies$ 
  intruder_synth_mod_eq_timpls M TI' t  $\longleftrightarrow$ 
  ( $\exists s \in \text{timpl\_closure } t (\text{set } TI). \text{timpl\_closure\_set } M (\text{set } TI) \vdash_c s$ )"
  (is "?Q TI TI'  $\implies$  ?C t  $\longleftrightarrow$  ?D t")
<proof>

lemma timpl_closure_finite:
  assumes t: "wftrms t"
  shows "finite (timpl_closure t (set TI))"
<proof>

lemma timpl_closure_set_finite:
  fixes TI:: "('sets set  $\times$  'sets set) list"
  assumes M_finite: "finite M"
  and M_wf: "wftrms M"
  shows "finite (timpl_closure_set M (set TI))"
<proof>

lemma comp_timpl_closure_is_timpl_closure_set:
  fixes M and TI:: "('sets set  $\times$  'sets set) list"
  assumes M_finite: "finite M"
  and M_wf: "wftrms M"
  shows "comp_timpl_closure M (set TI) = timpl_closure_set M (set TI)"
<proof>

context
begin

private lemma analyzed_closed_mod_timpls_is_analyzed_closed_timpl_closure_set_aux1:
  fixes M:: "('fun, 'atom, 'sets) prot_terms"
  assumes f: "arityf f = length T" "arityf f > 0" "Anaf f = (K, R)"
  and i: "i < length R"
  and M: "timpl_closure_set M TI  $\vdash_c$  T ! (R ! i)"
  and m: "Fun (Fu f) T  $\in$  M"
  and t: "Fun (Fu f) S  $\in$  timpl_closure (Fun (Fu f) T) TI"
  shows "timpl_closure_set M TI  $\vdash_c$  S ! (R ! i)"
<proof> lemma analyzed_closed_mod_timpls_is_analyzed_closed_timpl_closure_set_aux2:
  fixes M:: "('fun, 'atom, 'sets) prot_terms"
  assumes M: " $\forall s \in \text{set } (\text{snd } (\text{Ana } m)). \text{timpl\_closure\_set } M \text{ TI } \vdash_c s$ "
  and m: "m  $\in$  M"
  and t: "t  $\in$  timpl_closure m TI"
  and s: "s  $\in$  set (snd (Ana t))"
  shows "timpl_closure_set M TI  $\vdash_c$  s"
<proof>

lemma analyzed_closed_mod_timpls_is_analyzed_timpl_closure_set:
  fixes M:: "('fun, 'atom, 'sets) prot_term list"
  assumes TI': "set TI' =  $\{(a,b) \in (\text{set } TI)^+. a \neq b\}$ "
  and M_wf: "wftrms (set M)"
  shows "analyzed_closed_mod_timpls M TI'  $\longleftrightarrow$  analyzed (timpl_closure_set (set M) (set TI))"
  (is "?A  $\longleftrightarrow$  ?B")
<proof>

lemma analyzed_closed_mod_timpls'_is_analyzed_timpl_closure_set:
  fixes M:: "('fun, 'atom, 'sets) prot_term list"
  assumes M_wf: "wftrms (set M)"
  shows "analyzed_closed_mod_timpls' M TI  $\longleftrightarrow$  analyzed (timpl_closure_set (set M) (set TI))"
  (is "?A  $\longleftrightarrow$  ?B")
<proof>

end

```

end

end

2.6 Stateful Protocol Verification (Stateful_Protocol_Verification)

```
theory Stateful_Protocol_Verification
imports Stateful_Protocol_Model Term_Implication
begin
```

2.6.1 Fixed-Point Intruder Deduction Lemma

```
context stateful_protocol_model
begin
```

```
abbreviation pubval_terms:: "('fun, 'atom, 'sets) prot_terms" where
  "pubval_terms  $\equiv$  {t.  $\exists f \in$  funs_term t. is_Val f  $\wedge$  public f}"
```

```
abbreviation abs_terms:: "('fun, 'atom, 'sets) prot_terms" where
  "abs_terms  $\equiv$  {t.  $\exists f \in$  funs_term t. is_Abs f}"
```

```
definition intruder_deduct_GSMP::
  "[('fun, 'atom, 'sets) prot_terms,
   ('fun, 'atom, 'sets) prot_terms,
   ('fun, 'atom, 'sets) prot_term]
   $\Rightarrow$  bool" ("⟨_;_⟩  $\vdash_{GSMP}$  _" 50)
```

where

```
"⟨M; T⟩  $\vdash_{GSMP}$  t  $\equiv$  intruder_deduct_restricted M ( $\lambda t. t \in GSMP T - (pubval\_terms \cup abs\_terms)$ ) t"
```

```
lemma intruder_deduct_GSMP_induct[consumes 1, case_names AxiomH ComposeH DecomposeH]:
```

```
  assumes "⟨M; T⟩  $\vdash_{GSMP}$  t" " $\wedge t. t \in M \Rightarrow P M t$ "
```

```
  " $\wedge S f. \llbracket$ length S = arity f; public f;
```

```
   $\wedge s. s \in$  set S  $\Rightarrow$  ⟨M; T⟩  $\vdash_{GSMP}$  s;
```

```
   $\wedge s. s \in$  set S  $\Rightarrow P M s$ ;
```

```
  Fun f S  $\in GSMP T - (pubval\_terms \cup abs\_terms)$ 
```

```
   $\rrbracket \Rightarrow P M (Fun f S)$ "
```

```
  " $\wedge t K T' t_i. \llbracket$ ⟨M; T⟩  $\vdash_{GSMP}$  t; P M t; Ana t = (K, T');  $\wedge k. k \in$  set K  $\Rightarrow$  ⟨M; T⟩  $\vdash_{GSMP}$  k;
```

```
   $\wedge k. k \in$  set K  $\Rightarrow P M k$ ;  $t_i \in$  set T' $\rrbracket \Rightarrow P M t_i$ "
```

```
  shows "P M t"
```

⟨proof⟩

```
lemma pubval_terms_subst:
```

```
  assumes "t ·  $\vartheta \in$  pubval_terms" " $\vartheta$  ' fv t  $\cap$  pubval_terms = {}"
```

```
  shows "t  $\in$  pubval_terms"
```

⟨proof⟩

```
lemma abs_terms_subst:
```

```
  assumes "t ·  $\vartheta \in$  abs_terms" " $\vartheta$  ' fv t  $\cap$  abs_terms = {}"
```

```
  shows "t  $\in$  abs_terms"
```

⟨proof⟩

```
lemma pubval_terms_subst':
```

```
  assumes "t ·  $\vartheta \in$  pubval_terms" " $\forall n. Val (n, True) \notin \bigcup (funs\_term ' (\vartheta ' fv t))$ "
```

```
  shows "t  $\in$  pubval_terms"
```

⟨proof⟩

```
lemma abs_terms_subst':
```

```
  assumes "t ·  $\vartheta \in$  abs_terms" " $\forall n. Abs n \notin \bigcup (funs\_term ' (\vartheta ' fv t))$ "
```

```
  shows "t  $\in$  abs_terms"
```

⟨proof⟩

```
lemma pubval_terms_subst_range_disj:
```

"subst_range $\vartheta \cap \text{pubval_terms} = \{\}$ $\implies \vartheta \text{ 'fv t } \cap \text{pubval_terms} = \{\}$ "
 <proof>

lemma abs_terms_subst_range_disj:
 "subst_range $\vartheta \cap \text{abs_terms} = \{\}$ $\implies \vartheta \text{ 'fv t } \cap \text{abs_terms} = \{\}$ "
 <proof>

lemma pubval_terms_subst_range_comp:
 assumes "subst_range $\vartheta \cap \text{pubval_terms} = \{\}$ " "subst_range $\delta \cap \text{pubval_terms} = \{\}$ "
 shows "subst_range $(\vartheta \circ_s \delta) \cap \text{pubval_terms} = \{\}$ "
 <proof>

lemma pubval_terms_subst_range_comp':
 assumes " $(\vartheta \text{ 'X}) \cap \text{pubval_terms} = \{\}$ " " $(\delta \text{ 'fv}_{\text{set}} (\vartheta \text{ 'X})) \cap \text{pubval_terms} = \{\}$ "
 shows " $((\vartheta \circ_s \delta) \text{ 'X}) \cap \text{pubval_terms} = \{\}$ "
 <proof>

lemma abs_terms_subst_range_comp:
 assumes "subst_range $\vartheta \cap \text{abs_terms} = \{\}$ " "subst_range $\delta \cap \text{abs_terms} = \{\}$ "
 shows "subst_range $(\vartheta \circ_s \delta) \cap \text{abs_terms} = \{\}$ "
 <proof>

lemma abs_terms_subst_range_comp':
 assumes " $(\vartheta \text{ 'X}) \cap \text{abs_terms} = \{\}$ " " $(\delta \text{ 'fv}_{\text{set}} (\vartheta \text{ 'X})) \cap \text{abs_terms} = \{\}$ "
 shows " $((\vartheta \circ_s \delta) \text{ 'X}) \cap \text{abs_terms} = \{\}$ "
 <proof>

context

begin

private lemma Ana_abs_aux1:

fixes $\delta::('fun, 'atom, 'sets) \text{ prot_fun}, \text{ nat}, ('fun, 'atom, 'sets) \text{ prot_var} \text{ gsubst}$ "
 and $\alpha::\text{nat} \times \text{bool} \Rightarrow \text{'sets set}$ "

assumes "Ana_f f = (K, T)"

shows " $(K \cdot_{\text{list}} \delta) \cdot_{\alpha \text{list}} \alpha = K \cdot_{\text{list}} (\lambda n. \delta \ n \cdot_{\alpha} \alpha)$ "

<proof> **lemma** Ana_abs_aux2:

fixes $\alpha::\text{nat} \times \text{bool} \Rightarrow \text{'sets set}$ "

and $K::('fun, 'atom, 'sets) \text{ prot_fun}, \text{ nat}) \text{ term list}$ "

and $M::\text{nat list}$ "

and $T::('fun, 'atom, 'sets) \text{ prot_term list}$ "

assumes " $\forall i \in \text{fv}_{\text{set}} (\text{set } K) \cup \text{set } M. i < \text{length } T$ "

and " $(K \cdot_{\text{list}} (!) T) \cdot_{\alpha \text{list}} \alpha = K \cdot_{\text{list}} (\lambda n. T \ ! \ n \cdot_{\alpha} \alpha)$ "

shows " $(K \cdot_{\text{list}} (!) T) \cdot_{\alpha \text{list}} \alpha = K \cdot_{\text{list}} (!) (\text{map } (\lambda s. s \cdot_{\alpha} \alpha) T)$ " (is "?A1 = ?A2")

and " $(\text{map } (!) T) M) \cdot_{\alpha \text{list}} \alpha = \text{map } (!) (\text{map } (\lambda s. s \cdot_{\alpha} \alpha) T) M$ " (is "?B1 = ?B2")

<proof> **lemma** Ana_abs_aux1_set:

fixes $\delta::('fun, 'atom, 'sets) \text{ prot_fun}, \text{ nat}, ('fun, 'atom, 'sets) \text{ prot_var} \text{ gsubst}$ "

and $\alpha::\text{nat} \times \text{bool} \Rightarrow \text{'sets set}$ "

assumes "Ana_f f = (K, T)"

shows " $(\text{set } K \cdot_{\text{set}} \delta) \cdot_{\alpha \text{set}} \alpha = \text{set } K \cdot_{\text{set}} (\lambda n. \delta \ n \cdot_{\alpha} \alpha)$ "

<proof> **lemma** Ana_abs_aux2_set:

fixes $\alpha::\text{nat} \times \text{bool} \Rightarrow \text{'sets set}$ "

and $K::('fun, 'atom, 'sets) \text{ prot_fun}, \text{ nat}) \text{ terms}$ "

and $M::\text{nat set}$ "

and $T::('fun, 'atom, 'sets) \text{ prot_term list}$ "

assumes " $\forall i \in \text{fv}_{\text{set}} K \cup M. i < \text{length } T$ "

and " $(K \cdot_{\text{set}} (!) T) \cdot_{\alpha \text{set}} \alpha = K \cdot_{\text{set}} (\lambda n. T \ ! \ n \cdot_{\alpha} \alpha)$ "

shows " $(K \cdot_{\text{set}} (!) T) \cdot_{\alpha \text{set}} \alpha = K \cdot_{\text{set}} (!) (\text{map } (\lambda s. s \cdot_{\alpha} \alpha) T)$ " (is "?A1 = ?A2")

and " $((!) T \text{ 'M}) \cdot_{\alpha \text{set}} \alpha = (!) (\text{map } (\lambda s. s \cdot_{\alpha} \alpha) T) \text{ 'M}$ " (is "?B1 = ?B2")

<proof>

lemma Ana_abs:

fixes $t::('fun, 'atom, 'sets) \text{ prot_term}$ "

assumes "Ana t = (K, T)"

shows "Ana (t $\cdot_{\alpha} \alpha$) = (K $\cdot_{\alpha \text{list}} \alpha$, T $\cdot_{\alpha \text{list}} \alpha$)"


```

  <proof>
end

lemma deduct_FP_if_deduct:
  fixes M IK FP::("('fun,'atom,'sets) prot_terms"
  assumes IK: "IK  $\subseteq$  GSMP M - (pubval_terms  $\cup$  abs_terms)" " $\forall t \in IK \cdot_{\alpha_{set}} \alpha. FP \vdash_c t$ "
    and t: "IK  $\vdash t$ " " $t \in GSMP M - (pubval_terms \cup abs\_terms)$ "
  shows "FP  $\vdash t \cdot_{\alpha} \alpha$ "
<proof>

end

```

2.6.2 Computing and Checking Term Implications and Messages

```

context stateful_protocol_model
begin

```

```

abbreviation (input) "abs s  $\equiv$  (Fun (Abs s) []::('fun, 'atom, 'sets) prot_term)"

```

```

fun absdbupd where
  "absdbupd [] _ a = a"
| "absdbupd (insert(Var y, Fun (Set s) T)#D) x a = (
  if x = y then absdbupd D x (insert s a) else absdbupd D x a)"
| "absdbupd (delete(Var y, Fun (Set s) T)#D) x a = (
  if x = y then absdbupd D x (a - {s}) else absdbupd D x a)"
| "absdbupd (_#D) x a = absdbupd D x a"

```

```

lemma absdbupd_cons_cases:
  "absdbupd (insert(Var x, Fun (Set s) T)#D) x d = absdbupd D x (insert s d)"
  "absdbupd (delete(Var x, Fun (Set s) T)#D) x d = absdbupd D x (d - {s})"
  " $t \neq \text{Var } x \vee (\nexists s T. u = \text{Fun } (Set s) T) \implies \text{absdbupd } (\text{insert}\langle t,u \rangle\#D) x d = \text{absdbupd } D x d$ "
  " $t \neq \text{Var } x \vee (\nexists s T. u = \text{Fun } (Set s) T) \implies \text{absdbupd } (\text{delete}\langle t,u \rangle\#D) x d = \text{absdbupd } D x d$ "
<proof>

```

```

lemma absdbupd_filter: "absdbupd S x d = absdbupd (filter is_Update S) x d"
<proof>

```

```

lemma absdbupd_append:
  "absdbupd (A@B) x d = absdbupd B x (absdbupd A x d)"
<proof>

```

```

lemma absdbupd_wellformed_transaction:
  assumes T: "wellformed_transaction T"
  shows "absdbupd (unlabel (transaction_strand T)) = absdbupd (unlabel (transaction_updates T))"
<proof>

```

```

fun abs_substs_set::
  "[('fun,'atom,'sets) prot_var list,
  'sets set list,
  ('fun,'atom,'sets) prot_var  $\Rightarrow$  'sets set,
  ('fun,'atom,'sets) prot_var  $\Rightarrow$  'sets set]
 $\Rightarrow$  (((('fun,'atom,'sets) prot_var  $\times$  'sets set) list) list"
where
  "abs_substs_set [] _ _ = [[]]"
| "abs_substs_set (x#xs) as posconstrs negconstrs = (
  let bs = filter ( $\lambda a. \text{posconstrs } x \subseteq a \wedge a \cap \text{negconstrs } x = \{\}$ ) as
  in concat (map ( $\lambda b. \text{map } (\lambda \delta. (x, b)\#\delta$ ) (abs_substs_set xs as posconstrs negconstrs)) bs))"

```

```

definition abs_substs_fun::
  "[((('fun,'atom,'sets) prot_var  $\times$  'sets set) list,
  ('fun,'atom,'sets) prot_var]
 $\Rightarrow$  'sets set"
where

```

2 Stateful Protocol Verification

```

"abs_substs_fun  $\delta$  x = (case find ( $\lambda$ b. fst b = x)  $\delta$  of Some (_,a)  $\Rightarrow$  a | None  $\Rightarrow$  {})"

lemmas abs_substs_set_induct = abs_substs_set.induct[case_names Nil Cons]

fun transaction_poschecks_comp::
  "((('fun,'atom,'sets) prot_fun, ('fun,'atom,'sets) prot_var) stateful_strand
 $\Rightarrow$  (('fun,'atom,'sets) prot_var  $\Rightarrow$  'sets set))"
where
  "transaction_poschecks_comp [] = ( $\lambda$ _. {})"
| "transaction_poschecks_comp ((_: Var x  $\in$  Fun (Set s) [])#T) = (
  let f = transaction_poschecks_comp T in f(x := insert s (f x)))"
| "transaction_poschecks_comp (_#T) = transaction_poschecks_comp T"

fun transaction_negchecks_comp::
  "((('fun,'atom,'sets) prot_fun, ('fun,'atom,'sets) prot_var) stateful_strand
 $\Rightarrow$  (('fun,'atom,'sets) prot_var  $\Rightarrow$  'sets set))"
where
  "transaction_negchecks_comp [] = ( $\lambda$ _. {})"
| "transaction_negchecks_comp ((Var x not in Fun (Set s) [])#T) = (
  let f = transaction_negchecks_comp T in f(x := insert s (f x)))"
| "transaction_negchecks_comp (_#T) = transaction_negchecks_comp T"

definition transaction_check_pre where
  "transaction_check_pre FP TI T  $\delta$   $\equiv$ 
  let C = set (unlabel (transaction_checks T));
      S = set (unlabel (transaction_selects T));
      xs = fv_listsst (unlabel (transaction_strand T));
       $\vartheta$  =  $\lambda$  $\delta$  x. if fst x = TAtom Value then (absc  $\circ$   $\delta$ ) x else Var x
  in ( $\forall$ x  $\in$  set (transaction_fresh T).  $\delta$  x = {})  $\wedge$ 
  ( $\forall$ t  $\in$  trmsisst (transaction_receive T). intruder_synth_mod_timpls FP TI (t  $\cdot$   $\vartheta$   $\delta$ ))  $\wedge$ 
  ( $\forall$ u  $\in$  S  $\cup$  C.
    (is_InSet u  $\longrightarrow$  (
      let x = the_elem_term u; s = the_set_term u
      in (is_Var x  $\wedge$  is_Fun_Set s)  $\longrightarrow$  the_Set (the_Fun s)  $\in$   $\delta$  (the_Var x)))  $\wedge$ 
      ((is_NegChecks u  $\wedge$  bvarssstp u = []  $\wedge$  the_eqs u = []  $\wedge$  length (the_ins u) = 1)  $\longrightarrow$  (
        let x = fst (hd (the_ins u)); s = snd (hd (the_ins u))
        in (is_Var x  $\wedge$  is_Fun_Set s)  $\longrightarrow$  the_Set (the_Fun s)  $\notin$   $\delta$  (the_Var x))))))"

definition transaction_check_post where
  "transaction_check_post FP TI T  $\delta$   $\equiv$ 
  let xs = fv_listsst (unlabel (transaction_strand T));
       $\vartheta$  =  $\lambda$  $\delta$  x. if fst x = TAtom Value then (absc  $\circ$   $\delta$ ) x else Var x;
      u =  $\lambda$  $\delta$  x. absdbupd (unlabel (transaction_updates T)) x ( $\delta$  x)
  in ( $\forall$ x  $\in$  set xs - set (transaction_fresh T).  $\delta$  x  $\neq$  u  $\delta$  x  $\longrightarrow$  List.member TI ( $\delta$  x, u  $\delta$  x))  $\wedge$ 
  ( $\forall$ t  $\in$  trmsisst (transaction_send T). intruder_synth_mod_timpls FP TI (t  $\cdot$   $\vartheta$  (u  $\delta$ )))"

definition transaction_check_comp::
  "[('fun,'atom,'sets) prot_term list,
  'sets set list,
  ('sets set  $\times$  'sets set) list,
  ('fun,'atom,'sets,'lbl) prot_transaction]
 $\Rightarrow$  (('fun,'atom,'sets) prot_var  $\times$  'sets set) list"
where
  "transaction_check_comp FP OCC TI T  $\equiv$ 
  let S = unlabel (transaction_strand T);
      C = unlabel (transaction_selects T@transaction_checks T);
      xs = filter ( $\lambda$ x. x  $\notin$  set (transaction_fresh T)  $\wedge$  fst x = TAtom Value) (fv_listsst S);
      posconstrs = transaction_poschecks_comp C;
      negconstrs = transaction_negchecks_comp C;
      pre_check = transaction_check_pre FP TI T
  in filter ( $\lambda$  $\delta$ . pre_check (abs_substs_fun  $\delta$ )) (abs_substs_set xs OCC posconstrs negconstrs)"

definition transaction_check::

```

```

"[('fun, 'atom, 'sets) prot_term list,
  'sets set list,
  ('sets set × 'sets set) list,
  ('fun, 'atom, 'sets, 'lbl) prot_transaction]
⇒ bool"
where
  "transaction_check FP OCC TI T ≡
    list_all (λδ. transaction_check_post FP TI T (abs_substs_fun δ)) (transaction_check_comp FP OCC TI T)"

lemma abs_subst_fun_cons:
  "abs_substs_fun ((x,b)#δ) = (abs_substs_fun δ)(x := b)"
⟨proof⟩

lemma abs_substs_cons:
  assumes "δ ∈ set (abs_substs_set xs as poss negs)" "b ∈ set as" "poss x ⊆ b" "b ∩ negs x = {}"
  shows "(x,b)#δ ∈ set (abs_substs_set (x#xs) as poss negs)"
⟨proof⟩

lemma abs_substs_cons':
  assumes δ: "δ ∈ abs_substs_fun ' set (abs_substs_set xs as poss negs)"
  and b: "b ∈ set as" "poss x ⊆ b" "b ∩ negs x = {}"
  shows "δ(x := b) ∈ abs_substs_fun ' set (abs_substs_set (x#xs) as poss negs)"
⟨proof⟩

lemma abs_substs_has_all_abs:
  assumes "∀x. x ∈ set xs → δ x ∈ set as"
  and "∀x. x ∈ set xs → poss x ⊆ δ x"
  and "∀x. x ∈ set xs → δ x ∩ negs x = {}"
  and "∀x. x ∉ set xs → δ x = {}"
  shows "δ ∈ abs_substs_fun ' set (abs_substs_set xs as poss negs)"
⟨proof⟩

lemma abs_substs_abss_bounded:
  assumes "δ ∈ abs_substs_fun ' set (abs_substs_set xs as poss negs)"
  and "x ∈ set xs"
  shows "δ x ∈ set as"
  and "poss x ⊆ δ x"
  and "δ x ∩ negs x = {}"
⟨proof⟩

lemma transaction_poschecks_comp_unfold:
  "transaction_poschecks_comp C x = {s. ∃a. ⟨a: Var x ∈ Fun (Set s) []⟩ ∈ set C}"
⟨proof⟩

lemma transaction_poschecks_comp_notin_fv_empty:
  assumes "x ∉ fv_sst C"
  shows "transaction_poschecks_comp C x = {}"
⟨proof⟩

lemma transaction_negchecks_comp_unfold:
  "transaction_negchecks_comp C x = {s. ⟨Var x not in Fun (Set s) []⟩ ∈ set C}"
⟨proof⟩

lemma transaction_negchecks_comp_notin_fv_empty:
  assumes "x ∉ fv_sst C"
  shows "transaction_negchecks_comp C x = {}"
⟨proof⟩

lemma transaction_check_preI[intro]:
  fixes T
  defines "∅ ≡ λδ x. if fst x = TAtom Value then (absc ∘ δ) x else Var x"
  and "S ≡ set (unlabel (transaction_selects T))"
  and "C ≡ set (unlabel (transaction_checks T))"

```

```

assumes a0: "∀x ∈ set (transaction_fresh T). δ x = {}"
and a1: "∀x ∈ fv_transaction T - set (transaction_fresh T). fst x = TAtom Value → δ x ∈ set OCC"
and a2: "∀t ∈ trmslsst (transaction_receive T). intruder_synth_mod_timpls FP TI (t · ∅ δ)"
and a3: "∀a x s. ⟨a: Var x ∈ Fun (Set s) []⟩ ∈ S ∪ C → s ∈ δ x"
and a4: "∀x s. ⟨Var x not in Fun (Set s) []⟩ ∈ S ∪ C → s ∉ δ x"
shows "transaction_check_pre FP TI T δ"
⟨proof⟩

```

```

lemma transaction_check_pre_InSetE:
assumes T: "transaction_check_pre FP TI T δ"
and u: "u = ⟨a: Var x ∈ Fun (Set s) []⟩"
      "u ∈ set (unlabel (transaction_selects T)) ∪ set (unlabel (transaction_checks T))"
shows "s ∈ δ x"
⟨proof⟩

```

```

lemma transaction_check_pre_NotInSetE:
assumes T: "transaction_check_pre FP TI T δ"
and u: "u = ⟨Var x not in Fun (Set s) []⟩"
      "u ∈ set (unlabel (transaction_selects T)) ∪ set (unlabel (transaction_checks T))"
shows "s ∉ δ x"
⟨proof⟩

```

```

lemma transaction_check_compI[intro]:
assumes T: "transaction_check_pre FP TI T δ"
and Tadm: "admissible_transaction T"
and x1: "∀x. (x ∈ fv_transaction T - set (transaction_fresh T) ∧ fst x = TAtom Value)
          → δ x ∈ set OCC"
and x2: "∀x. (x ∉ fv_transaction T - set (transaction_fresh T) ∨ fst x ≠ TAtom Value)
          → δ x = {}"
shows "δ ∈ abs_substs_fun ' set (transaction_check_comp FP OCC TI T)"
⟨proof⟩

```

context

begin

```

private lemma transaction_check_comp_in_aux:
fixes T
defines "S ≡ set (unlabel (transaction_selects T))"
and "C ≡ set (unlabel (transaction_checks T))"
assumes Tadm: "admissible_transaction T"
and a1: "∀x ∈ fv_transaction T - set (transaction_fresh T). fst x = TAtom Value → (∀s.
          select⟨Var x, Fun (Set s) []⟩ ∈ S → s ∈ α x)"
and a2: "∀x ∈ fv_transaction T - set (transaction_fresh T). fst x = TAtom Value → (∀s.
          ⟨Var x in Fun (Set s) []⟩ ∈ C → s ∈ α x)"
and a3: "∀x ∈ fv_transaction T - set (transaction_fresh T). fst x = TAtom Value → (∀s.
          ⟨Var x not in Fun (Set s) []⟩ ∈ C → s ∉ α x)"
shows "∀a x s. ⟨a: Var x ∈ Fun (Set s) []⟩ ∈ S ∪ C → s ∈ α x" (is ?A)
and "∀x s. ⟨Var x not in Fun (Set s) []⟩ ∈ S ∪ C → s ∉ α x" (is ?B)
⟨proof⟩

```

```

lemma transaction_check_comp_in:
fixes T
defines "∅ ≡ λδ x. if fst x = TAtom Value then (absc o δ) x else Var x"
and "S ≡ set (unlabel (transaction_selects T))"
and "C ≡ set (unlabel (transaction_checks T))"
assumes Tadm: "admissible_transaction T"
and a1: "∀x ∈ set (transaction_fresh T). α x = {}"
and a2: "∀t ∈ trmslsst (transaction_receive T). intruder_synth_mod_timpls FP TI (t · ∅ α)"
and a3: "∀x ∈ fv_transaction T - set (transaction_fresh T). ∀s.
          select⟨Var x, Fun (Set s) []⟩ ∈ S → s ∈ α x"
and a4: "∀x ∈ fv_transaction T - set (transaction_fresh T). ∀s.
          ⟨Var x in Fun (Set s) []⟩ ∈ C → s ∈ α x"
and a5: "∀x ∈ fv_transaction T - set (transaction_fresh T). ∀s.
          ⟨Var x not in Fun (Set s) []⟩ ∈ C → s ∉ α x"

```

```

and a6: "∀ x ∈ fv_transaction T - set (transaction_fresh T).
  fst x = TAtom Value → α x ∈ set OCC"
shows "∃ δ ∈ abs_substs_fun ' set (transaction_check_comp FP OCC TI T). ∀ x ∈ fv_transaction T.
  fst x = TAtom Value → α x = δ x"
⟨proof⟩
end

end

```

2.6.3 Automatically Checking Protocol Security in a Typed Model

```

context stateful_protocol_model
begin

```

```

definition abs_intruder_knowledge ("αik") where
  "αik S I ≡ (iklsst S ·set I) ·αset α0 (dblsst S I)"

```

```

definition abs_value_constants ("αvals") where
  "αvals S I ≡ {t ∈ subtermsset (trmslsst S) ·set I. ∃ n. t = Fun (Val n) []} ·αset α0 (dblsst S I)"

```

```

definition abs_term_implications ("αti") where
  "αti A T σ α I ≡ {(s,t) | s t x.
  s ≠ t ∧ x ∈ fv_transaction T ∧ x ∉ set (transaction_fresh T) ∧
  Fun (Abs s) [] = (σ ◦s α) x · I ·α α0 (dblsst A I) ∧
  Fun (Abs t) [] = (σ ◦s α) x · I ·α α0 (dblsst (A@duallsst (transaction_strand T ·lsst σ ◦s α)) I)}"

```

```

lemma abs_intruder_knowledge_append:
  "αik (A@B) I =
  (iklsst A ·set I) ·αset α0 (dblsst (A@B) I) ∪
  (iklsst B ·set I) ·αset α0 (dblsst (A@B) I)"

```

⟨proof⟩

```

lemma abs_value_constants_append:
  fixes A B::('a,'b,'c,'d) prot_strand
  shows "αvals (A@B) I =
  {t ∈ subtermsset (trmslsst A) ·set I. ∃ n. t = Fun (Val n) []} ·αset α0 (dblsst (A@B) I) ∪
  {t ∈ subtermsset (trmslsst B) ·set I. ∃ n. t = Fun (Val n) []} ·αset α0 (dblsst (A@B) I)"

```

⟨proof⟩

```

lemma transaction_renaming_subst_has_no_pubconsts_abss:
  fixes α::('fun,'atom,'sets) prot_subst
  assumes "transaction_renaming_subst α P A"
  shows "subst_range α ∩ pubval_terms = {}" (is ?A)
  and "subst_range α ∩ abs_terms = {}" (is ?B)

```

⟨proof⟩

```

lemma transaction_fresh_subst_has_no_pubconsts_abss:
  fixes σ::('fun,'atom,'sets) prot_subst
  assumes "transaction_fresh_subst σ T A"
  shows "subst_range σ ∩ pubval_terms = {}" (is ?A)
  and "subst_range σ ∩ abs_terms = {}" (is ?B)

```

⟨proof⟩

```

lemma reachable_constraints_no_pubconsts_abss:
  assumes "A ∈ reachable_constraints P"
  and P: "∀ T ∈ set P. ∀ n. Val (n,True) ∉ ∪ (funs_term ' trms_transaction T)"
  "∀ T ∈ set P. ∀ n. Abs n ∉ ∪ (funs_term ' trms_transaction T)"
  "∀ T ∈ set P. ∀ x ∈ set (transaction_fresh T). Γv x = TAtom Value"
  "∀ T ∈ set P. bvarslsst (transaction_strand T) = {}"
  and I: "interpretationsubst I" "wtsubst I" "wftrms (subst_range I)"
  "∀ n. Val (n,True) ∉ ∪ (funs_term ' (I ' fvlsst A))"
  "∀ n. Abs n ∉ ∪ (funs_term ' (I ' fvlsst A))"
  shows "trmslsst A ·set I ⊆ GSMP (∪ T ∈ set P. trms_transaction T) - (pubval_terms ∪ abs_terms)"

```

(is "?A \subseteq ?B")
 <proof>

lemma $\alpha_{ti_covers_}\alpha_0_aux$:

assumes $\mathcal{A_reach}$: " $\mathcal{A} \in reachable_constraints\ P$ "
 and T : " $T \in set\ P$ "
 and \mathcal{I} : " $welltyped_constraint_model\ \mathcal{I}\ (A@dual_{lsst}\ (transaction_strand\ T\ \cdot_{lsst}\ \sigma\ \circ_s\ \alpha))$ "
 and σ : " $transaction_fresh_subst\ \sigma\ T\ \mathcal{A}$ "
 and α : " $transaction_renaming_subst\ \alpha\ P\ \mathcal{A}$ "
 and P : " $\forall T \in set\ P. admissible_transaction\ T$ "
 and t : " $t \in subterms_{set}\ (trms_{lsst}\ \mathcal{A})$ "
 " $t = Fun\ (Val\ n)\ [] \vee t = Var\ x$ "
 and neq:
 " $t \cdot \mathcal{I} \cdot_{\alpha}\ \alpha_0\ (db_{lsst}\ \mathcal{A}\ \mathcal{I}) \neq$
 $t \cdot \mathcal{I} \cdot_{\alpha}\ \alpha_0\ (db_{lsst}\ (A@dual_{lsst}\ (transaction_strand\ T\ \cdot_{lsst}\ \sigma\ \circ_s\ \alpha))\ \mathcal{I})$ "
 shows " $\exists y \in fv_transaction\ T - set\ (transaction_fresh\ T).$
 $t \cdot \mathcal{I} = (\sigma \circ_s\ \alpha)\ y \cdot \mathcal{I} \wedge \Gamma_v\ y = TAtom\ Value$ "

<proof>

lemma $\alpha_{ti_covers_}\alpha_0_Var$:

assumes $\mathcal{A_reach}$: " $\mathcal{A} \in reachable_constraints\ P$ "
 and T : " $T \in set\ P$ "
 and \mathcal{I} : " $welltyped_constraint_model\ \mathcal{I}\ (A@dual_{lsst}\ (transaction_strand\ T\ \cdot_{lsst}\ \sigma\ \circ_s\ \alpha))$ "
 and σ : " $transaction_fresh_subst\ \sigma\ T\ \mathcal{A}$ "
 and α : " $transaction_renaming_subst\ \alpha\ P\ \mathcal{A}$ "
 and P : " $\forall T \in set\ P. admissible_transaction\ T$ "
 and x : " $x \in fv_{lsst}\ \mathcal{A}$ "
 shows " $\mathcal{I}\ x \cdot_{\alpha}\ \alpha_0\ (db_{lsst}\ (A@dual_{lsst}\ (transaction_strand\ T\ \cdot_{lsst}\ \sigma\ \circ_s\ \alpha))\ \mathcal{I}) \in$
 $tipl_closure_set\ \{\mathcal{I}\ x \cdot_{\alpha}\ \alpha_0\ (db_{lsst}\ \mathcal{A}\ \mathcal{I})\}\ (\alpha_{ti}\ \mathcal{A}\ T\ \sigma\ \alpha\ \mathcal{I})$ "

<proof>

lemma $\alpha_{ti_covers_}\alpha_0_Val$:

assumes $\mathcal{A_reach}$: " $\mathcal{A} \in reachable_constraints\ P$ "
 and T : " $T \in set\ P$ "
 and \mathcal{I} : " $welltyped_constraint_model\ \mathcal{I}\ (A@dual_{lsst}\ (transaction_strand\ T\ \cdot_{lsst}\ \sigma\ \circ_s\ \alpha))$ "
 and σ : " $transaction_fresh_subst\ \sigma\ T\ \mathcal{A}$ "
 and α : " $transaction_renaming_subst\ \alpha\ P\ \mathcal{A}$ "
 and P : " $\forall T \in set\ P. admissible_transaction\ T$ "
 and n : " $Fun\ (Val\ n)\ [] \in subterms_{set}\ (trms_{lsst}\ \mathcal{A})$ "
 shows " $Fun\ (Val\ n)\ [] \cdot_{\alpha}\ \alpha_0\ (db_{lsst}\ (A@dual_{lsst}\ (transaction_strand\ T\ \cdot_{lsst}\ \sigma\ \circ_s\ \alpha))\ \mathcal{I}) \in$
 $tipl_closure_set\ \{Fun\ (Val\ n)\ [] \cdot_{\alpha}\ \alpha_0\ (db_{lsst}\ \mathcal{A}\ \mathcal{I})\}\ (\alpha_{ti}\ \mathcal{A}\ T\ \sigma\ \alpha\ \mathcal{I})$ "

<proof>

lemma $\alpha_{ti_covers_}\alpha_0_ik$:

assumes $\mathcal{A_reach}$: " $\mathcal{A} \in reachable_constraints\ P$ "
 and T : " $T \in set\ P$ "
 and \mathcal{I} : " $welltyped_constraint_model\ \mathcal{I}\ (A@dual_{lsst}\ (transaction_strand\ T\ \cdot_{lsst}\ \sigma\ \circ_s\ \alpha))$ "
 and σ : " $transaction_fresh_subst\ \sigma\ T\ \mathcal{A}$ "
 and α : " $transaction_renaming_subst\ \alpha\ P\ \mathcal{A}$ "
 and P : " $\forall T \in set\ P. admissible_transaction\ T$ "
 and t : " $t \in ik_{lsst}\ \mathcal{A}$ "
 shows " $t \cdot \mathcal{I} \cdot_{\alpha}\ \alpha_0\ (db_{lsst}\ (A@dual_{lsst}\ (transaction_strand\ T\ \cdot_{lsst}\ \sigma\ \circ_s\ \alpha))\ \mathcal{I}) \in$
 $tipl_closure_set\ \{t \cdot \mathcal{I} \cdot_{\alpha}\ \alpha_0\ (db_{lsst}\ \mathcal{A}\ \mathcal{I})\}\ (\alpha_{ti}\ \mathcal{A}\ T\ \sigma\ \alpha\ \mathcal{I})$ "

<proof>

lemma $transaction_prop1$:

assumes " $\delta \in abs_substs_fun\ 'set\ (transaction_check_comp\ FP\ OCC\ TI\ T)$ "
 and " $x \in fv_transaction\ T$ "
 and " $x \notin set\ (transaction_fresh\ T)$ "
 and " $\delta\ x \neq absdbupd\ (unlabel\ (transaction_updates\ T))\ x\ (\delta\ x)$ "
 and " $transaction_check\ FP\ OCC\ TI\ T$ "
 and TI :
 " $set\ TI = \{(a,b) \in (set\ TI)^+. a \neq b\}$ "

shows " $(\delta x, \text{absdbupd} (\text{unlabel} (\text{transaction_updates } T)) x (\delta x)) \in (\text{set } TI)^+$ "
 <proof>

lemma transaction_prop2:

assumes δ : " $\delta \in \text{abs_subst_fun } ' \text{set} (\text{transaction_check_comp } FP \text{ OCC } TI \text{ } T)$ "
 and x : " $x \in \text{fv_transaction } T$ " " $\text{fst } x = \text{TAtom Value}$ "
 and T_check : " $\text{transaction_check } FP \text{ OCC } TI \text{ } T$ "
 and T_adm : " $\text{admissible_transaction } T$ "
 and FP :
 " $\text{analyzed} (\text{timpl_closure_set} (\text{set } FP) (\text{set } TI))$ "
 " $\text{wf}_{trms} (\text{set } FP)$ "
 and OCC :
 " $\forall t \in \text{timpl_closure_set} (\text{set } FP) (\text{set } TI). \forall f \in \text{funs_term } t. \text{is_Abs } f \longrightarrow f \in \text{Abs } ' \text{set } OCC$ "
 " $\text{timpl_closure_set} (\text{absc } ' \text{set } OCC) (\text{set } TI) \subseteq \text{absc } ' \text{set } OCC$ "
 and TI :
 " $\text{set } TI = \{(a,b) \in (\text{set } TI)^+. a \neq b\}$ "
 shows " $x \notin \text{set} (\text{transaction_fresh } T) \implies \delta x \in \text{set } OCC$ " (is "?A" \implies ?A)"
 and " $\text{absdbupd} (\text{unlabel} (\text{transaction_updates } T)) x (\delta x) \in \text{set } OCC$ " (is ?B)"
 <proof>

lemma transaction_prop3:

assumes A_reach : " $A \in \text{reachable_constraints } P$ "
 and T : " $T \in \text{set } P$ "
 and \mathcal{I} : " $\text{welltyped_constraint_model } \mathcal{I} (A \text{dual}_{l_{sst}} (\text{transaction_strand } T \cdot l_{sst} \sigma \circ_s \alpha))$ "
 and σ : " $\text{transaction_fresh_subst } \sigma \text{ } T \text{ } A$ "
 and α : " $\text{transaction_renaming_subst } \alpha \text{ } P \text{ } A$ "
 and FP :
 " $\text{analyzed} (\text{timpl_closure_set} (\text{set } FP) (\text{set } TI))$ "
 " $\text{wf}_{trms} (\text{set } FP)$ "
 " $\forall t \in \alpha_{ik} \mathcal{A} \mathcal{I}. \text{timpl_closure_set} (\text{set } FP) (\text{set } TI) \vdash_c t$ "
 and OCC :
 " $\forall t \in \text{timpl_closure_set} (\text{set } FP) (\text{set } TI). \forall f \in \text{funs_term } t. \text{is_Abs } f \longrightarrow f \in \text{Abs } ' \text{set } OCC$ "
 " $\text{timpl_closure_set} (\text{absc } ' \text{set } OCC) (\text{set } TI) \subseteq \text{absc } ' \text{set } OCC$ "
 " $\alpha_{vals} \mathcal{A} \mathcal{I} \subseteq \text{absc } ' \text{set } OCC$ "
 and TI :
 " $\text{set } TI = \{(a,b) \in (\text{set } TI)^+. a \neq b\}$ "
 and P :
 " $\forall T \in \text{set } P. \text{admissible_transaction } T$ "
 shows " $\forall x \in \text{set} (\text{transaction_fresh } T). (\sigma \circ_s \alpha) x \cdot \mathcal{I} \cdot \alpha \alpha_0 (\text{db}_{l_{sst}} \mathcal{A} \mathcal{I}) = \text{absc } \{\}$ " (is ?A)"
 and " $\forall t \in \text{trms}_{l_{sst}} (\text{transaction_receive } T).$
 $\text{intruder_synth_mod_timpls } FP \text{ } TI (t \cdot (\sigma \circ_s \alpha) \cdot \mathcal{I} \cdot \alpha \alpha_0 (\text{db}_{l_{sst}} \mathcal{A} \mathcal{I}))$ " (is ?B)"
 and " $\forall x \in \text{fv_transaction } T - \text{set} (\text{transaction_fresh } T).$
 $\forall s. \text{select} (\text{Var } x, \text{Fun} (\text{Set } s) []) \in \text{set} (\text{unlabel} (\text{transaction_selects } T))$
 $\longrightarrow (\exists ss. (\sigma \circ_s \alpha) x \cdot \mathcal{I} \cdot \alpha \alpha_0 (\text{db}_{l_{sst}} \mathcal{A} \mathcal{I}) = \text{absc } ss \wedge s \in ss)$ " (is ?C)"
 and " $\forall x \in \text{fv_transaction } T - \text{set} (\text{transaction_fresh } T).$
 $\forall s. (\text{Var } x \text{ in } \text{Fun} (\text{Set } s) []) \in \text{set} (\text{unlabel} (\text{transaction_checks } T))$
 $\longrightarrow (\exists ss. (\sigma \circ_s \alpha) x \cdot \mathcal{I} \cdot \alpha \alpha_0 (\text{db}_{l_{sst}} \mathcal{A} \mathcal{I}) = \text{absc } ss \wedge s \in ss)$ " (is ?D)"
 and " $\forall x \in \text{fv_transaction } T - \text{set} (\text{transaction_fresh } T).$
 $\forall s. (\text{Var } x \text{ not in } \text{Fun} (\text{Set } s) []) \in \text{set} (\text{unlabel} (\text{transaction_checks } T))$
 $\longrightarrow (\exists ss. (\sigma \circ_s \alpha) x \cdot \mathcal{I} \cdot \alpha \alpha_0 (\text{db}_{l_{sst}} \mathcal{A} \mathcal{I}) = \text{absc } ss \wedge s \notin ss)$ " (is ?E)"
 and " $\forall x \in \text{fv_transaction } T - \text{set} (\text{transaction_fresh } T). \Gamma_v x = \text{TAtom Value} \longrightarrow$
 $(\sigma \circ_s \alpha) x \cdot \mathcal{I} \cdot \alpha \alpha_0 (\text{db}_{l_{sst}} \mathcal{A} \mathcal{I}) \in \text{absc } ' \text{set } OCC$ " (is ?F)"
 <proof>

lemma transaction_prop4:

assumes A_reach : " $A \in \text{reachable_constraints } P$ "
 and T : " $T \in \text{set } P$ "
 and \mathcal{I} : " $\text{welltyped_constraint_model } \mathcal{I} (A \text{dual}_{l_{sst}} (\text{transaction_strand } T \cdot l_{sst} \sigma \circ_s \alpha))$ "
 and σ : " $\text{transaction_fresh_subst } \sigma \text{ } T \text{ } A$ "
 and α : " $\text{transaction_renaming_subst } \alpha \text{ } P \text{ } A$ "
 and P : " $\forall T \in \text{set } P. \text{admissible_transaction } T$ "
 and x : " $x \in \text{set} (\text{transaction_fresh } T)$ "
 and y : " $y \in \text{fv_transaction } T - \text{set} (\text{transaction_fresh } T)$ " " $\Gamma_v y = \text{TAtom Value}$ "

shows " $(\sigma \circ_s \alpha) x \cdot \mathcal{I} \notin \text{subterms}_{\text{set}} (\text{trms}_{\text{lsst}} (\mathcal{A} \cdot_{\text{lsst}} \mathcal{I}))$ " (is ?A)
 and " $(\sigma \circ_s \alpha) y \cdot \mathcal{I} \in \text{subterms}_{\text{set}} (\text{trms}_{\text{lsst}} (\mathcal{A} \cdot_{\text{lsst}} \mathcal{I}))$ " (is ?B)
 <proof>

lemma transaction_prop5:

fixes $T \sigma \alpha \mathcal{A} \mathcal{I} T' a0 a0' \vartheta$
 defines " $T' \equiv \text{dual}_{\text{lsst}} (\text{transaction_strand } T \cdot_{\text{lsst}} \sigma \circ_s \alpha)$ "
 and " $a0 \equiv \alpha_0 (\text{db}_{\text{lsst}} \mathcal{A} \mathcal{I})$ "
 and " $a0' \equiv \alpha_0 (\text{db}_{\text{lsst}} (\mathcal{A} @ T')) \mathcal{I}$ "
 and " $\vartheta \equiv \lambda \delta x. \text{if } \text{fst } x = \text{TAtom Value then } (\text{absc } \circ \delta) x \text{ else Var } x$ "
 assumes $\mathcal{A}_{\text{reach}}$: " $\mathcal{A} \in \text{reachable_constraints } P$ "
 and T : " $T \in \text{set } P$ "
 and \mathcal{I} : " $\text{welltyped_constraint_model } \mathcal{I} (\mathcal{A} @ T')$ "
 and σ : " $\text{transaction_fresh_subst } \sigma T \mathcal{A}$ "
 and α : " $\text{transaction_renaming_subst } \alpha P \mathcal{A}$ "
 and FP:
 " $\text{analyzed } (\text{timpl_closure_set } (\text{set } FP) (\text{set } TI))$ "
 " $\text{wf}_{\text{trms}} (\text{set } FP)$ "
 " $\forall t \in \alpha_{ik} \mathcal{A} \mathcal{I}. \text{timpl_closure_set } (\text{set } FP) (\text{set } TI) \vdash_c t$ "
 and OCC:
 " $\forall t \in \text{timpl_closure_set } (\text{set } FP) (\text{set } TI). \forall f \in \text{funs_term } t. \text{is_Abs } f \longrightarrow f \in \text{Abs ' set OCC}$ "
 " $\text{timpl_closure_set } (\text{absc ' set OCC}) (\text{set } TI) \subseteq \text{absc ' set OCC}$ "
 " $\alpha_{\text{vals}} \mathcal{A} \mathcal{I} \subseteq \text{absc ' set OCC}$ "
 and TI:
 " $\text{set } TI = \{(a,b) \in (\text{set } TI)^+. a \neq b\}$ "
 and P:
 " $\forall T \in \text{set } P. \text{admissible_transaction } T$ "
 and step: " $\text{list_all } (\text{transaction_check } FP \text{ OCC } TI) P$ "
 shows " $\exists \delta \in \text{abs_subst_fun ' set } (\text{transaction_check_comp } FP \text{ OCC } TI T).$
 $\forall x \in \text{fv_transaction } T. \Gamma_v x = \text{TAtom Value} \longrightarrow$
 $(\sigma \circ_s \alpha) x \cdot \mathcal{I} \cdot_{\alpha} a0 = \text{absc } (\delta x) \wedge$
 $(\sigma \circ_s \alpha) x \cdot \mathcal{I} \cdot_{\alpha} a0' = \text{absc } (\text{absdbupd } (\text{unlabel } (\text{transaction_updates } T)) x (\delta x))$ "
 <proof>

lemma transaction_prop6:

fixes $T \sigma \alpha \mathcal{A} \mathcal{I} T' a0 a0'$
 defines " $T' \equiv \text{dual}_{\text{lsst}} (\text{transaction_strand } T \cdot_{\text{lsst}} \sigma \circ_s \alpha)$ "
 and " $a0 \equiv \alpha_0 (\text{db}_{\text{lsst}} \mathcal{A} \mathcal{I})$ "
 and " $a0' \equiv \alpha_0 (\text{db}_{\text{lsst}} (\mathcal{A} @ T')) \mathcal{I}$ "
 assumes $\mathcal{A}_{\text{reach}}$: " $\mathcal{A} \in \text{reachable_constraints } P$ "
 and T : " $T \in \text{set } P$ "
 and \mathcal{I} : " $\text{welltyped_constraint_model } \mathcal{I} (\mathcal{A} @ T')$ "
 and σ : " $\text{transaction_fresh_subst } \sigma T \mathcal{A}$ "
 and α : " $\text{transaction_renaming_subst } \alpha P \mathcal{A}$ "
 and FP:
 " $\text{analyzed } (\text{timpl_closure_set } (\text{set } FP) (\text{set } TI))$ "
 " $\text{wf}_{\text{trms}} (\text{set } FP)$ "
 " $\forall t \in \alpha_{ik} \mathcal{A} \mathcal{I}. \text{timpl_closure_set } (\text{set } FP) (\text{set } TI) \vdash_c t$ "
 and OCC:
 " $\forall t \in \text{timpl_closure_set } (\text{set } FP) (\text{set } TI). \forall f \in \text{funs_term } t. \text{is_Abs } f \longrightarrow f \in \text{Abs ' set OCC}$ "
 " $\text{timpl_closure_set } (\text{absc ' set OCC}) (\text{set } TI) \subseteq \text{absc ' set OCC}$ "
 " $\alpha_{\text{vals}} \mathcal{A} \mathcal{I} \subseteq \text{absc ' set OCC}$ "
 and TI:
 " $\text{set } TI = \{(a,b) \in (\text{set } TI)^+. a \neq b\}$ "
 and P:
 " $\forall T \in \text{set } P. \text{admissible_transaction } T$ "
 and step: " $\text{list_all } (\text{transaction_check } FP \text{ OCC } TI) P$ "
 shows " $\forall t \in \text{timpl_closure_set } (\alpha_{ik} \mathcal{A} \mathcal{I}) (\alpha_{ti} \mathcal{A} T \sigma \alpha \mathcal{I}).$
 $\text{timpl_closure_set } (\text{set } FP) (\text{set } TI) \vdash_c t$ " (is ?A)
 and " $\text{timpl_closure_set } (\alpha_{\text{vals}} \mathcal{A} \mathcal{I}) (\alpha_{ti} \mathcal{A} T \sigma \alpha \mathcal{I}) \subseteq \text{absc ' set OCC}$ " (is ?B)
 and " $\forall t \in \text{trms}_{\text{lsst}} (\text{transaction_send } T). \text{is_Fun } (t \cdot (\sigma \circ_s \alpha) \cdot \mathcal{I} \cdot_{\alpha} a0') \longrightarrow$
 $\text{timpl_closure_set } (\text{set } FP) (\text{set } TI) \vdash_c t \cdot (\sigma \circ_s \alpha) \cdot \mathcal{I} \cdot_{\alpha} a0'$ " (is ?C)
 and " $\forall x \in \text{fv_transaction } T. \Gamma_v x = \text{TAtom Value} \longrightarrow$

$$(\sigma \circ_s \alpha) x \cdot \mathcal{I} \cdot_{\alpha} a0' \in \text{absc } ' \text{ set OCC}'' \text{ (is ?D)}$$

<proof>

lemma `reachable_constraints_covered_step:`

`fixes A::('fun,'atom,'sets,'lbl) prot_constr"`

`assumes A_reach: "A ∈ reachable_constraints P"`

`and T: "T ∈ set P"`

`and I: "welltyped_constraint_model I (A@duallsst (transaction_strand T ·lsst σ ∘s α))"`

`and σ: "transaction_fresh_subst σ T A"`

`and α: "transaction_renaming_subst α P A"`

`and FP:`

`"analyzed (timpl_closure_set (set FP) (set TI))"`

`"wftrms (set FP)"`

`"∀t ∈ αik A I. timpl_closure_set (set FP) (set TI) ⊢c t"`

`"ground (set FP)"`

`and OCC:`

`"∀t ∈ timpl_closure_set (set FP) (set TI). ∀f ∈ funs_term t. is_Abs f → f ∈ Abs ' set OCC"`

`"timpl_closure_set (absc ' set OCC) (set TI) ⊆ absc ' set OCC"`

`"αvals A I ⊆ absc ' set OCC"`

`and TI:`

`"set TI = {(a,b) ∈ (set TI)+. a ≠ b}"`

`and P:`

`"∀T ∈ set P. admissible_transaction T"`

`and transactions_covered: "list_all (transaction_check FP OCC TI) P"`

`shows "∀t ∈ αik (A@duallsst (transaction_strand T ·lsst σ ∘s α)) I.`

`timpl_closure_set (set FP) (set TI) ⊢c t" (is ?A)`

`and "αvals (A@duallsst (transaction_strand T ·lsst σ ∘s α)) I ⊆ absc ' set OCC" (is ?B)`

<proof>

lemma `reachable_constraints_covered:`

`assumes A_reach: "A ∈ reachable_constraints P"`

`and I: "welltyped_constraint_model I A"`

`and FP:`

`"analyzed (timpl_closure_set (set FP) (set TI))"`

`"wftrms (set FP)"`

`"ground (set FP)"`

`and OCC:`

`"∀t ∈ timpl_closure_set (set FP) (set TI). ∀f ∈ funs_term t. is_Abs f → f ∈ Abs ' set OCC"`

`"timpl_closure_set (absc ' set OCC) (set TI) ⊆ absc ' set OCC"`

`and TI:`

`"set TI = {(a,b) ∈ (set TI)+. a ≠ b}"`

`and P:`

`"∀T ∈ set P. admissible_transaction T"`

`and transactions_covered: "list_all (transaction_check FP OCC TI) P"`

`shows "∀t ∈ αik A I. timpl_closure_set (set FP) (set TI) ⊢c t"`

`and "αvals A I ⊆ absc ' set OCC"`

<proof>

lemma `attack_in_fixpoint_if_attack_in_ik:`

`fixes FP::('fun,'atom,'sets) prot_terms"`

`assumes "∀t ∈ IK ·aset a. FP ⊢c t"`

`and "attack(n) ∈ IK"`

`shows "attack(n) ∈ FP"`

<proof>

lemma `attack_in_fixpoint_if_attack_in_timpl_closure_set:`

`fixes FP::('fun,'atom,'sets) prot_terms"`

`assumes "attack(n) ∈ timpl_closure_set FP TI"`

`shows "attack(n) ∈ FP"`

<proof>

theorem `prot_secure_if_fixpoint_covered_typed:`

`assumes FP:`

```

"analyzed (timpl_closure_set (set FP) (set TI))"
"wf_trms (set FP)"
"ground (set FP)"
and OCC:
  "∀t ∈ timpl_closure_set (set FP) (set TI). ∀f ∈ funs_term t. is_Abs f → f ∈ Abs ' set OCC"
  "timpl_closure_set (absc ' set OCC) (set TI) ⊆ absc ' set OCC"
and TI:
  "set TI = {(a,b) ∈ (set TI)+. a ≠ b}"
and P:
  "∀T ∈ set P. admissible_transaction T"
and transactions_covered: "list_all (transaction_check FP OCC TI) P"
and attack_notin_FP: "attack⟨n⟩ ∉ set FP"
and A: "A ∈ reachable_constraints P"
shows "‡I. welltyped_constraint_model I (A@[⟨1, send⟨attack⟨n⟩⟩])" (is "‡I. ?P I")
⟨proof⟩

end

```

2.6.4 Theorem: A Protocol is Secure if it is Covered by a Fixed-Point

```

context stateful_protocol_model
begin

```

```

theorem prot_secure_if_fixpoint_covered:

```

```

  fixes P
  assumes FP:
    "analyzed (timpl_closure_set (set FP) (set TI))"
    "wf_trms (set FP)"
    "ground (set FP)"
  and OCC:
    "∀t ∈ timpl_closure_set (set FP) (set TI). ∀f ∈ funs_term t. is_Abs f → f ∈ Abs ' set OCC"
    "timpl_closure_set (absc ' set OCC) (set TI) ⊆ absc ' set OCC"
  and TI:
    "set TI = {(a,b) ∈ (set TI)+. a ≠ b}"
  and M:
    "has_all_wt_instances_of Γ (⋃T ∈ set P. trms_transaction T) N"
    "finite N"
    "tfr_set N"
    "wf_trms N"
  and P:
    "∀T ∈ set P. admissible_transaction T"
    "∀T ∈ set P. list_all tfr_sstp (unlabel (transaction_strand T))"
  and transactions_covered: "list_all (transaction_check FP OCC TI) P"
  and attack_notin_FP: "attack⟨n⟩ ∉ set FP"
  and A: "A ∈ reachable_constraints P"
  shows "‡I. constraint_model I (A@[⟨1, send⟨attack⟨n⟩⟩])"
    (is "‡I. ?P A I")
  ⟨proof⟩

end

```

2.6.5 Automatic Fixed-Point Computation

```

context stateful_protocol_model
begin

```

```

definition compute_fixpoint_fun' where

```

```

  "compute_fixpoint_fun' P (n::nat option) enable_traces S0 ≡
  let sy = intruder_synth_mod_timpls;

      FP' = λS. fst (fst S);
      TI' = λS. snd (fst S);
      OCC' = λS. remdups (

```

```

(map (λt. the_Abs (the_Fun (args t ! 1)))
  (filter (λt. is_Fun t ∧ the_Fun t = OccursFact) (FP' S)))@
(map snd (TI' S)));

equal_states = λS S'. set (FP' S) = set (FP' S') ∧ set (TI' S) = set (TI' S');

trace' = λS. snd S;

close = λM f. let g = remdups ∘ f in while (λA. set (g A) ≠ set A) g M;
close' = λM f. let g = remdups ∘ f in while (λA. set (g A) ≠ set A) g M;
trancl_minus_refl = λTI.
  let aux = λts p. map (λq. (fst p, snd q)) (filter ((=) (snd p) ∘ fst) ts)
  in filter (λp. fst p ≠ snd p) (close' TI (λts. concat (map (aux ts) ts)@ts));
snd_Ana = λN M TI. let N' = filter (λt. ∀k ∈ set (fst (Ana t)). sy M TI k) N in
  filter (λt. ¬sy M TI t)
    (concat (map (λt. filter (λs. s ∈ set (snd (Ana t))) (args t)) N'));
Ana_cl = λFP TI.
  close FP (λM. (M@snd_Ana M M TI));
TI_cl = λFP TI.
  close FP (λM. (M@filter (λt. ¬sy M TI t)
    (concat (map (λm. concat (map (λ(a,b). ⟨a --> b⟩⟨m⟩) TI)) M))));
Ana_cl' = λFP TI.
  let N = λM. comp_timpl_closure_list (filter (λt. ∃k∈set (fst (Ana t)). ¬sy M TI k) M) TI
  in close FP (λM. M@snd_Ana (N M) M TI);

Δ = λS. transaction_check_comp (FP' S) (OCC' S) (TI' S);
result = λS T δ.
  let not_fresh = λx. x ∉ set (transaction_fresh T);
      xs = filter not_fresh (fv_listset (unlabel (transaction_strand T)));
      u = λδ x. absdbupd (unlabel (transaction_strand T)) x (δ x)
  in (remdups (filter (λt. ¬sy (FP' S) (TI' S) t)
    (map (λt. the_msg t · (absc ∘ u δ))
      (filter is_Send (unlabel (transaction_send T))))),
    remdups (filter (λs. fst s ≠ snd s) (map (λx. (δ x, u δ x)) xs)));
update_state = λS. if list_ex (λt. is_Fun t ∧ is_Attack (the_Fun t)) (FP' S) then S
  else let results = map (λT. map (λδ. result S T (abs_substs_fun δ)) (Δ S T)) P;
      newtraceflt = (λn. let x = results ! n; y = map fst x; z = map snd x
        in set (concat y) - set (FP' S) ≠ {} ∨ set (concat z) - set (TI' S) ≠ {});
      trace =
        if enable_traces
        then trace' S@[filter newtraceflt [0..

```

definition compute_fixpoint_fun where

"compute_fixpoint_fun P ≡ fst (compute_fixpoint_fun' P None False (([], []), []))"

end

2.6.6 Locales for Protocols Proven Secure through Fixed-Point Coverage

```

type_synonym ('f,'a,'s) fixpoint_triple =
  "('f,'a,'s) prot_term list × 's set list × ('s set × 's set) list"

context stateful_protocol_model
begin

definition "attack_notin_fixpoint (FPT::('fun,'atom,'sets) fixpoint_triple) ≡
  list_all (λt. ∀f ∈ funs_term t. ¬is_Attack f) (fst FPT)"

definition "protocol_covered_by_fixpoint (FPT::('fun,'atom,'sets) fixpoint_triple) P ≡
  let (FP, OCC, TI) = FPT
  in list_all (transaction_check FP OCC TI) P"

definition "analyzed_fixpoint (FPT::('fun,'atom,'sets) fixpoint_triple) ≡
  let (FP, _, TI) = FPT
  in analyzed_closed_mod_timpls FP TI"

definition "wellformed_protocol' (P::('fun,'atom,'sets,'lbl) prot) N ≡
  list_all admissible_transaction P ∧
  has_all_wt_instances_of Γ (⋃ T ∈ set P. trms_transaction T) (set N) ∧
  comp_tfr_set arity Ana Γ N ∧
  list_all (λT. list_all (comp_tfr_sstp Γ Pair) (unlabel (transaction_strand T))) P"

definition "wellformed_protocol (P::('fun,'atom,'sets,'lbl) prot) ≡
  let f = λM. remdups (concat (map subterms_list M@map (fst ∘ Ana) M));
      NO = remdups (concat (map (trms_list_sst ∘ unlabel ∘ transaction_strand) P));
      N = while (λA. set (f A) ≠ set A) f NO
  in wellformed_protocol' P N"

definition "wellformed_fixpoint (FPT::('fun,'atom,'sets) fixpoint_triple) ≡
  let (FP, OCC, TI) = FPT; OCC' = set OCC
  in list_all (λt. wf_trm' arity t ∧ fv t = {}) FP ∧
  list_all (λa. a ∈ OCC') (map snd TI) ∧
  list_all (λ(a,b). list_all (λ(c,d). b = c ∧ a ≠ d → List.member TI (a,d)) TI) TI ∧
  list_all (λp. fst p ≠ snd p) TI ∧
  list_all (λt. ∀f ∈ funs_term t. is_Abs f → the_Abs f ∈ OCC') FP"

lemma protocol_covered_by_fixpoint_I1[intro]:
  assumes "list_all (protocol_covered_by_fixpoint FPT) P"
  shows "protocol_covered_by_fixpoint FPT (concat P)"
⟨proof⟩

lemma protocol_covered_by_fixpoint_I2[intro]:
  assumes "protocol_covered_by_fixpoint FPT P1"
  and "protocol_covered_by_fixpoint FPT P2"
  shows "protocol_covered_by_fixpoint FPT (P1@P2)"
⟨proof⟩

lemma protocol_covered_by_fixpoint_I3[intro]:
  assumes "∀T ∈ set P. ∀δ::('fun,'atom,'sets) prot_var ⇒ 'sets set.
  transaction_check_pre FP TI T δ → transaction_check_post FP TI T δ"
  shows "protocol_covered_by_fixpoint (FP,OCC,TI) P"
⟨proof⟩

lemmas protocol_covered_by_fixpoint_intros =
  protocol_covered_by_fixpoint_I1
  protocol_covered_by_fixpoint_I2
  protocol_covered_by_fixpoint_I3

lemma prot_secure_if_prot_checks:
  fixes P::('fun,'atom,'sets,'lbl) prot_transaction list"

```

```

    and FP_OCC_TI:: "('fun, 'atom, 'sets) fixpoint_triple"
  assumes attack_notin_fixpoint: "attack_notin_fixpoint FP_OCC_TI"
    and transactions_covered: "protocol_covered_by_fixpoint FP_OCC_TI P"
    and analyzed_fixpoint: "analyzed_fixpoint FP_OCC_TI"
    and wellformed_protocol: "wellformed_protocol' P N"
    and wellformed_fixpoint: "wellformed_fixpoint FP_OCC_TI"
  shows "∀A ∈ reachable_constraints P. ∄I. constraint_model I (A@[1, send(attack(n))])"
<proof>

end

locale secure_stateful_protocol =
  pm: stateful_protocol_model arity_f arity_s public_f Ana_f Γ_f label_witness1 label_witness2
  for arity_f:: "'fun ⇒ nat"
    and arity_s:: "'sets ⇒ nat"
    and public_f:: "'fun ⇒ bool"
    and Ana_f:: "'fun ⇒ ((('fun, 'atom::finite, 'sets) prot_fun, nat) term list × nat list)"
    and Γ_f:: "'fun ⇒ 'atom option"
    and label_witness1:: "'lbl"
    and label_witness2:: "'lbl"
  +
  fixes P:: "('fun, 'atom, 'sets, 'lbl) prot_transaction list"
    and FP_OCC_TI:: "('fun, 'atom, 'sets) fixpoint_triple"
    and P_SMP:: "('fun, 'atom, 'sets) prot_term list"
  assumes attack_notin_fixpoint: "pm.attack_notin_fixpoint FP_OCC_TI"
    and transactions_covered: "pm.protocol_covered_by_fixpoint FP_OCC_TI P"
    and analyzed_fixpoint: "pm.analyzed_fixpoint FP_OCC_TI"
    and wellformed_protocol: "pm.wellformed_protocol' P P_SMP"
    and wellformed_fixpoint: "pm.wellformed_fixpoint FP_OCC_TI"
begin

theorem protocol_secure:
  "∀A ∈ pm.reachable_constraints P. ∄I. pm.constraint_model I (A@[1, send(attack(n))])"
<proof>

end

locale secure_stateful_protocol' =
  pm: stateful_protocol_model arity_f arity_s public_f Ana_f Γ_f label_witness1 label_witness2
  for arity_f:: "'fun ⇒ nat"
    and arity_s:: "'sets ⇒ nat"
    and public_f:: "'fun ⇒ bool"
    and Ana_f:: "'fun ⇒ ((('fun, 'atom::finite, 'sets) prot_fun, nat) term list × nat list)"
    and Γ_f:: "'fun ⇒ 'atom option"
    and label_witness1:: "'lbl"
    and label_witness2:: "'lbl"
  +
  fixes P:: "('fun, 'atom, 'sets, 'lbl) prot_transaction list"
    and FP_OCC_TI:: "('fun, 'atom, 'sets) fixpoint_triple"
  assumes attack_notin_fixpoint': "pm.attack_notin_fixpoint FP_OCC_TI"
    and transactions_covered': "pm.protocol_covered_by_fixpoint FP_OCC_TI P"
    and analyzed_fixpoint': "pm.analyzed_fixpoint FP_OCC_TI"
    and wellformed_protocol': "pm.wellformed_protocol P"
    and wellformed_fixpoint': "pm.wellformed_fixpoint FP_OCC_TI"
begin

sublocale secure_stateful_protocol
  arity_f arity_s public_f Ana_f Γ_f label_witness1 label_witness2 P
  FP_OCC_TI
  "let f = λM. remdups (concat (map subterms_list M@map (fst ∘ pm.Ana) M));
    NO = remdups (concat (map (trms_list_sst ∘ unlabel ∘ transaction_strand) P))
  in while (λA. set (f A) ≠ set A) f NO"
<proof>

```

end

```

locale secure_stateful_protocol'' =
  pm: stateful_protocol_model arity_f arity_s public_f Ana_f  $\Gamma_f$  label_witness1 label_witness2
  for arity_f::''fun  $\Rightarrow$  nat"
    and arity_s::''sets  $\Rightarrow$  nat"
    and public_f::''fun  $\Rightarrow$  bool"
    and Ana_f::''fun  $\Rightarrow$  ((('fun, 'atom::finite, 'sets) prot_fun, nat) term list  $\times$  nat list)"
    and  $\Gamma_f$ ::''fun  $\Rightarrow$  'atom option"
    and label_witness1::''lbl"
    and label_witness2::''lbl"
  +
  fixes P::('fun, 'atom, 'sets, 'lbl) prot_transaction list"
  assumes checks: "let FPT = pm.compute_fixpoint_fun P
    in pm.attack_notin_fixpoint FPT  $\wedge$  pm.protocol_covered_by_fixpoint FPT P  $\wedge$ 
    pm.analyzed_fixpoint FPT  $\wedge$  pm.wellformed_protocol P  $\wedge$  pm.wellformed_fixpoint FPT"
begin

  sublocale secure_stateful_protocol'
    arity_f arity_s public_f Ana_f  $\Gamma_f$  label_witness1 label_witness2 P "pm.compute_fixpoint_fun P"
    <proof>

```

end

```

locale secure_stateful_protocol'''' =
  pm: stateful_protocol_model arity_f arity_s public_f Ana_f  $\Gamma_f$  label_witness1 label_witness2
  for arity_f::''fun  $\Rightarrow$  nat"
    and arity_s::''sets  $\Rightarrow$  nat"
    and public_f::''fun  $\Rightarrow$  bool"
    and Ana_f::''fun  $\Rightarrow$  ((('fun, 'atom::finite, 'sets) prot_fun, nat) term list  $\times$  nat list)"
    and  $\Gamma_f$ ::''fun  $\Rightarrow$  'atom option"
    and label_witness1::''lbl"
    and label_witness2::''lbl"
  +
  fixes P::('fun, 'atom, 'sets, 'lbl) prot_transaction list"
    and FP_OCC_TI:: ('fun, 'atom, 'sets) fixpoint_triple"
    and P_SMP:: ('fun, 'atom, 'sets) prot_term list"
  assumes checks': "let P' = P; FPT = FP_OCC_TI; P'_SMP = P_SMP
    in pm.attack_notin_fixpoint FPT  $\wedge$ 
    pm.protocol_covered_by_fixpoint FPT P'  $\wedge$ 
    pm.analyzed_fixpoint FPT  $\wedge$ 
    pm.wellformed_protocol' P' P'_SMP  $\wedge$ 
    pm.wellformed_fixpoint FPT"
begin

```

```

  sublocale secure_stateful_protocol
    arity_f arity_s public_f Ana_f  $\Gamma_f$  label_witness1 label_witness2 P FP_OCC_TI P_SMP
    <proof>

```

end

```

locale secure_stateful_protocol'''''' =
  pm: stateful_protocol_model arity_f arity_s public_f Ana_f  $\Gamma_f$  label_witness1 label_witness2
  for arity_f::''fun  $\Rightarrow$  nat"
    and arity_s::''sets  $\Rightarrow$  nat"
    and public_f::''fun  $\Rightarrow$  bool"
    and Ana_f::''fun  $\Rightarrow$  ((('fun, 'atom::finite, 'sets) prot_fun, nat) term list  $\times$  nat list)"
    and  $\Gamma_f$ ::''fun  $\Rightarrow$  'atom option"
    and label_witness1::''lbl"
    and label_witness2::''lbl"
  +
  fixes P::('fun, 'atom, 'sets, 'lbl) prot_transaction list"

```

```

    and FP_OCC_TI:: "('fun, 'atom, 'sets) fixpoint_triple"
  assumes checks'': "let P' = P; FPT = FP_OCC_TI
    in pm.attack_notin_fixpoint FPT ^
      pm.protocol_covered_by_fixpoint FPT P' ^
      pm.analyzed_fixpoint FPT ^
      pm.wellformed_protocol P' ^
      pm.wellformed_fixpoint FPT"

begin

sublocale secure_stateful_protocol'
  arity_f arity_s public_f Ana_f  $\Gamma_f$  label_witness1 label_witness2 P FP_OCC_TI
  <proof>

end

```

2.6.7 Automatic Protocol Composition

```

context stateful_protocol_model
begin

```

```

definition wellformed_composable_protocols where
  "wellformed_composable_protocols (P::('fun, 'atom, 'sets, 'lbl) prot list) N  $\equiv$ 
  let
    Ts = concat P;
    steps = concat (map transaction_strand Ts);
    MPO =  $\bigcup T \in \text{set } Ts. \text{trms\_transaction } T \cup \text{pair}' \text{Pair ' setops\_transaction } T$ 
  in
    list_all (wftrm' arity) N ^
    has_all_wt_instances_of  $\Gamma$  MPO (set N) ^
    comp_tfrset arity Ana  $\Gamma$  N ^
    list_all (comp_tfrsstp  $\Gamma$  Pair  $\circ$  snd) steps ^
    list_all ( $\lambda T. \text{wellformed\_transaction } T$ ) Ts ^
    list_all ( $\lambda T. \text{wf}_{trms}' \text{arity} (\text{trms\_transaction } T)$ ) Ts ^
    list_all ( $\lambda T. \text{list\_all} (\lambda x. \Gamma_v x = \text{TAtom Value}) (\text{transaction\_fresh } T)$ ) Ts"

```

```

definition composable_protocols where
  "composable_protocols (P::('fun, 'atom, 'sets, 'lbl) prot list) Ms S  $\equiv$ 
  let
    Ts = concat P;
    steps = concat (map transaction_strand Ts);
    MPO =  $\bigcup T \in \text{set } Ts. \text{trms\_transaction } T \cup \text{pair}' \text{Pair ' setops\_transaction } T$ ;
    M_fun = ( $\lambda l. \text{case find } ((=) 1 \circ \text{fst}) \text{Ms of Some } M \Rightarrow \text{snd } M \mid \text{None} \Rightarrow []$ )
  in comp_par_complsst public arity Ana  $\Gamma$  Pair steps M_fun S"

```

```

lemma composable_protocols_par_comp_constr:
  fixes S f
  defines "f  $\equiv$   $\lambda M. \{t \cdot \delta \mid t \delta. t \in M \wedge \text{wt}_{subst} \delta \wedge \text{wf}_{trms} (\text{subst\_range } \delta) \wedge \text{fv } (t \cdot \delta) = \{\}\}$ "
  and "Sec  $\equiv$  (f (set S)) - {m. intruder_synth { } m}"
  assumes Ps_pc: "wellformed_composable_protocols Ps N" "composable_protocols Ps Ms S"
  shows " $\forall \mathcal{A} \in \text{reachable\_constraints} (\text{concat } Ps). \forall \mathcal{I}. \text{constraint\_model } \mathcal{I} \mathcal{A} \longrightarrow$ 
    ( $\exists \mathcal{I}_\tau. \text{welltyped\_constraint\_model } \mathcal{I}_\tau \mathcal{A} \wedge$ 
      ( $\forall n. \text{welltyped\_constraint\_model } \mathcal{I}_\tau (\text{proj } n \mathcal{A}) \vee$ 
        ( $\exists \mathcal{A}'. \text{prefix } \mathcal{A}' \mathcal{A} \wedge \text{strand\_leaks}_{lsst} \mathcal{A}' \text{Sec } \mathcal{I}_\tau$ )))"
    (is " $\forall \mathcal{A} \in \_ . \forall \_ . \_ \longrightarrow ?Q \mathcal{A} \mathcal{I}$ ")
  <proof>

```

```

end

```

```

end

```


3 Trac Support and Automation

3.1 Useful Eisbach Methods for Automating Protocol Verification (Eisbach_Protocol_Verification)

```
theory Eisbach_Protocol_Verification
  imports Main "HOL-Eisbach.Eisbach_Tools"
begin

named_theorems exhausts
named_theorems type_class_instance_lemmata
named_theorems protocol_checks
named_theorems coverage_check_unfold_protocol_lemma
named_theorems coverage_check_unfold_lemmata
named_theorems coverage_check_intro_lemmata
named_theorems transaction_coverage_lemmata

method UNIV_lemma =
  (rule UNIV_eq_I; (subst insert_iff)+; subst empty_iff; smt exhausts)+

method type_class_instance =
  (intro_classes; auto simp add: type_class_instance_lemmata)

method protocol_model_subgoal =
  (((rule allI, case_tac f); (erule forw_subst)+)?; simp_all)

method protocol_model_interpretation =
  (unfold_locales; protocol_model_subgoal+)

method check_protocol_intro =
  (unfold_locales, unfold protocol_checks[symmetric])

method check_protocol_with methods meth =
  (check_protocol_intro, meth)

method check_protocol' =
  (check_protocol_with <code_simp+>)

method check_protocol_unsafe' =
  (check_protocol_with <eval+>)

method check_protocol =
  (check_protocol_with <
    code_simp,
    code_simp,
    code_simp,
    code_simp,
    code_simp>)

method check_protocol_unsafe =
  (check_protocol_with <
    eval,
    eval,
    eval,
    eval,
    eval>)
```

```
method coverage_check_intro =
  ((unfold coverage_check_unfold_protocol_lemma)?;
   intro coverage_check_intro_lemmata;
   simp only: list_all_simps list_all_append list.map concat.simps map_append product_concat_map;
   intro conjI TrueI);
  (clarsimp+)?;
  ((rule transaction_coverage_lemmata)+)?

method coverage_check_unfold =
  (unfold coverage_check_unfold_protocol_lemma coverage_check_unfold_lemmata
   list_all_iff Let_def case_prod_unfold Product_Type.fst_conv Product_Type.snd_conv)

end
```

3.2 ML Yacc Library (ml_yacc_lib)

```
theory
  "ml_yacc_lib"
  imports
    Main
begin
  <ML>

end
```

3.3 Abstract Syntax for Trac Terms (trac_term)

```
theory
  trac_term
  imports
    "First_Order_Terms.Term"
    "ml_yacc_lib"

begin
  datatype cMsg = cVar "string * string"
               | cConst string
               | cFun "string * cMsg list"

  <ML>

end
```

3.4 Parser for Trac FP definitions (trac_fp_parser)

```
theory
  trac_fp_parser
  imports
    "trac_term"
begin
  <ML>

end
```

3.5 Parser for the Trac Format (*trac_protocol_parser*)

```
theory
  trac_protocol_parser
  imports
    "trac_term"
begin

⟨ML⟩

end
```

3.6 Support for the Trac Format (*trac*)

```
theory
  "trac"
  imports
    trac_fp_parser
    trac_protocol_parser
keywords
  "trac" :: thy_decl
  and "trac_import" :: thy_decl
  and "trac_trac" :: thy_decl
  and "trac_import_trac" :: thy_decl
  and "protocol_model_setup" :: thy_decl
  and "protocol_security_proof" :: thy_decl
  and "manual_protocol_model_setup" :: thy_decl
  and "manual_protocol_security_proof" :: thy_decl
  and "compute_fixpoint" :: thy_decl
  and "compute_SMP" :: thy_decl
  and "setup_protocol_model'" :: thy_decl
  and "protocol_security_proof'" :: thy_decl
  and "setup_protocol_checks" :: thy_decl
begin

⟨ML⟩

end
```


4 Examples

4.1 The Keyserver Protocol (Keyserver)

```
theory Keyserver
  imports "../PSPSP"
begin

declare [[code_timing]]

trac<
Protocol: keyserver

Types:
honest = {a,b,c}
server = {s}
agents = honest ++ server

Sets:
ring/1 valid/2 revoked/2

Functions:
Public sign/2 crypt/2 pair/2
Private inv/1

Analysis:
sign(X,Y) -> Y
crypt(X,Y) ? inv(X) -> Y
pair(X,Y) -> X,Y

Transactions:
# Out-of-band registration
outOfBand(A:honest,S:server)
  new NPK
  insert NPK ring(A)
  insert NPK valid(A,S)
  send NPK.

# User update key
keyUpdateUser(A:honest,PK:value)
  PK in ring(A)
  new NPK
  delete PK ring(A)
  insert NPK ring(A)
  send sign(inv(PK),pair(A,NPK)).

# Server update key
keyUpdateServer(A:honest,S:server,PK:value,NPK:value)
  receive sign(inv(PK),pair(A,NPK))
  PK in valid(A,S)
  NPK notin valid(_)
  NPK notin revoked(_)
  delete PK valid(A,S)
  insert PK revoked(A,S)
  insert NPK valid(A,S)
  send inv(PK).
```

4 Examples

```
# Attack definition
authAttack(A:honest,S:server,PK:value)
  receive inv(PK)
  PK in valid(A,S)
  attack.
><
val(ring(A)) where A:honest
sign(inv(val(0)),pair(A,val(ring(A)))) where A:honest
inv(val(revoked(A,S))) where A:honest S:server
pair(A,val(ring(A))) where A:honest

occurs(val(ring(A))) where A:honest

timplies(val(ring(A)),val(ring(A),valid(A,S))) where A:honest S:server
timplies(val(ring(A)),val(0)) where A:honest
timplies(val(ring(A),valid(A,S)),val(valid(A,S))) where A:honest S:server
timplies(val(0),val(valid(A,S))) where A:honest S:server
timplies(val(valid(A,S)),val(revoked(A,S))) where A:honest S:server
>
```

4.1.1 Proof of security

```
protocol_model_setup spm: keyserver
compute_SMP [optimized] keyserver_protocol keyserver_SMP
manual_protocol_security_proof ssp: keyserver
  for keyserver_protocol keyserver_fixpoint keyserver_SMP
  ⟨proof⟩
end
```

4.2 A Variant of the Keyserver Protocol (Keyserver2)

```
theory Keyserver2
  imports "../PSPSP"
begin

declare [[code_timing]]

trac<
Protocol: keyserver2

Types:
honest = {a,b,c}
dishonest = {i}
agent = honest ++ dishonest

Sets:
ring'/1 seen/1 pubkeys/0 valid/1

Functions:
Public h/1 sign/2 crypt/2 scrypt/2 pair/2 update/3
Private inv/1 pw/1

Analysis:
sign(X,Y) -> Y
crypt(X,Y) ? inv(X) -> Y
scrypt(X,Y) ? X -> Y
pair(X,Y) -> X,Y
update(X,Y,Z) -> X,Y,Z

Transactions:
passwordGenD(A:dishonest)
```

```

send pw(A).

pubkeysGen()
  new PK
  insert PK pubkeys
  send PK.

updateKeyPw(A:honest,PK:value)
  PK in pubkeys
  new NPK
  insert NPK ring'(A)
  send NPK
  send crypt(PK,update(A,NPK,pw(A))).

updateKeyServerPw(A:agent,PK:value,NPK:value)
  receive crypt(PK,update(A,NPK,pw(A)))
  PK in pubkeys
  NPK notin pubkeys
  NPK notin seen(_)
  insert NPK valid(A)
  insert NPK seen(A).

authAttack2(A:honest,PK:value)
  receive inv(PK)
  PK in valid(A)
  attack.
>

```

4.2.1 Proof of security

```

protocol_model_setup spm: keyserver2
compute_fixpoint keyserver2_protocol keyserver2_fixpoint
protocol_security_proof ssp: keyserver2

```

4.2.2 The generated theorems and definitions

```

thm ssp.protocol_secure

thm keyserver2_enum_consts.nchotomy
thm keyserver2_sets.nchotomy
thm keyserver2_fun.nchotomy
thm keyserver2_atom.nchotomy
thm keyserver2_arity.simps
thm keyserver2_public.simps
thm keyserver2_Γ.simps
thm keyserver2_Ana.simps

thm keyserver2_transaction_passwordGenD_def
thm keyserver2_transaction_pubkeysGen_def
thm keyserver2_transaction_updateKeyPw_def
thm keyserver2_transaction_updateKeyServerPw_def
thm keyserver2_transaction_authAttack2_def
thm keyserver2_protocol_def

thm keyserver2_fixpoint_def

end

```

4.3 The Composition of the Two Keyserver Protocols (Keyserver_Composition)

```

theory Keyserver_Composition

```

4 Examples

```
imports "../PSPSP"
begin

declare [[code_timing]]

trac<
Protocol: kscomp

Types:
honest = {a,b,c}
dishonest = {i}
agent = honest ++ dishonest

Sets:
ring/1 valid/1 revoked/1 deleted/1
ring'/1 seen/1 pubkeys/0

Functions:
Public h/1 sign/2 crypt/2 scrypt/2 pair/2 update/3
Private inv/1 pw/1

Analysis:
sign(X,Y) -> Y
crypt(X,Y) ? inv(X) -> Y
scrypt(X,Y) ? X -> Y
pair(X,Y) -> X,Y
update(X,Y,Z) -> X,Y,Z

Transactions:
### The signature-based keyserver protocol
p1_outOfBand(A:honest)
  new PK
  insert PK ring(A)
  * insert PK valid(A)
  send PK.

p1_oufOfBandD(A:dishonest)
  new PK
  * insert PK valid(A)
  send PK
  send inv(PK).

p1_updateKey(A:honest,PK:value)
  PK in ring(A)
  new NPK
  delete PK ring(A)
  insert PK deleted(A)
  insert NPK ring(A)
  send sign(inv(PK),pair(A,NPK)).

p1_updateKeyServer(A:agent,PK:value,NPK:value)
  receive sign(inv(PK),pair(A,NPK))
  * PK in valid(A)
  * NPK notin valid(_)
  NPK notin revoked(_)
  * delete PK valid(A)
  insert PK revoked(A)
  * insert NPK valid(A)
  send inv(PK).

p1_authAttack(A:honest,PK:value)
  receive inv(PK)
  * PK in valid(A)
```



```

attack.

### The password-based keyserver protocol
p2_passwordGenD(A:dishonest)
  send pw(A).

p2_pubkeysGen()
  new PK
  insert PK pubkeys
  send PK.

p2_updateKeyPw(A:honest,PK:value)
  PK in pubkeys
  new NPK
# NOTE: The ring' sets are not used elsewhere, but we have to avoid that the fresh keys generated
#       by this rule are abstracted to the empty abstraction, and so we insert them into a ring'
#       set. Otherwise the two protocols would have too many abstractions in common (in particular,
#       the empty abstraction) which leads to false attacks in the composed protocol (probably
#       because the term implication graphs of the two protocols then become 'linked' through the
#       empty abstraction)
  insert NPK ring'(A)
  send NPK
  send crypt(PK,update(A,NPK,pw(A))).

#Transactions of p2:
p2_updateKeyServerPw(A:agent,PK:value,NPK:value)
receive crypt(PK,update(A,NPK,pw(A)))
  PK in pubkeys
  NPK notin pubkeys
  NPK notin seen(_)
* insert NPK valid(A)
  insert NPK seen(A).

p2_authAttack2(A:honest,PK:value)
  receive inv(PK)
* PK in valid(A)
  attack.
> <
sign(inv(val(deleted(A))),pair(A,val(ring(A)))) where A:honest
sign(inv(val(deleted(A),valid(B))),pair(A,val(ring(A)))) where A:honest B:dishonest
sign(inv(val(deleted(A),seen(B),valid(B))),pair(A,val(ring(A)))) where A:honest B:dishonest
sign(inv(val(deleted(A),valid(A))),pair(A,val(ring(A)))) where A:honest B:dishonest
sign(inv(val(deleted(A),seen(B),valid(B),valid(A))),pair(A,val(ring(A)))) where A:honest B:dishonest
pair(A,val(ring(A))) where A:honest
inv(val(deleted(A),revoked(A))) where A:honest
inv(val(valid(A))) where A:dishonest
inv(val(revoked(A))) where A:dishonest
inv(val(revoked(A),seen(A))) where A:dishonest
inv(val(revoked(B),seen(B),revoked(A),deleted(A))) where A:honest B:dishonest
inv(val(revoked(A),deleted(A),seen(B),valid(B))) where A:honest B:dishonest
occurs(val(ring(A))) where A:honest
occurs(val(valid(A))) where A:dishonest
occurs(val(ring'(A))) where A:honest
occurs(val(pubkeys))
occurs(val(valid(A),ring(A))) where A:honest
pw(A) where A:dishonest
crypt(val(pubkeys),update(A,val(ring'(A)),pw(A))) where A:honest
val(ring(A)) where A:honest
val(valid(A)) where A:dishonest
val(ring'(A)) where A:honest
val(pubkeys)
val(valid(A),ring(A)) where A:honest

```



```

timplies(val(valid(B),deleted(A)),val(seen(B),valid(B),deleted(A))) where A:honest B:dishonest
timplies(val(ring(A),valid(B)),val(deleted(A),seen(B),valid(B))) where A:honest B:dishonest
timplies(val(ring(A),valid(B)),val(seen(B),valid(B),ring(A))) where A:honest B:dishonest

timplies(val(valid(A)),val(seen(A),valid(A))) where A:dishonest
>

```

4.3.1 Proof: The composition of the two keyserver protocols is secure

```

protocol_model_setup spm: kscomp
setup_protocol_checks spm kscomp_protocol
manual_protocol_security_proof ssp: kscomp
  ⟨proof⟩

```

4.3.2 The generated theorems and definitions

```

thm ssp.protocol_secure

thm kscomp_enum_consts.nchotomy
thm kscomp_sets.nchotomy
thm kscomp_fun.nchotomy
thm kscomp_atom.nchotomy
thm kscomp_arity.simps
thm kscomp_public.simps
thm kscomp_Γ.simps
thm kscomp_Ana.simps

thm kscomp_transaction_p1_outOfBand_def
thm kscomp_transaction_p1_outOfBandD_def
thm kscomp_transaction_p1_updateKey_def
thm kscomp_transaction_p1_updateKeyServer_def
thm kscomp_transaction_p1_authAttack_def
thm kscomp_transaction_p2_passwordGenD_def
thm kscomp_transaction_p2_pubkeysGen_def
thm kscomp_transaction_p2_updateKeyPw_def
thm kscomp_transaction_p2_updateKeyServerPw_def
thm kscomp_transaction_p2_authAttack2_def
thm kscomp_protocol_def

thm kscomp_fixpoint_def

end

```

4.4 The PKCS Model, Scenario 3 (PKCS_Model03)

```

theory PKCS_Model03
  imports "../..//PSPSP"

begin

declare [[code_timing]]

trac<
Protocol: ATTACK_UNSET

Types:
token = {token1}

Sets:
extract/1 wrap/1 decrypt/1 sensitive/1

Functions:

```

4 Examples

```
Public senc/2 h/1
Private inv/1
```

Analysis:

```
senc(M,K2) ? K2 -> M #This analysis rule corresponds to the decrypt2 rule in the AIF-omega specification.
                    #M was type untyped
```

Transactions:

```
iik1()
new K1
insert K1 sensitive(token1)
insert K1 extract(token1)
send h(K1).

iik2()
new K2
insert K2 wrap(token1)
send h(K2).

# =====wrap=====
wrap(K1:value,K2:value)
receive h(K1)
receive h(K2)
K1 in extract(token1)
K2 in wrap(token1)
send senc(K1,K2).

# =====set wrap=====
setwrap(K2:value)
receive h(K2)
K2 notin decrypt(token1)
insert K2 wrap(token1).

# =====set decrypt=====
setdecrypt(K2:value)
receive h(K2)
K2 notin wrap(token1)
insert K2 decrypt(token1).

# =====decrypt=====
decrypt1(K2:value,M:value) #M was untyped in the AIF-omega specification.
receive h(K2)
receive senc(M,K2)
K2 in decrypt(token1)
send M.

# =====attacks=====
attack1(K1:value)
receive K1
K1 in sensitive(token1)
attack.
>
```

4.4.1 Protocol model setup

```
protocol_model_setup spm: ATTACK_UNSET
```

4.4.2 Fixpoint computation

```
compute_fixpoint ATTACK_UNSET_protocol ATTACK_UNSET_fixpoint
compute_SMP [optimized] ATTACK_UNSET_protocol ATTACK_UNSET_SMP
```

4.4.3 Proof of security

```
manual_protocol_security_proof ssp: ATTACK_UNSET
  for ATTACK_UNSET_protocol ATTACK_UNSET_fixpoint ATTACK_UNSET_SMP
  <proof>
```

4.4.4 The generated theorems and definitions

```
thm ssp.protocol_secure

thm ATTACK_UNSET_enum_consts.nchotomy
thm ATTACK_UNSET_sets.nchotomy
thm ATTACK_UNSET_fun.nchotomy
thm ATTACK_UNSET_atom.nchotomy
thm ATTACK_UNSET_arity.simps
thm ATTACK_UNSET_public.simps
thm ATTACK_UNSET_Γ.simps
thm ATTACK_UNSET_Ana.simps

thm ATTACK_UNSET_transaction_iik1_def
thm ATTACK_UNSET_transaction_iik2_def
thm ATTACK_UNSET_transaction_wrap_def
thm ATTACK_UNSET_transaction_setwrap_def
thm ATTACK_UNSET_transaction_setdecrypt_def
thm ATTACK_UNSET_transaction_decrypt1_def
thm ATTACK_UNSET_transaction_attack1_def

thm ATTACK_UNSET_protocol_def

thm ATTACK_UNSET_fixpoint_def
thm ATTACK_UNSET_SMP_def

end
```

4.5 The PKCS Protocol, Scenario 7 (PKCS_Model07)

```
theory PKCS_Model07
  imports "../PSPSP"

begin

declare [[code_timing]]

trac<
Protocol: RE_IMPORT_ATT

Types:
token = {token1}

Sets:
extract/1 wrap/1 unwrap/1 decrypt/1 sensitive/1

Functions:
Public senc/2 h/2 bind/2
Private inv/1

Analysis:
senc(M1,K2) ? K2 -> M1 #This analysis rule corresponds to the decrypt2 rule in the AIF-omega specification.
#M1 was type untyped

Transactions:

iik1()
```

4 Examples

```
new K1
new N1
insert N1 sensitive(token1)
insert N1 extract(token1)
insert K1 sensitive(token1)
send h(N1,K1).

iik2()
new K2
new N2
insert N2 wrap(token1)
insert N2 extract(token1)
send h(N2,K2).

# =====set wrap=====
setwrap(N2:value,K2:value)
receive h(N2,K2)
N2 notin sensitive(token1)
N2 notin decrypt(token1)
insert N2 wrap(token1).

# =====set unwrap====
setunwrap(N2:value,K2:value)
receive h(N2,K2)
N2 notin sensitive(token1)
insert N2 unwrap(token1).

# =====unwrap, generate new handler=====
#-----the sensitive attr copy-----
unwrapsensitive(M2:value, K2:value, N1:value, N2:value) #M2 was untyped in the AIF-omega specification.
receive senc(M2,K2)
receive bind(N1,M2)
receive h(N2,K2)
N1 in sensitive(token1)
N2 in unwrap(token1)
new Nnew
insert Nnew sensitive(token1)
send h(Nnew,M2).

#-----the wrap attr copy-----
wrapattr(M2:value, K2:value, N1:value, N2:value) #M2 was untyped in the AIF-omega specification.
receive senc(M2,K2)
receive bind(N1,M2)
receive h(N2,K2)
N1 in wrap(token1)
N2 in unwrap(token1)
new Nnew
insert Nnew wrap(token1)
send h(Nnew,M2).

#-----the decrypt attr copy-----
decrypt1attr(M2:value,K2:value,N1:value,N2:value) #M2 was untyped in the AIF-omega specification.
receive senc(M2,K2)
receive bind(N1,M2)
receive h(N2,K2)
N1 in decrypt(token1)
N2 in unwrap(token1)
new Nnew
insert Nnew decrypt(token1)
send h(Nnew,M2).

decrypt2attr(M2:value,K2:value,N1:value,N2:value) #M2 was untyped in the AIF-omega specification.
receive senc(M2,K2)
```

```

receive bind(N1,M2)
receive h(N2,K2)
N1 notin sensitive(token1)
N1 notin wrap(token1)
N1 notin decrypt(token1)
N2 in unwrap(token1)
new Nnew
send h(Nnew,M2).

# =====wrap=====
wrap(N1:value,K1:value,N2:value,K2:value)
receive h(N1,K1)
receive h(N2,K2)
N1 in extract(token1)
N2 in wrap(token1)
send senc(K1,K2)
send bind(N1,K1).

# =====set decrypt===
setdecrypt(Nnew:value, K2:value)
receive h(Nnew,K2)
Nnew notin wrap(token1)
insert Nnew decrypt(token1).

# =====decrypt=====
decrypt1(Nnew:value, K2:value,M1:value) #M1 was untyped in the AIF-omega specification.
receive h(Nnew,K2)
receive senc(M1,K2)
Nnew in decrypt(token1)
delete Nnew decrypt(token1)
send M1.

# =====attacks=====
attack1(K1:value)
receive K1
K1 in sensitive(token1)
attack.
>

```

4.5.1 Protocol model setup

```
protocol_model_setup spm: RE_IMPORT_ATT
```

4.5.2 Fixpoint computation

```
compute_fixpoint RE_IMPORT_ATT_protocol RE_IMPORT_ATT_fixpoint
compute_SMP [optimized] RE_IMPORT_ATT_protocol RE_IMPORT_ATT_SMP
```

4.5.3 Proof of security

```
protocol_security_proof [unsafe] ssp: RE_IMPORT_ATT
  for RE_IMPORT_ATT_protocol RE_IMPORT_ATT_fixpoint RE_IMPORT_ATT_SMP
```

4.5.4 The generated theorems and definitions

```
thm ssp.protocol_secure

thm RE_IMPORT_ATT_enum_consts.nchotomy
thm RE_IMPORT_ATT_sets.nchotomy
thm RE_IMPORT_ATT_fun.nchotomy
thm RE_IMPORT_ATT_atom.nchotomy
thm RE_IMPORT_ATT_arity.simps
thm RE_IMPORT_ATT_public.simps
```

4 Examples

```
thm RE_IMPORT_ATT_Γ.simps
thm RE_IMPORT_ATT_Ana.simps

thm RE_IMPORT_ATT_transaction_iik1_def
thm RE_IMPORT_ATT_transaction_iik2_def
thm RE_IMPORT_ATT_transaction_setwrap_def
thm RE_IMPORT_ATT_transaction_setunwrap_def
thm RE_IMPORT_ATT_transaction_unwrapsensitive_def
thm RE_IMPORT_ATT_transaction_wrapattr_def
thm RE_IMPORT_ATT_transaction_decrypt1attr_def
thm RE_IMPORT_ATT_transaction_decrypt2attr_def
thm RE_IMPORT_ATT_transaction_wrap_def
thm RE_IMPORT_ATT_transaction_setdecrypt_def
thm RE_IMPORT_ATT_transaction_decrypt1_def
thm RE_IMPORT_ATT_transaction_attack1_def

thm RE_IMPORT_ATT_protocol_def

thm RE_IMPORT_ATT_fixpoint_def
thm RE_IMPORT_ATT_SMP_def

end
```

4.6 The PKCS Protocol, Scenario 9 (PKCS_Model09)

```
theory PKCS_Model09
  imports "../..//PSPSP"

begin

declare [[code_timing]]

trac<
Protocol: LOSS_KEY_ATT

Types:
token = {token1}

Sets:
extract/1 wrap/1 unwrap/1 decrypt/1 sensitive/1

Functions:
Public senc/2 h/2 bind/3
Private inv/1

Analysis:
senc(M1,K2) ? K2 -> M1 #This analysis rule corresponds to the decrypt2 rule in the AIF-omega specification.
                        #M1 was type untyped

Transactions:
iik1()
new K1
new N1
insert N1 sensitive(token1)
insert N1 extract(token1)
insert K1 sensitive(token1)
send h(N1,K1).

iik2()
new K2
new N2
insert N2 wrap(token1)
```



```

insert N2 extract(token1)
send h(N2,K2).

iik3()
new K3
new N3
insert N3 extract(token1)
insert N3 decrypt(token1)
insert K3 decrypt(token1)
send h(N3,K3)
send K3.

# =====set wrap=====
setwrap(N2:value,K2:value) where N2 != K2
receive h(N2,K2)
N2 notin sensitive(token1)
N2 notin decrypt(token1)
insert N2 wrap(token1).

# =====set unwrap===
setunwrap(N2:value,K2:value) where N2 != K2
receive h(N2,K2)
N2 notin sensitive(token1)
insert N2 unwrap(token1).

# =====unwrap, generate new handler=====
#-----add the wrap attr copy-----
unwrapWrap(M2:value,K2:value,N1:value,N2:value) where M2 != K2, M2 != N1, M2 != N2, K2 != N1, K2 != N2,
N1 != N2 #M2 was untyped in the AIF-omega specification.
receive senc(M2,K2)
receive bind(N1,M2,K2)
receive h(N2,K2)
N1 in wrap(token1)
N2 in unwrap(token1)
new Nnew
insert Nnew wrap(token1)
send h(Nnew,M2).

#-----add the sensitive attr copy-----
unwrapSens(M2:value,K2:value,N1:value,N2:value) where M2 != K2, M2 != N1, M2 != N2, K2 != N1, K2 != N2,
N1 != N2 #M2 was untyped in the AIF-omega specification.
receive senc(M2,K2)
receive bind(N1,M2,K2)
receive h(N2,K2)
N1 in sensitive(token1)
N2 in unwrap(token1)
new Nnew
insert Nnew sensitive(token1)
send h(Nnew,M2).

#-----add the decrypt attr copy-----
decrypt1Attr(M2:value, K2:value,N1:value,N2:value) where M2 != K2, M2 != N1, M2 != N2, K2 != N1, K2 != N2,
N1 != N2 #M2 was untyped in the AIF-omega specification.
receive senc(M2,K2)
receive bind(N1,M2,K2)
receive h(N2,K2)
N1 in decrypt(token1)
N2 in unwrap(token1)
new Nnew
insert Nnew decrypt(token1)
send h(Nnew,M2).

decrypt2Attr(M2:value, K2:value,N1:value,N2:value) where M2 != K2, M2 != N1, M2 != N2, K2 != N1, K2 != N2,

```

4 Examples

```
N1 != N2 #M2 was untyped in the AIF-omega specification.
receive senc(M2,K2)
receive bind(N1,M2,K2)
receive h(N2,K2)
N1 notin wrap(token1)
N1 notin sensitive(token1)
N1 notin decrypt(token1)
N2 in unwrap(token1)
new Nnew
send h(Nnew,M2).

# =====wrap=====
wrap(N1:value,K1:value, N2:value, K2:value) where N1 != N2, N1 != K2, N1 != K1, N2 != K2, N2 != K1, K2 !=
K1
receive h(N1,K1)
receive h(N2,K2)
N1 in extract(token1)
N2 in wrap(token1)
send senc(K1,K2)
send bind(N1,K1,K2).

# =====bind generation=====
bind1(K3:value,N2:value,K2:value, K1:value) where K3 != N2, K3 != K2, K3 != K1, N2 != K2, N2 != K1, K2 !=
K1
receive K3
receive h(N2,K2)
send bind(N2,K3,K3).

bind2(K3:value,N2:value,K2:value, K1:value) where K3 != N2, K3 != K2, K3 != K1, N2 != K2, N2 != K1, K2 !=
K1
receive K3
receive K1
receive h(N2,K2)
send bind(N2,K1,K3)
send bind(N2,K3,K1).

# =====set decrypt===
setdecrypt(Nnew:value,K2:value) where Nnew != K2
receive h(Nnew,K2)
Nnew notin wrap(token1)
insert Nnew decrypt(token1).

# =====decrypt=====
decrypt1(Nnew:value,K2:value,M1:value) where Nnew != K2, Nnew != M1, K2 != M1 #M1 was untyped in the AIF-omega
specification.
receive h(Nnew,K2)
receive senc(M1,K2)
Nnew in decrypt(token1)
send M1.

# =====attacks=====
attack1(K1:value)
receive K1
K1 in sensitive(token1)
attack.

>
```

4.6.1 Protocol model setup

protocol_model_setup spm: LOSS_KEY_ATT

4.6.2 Fixpoint computation

```
compute_fixpoint LOSS_KEY_ATT_protocol LOSS_KEY_ATT_fixpoint
```

The fixpoint contains an attack signal

```
value "attack_notin_fixpoint LOSS_KEY_ATT_fixpoint"
```

4.6.3 The generated theorems and definitions

```
thm LOSS_KEY_ATT_enum_consts.nchotomy
thm LOSS_KEY_ATT_sets.nchotomy
thm LOSS_KEY_ATT_fun.nchotomy
thm LOSS_KEY_ATT_atom.nchotomy
thm LOSS_KEY_ATT_arity.simps
thm LOSS_KEY_ATT_public.simps
thm LOSS_KEY_ATT_Γ.simps
thm LOSS_KEY_ATT_Ana.simps

thm LOSS_KEY_ATT_transaction_iik1_def
thm LOSS_KEY_ATT_transaction_iik2_def
thm LOSS_KEY_ATT_transaction_iik3_def
thm LOSS_KEY_ATT_transaction_setwrap_def
thm LOSS_KEY_ATT_transaction_setunwrap_def
thm LOSS_KEY_ATT_transaction_unwrapWrap_def
thm LOSS_KEY_ATT_transaction_unwrapSens_def
thm LOSS_KEY_ATT_transaction_decrypt1Attr_def
thm LOSS_KEY_ATT_transaction_decrypt2Attr_def
thm LOSS_KEY_ATT_transaction_wrap_def
thm LOSS_KEY_ATT_transaction_bind1_def
thm LOSS_KEY_ATT_transaction_bind2_def
thm LOSS_KEY_ATT_transaction_setdecrypt_def
thm LOSS_KEY_ATT_transaction_decrypt1_def
thm LOSS_KEY_ATT_transaction_attack1_def

thm LOSS_KEY_ATT_protocol_def
thm LOSS_KEY_ATT_fixpoint_def

end
```


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