

Arrow and Gibbard-Satterthwaite

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December 14, 2021

Abstract

This article formalizes two proofs of Arrow's impossibility theorem due to Geanakoplos and derives the Gibbard-Satterthwaite theorem as a corollary. One formalization is based on utility functions, the other one on strict partial orders.

For an article about these proofs see <http://www.in.tum.de/~nipkow/pubs/arrow.pdf>.

1 Arrow's Theorem for Utility Functions

theory *Arrow-Utility* **imports** *Complex-Main*
begin

This theory formalizes the first proof due to Geanakoplos [1]. In contrast to the standard model of preferences as linear orders, we model preferences as *utility functions* mapping each alternative to a real number. The type of alternatives and voters is assumed to be finite.

typedecl *alt*
typedecl *indi*

axiomatization **where**

alt3: $\exists a b c :: alt. distinct[a,b,c]$ **and**
finite-alt: *finite*(UNIV:: *alt set*) **and**

finite-indi: *finite*(UNIV:: *indi set*)

lemma *third-alt*: $a \neq b \implies \exists c :: alt. distinct[a,b,c]$
(*proof*)

lemma *alt2*: $\exists b :: alt. b \neq a$
(*proof*)

type-synonym *pref* = *alt* \Rightarrow *real*
type-synonym *prof* = *indi* \Rightarrow *pref*

definition

$top :: pref \Rightarrow alt \Rightarrow bool$ (**infixr** $<\cdot 60$) **where**
 $p <\cdot b \equiv \forall a. a \neq b \longrightarrow p a < p b$

definition

$bot :: alt \Rightarrow pref \Rightarrow bool$ (**infixr** $\cdot < 60$) **where**
 $b \cdot < p \equiv \forall a. a \neq b \longrightarrow p b < p a$

definition

$extreme :: pref \Rightarrow alt \Rightarrow bool$ **where**
 $extreme p b \equiv b \cdot < p \vee p <\cdot b$

abbreviation

$Extreme P b == \forall i. extreme (P i) b$

lemma $[simp]: r <= s \Longrightarrow r < s+(1::real)$

$\langle proof \rangle$

lemma $[simp]: r < s \Longrightarrow r < s+(1::real)$

$\langle proof \rangle$

lemma $[simp]: r <= s \Longrightarrow \neg s+(1::real) < r$

$\langle proof \rangle$

lemma $[simp]: (r < s-(1::real)) = (r+1 < s)$

$\langle proof \rangle$

lemma $[simp]: (s-(1::real) < r) = (s < r+1)$

$\langle proof \rangle$

lemma $less-if-bot[simp]: \llbracket b \cdot < p; x \neq b \rrbracket \Longrightarrow p b < p x$

$\langle proof \rangle$

lemma $[simp]: \llbracket p <\cdot b; x \neq b \rrbracket \Longrightarrow p x < p b$

$\langle proof \rangle$

lemma $[simp]: \text{assumes } top: p <\cdot b \text{ shows } \neg p b < p c$

$\langle proof \rangle$

lemma $not-less-if-bot[simp]:$

assumes $bot: b \cdot < p$ **shows** $\neg p c < p b$

$\langle proof \rangle$

lemma $top-impl-not-bot[simp]: p <\cdot b \Longrightarrow \neg b \cdot < p$

$\langle proof \rangle$

lemma $[simp]: extreme p b \Longrightarrow (\neg p <\cdot b) = (b \cdot < p)$

$\langle proof \rangle$

lemma $[simp]: extreme p b \Longrightarrow (\neg b \cdot < p) = (p <\cdot b)$

$\langle proof \rangle$

Auxiliary construction to hide details of preference model.

definition

$mktop :: pref \Rightarrow alt \Rightarrow pref$ **where**
 $mktop\ p\ b \equiv p(b := Max(range\ p) + 1)$

definition

$mkbob :: pref \Rightarrow alt \Rightarrow pref$ **where**
 $mkbob\ p\ b \equiv p(b := Min(range\ p) - 1)$

definition

$between :: pref \Rightarrow alt \Rightarrow alt \Rightarrow alt \Rightarrow pref$ **where**
 $between\ p\ a\ b\ c \equiv p(b := (p\ a + p\ c)/2)$

To make things simpler:

declare *between-def*[simp]

lemma [simp]: $a \neq b \implies mktop\ p\ b\ a = p\ a$
 $\langle proof \rangle$

lemma [simp]: $a \neq b \implies mkbob\ p\ b\ a = p\ a$
 $\langle proof \rangle$

lemma [simp]: $a \neq b \implies p\ a < mktop\ p\ b\ b$
 $\langle proof \rangle$

lemma [simp]: $a \neq b \implies mkbob\ p\ b\ b < p\ a$
 $\langle proof \rangle$

lemma [simp]: $mktop\ p\ b < \cdot\ b$
 $\langle proof \rangle$

lemma [simp]: $\neg b \cdot < mktop\ p\ b$
 $\langle proof \rangle$

lemma [simp]: $a \neq b \implies \neg P\ p\ a < mkbob\ (P\ p)\ b\ b$
 $\langle proof \rangle$

The proof starts here.

locale *arrow* =

fixes $F :: pref \Rightarrow pref$

assumes *unanimity*: $(\bigwedge i. P\ i\ a < P\ i\ b) \implies F\ P\ a < F\ P\ b$

and *IIA*:

$(\bigwedge i. (P\ i\ a < P\ i\ b) = (P'\ i\ a < P'\ i\ b)) \implies$
 $(F\ P\ a < F\ P\ b) = (F\ P'\ a < F\ P'\ b)$

begin

lemmas $IIA' = IIA[THEN\ iffD1]$

definition

$dictates :: indi \Rightarrow alt \Rightarrow alt \Rightarrow bool$ ($-$ dictates $- < -$) **where**

$(i \text{ dictates } a < b) \equiv \forall P. P i a < P i b \longrightarrow F P a < F P b$

definition

$\text{dictates2} :: \text{indi} \Rightarrow \text{alt} \Rightarrow \text{alt} \Rightarrow \text{bool} \text{ (- dictates -,)} \text{ where}$
 $(i \text{ dictates } a, b) \equiv (i \text{ dictates } a < b) \wedge (i \text{ dictates } b < a)$

definition

$\text{dictatesx} :: \text{indi} \Rightarrow \text{alt} \Rightarrow \text{bool} \text{ (- dictates'-except -)} \text{ where}$
 $(i \text{ dictates-except } c) \equiv \forall a b. c \notin \{a, b\} \longrightarrow (i \text{ dictates } a < b)$

definition

$\text{dictator} :: \text{indi} \Rightarrow \text{bool} \text{ where}$
 $\text{dictator } i \equiv \forall a b. (i \text{ dictates } a < b)$

definition

$\text{pivotal} :: \text{indi} \Rightarrow \text{alt} \Rightarrow \text{bool} \text{ where}$
 $\text{pivotal } i b \equiv$
 $\exists P. \text{Extreme } P b \wedge b \cdot < P i \wedge b \cdot < F P \wedge$
 $F (P(i := \text{mktop } (P i) b)) < \cdot b$

lemma *all-top[simp]*: $\forall i. P i < \cdot b \Longrightarrow F P < \cdot b$
<proof>

lemma *not-extreme*:

assumes *nex*: $\neg \text{extreme } p b$
shows $\exists a c. \text{distinct}[a, b, c] \wedge \neg p a < p b \wedge \neg p b < p c$
<proof>

lemma *extremal*:

assumes *extremes*: $\text{Extreme } P b$ **shows** $\text{extreme } (F P) b$
<proof>

lemma *pivotal-ind*: **assumes** *fin*: *finite D*

shows $\bigwedge P. \llbracket D = \{i. b \cdot < P i\}; \text{Extreme } P b; b \cdot < F P \rrbracket$
 $\Longrightarrow \exists i. \text{pivotal } i b \text{ (is } \bigwedge P. ?D D P \Longrightarrow ?E P \Longrightarrow ?B P \Longrightarrow \cdot)$
<proof>

lemma *pivotal-exists*: $\exists i. \text{pivotal } i b$
<proof>

lemma *pivotal-xdictates*: **assumes** *pivo*: *pivotal i b*

shows $i \text{ dictates-except } b$
<proof>

lemma *pivotal-is-dictator*:

assumes *pivo*: *pivotal i b* **and** *ab*: $a \neq b$ **and** *d*: $j \text{ dictates } a, b$
shows $i = j$
<proof>

theorem *dictator*: $\exists i. \text{dictator } i$
<proof>

end

end

2 Arrow's Theorem for Strict Linear Orders

theory *Arrow-Order* **imports** *Main HOL-Library.FuncSet*
begin

This theory formalizes the third proof due to Geanakoplos [1]. Preferences are modeled as strict linear orderings. The set of alternatives need not be finite.

Individuals are assumed to be finite but are not a priori identified with an initial segment of the naturals. In retrospect this generality appears gratuitous and complicates some of the low-level reasoning where we use a bijection with such an initial segment.

typedecl *alt*
typedecl *indi*

abbreviation $I == (UNIV::indi \text{ set})$

axiomatization **where**
alt3: $\exists a \ b \ c::alt. \text{distinct}[a,b,c]$ **and**
finite-indi: *finite* I

abbreviation $N == \text{card } I$

lemma *third-alt*: $a \neq b \implies \exists c::alt. \text{distinct}[a,b,c]$
<proof>

lemma *alt2*: $\exists b::alt. b \neq a$
<proof>

type-synonym $\text{pref} = (\text{alt} * \text{alt})\text{set}$

definition $\text{Lin} == \{L::\text{pref}. \text{strict-linear-order } L\}$

lemmas *slo-defs* = *Lin-def strict-linear-order-on-def total-on-def irrefl-def*

lemma *notin-Lin-iff*: $L : \text{Lin} \implies x \neq y \implies (x,y) \notin L \iff (y,x) : L$
<proof>

lemma *converse-in-Lin[simp]*: $L^{-1} : \text{Lin} \iff L : \text{Lin}$
<proof>

lemma *Lin-irrefl*: $L:Lin \implies (a,b):L \implies (b,a):L \implies False$
 ⟨proof⟩

corollary *linear-alt*: $\exists L::pref. L : Lin$
 ⟨proof⟩

abbreviation

$rem :: pref \Rightarrow alt \Rightarrow pref$ **where**
 $rem\ L\ a \equiv \{(x,y). (x,y) \in L \wedge x \neq a \wedge y \neq a\}$

definition

$mktop :: pref \Rightarrow alt \Rightarrow pref$ **where**
 $mktop\ L\ b \equiv rem\ L\ b \cup \{(x,b)|x. x \neq b\}$

definition

$mkbot :: pref \Rightarrow alt \Rightarrow pref$ **where**
 $mkbot\ L\ b \equiv rem\ L\ b \cup \{(b,y)|y. y \neq b\}$

definition

$below :: pref \Rightarrow alt \Rightarrow alt \Rightarrow pref$ **where**
 $below\ L\ a\ b \equiv rem\ L\ a \cup$
 $\{(a,b)\} \cup \{(x,a)|x. (x,b) : L \wedge x \neq a\} \cup \{(a,y)|y. (b,y) : L \wedge y \neq a\}$

definition

$above :: pref \Rightarrow alt \Rightarrow alt \Rightarrow pref$ **where**
 $above\ L\ a\ b \equiv rem\ L\ b \cup$
 $\{(a,b)\} \cup \{(x,b)|x. (x,a) : L \wedge x \neq b\} \cup \{(b,y)|y. (a,y) : L \wedge y \neq b\}$

lemma *in-mktop*: $(x,y) \in mktop\ L\ z \longleftrightarrow x \neq z \wedge (if\ y=z\ then\ x \neq y\ else\ (x,y) \in L)$
 ⟨proof⟩

lemma *in-mkbot*: $(x,y) \in mkbot\ L\ z \longleftrightarrow y \neq z \wedge (if\ x=z\ then\ x \neq y\ else\ (x,y) \in L)$
 ⟨proof⟩

lemma *in-above*: $a \neq b \implies L:Lin \implies$

$(x,y) : above\ L\ a\ b \longleftrightarrow x \neq y \wedge$
 $(if\ x=b\ then\ (a,y) : L\ else$
 $if\ y=b\ then\ x=a \mid (x,a) : L\ else\ (x,y) : L)$
 ⟨proof⟩

lemma *in-below*: $a \neq b \implies L:Lin \implies$

$(x,y) : below\ L\ a\ b \longleftrightarrow x \neq y \wedge$
 $(if\ y=a\ then\ (x,b) : L\ else$
 $if\ x=a\ then\ y=b \mid (b,y) : L\ else\ (x,y) : L)$
 ⟨proof⟩

declare $[[simp-depth-limit = 2]]$

lemma *mktop-Lin*: $L : Lin \implies mktop\ L\ x : Lin$
 ⟨proof⟩

lemma *mkbot-Lin*: $L : Lin \implies mkbot\ L\ x : Lin$
 ⟨proof⟩

lemma *below-Lin*: $x \neq y \implies L : Lin \implies \text{below } L \ x \ y : Lin$
 $\langle \text{proof} \rangle$

lemma *above-Lin*: $x \neq y \implies L : Lin \implies \text{above } L \ x \ y : Lin$
 $\langle \text{proof} \rangle$

declare $[[\text{simp-depth-limit} = 50]]$

abbreviation *lessLin* :: $\text{alt} \Rightarrow \text{pref} \Rightarrow \text{alt} \Rightarrow \text{bool} \ ((- <_L -) [51, 51] 50)$
where $a <_L b == (a, b) : L$

definition $\text{Prof} = I \rightarrow Lin$

abbreviation $\text{SWF} == \text{Prof} \rightarrow Lin$

definition *unanimity* $F == \forall P \in \text{Prof}. \forall a \ b. (\forall i. a <_P i \ b) \longrightarrow a <_F P \ b$

definition *IIA* $F == \forall P \in \text{Prof}. \forall P' \in \text{Prof}. \forall a \ b.$
 $(\forall i. a <_P i \ b \longleftrightarrow a <_{P'} i \ b) \longrightarrow (a <_F P \ b \longleftrightarrow a <_F P' \ b)$

definition *dictator* $F \ i == \forall P \in \text{Prof}. F \ P = P \ i$

lemma *dictatorI*: $F : \text{SWF} \implies$
 $\forall P \in \text{Prof}. \forall a \ b. a \neq b \longrightarrow (a, b) : P \ i \longrightarrow (a, b) : F \ P \implies \text{dictator } F \ i$
 $\langle \text{proof} \rangle$

lemma *const-Lin-Prof*: $L : Lin \implies (\%p. L) : \text{Prof}$
 $\langle \text{proof} \rangle$

lemma *complete-Lin*: **assumes** $a \neq b$ **shows** $\exists L \in Lin. (a, b) : L$
 $\langle \text{proof} \rangle$

declare $\text{Let-def}[\text{simp}]$

theorem *Arrow*: **assumes** $F : \text{SWF}$ **and** $u : \text{unanimity } F$ **and** *IIA* F
shows $\exists i. \text{dictator } F \ i$
 $\langle \text{proof} \rangle$

end

3 The Gibbard-Satterthwaite Theorem

theory *GS* **imports** *Arrow-Order*
begin

The Gibbard-Satterthwaite theorem as a corollary to Arrow's theorem.
The proof follows Nisan [2].

definition *manipulable* $f == \exists P \in \text{Prof}. \exists i. \exists L \in Lin. (f \ P, f(P(i:=L))) : P \ i$

definition $dict\ f\ i == \forall P \in Prof. \forall a. a \neq f\ P \longrightarrow (a, f\ P) : P\ i$

definition

$Top :: alt\ set \Rightarrow pref \Rightarrow pref$ **where**
 $Top\ S\ L \equiv \{(a,b). (a,b) \in L \wedge (a \in S \wedge b \in S \vee a \notin S \wedge b \notin S)\} \cup$
 $\{(a,b). a \notin S \wedge b \in S\}$

lemma $Top\text{-in-Lin}: L:Lin \Longrightarrow Top\ S\ L : Lin$
 $\langle proof \rangle$

lemma $Top\text{-in-Prof}: P:Prof \Longrightarrow Top\ S\ o\ P : Prof$
 $\langle proof \rangle$

lemma $not\text{-manipulable}: \neg\ manipulable\ f \longleftrightarrow$
 $(\forall P \in Prof. \forall i. \forall L \in Lin. f\ P \neq f(P(i:=L)) \longrightarrow$
 $(f(P(i:=L)), f\ P) : P\ i \wedge (f\ P, f(P(i:=L))) : L) \text{ (is } ?A = ?B)$
 $\langle proof \rangle$

definition $swf(f) \equiv \lambda P. \{(a,b). a \neq b \wedge f(Top\ \{a,b\}\ o\ P) = b\}$

locale $GS =$

fixes f

assumes $not\text{-manip}: \neg\ manipulable\ f$

and $onto: f\ ' Prof = UNIV$

begin

lemma $nonmanip:$

$P:Prof \Longrightarrow L:Lin \Longrightarrow f(P(i:=L)) \neq f\ P \Longrightarrow$
 $(f(P(i:=L)), f\ P) : P\ i \wedge (f\ P, f(P(i:=L))) : L$
 $\langle proof \rangle$

lemma $mono:$

assumes $P \in Prof\ P' \in Prof\ \forall i\ a. (a, f\ P) : P\ i \longrightarrow (a, f\ P) : P'\ i$
shows $f\ P' = f\ P$
 $\langle proof \rangle$

lemma $una\text{-}Top: \text{assumes } P:Prof\ S \neq \{\} \text{ shows } f(Top\ S\ o\ P) : S$
 $\langle proof \rangle$

lemma $SWF\text{-}swf: swf\ f : SWF$
 $\langle proof \rangle$

lemma $Top\text{-}top: L:Lin \Longrightarrow (!!a. a \neq b \Longrightarrow (a,b) : L) \Longrightarrow Top\ \{b\}\ L = L$
 $\langle proof \rangle$

lemma $una\text{-}swf: unanimity(swf\ f)$
 $\langle proof \rangle$

lemma $IIA\text{-}swf: IIA(swf\ f)$

<proof>

lemma *dict-swf*: **assumes** *dictator (swf f) i* **shows** *dict f i*
<proof>

theorem *Gibbard-Satterthwaite*:

$\exists i. \text{dict } f \ i$
<proof>

end

theorem *Gibbard-Satterthwaite*:

$\neg \text{manipulable } f \implies \forall a. \exists P \in \text{Prof}. a = f \ P \implies \exists i. \text{dict } f \ i$
<proof>

end

References

- [1] J. Geanakoplos. Three brief proofs of Arrow's impossibility theorem. *Economic Theory*, 26:211–215, 2005.
- [2] N. Nisan. Introduction to mechanism design (for computer scientists). In N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani, editors, *Algorithmic Game Theory*. Cambridge University Press, 2007.