The Akra–Bazzi theorem and the Master theorem

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Abstract

This article contains a formalisation of the Akra–Bazzi method [1] based on a proof by Leighton [2]. It is a generalisation of the well-known Master Theorem for analysing the complexity of Divide & Conquer algorithms. We also include a generalised version of the Master theorem based on the Akra–Bazzi theorem, which is easier to apply than the Akra–Bazzi theorem itself.

Some proof methods that facilitate applying the Master theorem are also included. For a more detailed explanation of the formalisation and the proof methods, see the accompanying paper (publication forthcoming).

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1 Auxiliary lemmas

theory *Akra-Bazzi-Library*
imports
Complex-Main
Landau-Symbols.Landau-More
Pure−ex.Guess
begin

lemma *ln-mono*: $0 < x \Longrightarrow 0 < y \Longrightarrow x \leq y \Longrightarrow ln\ (x{::}real) \leq ln\ y$
⟨*proof*⟩

lemma *ln-mono-strict*: $0 < x \Longrightarrow 0 < y \Longrightarrow x < y \Longrightarrow ln\ (x{::}real) < ln\ y$
⟨*proof*⟩

declare *DERIV-powr*[*THEN DERIV-chain2*, *derivative-intros*]

lemma *sum-pos'*:
assumes *finite I*
assumes $\exists x \in I.\ f\ x > (0 :: \text{-} :: linordered\text{-}ab\text{-}group\text{-}add)$
assumes $\bigwedge x.\ x \in I \Longrightarrow f\ x \geq 0$
shows $sum\ f\ I > 0$
⟨*proof*⟩

lemma *min-mult-left*:
assumes $(x{::}real) > 0$
shows $x * min\ y\ z = min\ (x{*}y)\ (x{*}z)$
⟨*proof*⟩

lemma *max-mult-left*:
assumes $(x{::}real) > 0$
shows $x * max\ y\ z = max\ (x{*}y)\ (x{*}z)$
⟨*proof*⟩

lemma *DERIV-nonneg-imp-mono*:
assumes $\bigwedge t.\ t \in \{x..y\} \Longrightarrow (f\ has\text{-}field\text{-}derivative\ f'\ t)\ (at\ t)$
assumes $\bigwedge t.\ t \in \{x..y\} \Longrightarrow f'\ t \geq 0$
assumes $(x{::}real) \leq y$
shows $(f\ x :: real) \leq f\ y$

$\langle proof \rangle$

lemma *eventually-conjE*: *eventually* $(\lambda x.\ P\ x \wedge Q\ x)\ F \Longrightarrow (eventually\ P\ F \Longrightarrow eventually\ Q\ F \Longrightarrow R) \Longrightarrow R$
$\langle proof \rangle$

lemma *real-natfloor-nat*: $x \in \mathbb{N} \Longrightarrow real\ (nat\ \lfloor x \rfloor) = x$ $\langle proof \rangle$

lemma *eventually-natfloor*:
assumes *eventually* P $(at\text{-}top :: nat\ filter)$
shows *eventually* $(\lambda x.\ P\ (nat\ \lfloor x \rfloor))\ (at\text{-}top :: real\ filter)$
$\langle proof \rangle$

lemma *tendsto-0-smallo-1*: $f \in o(\lambda x.\ 1 :: real) \Longrightarrow (f \longrightarrow 0)\ at\text{-}top$
$\langle proof \rangle$

lemma *smallo-1-tendsto-0*: $(f \longrightarrow 0)\ at\text{-}top \Longrightarrow f \in o(\lambda x.\ 1 :: real)$
$\langle proof \rangle$

lemma *filterlim-at-top-smallomega-1*:
assumes $f \in \omega[F](\lambda x.\ 1 :: real)$ *eventually* $(\lambda x.\ f\ x > 0)\ F$
shows *filterlim f at-top F*
$\langle proof \rangle$

lemma *smallo-imp-abs-less-real*:
assumes $f \in o[F](g)$ *eventually* $(\lambda x.\ g\ x > (0::real))\ F$
shows *eventually* $(\lambda x.\ |f\ x| < g\ x)\ F$
$\langle proof \rangle$

lemma *smallo-imp-less-real*:
assumes $f \in o[F](g)$ *eventually* $(\lambda x.\ g\ x > (0::real))\ F$
shows *eventually* $(\lambda x.\ f\ x < g\ x)\ F$
$\langle proof \rangle$

lemma *smallo-imp-le-real*:
assumes $f \in o[F](g)$ *eventually* $(\lambda x.\ g\ x \geq (0::real))\ F$
shows *eventually* $(\lambda x.\ f\ x \leq g\ x)\ F$
$\langle proof \rangle$

lemma *filterlim-at-right*:
filterlim f $(at\text{-}right\ a)\ F \longleftrightarrow eventually\ (\lambda x.\ f\ x > a)\ F \wedge filterlim\ f\ (nhds\ a)\ F$
$\langle proof \rangle$

lemma *one-plus-x-powr-approx-ex*:
assumes x: $abs\ (x::real) \leq 1/2$
obtains t **where** $abs\ t < 1/2$ $(1 + x)\ powr\ p =$
$1 + p * x + p * (p - 1) * (1 + t)\ powr\ (p - 2) / 2 * x \hat{}\ 2$

⟨*proof*⟩

lemma *powr-lower-bound*: $[\![(l{::}real) > 0;\ l \le x;\ x \le u]\!] \Longrightarrow min\ (l\ powr\ z)\ (u\ powr\ z) \le x\ powr\ z$
⟨*proof*⟩

lemma *powr-upper-bound*: $[\![(l{::}real) > 0;\ l \le x;\ x \le u]\!] \Longrightarrow max\ (l\ powr\ z)\ (u\ powr\ z) \ge x\ powr\ z$
⟨*proof*⟩

lemma *one-plus-x-powr-Taylor2*:
obtains *k* **where** $\bigwedge x.\ abs\ (x{::}real) \le 1/2 \Longrightarrow abs\ ((1 + x)\ powr\ p - 1 - p{*}x) \le k{*}x\hat{\ }2$
⟨*proof*⟩

lemma *one-plus-x-powr-Taylor2-bigo*:
assumes *lim*: $(f \longrightarrow 0)\ F$
shows $(\lambda x.\ (1 + f\ x)\ powr\ (p{::}real) - 1 - p * f\ x) \in O[F](\lambda x.\ f\ x\ \hat{}\ 2)$
⟨*proof*⟩

lemma *one-plus-x-powr-Taylor1-bigo*:
assumes *lim*: $(f \longrightarrow 0)\ F$
shows $(\lambda x.\ (1 + f\ x)\ powr\ (p{::}real) - 1) \in O[F](\lambda x.\ f\ x)$
⟨*proof*⟩

lemma *x-times-x-minus-1-nonneg*: $x \le 0 \lor x \ge 1 \Longrightarrow (x{::}\text{-}{::}linordered\text{-}idom) * (x - 1) \ge 0$
⟨*proof*⟩

lemma *x-times-x-minus-1-nonpos*: $x \ge 0 \Longrightarrow x \le 1 \Longrightarrow (x{::}\text{-}{::}linordered\text{-}idom) * (x - 1) \le 0$
⟨*proof*⟩

lemma *powr-mono'*:
assumes $(x{::}real) > 0\ x \le 1\ a \le b$
shows $x\ powr\ b \le x\ powr\ a$
⟨*proof*⟩

lemma *powr-less-mono'*:
assumes $(x{::}real) > 0\ x < 1\ a < b$
shows $x\ powr\ b < x\ powr\ a$
⟨*proof*⟩

lemma *real-powr-at-bot*:
assumes $(a{::}real) > 1$
shows $((\lambda x.\ a\ powr\ x) \longrightarrow 0)\ at\text{-}bot$
⟨*proof*⟩

lemma *real-powr-at-bot-neg*:

assumes $(a::real) > 0$ $a < 1$
shows *filterlim* $(\lambda x.\ a$ *powr* $x)$ *at-top at-bot*
⟨*proof*⟩

lemma *real-powr-at-top-neg*:
assumes $(a::real) > 0$ $a < 1$
shows $((\lambda x.\ a$ *powr* $x) \longrightarrow 0)$ *at-top*
⟨*proof*⟩

lemma *eventually-nat-real*:
assumes *eventually* P (*at-top* :: *real filter*)
shows *eventually* $(\lambda x.\ P$ (*real* x)) (*at-top* :: *nat filter*)
⟨*proof*⟩

end

2 Asymptotic bounds

theory *Akra-Bazzi-Asymptotics*
imports
Complex-Main
Akra-Bazzi-Library
HOL−Library.Landau-Symbols
begin

locale *akra-bazzi-asymptotics-bep* =
fixes b e p hb :: *real*
assumes *bep*: $b > 0$ $b < 1$ $e > 0$ $hb > 0$
begin

context
begin

Functions that are negligible w.r.t. $ln\ (b * x)$ *powr* $(e\ /\ 2 + 1)$.

private abbreviation (*input*) *negl* :: (*real* ⇒ *real*) ⇒ *bool* **where**
negl $f \equiv f \in o(\lambda x.\ ln\ (b{*}x)$ *powr* $(-(e/2 + 1)))$

private lemma *neglD*: *negl* $f \Longrightarrow c > 0 \Longrightarrow$ *eventually* $(\lambda x.\ |f\ x| \leq c\ /\ ln\ (b{*}x)$ *powr* $(e/2{+}1))$ *at-top*
⟨*proof*⟩ **lemma** *negl-mult*: *negl* $f \Longrightarrow$ *negl* $g \Longrightarrow$ *negl* $(\lambda x.\ f\ x * g\ x)$
⟨*proof*⟩ **lemma** *ev4*:
assumes *g*: *negl* g
shows *eventually* $(\lambda x.\ ln\ (b{*}x)$ *powr* $(-e/2) - ln\ x$ *powr* $(-e/2) \geq g\ x)$ *at-top*
⟨*proof*⟩ **lemma** *ev1*:
negl $(\lambda x.\ (1 + c *$ *inverse* $b * ln\ x$ *powr* $(-(1{+}e)))$ *powr* $p - 1)$
⟨*proof*⟩ **lemma** *ev2-aux*:
defines $f \equiv \lambda x.\ (1 + 1/ln\ (b{*}x) * ln\ (1 + hb\ /\ b * ln\ x$ *powr* $(-1{-}e)))$ *powr* $(-e/2)$
obtains h **where** *eventually* $(\lambda x.\ f\ x \geq 1 + h\ x)$ *at-top* $h \in o(\lambda x.\ 1\ /\ ln\ x)$

⟨*proof*⟩ **lemma** *ev2*:
defines $f \equiv \lambda x.\ ln\ (b * x + hb * x\ /\ ln\ x\ powr\ (1 + e))\ powr\ (-e/2)$
obtains h **where**
negl h
$eventually\ (\lambda x.\ f\ x \geq ln\ (b * x)\ powr\ (-e/2) + h\ x)\ at\text{-}top$
$eventually\ (\lambda x.\ |ln\ (b * x)\ powr\ (-e/2) + h\ x| < 1)\ at\text{-}top$
⟨*proof*⟩ **lemma** *ev21*:
obtains g **where**
negl g
$eventually\ (\lambda x.\ 1 + ln\ (b * x + hb * x\ /\ ln\ x\ powr\ (1 + e))\ powr\ (-e/2) \geq$
$1 + ln\ (b * x)\ powr\ (-e/2) + g\ x)\ at\text{-}top$
$eventually\ (\lambda x.\ 1 + ln\ (b * x)\ powr\ (-e/2) + g\ x > 0)\ at\text{-}top$
⟨*proof*⟩ **lemma** *ev22*:
obtains g **where**
negl g
$eventually\ (\lambda x.\ 1 - ln\ (b * x + hb * x\ /\ ln\ x\ powr\ (1 + e))\ powr\ (-e/2) \leq$
$1 - ln\ (b * x)\ powr\ (-e/2) - g\ x)\ at\text{-}top$
$eventually\ (\lambda x.\ 1 - ln\ (b * x)\ powr\ (-e/2) - g\ x > 0)\ at\text{-}top$
⟨*proof*⟩

lemma *asymptotics1*:
shows *eventually* $(\lambda x.$
$(1 + c * inverse\ b * ln\ x\ powr\ -(1+e))\ powr\ p\ *$
$(1 + ln\ (b * x + hb * x\ /\ ln\ x\ powr\ (1 + e))\ powr\ (-\ e\ /\ 2)) \geq$
$1 + (ln\ x\ powr\ (-e/2)))\ at\text{-}top$
⟨*proof*⟩

lemma *asymptotics2*:
shows *eventually* $(\lambda x.$
$(1 + c * inverse\ b * ln\ x\ powr\ -(1+e))\ powr\ p\ *$
$(1 - ln\ (b * x + hb * x\ /\ ln\ x\ powr\ (1 + e))\ powr\ (-\ e\ /\ 2)) \leq$
$1 - (ln\ x\ powr\ (-e/2)))\ at\text{-}top$
⟨*proof*⟩

lemma *asymptotics3*: $eventually\ (\lambda x.\ (1 + (ln\ x\ powr\ (-e/2)))\ /\ 2 \leq 1)\ at\text{-}top$
(**is** $eventually\ (\lambda x.\ ?f\ x \leq 1)$ -)
⟨*proof*⟩

lemma *asymptotics4*: $eventually\ (\lambda x.\ (1 - (ln\ x\ powr\ (-e/2))) * 2 \geq 1)\ at\text{-}top$
(**is** $eventually\ (\lambda x.\ ?f\ x \geq 1)$ -)
⟨*proof*⟩

lemma *asymptotics5*: $eventually\ (\lambda x.\ ln\ (b{*}x - hb{*}x{*}ln\ x\ powr\ -(1+e))\ powr\ (-e/2) < 1)\ at\text{-}top$
⟨*proof*⟩

lemma *asymptotics6*: $eventually\ (\lambda x.\ hb\ /\ ln\ x\ powr\ (1 + e) < b/2)\ at\text{-}top$
and *asymptotics7*: $eventually\ (\lambda x.\ hb\ /\ ln\ x\ powr\ (1 + e) < (1 - b)\ /\ 2)\ at\text{-}top$

and *asymptotics8*: *eventually* ($\lambda x.\ x*(1 - b - hb\ /\ ln\ x\ powr\ (1 + e)) > 1$) *at-top*
⟨*proof*⟩

end
end

definition *akra-bazzi-asymptotic1 b hb e p x* $\longleftrightarrow$
$(1 - hb * inverse\ b * ln\ x\ powr\ -(1+e))\ powr\ p * (1 + ln\ (b*x + hb*x/ln\ x\ powr\ (1+e))\ powr\ (-e/2))$
$\geq 1 + (ln\ x\ powr\ (-e/2) :: real)$
definition *akra-bazzi-asymptotic1' b hb e p x* $\longleftrightarrow$
$(1 + hb * inverse\ b * ln\ x\ powr\ -(1+e))\ powr\ p * (1 + ln\ (b*x + hb*x/ln\ x\ powr\ (1+e))\ powr\ (-e/2))$
$\geq 1 + (ln\ x\ powr\ (-e/2) :: real)$
definition *akra-bazzi-asymptotic2 b hb e p x* $\longleftrightarrow$
$(1 + hb * inverse\ b * ln\ x\ powr\ -(1+e))\ powr\ p * (1 - ln\ (b*x + hb*x/ln\ x\ powr\ (1+e))\ powr\ (-e/2))$
$\leq 1 - ln\ x\ powr\ (-e/2 :: real)$
definition *akra-bazzi-asymptotic2' b hb e p x* $\longleftrightarrow$
$(1 - hb * inverse\ b * ln\ x\ powr\ -(1+e))\ powr\ p * (1 - ln\ (b*x + hb*x/ln\ x\ powr\ (1+e))\ powr\ (-e/2))$
$\leq 1 - ln\ x\ powr\ (-e/2 :: real)$
definition *akra-bazzi-asymptotic3 e x* $\longleftrightarrow (1 + (ln\ x\ powr\ (-e/2)))\ /\ 2 \leq (1::real)$
definition *akra-bazzi-asymptotic4 e x* $\longleftrightarrow (1 - (ln\ x\ powr\ (-e/2))) * 2 \geq (1::real)$
definition *akra-bazzi-asymptotic5 b hb e x* $\longleftrightarrow$
$ln\ (b*x - hb*x*ln\ x\ powr\ -(1+e))\ powr\ (-e/2::real) < 1$

definition *akra-bazzi-asymptotic6 b hb e x* $\longleftrightarrow hb\ /\ ln\ x\ powr\ (1 + e :: real) < b/2$
definition *akra-bazzi-asymptotic7 b hb e x* $\longleftrightarrow hb\ /\ ln\ x\ powr\ (1 + e :: real) < (1 - b)\ /\ 2$
definition *akra-bazzi-asymptotic8 b hb e x* $\longleftrightarrow x*(1 - b - hb\ /\ ln\ x\ powr\ (1 + e :: real)) > 1$

definition *akra-bazzi-asymptotics b hb e p x* $\longleftrightarrow$
akra-bazzi-asymptotic1 b hb e p x ∧ *akra-bazzi-asymptotic1' b hb e p x* ∧
akra-bazzi-asymptotic2 b hb e p x ∧ *akra-bazzi-asymptotic2' b hb e p x* ∧
akra-bazzi-asymptotic3 e x ∧ *akra-bazzi-asymptotic4 e x* ∧ *akra-bazzi-asymptotic5 b hb e x* ∧
akra-bazzi-asymptotic6 b hb e x ∧ *akra-bazzi-asymptotic7 b hb e x* ∧
akra-bazzi-asymptotic8 b hb e x

lemmas *akra-bazzi-asymptotic-defs* =
akra-bazzi-asymptotic1-def akra-bazzi-asymptotic1'-def
akra-bazzi-asymptotic2-def akra-bazzi-asymptotic2'-def akra-bazzi-asymptotic3-def

akra-bazzi-asymptotic4-def akra-bazzi-asymptotic5-def akra-bazzi-asymptotic6-def akra-bazzi-asymptotic7-def akra-bazzi-asymptotic8-def akra-bazzi-asymptotics-def

lemma *akra-bazzi-asymptotics*:
  assumes $\bigwedge b.\ b \in \mathit{set}\ bs \Longrightarrow b \in \{0{<}..{<}1\}$
  assumes $hb > 0\ e > 0$
  shows *eventually* $(\lambda x.\ \forall b{\in}\mathit{set}\ bs.\ \textit{akra-bazzi-asymptotics}\ b\ hb\ e\ p\ x)$ *at-top*
⟨*proof*⟩

end

3 The continuous Akra-Bazzi theorem

theory *Akra-Bazzi-Real*
imports
  Complex-Main
  Akra-Bazzi-Asymptotics
begin

We want to be generic over the integral definition used; we fix some arbitrary notions of integrability and integral and assume just the properties we need. The user can then instantiate the theorems with any desired integral definition.

locale *akra-bazzi-integral* =
  fixes *integrable* :: $(\mathit{real} \Rightarrow \mathit{real}) \Rightarrow \mathit{real} \Rightarrow \mathit{real} \Rightarrow \mathit{bool}$
    and *integral* :: $(\mathit{real} \Rightarrow \mathit{real}) \Rightarrow \mathit{real} \Rightarrow \mathit{real} \Rightarrow \mathit{real}$
  assumes *integrable-const*: $c \geq 0 \Longrightarrow \mathit{integrable}\ (\lambda\text{-}.\ c)\ a\ b$
      and *integral-const*: $c \geq 0 \Longrightarrow a \leq b \Longrightarrow \mathit{integral}\ (\lambda\text{-}.\ c)\ a\ b = (b - a) * c$
      and *integrable-subinterval*:
          $\mathit{integrable}\ f\ a\ b \Longrightarrow a \leq a' \Longrightarrow b' \leq b \Longrightarrow \mathit{integrable}\ f\ a'\ b'$
      and *integral-le*:
          $\mathit{integrable}\ f\ a\ b \Longrightarrow \mathit{integrable}\ g\ a\ b \Longrightarrow (\bigwedge x.\ x \in \{a..b\} \Longrightarrow f\ x \leq g\ x)$
$\Longrightarrow$
              $\mathit{integral}\ f\ a\ b \leq \mathit{integral}\ g\ a\ b$
      and *integral-combine*:
          $a \leq c \Longrightarrow c \leq b \Longrightarrow \mathit{integrable}\ f\ a\ b \Longrightarrow$
              $\mathit{integral}\ f\ a\ c + \mathit{integral}\ f\ c\ b = \mathit{integral}\ f\ a\ b$
begin
lemma *integral-nonneg*:
  $a \leq b \Longrightarrow \mathit{integrable}\ f\ a\ b \Longrightarrow (\bigwedge x.\ x \in \{a..b\} \Longrightarrow f\ x \geq 0) \Longrightarrow \mathit{integral}\ f\ a\ b \geq 0$
  ⟨*proof*⟩
end

declare *sum.cong*[*fundef-cong*]

lemma *strict-mono-imp-ex1-real*:

fixes f :: *real* $\Rightarrow$ *real*
assumes *lim-neg-inf*: *LIM* x *at-bot*. $f\ x$:> *at-top*
assumes *lim-inf*: $(f \longrightarrow z)$ *at-top*
assumes *mono*: $\bigwedge a\ b.\ a < b \Longrightarrow f\ b < f\ a$
assumes *cont*: $\bigwedge x.$ *isCont* $f\ x$
assumes *y-greater-z*: $z < y$
shows $\exists!x.\ f\ x = y$
⟨*proof*⟩

The parameter p in the Akra-Bazzi theorem always exists and is unique.

definition *akra-bazzi-exponent* :: *real list* $\Rightarrow$ *real list* $\Rightarrow$ *real* **where**
akra-bazzi-exponent as bs $\equiv$ (*THE* p. ($\sum i<$*length as*. *as*!i $*$ *bs*!i *powr* p) = *1*)

locale *akra-bazzi-params* =
fixes k :: *nat* **and** *as bs* :: *real list*
assumes *length-as*: *length as* = k
and *length-bs*: *length bs* = k
and *k-not-0*: $k \neq 0$
and *a-ge-0*: $a \in$ *set as* $\Longrightarrow a \geq 0$
and *b-bounds*: $b \in$ *set bs* $\Longrightarrow b \in \{0<..<1\}$
begin

abbreviation p :: *real* **where** $p \equiv$ *akra-bazzi-exponent as bs*

lemma *p-def*: p = (*THE* p. ($\sum i<k$. *as*!i $*$ *bs*!i *powr* p) = *1*)
⟨*proof*⟩

lemma *b-pos*: $b \in$ *set bs* $\Longrightarrow b > 0$ **and** *b-less-1*: $b \in$ *set bs* $\Longrightarrow b < 1$
⟨*proof*⟩

lemma *as-nonempty* [*simp*]: *as* $\neq$ [] **and** *bs-nonempty* [*simp*]: *bs* $\neq$ []
⟨*proof*⟩

lemma *a-in-as*[*intro*, *simp*]: $i < k \Longrightarrow$ *as* ! $i \in$ *set as*
⟨*proof*⟩

lemma *b-in-bs*[*intro*, *simp*]: $i < k \Longrightarrow$ *bs* ! $i \in$ *set bs*
⟨*proof*⟩

end

locale *akra-bazzi-params-nonzero* =
fixes k :: *nat* **and** *as bs* :: *real list*
assumes *length-as*: *length as* = k
and *length-bs*: *length bs* = k
and *a-ge-0*: $a \in$ *set as* $\Longrightarrow a \geq 0$
and *ex-a-pos*: $\exists a \in$ *set as*. $a > 0$
and *b-bounds*: $b \in$ *set bs* $\Longrightarrow b \in \{0<..<1\}$

begin

sublocale *akra-bazzi-params k as bs*
⟨*proof*⟩

lemma *akra-bazzi-p-strict-mono*:
assumes $x < y$
shows $(\sum i<k.\ as!i * bs!i\ powr\ y) < (\sum i<k.\ as!i * bs!i\ powr\ x)$
⟨*proof*⟩

lemma *akra-bazzi-p-mono*:
assumes $x \leq y$
shows $(\sum i<k.\ as!i * bs!i\ powr\ y) \leq (\sum i<k.\ as!i * bs!i\ powr\ x)$
⟨*proof*⟩

lemma *akra-bazzi-p-unique*:
$\exists!p.\ (\sum i<k.\ as!i * bs!i\ powr\ p) = 1$
⟨*proof*⟩

lemma *p-props*: $(\sum i<k.\ as!i * bs!i\ powr\ p) = 1$
and *p-unique*: $(\sum i<k.\ as!i * bs!i\ powr\ p') = 1 \Longrightarrow p = p'$
⟨*proof*⟩

lemma *p-greaterI*: $1 < (\sum i<k.\ as!i * bs!i\ powr\ p') \Longrightarrow p' < p$
⟨*proof*⟩

lemma *p-lessI*: $1 > (\sum i<k.\ as!i * bs!i\ powr\ p') \Longrightarrow p' > p$
⟨*proof*⟩

lemma *p-geI*: $1 \leq (\sum i<k.\ as!i * bs!i\ powr\ p') \Longrightarrow p' \leq p$
⟨*proof*⟩

lemma *p-leI*: $1 \geq (\sum i<k.\ as!i * bs!i\ powr\ p') \Longrightarrow p' \geq p$
⟨*proof*⟩

lemma *p-boundsI*: $(\sum i<k.\ as!i * bs!i\ powr\ x) \leq 1 \land (\sum i<k.\ as!i * bs!i\ powr\ y) \geq 1 \Longrightarrow p \in \{y..x\}$
⟨*proof*⟩

lemma *p-boundsI'*: $(\sum i<k.\ as!i * bs!i\ powr\ x) < 1 \land (\sum i<k.\ as!i * bs!i\ powr\ y) > 1 \Longrightarrow p \in \{y<..<x\}$
⟨*proof*⟩

lemma *p-nonneg*: *sum-list as* $\geq 1 \Longrightarrow p \geq 0$
⟨*proof*⟩

end

locale *akra-bazzi-real-recursion* =
fixes *as bs* :: *real list* **and** *hs* :: (*real* ⇒ *real*) *list* **and** *k* :: *nat* **and** x_0 x_1 *hb e p* :: *real*
assumes *length-as*: *length as* = *k*
and *length-bs*: *length bs* = *k*
and *length-hs*: *length hs* = *k*
and *k-not-0*: $k \neq 0$
and *a-ge-0*: $a \in set\ as \Longrightarrow a \geq 0$
and *b-bounds*: $b \in set\ bs \Longrightarrow b \in \{0{<}..{<}1\}$

and *x0-ge-1*: $x_0 \geq 1$
and *x0-le-x1*: $x_0 \leq x_1$
and *x1-ge*: $b \in set\ bs \Longrightarrow x_1 \geq 2 * x_0 * inverse\ b$

and *e-pos*: $e > 0$
and *h-bounds*: $x \geq x_1 \Longrightarrow h \in set\ hs \Longrightarrow |h\ x| \leq hb * x\ /\ ln\ x\ powr\ (1 + e)$

and *asymptotics*: $x \geq x_0 \Longrightarrow b \in set\ bs \Longrightarrow$ *akra-bazzi-asymptotics b hb e p x*
begin

sublocale *akra-bazzi-params k as bs*
⟨*proof*⟩

lemma *h-in-hs*[*intro*, *simp*]: $i < k \Longrightarrow hs\ !\ i \in set\ hs$
⟨*proof*⟩

lemma *x1-gt-1*: $x_1 > 1$
⟨*proof*⟩

lemma *x1-ge-1*: $x_1 \geq 1$ ⟨*proof*⟩

lemma *x1-pos*: $x_1 > 0$ ⟨*proof*⟩

lemma *bx-le-x*: $x \geq 0 \Longrightarrow b \in set\ bs \Longrightarrow b * x \leq x$
⟨*proof*⟩

lemma *x0-pos*: $x_0 > 0$ ⟨*proof*⟩

lemma
assumes $x \geq x_0$ $b \in set\ bs$
shows *x0-hb-bound0*: $hb\ /\ ln\ x\ powr\ (1 + e) < b/2$
and *x0-hb-bound1*: $hb\ /\ ln\ x\ powr\ (1 + e) < (1 - b)\ /\ 2$
and *x0-hb-bound2*: $x*(1 - b - hb\ /\ ln\ x\ powr\ (1 + e)) > 1$
⟨*proof*⟩

lemma *step-diff*:

assumes $i < k$ $x \geq x_1$
shows $bs\ !\ i * x + (hs\ !\ i)\ x + 1 < x$
⟨*proof*⟩

lemma *step-le-x*: $i < k \Longrightarrow x \geq x_1 \Longrightarrow bs\ !\ i * x + (hs\ !\ i)\ x \leq x$
⟨*proof*⟩

lemma *x0-hb-bound0'*: $\bigwedge x\ b.\ x \geq x_0 \Longrightarrow b \in set\ bs \Longrightarrow hb\ /\ ln\ x\ powr\ (1 + e) < b$
⟨*proof*⟩

lemma *step-pos*:
assumes $i < k$ $x \geq x_1$
shows $bs\ !\ i * x + (hs\ !\ i)\ x > 0$
⟨*proof*⟩

lemma *step-nonneg*: $i < k \Longrightarrow x \geq x_1 \Longrightarrow bs\ !\ i * x + (hs\ !\ i)\ x \geq 0$
⟨*proof*⟩

lemma *step-nonneg'*: $i < k \Longrightarrow x \geq x_1 \Longrightarrow bs\ !\ i + (hs\ !\ i)\ x\ /\ x \geq 0$
⟨*proof*⟩

lemma *hb-nonneg*: $hb \geq 0$
⟨*proof*⟩

lemma *x0-hb-bound3*:
assumes $x \geq x_1$ $i < k$
shows $x - (bs\ !\ i * x + (hs\ !\ i)\ x) \leq x$
⟨*proof*⟩

lemma *x0-hb-bound4*:
assumes $x \geq x_1$ $i < k$
shows $(bs\ !\ i + (hs\ !\ i)\ x\ /\ x) > bs\ !\ i\ /\ 2$
⟨*proof*⟩

lemma *x0-hb-bound4'*: $x \geq x_1 \Longrightarrow i < k \Longrightarrow (bs\ !\ i + (hs\ !\ i)\ x\ /\ x) \geq bs\ !\ i\ /\ 2$
⟨*proof*⟩

lemma *x0-hb-bound5*:
assumes $x \geq x_1$ $i < k$
shows $(bs\ !\ i + (hs\ !\ i)\ x\ /\ x) \leq bs\ !\ i * 3/2$
⟨*proof*⟩

lemma *x0-hb-bound6*:
assumes $x \geq x_1$ $i < k$
shows $x * ((1 - bs\ !\ i)\ /\ 2) \leq x - (bs\ !\ i * x + (hs\ !\ i)\ x)$
⟨*proof*⟩

lemma *x0-hb-bound7*:

assumes $x \geq x_1$ $i < k$
shows $bs!i*x + (hs!i)\ x > x_0$
⟨*proof*⟩

lemma *x0-hb-bound7'*: $x \geq x_1 \Longrightarrow i < k \Longrightarrow bs!i*x + (hs!i)\ x > 1$
⟨*proof*⟩

lemma *x0-hb-bound8*:
assumes $x \geq x_1$ $i < k$
shows $bs!i*x - hb * x / ln\ x\ powr\ (1+e) > x_0$
⟨*proof*⟩

lemma *x0-hb-bound8'*:
assumes $x \geq x_1$ $i < k$
shows $bs!i*x + hb * x / ln\ x\ powr\ (1+e) > x_0$
⟨*proof*⟩

lemma
assumes $x \geq x_0$
shows *asymptotics1*: $i < k \Longrightarrow 1 + ln\ x\ powr\ (-\ e\ /\ 2) \leq$
$(1 - hb * inverse\ (bs!i) * ln\ x\ powr\ -(1+e))\ powr\ p *$
$(1 + ln\ (bs!i*x + hb*x/ln\ x\ powr\ (1+e))\ powr\ (-e/2))$
and *asymptotics2*: $i < k \Longrightarrow 1 - ln\ x\ powr\ (-\ e\ /\ 2) \geq$
$(1 + hb * inverse\ (bs!i) * ln\ x\ powr\ -(1+e))\ powr\ p *$
$(1 - ln\ (bs!i*x + hb*x/ln\ x\ powr\ (1+e))\ powr\ (-e/2))$
and *asymptotics1'*: $i < k \Longrightarrow 1 + ln\ x\ powr\ (-\ e\ /\ 2) \leq$
$(1 + hb * inverse\ (bs!i) * ln\ x\ powr\ -(1+e))\ powr\ p *$
$(1 + ln\ (bs!i*x + hb*x/ln\ x\ powr\ (1+e))\ powr\ (-e/2))$
and *asymptotics2'*: $i < k \Longrightarrow 1 - ln\ x\ powr\ (-\ e\ /\ 2) \geq$
$(1 - hb * inverse\ (bs!i) * ln\ x\ powr\ -(1+e))\ powr\ p *$
$(1 - ln\ (bs!i*x + hb*x/ln\ x\ powr\ (1+e))\ powr\ (-e/2))$
and *asymptotics3*: $(1 + ln\ x\ powr\ (-\ e\ /\ 2))\ /\ 2 \leq 1$
and *asymptotics4*: $(1 - ln\ x\ powr\ (-\ e\ /\ 2)) * 2 \geq 1$
and *asymptotics5*: $i < k \Longrightarrow ln\ (bs!i*x - hb*x*ln\ x\ powr\ -(1+e))\ powr$
$(-e/2) < 1$
⟨*proof*⟩

lemma *x0-hb-bound9*:
assumes $x \geq x_1$ $i < k$
shows $ln\ (bs!i*x + (hs!i)\ x)\ powr\ -(e/2) < 1$
⟨*proof*⟩

definition *akra-bazzi-measure* :: *real* ⇒ *nat* **where**
akra-bazzi-measure x = *nat* $\lceil x \rceil$

lemma *akra-bazzi-measure-decreases*:
assumes $x \geq x_1$ $i < k$

shows $akra\text{-}bazzi\text{-}measure\ (bs!i{*}x + (hs!i)\ x) < akra\text{-}bazzi\text{-}measure\ x$
⟨*proof*⟩

lemma *akra-bazzi-induct*[*consumes 1*, *case-names base rec*]:
assumes $x \geq x_0$
assumes *base*: $\bigwedge x.\ x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow P\ x$
assumes *rec*: $\bigwedge x.\ x > x_1 \Longrightarrow (\bigwedge i.\ i < k \Longrightarrow P\ (bs!i{*}x + (hs!i)\ x)) \Longrightarrow P\ x$
shows $P\ x$
⟨*proof*⟩

end

locale *akra-bazzi-real* = *akra-bazzi-real-recursion* +
fixes *integrable integral*
assumes *integral*: *akra-bazzi-integral integrable integral*
fixes $f :: real \Rightarrow real$
and $g :: real \Rightarrow real$
and $C :: real$
assumes *p-props*: $(\sum i{<}k.\ as!i * bs!i\ powr\ p) = 1$
and *f-base*: $x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f\ x \geq 0$
and *f-rec*: $x > x_1 \Longrightarrow f\ x = g\ x + (\sum i{<}k.\ as!i * f\ (bs!i * x + (hs!i)\ x))$
and *g-nonneg*: $x \geq x_0 \Longrightarrow g\ x \geq 0$
and *C-bound*: $b \in set\ bs \Longrightarrow x \geq x_1 \Longrightarrow C{*}x \leq b{*}x - hb{*}x/ln\ x\ powr\ (1{+}e)$
and *g-integrable*: $x \geq x_0 \Longrightarrow integrable\ (\lambda u.\ g\ u\ /\ u\ powr\ (p + 1))\ x_0\ x$
begin

interpretation *akra-bazzi-integral integrable integral* ⟨*proof*⟩

lemma *akra-bazzi-integrable*:
$a \geq x_0 \Longrightarrow a \leq b \Longrightarrow integrable\ (\lambda x.\ g\ x\ /\ x\ powr\ (p + 1))\ a\ b$
⟨*proof*⟩

definition $g\text{-}approx :: nat \Rightarrow real \Rightarrow real$ **where**
$g\text{-}approx\ i\ x = x\ powr\ p * integral\ (\lambda u.\ g\ u\ /\ u\ powr\ (p + 1))\ (bs!i * x + (hs!i)\ x)\ x$

lemma *f-nonneg*: $x \geq x_0 \Longrightarrow f\ x \geq 0$
⟨*proof*⟩

definition $f\text{-}approx :: real \Rightarrow real$ **where**
$f\text{-}approx\ x = x\ powr\ p * (1 + integral\ (\lambda u.\ g\ u\ /\ u\ powr\ (p + 1))\ x_0\ x)$

lemma *f-approx-aux*:
assumes $x \geq x_0$

shows $1 + integral\ (\lambda u.\ g\ u\ /\ u\ powr\ (p + 1))\ x_0\ x \geq 1$
⟨*proof*⟩

lemma *f-approx-pos*: $x \geq x_0 \Longrightarrow f\text{-}approx\ x > 0$
⟨*proof*⟩

lemma *f-approx-nonneg*: $x \geq x_0 \Longrightarrow f\text{-}approx\ x \geq 0$
⟨*proof*⟩

lemma *f-approx-bounded-below*:
obtains c **where** $\bigwedge x.\ x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f\text{-}approx\ x \geq c\ \ c > 0$
⟨*proof*⟩

lemma *asymptotics-aux*:
assumes $x \geq x_1\ \ i < k$
assumes $s \equiv (if\ p \geq 0\ then\ 1\ else\ -1)$
shows $(bs!i{*}x - s{*}hb{*}x{*}ln\ x\ powr\ -(1{+}e))\ powr\ p \leq (bs!i{*}x + (hs!i)\ x)\ powr\ p$ (**is** *?thesis1*)
and $(bs!i{*}x + (hs!i)\ x)\ powr\ p \leq (bs!i{*}x + s{*}hb{*}x{*}ln\ x\ powr\ -(1{+}e))\ powr\ p$ (**is** *?thesis2*)
⟨*proof*⟩

lemma *asymptotics1'*:
assumes $x \geq x_1\ \ i < k$
shows $(bs!i{*}x)\ powr\ p * (1 + ln\ x\ powr\ (-e/2)) \leq$
$(bs!i{*}x + (hs!i)\ x)\ powr\ p * (1 + ln\ (bs!i{*}x + (hs!i)\ x)\ powr\ (-e/2))$
⟨*proof*⟩

lemma *asymptotics2'*:
assumes $x \geq x_1\ \ i < k$
shows $(bs!i{*}x + (hs!i)\ x)\ powr\ p * (1 - ln\ (bs!i{*}x + (hs!i)\ x)\ powr\ (-e/2)) \leq$
$(bs!i{*}x)\ powr\ p * (1 - ln\ x\ powr\ (-e/2))$
⟨*proof*⟩

lemma *Cx-le-step*:
assumes $i < k\ \ x \geq x_1$
shows $C{*}x \leq bs!i{*}x + (hs!i)\ x$
⟨*proof*⟩

end

locale *akra-bazzi-nat-to-real* = *akra-bazzi-real-recursion* +
fixes $f :: nat \Rightarrow real$
and $g :: real \Rightarrow real$
assumes *f-base*: $real\ x \geq x_0 \Longrightarrow real\ x \leq x_1 \Longrightarrow f\ x \geq 0$

and *f-rec*: $real\ x > x_1 \Longrightarrow$
$f\ x = g\ (real\ x) + (\sum i{<}k.\ as!i * f\ (nat\ \lfloor bs!i * x + (hs!i)\ (real\ x)\rfloor))$
and *x0-int*: $real\ (nat\ \lfloor x_0 \rfloor) = x_0$
begin

function f' :: *real* ⇒ *real* **where**
$x \leq x_1 \Longrightarrow f'\ x = f\ (nat\ \lfloor x \rfloor)$
$|\ x > x_1 \Longrightarrow f'\ x = g\ x + (\sum i{<}k.\ as!i * f'\ (bs!i * x + (hs!i)\ x))$
⟨*proof*⟩
termination ⟨*proof*⟩

lemma *f′-base*: $x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f'\ x \geq 0$
⟨*proof*⟩

lemmas *f′-rec* = *f′.simps(2)*

end

locale *akra-bazzi-real-lower* = *akra-bazzi-real* +
fixes *fb2 gb2 c2* :: *real*
assumes *f-base2*: $x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f\ x \geq fb2$
and *fb2-pos*: $fb2 > 0$
and *g-growth2*: $\forall x{\geq}x_1.\ \forall u{\in}\{C{*}x..x\}.\ c2 * g\ x \geq g\ u$
and *c2-pos*: $c2 > 0$
and *g-bounded*: $x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow g\ x \leq gb2$
begin

interpretation *akra-bazzi-integral integrable integral* ⟨*proof*⟩

lemma *gb2-nonneg*: $gb2 \geq 0$ ⟨*proof*⟩

lemma *g-growth2′*:
assumes $x \geq x_1\ i < k\ u \in \{bs!i{*}x{+}(hs!i)\ x..x\}$
shows $c2 * g\ x \geq g\ u$
⟨*proof*⟩

lemma *g-bounds2*:
obtains *c4* **where** $\bigwedge x\ i.\ x \geq x_1 \Longrightarrow i < k \Longrightarrow g\text{-}approx\ i\ x \leq c4 * g\ x\ c4 > 0$
⟨*proof*⟩

lemma *f-approx-bounded-above*:
obtains *c* **where** $\bigwedge x.\ x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f\text{-}approx\ x \leq c\ c > 0$
⟨*proof*⟩

lemma *f-bounded-below*:
assumes *c′*: $c' > 0$
obtains *c* **where** $\bigwedge x.\ x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow 2 * (c * f\text{-}approx\ x) \leq f\ x\ c \leq c'$

$c > 0$
⟨*proof*⟩

lemma *akra-bazzi-lower*:
obtains *c5* **where** $\bigwedge x.\ x \geq x_0 \Longrightarrow f\ x \geq c5 * f\text{-}approx\ x\ c5 > 0$
⟨*proof*⟩

lemma *akra-bazzi-bigomega*:
$f \in \Omega(\lambda x.\ x\ powr\ p * (1 + integral\ (\lambda u.\ g\ u\ /\ u\ powr\ (p + 1))\ x_0\ x))$
⟨*proof*⟩

end

locale *akra-bazzi-real-upper* = *akra-bazzi-real* +
fixes *fb1 c1* :: *real*
assumes *f-base1*: $x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f\ x \leq fb1$
and *g-growth1*: $\forall x \geq x_1.\ \forall u \in \{C*x..x\}.\ c1 * g\ x \leq g\ u$
and *c1-pos*: $c1 > 0$
begin

interpretation *akra-bazzi-integral integrable integral* ⟨*proof*⟩

lemma *g-growth1'*:
assumes $x \geq x_1\ i < k\ u \in \{bs!i*x+(hs!i)\ x..x\}$
shows $c1 * g\ x \leq g\ u$
⟨*proof*⟩

lemma *g-bounds1*:
obtains *c3* **where**
$\bigwedge x\ i.\ x \geq x_1 \Longrightarrow i < k \Longrightarrow c3 * g\ x \leq g\text{-}approx\ i\ x\ c3 > 0$
⟨*proof*⟩

lemma *f-bounded-above*:
assumes *c'*: $c' > 0$
obtains *c* **where** $\bigwedge x.\ x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f\ x \leq (1/2) * (c * f\text{-}approx\ x)\ c$
$\geq c'\ c > 0$
⟨*proof*⟩

lemma *akra-bazzi-upper*:
obtains *c6* **where** $\bigwedge x.\ x \geq x_0 \Longrightarrow f\ x \leq c6 * f\text{-}approx\ x\ c6 > 0$
⟨*proof*⟩

lemma *akra-bazzi-bigo*:
$f \in O(\lambda x.\ x\ powr\ p *(1 + integral\ (\lambda u.\ g\ u\ /\ u\ powr\ (p + 1))\ x_0\ x))$
⟨*proof*⟩

end

end

4 The discrete Akra-Bazzi theorem

theory *Akra-Bazzi*
imports
  Complex-Main
  HOL−Library.Landau-Symbols
  Akra-Bazzi-Real
begin

lemma *ex-mono*: $(\exists x.\ P\ x) \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow (\exists x.\ Q\ x)$ ⟨*proof*⟩

lemma *x-over-ln-mono*:
  assumes $(e{::}real) > 0$
  assumes $x > exp\ e$
  assumes $x \leq y$
  shows $x\ /\ ln\ x\ powr\ e \leq y\ /\ ln\ y\ powr\ e$
⟨*proof*⟩

definition *akra-bazzi-term* :: $nat \Rightarrow nat \Rightarrow real \Rightarrow (nat \Rightarrow nat) \Rightarrow bool$ **where**
  $akra\text{-}bazzi\text{-}term\ x_0\ x_1\ b\ t =$
    $(\exists\ e\ h.\ e > 0 \wedge (\lambda x.\ h\ x) \in O(\lambda x.\ real\ x\ /\ ln\ (real\ x)\ powr\ (1{+}e)) \wedge$
      $(\forall\ x{\geq}x_1.\ t\ x \geq x_0 \wedge t\ x < x \wedge b{*}x + h\ x = real\ (t\ x)))$

lemma *akra-bazzi-termI* [*intro?*]:
  assumes $e > 0$ $(\lambda x.\ h\ x) \in O(\lambda x.\ real\ x\ /\ ln\ (real\ x)\ powr\ (1{+}e))$
      $\bigwedge x.\ x \geq x_1 \Longrightarrow t\ x \geq x_0$ $\bigwedge x.\ x \geq x_1 \Longrightarrow t\ x < x$
      $\bigwedge x.\ x \geq x_1 \Longrightarrow b{*}x + h\ x = real\ (t\ x)$
  shows $akra\text{-}bazzi\text{-}term\ x_0\ x_1\ b\ t$
  ⟨*proof*⟩

lemma *akra-bazzi-term-imp-less*:
  assumes $akra\text{-}bazzi\text{-}term\ x_0\ x_1\ b\ t$ $x \geq x_1$
  shows $t\ x < x$
  ⟨*proof*⟩

lemma *akra-bazzi-term-imp-less′*:
  assumes $akra\text{-}bazzi\text{-}term\ x_0\ (Suc\ x_1)\ b\ t$ $x > x_1$
  shows $t\ x < x$
  ⟨*proof*⟩

locale *akra-bazzi-recursion* =
  fixes $x_0\ x_1\ k :: nat$ **and** $as\ bs :: real\ list$ **and** $ts :: (nat \Rightarrow nat)\ list$ **and** $f :: nat \Rightarrow real$

assumes *k-not-0*: $k \neq 0$
and *length-as*: *length as* $=$ *k*
and *length-bs*: *length bs* $=$ *k*
and *length-ts*: *length ts* $=$ *k*
and *a-ge-0*: $a \in$ *set as* $\Longrightarrow a \geq 0$
and *b-bounds*: $b \in$ *set bs* $\Longrightarrow b \in \{0{<}..{<}1\}$
and *ts*: $i <$ *length bs* $\Longrightarrow$ *akra-bazzi-term* x_0 x_1 $(bs!i)$ $(ts!i)$
begin

sublocale *akra-bazzi-params k as bs*
⟨*proof*⟩

lemma *ts-nonempty*: $ts \neq []$ ⟨*proof*⟩

definition *e-hs* :: *real* × (*nat* ⇒ *real*) *list* **where**
e-hs = (*SOME* (*e*,*hs*).
$e > 0 \wedge$ *length hs* $= k \wedge (\forall h \in$ *set hs*. $(\lambda x.\ h\ x) \in O(\lambda x.$ *real x* / *ln* (*real x*) *powr* $(1+e))) \wedge$
$(\forall t \in$ *set ts*. $\forall x \geq x_1.\ t\ x \geq x_0 \wedge t\ x < x) \wedge$
$(\forall i<k.\ \forall x \geq x_1.\ (bs!i)*x + (hs!i)\ x =$ *real* $((ts!i)\ x))$
)

definition *e* = (*case e-hs of* (*e*,-) ⇒ *e*)
definition *hs* = (*case e-hs of* (-,*hs*) ⇒ *hs*)

lemma *filterlim-powr-zero-cong*:
filterlim (λ*x*. *P* (*x*::*real*) (*x powr* (*0*::*real*))) *F at-top* = *filterlim* (λ*x*. *P x 1*) *F at-top*
⟨*proof*⟩

lemma *e-hs-aux*:
$0 < e \wedge$ *length hs* $= k \wedge$
$(\forall h \in$ *set hs*. $(\lambda x.\ h\ x) \in O(\lambda x.$ *real x* / *ln* (*real x*) *powr* $(1 + e))) \wedge$
$(\forall t \in$ *set ts*. $\forall x \geq x_1.\ x_0 \leq t\ x \wedge t\ x < x) \wedge$
$(\forall i<k.\ \forall x \geq x_1.\ (bs!i)*x + (hs!i)\ x =$ *real* $((ts!i)\ x))$
⟨*proof*⟩

lemma
e-pos: $e > 0$ **and** *length-hs*: *length hs* $= k$ **and**
hs-growth: $\bigwedge h.\ h \in$ *set hs* $\Longrightarrow (\lambda x.\ h\ x) \in O(\lambda x.$ *real x* / *ln* (*real x*) *powr* $(1 + e))$ **and**
step-ge-x0: $\bigwedge t\ x.\ t \in$ *set ts* $\Longrightarrow x \geq x_1 \Longrightarrow x_0 \leq t\ x$ **and**
step-less: $\bigwedge t\ x.\ t \in$ *set ts* $\Longrightarrow x \geq x_1 \Longrightarrow t\ x < x$ **and**
decomp: $\bigwedge i\ x.\ i < k \Longrightarrow x \geq x_1 \Longrightarrow (bs!i)*x + (hs!i)\ x =$ *real* $((ts!i)\ x)$
⟨*proof*⟩

lemma *h-in-hs* [*intro*, *simp*]: $i < k \Longrightarrow hs\ !\ i \in$ *set hs*
⟨*proof*⟩

lemma *t-in-ts* [*intro*, *simp*]: $i < k \Longrightarrow ts\ !\ i \in set\ ts$
⟨*proof*⟩

lemma *x0-less-x1*: $x_0 < x_1$ **and** *x0-le-x1*: $x_0 \leq x_1$
⟨*proof*⟩

lemma *akra-bazzi-induct* [*consumes 1*, *case-names base rec*]:
assumes $x \geq x_0$
assumes *base*: $\bigwedge x.\ x \geq x_0 \Longrightarrow x < x_1 \Longrightarrow P\ x$
assumes *rec*: $\bigwedge x.\ x \geq x_1 \Longrightarrow (\bigwedge t.\ t \in set\ ts \Longrightarrow P\ (t\ x)) \Longrightarrow P\ x$
shows $P\ x$
⟨*proof*⟩

end

locale *akra-bazzi-function* = *akra-bazzi-recursion* +
fixes *integrable integral*
assumes *integral*: *akra-bazzi-integral integrable integral*
fixes g :: *nat* ⇒ *real*
assumes *f-nonneg-base*: $x \geq x_0 \Longrightarrow x < x_1 \Longrightarrow f\ x \geq 0$
and *f-rec*: $x \geq x_1 \Longrightarrow f\ x = g\ x + (\sum i<k.\ as!i * f\ ((ts!i)\ x))$
and *g-nonneg*: $x \geq x_1 \Longrightarrow g\ x \geq 0$
and *ex-pos-a*: $\exists\, a \in set\ as.\ a > 0$
begin

lemma *ex-pos-a'*: $\exists\, i<k.\ as!i > 0$
⟨*proof*⟩

sublocale *akra-bazzi-params-nonzero*
⟨*proof*⟩

definition *g-real* :: *real* ⇒ *real* **where** $g\text{-}real\ x = g\ (nat\ \lfloor x \rfloor)$

lemma *g-real-real*[*simp*]: $g\text{-}real\ (real\ x) = g\ x$ ⟨*proof*⟩

lemma *f-nonneg*: $x \geq x_0 \Longrightarrow f\ x \geq 0$
⟨*proof*⟩

definition $hs' = map\ (\lambda h\ x.\ h\ (nat\ \lfloor x{::}real \rfloor))\ hs$

lemma *length-hs'*: $length\ hs' = k$ ⟨*proof*⟩

lemma *hs'-real*: $i < k \Longrightarrow (hs'!i)\ (real\ x) = (hs!i)\ x$
⟨*proof*⟩

lemma *h-bound*:
obtains *hb* **where** $hb > 0$ **and**

eventually $(\lambda x.\ \forall h{\in}set\ hs'.\ |h\ x| \leq hb * x\ /\ ln\ x\ powr\ (1 + e))$ *at-top*
⟨*proof*⟩

lemma *C-bound*:
assumes $\bigwedge b.\ b \in set\ bs \Longrightarrow C < b\ hb > 0$
shows *eventually* $(\lambda x{::}real.\ \forall b{\in}set\ bs.\ C{*}x \leq b{*}x - hb{*}x/ln\ x\ powr\ (1{+}e))$ *at-top*
⟨*proof*⟩

end

locale *akra-bazzi-lower* = *akra-bazzi-function* +
fixes $g' :: real \Rightarrow real$
assumes *f-pos*: *eventually* $(\lambda x.\ f\ x > 0)$ *at-top*
and *g-growth2*: $\exists C\ c2.\ c2 > 0 \wedge C < Min\ (set\ bs) \wedge$ *eventually* $(\lambda x.\ \forall u{\in}\{C{*}x..x\}.\ g'\ u \leq c2 * g'\ x)$ *at-top*
and *g'-integrable*: $\exists a.\ \forall b{\geq}a.\ integrable\ (\lambda u.\ g'\ u\ /\ u\ powr\ (p + 1))\ a\ b$
and *g'-bounded*: *eventually* $(\lambda a{::}real.\ (\forall b{>}a.\ \exists c.\ \forall x{\in}\{a..b\}.\ g'\ x \leq c))$ *at-top*
and *g-bigomega*: $g \in \Omega(\lambda x.\ g'\ (real\ x))$
and *g'-nonneg*: *eventually* $(\lambda x.\ g'\ x \geq 0)$ *at-top*
begin

definition $gc2 \equiv SOME\ gc2.\ gc2 > 0 \wedge$ *eventually* $(\lambda x.\ g\ x \geq gc2 * g'\ (real\ x))$ *at-top*

lemma *gc2*: $gc2 > 0$ *eventually* $(\lambda x.\ g\ x \geq gc2 * g'\ (real\ x))$ *at-top*
⟨*proof*⟩

definition $gx0 \equiv max\ x_1\ (SOME\ gx0.\ \forall x{\geq}gx0.\ g\ x \geq gc2 * g'\ (real\ x) \wedge f\ x > 0 \wedge g'\ (real\ x) \geq 0)$
definition $gx1 \equiv max\ gx0\ (SOME\ gx1.\ \forall x{\geq}gx1.\ \forall i{<}k.\ (ts!i)\ x \geq gx0)$

lemma *gx0*:
assumes $x \geq gx0$
shows $g\ x \geq gc2 * g'\ (real\ x)\ f\ x > 0\ g'\ (real\ x) \geq 0$
⟨*proof*⟩

lemma *gx1*:
assumes $x \geq gx1\ i < k$
shows $(ts!i)\ x \geq gx0$
⟨*proof*⟩

lemma *gx0-ge-x1*: $gx0 \geq x_1$ ⟨*proof*⟩
lemma *gx0-le-gx1*: $gx0 \leq gx1$ ⟨*proof*⟩

function $f2' :: nat \Rightarrow real$ **where**
$x < gx1 \Longrightarrow f2'\ x = max\ 0\ (f\ x\ /\ gc2)$

| $x \geq gx1 \Longrightarrow f2'\ x = g'\ (real\ x) + (\sum i<k.\ as!i*f2'\ ((ts!i)\ x))$
⟨proof⟩
termination ⟨proof⟩

lemma *f2'-nonneg*: $x \geq gx0 \Longrightarrow f2'\ x \geq 0$
⟨proof⟩

lemma *f2'-le-f*: $x \geq x_0 \Longrightarrow gc2 * f2'\ x \leq f\ x$
⟨proof⟩

lemma *f2'-pos*: *eventually* $(\lambda x.\ f2'\ x > 0)$ *at-top*
⟨proof⟩

lemma *bigomega-f-aux*:
obtains *a* **where** $a \geq A\ \forall a' \geq a.\ a' \in \mathbb{N} \longrightarrow$
$f \in \Omega(\lambda x.\ x\ powr\ p *(1 + integral\ (\lambda u.\ g'\ u\ /\ u\ powr\ (p + 1))\ a'\ x))$
⟨proof⟩

lemma *bigomega-f*:
obtains *a* **where** $a \geq A\ f \in \Omega(\lambda x.\ x\ powr\ p *(1 + integral\ (\lambda u.\ g'\ u\ /\ u\ powr\ (p+1))\ a\ x))$
⟨proof⟩

end

locale *akra-bazzi-upper* = *akra-bazzi-function* +
fixes $g' :: real \Rightarrow real$
assumes *g'-integrable*: $\exists a.\ \forall b \geq a.\ integrable\ (\lambda u.\ g'\ u\ /\ u\ powr\ (p + 1))\ a\ b$
and *g-growth1*: $\exists C\ c1.\ c1 > 0 \wedge C < Min\ (set\ bs) \wedge$
eventually $(\lambda x.\ \forall u \in \{C*x..x\}.\ g'\ u \geq c1 * g'\ x)$ *at-top*
and *g-bigo*: $g \in O(g')$
and *g'-nonneg*: *eventually* $(\lambda x.\ g'\ x \geq 0)$ *at-top*
begin

definition $gc1 \equiv SOME\ gc1.\ gc1 > 0 \wedge eventually\ (\lambda x.\ g\ x \leq gc1 * g'\ (real\ x))$ *at-top*

lemma *gc1*: $gc1 > 0$ *eventually* $(\lambda x.\ g\ x \leq gc1 * g'\ (real\ x))$ *at-top*
⟨proof⟩

definition $gx3 \equiv max\ x_1\ (SOME\ gx0.\ \forall x \geq gx0.\ g\ x \leq gc1 * g'\ (real\ x))$

lemma *gx3*:
assumes $x \geq gx3$
shows $g\ x \leq gc1 * g'\ (real\ x)$

⟨*proof*⟩

lemma *gx3-ge-x1*: $gx3 \geq x_1$ ⟨*proof*⟩

function f' :: *nat* ⇒ *real* **where**
$x < gx3 \Longrightarrow f'\ x = max\ 0\ (f\ x\ /\ gc1)$
| $x \geq gx3 \Longrightarrow f'\ x = g'\ (real\ x) + (\sum i{<}k.\ as!i * f'\ ((ts!i)\ x))$
⟨*proof*⟩
termination ⟨*proof*⟩

lemma *f′-ge-f*: $x \geq x_0 \Longrightarrow gc1 * f'\ x \geq f\ x$
⟨*proof*⟩

lemma *bigo-f-aux*:
obtains *a* **where** $a \geq A\ \forall a'{\geq}a.\ a' \in \mathbb{N} \longrightarrow$
$f \in O(\lambda x.\ x\ powr\ p * (1 + integral\ (\lambda u.\ g'\ u\ /\ u\ powr\ (p + 1))\ a'\ x))$
⟨*proof*⟩

lemma *bigo-f*:
obtains *a* **where** $a > A\ f \in O(\lambda x.\ x\ powr\ p * (1 + integral\ (\lambda u.\ g'\ u\ /\ u\ powr\ (p + 1))\ a\ x))$
⟨*proof*⟩

end

locale *akra-bazzi* = *akra-bazzi-function* +
fixes g' :: *real* ⇒ *real*
assumes *f-pos*: *eventually* $(\lambda x.\ f\ x > 0)$ *at-top*
and *g′-nonneg*: *eventually* $(\lambda x.\ g'\ x \geq 0)$ *at-top*
assumes *g′-integrable*: $\exists a.\ \forall b{\geq}a.\ integrable\ (\lambda u.\ g'\ u\ /\ u\ powr\ (p + 1))\ a\ b$
and *g-growth1*: $\exists C\ c1.\ c1 > 0 \land C < Min\ (set\ bs) \land$
eventually $(\lambda x.\ \forall u{\in}\{C*x..x\}.\ g'\ u \geq c1 * g'\ x)$ *at-top*
and *g-growth2*: $\exists C\ c2.\ c2 > 0 \land C < Min\ (set\ bs) \land$
eventually $(\lambda x.\ \forall u{\in}\{C*x..x\}.\ g'\ u \leq c2 * g'\ x)$ *at-top*
and *g-bounded*: *eventually* $(\lambda a{::}real.\ (\forall b{>}a.\ \exists c.\ \forall x{\in}\{a..b\}.\ g'\ x \leq c))$ *at-top*
and *g-bigtheta*: $g \in \Theta(g')$
begin

sublocale *akra-bazzi-lower* ⟨*proof*⟩
sublocale *akra-bazzi-upper* ⟨*proof*⟩

lemma *bigtheta-f*:
obtains *a* **where** $a > A\ f \in \Theta(\lambda x.\ x\ powr\ p * (1 + integral\ (\lambda u.\ g'\ u\ /\ u\ powr\ (p + 1))\ a\ x))$
⟨*proof*⟩

end

named-theorems *akra-bazzi-term-intros introduction rules for Akra−Bazzi terms*

lemma *akra-bazzi-term-floor-add* [*akra-bazzi-term-intros*]:
assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \le b * real\ x_1 + c$ $c < (1 - b) * real\ x_1$ $x_1 > 0$
shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x.\ nat\ \lfloor b*real\ x + c\rfloor)$
⟨*proof*⟩

lemma *akra-bazzi-term-floor-add′* [*akra-bazzi-term-intros*]:
assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \le b * real\ x_1 + real\ c$ $real\ c < (1 - b) * real\ x_1$ $x_1 > 0$
shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x.\ nat\ \lfloor b*real\ x\rfloor + c)$
⟨*proof*⟩

lemma *akra-bazzi-term-floor-subtract* [*akra-bazzi-term-intros*]:
assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \le b * real\ x_1 - c$ $0 < c + (1 - b) * real\ x_1$ $x_1 > 0$
shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x.\ nat\ \lfloor b*real\ x - c\rfloor)$
⟨*proof*⟩

lemma *akra-bazzi-term-floor-subtract′* [*akra-bazzi-term-intros*]:
assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \le b * real\ x_1 - real\ c$ $0 < real\ c + (1 - b) * real\ x_1$ $x_1 > 0$
shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x.\ nat\ \lfloor b*real\ x\rfloor - c)$
⟨*proof*⟩

lemma *akra-bazzi-term-floor* [*akra-bazzi-term-intros*]:
assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \le b * real\ x_1$ $0 < (1 - b) * real\ x_1$ $x_1 > 0$
shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x.\ nat\ \lfloor b*real\ x\rfloor)$
⟨*proof*⟩

lemma *akra-bazzi-term-ceiling-add* [*akra-bazzi-term-intros*]:
assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \le b * real\ x_1 + c$ $c + 1 \le (1 - b) * x_1$
shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x.\ nat\ \lceil b*real\ x + c\rceil)$
⟨*proof*⟩

lemma *akra-bazzi-term-ceiling-add′* [*akra-bazzi-term-intros*]:
assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \le b * real\ x_1 + real\ c$ $real\ c + 1 \le (1 - b) * x_1$
shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x.\ nat\ \lceil b*real\ x\rceil + c)$
⟨*proof*⟩

lemma *akra-bazzi-term-ceiling-subtract* [*akra-bazzi-term-intros*]:
assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \le b * real\ x_1 - c$ $1 \le c + (1 - b) * x_1$
shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x.\ nat\ \lceil b*real\ x - c\rceil)$
⟨*proof*⟩

lemma *akra-bazzi-term-ceiling-subtract′* [*akra-bazzi-term-intros*]:
assumes $(b{::}real) > 0$ $b < 1$ $real\ x_0 \leq b * real\ x_1 - real\ c$ $1 \leq real\ c + (1 - b) * x_1$
shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x.\ nat\ \lceil b{*}real\ x\rceil - c)$
⟨*proof*⟩

lemma *akra-bazzi-term-ceiling* [*akra-bazzi-term-intros*]:
assumes $(b{::}real) > 0$ $b < 1$ $real\ x_0 \leq b * real\ x_1$ $1 \leq (1 - b) * x_1$
shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x.\ nat\ \lceil b{*}real\ x\rceil)$
⟨*proof*⟩

end

5 The Master theorem

theory *Master-Theorem*
imports
HOL−Analysis.Equivalence-Lebesgue-Henstock-Integration
Akra-Bazzi-Library
Akra-Bazzi
begin

lemma *fundamental-theorem-of-calculus-real*:
$a \leq b \Longrightarrow \forall x{\in}\{a..b\}.\ (f\ has\text{-}real\text{-}derivative\ f'\ x)\ (at\ x\ within\ \{a..b\}) \Longrightarrow$
$(f'\ has\text{-}integral\ (f\ b - f\ a))\ \{a..b\}$
⟨*proof*⟩

lemma *integral-powr*:
$y \neq -1 \Longrightarrow a \leq b \Longrightarrow a > 0 \Longrightarrow integral\ \{a..b\}\ (\lambda x.\ x\ powr\ y :: real) =$
$inverse\ (y + 1) * (b\ powr\ (y + 1) - a\ powr\ (y + 1))$
⟨*proof*⟩

lemma *integral-ln-powr-over-x*:
$y \neq -1 \Longrightarrow a \leq b \Longrightarrow a > 1 \Longrightarrow integral\ \{a..b\}\ (\lambda x.\ ln\ x\ powr\ y\ /\ x :: real) =$
$inverse\ (y + 1) * (ln\ b\ powr\ (y + 1) - ln\ a\ powr\ (y + 1))$
⟨*proof*⟩

lemma *integral-one-over-x-ln-x*:
$a \leq b \Longrightarrow a > 1 \Longrightarrow integral\ \{a..b\}\ (\lambda x.\ inverse\ (x * ln\ x) :: real) = ln\ (ln\ b) - ln\ (ln\ a)$
⟨*proof*⟩

lemma *akra-bazzi-integral-kurzweil-henstock*:
akra-bazzi-integral $(\lambda f\ a\ b.\ f\ integrable\text{-}on\ \{a..b\})\ (\lambda f\ a\ b.\ integral\ \{a..b\}\ f)$
⟨*proof*⟩

locale *master-theorem-function* = *akra-bazzi-recursion* +
fixes g :: *nat* ⇒ *real*
assumes *f-nonneg-base*: $x \geq x_0 \Longrightarrow x < x_1 \Longrightarrow f\ x \geq 0$
and *f-rec*: $x \geq x_1 \Longrightarrow f\ x = g\ x + (\sum i{<}k.\ as!i * f\ ((ts!i)\ x))$
and *g-nonneg*: $x \geq x_1 \Longrightarrow g\ x \geq 0$
and *ex-pos-a*: $\exists\, a{\in}set\ as.\ a > 0$
begin

interpretation *akra-bazzi-integral* $\lambda f\ a\ b.\ f$ *integrable-on* $\{a..b\}$ $\lambda f\ a\ b.$ *integral* $\{a..b\}\ f$
⟨*proof*⟩

sublocale *akra-bazzi-function* $x_0\ x_1\ k\ as\ bs\ ts\ f\ \lambda f\ a\ b.\ f$ *integrable-on* $\{a..b\}$
$\lambda f\ a\ b.$ *integral* $\{a..b\}\ f\ g$
⟨*proof*⟩

context
begin

private lemma *g-nonneg′*: *eventually* $(\lambda x.\ g\ x \geq 0)$ *at-top*
⟨*proof*⟩ **lemma** *g-pos*:
assumes $g \in \Omega(h)$
assumes *eventually* $(\lambda x.\ h\ x > 0)$ *at-top*
shows *eventually* $(\lambda x.\ g\ x > 0)$ *at-top*
⟨*proof*⟩ **lemma** *f-pos*:
assumes $g \in \Omega(h)$
assumes *eventually* $(\lambda x.\ h\ x > 0)$ *at-top*
shows *eventually* $(\lambda x.\ f\ x > 0)$ *at-top*
⟨*proof*⟩

lemma *bs-lower-bound*: $\exists\, C{>}0.\ \forall\, b{\in}set\ bs.\ C < b$
⟨*proof*⟩ **lemma** *powr-growth2*:
$\exists\, C\ c2.\ 0 < c2 \wedge C < Min\ (set\ bs) \wedge$
eventually $(\lambda x.\ \forall\, u{\in}\{C * x..x\}.\ c2 * x$ *powr* $p' \geq u$ *powr* $p')$ *at-top*
⟨*proof*⟩ **lemma** *powr-growth1*:
$\exists\, C\ c1.\ 0 < c1 \wedge C < Min\ (set\ bs) \wedge$
eventually $(\lambda x.\ \forall\, u{\in}\{C * x..x\}.\ c1 * x$ *powr* $p' \leq u$ *powr* $p')$ *at-top*
⟨*proof*⟩ **lemma** *powr-ln-powr-lower-bound*:
$a > 1 \Longrightarrow a \leq x \Longrightarrow x \leq b \Longrightarrow$
min (a *powr* p) (b *powr* p) $*$ *min* (*ln* a *powr* p') (*ln* b *powr* p') $\leq x$ *powr* $p *$ *ln* x *powr* p'
⟨*proof*⟩ **lemma** *powr-ln-powr-upper-bound*:
$a > 1 \Longrightarrow a \leq x \Longrightarrow x \leq b \Longrightarrow$
max (a *powr* p) (b *powr* p) $*$ *max* (*ln* a *powr* p') (*ln* b *powr* p') $\geq x$ *powr* $p *$ *ln* x *powr* p'
⟨*proof*⟩ **lemma** *powr-ln-powr-upper-bound′*:
eventually $(\lambda a.\ \forall\, b{>}a.\ \exists\, c.\ \forall\, x{\in}\{a..b\}.\ x$ *powr* $p *$ *ln* x *powr* $p' \leq c)$ *at-top*
⟨*proof*⟩ **lemma** *powr-upper-bound′*:
eventually $(\lambda a{::}real.\ \forall\, b{>}a.\ \exists\, c.\ \forall\, x{\in}\{a..b\}.\ x$ *powr* $p' \leq c)$ *at-top*

⟨*proof*⟩

lemmas *bounds* =
powr-ln-powr-lower-bound powr-ln-powr-upper-bound powr-ln-powr-upper-bound′ powr-upper-bound′

private lemma *eventually-ln-const*:
assumes $(C::real) > 0$
shows *eventually* $(\lambda x.\ ln\ (C*x)\ /\ ln\ x > 1/2)$ *at-top*
⟨*proof*⟩ **lemma** *powr-ln-powr-growth1*: $\exists\, C\ c1.\ 0 < c1 \wedge C < Min\ (set\ bs) \wedge$ *eventually* $(\lambda x.\ \forall u\in\{C * x..x\}.\ c1 * (x\ powr\ r * ln\ x\ powr\ r') \leq u\ powr\ r * ln\ u\ powr\ r')$ *at-top*
⟨*proof*⟩ **lemma** *powr-ln-powr-growth2*: $\exists\, C\ c1.\ 0 < c1 \wedge C < Min\ (set\ bs) \wedge$ *eventually* $(\lambda x.\ \forall u\in\{C * x..x\}.\ c1 * (x\ powr\ r * ln\ x\ powr\ r') \geq u\ powr\ r * ln\ u\ powr\ r')$ *at-top*
⟨*proof*⟩

lemmas *growths* = *powr-growth1 powr-growth2 powr-ln-powr-growth1 powr-ln-powr-growth2*

private lemma *master-integrable*:
$\exists\, a::real.\ \forall b\geq a.\ (\lambda u.\ u\ powr\ r * ln\ u\ powr\ s\ /\ u\ powr\ t)$ *integrable-on* $\{a..b\}$
$\exists\, a::real.\ \forall b\geq a.\ (\lambda u.\ u\ powr\ r\ /\ u\ powr\ s)$ *integrable-on* $\{a..b\}$
⟨*proof*⟩ **lemma** *master-integral*:
fixes $a\ p\ p' :: real$
assumes p: $p \neq p'$ **and** a: $a > 0$
obtains $c\ d$ **where** $c \neq 0\ p > p' \longrightarrow d \neq 0$
$(\lambda x::nat.\ x\ powr\ p * (1 + integral\ \{a..x\}\ (\lambda u.\ u\ powr\ p'\ /\ u\ powr\ (p+1)))) \in \Theta(\lambda x::nat.\ d * x\ powr\ p + c * x\ powr\ p')$
⟨*proof*⟩ **lemma** *master-integral′*:
fixes $a\ p\ p' :: real$
assumes p': $p' \neq 0$ **and** a: $a > 1$
obtains $c\ d :: real$ **where** $p' < 0 \longrightarrow c \neq 0\ d \neq 0$
$(\lambda x::nat.\ x\ powr\ p * (1 + integral\ \{a..x\}\ (\lambda u.\ u\ powr\ p * ln\ u\ powr\ (p'-1)\ /\ u\ powr\ (p+1)))) \in \Theta(\lambda x::nat.\ c * x\ powr\ p + d * x\ powr\ p * ln\ x\ powr\ p')$
⟨*proof*⟩ **lemma** *master-integral″*:
fixes $a\ p\ p' :: real$
assumes a: $a > 1$
shows $(\lambda x::nat.\ x\ powr\ p * (1 + integral\ \{a..x\}\ (\lambda u.\ u\ powr\ p * ln\ u\ powr\ -1/u\ powr\ (p+1)))) \in \Theta(\lambda x::nat.\ x\ powr\ p * ln\ (ln\ x))$
⟨*proof*⟩

lemma *master1-bigo*:
assumes *g-bigo*: $g \in O(\lambda x.\ real\ x\ powr\ p')$

assumes *less-p'*: $(\sum i{<}k.\ as!i * bs!i\ powr\ p') > 1$
shows $f \in O(\lambda x.\ real\ x\ powr\ p)$
⟨*proof*⟩

lemma *master1*:
assumes *g-bigo*: $g \in O(\lambda x.\ real\ x\ powr\ p')$
assumes *less-p'*: $(\sum i{<}k.\ as!i * bs!i\ powr\ p') > 1$
assumes *f-pos*: *eventually* $(\lambda x.\ f\ x > 0)$ *at-top*
shows $f \in \Theta(\lambda x.\ real\ x\ powr\ p)$
⟨*proof*⟩

lemma *master2-3*:
assumes *g-bigtheta*: $g \in \Theta(\lambda x.\ real\ x\ powr\ p * ln\ (real\ x)\ powr\ (p' - 1))$
assumes *p'*: $p' > 0$
shows $f \in \Theta(\lambda x.\ real\ x\ powr\ p * ln\ (real\ x)\ powr\ p')$
⟨*proof*⟩

lemma *master2-1*:
assumes *g-bigtheta*: $g \in \Theta(\lambda x.\ real\ x\ powr\ p * ln\ (real\ x)\ powr\ p')$
assumes *p'*: $p' < -1$
shows $f \in \Theta(\lambda x.\ real\ x\ powr\ p)$
⟨*proof*⟩

lemma *master2-2*:
assumes *g-bigtheta*: $g \in \Theta(\lambda x.\ real\ x\ powr\ p\ /\ ln\ (real\ x))$
shows $f \in \Theta(\lambda x.\ real\ x\ powr\ p * ln\ (ln\ (real\ x)))$
⟨*proof*⟩

lemma *master3*:
assumes *g-bigtheta*: $g \in \Theta(\lambda x.\ real\ x\ powr\ p')$
assumes *p'-greater'*: $(\sum i{<}k.\ as!i * bs!i\ powr\ p') < 1$
shows $f \in \Theta(\lambda x.\ real\ x\ powr\ p')$
⟨*proof*⟩

end
end

end

6 Evaluating expressions with rational numerals

theory *Eval-Numeral*
imports
Complex-Main
begin

lemma *real-numeral-to-Ratreal*:
$(0{::}real) = Ratreal\ (Frct\ (0,\ 1))$

$(1::real) = Ratreal\ (Frct\ (1,\ 1))$
$(numeral\ x :: real) = Ratreal\ (Frct\ (numeral\ x,\ 1))$
$(1::int) = numeral\ Num.One$
$\langle proof \rangle$

lemma *real-equals-code*: $Ratreal\ x = Ratreal\ y \longleftrightarrow x = y$
$\langle proof \rangle$

lemma *Rat-normalize-idempotent*: $Rat.normalize\ (Rat.normalize\ x) = Rat.normalize\ x$
$\langle proof \rangle$

lemma *uminus-pow-Numeral1*: $(-(x::-::monoid\text{-}mult))\ \hat{}\ Numeral1 = -x$ $\langle proof \rangle$

lemmas *power-numeral-simps* = *power-0 uminus-pow-Numeral1 power-minus-Bit0 power-minus-Bit1*

lemma *Fract-normalize*: $Fract\ (fst\ (Rat.normalize\ (x,y)))\ (snd\ (Rat.normalize\ (x,y))) = Fract\ x\ y$
$\langle proof \rangle$

lemma *Frct-add*: $Frct\ (a,\ numeral\ b) + Frct\ (c,\ numeral\ d) =$
$Frct\ (Rat.normalize\ (a * numeral\ d + c * numeral\ b,\ numeral\ (b*d)))$
$\langle proof \rangle$

lemma *Frct-uminus*: $-(Frct\ (a,b)) = Frct\ (-a,b)$ $\langle proof \rangle$

lemma *Frct-diff*: $Frct\ (a,\ numeral\ b) - Frct\ (c,\ numeral\ d) =$
$Frct\ (Rat.normalize\ (a * numeral\ d - c * numeral\ b,\ numeral\ (b*d)))$
$\langle proof \rangle$

lemma *Frct-mult*: $Frct\ (a,\ numeral\ b) * Frct\ (c,\ numeral\ d) = Frct\ (a*c,\ numeral\ (b*d))$
$\langle proof \rangle$

lemma *Frct-inverse*: $inverse\ (Frct\ (a,\ b)) = Frct\ (b,\ a)$ $\langle proof \rangle$

lemma *Frct-divide*: $Frct\ (a,\ numeral\ b)\ /\ Frct\ (c,\ numeral\ d) = Frct\ (a*numeral\ d,\ numeral\ b * c)$
$\langle proof \rangle$

lemma *Frct-pow*: $Frct\ (a,\ numeral\ b)\ \hat{}\ c = Frct\ (a\ \hat{}\ c,\ numeral\ b\ \hat{}\ c)$
$\langle proof \rangle$

lemma *Frct-less*: $Frct\ (a,\ numeral\ b) < Frct\ (c,\ numeral\ d) \longleftrightarrow a * numeral\ d < c * numeral\ b$

⟨*proof*⟩

lemma *Frct-le*: $Frct\ (a,\ numeral\ b) \le Frct\ (c,\ numeral\ d) \longleftrightarrow a * numeral\ d \le c * numeral\ b$
⟨*proof*⟩

lemma *Frct-equals*: $Frct\ (a,\ numeral\ b) = Frct\ (c,\ numeral\ d) \longleftrightarrow a * numeral\ d = c * numeral\ b$
⟨*proof*⟩

lemma *real-power-code*: $(Ratreal\ x)\ \widehat{}\ y = Ratreal\ (x\ \widehat{}\ y)$ ⟨*proof*⟩

lemmas *real-arith-code* =
real-plus-code real-minus-code real-times-code real-uminus-code real-inverse-code
real-divide-code real-power-code real-less-code real-less-eq-code real-equals-code

lemmas *rat-arith-code* =
Frct-add Frct-uminus Frct-diff Frct-mult Frct-inverse Frct-divide Frct-pow
Frct-less Frct-le Frct-equals

lemma *gcd-numeral-red*: $gcd\ (numeral\ x::int)\ (numeral\ y) = gcd\ (numeral\ y)\ (numeral\ x\ mod\ numeral\ y)$
⟨*proof*⟩

lemma *divmod-one*:
$divmod\ (Num.One)\ (Num.One) = (Numeral1,\ 0)$
$divmod\ (Num.One)\ (Num.Bit0\ x) = (0,\ Numeral1)$
$divmod\ (Num.One)\ (Num.Bit1\ x) = (0,\ Numeral1)$
$divmod\ x\ (Num.One) = (numeral\ x,\ 0)$
⟨*proof*⟩

lemmas *divmod-numeral-simps* =
div-0 div-by-0 mod-0 mod-by-0
fst-divmod [*symmetric*]
snd-divmod [*symmetric*]
divmod-cancel
divmod-steps [*simplified rel-simps if-True*] *divmod-trivial*
rel-simps

lemma *Suc-0-to-numeral*: $Suc\ 0 = Numeral1$ ⟨*proof*⟩
lemmas *Suc-to-numeral* = *Suc-0-to-numeral Num.Suc-1 Num.Suc-numeral*

lemma *rat-powr*:
$0\ powr\ y = 0$
$x > 0 \Longrightarrow x\ powr\ Ratreal\ (Frct\ (0,\ Numeral1)) = Ratreal\ (Frct\ (Numeral1,\ Numeral1))$
$x > 0 \Longrightarrow x\ powr\ Ratreal\ (Frct\ (numeral\ a,\ Numeral1)) = x\ \widehat{}\ numeral\ a$
$x > 0 \Longrightarrow x\ powr\ Ratreal\ (Frct\ (-numeral\ a,\ Numeral1)) = inverse\ (x\ \widehat{}\ numeral\ a)$

⟨*proof*⟩

lemmas *eval-numeral-simps* =
real-numeral-to-Ratreal real-arith-code rat-arith-code Num.arith-simps Rat.normalize-def fst-conv snd-conv gcd-0-int gcd-0-left-int gcd.bottom-right-bottom gcd.bottom-left-bottom
gcd-neg1-int gcd-neg2-int gcd-numeral-red zmod-numeral-Bit0 zmod-numeral-Bit1 power-numeral-simps
divmod-numeral-simps numeral-One [*symmetric*] *Groups.Let-0 Num.Let-numeral Suc-to-numeral power-numeral*
greaterThanLessThan-iff atLeastAtMost-iff atLeastLessThan-iff greaterThanAt-Most-iff rat-powr
Num.pow.simps Num.sqr.simps Product-Type.split of-int-numeral of-int-neg-numeral of-nat-numeral

⟨*ML*⟩

lemma *21254387548659589512*∗*314213523632464357453884361*∗*2342523623324234*∗*5643274385872417347474*
125617127386458243623293164829731643982142 86 powr 2 /
(*113024631297842312 3*+*23121237463108276484 2731842*∗*12247437838942436234745125126 3*)
\>
(*1231324451293124724354327976864574592947582931065120562384 4*::*real*)
⟨*proof*⟩

end

7 The proof methods

7.1 Master theorem and termination

theory *Akra-Bazzi-Method*
imports
Complex-Main
Akra-Bazzi
Master-Theorem
Eval-Numeral
begin

lemma *landau-symbol-ge-3-cong*:
assumes *landau-symbol L L′ Lr*
assumes $\bigwedge x$::′*a*::*linordered-semidom*. $x \geq 3 \Longrightarrow f\,x = g\,x$
shows *L at-top* (*f*) = *L at-top* (*g*)
⟨*proof*⟩

lemma *exp-1-lt-3*: *exp* (*1*::*real*) < *3*
⟨*proof*⟩

lemma *ln-ln-pos*:
assumes (*x*::*real*) ≥ *3*

shows $ln\ (ln\ x) > 0$
⟨*proof*⟩

definition *akra-bazzi-terms* **where**
akra-bazzi-terms $x_0\ x_1$ *bs ts* = (∀ *i*<*length bs*. *akra-bazzi-term* $x_0\ x_1$ (*bs*!*i*) (*ts*!*i*))

lemma *akra-bazzi-termsI*:
$(\bigwedge i.\ i <$ *length bs* $\Longrightarrow$ *akra-bazzi-term* $x_0\ x_1$ (*bs*!*i*) (*ts*!*i*)) $\Longrightarrow$ *akra-bazzi-terms* $x_0\ x_1$ *bs ts*
⟨*proof*⟩

lemma *master-theorem-functionI*:
assumes $\forall x \in \{x_0..<x_1\}.\ f\ x \geq 0$
assumes $\forall x \geq x_1.\ f\ x = g\ x + (\sum i<k.\ as\ !\ i * f\ ((ts\ !\ i)\ x))$
assumes $\forall x \geq x_1.\ g\ x \geq 0$
assumes ∀ *a*∈*set as*. $a \geq 0$
assumes *list-ex* ($\lambda a.\ a > 0$) *as*
assumes ∀ *b*∈*set bs*. $b \in \{0<..<1\}$
assumes $k \neq 0$
assumes *length as* = *k*
assumes *length bs* = *k*
assumes *length ts* = *k*
assumes *akra-bazzi-terms* $x_0\ x_1$ *bs ts*
shows *master-theorem-function* $x_0\ x_1$ *k as bs ts f g*
⟨*proof*⟩

lemma *akra-bazzi-term-measure*:
$x \geq x_1 \Longrightarrow$ *akra-bazzi-term* $0\ x_1\ b\ t \Longrightarrow (t\ x,\ x) \in$ *Wellfounded.measure* (λn::*nat*. *n*)
$x > x_1 \Longrightarrow$ *akra-bazzi-term* 0 (*Suc* x_1) $b\ t \Longrightarrow (t\ x,\ x) \in$ *Wellfounded.measure* (λn::*nat*. *n*)
⟨*proof*⟩

lemma *measure-prod-conv*:
((*a*, *b*), (*c*, *d*)) ∈ *Wellfounded.measure* (λx. *t* (*fst x*)) ⟷ (*a*, *c*) ∈ *Wellfounded.measure t*
((*e*, *f*), (*g*, *h*)) ∈ *Wellfounded.measure* (λx. *t* (*snd x*)) ⟷ (*f*, *h*) ∈ *Wellfounded.measure t*
⟨*proof*⟩

lemmas *measure-prod-conv′* = *measure-prod-conv*[**where** $t = \lambda x.\ x$]

lemma *akra-bazzi-termination-simps*:
fixes *x* :: *nat*
shows $a *$ *real x* / $b = a/b *$ *real x* *real x* / $b = 1/b *$ *real x*
⟨*proof*⟩

lemma *akra-bazzi-params-nonzeroI*:
length as = *length bs* $\Longrightarrow$

$(\forall a \in set\ as.\ a \geq 0) \Longrightarrow (\forall b \in set\ bs.\ b \in \{0<..<1\}) \Longrightarrow (\exists a \in set\ as.\ a > 0) \Longrightarrow$
akra-bazzi-params-nonzero (*length as*) *as bs* $\langle proof \rangle$

lemmas *akra-bazzi-p-rel-intros* =
akra-bazzi-params-nonzero.p-lessI[*rotated*, *OF - akra-bazzi-params-nonzeroI*]
akra-bazzi-params-nonzero.p-greaterI[*rotated*, *OF - akra-bazzi-params-nonzeroI*]
akra-bazzi-params-nonzero.p-leI[*rotated*, *OF - akra-bazzi-params-nonzeroI*]
akra-bazzi-params-nonzero.p-geI[*rotated*, *OF - akra-bazzi-params-nonzeroI*]
akra-bazzi-params-nonzero.p-boundsI[*rotated*, *OF - akra-bazzi-params-nonzeroI*]
akra-bazzi-params-nonzero.p-boundsI'[*rotated*, *OF - akra-bazzi-params-nonzeroI*]

lemma *eval-length*: $length\ [] = 0\ length\ (x \# xs) = Suc\ (length\ xs)$ $\langle proof \rangle$

lemma *eval-akra-bazzi-sum*:
$(\sum i<0.\ as!i * bs!i\ powr\ x) = 0$
$(\sum i<Suc\ 0.\ (a\#as)!i * (b\#bs)!i\ powr\ x) = a * b\ powr\ x$
$(\sum i<Suc\ k.\ (a\#as)!i * (b\#bs)!i\ powr\ x) = a * b\ powr\ x + (\sum i<k.\ as!i * bs!i\ powr\ x)$
$\langle proof \rangle$

lemma *eval-akra-bazzi-sum'*:
$(\sum i<0.\ as!i * f\ ((ts!i)\ x)) = 0$
$(\sum i<Suc\ 0.\ (a\#as)!i * f\ (((t\#ts)!i)\ x)) = a * f\ (t\ x)$
$(\sum i<Suc\ k.\ (a\#as)!i * f\ (((t\#ts)!i)\ x)) = a * f\ (t\ x) + (\sum i<k.\ as!i * f\ ((ts!i)\ x))$
$\langle proof \rangle$

lemma *akra-bazzi-termsI'*:
akra-bazzi-terms $x_0\ x_1$ [] []
akra-bazzi-term $x_0\ x_1\ b\ t \Longrightarrow$ *akra-bazzi-terms* $x_0\ x_1\ bs\ ts \Longrightarrow$ *akra-bazzi-terms* $x_0\ x_1\ (b\#bs)\ (t\#ts)$
$\langle proof \rangle$

lemma *ball-set-intros*: $(\forall x \in set\ [].\ P\ x)\ P\ x \Longrightarrow (\forall x \in set\ xs.\ P\ x) \Longrightarrow (\forall x \in set\ (x\#xs).\ P\ x)$
$\langle proof \rangle$

lemma *ball-set-simps*: $(\forall x \in set\ [].\ P\ x) = True\ (\forall x \in set\ (x\#xs).\ P\ x) = (P\ x \wedge (\forall x \in set\ xs.\ P\ x))$
$\langle proof \rangle$

lemma *bex-set-simps*: $(\exists x \in set\ [].\ P\ x) = False\ (\exists x \in set\ (x\#xs).\ P\ x) = (P\ x \vee (\exists x \in set\ xs.\ P\ x))$
$\langle proof \rangle$

lemma *eval-akra-bazzi-le-list-ex*:
$list\text{-}ex\ P\ (x\#y\#xs) \longleftrightarrow P\ x \vee list\text{-}ex\ P\ (y\#xs)$
$list\text{-}ex\ P\ [x] \longleftrightarrow P\ x$
$list\text{-}ex\ P\ [] \longleftrightarrow False$

$\langle proof \rangle$

lemma *eval-akra-bazzi-le-sum-list*:
$x \le sum\text{-}list\ [] \longleftrightarrow x \le 0$ $x \le sum\text{-}list\ (y\#ys) \longleftrightarrow x \le y + sum\text{-}list\ ys$
$x \le z + sum\text{-}list\ [] \longleftrightarrow x \le z$ $x \le z + sum\text{-}list\ (y\#ys) \longleftrightarrow x \le z + y + sum\text{-}list\ ys$
$\langle proof \rangle$

lemma *atLeastLessThanE*: $x \in \{a..<b\} \Longrightarrow (x \ge a \Longrightarrow x < b \Longrightarrow P) \Longrightarrow P$
$\langle proof \rangle$

lemma *master-theorem-preprocess*:
$\Theta(\lambda n::nat.\ 1) = \Theta(\lambda n::nat.\ real\ n\ powr\ 0)$
$\Theta(\lambda n::nat.\ real\ n) = \Theta(\lambda n::nat.\ real\ n\ powr\ 1)$
$O(\lambda n::nat.\ 1) = O(\lambda n::nat.\ real\ n\ powr\ 0)$
$O(\lambda n::nat.\ real\ n) = O(\lambda n::nat.\ real\ n\ powr\ 1)$

$\Theta(\lambda n::nat.\ ln\ (ln\ (real\ n))) = \Theta(\lambda n::nat.\ real\ n\ powr\ 0 * ln\ (ln\ (real\ n)))$
$\Theta(\lambda n::nat.\ real\ n * ln\ (ln\ (real\ n))) = \Theta(\lambda n::nat.\ real\ n\ powr\ 1 * ln\ (ln\ (real\ n)))$

$\Theta(\lambda n::nat.\ ln\ (real\ n)) = \Theta(\lambda n::nat.\ real\ n\ powr\ 0 * ln\ (real\ n)\ powr\ 1)$
$\Theta(\lambda n::nat.\ real\ n * ln\ (real\ n)) = \Theta(\lambda n::nat.\ real\ n\ powr\ 1 * ln\ (real\ n)\ powr\ 1)$
$\Theta(\lambda n::nat.\ real\ n\ powr\ p * ln\ (real\ n)) = \Theta(\lambda n::nat.\ real\ n\ powr\ p * ln\ (real\ n)\ powr\ 1)$
$\Theta(\lambda n::nat.\ ln\ (real\ n)\ powr\ p') = \Theta(\lambda n::nat.\ real\ n\ powr\ 0 * ln\ (real\ n)\ powr\ p')$
$\Theta(\lambda n::nat.\ real\ n * ln\ (real\ n)\ powr\ p') = \Theta(\lambda n::nat.\ real\ n\ powr\ 1 * ln\ (real\ n)\ powr\ p')$
$\langle proof \rangle$

lemma *akra-bazzi-term-imp-size-less*:
$x_1 \le x \Longrightarrow akra\text{-}bazzi\text{-}term\ 0\ x_1\ b\ t \Longrightarrow size\ (t\ x) < size\ x$
$x_1 < x \Longrightarrow akra\text{-}bazzi\text{-}term\ 0\ (Suc\ x_1)\ b\ t \Longrightarrow size\ (t\ x) < size\ x$
$\langle proof \rangle$

definition $CLAMP\ (f :: nat \Rightarrow real)\ x = (if\ x < 3\ then\ 0\ else\ f\ x)$
definition $CLAMP'\ (f :: nat \Rightarrow real)\ x = (if\ x < 3\ then\ 0\ else\ f\ x)$
definition $MASTER\text{-}BOUND\ a\ b\ c\ x = real\ x\ powr\ a * ln\ (real\ x)\ powr\ b * ln\ (ln\ (real\ x))\ powr\ c$
definition $MASTER\text{-}BOUND'\ a\ b\ x = real\ x\ powr\ a * ln\ (real\ x)\ powr\ b$
definition $MASTER\text{-}BOUND''\ a\ x = real\ x\ powr\ a$

lemma *ln-1-imp-less-3*:
$ln\ x = (1::real) \Longrightarrow x < 3$
$\langle proof \rangle$

lemma *ln-1-imp-less-3'*: $ln\ (real\ (x::nat)) = 1 \Longrightarrow x < 3$ $\langle proof \rangle$

lemma *ln-ln-nonneg*: $x \ge (3::real) \Longrightarrow ln\ (ln\ x) \ge 0$ $\langle proof \rangle$
lemma *ln-ln-nonneg'*: $x \ge (3::nat) \Longrightarrow ln\ (ln\ (real\ x)) \ge 0$ $\langle proof \rangle$

lemma *MASTER-BOUND-postproc*:
$CLAMP\ (MASTER\text{-}BOUND'\ a\ 0) = CLAMP\ (MASTER\text{-}BOUND''\ a)$
$CLAMP\ (MASTER\text{-}BOUND'\ a\ 1) = CLAMP\ (\lambda x.\ CLAMP\ (MASTER\text{-}BOUND''\ a)\ x * CLAMP\ (\lambda x.\ ln\ (real\ x))\ x)$
$CLAMP\ (MASTER\text{-}BOUND'\ a\ (numeral\ n)) =$
$CLAMP\ (\lambda x.\ CLAMP\ (MASTER\text{-}BOUND''\ a)\ x * CLAMP\ (\lambda x.\ ln\ (real\ x)\ \widehat{}\ numeral\ n)\ x)$
$CLAMP\ (MASTER\text{-}BOUND'\ a\ (-1)) =$
$CLAMP\ (\lambda x.\ CLAMP\ (MASTER\text{-}BOUND''\ a)\ x\ /\ CLAMP\ (\lambda x.\ ln\ (real\ x))\ x)$
$CLAMP\ (MASTER\text{-}BOUND'\ a\ (-numeral\ n)) =$
$CLAMP\ (\lambda x.\ CLAMP\ (MASTER\text{-}BOUND''\ a)\ x\ /\ CLAMP\ (\lambda x.\ ln\ (real\ x)\ \widehat{}\ numeral\ n)\ x)$
$CLAMP\ (MASTER\text{-}BOUND'\ a\ b) =$
$CLAMP\ (\lambda x.\ CLAMP\ (MASTER\text{-}BOUND''\ a)\ x * CLAMP\ (\lambda x.\ ln\ (real\ x)\ powr\ b)\ x)$

$CLAMP\ (MASTER\text{-}BOUND''\ 0) = CLAMP\ (\lambda x.\ 1)$
$CLAMP\ (MASTER\text{-}BOUND''\ 1) = CLAMP\ (\lambda x.\ (real\ x))$
$CLAMP\ (MASTER\text{-}BOUND''\ (numeral\ n)) = CLAMP\ (\lambda x.\ (real\ x)\ \widehat{}\ numeral\ n)$
$CLAMP\ (MASTER\text{-}BOUND''\ (-1)) = CLAMP\ (\lambda x.\ 1\ /\ (real\ x))$
$CLAMP\ (MASTER\text{-}BOUND''\ (-numeral\ n)) = CLAMP\ (\lambda x.\ 1\ /\ (real\ x)\ \widehat{}\ numeral\ n)$
$CLAMP\ (MASTER\text{-}BOUND''\ a) = CLAMP\ (\lambda x.\ (real\ x)\ powr\ a)$

and *MASTER-BOUND-UNCLAMP*:
$CLAMP\ (\lambda x.\ CLAMP\ f\ x * CLAMP\ g\ x) = CLAMP\ (\lambda x.\ f\ x * g\ x)$
$CLAMP\ (\lambda x.\ CLAMP\ f\ x\ /\ CLAMP\ g\ x) = CLAMP\ (\lambda x.\ f\ x\ /\ g\ x)$
$CLAMP\ (CLAMP\ f) = CLAMP\ f$
⟨*proof*⟩

context
begin

private lemma *CLAMP-*:
$landau\text{-}symbol\ L\ L'\ Lr \Longrightarrow L\ at\text{-}top\ (f{::}nat \Rightarrow real) \equiv L\ at\text{-}top\ (\lambda x.\ CLAMP\ f\ x)$
⟨*proof*⟩ **lemma** *UNCLAMP′-*:
$landau\text{-}symbol\ L\ L'\ Lr \Longrightarrow L\ at\text{-}top\ (CLAMP'\ (MASTER\text{-}BOUND\ a\ b\ c)) \equiv L\ at\text{-}top\ (MASTER\text{-}BOUND\ a\ b\ c)$
⟨*proof*⟩ **lemma** *UNCLAMP-*:
$landau\text{-}symbol\ L\ L'\ Lr \Longrightarrow L\ at\text{-}top\ (CLAMP\ f) \equiv L\ at\text{-}top\ (f)$
⟨*proof*⟩

lemmas *CLAMP* = *landau-symbols*[*THEN CLAMP-*]
lemmas *UNCLAMP′* = *landau-symbols*[*THEN UNCLAMP′-*]

lemmas *UNCLAMP* = *landau-symbols*[*THEN UNCLAMP-*]
end

lemma *propagate-CLAMP*:
  $CLAMP\ (\lambda x.\ f\ x * g\ x) = CLAMP'\ (\lambda x.\ CLAMP\ f\ x * CLAMP\ g\ x)$
  $CLAMP\ (\lambda x.\ f\ x\ /\ g\ x) = CLAMP'\ (\lambda x.\ CLAMP\ f\ x\ /\ CLAMP\ g\ x)$
  $CLAMP\ (\lambda x.\ inverse\ (f\ x)) = CLAMP'\ (\lambda x.\ inverse\ (CLAMP\ f\ x))$
  $CLAMP\ (\lambda x.\ real\ x) = CLAMP'\ (MASTER\text{-}BOUND\ 1\ 0\ 0)$
  $CLAMP\ (\lambda x.\ real\ x\ powr\ a) = CLAMP'\ (MASTER\text{-}BOUND\ a\ 0\ 0)$
  $CLAMP\ (\lambda x.\ real\ x\ \widehat{}\ a') = CLAMP'\ (MASTER\text{-}BOUND\ (real\ a')\ 0\ 0)$
  $CLAMP\ (\lambda x.\ ln\ (real\ x)) = CLAMP'\ (MASTER\text{-}BOUND\ 0\ 1\ 0)$
  $CLAMP\ (\lambda x.\ ln\ (real\ x)\ powr\ b) = CLAMP'\ (MASTER\text{-}BOUND\ 0\ b\ 0)$
  $CLAMP\ (\lambda x.\ ln\ (real\ x)\ \widehat{}\ b') = CLAMP'\ (MASTER\text{-}BOUND\ 0\ (real\ b')\ 0)$
  $CLAMP\ (\lambda x.\ ln\ (ln\ (real\ x))) = CLAMP'\ (MASTER\text{-}BOUND\ 0\ 0\ 1)$
  $CLAMP\ (\lambda x.\ ln\ (ln\ (real\ x))\ powr\ c) = CLAMP'\ (MASTER\text{-}BOUND\ 0\ 0\ c)$
  $CLAMP\ (\lambda x.\ ln\ (ln\ (real\ x))\ \widehat{}\ c') = CLAMP'\ (MASTER\text{-}BOUND\ 0\ 0\ (real\ c'))$
  $CLAMP'\ (CLAMP\ f) = CLAMP'\ f$
  $CLAMP'\ (\lambda x.\ CLAMP'\ (MASTER\text{-}BOUND\ a1\ b1\ c1)\ x * CLAMP'\ (MASTER\text{-}BOUND\ a2\ b2\ c2)\ x) =$
      $CLAMP'\ (MASTER\text{-}BOUND\ (a1{+}a2)\ (b1{+}b2)\ (c1{+}c2))$
  $CLAMP'\ (\lambda x.\ CLAMP'\ (MASTER\text{-}BOUND\ a1\ b1\ c1)\ x\ /\ CLAMP'\ (MASTER\text{-}BOUND\ a2\ b2\ c2)\ x) =$
      $CLAMP'\ (MASTER\text{-}BOUND\ (a1{-}a2)\ (b1{-}b2)\ (c1{-}c2))$
  $CLAMP'\ (\lambda x.\ inverse\ (MASTER\text{-}BOUND\ a1\ b1\ c1\ x)) = CLAMP'\ (MASTER\text{-}BOUND\ ({-}a1)\ ({-}b1)\ ({-}c1))$
$\langle proof \rangle$

lemma *numeral-assoc-simps*:
  $((a{::}real) + numeral\ b) + numeral\ c = a + numeral\ (b + c)$
  $(a + numeral\ b) - numeral\ c = a + neg\text{-}numeral\text{-}class.sub\ b\ c$
  $(a - numeral\ b) + numeral\ c = a + neg\text{-}numeral\text{-}class.sub\ c\ b$
  $(a - numeral\ b) - numeral\ c = a - numeral\ (b + c)$ $\langle proof \rangle$

lemmas *CLAMP-aux* =
  arith-simps numeral-assoc-simps of-nat-power of-nat-mult of-nat-numeral
  one-add-one numeral-One [*symmetric*]

lemmas *CLAMP-postproc* = *numeral-One*

context *master-theorem-function*
begin

lemma *master1-bigo-automation*:
  assumes $g \in O(\lambda x.\ real\ x\ powr\ p')$ $1 < (\sum i{<}k.\ as\ !\ i * bs\ !\ i\ powr\ p')$
  shows $f \in O(MASTER\text{-}BOUND\ p\ 0\ 0)$
$\langle proof \rangle$

lemma *master1-automation*:
  assumes $g \in O(MASTER\text{-}BOUND''\ p')$ $1 < (\sum i{<}k.\ as\ !\ i * bs\ !\ i\ powr\ p')$

eventually $(\lambda x.\ f\ x > 0)$ *at-top*
shows $f \in \Theta(\textit{MASTER-BOUND}\ p\ 0\ 0)$
⟨*proof*⟩

lemma *master2-1-automation*:
assumes $g \in \Theta(\textit{MASTER-BOUND}'\ p\ p')$ $p' < -1$
shows $f \in \Theta(\textit{MASTER-BOUND}\ p\ 0\ 0)$
⟨*proof*⟩

lemma *master2-2-automation*:
assumes $g \in \Theta(\textit{MASTER-BOUND}'\ p\ (-1))$
shows $f \in \Theta(\textit{MASTER-BOUND}\ p\ 0\ 1)$
⟨*proof*⟩

lemma *master2-3-automation*:
assumes $g \in \Theta(\textit{MASTER-BOUND}'\ p\ (p' - 1))$ $p' > 0$
shows $f \in \Theta(\textit{MASTER-BOUND}\ p\ p'\ 0)$
⟨*proof*⟩

lemma *master3-automation*:
assumes $g \in \Theta(\textit{MASTER-BOUND}''\ p')$ $1 > (\sum i{<}k.\ as\ !\ i * bs\ !\ i\ \textit{powr}\ p')$
shows $f \in \Theta(\textit{MASTER-BOUND}\ p'\ 0\ 0)$
⟨*proof*⟩

lemmas *master-automation* =
master1-automation master2-1-automation master2-2-automation
master2-2-automation master3-automation

⟨*ML*⟩

end

definition *arith-consts* $(x :: \textit{real})$ $(y :: \textit{nat})$ =
$(\textit{if}\ \neg\ (-x) + 3\ /\ x * 5 - 1 \le x \wedge \textit{True} \vee \textit{True} \longrightarrow \textit{True then}$
$x < \textit{inverse}\ 3\ \textit{powr}\ 21\ \textit{else}\ x = \textit{real}\ (\textit{Suc}\ 0\ \hat{}\ 2\ +$
$(\textit{if}\ 42 - x \le 1 \wedge 1\ \textit{div}\ y = y\ \textit{mod}\ 2 \vee y < \textit{Numeral1 then}\ 0\ \textit{else}\ 0)) + \textit{Numeral1})$

⟨*ML*⟩

hide-const *arith-consts*

⟨*ML*⟩

hide-const *CLAMP CLAMP′ MASTER-BOUND MASTER-BOUND′ MASTER-BOUND″*

end
theory *Akra-Bazzi-Approximation*

imports
Complex-Main
Akra-Bazzi-Method
HOL−Decision-Procs.Approximation
begin

context *akra-bazzi-params-nonzero*
begin

lemma *sum-alt*: $(\sum i<k.\ as!i * bs!i\ powr\ p') = (\sum i<k.\ as!i * exp\ (p' * ln\ (bs!i)))$
⟨*proof*⟩

lemma *akra-bazzi-p-rel-intros-aux*:
$1 < (\sum i<k.\ as!i * exp\ (p' * ln\ (bs!i))) \Longrightarrow p' < p$
$1 > (\sum i<k.\ as!i * exp\ (p' * ln\ (bs!i))) \Longrightarrow p' > p$
$1 \leq (\sum i<k.\ as!i * exp\ (p' * ln\ (bs!i))) \Longrightarrow p' \leq p$
$1 \geq (\sum i<k.\ as!i * exp\ (p' * ln\ (bs!i))) \Longrightarrow p' \geq p$
$(\sum i<k.\ as!i * exp\ (x * ln\ (bs!i))) \leq 1 \wedge (\sum i<k.\ as!i * exp\ (y * ln\ (bs!i))) \geq 1 \Longrightarrow p \in \{y..x\}$
$(\sum i<k.\ as!i * exp\ (x * ln\ (bs!i))) < 1 \wedge (\sum i<k.\ as!i * exp\ (y * ln\ (bs!i))) > 1 \Longrightarrow p \in \{y<..<x\}$
⟨*proof*⟩
end

lemmas *akra-bazzi-p-rel-intros-exp* =
akra-bazzi-params-nonzero.akra-bazzi-p-rel-intros-aux[*rotated*, *OF* - *akra-bazzi-params-nonzeroI*]

lemma *eval-akra-bazzi-sum*:
$(\sum i<0.\ as!i * exp\ (x * ln\ (bs!i))) = 0$
$(\sum i<Suc\ 0.\ (a\#as)!i * exp\ (x * ln\ ((b\#bs)!i))) = a * exp\ (x * ln\ b)$
$(\sum i<Suc\ k.\ (a\#as)!i * exp\ (x * ln\ ((b\#bs)!i))) = a * exp\ (x * ln\ b) +$
$(\sum i<k.\ as!i * exp\ (x * ln\ (bs!i)))$
⟨*proof*⟩

⟨*ML*⟩

end

8 Examples

theory *Master-Theorem-Examples*
imports
Complex-Main
Akra-Bazzi-Method

Akra-Bazzi-Approximation
begin

8.1 Merge sort

function *merge-sort-cost* :: $(nat \Rightarrow real) \Rightarrow nat \Rightarrow real$ **where**
merge-sort-cost t $0 = 0$
| *merge-sort-cost* t $1 = 1$
| $n \geq 2 \Longrightarrow$ *merge-sort-cost* t $n =$
merge-sort-cost t $(nat\ \lfloor real\ n\ /\ 2\rfloor)$ + *merge-sort-cost* t $(nat\ \lceil real\ n\ /\ 2\rceil) + t$
n
⟨*proof*⟩
termination ⟨*proof*⟩

lemma *merge-sort-nonneg*[*simp*]: $(\bigwedge n.\ t\ n \geq 0) \Longrightarrow$ *merge-sort-cost* $t\ x \geq 0$
⟨*proof*⟩

lemma $t \in \Theta(\lambda n.\ real\ n) \Longrightarrow (\bigwedge n.\ t\ n \geq 0) \Longrightarrow$ *merge-sort-cost* $t \in \Theta(\lambda n.\ real$
$n * ln\ (real\ n))$
⟨*proof*⟩

8.2 Karatsuba multiplication

function *karatsuba-cost* :: $nat \Rightarrow real$ **where**
karatsuba-cost $0 = 0$
| *karatsuba-cost* $1 = 1$
| $n \geq 2 \Longrightarrow$ *karatsuba-cost* $n =$
$3 *$ *karatsuba-cost* $(nat\ \lceil real\ n\ /\ 2\rceil) + real\ n$
⟨*proof*⟩
termination ⟨*proof*⟩

lemma *karatsuba-cost-nonneg*[*simp*]: *karatsuba-cost* $n \geq 0$
⟨*proof*⟩

lemma *karatsuba-cost* $\in O(\lambda n.\ real\ n\ powr\ log\ 2\ 3)$
⟨*proof*⟩

lemma *karatsuba-cost-pos*: $n \geq 1 \Longrightarrow$ *karatsuba-cost* $n > 0$
⟨*proof*⟩

lemma *karatsuba-cost* $\in \Theta(\lambda n.\ real\ n\ powr\ log\ 2\ 3)$
⟨*proof*⟩

8.3 Strassen matrix multiplication

function *strassen-cost* :: $nat \Rightarrow real$ **where**
strassen-cost $0 = 0$
| *strassen-cost* $1 = 1$
| $n \geq 2 \Longrightarrow$ *strassen-cost* $n = 7 *$ *strassen-cost* $(nat\ \lceil real\ n\ /\ 2\rceil) + real\ (n\hat{\ }2)$
⟨*proof*⟩

termination ⟨*proof*⟩

lemma *strassen-cost-nonneg*[*simp*]: $strassen\text{-}cost\ n \geq 0$
⟨*proof*⟩

lemma $strassen\text{-}cost \in O(\lambda n.\ real\ n\ powr\ log\ 2\ 7)$
⟨*proof*⟩

lemma *strassen-cost-pos*: $n \geq 1 \Longrightarrow strassen\text{-}cost\ n > 0$
⟨*proof*⟩

lemma $strassen\text{-}cost \in \Theta(\lambda n.\ real\ n\ powr\ log\ 2\ 7)$
⟨*proof*⟩

8.4 Deterministic select

function *select-cost* :: *nat* ⇒ *real* **where**
$n \leq 20 \Longrightarrow select\text{-}cost\ n = 0$
$|\ n > 20 \Longrightarrow select\text{-}cost\ n =$
$select\text{-}cost\ (nat\ \lfloor real\ n\ /\ 5\rfloor) + select\text{-}cost\ (nat\ \lfloor 7 * real\ n\ /\ 10\rfloor + 6) + 12 * real\ n\ /\ 5$
⟨*proof*⟩
termination ⟨*proof*⟩

lemma $select\text{-}cost \in \Theta(\lambda n.\ real\ n)$
⟨*proof*⟩

8.5 Decreasing function

function *dec-cost* :: *nat* ⇒ *real* **where**
$n \leq 2 \Longrightarrow dec\text{-}cost\ n = 1$
$|\ n > 2 \Longrightarrow dec\text{-}cost\ n = 0.5 * dec\text{-}cost\ (nat\ \lfloor real\ n\ /\ 2\rfloor) + 1\ /\ real\ n$
⟨*proof*⟩
termination ⟨*proof*⟩

lemma $dec\text{-}cost \in \Theta(\lambda x{::}nat.\ ln\ x\ /\ x)$
⟨*proof*⟩

8.6 Example taken from Drmota and Szpakowski

function *drmota1* :: *nat* ⇒ *real* **where**
$n < 20 \Longrightarrow drmota1\ n = 1$
$|\ n \geq 20 \Longrightarrow drmota1\ n = 2 * drmota1\ (nat\ \lfloor real\ n/2\rfloor) + 8/9 * drmota1\ (nat\ \lfloor 3*real\ n/4\rfloor) + real\ n\hat{\ }2\ /\ ln\ (real\ n)$
⟨*proof*⟩
termination ⟨*proof*⟩

lemma $drmota1 \in \Theta(\lambda n{::}real.\ n\hat{\ }2 * ln\ (ln\ n))$
⟨*proof*⟩

function *drmota2* :: *nat* ⇒ *real* **where**
  $n < 20 \Longrightarrow$ *drmota2* $n = 1$
| $n \geq 20 \Longrightarrow$ *drmota2* $n = 1/3 * $ *drmota2* (*nat* ⌊*real* $n/3 + 1/2$⌋) $+ 2/3 *$ *drmota2* (*nat* ⌊$2*$*real* $n/3 - 1/2$⌋) $+ 1$
⟨*proof*⟩
termination ⟨*proof*⟩

lemma *drmota2* $\in \Theta(\lambda x.$ *ln* (*real* x))
  ⟨*proof*⟩

lemma *boncelet-phrase-length*:
  fixes $p\ \delta$ :: *real* **assumes** p: $p > 0$ $p < 1$ **and** δ: $\delta > 0$ $\delta < 1$ $2*p + \delta < 2$
  fixes d :: *nat* ⇒ *real*
  defines $q \equiv 1 - p$
  assumes *d-nonneg*: $\bigwedge n.\ d\ n \geq 0$
  assumes *d-rec*: $\bigwedge n.\ n \geq 2 \Longrightarrow d\ n = 1 + p * d$ (*nat* ⌊$p *$ *real* $n + \delta$⌋) $+ q * d$ (*nat* ⌊$q *$ *real* $n - \delta$⌋)
  shows $d \in \Theta(\lambda x.$ *ln* $x)$
  ⟨*proof*⟩

8.7 Transcendental exponents

function *foo-cost* :: *nat* ⇒ *real* **where**
  $n < 200 \Longrightarrow$ *foo-cost* $n = 0$
| $n \geq 200 \Longrightarrow$ *foo-cost* $n =$
    foo-cost (*nat* ⌊*real* n / 3⌋) + *foo-cost* (*nat* ⌊$3 *$ *real* n / 4⌋ + 42) + *real* n
⟨*proof*⟩
termination ⟨*proof*⟩

lemma *foo-cost-nonneg* [*simp*]: *foo-cost* $n \geq 0$
  ⟨*proof*⟩

lemma *foo-cost* $\in \Theta(\lambda n.$ *real* n *powr* *akra-bazzi-exponent* [1,1] [1/3,3/4])
⟨*proof*⟩

lemma *akra-bazzi-exponent* [1,1] [1/3,3/4] ∈ {1.1519623..1.1519624}
  ⟨*proof*⟩

8.8 Functions in locale contexts

locale *det-select* =
  fixes b :: *real*
  assumes b: $b > 0$ $b < 7/10$
begin

function *select-cost'* :: *nat* ⇒ *real* **where**
  $n \leq 20 \Longrightarrow$ *select-cost'* $n = 0$
| $n > 20 \Longrightarrow$ *select-cost'* $n =$

$\quad$ *select-cost'* (*nat* ⌊*real n* / *5*⌋) + *select-cost'* (*nat* ⌊*b* ∗ *real n*⌋ + *6*) + *6* ∗ *real n* + *5*
⟨*proof*⟩
termination ⟨*proof*⟩

lemma $a \geq 0 \Longrightarrow$ *select-cost'* $\in \Theta(\lambda n.\ real\ n)$
⟨*proof*⟩

end

8.9 Non-curried functions

function *baz-cost* :: *nat* × *nat* ⇒ *real* **where**
$n \leq 2 \Longrightarrow$ *baz-cost* (*a*, *n*) = *0*
| $n > 2 \Longrightarrow$ *baz-cost* (*a*, *n*) = *3* ∗ *baz-cost* (*a*, *nat* ⌊*real n* / *2*⌋) + *real a*
⟨*proof*⟩
termination ⟨*proof*⟩

lemma *baz-cost-nonneg* [*simp*]: $a \geq 0 \Longrightarrow$ *baz-cost* (*a*, *n*) ≥ *0*
⟨*proof*⟩

lemma
assumes $a > 0$
shows (λ*x*. *baz-cost* (*a*, *x*)) ∈ Θ(λ*x*. *x powr log 2 3*)
⟨*proof*⟩

function *bar-cost* :: *nat* × *nat* ⇒ *real* **where**
$n \leq 2 \Longrightarrow$ *bar-cost* (*a*, *n*) = *0*
| $n > 2 \Longrightarrow$ *bar-cost* (*a*, *n*) = *3* ∗ *bar-cost* (*2* ∗ *a*, *nat* ⌊*real n* / *2*⌋) + *real a*
⟨*proof*⟩
termination ⟨*proof*⟩

8.10 Ham-sandwich trees

function *ham-sandwich-cost* :: *nat* ⇒ *real* **where**
$n < 4 \Longrightarrow$ *ham-sandwich-cost n* = *1*
| $n \geq 4 \Longrightarrow$ *ham-sandwich-cost n* =
ham-sandwich-cost (*nat* ⌊*n*/*4*⌋) + *ham-sandwich-cost* (*nat* ⌊*n*/*2*⌋) + *1*
⟨*proof*⟩
termination ⟨*proof*⟩

lemma *ham-sandwich-cost-pos* [*simp*]: *ham-sandwich-cost n* > *0*
⟨*proof*⟩

The golden ratio

definition φ = ((*1* + *sqrt 5*) / *2* :: *real*)

lemma *φ-pos* [*simp*]: $\varphi > 0$ **and** *φ-nonneg* [*simp*]: $\varphi \geq 0$ **and** *φ-nonzero* [*simp*]: $\varphi \neq 0$

⟨*proof*⟩

lemma *ham-sandwich-cost* $\in \Theta(\lambda n.\ n\ powr\ (log\ 2\ \varphi))$
⟨*proof*⟩

end

References

[1] M. Akra and L. Bazzi. On the solution of linear recurrence equations. *Computational Optimization and Applications*, 10(2):195–210, 1998.

[2] T. Leighton. Notes on better Master theorems for divide-and-conquer recurrences. 1996.