Arithmetic progressions and relative primes

José Manuel Rodríguez Caballero

March 17, 2025

Abstract

This article provides a formalization of the solution obtained by the author of the Problem "ARITHMETIC PROGRESSIONS" from the Putnam exam problems [1] of 2002. The statement of the problem is as follows: For which integers $n > 1$ does the set of positive integers less than and relatively prime to n constitute an arithmetic progression?

Contents

1 Problem ARITHMETIC PROGRESSIONS (Putnam exam problems 2002)

theory *Arith-Prog-Rel-Primes*
imports
Complex-Main
HOL−Number-Theory.Number-Theory
begin

Statement of the problem (from [1]): For which integers $n > 1$ does the set of positive integers less than and relatively prime to n constitute an arithmetic progression?

The solution of the above problem is theorem *arith-prog-rel-primes-solution*.

First, we will require some auxiliary material before we get started with the actual solution.

1.1 Auxiliary results

lemma *even-and-odd-parts*:
fixes *n*::*nat*

assumes ‹$n \neq 0$›
shows ‹$\exists\ k\ q{::}nat.\ n = (2{::}nat)\hat{}k{*}q \wedge odd\ q$›
⟨*proof*⟩

lemma *only-one-odd-div-power2*:
fixes $n{::}nat$
assumes ‹$n \neq 0$› **and** ‹$\bigwedge x.\ x\ dvd\ n \Longrightarrow odd\ x \Longrightarrow x = 1$›
shows ‹$\exists\ k.\ n = (2{::}nat)\hat{}k$›
⟨*proof*⟩

lemma *coprime-power2*:
fixes $n{::}nat$
assumes ‹$n \neq 0$› **and** ‹$\bigwedge x.\ x < n \Longrightarrow (coprime\ x\ n \longleftrightarrow odd\ x)$›
shows ‹$\exists\ k.\ n = (2{::}nat)\hat{}k$›
⟨*proof*⟩

1.2 Main result

The solution to the problem ARITHMETIC PROGRESSIONS (Putnam exam problems 2002)

theorem *arith-prog-rel-primes-solution*:
fixes $n :: nat$
assumes ‹$n > 1$›
shows ‹$(prime\ n \vee (\exists\ k.\ n = 2\hat{}k) \vee n = 6) \longleftrightarrow$
$(\exists\ a\ b\ m.\ m \neq 0 \wedge \{x \mid x.\ x < n \wedge coprime\ x\ n\} = \{a{+}j{*}b \mid j{::}nat.\ j < m\})$›
⟨*proof*⟩

end

References

[1] Problem “ARITHMETIC PROGRESSIONS", from Putnam exam problems 2002, https://www.ocf.berkeley.edu/ wwu/riddles/putnam.shtml.