

The Akra–Bazzi theorem and the Master theorem

Manuel Eberl

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Abstract

This article contains a formalisation of the Akra–Bazzi method [1] based on a proof by Leighton [2]. It is a generalisation of the well-known Master Theorem for analysing the complexity of Divide & Conquer algorithms. We also include a generalised version of the Master theorem based on the Akra–Bazzi theorem, which is easier to apply than the Akra–Bazzi theorem itself.

Some proof methods that facilitate applying the Master theorem are also included. For a more detailed explanation of the formalisation and the proof methods, see the accompanying paper (publication forthcoming).

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1 Auxiliary lemmas

theory *Akra-Bazzi-Library*

imports

Complex-Main

Landau-Symbols.Landau-More

Pure-ex.Guess

begin

lemma *ln-mono*: $0 < x \implies 0 < y \implies x \leq y \implies \ln (x::real) \leq \ln y$
 <proof>

lemma *ln-mono-strict*: $0 < x \implies 0 < y \implies x < y \implies \ln (x::real) < \ln y$
 <proof>

declare *DERIV-pow*[*THEN DERIV-chain2, derivative-intros*]

lemma *sum-pos'*:

assumes *finite I*

assumes $\exists x \in I. f x > (0 :: - :: \text{linordered-ab-group-add})$

assumes $\bigwedge x. x \in I \implies f x \geq 0$

shows $\text{sum } f I > 0$

<proof>

lemma *min-mult-left*:

assumes $(x::real) > 0$

shows $x * \text{min } y z = \text{min } (x*y) (x*z)$

<proof>

lemma *max-mult-left*:

assumes $(x::real) > 0$

shows $x * \text{max } y z = \text{max } (x*y) (x*z)$

<proof>

lemma *DERIV-nonneg-imp-mono*:

assumes $\bigwedge t. t \in \{x..y\} \implies (f \text{ has-field-derivative } f' t) (at t)$

assumes $\bigwedge t. t \in \{x..y\} \implies f' t \geq 0$

assumes $(x::real) \leq y$

shows $(f x :: real) \leq f y$

<proof>

lemma *eventually-conjE*: *eventually* $(\lambda x. P x \wedge Q x) F \implies (\text{eventually } P F \implies \text{eventually } Q F \implies R) \implies R$
<proof>

lemma *real-natfloor-nat*: $x \in \mathbf{N} \implies \text{real } (\text{nat } \lfloor x \rfloor) = x$ *<proof>*

lemma *eventually-natfloor*:
assumes *eventually* P (*at-top* :: *nat filter*)
shows *eventually* $(\lambda x. P (\text{nat } \lfloor x \rfloor))$ (*at-top* :: *real filter*)
<proof>

lemma *tendsto-0-smallo-1*: $f \in o(\lambda x. 1 :: \text{real}) \implies (f \longrightarrow 0)$ *at-top*
<proof>

lemma *smallo-1-tendsto-0*: $(f \longrightarrow 0)$ *at-top* $\implies f \in o(\lambda x. 1 :: \text{real})$
<proof>

lemma *filterlim-at-top-smallomega-1*:
assumes $f \in \omega[F](\lambda x. 1 :: \text{real})$ *eventually* $(\lambda x. f x > 0)$ F
shows *filterlim* f *at-top* F
<proof>

lemma *smallo-imp-abs-less-real*:
assumes $f \in o[F](g)$ *eventually* $(\lambda x. g x > (0 :: \text{real}))$ F
shows *eventually* $(\lambda x. |f x| < g x)$ F
<proof>

lemma *smallo-imp-less-real*:
assumes $f \in o[F](g)$ *eventually* $(\lambda x. g x > (0 :: \text{real}))$ F
shows *eventually* $(\lambda x. f x < g x)$ F
<proof>

lemma *smallo-imp-le-real*:
assumes $f \in o[F](g)$ *eventually* $(\lambda x. g x \geq (0 :: \text{real}))$ F
shows *eventually* $(\lambda x. f x \leq g x)$ F
<proof>

lemma *filterlim-at-right*:
filterlim f (*at-right* a) $F \iff \text{eventually } (\lambda x. f x > a) F \wedge \text{filterlim } f$ (*nhds* a) F
<proof>

lemma *one-plus-x-powr-approx-ex*:
assumes $x: \text{abs } (x :: \text{real}) \leq 1/2$
obtains t **where** $\text{abs } t < 1/2$ $(1 + x)$ *powr* $p =$
 $1 + p * x + p * (p - 1) * (1 + t)$ *powr* $(p - 2) / 2 * x ^ 2$

<proof>

lemma *powr-lower-bound*: $\llbracket (l::\text{real}) > 0; l \leq x; x \leq u \rrbracket \implies \min (l \text{ powr } z) (u \text{ powr } z) \leq x \text{ powr } z$
<proof>

lemma *powr-upper-bound*: $\llbracket (l::\text{real}) > 0; l \leq x; x \leq u \rrbracket \implies \max (l \text{ powr } z) (u \text{ powr } z) \geq x \text{ powr } z$
<proof>

lemma *one-plus-x-powr-Taylor2*:

obtains *k* **where** $\bigwedge x. \text{abs } (x::\text{real}) \leq 1/2 \implies \text{abs } ((1 + x) \text{ powr } p - 1 - p*x) \leq k*x^2$
<proof>

lemma *one-plus-x-powr-Taylor2-bigo*:

assumes *lim*: $(f \longrightarrow 0) F$
shows $(\lambda x. (1 + f x) \text{ powr } (p::\text{real}) - 1 - p * f x) \in O[F](\lambda x. f x^2)$
<proof>

lemma *one-plus-x-powr-Taylor1-bigo*:

assumes *lim*: $(f \longrightarrow 0) F$
shows $(\lambda x. (1 + f x) \text{ powr } (p::\text{real}) - 1) \in O[F](\lambda x. f x)$
<proof>

lemma *x-times-x-minus-1-nonneg*: $x \leq 0 \vee x \geq 1 \implies (x:::\text{linordered-idom}) * (x - 1) \geq 0$
<proof>

lemma *x-times-x-minus-1-nonpos*: $x \geq 0 \implies x \leq 1 \implies (x:::\text{linordered-idom}) * (x - 1) \leq 0$
<proof>

lemma *real-powr-at-bot*:

assumes $(a::\text{real}) > 1$
shows $((\lambda x. a \text{ powr } x) \longrightarrow 0) \text{ at-bot}$
<proof>

lemma *real-powr-at-bot-neg*:

assumes $(a::\text{real}) > 0 \ a < 1$
shows *filterlim* $(\lambda x. a \text{ powr } x) \text{ at-top at-bot}$
<proof>

lemma *real-powr-at-top-neg*:

assumes $(a::\text{real}) > 0 \ a < 1$
shows $((\lambda x. a \text{ powr } x) \longrightarrow 0) \text{ at-top}$
<proof>

lemma *eventually-nat-real*:

```

assumes eventually  $P$  (at-top :: real filter)
shows eventually  $(\lambda x. P (\text{real } x))$  (at-top :: nat filter)
  <proof>

```

end

2 Asymptotic bounds

theory *Akra-Bazzi-Asymptotics*

imports

Complex-Main

Akra-Bazzi-Library

HOL-Library.Landau-Symbols

begin

locale *akra-bazzi-asymptotics-bep* =

fixes $b\ e\ p\ hb$:: *real*

assumes *bep*: $b > 0\ b < 1\ e > 0\ hb > 0$

begin

context

begin

Functions that are negligible w.r.t. $\ln (b * x)\ \text{powr } (e / 2 + 1)$.

private abbreviation (*input*) *negl* :: (*real* \Rightarrow *real*) \Rightarrow *bool* **where**

negl $f \equiv f \in o(\lambda x. \ln (b*x)\ \text{powr } (-(e/2 + 1)))$

private lemma *neglD*: $\text{negl } f \Longrightarrow c > 0 \Longrightarrow \text{eventually } (\lambda x. |f\ x| \leq c / \ln (b*x)\ \text{powr } (e/2+1))\ \text{at-top}$

<proof> **lemma** *negl-mult*: $\text{negl } f \Longrightarrow \text{negl } g \Longrightarrow \text{negl } (\lambda x. f\ x * g\ x)$

<proof> **lemma** *ev4*:

assumes *g*: *negl* *g*

shows eventually $(\lambda x. \ln (b*x)\ \text{powr } (-e/2) - \ln x\ \text{powr } (-e/2) \geq g\ x)\ \text{at-top}$

<proof> **lemma** *ev1*:

negl $(\lambda x. (1 + c * \text{inverse } b * \ln x\ \text{powr } (-(1+e)))\ \text{powr } p - 1)$

<proof> **lemma** *ev2-aux*:

defines $f \equiv \lambda x. (1 + 1/\ln (b*x) * \ln (1 + hb / b * \ln x\ \text{powr } (-1-e)))\ \text{powr } (-e/2)$

obtains *h* **where** eventually $(\lambda x. f\ x \geq 1 + h\ x)\ \text{at-top } h \in o(\lambda x. 1 / \ln x)$

<proof> **lemma** *ev2*:

defines $f \equiv \lambda x. \ln (b * x + hb * x / \ln x\ \text{powr } (1 + e))\ \text{powr } (-e/2)$

obtains *h* **where**

negl *h*

eventually $(\lambda x. f\ x \geq \ln (b * x)\ \text{powr } (-e/2) + h\ x)\ \text{at-top}$

eventually $(\lambda x. |\ln (b * x)\ \text{powr } (-e/2) + h\ x| < 1)\ \text{at-top}$

<proof> **lemma** *ev21*:

obtains *g* **where**

negl *g*

eventually $(\lambda x. 1 + \ln (b * x + hb * x / \ln x\ \text{powr } (1 + e))\ \text{powr } (-e/2) \geq$

$1 + \ln (b * x) \text{ powr } (-e/2) + g x) \text{ at-top}$
 $\text{eventually } (\lambda x. 1 + \ln (b * x) \text{ powr } (-e/2) + g x > 0) \text{ at-top}$
 <proof> **lemma ev22:**
obtains g where
 $\text{negl } g$
 $\text{eventually } (\lambda x. 1 - \ln (b * x + hb * x / \ln x \text{ powr } (1 + e)) \text{ powr } (-e/2) \leq$
 $1 - \ln (b * x) \text{ powr } (-e/2) - g x) \text{ at-top}$
 $\text{eventually } (\lambda x. 1 - \ln (b * x) \text{ powr } (-e/2) - g x > 0) \text{ at-top}$
 <proof>

lemma asymptotics1:
shows eventually $(\lambda x.$
 $(1 + c * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p *$
 $(1 + \ln (b * x + hb * x / \ln x \text{ powr } (1 + e)) \text{ powr } (- e / 2)) \geq$
 $1 + (\ln x \text{ powr } (-e/2))) \text{ at-top}$
 <proof>

lemma asymptotics2:
shows eventually $(\lambda x.$
 $(1 + c * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p *$
 $(1 - \ln (b * x + hb * x / \ln x \text{ powr } (1 + e)) \text{ powr } (- e / 2)) \leq$
 $1 - (\ln x \text{ powr } (-e/2))) \text{ at-top}$
 <proof>

lemma asymptotics3: $\text{eventually } (\lambda x. (1 + (\ln x \text{ powr } (-e/2))) / 2 \leq 1) \text{ at-top}$
 (is eventually $(\lambda x. ?f x \leq 1) -$)
 <proof>

lemma asymptotics4: $\text{eventually } (\lambda x. (1 - (\ln x \text{ powr } (-e/2))) * 2 \geq 1) \text{ at-top}$
 (is eventually $(\lambda x. ?f x \geq 1) -$)
 <proof>

lemma asymptotics5: $\text{eventually } (\lambda x. \ln (b*x - hb*x*\ln x \text{ powr } -(1+e)) \text{ powr } (-e/2) < 1) \text{ at-top}$
 <proof>

lemma asymptotics6: $\text{eventually } (\lambda x. hb / \ln x \text{ powr } (1 + e) < b/2) \text{ at-top}$
and asymptotics7: $\text{eventually } (\lambda x. hb / \ln x \text{ powr } (1 + e) < (1 - b) / 2) \text{ at-top}$
and asymptotics8: $\text{eventually } (\lambda x. x*(1 - b - hb / \ln x \text{ powr } (1 + e)) > 1)$
 at-top
 <proof>

end
end

definition akra-bazzi-asymptotic1 $b \ hb \ e \ p \ x \longleftrightarrow$
 $(1 - hb * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 + \ln (b*x + hb*x/\ln x$

$\text{powr } (1+e) \text{ powr } (-e/2)$
 $\geq 1 + (\ln x \text{ powr } (-e/2) :: \text{real})$
definition *akra-bazzi-asymptotic1'* $b \text{ hb } e \text{ p } x \longleftrightarrow$
 $(1 + \text{hb} * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 + \ln (b*x + \text{hb}*x/\ln x$
 $\text{powr } (1+e) \text{ powr } (-e/2))$
 $\geq 1 + (\ln x \text{ powr } (-e/2) :: \text{real})$
definition *akra-bazzi-asymptotic2* $b \text{ hb } e \text{ p } x \longleftrightarrow$
 $(1 + \text{hb} * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 - \ln (b*x + \text{hb}*x/\ln x$
 $\text{powr } (1+e) \text{ powr } (-e/2))$
 $\leq 1 - \ln x \text{ powr } (-e/2 :: \text{real})$
definition *akra-bazzi-asymptotic2'* $b \text{ hb } e \text{ p } x \longleftrightarrow$
 $(1 - \text{hb} * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 - \ln (b*x + \text{hb}*x/\ln x$
 $\text{powr } (1+e) \text{ powr } (-e/2))$
 $\leq 1 - \ln x \text{ powr } (-e/2 :: \text{real})$
definition *akra-bazzi-asymptotic3* $e \text{ x} \longleftrightarrow (1 + (\ln x \text{ powr } (-e/2))) / 2 \leq$
 $(1 :: \text{real})$
definition *akra-bazzi-asymptotic4* $e \text{ x} \longleftrightarrow (1 - (\ln x \text{ powr } (-e/2))) * 2 \geq$
 $(1 :: \text{real})$
definition *akra-bazzi-asymptotic5* $b \text{ hb } e \text{ x} \longleftrightarrow$
 $\ln (b*x - \text{hb}*x*\ln x \text{ powr } -(1+e)) \text{ powr } (-e/2 :: \text{real}) < 1$

definition *akra-bazzi-asymptotic6* $b \text{ hb } e \text{ x} \longleftrightarrow \text{hb} / \ln x \text{ powr } (1 + e :: \text{real}) <$
 $b/2$
definition *akra-bazzi-asymptotic7* $b \text{ hb } e \text{ x} \longleftrightarrow \text{hb} / \ln x \text{ powr } (1 + e :: \text{real}) <$
 $(1 - b) / 2$
definition *akra-bazzi-asymptotic8* $b \text{ hb } e \text{ x} \longleftrightarrow x*(1 - b - \text{hb} / \ln x \text{ powr } (1 +$
 $e :: \text{real})) > 1$

definition *akra-bazzi-asymptotics* $b \text{ hb } e \text{ p } x \longleftrightarrow$
 $\text{akra-bazzi-asymptotic1 } b \text{ hb } e \text{ p } x \wedge \text{akra-bazzi-asymptotic1' } b \text{ hb } e \text{ p } x \wedge$
 $\text{akra-bazzi-asymptotic2 } b \text{ hb } e \text{ p } x \wedge \text{akra-bazzi-asymptotic2' } b \text{ hb } e \text{ p } x \wedge$
 $\text{akra-bazzi-asymptotic3 } e \text{ x} \wedge \text{akra-bazzi-asymptotic4 } e \text{ x} \wedge \text{akra-bazzi-asymptotic5}$
 $b \text{ hb } e \text{ x} \wedge$
 $\text{akra-bazzi-asymptotic6 } b \text{ hb } e \text{ x} \wedge \text{akra-bazzi-asymptotic7 } b \text{ hb } e \text{ x} \wedge$
 $\text{akra-bazzi-asymptotic8 } b \text{ hb } e \text{ x}$

lemmas *akra-bazzi-asymptotic-defs* =
 $\text{akra-bazzi-asymptotic1-def } \text{akra-bazzi-asymptotic1'-def}$
 $\text{akra-bazzi-asymptotic2-def } \text{akra-bazzi-asymptotic2'-def } \text{akra-bazzi-asymptotic3-def}$
 $\text{akra-bazzi-asymptotic4-def } \text{akra-bazzi-asymptotic5-def } \text{akra-bazzi-asymptotic6-def}$
 $\text{akra-bazzi-asymptotic7-def } \text{akra-bazzi-asymptotic8-def } \text{akra-bazzi-asymptotics-def}$

lemma *akra-bazzi-asymptotics*:
assumes $\bigwedge b. b \in \text{set } bs \implies b \in \{0 < .. < 1\}$
assumes $\text{hb} > 0 \text{ e} > 0$
shows *eventually* $(\lambda x. \forall b \in \text{set } bs. \text{akra-bazzi-asymptotics } b \text{ hb } e \text{ p } x)$ *at-top*
<proof>

end

3 The continuous Akra-Bazzi theorem

```

theory Akra-Bazzi-Real
imports
  Complex-Main
  Akra-Bazzi-Asymptotics
begin

```

We want to be generic over the integral definition used; we fix some arbitrary notions of integrability and integral and assume just the properties we need. The user can then instantiate the theorems with any desired integral definition.

```

locale akra-bazzi-integral =
  fixes integrable :: (real  $\Rightarrow$  real)  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  bool
  and integral :: (real  $\Rightarrow$  real)  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  real
  assumes integrable-const:  $c \geq 0 \implies \text{integrable } (\lambda-. c) a b$ 
  and integral-const:  $c \geq 0 \implies a \leq b \implies \text{integral } (\lambda-. c) a b = (b - a) * c$ 
  and integrable-subinterval:
     $\text{integrable } f a b \implies a \leq a' \implies b' \leq b \implies \text{integrable } f a' b'$ 
  and integral-le:
     $\text{integrable } f a b \implies \text{integrable } g a b \implies (\bigwedge x. x \in \{a..b\} \implies f x \leq g x)$ 
 $\implies$ 
     $\text{integral } f a b \leq \text{integral } g a b$ 
  and integral-combine:
     $a \leq c \implies c \leq b \implies \text{integrable } f a b \implies$ 
     $\text{integral } f a c + \text{integral } f c b = \text{integral } f a b$ 
begin
lemma integral-nonneg:
   $a \leq b \implies \text{integrable } f a b \implies (\bigwedge x. x \in \{a..b\} \implies f x \geq 0) \implies \text{integral } f a b \geq 0$ 
   $\langle \text{proof} \rangle$ 
end

```

```

declare sum.cong[fundef-cong]

```

```

lemma strict-mono-imp-ex1-real:
  fixes f :: real  $\Rightarrow$  real
  assumes lim-neg-inf: LIM  $x$  at-bot.  $f x \text{ :> at-top}$ 
  assumes lim-inf:  $(f \longrightarrow z)$  at-top
  assumes mono:  $\bigwedge a b. a < b \implies f b < f a$ 
  assumes cont:  $\bigwedge x. \text{isCont } f x$ 
  assumes y-greater-z:  $z < y$ 
  shows  $\exists !x. f x = y$ 
   $\langle \text{proof} \rangle$ 

```

The parameter p in the Akra-Bazzi theorem always exists and is unique.

```

definition akra-bazzi-exponent :: real list  $\Rightarrow$  real list  $\Rightarrow$  real where
  akra-bazzi-exponent  $as\ bs \equiv (\text{THE } p. (\sum_{i < \text{length } as} as!i * bs!i \text{ powr } p) = 1)$ 

```



```

locale akra-bazzi-params =
  fixes  $k :: \text{nat}$  and  $as\ bs :: \text{real list}$ 
  assumes  $\text{length-as: length } as = k$ 
  and  $\text{length-bs: length } bs = k$ 
  and  $k\text{-not-0: } k \neq 0$ 
  and  $a\text{-ge-0: } a \in \text{set } as \implies a \geq 0$ 
  and  $b\text{-bounds: } b \in \text{set } bs \implies b \in \{0 < .. < 1\}$ 
begin

abbreviation  $p :: \text{real}$  where  $p \equiv \text{akra-bazzi-exponent } as\ bs$ 

lemma  $p\text{-def: } p = (\text{THE } p. (\sum_{i < k. as!i * bs!i \text{ powr } p} = 1))$ 
   $\langle \text{proof} \rangle$ 

lemma  $b\text{-pos: } b \in \text{set } bs \implies b > 0$  and  $b\text{-less-1: } b \in \text{set } bs \implies b < 1$ 
   $\langle \text{proof} \rangle$ 

lemma  $as\text{-nonempty [simp]: } as \neq []$  and  $bs\text{-nonempty [simp]: } bs \neq []$ 
   $\langle \text{proof} \rangle$ 

lemma  $a\text{-in-as [intro, simp]: } i < k \implies as ! i \in \text{set } as$ 
   $\langle \text{proof} \rangle$ 

lemma  $b\text{-in-bs [intro, simp]: } i < k \implies bs ! i \in \text{set } bs$ 
   $\langle \text{proof} \rangle$ 

end

locale akra-bazzi-params-nonzero =
  fixes  $k :: \text{nat}$  and  $as\ bs :: \text{real list}$ 
  assumes  $\text{length-as: length } as = k$ 
  and  $\text{length-bs: length } bs = k$ 
  and  $a\text{-ge-0: } a \in \text{set } as \implies a \geq 0$ 
  and  $ex\text{-a-pos: } \exists a \in \text{set } as. a > 0$ 
  and  $b\text{-bounds: } b \in \text{set } bs \implies b \in \{0 < .. < 1\}$ 
begin

sublocale akra-bazzi-params  $k\ as\ bs$ 
   $\langle \text{proof} \rangle$ 

lemma  $akra\text{-bazzi-p-strict-mono:}$ 
  assumes  $x < y$ 
  shows  $(\sum_{i < k. as!i * bs!i \text{ powr } y} < (\sum_{i < k. as!i * bs!i \text{ powr } x})$ 
   $\langle \text{proof} \rangle$ 

lemma  $akra\text{-bazzi-p-mono:}$ 
  assumes  $x \leq y$ 

```

shows $(\sum_{i < k}. as!i * bs!i \text{ powr } y) \leq (\sum_{i < k}. as!i * bs!i \text{ powr } x)$
 ⟨proof⟩

lemma akra-bazzi-p-unique:
 $\exists! p. (\sum_{i < k}. as!i * bs!i \text{ powr } p) = 1$
 ⟨proof⟩

lemma p-props: $(\sum_{i < k}. as!i * bs!i \text{ powr } p) = 1$
and p-unique: $(\sum_{i < k}. as!i * bs!i \text{ powr } p') = 1 \implies p = p'$
 ⟨proof⟩

lemma p-greaterI: $1 < (\sum_{i < k}. as!i * bs!i \text{ powr } p') \implies p' < p$
 ⟨proof⟩

lemma p-lessI: $1 > (\sum_{i < k}. as!i * bs!i \text{ powr } p') \implies p' > p$
 ⟨proof⟩

lemma p-geI: $1 \leq (\sum_{i < k}. as!i * bs!i \text{ powr } p') \implies p' \leq p$
 ⟨proof⟩

lemma p-leI: $1 \geq (\sum_{i < k}. as!i * bs!i \text{ powr } p') \implies p' \geq p$
 ⟨proof⟩

lemma p-boundsI: $(\sum_{i < k}. as!i * bs!i \text{ powr } x) \leq 1 \wedge (\sum_{i < k}. as!i * bs!i \text{ powr } y) \geq 1 \implies p \in \{y..x\}$
 ⟨proof⟩

lemma p-boundsI': $(\sum_{i < k}. as!i * bs!i \text{ powr } x) < 1 \wedge (\sum_{i < k}. as!i * bs!i \text{ powr } y) > 1 \implies p \in \{y<..
 ⟨proof⟩$

lemma p-nonneg: $\text{sum-list } as \geq 1 \implies p \geq 0$
 ⟨proof⟩

end

locale akra-bazzi-real-recursion =

fixes $as \ bs :: \text{real list}$ **and** $hs :: (\text{real} \implies \text{real}) \text{ list}$ **and** $k :: \text{nat}$ **and** $x_0 \ x_1 \ hb \ e \ p$
 :: real

assumes $\text{length-as: length } as = k$

and $\text{length-bs: length } bs = k$

and $\text{length-hs: length } hs = k$

and $k\text{-not-0: } k \neq 0$

and $a\text{-ge-0: } a \in \text{set } as \implies a \geq 0$

and $b\text{-bounds: } b \in \text{set } bs \implies b \in \{0<..$

and $x0\text{-ge-1}$: $x_0 \geq 1$
and $x0\text{-le-x1}$: $x_0 \leq x_1$
and $x1\text{-ge}$: $b \in \text{set } bs \implies x_1 \geq 2 * x_0 * \text{inverse } b$

and $e\text{-pos}$: $e > 0$
and $h\text{-bounds}$: $x \geq x_1 \implies h \in \text{set } hs \implies |h x| \leq hb * x / \ln x \text{ powr } (1 + e)$

and $asymptotics$: $x \geq x_0 \implies b \in \text{set } bs \implies \text{akra-bazzi-asymptotics } b \text{ } hb \text{ } e \text{ } p \text{ } x$
begin

sublocale $\text{akra-bazzi-params } k \text{ } as \text{ } bs$
 $\langle \text{proof} \rangle$

lemma $h\text{-in-hs}[\text{intro}, \text{simp}]$: $i < k \implies hs ! i \in \text{set } hs$
 $\langle \text{proof} \rangle$

lemma $x1\text{-gt-1}$: $x_1 > 1$
 $\langle \text{proof} \rangle$

lemma $x1\text{-ge-1}$: $x_1 \geq 1$ $\langle \text{proof} \rangle$

lemma $x1\text{-pos}$: $x_1 > 0$ $\langle \text{proof} \rangle$

lemma $bx\text{-le-x}$: $x \geq 0 \implies b \in \text{set } bs \implies b * x \leq x$
 $\langle \text{proof} \rangle$

lemma $x0\text{-pos}$: $x_0 > 0$ $\langle \text{proof} \rangle$

lemma
assumes $x \geq x_0 \text{ } b \in \text{set } bs$
shows $x0\text{-hb-bound0}$: $hb / \ln x \text{ powr } (1 + e) < b/2$
and $x0\text{-hb-bound1}$: $hb / \ln x \text{ powr } (1 + e) < (1 - b) / 2$
and $x0\text{-hb-bound2}$: $x*(1 - b - hb / \ln x \text{ powr } (1 + e)) > 1$
 $\langle \text{proof} \rangle$

lemma step-diff :
assumes $i < k \text{ } x \geq x_1$
shows $bs ! i * x + (hs ! i) x + 1 < x$
 $\langle \text{proof} \rangle$

lemma step-le-x : $i < k \implies x \geq x_1 \implies bs ! i * x + (hs ! i) x \leq x$
 $\langle \text{proof} \rangle$

lemma $x0\text{-hb-bound0'}$: $\bigwedge x \text{ } b. x \geq x_0 \implies b \in \text{set } bs \implies hb / \ln x \text{ powr } (1 + e) < b$
 $\langle \text{proof} \rangle$

lemma step-pos :

assumes $i < k$ $x \geq x_1$
shows $bs ! i * x + (hs ! i) x > 0$
 ⟨proof⟩

lemma *step-nonneg*: $i < k \implies x \geq x_1 \implies bs ! i * x + (hs ! i) x \geq 0$
 ⟨proof⟩

lemma *step-nonneg'*: $i < k \implies x \geq x_1 \implies bs ! i + (hs ! i) x / x \geq 0$
 ⟨proof⟩

lemma *hb-nonneg*: $hb \geq 0$
 ⟨proof⟩

lemma *x0-hb-bound3*:
assumes $x \geq x_1$ $i < k$
shows $x - (bs ! i * x + (hs ! i) x) \leq x$
 ⟨proof⟩

lemma *x0-hb-bound4*:
assumes $x \geq x_1$ $i < k$
shows $(bs ! i + (hs ! i) x / x) > bs ! i / 2$
 ⟨proof⟩

lemma *x0-hb-bound4'*: $x \geq x_1 \implies i < k \implies (bs ! i + (hs ! i) x / x) \geq bs ! i / 2$
 ⟨proof⟩

lemma *x0-hb-bound5*:
assumes $x \geq x_1$ $i < k$
shows $(bs ! i + (hs ! i) x / x) \leq bs ! i * 3/2$
 ⟨proof⟩

lemma *x0-hb-bound6*:
assumes $x \geq x_1$ $i < k$
shows $x * ((1 - bs ! i) / 2) \leq x - (bs ! i * x + (hs ! i) x)$
 ⟨proof⟩

lemma *x0-hb-bound7*:
assumes $x \geq x_1$ $i < k$
shows $bs ! i * x + (hs ! i) x > x_0$
 ⟨proof⟩

lemma *x0-hb-bound7'*: $x \geq x_1 \implies i < k \implies bs ! i * x + (hs ! i) x > 1$
 ⟨proof⟩

lemma *x0-hb-bound8*:
assumes $x \geq x_1$ $i < k$
shows $bs ! i * x - hb * x / \ln x \text{ powr } (1+e) > x_0$
 ⟨proof⟩

lemma *x0-hb-bound8'*:

assumes $x \geq x_1$ $i < k$

shows $bs!i*x + hb * x / \ln x \text{ powr } (1+e) > x_0$

<proof>

lemma

assumes $x \geq x_0$

shows *asymptotics1*: $i < k \implies 1 + \ln x \text{ powr } (-e/2) \leq$
 $(1 - hb * \text{inverse } (bs!i) * \ln x \text{ powr } -(1+e)) \text{ powr } p *$
 $(1 + \ln (bs!i*x + hb*x/\ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$

and *asymptotics2*: $i < k \implies 1 - \ln x \text{ powr } (-e/2) \geq$
 $(1 + hb * \text{inverse } (bs!i) * \ln x \text{ powr } -(1+e)) \text{ powr } p *$
 $(1 - \ln (bs!i*x + hb*x/\ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$

and *asymptotics1'*: $i < k \implies 1 + \ln x \text{ powr } (-e/2) \leq$
 $(1 + hb * \text{inverse } (bs!i) * \ln x \text{ powr } -(1+e)) \text{ powr } p *$
 $(1 + \ln (bs!i*x + hb*x/\ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$

and *asymptotics2'*: $i < k \implies 1 - \ln x \text{ powr } (-e/2) \geq$
 $(1 - hb * \text{inverse } (bs!i) * \ln x \text{ powr } -(1+e)) \text{ powr } p *$
 $(1 - \ln (bs!i*x + hb*x/\ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$

and *asymptotics3*: $(1 + \ln x \text{ powr } (-e/2)) / 2 \leq 1$

and *asymptotics4*: $(1 - \ln x \text{ powr } (-e/2)) * 2 \geq 1$

and *asymptotics5*: $i < k \implies \ln (bs!i*x - hb*x*\ln x \text{ powr } -(1+e)) \text{ powr } (-e/2) < 1$
<proof>

lemma *x0-hb-bound9*:

assumes $x \geq x_1$ $i < k$

shows $\ln (bs!i*x + (hs!i) x) \text{ powr } -(e/2) < 1$

<proof>

definition *akra-bazzi-measure* :: *real* \implies *nat* **where**

akra-bazzi-measure $x = \text{nat } \lceil x \rceil$

lemma *akra-bazzi-measure-decreases*:

assumes $x \geq x_1$ $i < k$

shows *akra-bazzi-measure* $(bs!i*x + (hs!i) x) < \text{akra-bazzi-measure } x$

<proof>

lemma *akra-bazzi-induct*[*consumes 1, case-names base rec*]:

assumes $x \geq x_0$

assumes *base*: $\bigwedge x. x \geq x_0 \implies x \leq x_1 \implies P x$

assumes *rec*: $\bigwedge x. x > x_1 \implies (\bigwedge i. i < k \implies P (bs!i*x + (hs!i) x)) \implies P x$

shows $P x$

<proof>

end

locale *akra-bazzi-real* = *akra-bazzi-real-recursion* +
fixes *integrable integral*
assumes *integral: akra-bazzi-integral integrable integral*
fixes $f :: \text{real} \Rightarrow \text{real}$
and $g :: \text{real} \Rightarrow \text{real}$
and $C :: \text{real}$
assumes *p-props:* $(\sum_{i < k}. \text{as!}i * \text{bs!}i \text{ powr } p) = 1$
and *f-base:* $x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f\ x \geq 0$
and *f-rec:* $x > x_1 \Longrightarrow f\ x = g\ x + (\sum_{i < k}. \text{as!}i * f\ (\text{bs!}i * x + (\text{hs!}i) x))$
and *g-nonneg:* $x \geq x_0 \Longrightarrow g\ x \geq 0$
and *C-bound:* $b \in \text{set } \text{bs} \Longrightarrow x \geq x_1 \Longrightarrow C * x \leq b * x - \text{hb} * x / \ln\ x \text{ powr } (1 + e)$
and *g-integrable:* $x \geq x_0 \Longrightarrow \text{integrable } (\lambda u. g\ u / u \text{ powr } (p + 1))\ x_0\ x$
begin

interpretation *akra-bazzi-integral integrable integral* $\langle \text{proof} \rangle$

lemma *akra-bazzi-integrable:*

$a \geq x_0 \Longrightarrow a \leq b \Longrightarrow \text{integrable } (\lambda x. g\ x / x \text{ powr } (p + 1))\ a\ b$
 $\langle \text{proof} \rangle$

definition *g-approx* :: $\text{nat} \Rightarrow \text{real} \Rightarrow \text{real}$ **where**

$g\text{-approx } i\ x = x \text{ powr } p * \text{integral } (\lambda u. g\ u / u \text{ powr } (p + 1))\ (\text{bs!}i * x + (\text{hs!}i) x)$

lemma *f-nonneg:* $x \geq x_0 \Longrightarrow f\ x \geq 0$

$\langle \text{proof} \rangle$

definition *f-approx* :: $\text{real} \Rightarrow \text{real}$ **where**

$f\text{-approx } x = x \text{ powr } p * (1 + \text{integral } (\lambda u. g\ u / u \text{ powr } (p + 1))\ x_0\ x)$

lemma *f-approx-aux:*

assumes $x \geq x_0$

shows $1 + \text{integral } (\lambda u. g\ u / u \text{ powr } (p + 1))\ x_0\ x \geq 1$

$\langle \text{proof} \rangle$

lemma *f-approx-pos:* $x \geq x_0 \Longrightarrow f\text{-approx } x > 0$

$\langle \text{proof} \rangle$

lemma *f-approx-nonneg:* $x \geq x_0 \Longrightarrow f\text{-approx } x \geq 0$

$\langle \text{proof} \rangle$

lemma *f-approx-bounded-below:*

obtains c **where** $\bigwedge x. x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f\text{-approx } x \geq c\ c > 0$

<proof>

lemma *asymptotics-aux*:

assumes $x \geq x_1$ $i < k$

assumes $s \equiv (\text{if } p \geq 0 \text{ then } 1 \text{ else } -1)$

shows $(bs!i*x - s*hb*x*ln\ x\ \text{powr}\ -(1+e))\ \text{powr}\ p \leq (bs!i*x + (hs!i)\ x)\ \text{powr}\ p$
(is ?thesis1)

and $(bs!i*x + (hs!i)\ x)\ \text{powr}\ p \leq (bs!i*x + s*hb*x*ln\ x\ \text{powr}\ -(1+e))\ \text{powr}\ p$
(is ?thesis2)

<proof>

lemma *asymptotics1'*:

assumes $x \geq x_1$ $i < k$

shows $(bs!i*x)\ \text{powr}\ p * (1 + ln\ x\ \text{powr}\ (-e/2)) \leq$

$(bs!i*x + (hs!i)\ x)\ \text{powr}\ p * (1 + ln\ (bs!i*x + (hs!i)\ x)\ \text{powr}\ (-e/2))$

<proof>

lemma *asymptotics2'*:

assumes $x \geq x_1$ $i < k$

shows $(bs!i*x + (hs!i)\ x)\ \text{powr}\ p * (1 - ln\ (bs!i*x + (hs!i)\ x)\ \text{powr}\ (-e/2))$
 \leq

$(bs!i*x)\ \text{powr}\ p * (1 - ln\ x\ \text{powr}\ (-e/2))$

<proof>

lemma *Cx-le-step*:

assumes $i < k$ $x \geq x_1$

shows $C*x \leq bs!i*x + (hs!i)\ x$

<proof>

end

locale *akra-bazzi-nat-to-real = akra-bazzi-real-recursion +*

fixes $f :: \text{nat} \Rightarrow \text{real}$

and $g :: \text{real} \Rightarrow \text{real}$

assumes $f\text{-base}: \text{real } x \geq x_0 \Longrightarrow \text{real } x \leq x_1 \Longrightarrow f\ x \geq 0$

and $f\text{-rec}: \text{real } x > x_1 \Longrightarrow$

$f\ x = g\ (\text{real } x) + (\sum\ i < k. as!i * f\ (\text{nat } \lfloor bs!i * x + (hs!i)$

$(\text{real } x) \rfloor))$

and $x0\text{-int}: \text{real } (\text{nat } \lfloor x_0 \rfloor) = x_0$

begin

function $f' :: \text{real} \Rightarrow \text{real}$ **where**

$x \leq x_1 \Longrightarrow f'\ x = f\ (\text{nat } \lfloor x \rfloor)$

$| x > x_1 \Longrightarrow f'\ x = g\ x + (\sum\ i < k. as!i * f'\ (bs!i * x + (hs!i)\ x))$

<proof>

termination *<proof>*

lemma *f'-base*: $x \geq x_0 \implies x \leq x_1 \implies f' x \geq 0$
<proof>

lemmas *f'-rec* = *f'.simps(2)*

end

locale *akra-bazzi-real-lower* = *akra-bazzi-real* +
 fixes *fb2 gb2 c2* :: *real*
 assumes *f-base2*: $x \geq x_0 \implies x \leq x_1 \implies f x \geq fb2$
 and *fb2-pos*: $fb2 > 0$
 and *g-growth2*: $\forall x \geq x_1. \forall u \in \{C * x..x\}. c2 * g x \geq g u$
 and *c2-pos*: $c2 > 0$
 and *g-bounded*: $x \geq x_0 \implies x \leq x_1 \implies g x \leq gb2$
begin

interpretation *akra-bazzi-integral integrable integral* *<proof>*

lemma *gb2-nonneg*: $gb2 \geq 0$ *<proof>*

lemma *g-growth2'*:
 assumes $x \geq x_1 \ i < k \ u \in \{bs!i*x+(hs!i) x..x\}$
 shows $c2 * g x \geq g u$
<proof>

lemma *g-bounds2*:
 obtains *c4* **where** $\bigwedge x \ i. x \geq x_1 \implies i < k \implies g\text{-approx } i \ x \leq c4 * g x \ c4 > 0$
<proof>

lemma *f-approx-bounded-above*:
 obtains *c* **where** $\bigwedge x. x \geq x_0 \implies x \leq x_1 \implies f\text{-approx } x \leq c \ c > 0$
<proof>

lemma *f-bounded-below*:
 assumes *c'*: $c' > 0$
 obtains *c* **where** $\bigwedge x. x \geq x_0 \implies x \leq x_1 \implies 2 * (c * f\text{-approx } x) \leq f x \ c \leq c'$
 $c > 0$
<proof>

lemma *akra-bazzi-lower*:
 obtains *c5* **where** $\bigwedge x. x \geq x_0 \implies f x \geq c5 * f\text{-approx } x \ c5 > 0$
<proof>

lemma *akra-bazzi-bigomega*:
 $f \in \Omega(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x))$
<proof>

end


```

locale akra-bazzi-real-upper = akra-bazzi-real +
  fixes fb1 c1 :: real
  assumes f-base1:  $x \geq x_0 \implies x \leq x_1 \implies f x \leq fb1$ 
  and g-growth1:  $\forall x \geq x_1. \forall u \in \{C * x..x\}. c1 * g x \leq g u$ 
  and c1-pos:  $c1 > 0$ 
begin

interpretation akra-bazzi-integral integrable integral <proof>

lemma g-growth1':
  assumes  $x \geq x_1 \ i < k \ u \in \{bs!i*x+(hs!i) x..x\}$ 
  shows  $c1 * g x \leq g u$ 
  <proof>

lemma g-bounds1:
  obtains c3 where
     $\bigwedge x \ i. x \geq x_1 \implies i < k \implies c3 * g x \leq g\text{-approx } i \ x \ c3 > 0$ 
  <proof>

lemma f-bounded-above:
  assumes c':  $c' > 0$ 
  obtains c where  $\bigwedge x. x \geq x_0 \implies x \leq x_1 \implies f x \leq (1/2) * (c * f\text{-approx } x) \ c$ 
   $\geq c' \ c > 0$ 
  <proof>

lemma akra-bazzi-upper:
  obtains c6 where  $\bigwedge x. x \geq x_0 \implies f x \leq c6 * f\text{-approx } x \ c6 > 0$ 
  <proof>

lemma akra-bazzi-bigo:
   $f \in O(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x))$ 
  <proof>

end

end

```

4 The discrete Akra-Bazzi theorem

```

theory Akra-Bazzi
imports
  Complex-Main
  HOL-Library.Landau-Symbols
  Akra-Bazzi-Real
begin

```

lemma *ex-mono*: $(\exists x. P x) \implies (\bigwedge x. P x \implies Q x) \implies (\exists x. Q x)$ *<proof>*

lemma *x-over-ln-mono*:

assumes $(e::real) > 0$

assumes $x > \exp e$

assumes $x \leq y$

shows $x / \ln x \text{ powr } e \leq y / \ln y \text{ powr } e$

<proof>

definition *akra-bazzi-term* :: $nat \Rightarrow nat \Rightarrow real \Rightarrow (nat \Rightarrow nat) \Rightarrow bool$ **where**

akra-bazzi-term $x_0 x_1 b t =$

$(\exists e h. e > 0 \wedge (\lambda x. h x) \in O(\lambda x. real\ x / \ln (real\ x) \text{ powr } (1+e))) \wedge$
 $(\forall x \geq x_1. t\ x \geq x_0 \wedge t\ x < x \wedge b*x + h\ x = real\ (t\ x))$

lemma *akra-bazzi-termI* [*intro?*]:

assumes $e > 0 (\lambda x. h x) \in O(\lambda x. real\ x / \ln (real\ x) \text{ powr } (1+e))$

$\bigwedge x. x \geq x_1 \implies t\ x \geq x_0 \bigwedge x. x \geq x_1 \implies t\ x < x$

$\bigwedge x. x \geq x_1 \implies b*x + h\ x = real\ (t\ x)$

shows *akra-bazzi-term* $x_0 x_1 b t$

<proof>

lemma *akra-bazzi-term-imp-less*:

assumes *akra-bazzi-term* $x_0 x_1 b t x \geq x_1$

shows $t\ x < x$

<proof>

lemma *akra-bazzi-term-imp-less'*:

assumes *akra-bazzi-term* $x_0 (Suc\ x_1) b t x > x_1$

shows $t\ x < x$

<proof>

locale *akra-bazzi-recursion* =

fixes $x_0 x_1 k :: nat$ **and** *as* *bs* :: *real list* **and** *ts* :: $(nat \Rightarrow nat)$ *list* **and** *f* :: $nat \Rightarrow real$

assumes *k-not-0*: $k \neq 0$

and *length-as*: $length\ as = k$

and *length-bs*: $length\ bs = k$

and *length-ts*: $length\ ts = k$

and *a-ge-0*: $a \in set\ as \implies a \geq 0$

and *b-bounds*: $b \in set\ bs \implies b \in \{0 < .. < 1\}$

and *ts*: $i < length\ bs \implies \text{akra-bazzi-term } x_0 x_1 (bs!i) (ts!i)$

begin

sublocale *akra-bazzi-params* $k as bs$

<proof>

lemma *ts-nonempty*: $ts \neq []$ \langle proof \rangle

definition *e-hs* :: $real \times (nat \Rightarrow real)$ list **where**

e -*hs* = (SOME (e,hs).
 $e > 0 \wedge length\ hs = k \wedge (\forall h \in set\ hs. (\lambda x. h\ x) \in O(\lambda x. real\ x / ln\ (real\ x)$
powr (1+e))) \wedge
 $(\forall t \in set\ ts. \forall x \geq x_1. t\ x \geq x_0 \wedge t\ x < x) \wedge$
 $(\forall i < k. \forall x \geq x_1. (bs!i)*x + (hs!i)\ x = real\ ((ts!i)\ x))$
 $)$

definition $e = (case\ e\text{-}hs\ of\ (e,-) \Rightarrow e)$

definition $hs = (case\ e\text{-}hs\ of\ (-,hs) \Rightarrow hs)$

lemma *filterlim-powr-zero-cong*:

$filterlim\ (\lambda x. P\ (x::real)\ (x\ powr\ (0::real)))\ F\ at\text{-}top = filterlim\ (\lambda x. P\ x\ 1)\ F$
at-top
 \langle proof \rangle

lemma *e-hs-aux*:

$0 < e \wedge length\ hs = k \wedge$
 $(\forall h \in set\ hs. (\lambda x. h\ x) \in O(\lambda x. real\ x / ln\ (real\ x)\ powr\ (1 + e))) \wedge$
 $(\forall t \in set\ ts. \forall x \geq x_1. x_0 \leq t\ x \wedge t\ x < x) \wedge$
 $(\forall i < k. \forall x \geq x_1. (bs!i)*x + (hs!i)\ x = real\ ((ts!i)\ x))$
 \langle proof \rangle

lemma

e-pos: $e > 0$ **and** *length-hs*: $length\ hs = k$ **and**
hs-growth: $\bigwedge h. h \in set\ hs \Longrightarrow (\lambda x. h\ x) \in O(\lambda x. real\ x / ln\ (real\ x)\ powr\ (1 + e))$ **and**
step-ge-x0: $\bigwedge t\ x. t \in set\ ts \Longrightarrow x \geq x_1 \Longrightarrow x_0 \leq t\ x$ **and**
step-less: $\bigwedge t\ x. t \in set\ ts \Longrightarrow x \geq x_1 \Longrightarrow t\ x < x$ **and**
decomp: $\bigwedge i\ x. i < k \Longrightarrow x \geq x_1 \Longrightarrow (bs!i)*x + (hs!i)\ x = real\ ((ts!i)\ x)$
 \langle proof \rangle

lemma *h-in-hs* [*intro*, *simp*]: $i < k \Longrightarrow hs\ !\ i \in set\ hs$

\langle proof \rangle

lemma *t-in-ts* [*intro*, *simp*]: $i < k \Longrightarrow ts\ !\ i \in set\ ts$

\langle proof \rangle

lemma *x0-less-x1*: $x_0 < x_1$ **and** *x0-le-x1*: $x_0 \leq x_1$

\langle proof \rangle

lemma *akra-bazzi-induct* [*consumes 1*, *case-names base rec*]:

assumes $x \geq x_0$

assumes *base*: $\bigwedge x. x \geq x_0 \Longrightarrow x < x_1 \Longrightarrow P\ x$

assumes *rec*: $\bigwedge x. x \geq x_1 \Longrightarrow (\bigwedge t. t \in set\ ts \Longrightarrow P\ (t\ x)) \Longrightarrow P\ x$

shows $P\ x$

<proof>

end

locale *akra-bazzi-function* = *akra-bazzi-recursion* +
 fixes *integrable integral*
 assumes *integral: akra-bazzi-integral integrable integral*
 fixes *g :: nat ⇒ real*
 assumes *f-nonneg-base: x ≥ x₀ ⇒ x < x₁ ⇒ f x ≥ 0*
 and *f-rec: x ≥ x₁ ⇒ f x = g x + (∑ i<k. as!i * f ((ts!i) x))*
 and *g-nonneg: x ≥ x₁ ⇒ g x ≥ 0*
 and *ex-pos-a: ∃ a ∈ set as. a > 0*
begin

lemma *ex-pos-a'*: $\exists i < k. as!i > 0$
<proof>

sublocale *akra-bazzi-params-nonzero*
<proof>

definition *g-real :: real ⇒ real where g-real x = g (nat [x])*

lemma *g-real-real[simp]*: $g\text{-real } (real\ x) = g\ x$ *<proof>*

lemma *f-nonneg*: $x \geq x_0 \implies f\ x \geq 0$
<proof>

definition *hs' = map (λh x. h (nat [x::real])) hs*

lemma *length-hs'*: $length\ hs' = k$ *<proof>*

lemma *hs'-real*: $i < k \implies (hs'!i)\ (real\ x) = (hs!i)\ x$
<proof>

lemma *h-bound*:
 obtains *hb where hb > 0 and*
 *eventually (λx. ∀ h ∈ set hs'. |h x| ≤ hb * x / ln x powr (1 + e)) at-top*
<proof>

lemma *C-bound*:
 assumes $\bigwedge b. b \in set\ bs \implies C < b\ hb > 0$
 shows *eventually (λx::real. ∀ b ∈ set bs. C*x ≤ b*x - hb*x/ln x powr (1+e))*
at-top
<proof>

end

locale *akra-bazzi-lower* = *akra-bazzi-function* +
fixes $g' :: \text{real} \Rightarrow \text{real}$
assumes *f-pos*: *eventually* $(\lambda x. f\ x > 0)$ *at-top*
and *g-growth2*: $\exists C\ c2. c2 > 0 \wedge C < \text{Min}(\text{set } bs) \wedge$
eventually $(\lambda x. \forall u \in \{C * x..x\}. g'\ u \leq c2 * g'\ x)$ *at-top*
and *g'-integrable*: $\exists a. \forall b \geq a. \text{integrable}(\lambda u. g'\ u / u^{\text{powr}(p+1)})\ a\ b$
and *g'-bounded*: *eventually* $(\lambda a :: \text{real}. (\forall b > a. \exists c. \forall x \in \{a..b\}. g'\ x \leq c))$ *at-top*
and *g-bigomega*: $g \in \Omega(\lambda x. g'\ (\text{real } x))$
and *g'-nonneg*: *eventually* $(\lambda x. g'\ x \geq 0)$ *at-top*
begin

definition *gc2* $\equiv \text{SOME } gc2. gc2 > 0 \wedge \text{eventually}(\lambda x. g\ x \geq gc2 * g'\ (\text{real } x))$
at-top

lemma *gc2*: $gc2 > 0$ *eventually* $(\lambda x. g\ x \geq gc2 * g'\ (\text{real } x))$ *at-top*
 $\langle \text{proof} \rangle$

definition *gx0* $\equiv \max\ x_1\ (\text{SOME } gx0. \forall x \geq gx0. g\ x \geq gc2 * g'\ (\text{real } x) \wedge f\ x > 0 \wedge g'\ (\text{real } x) \geq 0)$

definition *gx1* $\equiv \max\ gx0\ (\text{SOME } gx1. \forall x \geq gx1. \forall i < k. (ts!i)\ x \geq gx0)$

lemma *gx0*:
assumes $x \geq gx0$
shows $g\ x \geq gc2 * g'\ (\text{real } x)$ $f\ x > 0$ $g'\ (\text{real } x) \geq 0$
 $\langle \text{proof} \rangle$

lemma *gx1*:
assumes $x \geq gx1$ $i < k$
shows $(ts!i)\ x \geq gx0$
 $\langle \text{proof} \rangle$

lemma *gx0-ge-x1*: $gx0 \geq x_1$ $\langle \text{proof} \rangle$

lemma *gx0-le-gx1*: $gx0 \leq gx1$ $\langle \text{proof} \rangle$

function *f2'* $:: \text{nat} \Rightarrow \text{real}$ **where**
 $x < gx1 \implies f2'\ x = \max\ 0\ (f\ x / gc2)$
 $| x \geq gx1 \implies f2'\ x = g'\ (\text{real } x) + (\sum\ i < k. as!i * f2'\ ((ts!i)\ x))$
 $\langle \text{proof} \rangle$
termination $\langle \text{proof} \rangle$

lemma *f2'-nonneg*: $x \geq gx0 \implies f2'\ x \geq 0$
 $\langle \text{proof} \rangle$

lemma *f2'-le-f*: $x \geq x_0 \implies gc2 * f2'\ x \leq f\ x$
 $\langle \text{proof} \rangle$

lemma *f2'-pos*: *eventually* $(\lambda x. f2'\ x > 0)$ *at-top*
 $\langle \text{proof} \rangle$

lemma *bigomega-f-aux*:

obtains a where $a \geq A \forall a' \geq a. a' \in \mathbf{N} \longrightarrow$

$f \in \Omega(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) a' x))$

<proof>

lemma *bigomega-f*:

obtains a where $a \geq A f \in \Omega(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p+1)) a x))$

<proof>

end

locale *akra-bazzi-upper* = *akra-bazzi-function* +

fixes $g' :: \text{real} \Rightarrow \text{real}$

assumes *g'-integrable*: $\exists a. \forall b \geq a. \text{integrable } (\lambda u. g' u / u \text{ powr } (p + 1)) a b$

and *g-growth1*: $\exists C c1. c1 > 0 \wedge C < \text{Min } (\text{set } bs) \wedge$

$\text{eventually } (\lambda x. \forall u \in \{C * x..x\}. g' u \geq c1 * g' x) \text{ at-top}$

and *g-bigo*: $g \in O(g')$

and *g'-nonneg*: $\text{eventually } (\lambda x. g' x \geq 0) \text{ at-top}$

begin

definition *gc1* $\equiv \text{SOME } gc1. gc1 > 0 \wedge \text{eventually } (\lambda x. g x \leq gc1 * g' (\text{real } x)) \text{ at-top}$

lemma *gc1*: $gc1 > 0 \text{ eventually } (\lambda x. g x \leq gc1 * g' (\text{real } x)) \text{ at-top}$

<proof>

definition *gx3* $\equiv \max x_1 (\text{SOME } gx0. \forall x \geq gx0. g x \leq gc1 * g' (\text{real } x))$

lemma *gx3*:

assumes $x \geq gx3$

shows $g x \leq gc1 * g' (\text{real } x)$

<proof>

lemma *gx3-ge-x1*: $gx3 \geq x_1$ *<proof>*

function *f'* $:: \text{nat} \Rightarrow \text{real}$ **where**

$x < gx3 \Longrightarrow f' x = \max 0 (f x / gc1)$

$| x \geq gx3 \Longrightarrow f' x = g' (\text{real } x) + (\sum_{i < k. as!i * f' ((ts!i) x))$

<proof>

termination *<proof>*

lemma *f'-ge-f*: $x \geq x_0 \Longrightarrow gc1 * f' x \geq f x$

<proof>

lemma *bigo-f-aux*:

obtains a where $a \geq A \forall a' \geq a. a' \in \mathbf{N} \longrightarrow$

$f \in O(\lambda x. x \text{ powr } p *(1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) a' x))$

<proof>

lemma *bigo-f*:

obtains a where $a > A f \in O(\lambda x. x \text{ powr } p *(1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) a x))$

<proof>

end

locale *akra-bazzi* = *akra-bazzi-function* +

fixes $g' :: \text{real} \Rightarrow \text{real}$

assumes *f-pos*: *eventually* $(\lambda x. f x > 0)$ *at-top*

and *g'-nonneg*: *eventually* $(\lambda x. g' x \geq 0)$ *at-top*

assumes *g'-integrable*: $\exists a. \forall b \geq a. \text{integrable } (\lambda u. g' u / u \text{ powr } (p + 1)) a b$

and *g-growth1*: $\exists C c1. c1 > 0 \wedge C < \text{Min } (\text{set } bs) \wedge$
eventually $(\lambda x. \forall u \in \{C*x..x\}. g' u \geq c1 * g' x)$ *at-top*

and *g-growth2*: $\exists C c2. c2 > 0 \wedge C < \text{Min } (\text{set } bs) \wedge$
eventually $(\lambda x. \forall u \in \{C*x..x\}. g' u \leq c2 * g' x)$ *at-top*

and *g-bounded*: *eventually* $(\lambda a :: \text{real}. (\forall b > a. \exists c. \forall x \in \{a..b\}. g' x \leq c))$ *at-top*

and *g-bigtheta*: $g \in \Theta(g')$

begin

sublocale *akra-bazzi-lower* *<proof>*

sublocale *akra-bazzi-upper* *<proof>*

lemma *bigtheta-f*:

obtains a where $a > A f \in \Theta(\lambda x. x \text{ powr } p *(1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) a x))$

<proof>

end

named-theorems *akra-bazzi-term-intros* *introduction rules for Akra–Bazzi terms*

lemma *akra-bazzi-term-floor-add* [*akra-bazzi-term-intros*]:

assumes $(b :: \text{real}) > 0 b < 1 \text{ real } x_0 \leq b * \text{real } x_1 + c c < (1 - b) * \text{real } x_1 x_1 > 0$

shows *akra-bazzi-term* $x_0 x_1 b (\lambda x. \text{nat } \lfloor b * \text{real } x + c \rfloor)$

<proof>

lemma *akra-bazzi-term-floor-add'* [*akra-bazzi-term-intros*]:

assumes $(b :: \text{real}) > 0 b < 1 \text{ real } x_0 \leq b * \text{real } x_1 + \text{real } c \text{ real } c < (1 - b) * \text{real } x_1 x_1 > 0$

shows *akra-bazzi-term* $x_0 x_1 b (\lambda x. \text{nat } \lfloor b * \text{real } x \rfloor + c)$

<proof>

lemma *akra-bazzi-term-floor-subtract* [*akra-bazzi-term-intros*]:

assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \leq b * real\ x_1 - c$ $0 < c + (1 - b) * real\ x_1$ $x_1 > 0$

shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x - c])$

<proof>

lemma *akra-bazzi-term-floor-subtract'* [*akra-bazzi-term-intros*]:

assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \leq b * real\ x_1 - real\ c$ $0 < real\ c + (1 - b) * real\ x_1$ $x_1 > 0$

shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x] - c)$

<proof>

lemma *akra-bazzi-term-floor* [*akra-bazzi-term-intros*]:

assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \leq b * real\ x_1$ $0 < (1 - b) * real\ x_1$ $x_1 > 0$

shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x])$

<proof>

lemma *akra-bazzi-term-ceiling-add* [*akra-bazzi-term-intros*]:

assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \leq b * real\ x_1 + c$ $c + 1 \leq (1 - b) * x_1$

shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x + c])$

<proof>

lemma *akra-bazzi-term-ceiling-add'* [*akra-bazzi-term-intros*]:

assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \leq b * real\ x_1 + real\ c$ $real\ c + 1 \leq (1 - b) * x_1$

shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x] + c)$

<proof>

lemma *akra-bazzi-term-ceiling-subtract* [*akra-bazzi-term-intros*]:

assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \leq b * real\ x_1 - c$ $1 \leq c + (1 - b) * x_1$

shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x - c])$

<proof>

lemma *akra-bazzi-term-ceiling-subtract'* [*akra-bazzi-term-intros*]:

assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \leq b * real\ x_1 - real\ c$ $1 \leq real\ c + (1 - b) * x_1$

shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x] - c)$

<proof>

lemma *akra-bazzi-term-ceiling* [*akra-bazzi-term-intros*]:

assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \leq b * real\ x_1$ $1 \leq (1 - b) * x_1$

shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x. nat\ [b * real\ x])$

<proof>

end

5 The Master theorem

theory *Master-Theorem*

imports

HOL-Analysis.Equivalence-Lebesgue-Henstock-Integration

Akra-Bazzi-Library

Akra-Bazzi

begin

lemma *fundamental-theorem-of-calculus-real*:

$a \leq b \implies \forall x \in \{a..b\}. (f \text{ has-real-derivative } f' x) \text{ (at } x \text{ within } \{a..b\}) \implies$
 $(f' \text{ has-integral } (f b - f a)) \{a..b\}$

<proof>

lemma *integral-powr*:

$y \neq -1 \implies a \leq b \implies a > 0 \implies \text{integral } \{a..b\} (\lambda x. x \text{ powr } y :: \text{real}) =$
 $\text{inverse } (y + 1) * (b \text{ powr } (y + 1) - a \text{ powr } (y + 1))$

<proof>

lemma *integral-ln-powr-over-x*:

$y \neq -1 \implies a \leq b \implies a > 1 \implies \text{integral } \{a..b\} (\lambda x. \ln x \text{ powr } y / x :: \text{real}) =$
 $\text{inverse } (y + 1) * (\ln b \text{ powr } (y + 1) - \ln a \text{ powr } (y + 1))$

<proof>

lemma *integral-one-over-x-ln-x*:

$a \leq b \implies a > 1 \implies \text{integral } \{a..b\} (\lambda x. \text{inverse } (x * \ln x) :: \text{real}) = \ln (\ln b)$
 $- \ln (\ln a)$

<proof>

lemma *akra-bazzi-integral-kurzweil-henstock*:

$\text{akra-bazzi-integral } (\lambda f a b. f \text{ integrable-on } \{a..b\}) (\lambda f a b. \text{integral } \{a..b\} f)$

<proof>

locale *master-theorem-function = akra-bazzi-recursion +*

fixes $g :: \text{nat} \Rightarrow \text{real}$

assumes *f-nonneg-base*: $x \geq x_0 \implies x < x_1 \implies f x \geq 0$

and *f-rec*: $x \geq x_1 \implies f x = g x + (\sum_{i < k}. \text{as! } i * f ((\text{ts! } i) x))$

and *g-nonneg*: $x \geq x_1 \implies g x \geq 0$

and *ex-pos-a*: $\exists a \in \text{set } \text{as}. a > 0$

begin

interpretation *akra-bazzi-integral* $\lambda f a b. f \text{ integrable-on } \{a..b\} \lambda f a b. \text{integral } \{a..b\} f$

<proof>

sublocale *akra-bazzi-function* $x_0 x_1 k$ as bs ts f $\lambda f a b. f$ *integrable-on* $\{a..b\}$
 $\lambda f a b. \text{integral } \{a..b\} f g$
 ⟨proof⟩

context
begin

private lemma *g-nonneg'*: *eventually* $(\lambda x. g x \geq 0)$ *at-top*

⟨proof⟩ **lemma** *g-pos*:

assumes $g \in \Omega(h)$

assumes *eventually* $(\lambda x. h x > 0)$ *at-top*

shows *eventually* $(\lambda x. g x > 0)$ *at-top*

⟨proof⟩ **lemma** *f-pos*:

assumes $g \in \Omega(h)$

assumes *eventually* $(\lambda x. h x > 0)$ *at-top*

shows *eventually* $(\lambda x. f x > 0)$ *at-top*

⟨proof⟩

lemma *bs-lower-bound*: $\exists C > 0. \forall b \in \text{set } bs. C < b$

⟨proof⟩ **lemma** *powr-growth2*:

$\exists C c2. 0 < c2 \wedge C < \text{Min } (\text{set } bs) \wedge$

eventually $(\lambda x. \forall u \in \{C * x..x\}. c2 * x \text{ powr } p' \geq u \text{ powr } p')$ *at-top*

⟨proof⟩ **lemma** *powr-growth1*:

$\exists C c1. 0 < c1 \wedge C < \text{Min } (\text{set } bs) \wedge$

eventually $(\lambda x. \forall u \in \{C * x..x\}. c1 * x \text{ powr } p' \leq u \text{ powr } p')$ *at-top*

⟨proof⟩ **lemma** *powr-ln-powr-lower-bound*:

$a > 1 \implies a \leq x \implies x \leq b \implies$

$\min (a \text{ powr } p) (b \text{ powr } p) * \min (\ln a \text{ powr } p') (\ln b \text{ powr } p') \leq x \text{ powr } p * \ln x \text{ powr } p'$

⟨proof⟩ **lemma** *powr-ln-powr-upper-bound*:

$a > 1 \implies a \leq x \implies x \leq b \implies$

$\max (a \text{ powr } p) (b \text{ powr } p) * \max (\ln a \text{ powr } p') (\ln b \text{ powr } p') \geq x \text{ powr } p * \ln x \text{ powr } p'$

⟨proof⟩ **lemma** *powr-ln-powr-upper-bound'*:

eventually $(\lambda a. \forall b > a. \exists c. \forall x \in \{a..b\}. x \text{ powr } p * \ln x \text{ powr } p' \leq c)$ *at-top*

⟨proof⟩ **lemma** *powr-upper-bound'*:

eventually $(\lambda a::\text{real}. \forall b > a. \exists c. \forall x \in \{a..b\}. x \text{ powr } p' \leq c)$ *at-top*

⟨proof⟩

lemmas *bounds* =

powr-ln-powr-lower-bound powr-ln-powr-upper-bound powr-ln-powr-upper-bound'
powr-upper-bound'

private lemma *eventually-ln-const*:

assumes $(C::\text{real}) > 0$

shows *eventually* $(\lambda x. \ln (C*x) / \ln x > 1/2)$ *at-top*

⟨proof⟩ **lemma** *powr-ln-powr-growth1*: $\exists C c1. 0 < c1 \wedge C < \text{Min } (\text{set } bs) \wedge$

eventually $(\lambda x. \forall u \in \{C * x..x\}. c1 * (x \text{ powr } r * \ln x \text{ powr } r') \leq u \text{ powr } r * \ln$

$u \text{ powr } r')$ at-top
 ⟨proof⟩ **lemma** *powr-ln-powr-growth2*: $\exists C c1. 0 < c1 \wedge C < \text{Min } (\text{set } bs) \wedge$
 eventually $(\lambda x. \forall u \in \{C * x..x\}. c1 * (x \text{ powr } r * \ln x \text{ powr } r') \geq u \text{ powr } r * \ln$
 $u \text{ powr } r')$ at-top
 ⟨proof⟩

lemmas *growths* = *powr-growth1 powr-growth2 powr-ln-powr-growth1 powr-ln-powr-growth2*

private lemma *master-integrable*:

$\exists a::\text{real}. \forall b \geq a. (\lambda u. u \text{ powr } r * \ln u \text{ powr } s / u \text{ powr } t) \text{ integrable-on } \{a..b\}$

$\exists a::\text{real}. \forall b \geq a. (\lambda u. u \text{ powr } r / u \text{ powr } s) \text{ integrable-on } \{a..b\}$

⟨proof⟩ **lemma** *master-integral*:

fixes $a p p' :: \text{real}$

assumes $p: p \neq p'$ **and** $a: a > 0$

obtains $c d$ **where** $c \neq 0 p > p' \longrightarrow d \neq 0$

$(\lambda x::\text{nat}. x \text{ powr } p * (1 + \text{integral } \{a..x\} (\lambda u. u \text{ powr } p' / u \text{ powr } (p+1)))) \in$
 $\Theta(\lambda x::\text{nat}. d * x \text{ powr } p + c * x \text{ powr } p')$

⟨proof⟩ **lemma** *master-integral'*:

fixes $a p p' :: \text{real}$

assumes $p': p' \neq 0$ **and** $a: a > 1$

obtains $c d :: \text{real}$ **where** $p' < 0 \longrightarrow c \neq 0 d \neq 0$

$(\lambda x::\text{nat}. x \text{ powr } p * (1 + \text{integral } \{a..x\} (\lambda u. u \text{ powr } p * \ln u \text{ powr } (p'-1) / u$
 $\text{powr } (p+1)))) \in$
 $\Theta(\lambda x::\text{nat}. c * x \text{ powr } p + d * x \text{ powr } p * \ln x \text{ powr } p')$

⟨proof⟩ **lemma** *master-integral''*:

fixes $a p p' :: \text{real}$

assumes $a: a > 1$

shows $(\lambda x::\text{nat}. x \text{ powr } p * (1 + \text{integral } \{a..x\} (\lambda u. u \text{ powr } p * \ln u \text{ powr } -$
 $1 / u \text{ powr } (p+1)))) \in$

$\Theta(\lambda x::\text{nat}. x \text{ powr } p * \ln (\ln x))$

⟨proof⟩

lemma *master1-bigo*:

assumes *g-bigo*: $g \in O(\lambda x. \text{real } x \text{ powr } p')$

assumes *less-p'*: $(\sum i < k. \text{as}!i * \text{bs}!i \text{ powr } p') > 1$

shows $f \in O(\lambda x. \text{real } x \text{ powr } p)$

⟨proof⟩

lemma *master1*:

assumes *g-bigo*: $g \in O(\lambda x. \text{real } x \text{ powr } p')$

assumes *less-p'*: $(\sum i < k. \text{as}!i * \text{bs}!i \text{ powr } p') > 1$

assumes *f-pos*: *eventually* $(\lambda x. f x > 0)$ at-top

shows $f \in \Theta(\lambda x. \text{real } x \text{ powr } p)$

⟨proof⟩

lemma *master2-3*:
assumes *g-bigtheta*: $g \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln (\text{real } x) \text{ powr } (p' - 1))$
assumes *p'*: $p' > 0$
shows $f \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln (\text{real } x) \text{ powr } p')$
 $\langle \text{proof} \rangle$

lemma *master2-1*:
assumes *g-bigtheta*: $g \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln (\text{real } x) \text{ powr } p')$
assumes *p'*: $p' < -1$
shows $f \in \Theta(\lambda x. \text{real } x \text{ powr } p)$
 $\langle \text{proof} \rangle$

lemma *master2-2*:
assumes *g-bigtheta*: $g \in \Theta(\lambda x. \text{real } x \text{ powr } p / \ln (\text{real } x))$
shows $f \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln (\ln (\text{real } x)))$
 $\langle \text{proof} \rangle$

lemma *master3*:
assumes *g-bigtheta*: $g \in \Theta(\lambda x. \text{real } x \text{ powr } p')$
assumes *p'-greater'*: $(\sum_{i < k}. \text{as! } i * \text{bs! } i \text{ powr } p') < 1$
shows $f \in \Theta(\lambda x. \text{real } x \text{ powr } p')$
 $\langle \text{proof} \rangle$

end
end

end

6 Evaluating expressions with rational numerals

theory *Eval-Numeral*
imports
Complex-Main
begin

lemma *real-numeral-to-Ratreal*:
 $(0::\text{real}) = \text{Ratreal } (\text{Frct } (0, 1))$
 $(1::\text{real}) = \text{Ratreal } (\text{Frct } (1, 1))$
 $(\text{numeral } x :: \text{real}) = \text{Ratreal } (\text{Frct } (\text{numeral } x, 1))$
 $(1::\text{int}) = \text{numeral } \text{Num.One}$
 $\langle \text{proof} \rangle$

lemma *real-equals-code*: $\text{Ratreal } x = \text{Ratreal } y \longleftrightarrow x = y$
 $\langle \text{proof} \rangle$

lemma *Rat-normalize-idempotent*: $\text{Rat.normalize } (\text{Rat.normalize } x) = \text{Rat.normalize } x$
 $\langle \text{proof} \rangle$

lemma *uminus-pow-Numerals1*: $(-(x:::\text{monoid-mult})) \wedge \text{Numerals1} = -x$ *<proof>*

lemmas *power-numeral-simps* = *power-0 uminus-pow-Numerals1 power-minus-Bit0 power-minus-Bit1*

lemma *Fract-normalize*: $\text{Fract} (\text{fst} (\text{Rat.normalize} (x,y))) (\text{snd} (\text{Rat.normalize} (x,y))) = \text{Fract} x y$
<proof>

lemma *Fract-add*: $\text{Frct} (a, \text{numeral } b) + \text{Frct} (c, \text{numeral } d) = \text{Frct} (\text{Rat.normalize} (a * \text{numeral } d + c * \text{numeral } b, \text{numeral } (b*d)))$
<proof>

lemma *Frct-uminus*: $-(\text{Frct} (a,b)) = \text{Frct} (-a,b)$ *<proof>*

lemma *Frct-diff*: $\text{Frct} (a, \text{numeral } b) - \text{Frct} (c, \text{numeral } d) = \text{Frct} (\text{Rat.normalize} (a * \text{numeral } d - c * \text{numeral } b, \text{numeral } (b*d)))$
<proof>

lemma *Frct-mult*: $\text{Frct} (a, \text{numeral } b) * \text{Frct} (c, \text{numeral } d) = \text{Frct} (a*c, \text{numeral } (b*d))$
<proof>

lemma *Frct-inverse*: $\text{inverse} (\text{Frct} (a, b)) = \text{Frct} (b, a)$ *<proof>*

lemma *Frct-divide*: $\text{Frct} (a, \text{numeral } b) / \text{Frct} (c, \text{numeral } d) = \text{Frct} (a*\text{numeral } d, \text{numeral } b * c)$
<proof>

lemma *Frct-pow*: $\text{Frct} (a, \text{numeral } b) \wedge c = \text{Frct} (a \wedge c, \text{numeral } b \wedge c)$
<proof>

lemma *Frct-less*: $\text{Frct} (a, \text{numeral } b) < \text{Frct} (c, \text{numeral } d) \iff a * \text{numeral } d < c * \text{numeral } b$
<proof>

lemma *Frct-le*: $\text{Frct} (a, \text{numeral } b) \leq \text{Frct} (c, \text{numeral } d) \iff a * \text{numeral } d \leq c * \text{numeral } b$
<proof>

lemma *Frct-equals*: $\text{Frct} (a, \text{numeral } b) = \text{Frct} (c, \text{numeral } d) \iff a * \text{numeral } d = c * \text{numeral } b$
<proof>

lemma *real-power-code*: $(\text{Ratreal } x) \wedge y = \text{Ratreal} (x \wedge y)$ *<proof>*

lemmas *real-arith-code* =
real-plus-code real-minus-code real-times-code real-uminus-code real-inverse-code
real-divide-code real-power-code real-less-code real-less-eq-code real-equals-code

lemmas *rat-arith-code* =
Frct-add Frct-uminus Frct-diff Frct-mult Frct-inverse Frct-divide Frct-pow
Frct-less Frct-le Frct-equals

lemma *gcd-numeral-red*: $\text{gcd} (\text{numeral } x :: \text{int}) (\text{numeral } y) = \text{gcd} (\text{numeral } y)$
 $(\text{numeral } x \text{ mod numeral } y)$
 $\langle \text{proof} \rangle$

lemma *divmod-one*:
 $\text{divmod} (\text{Num.One}) (\text{Num.One}) = (\text{Numeral1}, 0)$
 $\text{divmod} (\text{Num.One}) (\text{Num.Bit0 } x) = (0, \text{Numeral1})$
 $\text{divmod} (\text{Num.One}) (\text{Num.Bit1 } x) = (0, \text{Numeral1})$
 $\text{divmod } x (\text{Num.One}) = (\text{numeral } x, 0)$
 $\langle \text{proof} \rangle$

lemmas *divmod-numeral-simps* =
div-0 div-by-0 mod-0 mod-by-0
fst-divmod [symmetric]
snd-divmod [symmetric]
divmod-cancel
divmod-steps [simplified rel-simps if-True] divmod-trivial
rel-simps

lemma *Suc-0-to-numeral*: $\text{Suc } 0 = \text{Numeral1}$ $\langle \text{proof} \rangle$
lemmas *Suc-to-numeral* = *Suc-0-to-numeral Num.Suc-1 Num.Suc-numeral*

lemma *rat-powr*:
 $0 \text{ powr } y = 0$
 $x > 0 \implies x \text{ powr } \text{Ratreal} (\text{Frct} (0, \text{Numeral1})) = \text{Ratreal} (\text{Frct} (\text{Numeral1}, \text{Numeral1}))$
 $x > 0 \implies x \text{ powr } \text{Ratreal} (\text{Frct} (\text{numeral } a, \text{Numeral1})) = x \wedge \text{numeral } a$
 $x > 0 \implies x \text{ powr } \text{Ratreal} (\text{Frct} (-\text{numeral } a, \text{Numeral1})) = \text{inverse} (x \wedge \text{numeral } a)$
 $\langle \text{proof} \rangle$

lemmas *eval-numeral-simps* =
real-numeral-to-Ratreal real-arith-code rat-arith-code Num.arith-simps
Rat.normalize-def fst-conv snd-conv gcd-0-int gcd-0-left-int gcd.bottom-right-bottom
gcd.bottom-left-bottom
gcd-neg1-int gcd-neg2-int gcd-numeral-red zmod-numeral-Bit0 zmod-numeral-Bit1
power-numeral-simps
divmod-numeral-simps numeral-One [symmetric] Groups.Let-0 Num.Let-numeral
Suc-to-numeral power-numeral
greaterThanLessThan-iff atLeastAtMost-iff atLeastLessThan-iff greaterThanAt-
Most-iff rat-powr

Num.pow.simps Num.sqr.simps Product-Type.split of-int-numeral of-int-neg-numeral of-nat-numeral

<ML>

lemma $21254387548659589512 * 314213523632464357453884361 * 2342523623324234 * 56432743858724173474$
 $12561712738645824362329316482973164398214286 \text{ powr } 2 /$
 $(1130246312978423123 + 231212374631082764842731842 * 122474378389424362347451251263)$
 $>$
 $(12313244512931247243543279768645745929475829310651205623844 :: \text{real})$
<proof>

end

7 The proof methods

7.1 Master theorem and termination

theory *Akra-Bazzi-Method*

imports

Complex-Main

Akra-Bazzi

Master-Theorem

Eval-Numeral

begin

lemma *landau-symbol-ge-3-cong*:

assumes *landau-symbol* $L L' Lr$

assumes $\bigwedge x :: 'a :: \text{linordered-semidom. } x \geq 3 \implies f x = g x$

shows $L \text{ at-top } (f) = L \text{ at-top } (g)$

<proof>

lemma *exp-1-lt-3*: $\text{exp } (1 :: \text{real}) < 3$

<proof>

lemma *ln-ln-pos*:

assumes $(x :: \text{real}) \geq 3$

shows $\ln (\ln x) > 0$

<proof>

definition *akra-bazzi-terms* **where**

akra-bazzi-terms $x_0 x_1 bs ts = (\forall i < \text{length } bs. \text{akra-bazzi-term } x_0 x_1 (bs!i) (ts!i))$

lemma *akra-bazzi-termsI*:

$(\bigwedge i. i < \text{length } bs \implies \text{akra-bazzi-term } x_0 x_1 (bs!i) (ts!i)) \implies \text{akra-bazzi-terms } x_0 x_1 bs ts$

<proof>

lemma *master-theorem-functionI*:

assumes $\forall x \in \{x_0..<x_1\}. f\ x \geq 0$
assumes $\forall x \geq x_1. f\ x = g\ x + (\sum i < k. as\ !\ i * f\ ((ts\ !\ i)\ x))$
assumes $\forall x \geq x_1. g\ x \geq 0$
assumes $\forall a \in set\ as. a \geq 0$
assumes $list-ex\ (\lambda a. a > 0)\ as$
assumes $\forall b \in set\ bs. b \in \{0 < .. < 1\}$
assumes $k \neq 0$
assumes $length\ as = k$
assumes $length\ bs = k$
assumes $length\ ts = k$
assumes $akra-bazzi-terms\ x_0\ x_1\ bs\ ts$
shows $master-theorem-function\ x_0\ x_1\ k\ as\ bs\ ts\ f\ g$
 $\langle proof \rangle$

lemma *akra-bazzi-term-measure*:

$x \geq x_1 \implies akra-bazzi-term\ 0\ x_1\ b\ t \implies (t\ x, x) \in Wellfounded.measure\ (\lambda n :: nat. n)$
 $x > x_1 \implies akra-bazzi-term\ 0\ (Suc\ x_1)\ b\ t \implies (t\ x, x) \in Wellfounded.measure\ (\lambda n :: nat. n)$
 $\langle proof \rangle$

lemma *measure-prod-conv*:

$((a, b), (c, d)) \in Wellfounded.measure\ (\lambda x. t\ (fst\ x)) \longleftrightarrow (a, c) \in Wellfounded.measure\ t$
 $((e, f), (g, h)) \in Wellfounded.measure\ (\lambda x. t\ (snd\ x)) \longleftrightarrow (f, h) \in Wellfounded.measure\ t$
 $\langle proof \rangle$

lemmas $measure-prod-conv' = measure-prod-conv$ [where $t = \lambda x. x$]

lemma *akra-bazzi-termination-simps*:

fixes $x :: nat$
shows $a * real\ x / b = a/b * real\ x$ $real\ x / b = 1/b * real\ x$
 $\langle proof \rangle$

lemma *akra-bazzi-params-nonzeroI*:

$length\ as = length\ bs \implies$
 $(\forall a \in set\ as. a \geq 0) \implies (\forall b \in set\ bs. b \in \{0 < .. < 1\}) \implies (\exists a \in set\ as. a > 0) \implies$
 $akra-bazzi-params-nonzero\ (length\ as)\ as\ bs$ $\langle proof \rangle$

lemmas *akra-bazzi-p-rel-intros* =

$akra-bazzi-params-nonzero.p-lessI$ [rotated, OF - $akra-bazzi-params-nonzeroI$]
 $akra-bazzi-params-nonzero.p-greaterI$ [rotated, OF - $akra-bazzi-params-nonzeroI$]
 $akra-bazzi-params-nonzero.p-leI$ [rotated, OF - $akra-bazzi-params-nonzeroI$]
 $akra-bazzi-params-nonzero.p-geI$ [rotated, OF - $akra-bazzi-params-nonzeroI$]
 $akra-bazzi-params-nonzero.p-boundsI$ [rotated, OF - $akra-bazzi-params-nonzeroI$]
 $akra-bazzi-params-nonzero.p-boundsI'$ [rotated, OF - $akra-bazzi-params-nonzeroI$]

lemma *eval-length*: $length\ [] = 0$ $length\ (x \# xs) = Suc\ (length\ xs)$ $\langle proof \rangle$

lemma *eval-akra-bazzi-sum*:

$$\begin{aligned} & (\sum i < 0. as!i * bs!i \text{ powr } x) = 0 \\ & (\sum i < \text{Suc } 0. (a\#as)!i * (b\#bs)!i \text{ powr } x) = a * b \text{ powr } x \\ & (\sum i < \text{Suc } k. (a\#as)!i * (b\#bs)!i \text{ powr } x) = a * b \text{ powr } x + (\sum i < k. as!i * bs!i \\ & \text{ powr } x) \\ & \langle \text{proof} \rangle \end{aligned}$$

lemma *eval-akra-bazzi-sum'*:

$$\begin{aligned} & (\sum i < 0. as!i * f ((ts!i) x)) = 0 \\ & (\sum i < \text{Suc } 0. (a\#as)!i * f (((t\#ts)!i) x)) = a * f (t x) \\ & (\sum i < \text{Suc } k. (a\#as)!i * f (((t\#ts)!i) x)) = a * f (t x) + (\sum i < k. as!i * f ((ts!i) \\ & x)) \\ & \langle \text{proof} \rangle \end{aligned}$$

lemma *akra-bazzi-termsI'*:

$$\begin{aligned} & \text{akra-bazzi-terms } x_0 \ x_1 \ [] \ [] \\ & \text{akra-bazzi-term } x_0 \ x_1 \ b \ t \implies \text{akra-bazzi-terms } x_0 \ x_1 \ bs \ ts \implies \text{akra-bazzi-terms} \\ & x_0 \ x_1 \ (b\#bs) \ (t\#ts) \\ & \langle \text{proof} \rangle \end{aligned}$$

lemma *ball-set-intros*: $(\forall x \in \text{set } []. P \ x) \implies (\forall x \in \text{set } xs. P \ x) \implies (\forall x \in \text{set} (x\#xs). P \ x)$
 $\langle \text{proof} \rangle$

lemma *ball-set-simps*: $(\forall x \in \text{set } []. P \ x) = \text{True} \ (\forall x \in \text{set } (x\#xs). P \ x) = (P \ x \wedge (\forall x \in \text{set } xs. P \ x))$
 $\langle \text{proof} \rangle$

lemma *beX-set-simps*: $(\exists x \in \text{set } []. P \ x) = \text{False} \ (\exists x \in \text{set } (x\#xs). P \ x) = (P \ x \vee (\exists x \in \text{set } xs. P \ x))$
 $\langle \text{proof} \rangle$

lemma *eval-akra-bazzi-le-list-ex*:

$$\begin{aligned} & \text{list-ex } P \ (x\#y\#xs) \longleftrightarrow P \ x \vee \text{list-ex } P \ (y\#xs) \\ & \text{list-ex } P \ [x] \longleftrightarrow P \ x \\ & \text{list-ex } P \ [] \longleftrightarrow \text{False} \\ & \langle \text{proof} \rangle \end{aligned}$$

lemma *eval-akra-bazzi-le-sum-list*:

$$\begin{aligned} & x \leq \text{sum-list } [] \longleftrightarrow x \leq 0 \ x \leq \text{sum-list } (y\#ys) \longleftrightarrow x \leq y + \text{sum-list } ys \\ & x \leq z + \text{sum-list } [] \longleftrightarrow x \leq z \ x \leq z + \text{sum-list } (y\#ys) \longleftrightarrow x \leq z + y + \text{sum-list} \\ & ys \\ & \langle \text{proof} \rangle \end{aligned}$$

lemma *atLeastLessThanE*: $x \in \{a..<b\} \implies (x \geq a \implies x < b \implies P) \implies P$
 $\langle \text{proof} \rangle$

lemma *master-theorem-preprocess*:

$$\begin{aligned}
\Theta(\lambda n::nat. 1) &= \Theta(\lambda n::nat. real\ n\ powr\ 0) \\
\Theta(\lambda n::nat. real\ n) &= \Theta(\lambda n::nat. real\ n\ powr\ 1) \\
O(\lambda n::nat. 1) &= O(\lambda n::nat. real\ n\ powr\ 0) \\
O(\lambda n::nat. real\ n) &= O(\lambda n::nat. real\ n\ powr\ 1)
\end{aligned}$$

$$\begin{aligned}
\Theta(\lambda n::nat. \ln(\ln(\text{real } n))) &= \Theta(\lambda n::nat. real\ n\ powr\ 0 * \ln(\ln(\text{real } n))) \\
\Theta(\lambda n::nat. real\ n * \ln(\ln(\text{real } n))) &= \Theta(\lambda n::nat. real\ n\ powr\ 1 * \ln(\ln(\text{real } n)))
\end{aligned}$$

$$\begin{aligned}
\Theta(\lambda n::nat. \ln(\text{real } n)) &= \Theta(\lambda n::nat. real\ n\ powr\ 0 * \ln(\text{real } n)\ powr\ 1) \\
\Theta(\lambda n::nat. real\ n * \ln(\text{real } n)) &= \Theta(\lambda n::nat. real\ n\ powr\ 1 * \ln(\text{real } n)\ powr\ 1) \\
\Theta(\lambda n::nat. real\ n\ powr\ p * \ln(\text{real } n)) &= \Theta(\lambda n::nat. real\ n\ powr\ p * \ln(\text{real } n)\ powr\ 1) \\
\Theta(\lambda n::nat. \ln(\text{real } n)\ powr\ p') &= \Theta(\lambda n::nat. real\ n\ powr\ 0 * \ln(\text{real } n)\ powr\ p') \\
\Theta(\lambda n::nat. real\ n * \ln(\text{real } n)\ powr\ p') &= \Theta(\lambda n::nat. real\ n\ powr\ 1 * \ln(\text{real } n)\ powr\ p') \\
\langle proof \rangle
\end{aligned}$$

lemma *akra-bazzi-term-imp-size-less*:

$$\begin{aligned}
x_1 \leq x &\implies \text{akra-bazzi-term } 0\ x_1\ b\ t \implies \text{size } (t\ x) < \text{size } x \\
x_1 < x &\implies \text{akra-bazzi-term } 0\ (\text{Suc } x_1)\ b\ t \implies \text{size } (t\ x) < \text{size } x \\
\langle proof \rangle
\end{aligned}$$

definition *CLAMP* ($f :: nat \Rightarrow real$) $x = (\text{if } x < 3 \text{ then } 0 \text{ else } f\ x)$

definition *CLAMP'* ($f :: nat \Rightarrow real$) $x = (\text{if } x < 3 \text{ then } 0 \text{ else } f\ x)$

definition *MASTER-BOUND* $a\ b\ c\ x = real\ x\ powr\ a * \ln(\text{real } x)\ powr\ b * \ln(\ln(\text{real } x))\ powr\ c$

definition *MASTER-BOUND'* $a\ b\ x = real\ x\ powr\ a * \ln(\text{real } x)\ powr\ b$

definition *MASTER-BOUND''* $a\ x = real\ x\ powr\ a$

lemma *ln-1-imp-less-3*:

$$\ln\ x = (1::real) \implies x < 3 \\
\langle proof \rangle$$

lemma *ln-1-imp-less-3'*: $\ln(\text{real } (x::nat)) = 1 \implies x < 3$ $\langle proof \rangle$

lemma *ln-ln-nonneg*: $x \geq (3::real) \implies \ln(\ln\ x) \geq 0$ $\langle proof \rangle$

lemma *ln-ln-nonneg'*: $x \geq (3::nat) \implies \ln(\ln(\text{real } x)) \geq 0$ $\langle proof \rangle$

lemma *MASTER-BOUND-postproc*:

$$\begin{aligned}
CLAMP(\text{MASTER-BOUND}'\ a\ 0) &= CLAMP(\text{MASTER-BOUND}''\ a) \\
CLAMP(\text{MASTER-BOUND}'\ a\ 1) &= CLAMP(\lambda x. CLAMP(\text{MASTER-BOUND}''\ a)\ x * CLAMP(\lambda x. \ln(\text{real } x))\ x) \\
CLAMP(\text{MASTER-BOUND}'\ a\ (\text{numeral } n)) &= \\
&\quad CLAMP(\lambda x. CLAMP(\text{MASTER-BOUND}''\ a)\ x * CLAMP(\lambda x. \ln(\text{real } x))\ x) \\
&\quad \wedge\ \text{numeral } n\ x) \\
CLAMP(\text{MASTER-BOUND}'\ a\ (-1)) &= \\
&\quad CLAMP(\lambda x. CLAMP(\text{MASTER-BOUND}''\ a)\ x / CLAMP(\lambda x. \ln(\text{real } x))\ x) \\
CLAMP(\text{MASTER-BOUND}'\ a\ (-\text{numeral } n)) &=
\end{aligned}$$

$$\begin{aligned} & \text{CLAMP } (\lambda x. \text{CLAMP } (\text{MASTER-BOUND'' } a) x / \text{CLAMP } (\lambda x. \text{ln } (\text{real } x) \\ & \text{^ numeral } n) x) \\ & \text{CLAMP } (\text{MASTER-BOUND' } a b) = \\ & \text{CLAMP } (\lambda x. \text{CLAMP } (\text{MASTER-BOUND'' } a) x * \text{CLAMP } (\lambda x. \text{ln } (\text{real } x) \\ & \text{powr } b) x) \end{aligned}$$

$$\begin{aligned} & \text{CLAMP } (\text{MASTER-BOUND'' } 0) = \text{CLAMP } (\lambda x. 1) \\ & \text{CLAMP } (\text{MASTER-BOUND'' } 1) = \text{CLAMP } (\lambda x. (\text{real } x)) \\ & \text{CLAMP } (\text{MASTER-BOUND'' } (\text{numeral } n)) = \text{CLAMP } (\lambda x. (\text{real } x) \text{^ numeral } \\ & n) \\ & \text{CLAMP } (\text{MASTER-BOUND'' } (-1)) = \text{CLAMP } (\lambda x. 1 / (\text{real } x)) \\ & \text{CLAMP } (\text{MASTER-BOUND'' } (-\text{numeral } n)) = \text{CLAMP } (\lambda x. 1 / (\text{real } x) \text{^ } \\ & \text{numeral } n) \\ & \text{CLAMP } (\text{MASTER-BOUND'' } a) = \text{CLAMP } (\lambda x. (\text{real } x) \text{ powr } a) \end{aligned}$$

and MASTER-BOUND-UNCLAMP:

$$\begin{aligned} & \text{CLAMP } (\lambda x. \text{CLAMP } f x * \text{CLAMP } g x) = \text{CLAMP } (\lambda x. f x * g x) \\ & \text{CLAMP } (\lambda x. \text{CLAMP } f x / \text{CLAMP } g x) = \text{CLAMP } (\lambda x. f x / g x) \\ & \text{CLAMP } (\text{CLAMP } f) = \text{CLAMP } f \\ & \langle \text{proof} \rangle \end{aligned}$$

context

begin

private lemma CLAMP-:

$$\text{landau-symbol } L L' Lr \implies L \text{ at-top } (f :: \text{nat} \Rightarrow \text{real}) \equiv L \text{ at-top } (\lambda x. \text{CLAMP } f x)$$

$\langle \text{proof} \rangle$ **lemma UNCLAMP'-:**

$$\text{landau-symbol } L L' Lr \implies L \text{ at-top } (\text{CLAMP' } (\text{MASTER-BOUND } a b c)) \equiv L \text{ at-top } (\text{MASTER-BOUND } a b c)$$

$\langle \text{proof} \rangle$ **lemma UNCLAMP-:**

$$\text{landau-symbol } L L' Lr \implies L \text{ at-top } (\text{CLAMP } f) \equiv L \text{ at-top } (f)$$

$\langle \text{proof} \rangle$

lemmas CLAMP = landau-symbols[THEN CLAMP-]

lemmas UNCLAMP' = landau-symbols[THEN UNCLAMP'-]

lemmas UNCLAMP = landau-symbols[THEN UNCLAMP-]

end

lemma propagate-CLAMP:

$$\text{CLAMP } (\lambda x. f x * g x) = \text{CLAMP' } (\lambda x. \text{CLAMP } f x * \text{CLAMP } g x)$$

$$\text{CLAMP } (\lambda x. f x / g x) = \text{CLAMP' } (\lambda x. \text{CLAMP } f x / \text{CLAMP } g x)$$

$$\text{CLAMP } (\lambda x. \text{inverse } (f x)) = \text{CLAMP' } (\lambda x. \text{inverse } (\text{CLAMP } f x))$$

$$\text{CLAMP } (\lambda x. \text{real } x) = \text{CLAMP' } (\text{MASTER-BOUND } 1 0 0)$$

$$\text{CLAMP } (\lambda x. \text{real } x \text{ powr } a) = \text{CLAMP' } (\text{MASTER-BOUND } a 0 0)$$

$$\text{CLAMP } (\lambda x. \text{real } x \text{^ } a) = \text{CLAMP' } (\text{MASTER-BOUND } (\text{real } a) 0 0)$$

$$\text{CLAMP } (\lambda x. \text{ln } (\text{real } x)) = \text{CLAMP' } (\text{MASTER-BOUND } 0 1 0)$$

$$\text{CLAMP } (\lambda x. \text{ln } (\text{real } x) \text{ powr } b) = \text{CLAMP' } (\text{MASTER-BOUND } 0 b 0)$$

$CLAMP (\lambda x. \ln (\text{real } x) \wedge b') = CLAMP' (MASTER-BOUND 0 (\text{real } b') 0)$
 $CLAMP (\lambda x. \ln (\ln (\text{real } x))) = CLAMP' (MASTER-BOUND 0 0 1)$
 $CLAMP (\lambda x. \ln (\ln (\text{real } x)) \text{ powr } c) = CLAMP' (MASTER-BOUND 0 0 c)$
 $CLAMP (\lambda x. \ln (\ln (\text{real } x)) \wedge c') = CLAMP' (MASTER-BOUND 0 0 (\text{real } c'))$
 $CLAMP' (CLAMP f) = CLAMP' f$
 $CLAMP' (\lambda x. CLAMP' (MASTER-BOUND a1 b1 c1) x * CLAMP' (MASTER-BOUND a2 b2 c2) x) =$
 $CLAMP' (MASTER-BOUND (a1+a2) (b1+b2) (c1+c2))$
 $CLAMP' (\lambda x. CLAMP' (MASTER-BOUND a1 b1 c1) x / CLAMP' (MASTER-BOUND a2 b2 c2) x) =$
 $CLAMP' (MASTER-BOUND (a1-a2) (b1-b2) (c1-c2))$
 $CLAMP' (\lambda x. \text{inverse } (MASTER-BOUND a1 b1 c1) x) = CLAMP' (MASTER-BOUND (-a1) (-b1) (-c1))$
 <proof>

lemma *numeral-assoc-simps*:

$((a::\text{real}) + \text{numeral } b) + \text{numeral } c = a + \text{numeral } (b + c)$
 $(a + \text{numeral } b) - \text{numeral } c = a + \text{neg-numeral-class.sub } b c$
 $(a - \text{numeral } b) + \text{numeral } c = a + \text{neg-numeral-class.sub } c b$
 $(a - \text{numeral } b) - \text{numeral } c = a - \text{numeral } (b + c)$ <proof>

lemmas *CLAMP-aux* =

arith-simps numeral-assoc-simps of-nat-power of-nat-mult of-nat-numeral
one-add-one numeral-One [symmetric]

lemmas *CLAMP-postproc* = *numeral-One*

context *master-theorem-function*

begin

lemma *master1-bigo-automation*:

assumes $g \in O(\lambda x. \text{real } x \text{ powr } p')$ $1 < (\sum i < k. a_s ! i * b_s ! i \text{ powr } p')$
shows $f \in O(MASTER-BOUND p 0 0)$
 <proof>

lemma *master1-automation*:

assumes $g \in O(MASTER-BOUND'' p')$ $1 < (\sum i < k. a_s ! i * b_s ! i \text{ powr } p')$
eventually $(\lambda x. f x > 0)$ *at-top*
shows $f \in \Theta(MASTER-BOUND p 0 0)$
 <proof>

lemma *master2-1-automation*:

assumes $g \in \Theta(MASTER-BOUND' p p')$ $p' < -1$
shows $f \in \Theta(MASTER-BOUND p 0 0)$
 <proof>

lemma *master2-2-automation*:

assumes $g \in \Theta(MASTER-BOUND' p (-1))$
shows $f \in \Theta(MASTER-BOUND p 0 1)$

<proof>

lemma *master2-3-automation*:

assumes $g \in \Theta(\text{MASTER-BOUND}' p (p' - 1))$ $p' > 0$

shows $f \in \Theta(\text{MASTER-BOUND } p p' 0)$

<proof>

lemma *master3-automation*:

assumes $g \in \Theta(\text{MASTER-BOUND}'' p')$ $1 > (\sum_{i < k}. as ! i * bs ! i \text{ powr } p')$

shows $f \in \Theta(\text{MASTER-BOUND } p' 0 0)$

<proof>

lemmas *master-automation =*

master1-automation master2-1-automation master2-2-automation

master2-2-automation master3-automation

<ML>

end

definition *arith-consts* ($x :: \text{real}$) ($y :: \text{nat}$) =

*(if $\neg (-x) + 3 / x * 5 - 1 \leq x \wedge \text{True} \vee \text{True} \longrightarrow \text{True}$ then*

$x < \text{inverse } 3 \text{ powr } 21$ else $x = \text{real } (\text{Suc } 0 \wedge 2 +$

$(\text{if } 42 - x \leq 1 \wedge 1 \text{ div } y = y \text{ mod } 2 \vee y < \text{Numeral1}$ then 0 else 0)) + Numeral1)

<ML>

hide-const *arith-consts*

<ML>

hide-const *CLAMP CLAMP' MASTER-BOUND MASTER-BOUND' MASTER-BOUND''*

end

theory *Akra-Bazzi-Approximation*

imports

Complex-Main

Akra-Bazzi-Method

HOL-Decision-Procs.Approximation

begin

context *akra-bazzi-params-nonzero*

begin

lemma *sum-alt*: $(\sum_{i < k}. as ! i * bs ! i \text{ powr } p') = (\sum_{i < k}. as ! i * \text{exp } (p' * \ln (bs ! i)))$

<proof>

lemma *akra-bazzi-p-rel-intros-aux*:

$1 < (\sum_{i < k}. as!i * \exp (p' * \ln (bs!i))) \implies p' < p$
 $1 > (\sum_{i < k}. as!i * \exp (p' * \ln (bs!i))) \implies p' > p$
 $1 \leq (\sum_{i < k}. as!i * \exp (p' * \ln (bs!i))) \implies p' \leq p$
 $1 \geq (\sum_{i < k}. as!i * \exp (p' * \ln (bs!i))) \implies p' \geq p$
 $(\sum_{i < k}. as!i * \exp (x * \ln (bs!i))) \leq 1 \wedge (\sum_{i < k}. as!i * \exp (y * \ln (bs!i))) \geq 1 \implies p \in \{y..x\}$
 $(\sum_{i < k}. as!i * \exp (x * \ln (bs!i))) < 1 \wedge (\sum_{i < k}. as!i * \exp (y * \ln (bs!i))) > 1 \implies p \in \{y < .. < x\}$
<proof>

end

lemmas *akra-bazzi-p-rel-intros-exp* =

akra-bazzi-params-nonzero.akra-bazzi-p-rel-intros-aux[rotated, OF - akra-bazzi-params-nonzeroI]

lemma *eval-akra-bazzi-sum*:

$(\sum_{i < 0}. as!i * \exp (x * \ln (bs!i))) = 0$
 $(\sum_{i < \text{Suc } 0}. (a\#as)!i * \exp (x * \ln ((b\#bs)!i))) = a * \exp (x * \ln b)$
 $(\sum_{i < \text{Suc } k}. (a\#as)!i * \exp (x * \ln ((b\#bs)!i))) = a * \exp (x * \ln b) +$
 $(\sum_{i < k}. as!i * \exp (x * \ln (bs!i)))$
<proof>

<ML>

end

8 Examples

theory *Master-Theorem-Examples*

imports

Complex-Main

Akra-Bazzi-Method

Akra-Bazzi-Approximation

begin

8.1 Merge sort

function *merge-sort-cost* :: $(\text{nat} \Rightarrow \text{real}) \Rightarrow \text{nat} \Rightarrow \text{real}$ **where**

merge-sort-cost *t* 0 = 0

| *merge-sort-cost* *t* 1 = 1

| $n \geq 2 \implies \text{merge-sort-cost } t \ n =$

merge-sort-cost *t* ($\text{nat } \lfloor \text{real } n / 2 \rfloor$) + *merge-sort-cost* *t* ($\text{nat } \lceil \text{real } n / 2 \rceil$) + *t*

n

<proof>

termination $\langle proof \rangle$

lemma *merge-sort-nonneg[simp]*: $(\bigwedge n. t\ n \geq 0) \implies \text{merge-sort-cost } t\ x \geq 0$
 $\langle proof \rangle$

lemma $t \in \Theta(\lambda n. \text{real } n) \implies (\bigwedge n. t\ n \geq 0) \implies \text{merge-sort-cost } t \in \Theta(\lambda n. \text{real } n * \ln(\text{real } n))$
 $\langle proof \rangle$

8.2 Karatsuba multiplication

function *karatsuba-cost* :: $\text{nat} \Rightarrow \text{real}$ **where**

karatsuba-cost 0 = 0
| *karatsuba-cost* 1 = 1
| $n \geq 2 \implies \text{karatsuba-cost } n =$
 $3 * \text{karatsuba-cost } (\text{nat } \lceil \text{real } n / 2 \rceil) + \text{real } n$
 $\langle proof \rangle$

termination $\langle proof \rangle$

lemma *karatsuba-cost-nonneg[simp]*: $\text{karatsuba-cost } n \geq 0$
 $\langle proof \rangle$

lemma $\text{karatsuba-cost} \in O(\lambda n. \text{real } n \text{ powr } \log 2 3)$
 $\langle proof \rangle$

lemma *karatsuba-cost-pos*: $n \geq 1 \implies \text{karatsuba-cost } n > 0$
 $\langle proof \rangle$

lemma $\text{karatsuba-cost} \in \Theta(\lambda n. \text{real } n \text{ powr } \log 2 3)$
 $\langle proof \rangle$

8.3 Strassen matrix multiplication

function *strassen-cost* :: $\text{nat} \Rightarrow \text{real}$ **where**

strassen-cost 0 = 0
| *strassen-cost* 1 = 1
| $n \geq 2 \implies \text{strassen-cost } n = 7 * \text{strassen-cost } (\text{nat } \lceil \text{real } n / 2 \rceil) + \text{real } (n^2)$
 $\langle proof \rangle$

termination $\langle proof \rangle$

lemma *strassen-cost-nonneg[simp]*: $\text{strassen-cost } n \geq 0$
 $\langle proof \rangle$

lemma $\text{strassen-cost} \in O(\lambda n. \text{real } n \text{ powr } \log 2 7)$
 $\langle proof \rangle$

lemma *strassen-cost-pos*: $n \geq 1 \implies \text{strassen-cost } n > 0$
 $\langle proof \rangle$

lemma $\text{strassen-cost} \in \Theta(\lambda n. \text{real } n \text{ powr } \log 2 7)$

<proof>

8.4 Deterministic select

function *select-cost* :: *nat* \Rightarrow *real* **where**

$n \leq 20 \implies \text{select-cost } n = 0$

| $n > 20 \implies \text{select-cost } n =$

$\text{select-cost } (\text{nat } \lfloor \text{real } n / 5 \rfloor) + \text{select-cost } (\text{nat } \lfloor 7 * \text{real } n / 10 \rfloor + 6) + 12$
 $* \text{real } n / 5$

<proof>

termination *<proof>*

lemma *select-cost* $\in \Theta(\lambda n. \text{real } n)$

<proof>

8.5 Decreasing function

function *dec-cost* :: *nat* \Rightarrow *real* **where**

$n \leq 2 \implies \text{dec-cost } n = 1$

| $n > 2 \implies \text{dec-cost } n = 0.5 * \text{dec-cost } (\text{nat } \lfloor \text{real } n / 2 \rfloor) + 1 / \text{real } n$

<proof>

termination *<proof>*

lemma *dec-cost* $\in \Theta(\lambda x :: \text{nat}. \ln x / x)$

<proof>

8.6 Example taken from Drmota and Szpakowski

function *drmota1* :: *nat* \Rightarrow *real* **where**

$n < 20 \implies \text{drmota1 } n = 1$

| $n \geq 20 \implies \text{drmota1 } n = 2 * \text{drmota1 } (\text{nat } \lfloor \text{real } n / 2 \rfloor) + 8/9 * \text{drmota1 } (\text{nat } \lfloor 3 * \text{real } n / 4 \rfloor) + \text{real } n^2 / \ln (\text{real } n)$

<proof>

termination *<proof>*

lemma *drmota1* $\in \Theta(\lambda n :: \text{real}. n^2 * \ln (\ln n))$

<proof>

function *drmota2* :: *nat* \Rightarrow *real* **where**

$n < 20 \implies \text{drmota2 } n = 1$

| $n \geq 20 \implies \text{drmota2 } n = 1/3 * \text{drmota2 } (\text{nat } \lfloor \text{real } n / 3 + 1/2 \rfloor) + 2/3 * \text{drmota2 } (\text{nat } \lfloor 2 * \text{real } n / 3 - 1/2 \rfloor) + 1$

<proof>

termination *<proof>*

lemma *drmota2* $\in \Theta(\lambda x. \ln (\text{real } x))$

<proof>

lemma *boncelet-phrase-length*:
fixes $p \delta :: \text{real}$ **assumes** $p: p > 0 \ p < 1$ **and** $\delta: \delta > 0 \ \delta < 1 \ 2*p + \delta < 2$
fixes $d :: \text{nat} \Rightarrow \text{real}$
defines $q \equiv 1 - p$
assumes $d\text{-nonneg}: \bigwedge n. d \ n \geq 0$
assumes $d\text{-rec}: \bigwedge n. n \geq 2 \implies d \ n = 1 + p * d \ (\text{nat} \lfloor p * \text{real } n + \delta \rfloor) + q * d$
 $(\text{nat} \lfloor q * \text{real } n - \delta \rfloor)$
shows $d \in \Theta(\lambda x. \ln x)$
 $\langle \text{proof} \rangle$

8.7 Transcendental exponents

function *foo-cost* $:: \text{nat} \Rightarrow \text{real}$ **where**
 $n < 200 \implies \text{foo-cost } n = 0$
 $| \ n \geq 200 \implies \text{foo-cost } n =$
 $\text{foo-cost } (\text{nat} \lfloor \text{real } n / 3 \rfloor) + \text{foo-cost } (\text{nat} \lfloor 3 * \text{real } n / 4 \rfloor + 42) + \text{real } n$
 $\langle \text{proof} \rangle$
termination $\langle \text{proof} \rangle$

lemma *foo-cost-nonneg* [*simp*]: $\text{foo-cost } n \geq 0$
 $\langle \text{proof} \rangle$

lemma $\text{foo-cost} \in \Theta(\lambda n. \text{real } n \text{ powr } \text{akra-bazzi-exponent } [1,1] [1/3,3/4])$
 $\langle \text{proof} \rangle$

lemma $\text{akra-bazzi-exponent } [1,1] [1/3,3/4] \in \{1.1519623..1.1519624\}$
 $\langle \text{proof} \rangle$

8.8 Functions in locale contexts

locale *det-select* =
fixes $b :: \text{real}$
assumes $b: b > 0 \ b < 7/10$
begin

function *select-cost'* $:: \text{nat} \Rightarrow \text{real}$ **where**
 $n \leq 20 \implies \text{select-cost}' \ n = 0$
 $| \ n > 20 \implies \text{select-cost}' \ n =$
 $\text{select-cost}' \ (\text{nat} \lfloor \text{real } n / 5 \rfloor) + \text{select-cost}' \ (\text{nat} \lfloor b * \text{real } n \rfloor + 6) + 6 * \text{real}$
 $n + 5$
 $\langle \text{proof} \rangle$
termination $\langle \text{proof} \rangle$

lemma $a \geq 0 \implies \text{select-cost}' \in \Theta(\lambda n. \text{real } n)$
 $\langle \text{proof} \rangle$

end

8.9 Non-curried functions

function *baz-cost* :: *nat* × *nat* ⇒ *real* **where**

$n \leq 2 \implies \text{baz-cost } (a, n) = 0$

$| n > 2 \implies \text{baz-cost } (a, n) = 3 * \text{baz-cost } (a, \text{nat } \lfloor \text{real } n / 2 \rfloor) + \text{real } a$
⟨*proof*⟩

termination ⟨*proof*⟩

lemma *baz-cost-nonneg* [*simp*]: $a \geq 0 \implies \text{baz-cost } (a, n) \geq 0$

⟨*proof*⟩

lemma

assumes $a > 0$

shows $(\lambda x. \text{baz-cost } (a, x)) \in \Theta(\lambda x. x \text{ powr } \log 2 3)$

⟨*proof*⟩

function *bar-cost* :: *nat* × *nat* ⇒ *real* **where**

$n \leq 2 \implies \text{bar-cost } (a, n) = 0$

$| n > 2 \implies \text{bar-cost } (a, n) = 3 * \text{bar-cost } (2 * a, \text{nat } \lfloor \text{real } n / 2 \rfloor) + \text{real } a$
⟨*proof*⟩

termination ⟨*proof*⟩

8.10 Ham-sandwich trees

function *ham-sandwich-cost* :: *nat* ⇒ *real* **where**

$n < 4 \implies \text{ham-sandwich-cost } n = 1$

$| n \geq 4 \implies \text{ham-sandwich-cost } n =$
 $\text{ham-sandwich-cost } (\text{nat } \lfloor n/4 \rfloor) + \text{ham-sandwich-cost } (\text{nat } \lfloor n/2 \rfloor) + 1$

⟨*proof*⟩

termination ⟨*proof*⟩

lemma *ham-sandwich-cost-pos* [*simp*]: $\text{ham-sandwich-cost } n > 0$

⟨*proof*⟩

The golden ratio

definition $\varphi = ((1 + \text{sqrt } 5) / 2 :: \text{real})$

lemma *φ-pos* [*simp*]: $\varphi > 0$ **and** *φ-nonneg* [*simp*]: $\varphi \geq 0$ **and** *φ-nonzero* [*simp*]:

$\varphi \neq 0$

⟨*proof*⟩

lemma *ham-sandwich-cost* ∈ $\Theta(\lambda n. n \text{ powr } (\log 2 \varphi))$

⟨*proof*⟩

end

References

- [1] M. Akra and L. Bazzi. On the solution of linear recurrence equations. *Computational Optimization and Applications*, 10(2):195–210, 1998.
- [2] T. Leighton. Notes on better Master theorems for divide-and-conquer recurrences. 1996.