Approximate Model Counting

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Abstract

Approximate model counting is the task of approximating the number of solutions to an input formula. This entry formalizes ApproxMC, an algorithm due to Chakraborty et al. [1] with a probably approximately correct (PAC) guarantee, i.e., ApproxMC returns a multiplicative $(1+\varepsilon)$-factor approximation of the model count with probability at least $1-\delta$, where $\varepsilon > 0$ and $0 < \delta \leq 1$. The algorithmic specification is further refined to a verified certificate checker that can be used to validate the results of untrusted ApproxMC implementations (assuming access to trusted randomness).

Contents

1 Preliminary probability/UHF lemmas

This section proves some simplified/specialized forms of lemmas that will be used in the algorithm's analysis later.

theory *ApproxMCPreliminaries* **imports**
Universal-Hash-Families.Carter-Wegman-Hash-Family
Concentration-Inequalities.Bienaymes-Identity
Concentration-Inequalities.Paley-Zygmund-Inequality
begin

lemma *card-inter-sum-indicat-real*:
assumes *finite A*
shows *card* $(A \cap B) = sum\ (indicat\text{-}real\ B)\ A$
⟨*proof*⟩

lemma *card-dom-ran*:
assumes *finite D*
shows $card\ \{w.\ dom\ w = D \wedge ran\ w \subseteq R\} = card\ R\ \widehat{}\ card\ D$
⟨*proof*⟩

lemma *finite-set-pmf-expectation-sum*:
fixes $f :: 'a \Rightarrow 'c \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes *finite* (*set-pmf A*)
shows *measure-pmf.expectation A* $(\lambda x.\ sum\ (f\ x)\ T) =$
$(\sum i \in T.\ measure\text{-}pmf.expectation\ A\ (\lambda x.\ f\ x\ i))$
⟨*proof*⟩

lemma (**in** *prob-space*) *k-universal-prob-unif*:
assumes *k-universal k H D R*
assumes $w \in D\ \alpha \in R$
shows $prob\ \{s \in space\ M.\ H\ w\ s = \alpha\} = 1\ /\ card\ R$
⟨*proof*⟩

lemma *k-universal-expectation-eq*:
assumes *p*: *finite* (*set-pmf p*)
assumes *ind*: *prob-space.k-universal p k H D R*
assumes *S*: *finite S* $S \subseteq D$
assumes *a*: $\alpha \in R$
shows
prob-space.expectation p
$(\lambda s.\ real\ (card\ (S \cap \{w.\ H\ w\ s = \alpha\})))$ =
$real\ (card\ S)\ /\ card\ R$
⟨*proof*⟩

lemma (**in** *prob-space*) *two-universal-indep-var*:
assumes *k-universal 2 H D R*
assumes $w \in D\ w' \in D\ w \neq w'$

shows *indep-var*
borel
$(\lambda x.\ \textit{indicat-real}\ \{w.\ H\ w\ x = \alpha\}\ w)$
borel
$(\lambda x.\ \textit{indicat-real}\ \{w.\ H\ w\ x = \alpha\}\ w')$
⟨*proof*⟩

lemma *two-universal-variance-bound*:
assumes p: *finite* (*set-pmf* p)
assumes *ind*: *prob-space.k-universal* (*measure-pmf* p) *2* H D R
assumes S: *finite* S $S \subseteq D$
assumes a: $\alpha \in R$
shows
measure-pmf.variance p
$(\lambda s.\ \textit{real}\ (\textit{card}\ (S \cap \{w.\ H\ w\ s = \alpha\}))) \leq$
measure-pmf.expectation p
$(\lambda s.\ \textit{real}\ (\textit{card}\ (S \cap \{w.\ H\ w\ s = \alpha\})))$
⟨*proof*⟩

lemma (**in** *prob-space*) *k-universal-mono*:
assumes $k' \leq k$
assumes*k-universal* k H D R
shows*k-universal* k' H D R
⟨*proof*⟩

lemma *finite-set-pmf-expectation-add*:
assumes *finite* (*set-pmf* S)
shows *measure-pmf.expectation* S $(\lambda x.\ ((f\ x)::\textit{real}) + g\ x) =$
measure-pmf.expectation S f + *measure-pmf.expectation* S g
⟨*proof*⟩

lemma *finite-set-pmf-expectation-add-const*:
assumes *finite* (*set-pmf* S)
shows *measure-pmf.expectation* S $(\lambda x.\ ((f\ x)::\textit{real}) + g) =$
measure-pmf.expectation S $f + g$
⟨*proof*⟩

lemma *finite-set-pmf-expectation-diff*:
assumes *finite* (*set-pmf* S)
shows *measure-pmf.expectation* S $(\lambda x.\ ((f\ x)::\textit{real}) - g\ x) =$
measure-pmf.expectation S f − *measure-pmf.expectation* S g
⟨*proof*⟩

lemma *spec-paley-zygmund-inequality*:
assumes *fin*: *finite* (*set-pmf* p)
assumes *Zpos*: $\bigwedge z.\ Z\ z \geq 0$
assumes t: $\vartheta \leq 1$

shows
(*measure-pmf.variance p Z* + (*1*−*ϑ*)^*2* ∗ (*measure-pmf.expectation p Z*)^*2*) ∗
measure-pmf.prob p {*z*. *Z z* > *ϑ* ∗ *measure-pmf.expectation p Z*}
≥
(*1*−*ϑ*)^*2* ∗ (*measure-pmf.expectation p Z*)^*2*
⟨*proof*⟩

lemma *spec-chebyshev-inequality*:
assumes *fin*: *finite* (*set-pmf p*)
assumes *pvar*: *measure-pmf.variance p Y* > *0*
assumes *k*: *k* > *0*
shows
measure-pmf.prob p
{*y*. (*Y y* − *measure-pmf.expectation p Y*)^*2* ≥
k^*2* ∗ *measure-pmf.variance p Y*} ≤ *1* / *k*^*2*
⟨*proof*⟩

end

2 Random XORs

The goal of this section is to prove that, for a randomly sampled XOR X from a set of variables V:

1. the probability of an assignment w satisfying X is $\frac{1}{2}$;
2. for any distinct assignments w, w' the probability of both satisfying X is equal to $\frac{1}{4}$ (2-wise independence); and
3. for any distinct assignments w, w', w'' the probability of all three satisfying X is equal to $\frac{1}{8}$ (3-wise independence).

theory *RandomXOR* **imports**
ApproxMCPreliminaries
Universal-Hash-Families.Universal-Hash-Families-More-Product-PMF
Monad-Normalisation.Monad-Normalisation
begin

A random XOR constraint is modeled as a random subset of variables and a randomly chosen RHS bit.

definition *random-xor* :: *'a set* ⇒ (*'a set* × *bool*) *pmf*
where *random-xor V* =
pair-pmf (*pmf-of-set* (*Pow V*)) (*bernoulli-pmf* (*1*/*2*))

lemma *pmf-of-set-Pow-fin-map*:
assumes *V*:*finite V*
shows *pmf-of-set* (*Pow V*) =
map-pmf (λ*b*. {*x* ∈ *V*. *b x* = *Some True*})

(*Pi-pmf V def* (λ-. *map-pmf Some* (*bernoulli-pmf* (*1 / 2*))))
⟨*proof*⟩

lemma *random-xor-from-bits*:
assumes *V*:*finite V*
shows *random-xor V* =
pair-pmf
(*map-pmf* (λ*b*. {*x* ∈ *V*. *b x* = *Some True*})
(*Pi-pmf V def* (λ-. *map-pmf Some* (*bernoulli-pmf* (*1*/*2*)))))
(*bernoulli-pmf* (*1*/*2*))
⟨*proof*⟩

fun *satisfies-xor* :: (′*a set* × *bool*) ⇒ ′*a set* ⇒ *bool*
where *satisfies-xor* (*x*,*b*) ω =
even (*card* (ω ∩ *x*) + *of-bool b*)

lemma *satisfies-xor-inter*:
shows *satisfies-xor* (ω ∩ *x* ,*b*) ω = *satisfies-xor* (*x*,*b*) ω
⟨*proof*⟩

lemma *prob-bernoulli-bind-pmf*:
assumes *0* ≤ *p p* ≤ *1*
assumes *finite E*
shows *measure-pmf.prob*
(*bernoulli-pmf p* ⤜ *x*) *E* =
p ∗ (*measure-pmf.prob* (*x True*) *E*) +
(*1* − *p*) ∗ (*measure-pmf.prob* (*x False*) *E*)
⟨*proof*⟩

lemma *set-pmf-random-xor*:
assumes *V*: *finite V*
shows *set-pmf* (*random-xor V*) = (*Pow V*) × *UNIV*
⟨*proof*⟩

lemma *pmf-of-set-prod*:
assumes *P* ≠ {} *Q* ≠ {}
assumes *finite P finite Q*
shows *pmf-of-set* (*P* × *Q*) = *pair-pmf* (*pmf-of-set P*) (*pmf-of-set Q*)
⟨*proof*⟩

lemma *random-xor-pmf-of-set*:
assumes *V*:*finite V*
shows *random-xor V* = *pmf-of-set* ((*Pow V*) × *UNIV*)
⟨*proof*⟩

lemma *prob-random-xor-with-set-pmf*:
assumes V: *finite* V
shows *prob-space.prob* (*random-xor* V) $\{c.\ P\ c\}$ =
prob-space.prob (*random-xor* V) $\{c.\ fst\ c \subseteq V \wedge P\ c\}$
⟨*proof*⟩

lemma *prob-set-parity*:
assumes *measure-pmf.prob* M
$\{c.\ P\ c\} = q$
shows *measure-pmf.prob* M
$\{c.\ P\ c = b\} = (if\ b\ then\ q\ else\ 1 - q)$
⟨*proof*⟩

lemma *satisfies-random-xor*:
assumes V: *finite* V
shows *prob-space.prob* (*random-xor* V)
$\{c.\ satisfies\text{-}xor\ c\ \omega\} = 1\ /\ 2$
⟨*proof*⟩

lemma *satisfies-random-xor-parity*:
assumes V: *finite* V
shows *prob-space.prob* (*random-xor* V)
$\{c.\ satisfies\text{-}xor\ c\ \omega = b\} = 1\ /\ 2$
⟨*proof*⟩

2.1 Independence properties of random XORs

lemma *pmf-of-set-powerset-split*:
assumes $S \subseteq V$ *finite* V
shows
map-pmf $(\lambda(x,y).\ x \cup y)$
(*pmf-of-set* (*Pow* $S \times$ *Pow* $(V - S)$)) =
pmf-of-set (*Pow* V)
⟨*proof*⟩

lemma *pmf-of-set-Pow-sing*:
shows*pmf-of-set* (*Pow* $\{x\}$) =
bernoulli-pmf $(1\ /\ 2) \gg\!=$
$(\lambda b.\ return\text{-}pmf\ (if\ b\ then\ \{x\}\ else\ \{\}))$
⟨*proof*⟩

lemma *pmf-of-set-sing-coin-flip*:
assumes *finite* V
shows *pmf-of-set* (*Pow* $\{x\} \times$ *Pow* V) =
map-pmf $(\lambda(r,c).\ (if\ c\ then\ \{x\}\ else\ \{\},\ r))$ (*random-xor* V)
⟨*proof*⟩
including *monad-normalisation*
⟨*proof*⟩

lemma *measure-pmf-prob-dependent-product-bound-eq*:
assumes *countable A* $\bigwedge i$. *countable* (*B i*)
assumes $\bigwedge a$. $a \in A \Longrightarrow$ *measure-pmf.prob N* (*B a*) = *r*
shows *measure-pmf.prob* (*pair-pmf M N*) (*Sigma A B*) =
measure-pmf.prob M A ∗ *r*
⟨*proof*⟩

lemma *measure-pmf-prob-dependent-product-bound-eq′*:
assumes *countable* (*A* ∩ *set-pmf M*) $\bigwedge i$. *countable* (*B i* ∩ *set-pmf N*)
assumes $\bigwedge a$. $a \in A \cap$ *set-pmf M* $\Longrightarrow$ *measure-pmf.prob N* (*B a* ∩ *set-pmf N*) = *r*
shows *measure-pmf.prob* (*pair-pmf M N*) (*Sigma A B*) = *measure-pmf.prob M A* ∗ *r*
⟨*proof*⟩

lemma *single-var-parity-coin-flip*:
assumes $x \in \omega$ *finite* ω
assumes *finite a* $x \notin a$
shows *measure-pmf.prob* (*pmf-of-set* (*Pow* {*x*}))
{*y*. *even* (*card* ((*a* ∪ *y*) ∩ ω)) = *b*} = *1/2*
⟨*proof*⟩

lemma *prob-pmf-of-set-nonempty-parity*:
assumes *V*: *finite V*
assumes $x \in \omega$ $\omega \subseteq V$
assumes $\bigwedge c$. $c \in E \longleftrightarrow c - \{x\} \in E$
shows *prob-space.prob* (*pmf-of-set* (*Pow V*))
(*E* ∩ {*c*. *even* (*card* (*c* ∩ ω)) = *b*}) =
1 / 2 ∗ *prob-space.prob* (*pmf-of-set* (*Pow* (*V* − {*x*}))) *E*
⟨*proof*⟩

lemma *prob-random-xor-split*:
assumes *V*: *finite V*
shows *prob-space.prob* (*random-xor V*) *E* =
1 / 2 ∗ *prob-space.prob* (*pmf-of-set* (*Pow V*)) {*e*. (*e*,*True*) ∈ *E*} +
1 / 2 ∗ *prob-space.prob* (*pmf-of-set* (*Pow V*)) {*e*. (*e*,*False*) ∈ *E*}
⟨*proof*⟩

lemma *prob-random-xor-nonempty-parity*:
assumes *V*: *finite V*
assumes ω: $x \in \omega$ $\omega \subseteq V$
assumes *E*: $\bigwedge c$. $c \in E \longleftrightarrow$ (*fst c* − {*x*},*snd c*) ∈ *E*
shows *prob-space.prob* (*random-xor V*)
(*E* ∩ {*c*. *satisfies-xor c* ω = *b*}) =

$1 / 2 * \textit{prob-space.prob}\ (\textit{random-xor}\ (V - \{x\}))\ E$
⟨*proof*⟩

lemma *pair-satisfies-random-xor-parity-1*:
assumes V:*finite* V
assumes x: $x \notin \omega\ x \in \omega'$
assumes ω: $\omega \subseteq V\ \omega' \subseteq V$
shows *prob-space.prob* (*random-xor* V)
$\{c.\ \textit{satisfies-xor}\ c\ \omega = b \wedge \textit{satisfies-xor}\ c\ \omega' = b'\} = 1 / 4$
⟨*proof*⟩

lemma *pair-satisfies-random-xor-parity*:
assumes V:*finite* V
assumes ω: $\omega \neq \omega'\ \omega \subseteq V\ \omega' \subseteq V$
shows *prob-space.prob* (*random-xor* V)
$\{c.\ \textit{satisfies-xor}\ c\ \omega = b \wedge \textit{satisfies-xor}\ c\ \omega' = b'\} = 1 / 4$
⟨*proof*⟩

lemma *prob-pmf-of-set-nonempty-parity-UNIV*:
assumes *finite* V
assumes $x \in \omega\ \omega \subseteq V$
shows *prob-space.prob* (*pmf-of-set* (*Pow* V))
$\{c.\ \textit{even}\ (\textit{card}\ (c \cap \omega)) = b\} = 1 / 2$
⟨*proof*⟩

lemma *prob-Pow-split*:
assumes $\omega \subseteq V$ *finite* V
shows *prob-space.prob* (*pmf-of-set* (*Pow* V))
$\{x.\ P\ (\omega \cap x) \wedge Q\ ((V - \omega) \cap x)\} =$
prob-space.prob (*pmf-of-set* (*Pow* ω))
$\{x.\ P\ x\} *$
prob-space.prob (*pmf-of-set* (*Pow* $(V - \omega)$))
$\{x.\ Q\ x\}$
⟨*proof*⟩

lemma *disjoint-prob-pmf-of-set-nonempty*:
assumes ω: $x \in \omega\ \omega \subseteq V$
assumes ω': $x' \in \omega'\ \omega' \subseteq V$
assumes $\omega \cap \omega' = \{\}$
assumes V: *finite* V
shows *prob-space.prob* (*pmf-of-set* (*Pow* V))
$\{c.\ \textit{even}\ (\textit{card}\ (\omega \cap c)) = b \wedge \textit{even}\ (\textit{card}\ (\omega' \cap c)) = b'\} = 1 / 4$
⟨*proof*⟩

lemma *measure-pmf-prob-product-finite-set-pmf*:
assumes *finite* (*set-pmf* M) *finite* (*set-pmf* N)
shows *measure-pmf.prob* (*pair-pmf* M N) $(A \times B) =$
measure-pmf.prob M $A *$ *measure-pmf.prob* N B

⟨*proof*⟩

lemma *prob-random-xor-split-space*:
assumes $\omega \subseteq V$ *finite* V
shows *prob-space.prob* (*random-xor* V)
$\{(x,b).\ P\ (\omega \cap x)\ b \wedge Q\ ((V - \omega) \cap x)\} =$
prob-space.prob (*random-xor* ω)
$\{(x,b).\ P\ x\ b\} *$
prob-space.prob (*pmf-of-set* (*Pow* $(V - \omega)$))
$\{x.\ Q\ x\}$
⟨*proof*⟩
including *monad-normalisation*
⟨*proof*⟩

lemma *three-disjoint-prob-random-xor-nonempty*:
assumes ω: $\omega \neq \{\}\ \omega \subseteq V$
assumes ω': $\omega' \neq \{\}\ \omega' \subseteq V$
assumes I: $I \subseteq V$
assumes *int*: $I \cap \omega = \{\}\ I \cap \omega' = \{\}\ \omega \cap \omega' = \{\}$
assumes V: *finite* V
shows *prob-space.prob* (*random-xor* V)
$\{c.$ *satisfies-xor* $c\ I = b \wedge$
even (*card* $(\omega \cap$ *fst* $c)) = b' \wedge$
even (*card* $(\omega' \cap$ *fst* $c)) = b''\} = 1\ /\ 8$
⟨*proof*⟩

lemma *three-disjoint-prob-pmf-of-set-nonempty*:
assumes ω: $x \in \omega\ \omega \subseteq V$
assumes ω': $x' \in \omega'\ \omega' \subseteq V$
assumes ω'': $x'' \in \omega''\ \omega'' \subseteq V$
assumes *int*: $\omega \cap \omega' = \{\}\ \omega' \cap \omega'' = \{\}\ \omega'' \cap \omega = \{\}$
assumes V: *finite* V
shows *prob-space.prob* (*pmf-of-set* (*Pow* V))
$\{c.$ *even* (*card* $(\omega \cap c)) = b \wedge$ *even* (*card* $(\omega' \cap c)) = b' \wedge$ *even* (*card* $(\omega'' \cap c)) = b''\} = 1\ /\ 8$
⟨*proof*⟩

lemma *four-disjoint-prob-random-xor-nonempty*:
assumes ω: $\omega \neq \{\}\ \omega \subseteq V$
assumes ω': $\omega' \neq \{\}\ \omega' \subseteq V$
assumes ω'': $\omega'' \neq \{\}\ \omega'' \subseteq V$
assumes I: $I \subseteq V$
assumes *int*: $I \cap \omega = \{\}\ I \cap \omega' = \{\}\ I \cap \omega'' = \{\}$
$\omega \cap \omega' = \{\}\ \omega' \cap \omega'' = \{\}\ \omega'' \cap \omega = \{\}$
assumes V: *finite* V
shows *prob-space.prob* (*random-xor* V)
$\{c.$ *satisfies-xor* $c\ I = b0 \wedge$
even (*card* $(\omega \cap$ *fst* $c)) = b \wedge$

even (*card* (ω' ∩ *fst c*)) = b' ∧
even (*card* (ω'' ∩ *fst c*)) = b''} = *1 / 16*
⟨*proof*⟩

lemma *three-satisfies-random-xor-parity-1*:
assumes *V*:*finite V*
assumes ω: $\omega \subseteq V$ $\omega' \subseteq V$ $\omega'' \subseteq V$
assumes *x*: $x \notin \omega$ $x \notin \omega'$ $x \in \omega''$
assumes *d*: $\omega \neq \omega'$
shows *prob-space.prob* (*random-xor V*)
{*c*.
satisfies-xor c ω = *b* ∧
satisfies-xor c ω' = b' ∧
satisfies-xor c ω'' = b''} = *1 / 8*
⟨*proof*⟩

lemma *split-boolean-eq*:
shows(*A* ⟷ *B*) = (*b* ⟷ *I*) ∧
(*B* ⟷ *C*) = (b' ⟷ *I*) ∧
(*C* ⟷ *A*) = (b'' ⟷ *I*)
⟷
I = *odd*(*of-bool b* + *of-bool* b' + *of-bool* b'') ∧
(*A* = *True* ∧
B = (*b*=b'') ∧
C = (*b* = b') ∨
A = *False* ∧
B = (b' ≠ b'') ∧
C = (*b* ≠ b'))
⟨*proof*⟩

lemma *three-satisfies-random-xor-parity*:
assumes *V*:*finite V*
assumes ω:
$\omega \neq \omega'$ $\omega \neq \omega''$ $\omega' \neq \omega''$
$\omega \subseteq V$ $\omega' \subseteq V$ $\omega'' \subseteq V$
shows *prob-space.prob* (*random-xor V*)
{*c. satisfies-xor c* ω = *b* ∧
satisfies-xor c ω' = b' ∧
satisfies-xor c ω'' = b''} = *1 / 8*
⟨*proof*⟩

2.2 Independence for repeated XORs

We can lift the previous result to a list of independent sampled XORs.

definition *random-xors* :: *'a set* ⇒ *nat* ⇒
(*nat* ⇀ *'a set* × *bool*) *pmf*
where *random-xors V n* =
Pi-pmf {..<(*n*::*nat*)} *None*

$(\lambda\text{-}.\ \textit{map-pmf Some}\ (\textit{random-xor}\ V))$

lemma *random-xors-set*:
assumes V:*finite* V
shows
$\textit{PiE-dflt}\ \{..<n\}\ \textit{None}$
$(\textit{set-pmf} \circ (\lambda\text{-}.\ \textit{map-pmf Some}\ (\textit{random-xor}\ V))) =$
$\{\textit{xors}.\ \textit{dom xors} = \{..<n\} \wedge$
$\textit{ran xors} \subseteq (\textit{Pow}\ V) \times \textit{UNIV}\}$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *random-xors-eq*:
assumes V:*finite* V
shows*random-xors* $V\ n =$
pmf-of-set
$\{\textit{xors}.\ \textit{dom xors} = \{..<n\} \wedge \textit{ran xors} \subseteq (\textit{Pow}\ V) \times \textit{UNIV}\}$
⟨*proof*⟩

definition *xor-hash* ::
$('a \rightharpoonup \textit{bool}) \Rightarrow$
$(\textit{nat} \rightharpoonup ('a\ \textit{set} \times \textit{bool})) \Rightarrow$
$(\textit{nat} \rightharpoonup \textit{bool})$
where *xor-hash* ω *xors* =
$(\textit{map-option}$
$(\lambda \textit{xor}.\ \textit{satisfies-xor xor}\ \{x.\ \omega\ x = \textit{Some True}\}) \circ \textit{xors})$

lemma *finite-map-set-nonempty*:
assumes $R \neq \{\}$
shows
$\{\textit{xors}.$
$\textit{dom xors} = D \wedge \textit{ran xors} \subseteq R\} \neq \{\}$
⟨*proof*⟩

lemma *random-xors-set-pmf*:
assumes V: *finite* V
shows
$\textit{set-pmf}\ (\textit{random-xors}\ V\ n) =$
$\{\textit{xors}.\ \textit{dom xors} = \{..<n\} \wedge$
$\textit{ran xors} \subseteq (\textit{Pow}\ V) \times \textit{UNIV}\}$
⟨*proof*⟩

lemma *finite-random-xors-set-pmf*:
assumes V: *finite* V
shows
$\textit{finite}\ (\textit{set-pmf}\ (\textit{random-xors}\ V\ n))$
⟨*proof*⟩

lemma *map-eq-1*:
assumes $dom\ f = dom\ g$
assumes $\bigwedge x.\ x \in dom\ f \Longrightarrow the\ (f\ x) = the\ (g\ x)$
shows $f = g$
⟨*proof*⟩

lemma *xor-hash-eq-iff*:
assumes $dom\ \alpha = \{..<n\}$
shows*xor-hash* $\omega\ x = \alpha \longleftrightarrow$
$(dom\ x = \{..<n\} \wedge$
$(\forall i.\ i < n \longrightarrow$
$(\exists xor.\ x\ i = Some\ xor \wedge$
satisfies-xor xor $\{x.\ \omega\ x = Some\ True\} = the\ (\alpha\ i))$
$))$
⟨*proof*⟩

lemma *xor-hash-eq-PiE-dflt*:
assumes $dom\ \alpha = \{..<n\}$
shows
$\{xors.\ xor\text{-}hash\ \omega\ xors = \alpha\} =$
PiE-dflt $\{..<n\}$ *None*
$(\lambda i.\ Some\ `$
$\{xor.\ satisfies\text{-}xor\ xor\ \{x.\ \omega\ x = Some\ True\} = the\ (\alpha\ i)\})$
⟨*proof*⟩

lemma *prob-random-xors-xor-hash*:
assumes V: *finite* V
assumes α: $dom\ \alpha = \{..<n\}$
shows
measure-pmf.prob $(random\text{-}xors\ V\ n)$
$\{xors.\ xor\text{-}hash\ \omega\ xors = \alpha\} = 1\ /\ 2\ \hat{}\ n$
⟨*proof*⟩

lemma *PiE-dflt-inter*:
shows*PiE-dflt A dflt B* $\cap$ *PiE-dflt A dflt B′* =
PiE-dflt A dflt $(\lambda b.\ B\ b \cap B'\ b)$
⟨*proof*⟩

lemma *random-xors-xor-hash-pair*:
assumes V: *finite* V
assumes α: $dom\ \alpha = \{..<n\}$
assumes α': $dom\ \alpha' = \{..<n\}$
assumes ω: $dom\ \omega = V$
assumes ω': $dom\ \omega' = V$
assumes *neq*: $\omega \neq \omega'$
shows
measure-pmf.prob $(random\text{-}xors\ V\ n)$
$\{xors.\ xor\text{-}hash\ \omega\ xors = \alpha \wedge xor\text{-}hash\ \omega'\ xors = \alpha'\} =$
$1\ /\ 4\ \hat{}\ n$

⟨*proof*⟩

lemma *random-xors-xor-hash-three*:
assumes V: *finite* V
assumes α: *dom* $\alpha = \{..<n\}$
assumes α': *dom* $\alpha' = \{..<n\}$
assumes α'': *dom* $\alpha'' = \{..<n\}$
assumes ω: *dom* $\omega = V$
assumes ω': *dom* $\omega' = V$
assumes ω': *dom* $\omega'' = V$
assumes *neq*: $\omega \neq \omega'$ $\omega' \neq \omega''$ $\omega'' \neq \omega$
shows
measure-pmf.prob (*random-xors* V n)
$\{$*xors*.
xor-hash ω *xors* $= \alpha$
$\wedge$ *xor-hash* ω' *xors* $= \alpha'$
$\wedge$ *xor-hash* ω'' *xors* $= \alpha''\} =$
1 / 8 $\hat{}$ n
⟨*proof*⟩

end

3 Random XOR hash family

This section defines a hash family based on random XORs and proves that this hash family is 3-universal.

theory *RandomXORHashFamily* **imports**
RandomXOR
begin

lemma *finite-dom*:
assumes *finite* V
shows *finite* $\{w :: 'a \rightharpoonup bool.\ dom\ w = V\}$
⟨*proof*⟩

lemma *xor-hash-eq-dom*:
assumes *xor-hash* ω *xors* $= \alpha$
shows *dom xors* $=$ *dom* α
⟨*proof*⟩

lemma *prob-random-xors-xor-hash-indicat-real*:
assumes V: *finite* V
shows
measure-pmf.prob (*random-xors* V n)
$\{$*xors*. *xor-hash* ω *xors* $= \alpha\} =$
indicat-real $\{\alpha$::*nat* $\rightharpoonup$ *bool*. *dom* $\alpha = \{0..<n\}\}$ α /
real (*card* $\{\alpha$::*nat* $\rightharpoonup$ *bool*. *dom* $\alpha = \{0..<n\}\}$)
⟨*proof*⟩

lemma *xor-hash-family-uniform*:
assumes V: *finite* V
assumes *dom* $\omega = V$
shows *prob-space.uniform-on*
(*random-xors* V n)
(*xor-hash* i) $\{\alpha.\ dom\ \alpha = \{0..<n\}\}$
⟨*proof*⟩

lemma *random-xors-xor-hash-pair-indicat*:
assumes V: *finite* V
assumes ω: *dom* $\omega = V$
assumes ω': *dom* $\omega' = V$
assumes *neq*: $\omega \neq \omega'$
shows
measure-pmf.prob (*random-xors* V n)
$\{xors.$
xor-hash ω *xors* $= \alpha \wedge$ *xor-hash* ω' *xors* $= \alpha'\} =$
(*measure-pmf.prob* (*random-xors* V n)
$\{xors.$
xor-hash ω *xors* $= \alpha\} *$
measure-pmf.prob (*random-xors* V n)
$\{xors.$
xor-hash ω' *xors* $= \alpha'\}$)
⟨*proof*⟩

lemma *prod-3-expand*:
assumes $a \neq b$ $b \neq c$ $c \neq a$
shows$(\prod \omega \in \{a,\ b,\ c\}.\ f\ \omega) = f\ a * (f\ b * f\ c)$
⟨*proof*⟩

lemma *random-xors-xor-hash-three-indicat*:
assumes V: *finite* V
assumes ω: *dom* $\omega = V$
assumes ω': *dom* $\omega' = V$
assumes ω'': *dom* $\omega'' = V$
assumes *neq*: $\omega \neq \omega'$ $\omega' \neq \omega''$ $\omega'' \neq \omega$
shows
measure-pmf.prob (*random-xors* V n)
$\{xors.$
xor-hash ω *xors* $= \alpha$
$\wedge$ *xor-hash* ω' *xors* $= \alpha'$
$\wedge$ *xor-hash* ω'' *xors* $= \alpha''\} =$
(*measure-pmf.prob* (*random-xors* V n)
$\{xors.$
xor-hash ω *xors* $= \alpha\} *$
measure-pmf.prob (*random-xors* V n)
$\{xors.$
xor-hash ω' *xors* $= \alpha'\} *$

measure-pmf.prob (*random-xors V n*)
{*xors*.
xor-hash ω″ xors = *α′*″})
⟨*proof*⟩

lemma *xor-hash-3-indep*:
assumes *V*: *finite V*
assumes *J*: *card J* ≤ *3 J* ⊆ {*α*. *dom α* = *V*}
shows
measure-pmf.prob (*random-xors V n*)
{*xors*. ∀*ω*∈*J*. *xor-hash ω xors* = *f ω*} =
(∏*ω*∈*J*.
measure-pmf.prob (*random-xors V n*)
{*xors*. *xor-hash ω xors* = *f ω*})
⟨*proof*⟩

lemma *xor-hash-3-wise-indep*:
assumes *finite V*
shows *prob-space.k-wise-indep-vars*
(*random-xors V n*) *3*
(*λ-*. *Universal-Hash-Families-More-Independent-Families.discrete*)
xor-hash
{*α*. *dom α* = *V*}
⟨*proof*⟩

theorem *xor-hash-family-3-universal*:
assumes *finite V*
shows*prob-space.k-universal*
(*random-xors V n*) *3 xor-hash*
{*α*::*'a* ⇀ *bool*. *dom α* = *V*}
{*α*::*nat* ⇀ *bool*. *dom α* = {*0*..<*n*}}
⟨*proof*⟩

corollary *xor-hash-family-2-universal*:
assumes *finite V*
shows*prob-space.k-universal*
(*random-xors V n*) *2 xor-hash*
{*α*::*'a* ⇀ *bool*. *dom α* = *V*}
{*α*::*nat* ⇀ *bool*. *dom α* = {*0*..<*n*}}
⟨*proof*⟩

end

4 ApproxMCCore definitions

This section defines the ApproxMCCore locale and various failure events to be used in its probabilistic analysis. The definitions closely follow Section 4.2 of Chakraborty et al. [1]. Some non-

probabilistic properties of the events are proved, most notably, the event inclusions of Lemma 3 [1]. Note that "events" here refer to subsets of hash functions.

theory *ApproxMCCore* **imports**
ApproxMCPreliminaries
begin

type-synonym *'a assg* = *'a* ⇀ *bool*

definition *restr* :: *'a set* ⇒ (*'a* ⇒ *bool*) ⇒ *'a assg*
where *restr S w* = (λ*x*. *if x* ∈ *S then Some* (*w x*) *else None*)

lemma *restrict-eq-mono*:
assumes *x* ⊆ *y*
assumes *f* |‘ *y* = *g* |‘ *y*
shows *f* |‘ *x* = *g* |‘ *x*
⟨*proof*⟩

definition *proj* :: *'a set* ⇒ (*'a* ⇒ *bool*) *set* ⇒ *'a assg set*
where *proj S W* = *restr S* ‘ *W*

lemma *card-proj*:
assumes *finite S*
shows *finite* (*proj S W*) *card* (*proj S W*) ≤ 2 ^ *card S*
⟨*proof*⟩

lemma *proj-mono*:
assumes *x* ⊆ *y*
shows *proj w x* ⊆ *proj w y*
⟨*proof*⟩

definition *aslice* :: *nat* ⇒ *nat assg* ⇒ *nat assg*
where *aslice i a* = *a* |‘ {..<*i*}

lemma *aslice-eq*:
assumes *i* ≥ *n*
assumes *dom a* = {..<*n*}
shows *aslice i a* = *aslice n a*
⟨*proof*⟩

definition *hslice* :: *nat* ⇒
(*'a assg* ⇒ *nat assg*) ⇒ (*'a assg* ⇒ *nat assg*)
where *hslice i h* = *aslice i* ∘ *h*

locale *ApproxMCCore* =
  fixes $W :: ('a \Rightarrow bool)\ set$
  fixes $S :: 'a\ set$
  fixes $\varepsilon :: real$
  fixes $\alpha :: nat\ assg$
  fixes *thresh* :: *nat*
  assumes α: $dom\ \alpha = \{0..<card\ S - 1\}$
  assumes ε: $\varepsilon > 0$
  assumes *thresh*:
    $thresh > 4$
    $card\ (proj\ S\ W) \geq thresh$
  assumes S: *finite* S
begin

lemma *finite-proj-S*:
  shows *finite* $(proj\ S\ W)$
  ⟨*proof*⟩

definition $\mu :: nat \Rightarrow real$
  where $\mu\ i = card\ (proj\ S\ W)\ /\ 2\ \hat{}\ i$

definition *card-slice* ::
  $('a\ assg \Rightarrow nat\ assg) \Rightarrow$
  $nat \Rightarrow nat$
  where *card-slice* $h\ i =$
    $card\ (proj\ S\ W \cap \{w.\ hslice\ i\ h\ w = aslice\ i\ \alpha\})$

lemma *card-slice-anti-mono*:
  assumes $i \leq j$
  shows *card-slice* $h\ j \leq$ *card-slice* $h\ i$
⟨*proof*⟩

lemma *hslice-eq*:
  assumes $n \leq i$
  assumes $\bigwedge w.\ dom\ (h\ w) = \{..<n\}$
  shows *hslice* $i\ h =$ *hslice* $n\ h$
  ⟨*proof*⟩

lemma *card-slice-lim*:
  assumes $card\ S - 1 \leq i$
  assumes $\bigwedge w.\ dom\ (h\ w) = \{..<(card\ S - 1)\}$
  shows *card-slice* $h\ i =$ *card-slice* $h\ (card\ S - 1)$
  ⟨*proof*⟩

definition $T :: nat \Rightarrow$
  $('a\ assg \Rightarrow nat\ assg)\ set$

where $T\ i = \{h.\ \textit{card-slice}\ h\ i < \textit{thresh}\}$

lemma *T-mono*:
assumes $i \leq j$
shows $T\ i \subseteq T\ j$
⟨*proof*⟩

lemma *μ-anti-mono*:
assumes $i \leq j$
shows $\mu\ i \geq \mu\ j$
⟨*proof*⟩

lemma *card-proj-witnesses*:
$\textit{card}\ (\textit{proj}\ S\ W) > 0$
⟨*proof*⟩

lemma *μ-strict-anti-mono*:
assumes $i < j$
shows $\mu\ i > \mu\ j$
⟨*proof*⟩

lemma *μ-gt-zero*:
shows $\mu\ i > 0$
⟨*proof*⟩

definition $L :: \textit{nat} \Rightarrow$
$('a\ \textit{assg} \Rightarrow \textit{nat}\ \textit{assg})\ \textit{set}$
where
$L\ i =$
$\{h.\ \textit{real}\ (\textit{card-slice}\ h\ i) < \mu\ i\ /\ (1 + \varepsilon)\}$

definition $U :: \textit{nat} \Rightarrow$
$('a\ \textit{assg} \Rightarrow \textit{nat}\ \textit{assg})\ \textit{set}$
where
$U\ i =$
$\{h.\ \textit{real}\ (\textit{card-slice}\ h\ i) \geq \mu\ i * (1 + \varepsilon\ /\ (1 + \varepsilon))\}$

definition *approxcore* ::
$('a\ \textit{assg} \Rightarrow \textit{nat}\ \textit{assg}) \Rightarrow$
$\textit{nat} \times \textit{nat}$
where
$\textit{approxcore}\ h =$
(*case List.find*
$(\lambda i.\ h \in T\ i)\ [1..<\textit{card}\ S]$ *of*
$\textit{None} \Rightarrow (2\ \widehat{}\ \textit{card}\ S,\ 1)$
| $\textit{Some}\ m \Rightarrow$
$(2\ \widehat{}\ m,\ \textit{card-slice}\ h\ m)$)

definition *approxcore-fail* ::
(*'a assg* ⇒ *nat assg*) *set*
where *approxcore-fail* =
{*h*.
let (*cells*,*sols*) = *approxcore h in*
cells ∗ *sols* ∉
{ *card* (*proj S W*) / (*1* + ε) ..
(*1* + ε::*real*) ∗ *card* (*proj S W*)}
}

lemma *T0-empty*:
shows *T 0* = {}
⟨*proof*⟩

lemma *L0-empty*:
shows *L 0* = {}
⟨*proof*⟩

lemma *U0-empty*:
shows *U 0* = {}
⟨*proof*⟩

lemma *real-divide-pos-left*:
assumes (*0*::*real*) < *a*
assumes *a* ∗ *b* < *c*
shows *b* < *c* / *a*
⟨*proof*⟩

lemma *real-divide-pos-right*:
assumes *a* > (*0*::*real*)
assumes *b* < *a* ∗ *c*
shows *b* / *a* < *c*
⟨*proof*⟩

lemma *failure-imp*:
shows *approxcore-fail* ⊆
(⋃ *i*∈{*1*..<*card S*}.
(*T i* − *T* (*i*−*1*)) ∩ (*L i* ∪ *U i*)) ∪
−*T* (*card S* − *1*)
⟨*proof*⟩

lemma *smallest-nat-exists*:
assumes *P i* ¬*P* (*0*::*nat*)
obtains *m* **where** *m* ≤ *i P m* ¬*P* (*m*−*1*)
⟨*proof*⟩

lemma *mstar-non-zero*:
shows $\neg\ \mu\ 0 * (1 + \varepsilon\ /\ (1 + \varepsilon)) \leq thresh$
⟨*proof*⟩

lemma *real-div-less*:
assumes $c > 0$
assumes $a \leq b * (c::nat)$
shows $real\ a\ /\ real\ c \leq b$
⟨*proof*⟩

lemma *mstar-exists*:
obtains m **where**
$\mu\ (m - 1) * (1 + \varepsilon\ /\ (1 + \varepsilon)) > thresh$
$\mu\ m * (1 + \varepsilon\ /\ (1 + \varepsilon)) \leq thresh$
$m \leq card\ S - 1$
⟨*proof*⟩

definition *mstar* :: *nat*
where $mstar = (@m.$
$\mu\ (m - 1) * (1 + \varepsilon\ /\ (1 + \varepsilon)) > thresh \wedge$
$\mu\ m * (1 + \varepsilon\ /\ (1 + \varepsilon)) \leq thresh \wedge$
$m \leq card\ S - 1)$

lemma *mstar-prop*:
shows
$\mu\ (mstar - 1) * (1 + \varepsilon\ /\ (1 + \varepsilon)) > thresh$
$\mu\ mstar * (1 + \varepsilon\ /\ (1 + \varepsilon)) \leq thresh$
$mstar \leq card\ S - 1$
⟨*proof*⟩

lemma *O1-lem*:
assumes $i \leq m$
shows $(T\ i - T\ (i-1)) \cap (L\ i \cup U\ i) \subseteq T\ m$
⟨*proof*⟩

lemma *O1*:
shows $(\bigcup i \in \{1..mstar-3\}.$
$(T\ i - T\ (i-1)) \cap (L\ i \cup U\ i)) \subseteq T\ (mstar-3)$
⟨*proof*⟩

lemma *T-anti-mono-neg*:
assumes $i \leq j$
shows $-\ T\ j \subseteq -\ T\ i$
⟨*proof*⟩

lemma *O2-lem*:

assumes *mstar* $<$ *i*
shows (*T i* $-$ *T* (*i*$-$*1*)) $\cap$ (*L i* $\cup$ *U i*) $\subseteq$ $-$*T mstar*
⟨*proof*⟩

lemma *O2*:
shows ($\bigcup$ *i*$\in${*mstar*..$<$*card S*}.
(*T i* $-$ *T* (*i*$-$*1*)) $\cap$ (*L i* $\cup$ *U i*)) $\cup$
$-$*T* (*card S* $-$ *1*) $\subseteq$ *L mstar* $\cup$ *U mstar*
⟨*proof*⟩

lemma *O3*:
assumes *i* $\leq$ *mstar* $-$ *1*
shows (*T i* $-$ *T* (*i*$-$*1*)) $\cap$ (*L i* $\cup$ *U i*) $\subseteq$ *L i*
⟨*proof*⟩

lemma *union-split-lem*:
assumes *x*: *x* $\in$ ($\bigcup$ *i*$\in${*1*..$<$*n*::*nat*}. *P i*)
shows *x* $\in$ ($\bigcup$ *i*$\in${*1*..*m*$-$*3*}. *P i*) $\cup$
P (*m*$-$*2*) $\cup$
P (*m*$-$*1*) $\cup$
($\bigcup$ *i*$\in${*m*..$<$*n*}. *P i*)
⟨*proof*⟩

lemma *union-split*:
($\bigcup$ *i*$\in${*1*..$<$*n*::*nat*}. *P i*) $\subseteq$
($\bigcup$ *i*$\in${*1*..*m*$-$*3*}. *P i*) $\cup$
P (*m*$-$*2*) $\cup$
P (*m*$-$*1*) $\cup$
($\bigcup$ *i*$\in${*m*..$<$*n*}. *P i*)
⟨*proof*⟩

lemma *failure-bound*:
shows *approxcore-fail* $\subseteq$
T (*mstar*$-$*3*) $\cup$
L (*mstar*$-$*2*) $\cup$
L (*mstar*$-$*1*) $\cup$
(*L mstar* $\cup$ *U mstar*)
⟨*proof*⟩

end

end

5 ApproxMCCore analysis

This section analyzes ApproxMCCore with respect to a universal hash family. The proof follows Lemmas 1 and 2 from Chakraborty et al. [1].

theory *ApproxMCCoreAnalysis* **imports**
HOL−Decision-Procs.Dense-Linear-Order
ApproxMCCore
begin

definition *Hslice* :: *nat* ⇒
(*'a assg* ⇒ *'b* ⇒ *nat assg*) ⇒ (*'a assg* ⇒ *'b* ⇒ *nat assg*)
where *Hslice i H* = (λ*w s*. *aslice i* (*H w s*))

context *prob-space*
begin

lemma *indep-vars-prefix*:
assumes *indep-vars* (λ-. *count-space UNIV*) *H J*
shows *indep-vars* (λ-. *count-space UNIV*) (*Hslice i H*) *J*
⟨*proof*⟩

lemma *assg-nonempty-dom*:
shows
(λ*x*. *if x* < *i then Some True else None*) ∈
{α::*nat assg*. *dom* α = {*0*..<*i*}}
⟨*proof*⟩

lemma *card-dom-ran-nat-assg*:
shows *card* {α::*nat assg*. *dom* α = {*0*..<*n*}} = *2*^*n*
⟨*proof*⟩

lemma *card-nat-assg-le*:
assumes *i* ≤ *n*
shows *card* {α::*nat assg*. *dom* α = {*0*..<*n*}} =
2^(*n*−*i*) ∗ *card* {α::*nat assg*. *dom* α = {*0*..<*i*}}
⟨*proof*⟩

lemma *empty-nat-assg-slice-notin*:
assumes *i* ≤ *n*
assumes *dom* β ≠ {*0*..<*i*}
shows{α::*nat assg*. *dom* α = {*0*..<*n*} ∧ *aslice i* α = β} = {}
⟨*proof*⟩

lemma *restrict-map-dom*:
shows α |‘ *dom* α = α
⟨*proof*⟩

lemma *aslice-refl*:
assumes *dom* α = {..<*i*}
shows *aslice i* α = α
⟨*proof*⟩

lemma *bij-betw-with-inverse*:
assumes f ‘ $A \subseteq B$
assumes $\bigwedge x.\ x \in A \Longrightarrow g\ (f\ x) = x$
assumes g ‘ $B \subseteq A$
assumes $\bigwedge x.\ x \in B \Longrightarrow f\ (g\ x) = x$
shows *bij-betw f A B*
⟨*proof*⟩

lemma *card-nat-assg-slice*:
assumes $i \leq n$
assumes $dom\ \beta = \{0..<i\}$
shows $card\ \{\alpha::nat\ assg.\ dom\ \alpha = \{0..<n\} \wedge aslice\ i\ \alpha = \beta\} = 2\ \widehat{}\ (n-i)$
⟨*proof*⟩

lemma *finite-dom*:
assumes *finite V*
shows *finite* $\{w :: 'a \rightharpoonup bool.\ dom\ w = V\}$
⟨*proof*⟩

lemma *universal-prefix-family-from-hash*:
assumes *M*: $M = measure\text{-}pmf\ p$
assumes *kH*: $k\text{-}universal\ k\ H\ D\ \{\alpha::nat\ assg.\ dom\ \alpha = \{0..<n\}\}$
assumes *i*: $i \leq n$
shows $k\text{-}universal\ k\ (Hslice\ i\ H)\ D\ \{\alpha.\ dom\ \alpha = \{0..<i\}\}$
⟨*proof*⟩

end

context *ApproxMCCore*
begin

definition *pivot* :: *real*
where $pivot = 9.84 * (\ 1 + 1\ /\ \varepsilon\ )\widehat{}2$

context
assumes *thresh*: $thresh \geq (1 + \varepsilon\ /\ (1 + \varepsilon)) * pivot$
begin

lemma *aux-1*:
assumes *fin*:*finite* $(set\text{-}pmf\ p)$
assumes σ: $\sigma > 0$
assumes *exp*: $\mu\ i = measure\text{-}pmf.expectation\ p\ Y$
assumes *var*: $\sigma\widehat{}2 = measure\text{-}pmf.variance\ p\ Y$
assumes *var-bound*: $\sigma\widehat{}2 \leq \mu\ i$
shows
$measure\text{-}pmf.prob\ p\ \{y.\ |\ Y\ y - \mu\ i\ | \geq \varepsilon\ /\ (1 + \varepsilon) * \mu\ i\ \}$
$\leq (1 + \varepsilon)\widehat{}2\ /\ (\varepsilon\widehat{}2 * \mu\ i)$

⟨*proof*⟩

lemma *analysis-1-1*:
assumes *p*: *finite* (*set-pmf p*)
assumes *ind*: *prob-space.k-universal* (*measure-pmf p*) *2 H*
{α::'*a assg. dom* α = *S*}
{α::*nat assg. dom* α = {*0*..<*card S* − *1*}}
assumes *i*: *i* ≤ *card S* − *1*
shows
measure-pmf.prob p
{*s.* | *card-slice* ((λ*w. H w s*)) *i* − μ *i* | ≥ ε / (*1* + ε) ∗ μ *i*}
≤ (*1* + ε)^*2* / (ε^*2* ∗ μ *i*)
⟨*proof*⟩

lemma *analysis-1-2*:
assumes *p*: *finite* (*set-pmf p*)
assumes *ind*: *prob-space.k-universal* (*measure-pmf p*) *2 H*
{α::'*a assg. dom* α = *S*}
{α::*nat assg. dom* α = {*0*..<*card S* − *1*}}
assumes *i*: *i* ≤ *card S* − *1*
assumes β: β ≤ *1*
shows *measure-pmf.prob p*
{*s. real*(*card-slice* ((λ*w. H w s*)) *i*) ≤ β ∗ μ *i* }
≤ *1* / (*1* + (*1* − β)^*2* ∗ μ *i*)
⟨*proof*⟩

lemma *shift-*μ:
assumes *k* ≤ *i*
showsμ *i* ∗ *2*^*k* = μ (*i*−*k*)
⟨*proof*⟩

lemma *analysis-2-1*:
assumes *p*: *finite* (*set-pmf p*)
assumes *ind*: *prob-space.k-universal* (*measure-pmf p*) *2 H*
{α::'*a assg. dom* α = *S*}
{α::*nat assg. dom* α = {*0*..<*card S* − *1*}}
assumes ε*-up*: ε ≤ *1*
shows
measure-pmf.prob (*map-pmf* (λ*s w. H w s*) *p*) (*T* (*mstar*−*3*))
≤ *1* / *62.5*
⟨*proof*⟩

lemma *analysis-2-1'*:
assumes *p*: *finite* (*set-pmf p*)
assumes *ind*: *prob-space.k-universal* (*measure-pmf p*) *2 H*

$\{\alpha::'a\ assg.\ dom\ \alpha = S\}$
$\{\alpha::nat\ assg.\ dom\ \alpha = \{0..<card\ S - 1\}\}$
shows
measure-pmf.prob (*map-pmf* ($\lambda s\ w.\ H\ w\ s$) *p*) (*T* (*mstar*−*3*))
$\leq$ *1 / 10.84*
⟨*proof*⟩

lemma *analysis-2-2*:
assumes *p*: *finite* (*set-pmf p*)
assumes *ind*: *prob-space.k-universal* (*measure-pmf p*) *2 H*
$\{\alpha::'a\ assg.\ dom\ \alpha = S\}$
$\{\alpha::nat\ assg.\ dom\ \alpha = \{0..<card\ S - 1\}\}$
shows
measure-pmf.prob (*map-pmf* ($\lambda s\ w.\ H\ w\ s$) *p*) (*L* (*mstar*−*2*)) $\leq$ *1 / 20.68*
⟨*proof*⟩

lemma *analysis-2-3*:
assumes *p*: *finite* (*set-pmf p*)
assumes *ind*: *prob-space.k-universal* (*measure-pmf p*) *2 H*
$\{\alpha::'a\ assg.\ dom\ \alpha = S\}$
$\{\alpha::nat\ assg.\ dom\ \alpha = \{0..<card\ S - 1\}\}$
shows
measure-pmf.prob
(*map-pmf* ($\lambda s\ w.\ H\ w\ s$) *p*) (*L* (*mstar*−*1*)) $\leq$ *1 / 10.84*
⟨*proof*⟩

lemma *analysis-2-4*:
assumes *p*: *finite* (*set-pmf p*)
assumes *ind*: *prob-space.k-universal* (*measure-pmf p*) *2 H*
$\{\alpha::'a\ assg.\ dom\ \alpha = S\}$
$\{\alpha::nat\ assg.\ dom\ \alpha = \{0..<card\ S - 1\}\}$
shows
measure-pmf.prob (*map-pmf* ($\lambda s\ w.\ H\ w\ s$) *p*) (*L mstar* $\cup$ *U mstar*)
$\leq$ *1 / 4.92*
⟨*proof*⟩

lemma *analysis-3*:
assumes *p*: *finite* (*set-pmf p*)
assumes *ind*: *prob-space.k-universal* (*measure-pmf p*) *2 H*
$\{\alpha::'a\ assg.\ dom\ \alpha = S\}$
$\{\alpha::nat\ assg.\ dom\ \alpha = \{0..<card\ S - 1\}\}$
assumes ε-*up*: $\varepsilon \leq 1$
shows
measure-pmf.prob (*map-pmf* ($\lambda s\ w.\ H\ w\ s$) *p*)

approxcore-fail $\leq$ *0.36*
⟨*proof*⟩

lemma *analysis-3'*:
assumes *p*: *finite* (*set-pmf p*)
assumes *ind*: *prob-space.k-universal* (*measure-pmf p*) *2 H*
{α::'*a assg. dom* α = *S*}
{α::*nat assg. dom* α = {*0*..<*card S* − *1*}}
shows
measure-pmf.prob (*map-pmf* (λ*s w. H w s*) *p*)
approxcore-fail $\leq$ *0.44*
⟨*proof*⟩
end

end

end

6 ApproxMC definition and analysis

This section puts together preceding results to formalize the PAC guarantee of ApproxMC.

theory *ApproxMCAnalysis* **imports**
ApproxMCCoreAnalysis
RandomXORHashFamily
Median-Method.Median
begin

lemma *replicate-pmf-Pi-pmf*:
assumes *distinct ls*
shows *replicate-pmf* (*length ls*) *P* =
map-pmf (λ*f. map f ls*)
(*Pi-pmf* (*set ls*) *def* (λ-. *P*))
⟨*proof*⟩

lemma *replicate-pmf-Pi-pmf'*:
assumes *finite V*
shows *replicate-pmf* (*card V*) *P* =
map-pmf (λ*f. map f* (*sorted-list-of-set V*))
(*Pi-pmf V def* (λ-. *P*))
⟨*proof*⟩

definition *map-of-default*::('*a* × '*b*) *list* ⇒ '*b* ⇒ '*a* ⇒ '*b*
where *map-of-default ls def* =
(*let m* = *map-of ls in*
(λ*x. case m x of None* ⇒ *def* | *Some v* ⇒ *v*))

lemma *Pi-pmf-replicate-pmf*:
assumes *finite V*
shows
(*Pi-pmf V def* (λ-. *p*)) =
map-pmf (λ*bs*.
map-of-default (*zip* (*sorted-list-of-set V*) *bs*) *def*)
(*replicate-pmf* (*card V*) *p*)
⟨*proof*⟩

lemma *proj-inter-neutral*:
assumes ⋀*w*. *w* ∈ *B* ⟷ *restr S w* ∈ *C*
shows *proj S* (*A* ∩ *B*) = *proj S A* ∩ *C*
⟨*proof*⟩

An abstract spec of ApproxMC for any Boolean theory. This locale must be instantiated with a theory implementing the two the functions below (and satisfying the assumption linking them).

locale *ApproxMC* =
fixes *sols* :: *'fml* ⇒ (*'a* ⇒ *bool*) *set*
fixes *enc-xor* :: *'a set* × *bool* ⇒ *'fml* ⇒ *'fml*
assumes *sols-enc-xor*:
⋀*F xor*. *finite* (*fst xor*) ⟹
sols (*enc-xor xor F*) =
sols F ∩ {ω. *satisfies-xor xor* {*x*. ω *x*}}
begin

definition *compute-thresh* :: *real* ⇒ *nat*
where *compute-thresh* ε =
nat ⌈1 + 9.84 ∗ (1 + ε / (1 + ε)) ∗ (1 + 1 / ε)^2⌉

definition *fix-t* :: *real* ⇒ *nat*
where *fix-t* δ =
nat ⌈*ln* (1 / δ) /(2 ∗ (0.5−0.36)^2)⌉

definition *raw-median-bound* :: *real* ⇒ *nat* ⇒ *real*
where *raw-median-bound* α *t* =
(∑ *i* = 0..*t div* 2.
(*t choose i*) ∗ (1 / 2 + α) ^ *i* ∗ (1 / 2 − α) ^ (*t* − *i*))

definition *compute-t* :: *real* ⇒ *nat* ⇒ *nat*
where *compute-t* δ *n* =
(*if raw-median-bound* 0.14 *n* < δ *then n*
else fix-t δ)

definition *size-xor* ::
'fml ⇒ *'a set* ⇒
(*nat* ⇒ (*'a set* × *bool*) *option*) ⇒
nat ⇒ *nat*

```
  where size-xor F S xorsf i = (
    let xors = map (the ∘ xorsf) [0..<i] in
    let Fenc = fold enc-xor xors F in
      card (proj S (sols Fenc))
    )

definition check-xor ::
  'fml ⇒ 'a set ⇒
  nat ⇒
  (nat ⇒ ('a set × bool) option) ⇒
  nat ⇒ bool
  where check-xor F S thresh xorsf i =
  (size-xor F S xorsf i < thresh)

definition approxcore-xors ::
  'fml ⇒ 'a set ⇒
  nat ⇒
  (nat ⇒ ('a set × bool) option) ⇒
  nat
  where
    approxcore-xors F S thresh xorsf =
    (case List.find
      (check-xor F S thresh xorsf) [1..<card S] of
      None ⇒ 2 ^ card S
    | Some m ⇒
      (2 ^ m * size-xor F S xorsf m))

definition approxmccore :: 'fml ⇒ 'a set ⇒ nat ⇒ nat pmf
where approxmccore F S thresh =
  map-pmf (approxcore-xors F S thresh) (random-xors S (card S − 1))

definition approxmc :: 'fml ⇒ 'a set ⇒ real ⇒ real ⇒ nat ⇒ nat
pmf
  where approxmc F S ε δ n = (
    let thresh = compute-thresh ε in
    if card (proj S (sols F)) < thresh then
      return-pmf (card (proj S (sols F)))
    else
      let t = compute-t δ n in
      map-pmf (median t)
        (prod-pmf {0..<t::nat} (λi. approxmccore F S thresh))
  )

lemma median-commute:
  assumes t ≥ 1
  shows (real ∘ median t) = (λw::nat ⇒ nat. median t (real ∘ w))
⟨proof⟩

lemma median-default:
```

shows *median t y = median t* (λx. *if x < t then y x else def*)
⟨*proof*⟩

definition *default-α*::*'a set* ⇒ *nat assg*
where *default-α S i* = (*if i < card S − 1 then Some True else None*)

lemma *dom-default-α*:
dom (*default-α S*) = {*0*..<*card S −1*}
⟨*proof*⟩

lemma *compute-thresh-bound-4*:
assumes *ε > 0*
shows*4 < compute-thresh ε*
⟨*proof*⟩

lemma *satisfies-xor-with-domain*:
assumes *fst x* ⊆ *S*
shows *satisfies-xor x* {*x. w x*} ⟷
satisfies-xor x ({*x. w x*} ∩ *S*)
⟨*proof*⟩

lemma *approxcore-xors-eq*:
assumes *thresh*:
thresh = compute-thresh ε
thresh ≤ *card* (*proj S* (*sols F*))
assumes *ε*: *ε* > (*0*::*real*) *ε* ≤ *1*
assumes *S*: *finite S*
shows *measure-pmf.prob* (*random-xors S* (*card S − 1*))
{*xors. real* (*approxcore-xors F S thresh xors*) ∈
{*real* (*card* (*proj S* (*sols F*))) / (*1* + *ε*)..
(*1* + *ε*) ∗ *real* (*card* (*proj S* (*sols F*)))}} ≥ *0.64*
⟨*proof*⟩

lemma *compute-t-ge1*:
assumes *0 < δ δ < 1*
shows*compute-t δ n* ≥ *1*
⟨*proof*⟩

lemma *success-arith-bound*:
assumes *s* ≤ (*f* ::*nat*)
assumes *p* ≤ (*1*::*real*) *q* ≤ *p 1 /2* ≤ *q*
shows *p* ^ *s* ∗ (*1 − p*) ^ *f* ≤ *q* ^ *s* ∗ (*1 − q*) ^ *f*
⟨*proof*⟩

lemma *prob-binomial-pmf-upto-mono*:
assumes *1/2* ≤ *q q* ≤ *p p* ≤ *1*
shows
measure-pmf.prob (*binomial-pmf n p*) {*..n div 2*} ≤

```
  measure-pmf.prob (binomial-pmf n q) {..n div 2}
⟨proof⟩


theorem approxmc-sound:
  assumes δ: δ > 0 δ < 1
  assumes ε: ε > 0 ε ≤ 1
  assumes S: finite S
  shows measure-pmf.prob (approxmc F S ε δ n)
      {c. real c ∈
        {real (card (proj S (sols F))) / (1 + ε)..
          (1 + ε) * real (card (proj S (sols F)))}}
      ≥ 1 − δ
⟨proof⟩
```

To simplify further analyses, we can remove the (required) upper bound on epsilon.

```
definition mk-eps :: real ⇒ real
  where mk-eps ε = (if ε > 1 then 1 else ε)

definition approxmc'::
  'fml ⇒ 'a set ⇒
  real ⇒ real ⇒ nat ⇒ nat pmf
  where approxmc' F S ε δ n =
    approxmc F S (mk-eps ε) δ n

corollary approxmc'-sound:
  assumes δ: δ > 0 δ < 1
  assumes ε: ε > 0
  assumes S: finite S
  shows prob-space.prob (approxmc' F S ε δ n)
      {c. real c ∈
        {real (card (proj S (sols F))) / (1 + ε)..
          (1 + ε) * real (card (proj S (sols F)))}}
      ≥ 1 − δ
⟨proof⟩
```

This shows we can lift all randomness to the top-level (i.e., eagerly sample it).

```
definition approxmc-map::
  'fml ⇒ 'a set ⇒
  real ⇒ real ⇒ nat ⇒
  (nat ⇒ nat ⇒ ('a set × bool) option) ⇒
  nat
  where approxmc-map F S ε δ n xorsFs = (
    let ε = mk-eps ε in
    let thresh = compute-thresh ε in
    if card (proj S (sols F)) < thresh then
      card (proj S (sols F))
```

else
let t = compute-t δ n in
median t (approxcore-xors F S thresh ∘ xorsFs))

lemma *approxmc-map-eq*:
shows
map-pmf (approxmc-map F S ε δ n)
(Pi-pmf {0..<compute-t δ n} def
(λi. random-xors S (card S − 1))) =
approxmc′ F S ε δ n
⟨*proof*⟩

end

end

7 Certificate-based ApproxMC

This turns the randomized algorithm into an executable certificate checker

theory *CertCheck*
imports *ApproxMCAnalysis*

begin

7.1 ApproxMC with lists instead of sets

type-synonym *'a xor = 'a list × bool*

definition *satisfies-xorL :: 'a xor ⇒ ('a ⇒ bool) ⇒ bool*
where *satisfies-xorL xb ω =*
even (sum-list (map (of-bool ∘ ω) (fst xb)) +
of-bool (snd xb)::nat)

definition *sublist-bits::'a list ⇒ bool list ⇒ 'a list*
where *sublist-bits ls bs =*
map fst (filter snd (zip ls bs))

definition *xor-from-bits::*
'a list ⇒ bool list × bool ⇒ 'a xor
where *xor-from-bits V xsb =*
(sublist-bits V (fst xsb), snd xsb)

locale *ApproxMCL =*
fixes *sols :: 'fml ⇒ ('a ⇒ bool) set*
fixes *enc-xor :: 'a xor ⇒ 'fml ⇒ 'fml*
assumes *sols-enc-xor*:

$\bigwedge F$ *xor*.
sols (*enc-xor xor F*) =
sols F ∩ {ω. *satisfies-xorL xor* ω}
begin

definition *list-of-set* ::'*a set* ⇒ '*a list*
where *list-of-set x* = (@*ls*. *set ls* = *x* ∧ *distinct ls*)

definition *xor-conc* :: '*a set* × *bool* ⇒ '*a xor*
where *xor-conc xsb* = (*list-of-set* (*fst xsb*), *snd xsb*)

definition *enc-xor-conc* :: '*a set* × *bool* ⇒ '*fml* ⇒ '*fml*
where *enc-xor-conc* = *enc-xor* ∘ *xor-conc*

lemma *distinct-count-list*:
assumes *distinct ls*
shows *count-list ls x* = *of-bool* (*x* ∈ *set ls*)
⟨*proof*⟩

lemma *list-of-set*:
assumes *finite x*
shows *distinct* (*list-of-set x*) *set* (*list-of-set x*) = *x*
⟨*proof*⟩

lemma *count-list-list-of-set*:
assumes *finite x*
shows *count-list* (*list-of-set x*) *y* = *of-bool* (*y* ∈ *x*)
⟨*proof*⟩

lemma *satisfies-xorL-xor-conc*:
assumes *finite x*
shows*satisfies-xorL* (*xor-conc* (*x*,*b*)) ω ⟷ *satisfies-xor* (*x*,*b*) {*x*. ω *x*}
⟨*proof*⟩

sublocale *appmc*: *ApproxMC sols enc-xor-conc*
⟨*proof*⟩

definition *size-xorL* ::
'*fml* ⇒ '*a list* ⇒
(*nat* ⇒ *bool list* × *bool*) ⇒
nat ⇒ *nat*
where *size-xorL F S xorsl i* = (
let xors = *map* (*xor-from-bits S* ∘ *xorsl*) [*0*..<*i*] *in*
let Fenc = *fold enc-xor xors F in*
card (*proj* (*set S*) (*sols Fenc*)))

definition *check-xorL* ::
'*fml* ⇒ '*a list* ⇒

nat ⇒
(*nat* ⇒ *bool list* × *bool*) ⇒
nat ⇒ *bool*
where *check-xorL F S thresh xorsl i* =
(*size-xorL F S xorsl i* < *thresh*)

definition *approxcore-xorsL* ::
'fml ⇒ *'a list* ⇒
nat ⇒
(*nat* ⇒ (*bool list* × *bool*)) ⇒
nat
where
approxcore-xorsL F S thresh xorsl =
(*case List.find*
(*check-xorL F S thresh xorsl*) [*1*..<*length S*] *of*
None ⇒ *2* ^ *length S*
| *Some m* ⇒
(*2* ^ *m* ∗ *size-xorL F S xorsl m*))

definition *xor-abs* :: *'a xor* ⇒ *'a set* × *bool*
where *xor-abs xsb* = (*set* (*fst xsb*), *snd xsb*)

lemma *sols-fold-enc-xor*:
assumes *list-all2* (λ*x y*.
∀ *w*. *satisfies-xorL x w* = *satisfies-xorL y w*) *xs ys*
assumes *sols F* = *sols G*
shows *sols* (*fold enc-xor xs F*) = *sols* (*fold enc-xor ys G*)
⟨*proof*⟩

lemma *satisfies-xor-xor-abs*:
assumes *distinct x*
shows*satisfies-xor* (*xor-abs* (*x*,*b*)) {*x*. ω *x*} ⟷ *satisfies-xorL* (*x*,*b*)
ω
⟨*proof*⟩

lemma *xor-conc-xor-abs-rel*:
assumes *distinct* (*fst x*)
shows*satisfies-xorL* (*xor-conc* (*xor-abs x*)) *w* ⟷
satisfies-xorL x w
⟨*proof*⟩

lemma *sorted-sublist-bits*:
assumes *sorted V*
shows*sorted* (*sublist-bits V bs*)
⟨*proof*⟩

lemma *distinct-sublist-bits*:
assumes *distinct V*
shows*distinct* (*sublist-bits V bs*)

⟨*proof*⟩

lemma *distinct-fst-xor-from-bits*:
assumes *distinct V*
shows*distinct* (*fst* (*xor-from-bits V bs*))
⟨*proof*⟩

lemma *size-xorL*:
assumes $\bigwedge j.\ j < i \Longrightarrow$
xorsf j =
Some (*xor-abs* (*xor-from-bits S* (*xorsl j*)))
assumes *distinct S*
shows *size-xorL F S xorsl i* =
appmc.size-xor F (*set S*) *xorsf i*
⟨*proof*⟩

lemma *fold-enc-xor-more*:
assumes $x \in$ *sols* (*fold enc-xor* (*xs* @ *rev ys*) *F*)
shows $x \in$ *sols* (*fold enc-xor xs F*)
⟨*proof*⟩

lemma *size-xorL-anti-mono*:
assumes $x \leq y$ *distinct S*
shows *size-xorL F S xorsl x* $\geq$ *size-xorL F S xorsl y*
⟨*proof*⟩

lemma *find-upto-SomeI*:
assumes $\bigwedge i.\ a \leq i \Longrightarrow i < x \Longrightarrow \neg P\ i$
assumes $P\ x\ a \leq x\ x < b$
shows *find P* $[a..<b]$ = *Some x*
⟨*proof*⟩

lemma *check-xorL*:
assumes $\bigwedge j.\ j < i \Longrightarrow$
xorsf j =
Some (*xor-abs* (*xor-from-bits S* (*xorsl j*)))
assumes *distinct S*
shows *check-xorL F S thresh xorsl i* =
appmc.check-xor F (*set S*) *thresh xorsf i*
⟨*proof*⟩

lemma *approxcore-xorsL*:
assumes $\bigwedge j.\ j <$ *length S* − *1* $\Longrightarrow$
xorsf j =
Some (*xor-abs* (*xor-from-bits S* (*xorsl j*)))
assumes *S*: *distinct S*
shows *approxcore-xorsL F S thresh xorsl* =
appmc.approxcore-xors F (*set S*) *thresh xorsf*
⟨*proof*⟩

definition *approxmc-mapL*::
  'fml ⇒ *'a list* ⇒
  real ⇒ *real* ⇒ *nat* ⇒
  (*nat* ⇒ *nat* ⇒ (*bool list* × *bool*)) ⇒
  nat
  where *approxmc-mapL F S ε δ n xorsLs* = (
    let ε = *appmc.mk-eps ε in*
    let thresh = *appmc.compute-thresh ε in*
    if card (*proj* (*set S*) (*sols F*)) < *thresh then*
      card (*proj* (*set S*) (*sols F*))
    else
      let t = *appmc.compute-t δ n in*
      median t (*approxcore-xorsL F S thresh* ∘ *xorsLs*))

definition *random-xorB* :: *nat* ⇒ (*bool list* × *bool*) *pmf*
  where *random-xorB n* =
    pair-pmf
      (*replicate-pmf n* (*bernoulli-pmf* (*1*/*2*)))
      (*bernoulli-pmf* (*1*/*2*))

lemma *approxmc-mapL*:
  assumes ⋀*i j*. *j* < *length S* − *1* ⟹
    xorsFs i j =
    Some (*xor-abs* (*xor-from-bits S* (*xorsLs i j*)))
  assumes *S*: *distinct S*
  shows
    approxmc-mapL F S ε δ n xorsLs =
    appmc.approxmc-map F (*set S*) *ε δ n xorsFs*
⟨*proof*⟩

lemma *approxmc-mapL'*:
  assumes *S*: *distinct S*
  shows
    approxmc-mapL F S ε δ n xorsLs =
    appmc.approxmc-map F (*set S*) *ε δ n*
      (λ*i j*. *if j* < *length S* − *1*
        then Some (*xor-abs* (*xor-from-bits S* (*xorsLs i j*)))
        else None)
  ⟨*proof*⟩

lemma *bits-to-random-xor*:
  assumes *distinct S*
  shows *map-pmf*
    (λ*x*. *xor-abs* (*xor-from-bits S x*))
    (*random-xorB* (*length S*)) =
    random-xor (*set S*)
⟨*proof*⟩

lemma *Pi-pmf-map-pmf-Some*:
assumes *finite S*
shows *Pi-pmf S None* (λ-. *map-pmf Some p*) =
map-pmf (λ*f v*. *if v* ∈ *S then Some* (*f v*) *else None*)
(*Pi-pmf S def* (λ-. *p*))
⟨*proof*⟩

lemma *bits-to-random-xors*:
assumes *distinct S*
shows
map-pmf
(λ*f j*.
if j < *n*
then Some (*xor-abs* (*xor-from-bits S* (*f j*)))
else None)
(*Pi-pmf* {..<(*n*::*nat*)} *def* (λ-. *random-xorB* (*length S*))) =
random-xors (*set S*) *n*
⟨*proof*⟩

lemma *bits-to-all-random-xors*:
assumes *distinct S*
assumes (λ*j*. *if j* < *n*
then Some (*xor-abs* (*xor-from-bits S* (*def1 j*)))
else None) = *def*
shows
map-pmf
((∘) (λ*f j*. *if j* < *n*
then Some (*xor-abs* (*xor-from-bits S* (*f j*)))
else None))
(*Pi-pmf* {*0*..<(*m*::*nat*)} *def1*
(λ-.
Pi-pmf {..<(*n*::*nat*)} *def2* (λ-. *random-xorB* (*length S*)))) =
Pi-pmf {*0*..<*m*} *def*
(λ*i*. *random-xors* (*set S*) *n*)
⟨*proof*⟩

definition *random-seed-xors*::*nat* ⇒ *nat* ⇒ (*nat* ⇒ *nat* ⇒ *bool list* × *bool*) *pmf*
where *random-seed-xors t l* =
(*prod-pmf* {*0*..<*t*}
(λ-. *prod-pmf* {..<*l*−*1*} (λ-. *random-xorB l*)))

lemma *approxmcL-sound*:
assumes δ: δ > *0* δ < *1*
assumes ε: ε > *0*
assumes *S*: *distinct S*
shows

prob-space.prob
(*map-pmf* (*approxmc-mapL F S ε δ n*)
(*random-seed-xors* (*appmc.compute-t δ n*) (*length S*)))
{*c*. *real c* ∈
{*real* (*card* (*proj* (*set S*) (*sols F*))) / (*1* + *ε*)..
(*1* + *ε*) ∗ *real* (*card* (*proj* (*set S*) (*sols F*)))}}
≥ *1* − *δ*
⟨*proof*⟩

lemma *approxmcL-sound′*:
assumes *δ*: *δ* > *0* *δ* < *1*
assumes *ε*: *ε* > *0*
assumes *S*: *distinct S*
shows
prob-space.prob
(*map-pmf* (*approxmc-mapL F S ε δ n*)
(*random-seed-xors* (*appmc.compute-t δ n*) (*length S*)))
{*c*. *real c* ∉
{*real* (*card* (*proj* (*set S*) (*sols F*))) / (*1* + *ε*)..
(*1* + *ε*) ∗ *real* (*card* (*proj* (*set S*) (*sols F*)))}} ≤ *δ*
⟨*proof*⟩

end

7.2 ApproxMC certificate checker

definition *str-of-bool* :: *bool* ⇒ *String.literal*
where *str-of-bool b* = (
if b then STR ″true″ else STR ″false″)

fun *str-of-nat-aux* :: *nat* ⇒ *char list* ⇒ *char list*
where *str-of-nat-aux n acc* = (
let c = *char-of-integer* (*of-nat* (*48* + *n mod 10*)) *in*
if n < *10 then c* # *acc*
else str-of-nat-aux (*n div 10*) (*c* # *acc*))

definition *str-of-nat* :: *nat* ⇒ *String.literal*
where *str-of-nat n* = *String.implode* (*str-of-nat-aux n* [])

type-synonym *′a sol* = (*′a* × *bool*) *list*

definition *canon-map-of* :: (*′a* × *bool*) *list* ⇒ (*′a* ⇒ *bool*)
where *canon-map-of ls* =
(*let m* = *map-of ls in*
(*λx*. *case m x of None* ⇒ *False* | *Some b* ⇒ *b*))

lemma *canon-map-of*[*code*]:
shows *canon-map-of ls* =
(*let m* = *Mapping.of-alist ls in*
Mapping.lookup-default False m)
⟨*proof*⟩

definition *proj-sol* :: *'a list* ⇒ (*'a* ⇒ *bool*) ⇒ *bool list*
where *proj-sol S w* = *map w S*

The following extended locale assumes additional support for syntactically working with solutions

locale *CertCheck* = *ApproxMCL sols enc-xor*
for *sols* :: *'fml* ⇒ (*'a* ⇒ *bool*) *set*
and *enc-xor* :: *'a list* × *bool* ⇒ *'fml* ⇒ *'fml* +
fixes *check-sol* :: *'fml* ⇒ (*'a* ⇒ *bool*) ⇒ *bool*
fixes *ban-sol* :: *'a sol* ⇒ *'fml* ⇒ *'fml*
assumes *sols-ban-sol*:
⋀*F vs*.
sols (*ban-sol vs F*) =
sols F ∩ {*ω*. *map ω* (*map fst vs*) ≠ *map snd vs*}
assumes *check-sol*:
⋀*F w*. *check-sol F w* ⟷ *w* ∈ *sols F*
begin

Assuming parameter access to an UNSAT checking oracle

context
fixes *check-unsat* :: *'fml* ⇒ *bool*
begin

Throughout this checker, INL indicates error, INR indicates success

definition *check-BSAT-sols*::
'fml ⇒ *'a list* ⇒ *nat* ⇒ (*'a* ⇒ *bool*) *list* ⇒ *String.literal* + *unit*
where *check-BSAT-sols F S thresh cms* = (
let ps = *map* (*proj-sol S*) *cms in*
let b1 = *list-all* (*check-sol F*) *cms in*
let b2 = *distinct ps in*
let b3 =
(*length cms* < *thresh* ⟶
check-unsat (*fold ban-sol* (*map* (*zip S*) *ps*) *F*)) *in*
if b1 ∧ *b2* ∧ *b3 then Inr* ()
else Inl (*STR* ''*checks* −−−'' +
STR '' *all valid sols*: '' + *str-of-bool b1* +
STR '', *all distinct sols*: '' + *str-of-bool b2* +
STR '', *unsat check* (< *thresh sols*): '' + *str-of-bool b3*)
)

definition *BSAT* ::
'fml ⇒ *'a list* ⇒ *nat* ⇒ (*'a* ⇒ *bool*) *list* ⇒ *String.literal* + *nat*

```
where BSAT F S thresh xs = (
  case check-BSAT-sols F S thresh xs of
    Inl err ⇒ Inl err
  | Inr - ⇒ Inr (length xs)
)


definition size-xorL-cert ::
  'fml ⇒ 'a list ⇒ nat ⇒
  (nat ⇒ (bool list × bool)) ⇒ nat ⇒
  (('a ⇒ bool) list) ⇒ String.literal + nat
  where size-xorL-cert F S thresh xorsl i xs = (
    let xors = map (xor-from-bits S ∘ xorsl) [0..<i]  in
    let Fenc = fold enc-xor xors F in
    BSAT Fenc S thresh xs
  )


fun approxcore-xorsL-cert ::
  'fml ⇒ 'a list ⇒ nat ⇒
  nat × ('a ⇒ bool) list × ('a ⇒ bool) list ⇒
  (nat ⇒ (bool list × bool))
  ⇒ String.literal + nat
where approxcore-xorsL-cert F S thresh (m,cert1,cert2) xorsl = (
  if 1 ≤ m ∧ m ≤ length S
  then
    case size-xorL-cert F S thresh xorsl (m−1) cert1 of
      Inl err ⇒ Inl (STR ''cert1 '' + err)
    | Inr n ⇒
    if n ≥ thresh
    then
      if m = length S
      then Inr (2 ^ length S)
      else
        case size-xorL-cert F S thresh xorsl m cert2 of
          Inl err ⇒ Inl (STR ''cert2 '' + err)
        | Inr c ⇒
          if c < thresh then Inr (2 ^ m * c)
          else Inl (STR ''too many solutions at m added XORs'')
    else Inl (STR ''too few solutions at m−1 added XORs'')
  else
     Inl (STR ''invalid value of m, need 1 <= m <= |S|''))


definition find-t :: real ⇒ nat
where find-t δ = (
  case find (λi. appmc.raw-median-bound 0.14 i < δ) [0..<256] of
    Some m ⇒ m
  | None ⇒ appmc.fix-t δ
```

```
  )

fun fold-approxcore-xorsL-cert::
  'fml ⇒ 'a list ⇒ nat ⇒
  nat ⇒ nat ⇒
  (nat ⇒ (nat × ('a ⇒ bool) list × ('a ⇒ bool) list)) ⇒
  (nat ⇒ nat ⇒ (bool list × bool))
  ⇒ String.literal + (nat list)
  where
  fold-approxcore-xorsL-cert F S thresh t 0 cs xorsLs = Inr []
| fold-approxcore-xorsL-cert F S thresh t (Suc i) cs xorsLs = (
    let it = t − Suc i in
    case approxcore-xorsL-cert F S thresh (cs it) (xorsLs it) of
      Inl err ⇒ Inl (STR ''round '' + str-of-nat it + STR '' '' + err)
    | Inr n ⇒
      (case fold-approxcore-xorsL-cert F S thresh t i cs xorsLs of
          Inl err ⇒ Inl err
        | Inr ns ⇒ Inr (n # ns)))

definition calc-median::
  'fml ⇒ 'a list ⇒ nat ⇒ nat ⇒
  (nat ⇒ (nat × ('a ⇒ bool) list × ('a ⇒ bool) list)) ⇒
  (nat ⇒ nat ⇒ (bool list × bool)) ⇒
  String.literal + nat
  where calc-median F S thresh t ms xorsLs = (
      case fold-approxcore-xorsL-cert F S thresh t t ms xorsLs of
        Inl err ⇒ Inl err
      | Inr ls ⇒ Inr (sort ls ! (t div 2))
  )

fun certcheck::
  'fml ⇒ 'a list ⇒
  real ⇒ real ⇒
  (('a ⇒ bool) list ×
  (nat ⇒ (nat × ('a ⇒ bool) list × ('a ⇒ bool) list))) ⇒
  (nat ⇒ nat ⇒ (bool list × bool)) ⇒
  String.literal + nat
  where certcheck F S ε δ (m0,ms) xorsLs = (
    let ε = appmc.mk-eps ε in
    let thresh = appmc.compute-thresh ε in
    case BSAT F S thresh m0 of Inl err ⇒ Inl err
    | Inr Y ⇒
    if Y < thresh then Inr Y
    else
      let t = find-t δ in
      calc-median F S thresh t ms xorsLs)


context
```

assumes *check-unsat*: $\bigwedge F.$ *check-unsat* $F \Longrightarrow$ *sols* $F = \{\}$
begin

lemma *sols-fold-ban-sol*:
shows*sols* (*fold ban-sol ls F*) =
sols F $\cap$ $\{\omega.$ $(\forall$ *vs* $\in$ *set ls. map* ω (*map fst vs*) $\neq$ *map snd vs*)$\}$
⟨*proof*⟩

lemma *inter-cong-right*:
assumes $\bigwedge x.$ $x \in A \Longrightarrow x \in B \longleftrightarrow x \in C$
shows $A \cap B = A \cap C$
⟨*proof*⟩

lemma *proj-sol-canon-map-of*:
assumes *distinct S length S* = *length w*
shows *proj-sol S* (*canon-map-of* (*zip S w*)) = *w*
⟨*proof*⟩

lemma *proj-sol-cong*:
assumes *restr* (*set S*) *A* = *restr* (*set S*) *B*
shows *proj-sol S A* = *proj-sol S B*
⟨*proof*⟩

lemma *canon-map-of-map-of*:
assumes *length S* = *length x*
assumes *canon-map-of* (*zip S x*) $\in$ *A*
shows *map-of* (*zip S x*) $\in$ *proj* (*set S*) *A*
⟨*proof*⟩

lemma *proj-proj-sol-map-of-zip-1*:
assumes *distinct S length S* = *length w*
assumes *w*: *w* $\in$ *rdb*
shows
map-of (*zip S w*) $\in$
proj (*set S*) $\{\omega.$ *proj-sol S* $\omega \in$ *rdb*$\}$
⟨*proof*⟩

lemma *proj-proj-sol-map-of-zip-2*:
assumes $\bigwedge bs.$ *bs* $\in$ *rdb* $\Longrightarrow$ *length bs* = *length S*
assumes *w*: *w* $\in$ *proj* (*set S*) $\{\omega.$ *proj-sol S* $\omega \in$ *rdb*$\}$
shows
w $\in$ (*map-of* $\circ$ *zip S*) ‘ *rdb*
⟨*proof*⟩

lemma *proj-proj-sol-map-of-zip*:
assumes *distinct S*
assumes $\bigwedge bs.$ *bs* $\in$ *rdb* $\Longrightarrow$ *length bs* = *length S*
shows
proj (*set S*) $\{\omega.$ *proj-sol S* $\omega \in$ *rdb*$\}$ =

(*map-of* ∘ *zip S*) ‘ *rdb*
⟨*proof*⟩

definition *ban-proj-sol* ::*'a list* ⇒ (*'a* ⇒ *bool*) *list* ⇒ *'fml* ⇒ *'fml*
where *ban-proj-sol S xs F* =
fold ban-sol (*map* (*zip S* ∘ *proj-sol S*) *xs*) *F*

lemma *check-sol-imp-proj*:
assumes *w* ∈ *sols F*
shows *map-of* (*zip S* (*proj-sol S w*)) ∈ *proj* (*set S*) (*sols F*)
⟨*proof*⟩

lemma *checked-BSAT-lower*:
assumes *S*: *distinct S*
assumes *check-BSAT-sols F S thresh xs* = *Inr* ()
shows *length xs* ≤ *card* (*proj* (*set S*) (*sols F*))
length xs < *thresh* ⟹
card (*proj* (*set S*) (*sols F*)) = *length xs*
⟨*proof*⟩

lemma *good-BSAT*:
assumes *distinct S*
assumes *BSAT F S thresh xs* = *Inr n*
shows *n* ≤ *card* (*proj* (*set S*) (*sols F*))
n < *thresh* ⟹
card (*proj* (*set S*) (*sols F*)) = *n*
⟨*proof*⟩

lemma *size-xorL-cert*:
assumes *distinct S*
assumes *size-xorL-cert F S thresh xorsl i xs* = *Inr n*
shows
size-xorL F S xorsl i ≥ *n*
n < *thresh* ⟶ *size-xorL F S xorsl i* = *n*
⟨*proof*⟩

lemma *approxcore-xorsL-cert*:
assumes *S*: *distinct S*
assumes *approxcore-xorsL-cert F S thresh mc xorsl* = *Inr n*
shows *approxcore-xorsL F S thresh xorsl* = *n*
⟨*proof*⟩

lemma *fold-approxcore-xorsL-cert*:
assumes *S*: *distinct S*
assumes *i* ≤ *t*
assumes *fold-approxcore-xorsL-cert F S thresh t i cs xorsLs* = *Inr ns*
shows *map* (*approxcore-xorsL F S thresh* ∘ *xorsLs*) [*t*−*i*..<*t*] = *ns*
⟨*proof*⟩

lemma *calc-median*:
assumes *S*: *distinct S*
assumes *calc-median F S thresh t ms xorsLs* = *Inr n*
shows *median t* (*approxcore-xorsL F S thresh* ∘ *xorsLs*) = *n*
⟨*proof*⟩

lemma *compute-t-find-t*[*simp*]:
shows *appmc.compute-t δ* (*find-t δ*) = *find-t δ*
⟨*proof*⟩

lemma *certcheck*:
assumes *distinct S*
assumes *certcheck F S ε δ* (*m0*,*ms*) *xorsLs* = *Inr n*
shows *approxmc-mapL F S ε δ* (*find-t δ*) *xorsLs* = *n*
⟨*proof*⟩

lemma *certcheck′*:
assumes *distinct S*
assumes ¬*isl* (*certcheck F S ε δ m xorsLs*)
shows *projr* (*certcheck F S ε δ m xorsLs*) =
approxmc-mapL F S ε δ (*find-t δ*) *xorsLs*
⟨*proof*⟩

lemma *certcheck-sound*:
assumes *δ*: *δ* > *0 δ* < *1*
assumes *ε*: *ε* > *0*
assumes *S*: *distinct S*
shows
measure-pmf.prob
(*map-pmf* (λ*r*. *certcheck F S ε δ* (*f r*) *r*)
(*random-seed-xors* (*find-t δ*) (*length S*)))
{*c*. ¬*isl c* ∧
real (*projr c*) ∉
{*real* (*card* (*proj* (*set S*) (*sols F*))) / (*1* + *ε*)..
(*1* + *ε*) ∗ *real* (*card* (*proj* (*set S*) (*sols F*)))}} ≤ *δ*
⟨*proof*⟩

lemma *certcheck-promise-complete*:
assumes *δ*: *δ* > *0 δ* < *1*
assumes *ε*: *ε* > *0*
assumes *S*: *distinct S*
assumes *r*: ⋀*r*.
r ∈ *set-pmf* (*random-seed-xors* (*find-t δ*) (*length S*)) ⟹
¬*isl* (*certcheck F S ε δ* (*f r*) *r*)
shows
measure-pmf.prob

```
    (map-pmf (λr. certcheck F S ε δ (f r) r)
      (random-seed-xors (find-t δ) (length S)))
    {c. real (projr c) ∈
      {real (card (proj (set S) (sols F))) / (1 + ε)..
        (1 + ε) * real (card (proj (set S) (sols F)))}} ≥ 1 − δ
⟨proof⟩

end

lemma certcheck-code[code]:
  certcheck F S ε δ (m0,ms) xorsLs = (
   if δ > 0 ∧ δ < 1 ∧ ε > 0 ∧ distinct S then
    (let ε = appmc.mk-eps ε in
    let thresh = appmc.compute-thresh ε in
    case BSAT F S thresh m0 of Inl err ⇒ Inl err
    | Inr Y ⇒
    if Y < thresh then Inr Y
    else
      let t = find-t δ in
      calc-median F S thresh t ms xorsLs)
   else Code.abort (STR ''invalid inputs'')
      (λ-. certcheck F S ε δ (m0,ms) xorsLs))
 ⟨proof⟩

end

end

end
```

8 ApproxMC certification for CNF-XOR

This concretely instantiates the locales with a syntax and semantics for CNF-XOR, giving us a certificate checker for approximate counting in this theory.

```
theory CertCheck-CNF-XOR imports
  ApproxMCAnalysis
  CertCheck
  HOL.String HOL−Library.Code-Target-Numeral
  Show.Show-Real
begin
```

This follows CryptoMiniSAT's CNF-XOR formula syntax. A clause is a list of literals (one of which must be satisfied). An XOR constraint has the form $l_1 + l_2 + \ldots + l_n = 1$ where addition is taken over F_2. Syntactically, they are specified by the list of LHS literals. Variables are natural numbers (in practice, variable

0 is never used)

datatype *lit* = *Pos nat* | *Neg nat*
type-synonym *clause* = *lit list*
type-synonym *cmsxor* = *lit list*
type-synonym *fml* = *clause list* × *cmsxor list*

type-synonym *assignment* = *nat* ⇒ *bool*

definition *sat-lit* :: *assignment* ⇒ *lit* ⇒ *bool* **where**
sat-lit w l = (*case l of Pos x* ⇒ *w x* | *Neg x* ⇒ ¬*w x*)

definition *sat-clause* :: *assignment* ⇒ *clause* ⇒ *bool* **where**
sat-clause w C = (∃ *l* ∈ *set C*. *sat-lit w l*)

definition *sat-cmsxor* :: *assignment* ⇒ *cmsxor* ⇒ *bool* **where**
sat-cmsxor w C = *odd* ((*sum-list* (*map* (*of-bool* ∘ (*sat-lit w*)) *C*))::*nat*)

definition *sat-fml* :: *assignment* ⇒ *fml* ⇒ *bool*
where
sat-fml w f = (
(∀ *C* ∈ *set* (*fst f*). *sat-clause w C*) ∧
(∀ *C* ∈ *set* (*snd f*). *sat-cmsxor w C*))

definition *sols* :: *fml* ⇒ *assignment set*
where *sols f* = {*w*. *sat-fml w f*}

lemma *sat-fml-cons*[*simp*]:
shows
sat-fml w (*FC*, *x* # *FX*) ⟷
sat-fml w (*FC*,*FX*) ∧ *sat-cmsxor w x*
sat-fml w (*c* # *FC*, *FX*) ⟷
sat-fml w (*FC*,*FX*) ∧ *sat-clause w c*
⟨*proof*⟩

fun *enc-xor* :: *nat xor* ⇒ *fml* ⇒ *fml*
where
enc-xor (*x*,*b*) (*FC*,*FX*) = (
if b then (*FC*, *map Pos x* # *FX*)
else
case x of
[] ⇒ (*FC*,*FX*)
| (*v*#*vs*) ⇒ (*FC*, (*Neg v* # *map Pos vs*) # *FX*))

lemma *sols-enc-xor*:
shows *sols* (*enc-xor* (*x*,*b*) (*FC*,*FX*)) =
sols (*FC*,*FX*) ∩ {*ω*. *satisfies-xorL* (*x*,*b*) *ω*}
⟨*proof*⟩

definition *check-sol* :: *fml* ⇒ (*nat* ⇒ *bool*) ⇒ *bool*
where *check-sol fml w* = (
list-all (*list-ex* (*sat-lit w*)) (*fst fml*) ∧
list-all (*sat-cmsxor w*) (*snd fml*))

definition *ban-sol* :: (*nat* × *bool*) *list* ⇒ *fml* ⇒ *fml*
where *ban-sol vs fml* =
((*map* (λ(*v*,*b*). *if b then Neg v else Pos v*) *vs*)#*fst fml*, *snd fml*)

lemma *check-sol-sol*:
shows *w* ∈ *sols F* ⟷
check-sol F w
⟨*proof*⟩

lemma *ban-sat-clause*:
shows *sat-clause w* (*map* (λ(*v*, *b*). *if b then Neg v else Pos v*) *vs*)
⟷
map w (*map fst vs*) ≠ *map snd vs*
⟨*proof*⟩

lemma *sols-ban-sol*:
shows*sols* (*ban-sol vs F*) =
sols F ∩
{ω. *map* ω (*map fst vs*) ≠ *map snd vs*}
⟨*proof*⟩

global-interpretation *CertCheck-CNF-XOR* :
CertCheck sols enc-xor check-sol ban-sol
defines
random-seed-xors = *CertCheck-CNF-XOR.random-seed-xors* **and**
fix-t = *CertCheck-CNF-XOR.appmc.fix-t* **and**
find-t = *CertCheck-CNF-XOR.find-t* **and**
BSAT = *CertCheck-CNF-XOR.BSAT* **and**
check-BSAT-sols = *CertCheck-CNF-XOR.check-BSAT-sols* **and**
size-xorL-cert = *CertCheck-CNF-XOR.size-xorL-cert* **and**
approxcore-xorsL = *CertCheck-CNF-XOR.approxcore-xorsL* **and**
fold-approxcore-xorsL-cert = *CertCheck-CNF-XOR.fold-approxcore-xorsL-cert*
and
approxcore-xorsL-cert = *CertCheck-CNF-XOR.approxcore-xorsL-cert*
and
calc-median = *CertCheck-CNF-XOR.calc-median* **and**
certcheck = *CertCheck-CNF-XOR.certcheck*
⟨*proof*⟩

8.1 Blasting XOR constraints to CNF

This formalizes the usual linear conversion from CNF-XOR into CNF. It is not necessary to use this conversion for solvers that support CNF-XOR formulas natively.

definition *negate-lit* :: *lit* ⇒ *lit*
where *negate-lit l* = (*case l of Pos x* ⇒ *Neg x* | *Neg x* ⇒ *Pos x*)

fun *xor-clauses* :: *cmsxor* ⇒ *bool* ⇒ *clause list*
where
xor-clauses [] *b* = (*if b then* [[]] *else* [])
| *xor-clauses* (*x*#*xs*) *b* =
(*let p-x* = *xor-clauses xs b in*
let n-x = *xor-clauses xs* (¬*b*) *in*
map (λ*c*. *x* # *c*) *p-x* @ *map* (λ*c*. *negate-lit x* # *c*) *n-x*)

lemma *sat-cmsxor-nil*[*simp*]:
shows¬ (*sat-cmsxor w* [])
⟨*proof*⟩

lemma *sat-cmsxor-cons*:
shows *sat-cmsxor w* (*x* # *xs*) =
(*if sat-lit w x then* ¬ (*sat-cmsxor w xs*) *else sat-cmsxor w xs*)
⟨*proof*⟩

lemma *sat-cmsxor-append*:
shows *sat-cmsxor w* (*xs* @ *ys*) =
(*if sat-cmsxor w xs then* ¬ (*sat-cmsxor w ys*) *else sat-cmsxor w ys*)
⟨*proof*⟩

definition *sat-clauses*:: *assignment* ⇒ *clause list* ⇒ *bool*
where *sat-clauses w cs* = (∀ *c* ∈ *set cs*. *sat-clause w c*)

lemma *sat-clauses-append*:
shows *sat-clauses w* (*xs* @ *ys*) =
(*sat-clauses w xs* ∧ *sat-clauses w ys*)
⟨*proof*⟩

lemma *sat-clauses-map*:
shows *sat-clauses w* (*map* ((#) *x*) *cs*) =
(*sat-lit w x* ∨ *sat-clauses w cs*)
⟨*proof*⟩

lemma *sat-lit-negate-lit*[*simp*]:
sat-lit w (*negate-lit l*) = (¬*sat-lit w l*)
⟨*proof*⟩

lemma *sols-xor-clauses*:

shows
sat-clauses w (xor-clauses xs b) $\longleftrightarrow$
(sat-cmsxor w xs = b)
⟨*proof*⟩

definition *var-lit* :: *lit* ⇒ *nat*
where *var-lit l* = (*case l of Pos x* ⇒ *x* | *Neg x* ⇒ *x*)

definition *var-lits* :: *lit list* ⇒ *nat*
where *var-lits ls* = *fold max* (*map var-lit ls*) *0*

lemma *sat-lit-same*:
assumes $\bigwedge x.\ x \leq$ *var-lit l* $\Longrightarrow$ *w x* = *w′ x*
shows *sat-lit w l* = *sat-lit w′ l*
⟨*proof*⟩

lemma *var-lits-eq*:
var-lits ls = *Max* (*set* (*0* # *map var-lit ls*))
⟨*proof*⟩

lemma *sat-lits-same*:
assumes $\bigwedge x.\ x \leq$ *var-lits c* $\Longrightarrow$ *w x* = *w′ x*
shows *sat-clause w c* = *sat-clause w′ c*
⟨*proof*⟩

lemma *le-var-lits-in*:
assumes *y* ∈ *set ys v* ≤ *var-lit y*
shows *v* ≤ *var-lits ys*
⟨*proof*⟩

lemma *sat-cmsxor-same*:
assumes $\bigwedge x.\ x \leq$ *var-lits xs* $\Longrightarrow$ *w x* = *w′ x*
shows *sat-cmsxor w xs* = *sat-cmsxor w′ xs*
⟨*proof*⟩

lemma *sat-cmsxor-split*:
assumes *u*: *var-lits xs* < *u var-lits ys* < *u*
assumes *w′*: *w′* = (λ*x*. *if x* = *u then* ¬ *sat-cmsxor w xs else w x*)
shows
(*sat-cmsxor w* (*xs* @ *ys*) =
(*sat-cmsxor w′* (*Pos u* # *xs*) ∧
sat-cmsxor w′ (*Neg u* # *ys*)))
⟨*proof*⟩

fun *split-xor* ::*nat* ⇒ *cmsxor* ⇒ *cmsxor list* × *nat* ⇒ *cmsxor list* × *nat*

where *split-xor k xs* (*acc*,*u*) = (
if length xs ≤ *k* + *3 then* (*xs* # *acc*, *u*)
else (
let xs1 = *take* (*k* + *2*) *xs in*
let xs2 = *drop* (*k* + *2*) *xs in*
split-xor k (*Neg u* # *xs2*) ((*Pos u* # *xs1*) # *acc*, *u*+*1*)
)
)

declare *split-xor.simps*[*simp del*]

lemma *split-xor-bound*:
assumes *split-xor k xs* (*acc*,*u*) = (*acc*′,*u*′)
shows *u* ≤ *u*′
⟨*proof*⟩

lemma *var-lits-append*:
shows *var-lits xs* ≤ *var-lits* (*xs* @ *ys*)
var-lits ys ≤ *var-lits* (*xs* @ *ys*)
⟨*proof*⟩

lemma *fold-max-eq*:
assumes *i* ≤ *u*
shows *fold max ls u* = *max u* (*fold max ls* (*i*::*nat*))
⟨*proof*⟩

lemma *split-xor-sound*:
assumes *sat-cmsxor w xs* ⋀*x*. *x* ∈ *set acc* ⟹ *sat-cmsxor w x*
assumes *u*: *var-lits xs* < *u* ⋀*x*. *x* ∈ *set acc* ⟹ *var-lits x* < *u*
assumes *split-xor k xs* (*acc*,*u*) = (*acc*′,*u*′)
obtains *w*′ **where**
⋀*x*. *x* < *u* ⟹ *w x* = *w*′ *x*
⋀*x*. *x* ∈ *set acc*′ ⟹ *sat-cmsxor w*′ *x*
⋀*x*. *x* ∈ *set acc*′ ⟹ *var-lits x* < *u*′
⟨*proof*⟩

definition *split-xors* ::*nat* ⇒ *nat* ⇒ *cmsxor list* ⇒ *cmsxor list*
where *split-xors k u xs* = *fst* (*fold* (*split-xor k*) *xs* ([],*u*))

lemma *split-xors-sound*:
assumes ⋀*x*. *x* ∈ *set xs* ⟹ *sat-cmsxor w x*
⋀*x*. *x* ∈ *set acc* ⟹ *sat-cmsxor w x*
assumes *u*: ⋀*x*. *x* ∈ *set xs* ⟹ *var-lits x* < *u*
⋀*x*. *x* ∈ *set acc* ⟹ *var-lits x* < *u*
assumes *fold* (*split-xor k*) *xs* (*acc*,*u*) = (*acc*′,*u*′)
obtains *w*′ **where**
⋀*x*. *x* < *u* ⟹ *w x* = *w*′ *x*
⋀*x*. *x* ∈ *set acc*′ ⟹ *sat-cmsxor w*′ *x*

$\bigwedge x.\ x \in set\ acc' \Longrightarrow var\text{-}lits\ x < u'$
⟨*proof*⟩

definition *var-fml* :: *fml* ⇒ *nat*
where *var-fml f* =
max (*fold max* (*map var-lits* (*fst f*)) *0*)
(*fold max* (*map var-lits* (*snd f*)) *0*)

lemma *var-fml-eq*:
var-fml f =
max (*Max* (*set* (*0* # *map var-lits* (*fst f*))))
(*Max* (*set* (*0* # *map var-lits* (*snd f*))))
⟨*proof*⟩

definition *split-fml* :: *nat* ⇒ *fml* ⇒ *fml*
where *split-fml k f* = (
let u = *var-fml f* + *1 in*
(*fst f*, (*split-xors k u* (*snd f*)))
)

lemma *var-lits-var-fml*:
shows $\bigwedge x.\ x \in set\ (snd\ F) \Longrightarrow var\text{-}lits\ x \leq var\text{-}fml\ F$
$\bigwedge x.\ x \in set\ (fst\ F) \Longrightarrow var\text{-}lits\ x \leq var\text{-}fml\ F$
⟨*proof*⟩

lemma *split-fml-satisfies*:
assumes *sat-fml w F*
obtains *w'* **where** *sat-fml w'* (*split-fml k F*)
⟨*proof*⟩

lemma *split-fml-sols*:
assumes *sols* (*split-fml k F*) = {}
shows *sols F* = {}
⟨*proof*⟩

definition *blast-xors* :: *cmsxor list* ⇒ *clause list*
where *blast-xors xors* = *concat* (*map* (λ*x*. *xor-clauses x True*) *xors*)

definition *blast-fml* :: *fml* ⇒ *clause list*
where *blast-fml f* =
fst f @ *blast-xors* (*snd f*)

lemma *sat-clauses-concat*:
sat-clauses w (*concat xs*) ⟷
(∀ *x* ∈ *set xs*. *sat-clauses w x*)
⟨*proof*⟩

lemma *blast-xors-sound*:

assumes $(\bigwedge x.\ x \in set\ xors \Longrightarrow sat\text{-}cmsxor\ w\ x)$
shows $sat\text{-}clauses\ w\ (blast\text{-}xors\ xors)$
⟨*proof*⟩

lemma *blast-fml-sound*:
assumes $sat\text{-}fml\ w\ F$
shows $sat\text{-}fml\ w\ (blast\text{-}fml\ F,[])$
⟨*proof*⟩

definition $blast\text{-}split\text{-}fml :: fml \Rightarrow clause\ list$
where $blast\text{-}split\text{-}fml\ f = blast\text{-}fml\ (split\text{-}fml\ 1\ f)$

lemma *blast-split-fml-sols*:
assumes $sols\ (blast\text{-}split\text{-}fml\ F,[]) = \{\}$
shows $sols\ F = \{\}$
⟨*proof*⟩

definition *certcheck-blast*::
$(clause\ list \Rightarrow bool) \Rightarrow$
$fml \Rightarrow nat\ list \Rightarrow$
$real \Rightarrow real \Rightarrow$
$((nat \Rightarrow bool)\ list \times$
$(nat \Rightarrow (nat \times (nat \Rightarrow bool)\ list \times (nat \Rightarrow bool)\ list))) \Rightarrow$
$(nat \Rightarrow nat \Rightarrow (bool\ list \times bool)) \Rightarrow$
$String.literal + nat$
where $certcheck\text{-}blast\ check\text{-}unsat\ F\ S\ \varepsilon\ \delta\ m0ms =$
$certcheck\ (check\text{-}unsat \circ blast\text{-}split\text{-}fml)\ F\ S\ \varepsilon\ \delta\ m0ms$

corollary *certcheck-blast-sound*:
assumes $\bigwedge F.\ check\text{-}unsat\ F \Longrightarrow sols\ (F,\ []) = \{\}$
assumes $0 < \delta\ \delta < 1$
assumes $0 < \varepsilon$
assumes $distinct\ S$
shows
$measure\text{-}pmf.prob$
$(map\text{-}pmf\ (\lambda r.\ certcheck\text{-}blast\ check\text{-}unsat\ F\ S\ \varepsilon\ \delta\ (f\ r)\ r)$
$(random\text{-}seed\text{-}xors\ (find\text{-}t\ \delta)\ (length\ S)))$
$\{c.\ \neg\ isl\ c \wedge$
$real\ (projr\ c) \notin$
$\{real\ (card\ (proj\ (set\ S)\ (sols\ F))) / (1 + \varepsilon)..$
$(1 + \varepsilon) * real\ (card\ (proj\ (set\ S)\ (sols\ F)))\}\} \leq \delta$
⟨*proof*⟩

corollary *certcheck-blast-promise-complete*:
assumes $\bigwedge F.\ check\text{-}unsat\ F \Longrightarrow sols\ (F,\ []) = \{\}$
assumes $0 < \delta\ \delta < 1$
assumes $0 < \varepsilon$
assumes $distinct\ S$

assumes r: $\bigwedge r$.
$r \in$ *set-pmf* (*random-seed-xors* (*find-t* δ) (*length* S)) $\Longrightarrow$
$\neg$*isl* (*certcheck-blast check-unsat* F S ε δ (f r) r)
shows
measure-pmf.prob
(*map-pmf* (λr. *certcheck-blast check-unsat* F S ε δ (f r) r)
(*random-seed-xors* (*find-t* δ) (*length* S)))
{c. *real* (*projr* c) $\in$
{*real* (*card* (*proj* (*set* S) (*sols* F))) / (1 + ε)..
(1 + ε) $*$ *real* (*card* (*proj* (*set* S) (*sols* F)))}} $\geq$ 1 $-$ δ
⟨*proof*⟩

8.2 Export code for a SML implementation.

definition *real-of-int* :: *integer* $\Rightarrow$ *real*
where *real-of-int* n = *real* (*nat-of-integer* n)

definition *real-mult* :: *real* $\Rightarrow$ *real* $\Rightarrow$ *real*
where *real-mult* n m = $n * m$

definition *real-div* :: *real* $\Rightarrow$ *real* $\Rightarrow$ *real*
where *real-div* n m = n / m

definition *real-plus* :: *real* $\Rightarrow$ *real* $\Rightarrow$ *real*
where *real-plus* n m = n + m

definition *real-minus* :: *real* $\Rightarrow$ *real* $\Rightarrow$ *real*
where *real-minus* n m = n $-$ m

declare [[*code abort*: *fix-t*]]

export-code
length
nat-of-integer int-of-integer
integer-of-nat integer-of-int
real-of-int real-mult real-div real-plus real-minus
quotient-of

Pos Neg
CertCheck-CNF-XOR.appmc.compute-thresh
find-t certcheck
certcheck-blast
in *SML*

end

References

[1] S. Chakraborty, K. S. Meel, and M. Y. Vardi. Algorithmic improvements in approximate counting for probabilistic inference: From linear to logarithmic SAT calls. In S. Kambhampati, editor, *IJCAI*, pages 3569–3576. IJCAI/AAAI Press, 2016.